Title

Determinants of intertwining operators between genuine principal series representations of nonlinear real split groups

Author
Lee, Seung Won

Publication Date
2012

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Determinants of Intertwining Operators between Genuine Principal Series Representations of Nonlinear Real Split Groups

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Seