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Abstract

In the current paper, distributional analyses were used to provide a detailed description of addition and multiplication latencies. University students (n = 32) solved single-digit problems and reported their solution methods. Mean response times were decomposed into mu, reflecting the position of the distribution, and tau, reflecting the skew of the distribution (mean RT = mu + tau) for retrievers, occasional procedure users, and frequent procedure users. By decomposing the mean we were able to determine if observed effects of problem size and group reflected an overall slowing of responses (reflected in mu) or a slowing on some trials only (reflected in tau). Findings provide evidence for tau as an index of procedure use and highlight differences across operations in the locus of the *problem-size effect*, a robust finding that larger problems (e.g., 8 x 7, 9 + 6) take longer to solve than smaller problems (e.g., 4 x 3, 2 + 4).

Introduction

North American adults use a variety of procedures to solve single-digit arithmetic problems, supplementing the direct retrieval of facts from memory. Studying adults' solution procedures is vital to an understanding of how we represent and process mathematical information. Research on adults' selection of procedures has provided insights into the most fundamental phenomena in mathematical cognition. Adults report a number of non-retrieval procedures for solving simple arithmetic problems including: *transformation* (e.g., 8 + 3 = 8 + 2 + 1; 3 x 4 = 3 x 3 + 3) and *counting* (e.g., 8 + 3 = 8, 9, 10, 11). The percentage of procedure use reported varies by operation and problem size. Participants report greater reliance on non-retrieval procedures for addition than for multiplication, and for large than for small problems (Campbell & Xue, 2001; Hecht, 1999).

The most robust effect in mathematical cognition is the *problem-size effect*, the finding that larger problems (e.g., 8 x 7, 9 + 6) take longer to solve and are more prone to errors than smaller problems (e.g., 4 x 3, 2 + 4). The problem-size effect is greater in multiplication than addition and greater for North American than Chinese educated adults (Campbell & Xue, 2001). LeFevre, Sadesky, and Bisanz (1996) found that the problem-size effect in addition was all but eliminated when they separately analyzed trials on which participants reported using direct retrieval of facts from memory. Similarly, LeFevre, Bisanz, et al. (1996) found that for multiplication, the problem-size effect was substantially reduced when only retrieval trials were considered. These findings suggest that the problem-size effect may be due in large part to the use of procedures other than direct retrieval on larger (versus smaller) problems. Moreover, the loci of the problem-size effect may be different across individuals who use different solution methods. Campbell & Xue (2001) outlined three potential sources of the problem-size effect: reduced efficiency of retrieval, reduced efficiency of non-retrieval solution procedures, and greater use of procedures for large problems.

Current models of adult arithmetic performance (Campbell, 1995; Verguts & Fias, 2005) are limited to retrieval processes and do not incorporate non-retrieval solution procedures. Thus, existing models are limited to the reduced retrieval efficiency account of the problem-size effect. Campbell's *network interference model* (1995) is an implemented model of single-digit addition and multiplication performance. In the network interference model, the problem-size effect in both multiplication and addition arises due to magnitude-related retrieval interference, such that the magnitudes of larger problem answers are less distinguishable. The model produces a problem-size effect in addition and multiplication, with a greater effect in multiplication.

Verguts and Fias' *interacting neighbors model* (2005) is an implemented neural network model of single-digit multiplication. In the interacting neighbors model, the problem-size effect arises due to the structure of the network. The network is structured similar to a multiplication table. Problem nodes receive excitation and inhibition from neighboring nodes based on the agreement or disagreement of answers in the ones and tens place. Large problems are subject to greater inhibition and reduced excitation because neighboring nodes have conflicting digits in the ones and tens place. The model produces a problem-size effect in multiplication.

Penner-Wilger and colleagues (Campbell & Penner-Wilger, 2006; LeFevre, DeStefano, Penner-Wilger, & Daley, 2006; Penner-Wilger, Leth-Steensen, & LeFevre, 2002) used distributional analyses of response latencies in arithmetic tasks to provide a detailed description of arithmetic performance. By applying the ex-Gaussian distributional model to latency distributions, mean response times can be decomposed into components mu and tau (Heathcote, 1996). Mu reflects the position or leading edge of the distribution; increases in mu reflect an *overall* slowing or shift of the distribution. Tau reflects the positive skew of the distribution; increases in tau reflect a slowing *on some trials* or skewing of the distribution. Mean response time is simply the sum of mu and tau. Distributional analyses provide more detailed descriptions than traditional measures (i.e., mean or median) and, like response times, distributional shapes can be used to make inferences about underlying processes. Thus, distributional analysis allows for the testing of hypotheses that are indistinguishable when comparing mean response times.

Penner-Wilger, Leth-Steensen, and LeFevre (2002) used distributional analyses to explore differences in the locus of the problem-size effect across cultures for multiplication. Participants had been educated in either Canada or China. Similar groups of individuals from Canada and China had previously been shown to solve problems using different methods on the basis of self-reports (Campbell & Xue, 2001). Chinese groups reported almost exclusive use of direct retrieval of answers from memory. In contrast, North American groups reported using a variety of solution procedures.

Penner-Wilger et al. (2002) predicted that mu would reflect differences in retrieval efficiency for small and large problems. They assumed that retrieval is slower as the strength of association between the problem and answer is reduced or as interference from competing answers is increased (Campbell, 1995). Furthermore, Penner-Wilger et al. predicted that tau would reflect differences in the proportion or efficiency of non-retrieval procedures for small and large problems. Non-retrieval procedures, such as repeated addition or transformation are generally slower than retrieval, especially for large problems (LeFevre et al., 1996; LeFevre, Bisanz et al., 1996). In accord with these predictions, ex-Gaussian analyses showed that the problem-size effect for participants educated in China was reflected only in mu. For participants educated in Canada the problem-size effect was reflected in both mu and tau. The different pattern of results for the two groups supported the assumption that the underlying mental processes of the two groups differed. Moreover, tau seemed to be indicative of the extent to which non-retrieval solution procedures were used. Although preliminary, Penner-Wilger et al.'s findings suggested that tau showed promise as an unbiased indicator of procedure use.

The primary goal of the current study was to explore whether differences across operations in the loci of the problem-size effect were evident at the detailed level afforded by distributional analyses. To this end, we (a) replicated the multiplication condition of Penner-Wilger et al. (2002) with North American participants, (b) extended the analysis to single-digit addition, and (c) collected self-reports of solution methods to further examine the relation between reported procedure use and tau. As in LeFevre et al. (2003), participants were grouped into three categories (retrievers, occasional procedure users, and frequent procedure users) based on their reported percentage of retrieval use for each operation. A secondary goal of the current study was to further evaluate the use of tau as an objective indicator of procedure use.

Method

Participants

Thirty-two graduate or upper-level undergraduate students (19 male, 13 female, median age 23 years) participated in this experiment and received a $20 honorarium. All participants had completed their primary and secondary education in North America.

Materials

Participants completed 36 trials within each of eight blocks per session for a total of 288 problems per session and 576 problems in total over two sessions. Stimuli consisted of all pairings of operands 2 through 9 and their reverse. Each operand pair appeared once in each block with its reverse appearing in the following block. Tie problems (e.g., 3 + 3) appeared once in each block. Within each block, problems for each trial were presented in random order. As commonly operationalized in the mathematical cognition literature, small problems were defined as those operand pairings with products less than or equal to 25, and large problems were defined as those with products greater than 25.

Procedure

Testing of multiplication and addition occurred at separate sessions one week apart, with the order of operation counterbalanced. Participants were seated comfortably in a quiet room in front of a computer monitor. Instructions were displayed on the monitor and read by the experimenter. Both accuracy and speed in the arithmetic task were stressed. Participants were also given instructions to indicate, after solving each problem, their solution procedure from a list of: transform, count, remember, and other. They were given examples of each solution method before the experiment began. This technique was previously used by Campbell & Timm (2000), who found that participants' self-reported use of solution methods was comparable to previous results from open-ended self-reports (e.g., "Tell me how you solved this problem") both in terms of amount and pattern of reported procedure use.

For each experimental trial, an asterisk was presented in the centre of the screen, and flashed twice, with the problem appearing on what would have been the third flash, one second from the start of the trial (e.g., 2 x 2 =). The operation sign appeared in the same position as the asterisk. The problem remained on the screen until the participant made a verbal response, or until a 10 s deadline was reached. The experimenter recorded the response. Immediately following, the participant was prompted to indicate how they solved the problem from a list of possible procedures: Transform, Count, Remember, Other. The experimenter recorded the response and the next trial began. The experimental trials lasted approximately 40 minutes, and participants were given a break halfway through.

Results

Participants were grouped using self-reports of solution methods. Separate groups were created for addition and multiplication based on percentile ranks of percentage reported use of retrieval. *Retrievers* were defined as individuals who reported a percentage of retrieval that was greater than or equal to the 75th percentile of the group of participants (i.e., 95% retrieval for addition, 9 participants; 99% for multiplication, 15 participants). *Occasional procedure users* were defined as individuals who reported retrieval less than the 75th percentile but greater than the 25th percentile (54-94% retrieval for addition, 15 participants; 77-98% for multiplication, 9 participants). *Frequent procedure users* were defined as individuals who reported retrieval less than the 25th percentile (53% retrieval for addition, 8 participants; 78% for multiplication, 8 participants).

All reported findings are significant at $p < .05$ unless otherwise noted. Dependent measures included percent error, mean response time, and the ex-Gaussian parameters mu and tau, obtained by fitting the ex-Gaussian distribution to the data of individual participants, separately for small and large problems. For each dependent measure, a 2 (problem size: large, small) x 3 (group: retrievers, occasional, frequent) repeated measures ANOVA was performed separately for addition and multiplication with group as a between-participants factor. For all analyses, post-hoc pairwise comparisons were used to compare across groups. Of the 18432 total trials, 191 (1.0 %) were invalid due to either a premature firing of the voice-activated relay or a failure of the participant to respond within the 10 s time limit, and 531 (2.9 %) of the responses were incorrect, leaving 17710 (96.1 %) correct trials available for latency analyses. Figure 1 shows the problem size x group interactions for all latency variables; 95% confidence intervals were calculated based on the approach recommended by Loftus & Masson (1994).

Addition

Percent error Participants made more errors on large problems than on small problems (3.3 vs. 1.1 %), F (1,29) = 12.7, MSE = 5.3. Percent error also varied by group, F (2,29) = 4.4, MSE = 8.0, such that frequent procedure users made significantly more errors (3.8 %) than occasional procedure users and retrievers (1.3 and 1.5 %, respectively). This finding is not surprising given the number of steps implemented in procedures on which to make an error. The interaction between problem size and group approached significance, F (2,29) = 3.2, MSE = 5.3, p =.05.

Mean response time Participants solved large problems more slowly than small problems (1181 vs. 935 ms), F (1,29) = 71.0, MSE = 12647. Response times also varied by group, F (2,29) = 11.9, MSE = 127278, such that frequent procedure users (1386 ms) solved problems significantly more slowly than occasional procedure users and retrievers (987 and 800 ms, respectively). This finding is expected given that procedures take longer to implement than retrieval. As hypothesized, there was also an interaction between problem size and group, F (2,29) = 12.4, MSE = 12647. As shown in Figure 1 (top left panel), retrievers did not show a significant problem-size effect. In contrast, procedure users (both frequent and occasional) solved large problems more slowly than small problems, reflecting their greater reliance on less-efficient procedures for large problems, consistent with the findings of LeFevre et al. (1996).

Mu In contrast to mean response time, there was no significant problem-size effect in mu. Mu varied by group, F (2,29) = 4.2, MSE = 65718, such that mu was significantly larger for frequent procedure users (895 ms) than occasional procedure users and retrievers (707 and 651 ms respectively), thus, frequent procedure users showed an overall slowing in addition response times. Also in contrast to mean response time, there was no significant interaction between problem size and group. As shown in Figure 1 (center left panel), the problem-size effect evident in mean response times was not significant for any of the groups in mu.

Tau As hypothesized, tau was greater for large problems than small problems, (406 vs. 206 ms), corresponding to the finding that participants use a greater proportion of non-retrieval procedures for large than small problems, F (1,29) = 40.1, MSE = 14921. Also as hypothesized, tau varied by group, F (2,29) = 17.4, MSE = 28739, such that tau was significantly greater for frequent procedure users (490 ms) than occasional procedure users (279 ms) and occasional users than retrievers (149 ms). Thus, tau reflected non-retrieval procedure use, supporting the position that tau values can be used to distinguish the three groups. As hypothesized, there was also an interaction between problem size and group, F (2,29) = 7.2, MSE = 14921. As shown in Figure 1 (bottom left panel), retrievers did not show a problem-size effect in tau. In contrast, procedure users (both frequent and occasional) showed significant problem-size effects, reflecting a greater reliance on non-retrieval procedures, decreased efficiency of such procedures, or both, for large problems.

In summary, for addition we found a problem-size effect, effect of group, and problem size x group interaction in mean response times. When mean response time was decomposed into mu and tau, the problem-size effect was isolated to tau, reflecting a slowing on some trials rather than overall slower latencies for large problems. This finding is consistent with the view that North American adults show an increased reliance on procedures other than retrieval for large problems as compared to small problems. The effect of group was reflected both in mu, suggesting an overall slowing for frequent procedure users, and in tau, suggesting an increased slowing on some trials as procedure use increases. The finding that retrievers do not show a problem-size effect provides additional support for these interpretations of performance. Tau values serve to distinguish all three groups in addition, thus, tau proves to be a strong indicator of non-retrieval procedure use for addition. The problem size x group interaction in mean response time arises in tau, such that procedure users show a problem-size effect in tau, whereas retrievers do not.

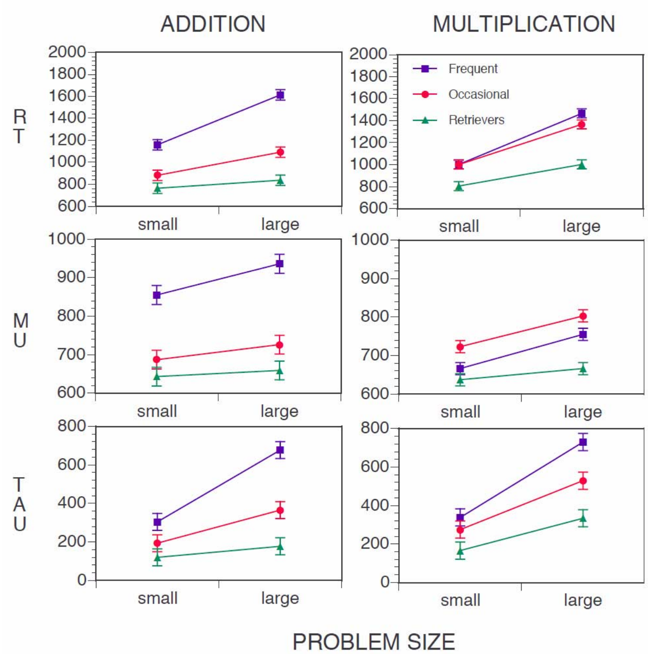

Figure 1: Mean response time and component ex-Gaussian parameters mu and tau (in ms), for addition and multiplication as a function of problem size for frequent procedure users, occasional procedure users, and retrievers.

Multiplication

Percent error Participants made more errors on large problems than on small problems (5.5 vs. 1.3 %), F (1,29) = 9.2, MSE = 27.5. There was no significant main effect of group and no significant interaction between problem size and group.

Mean response time Participants solved large problems more slowly than small problems (1278 vs. 936 ms), F (1,29) = 101.2, MSE = 17171. Response times also varied by group, F (2,29) = 6.5, MSE = 112740, such that frequent procedure users (1234 ms) solved problems significantly more slowly than retrievers (904 ms), which is expected given that procedures take longer to implement than retrieval. As hypothesized, there was also an interaction between problem size and group, F (2,29) = 5.9, MSE = 17170.5. As shown in Figure 1 (top right panel), all groups showed a problem-size effect but the effect is smaller for retrievers than for other participants, consistent with the findings of LeFevre, Bisanz, et al. (1996).

Mu Consistent with the findings of Penner-Wilger et al. (2002), mu was greater for large than for small problems (741 vs. 675 ms), suggesting an overall slowing due to greater interference or decreased associative strength between large problems and answers, F (1,29) = 34.5, MSE = 1896. Mu also varied with group, F (2,29) = 3.9, MSE = 18229, such that it was larger for occasional procedure users than retrievers (763 versus 651 ms respectively). Although it may seem unexpected that mu is larger for occasional procedure users, the performance of frequent and occasional procedure users is indistinguishable in mean response times yet reflects a trade-off between mu and tau such that mu is significantly higher for occasional procedure users but tau is significantly higher for frequent procedure users. The interaction between group and problem size approached significance, F (2,29) = 3.2, MSE = 1896.4, p = .05. As shown in Figure 1 (center right panel), retrievers did not show a problem-size effect. In contrast, procedure users (both frequent and occasional) showed significant problem-size effects suggesting that retrieval efficiency was reduced for large problems.

Tau As hypothesized, tau was greater for large problems than small problems (531 vs. 260 ms), corresponding to the finding that participants use a greater proportion of non-retrieval procedures for large than small problems, F (1,29) = 71.6, MSE = 15249.6. Also as hypothesized, tau varied with group, F (2,29) = 7.1, MSE = 61137.2, such that it was significantly smaller for retrievers (250 ms) than occasional procedure users and frequent procedure users (402 and 534 ms respectively). Thus, tau differentiated between retrievers and procedure users (both frequent and occasional). In multiplication, however, tau does not distinguish the groups as clearly as for addition, where all three groups could be differentiated based on tau values. The proportion of non-retrieval procedures reported for multiplication is notably less than for addition, making distinctions among groups more challenging. As hypothesized, there was an interaction between problem size and group, F (2,29) = 4.2, MSE = 15249. As shown in Figure 1 (bottom right panel), though all groups show a problem-size effect, the effect is reduced for retrievers.

In summary, for multiplication we again found a problem-size effect, effect of group, and problem size x group interaction in mean response times. When mean response time was decomposed into mu and tau, we found that the problem-size effect arises both in mu and in tau, thus reflecting an overall slowing as well as a slowing on some trials for large problems. This finding replicates Penner-Wilger et al. (2002) and is consistent with reduced retrieval efficiency, increased reliance on non-retrieval procedures, and reduced efficiency of such procedures for large problems. The effect of group arises both in mu, reflecting an overall slowing for occasional procedure users (compared to retrievers), and in tau, reflecting an increased slowing on some trials for procedure users. Tau values serve to distinguish retrievers from procedure users in multiplication. Thus, tau still proves to be an indicator of non-retrieval procedure use for multiplication, despite the lower levels of procedure use within the groups. The problem size x group interaction in mean response time arises in tau, such that the problem-size effect in tau increases with procedure use.

Discussion

Across operations, the same pattern of effects is evident in mean response times: a problem-size effect, effect of group, and problem size x group interaction. These same patterns of results, however, arise from different distributional effects.

The current experiment revealed a notable difference across operations; the problem-size effect in multiplication reflects both shifting and skewing of the response time distribution, whereas, the problem-size effect in addition reflects only skewing of the distribution. This difference is important, as the problem-size effect is a 'landmark finding' in mathematical cognition (Zbrodoff & Logan, 2005). Penner-Wilger et al. (2002) concluded that the problem-size effect in multiplication (for participants using a mix of solution methods) arises both from reduced retrieval efficiency (a mu or shift effect) and reduced procedural efficiency or increased reliance on procedures (tau or skew effect). The current findings support that conclusion.

For addition, however, retrievers do not show a problem-size effect in mu or tau, suggesting retrieval efficiency is not reduced for large addition problems. This crucial difference is an important criterion for models of arithmetic performance. The problem-size effect for addition is reflected only in tau suggesting that the problem-size effect arises from reduced procedural efficiency or increased reliance on procedures. Thus, our results suggest that the loci of the problem-size effect are different across operations.

All models of arithmetic performance must account for the problem-size effect. As discussed previously, current models of adult arithmetic performance do not incorporate non-retrieval solution procedures, thus, converging evidence suggests they cannot fully account for North American performance. We can, however, evaluate the current models based on the present findings. For addition, we found that the problem-size effect was reflected in tau, supporting the view that for addition the problem-size effect arises due to reduced efficiency of procedures and/or increased use of procedures. Additional support for this view comes from participants' self reports; procedure use increased in our sample from 21.4 % for small problems to 36.3% for large problems. Moreover, retrievers did not show a problem-size effect in addition. These findings suggest that the addition portion of Campbell's (1995) model may be inaccurate. If the problem-size effect in addition arises from procedure use, then there should not be an effect in a retrieval-only model. Campbell, however, does find a problem-size effect for addition. These findings also suggest that the Verguts and Fias (2005) model could not be extended to addition, as the authors themselves conclude.

For multiplication, we found that the problem-size effect was reflected in both mu and tau, supporting the view that for multiplication the problem-size effect arises due to reduced retrieval efficiency as well as reduced efficiency of procedures and/or increased use of procedures. Additional support for this view comes from participants self reports; procedure use increased in our sample from 5.3 % of small problems to 14.0% for large problems. Moreover, retrievers did show a problem-size effect in multiplication. Thus, current retrieval models can account for a portion of the problem-size effect in multiplication. Hecht (1999, 2006) advocates for the modification of Shrager and Seigler's (1998) model of children's arithmetic performance to a model of adult performance. The Strategy Choice and Discovery Simulation (SCADS) model incorporates both retrieval and non-retrieval solution procedures and, as such, holds promise for fully accounting for the problem-size effect.

Conclusion

In the current paper, distributional analyses were used to provide a detailed description of single-digit addition and multiplication latencies. The results provide detailed criteria that must be met by future models of arithmetic performance. Two findings are of particular interest. First, the distributional analyses revealed differences across operations in the locus of the problem-size effect. These differences, not evident in mean response times, may point to differences in the representation of addition and multiplication knowledge not yet reflected in models of arithmetic performance. Second, as posited by Penner-Wilger et al. (2002) tau served as an index of procedure use in addition and multiplication. Future work will continue to explore the use of tau as an objective indicator of procedure use, with the goal of providing researchers with another tool to examine the substantial individual differences in basic arithmetic. Thus, the current experiment highlights the usefulness of distributional analyses both in mathematical cognition and more generally in cognitive research.

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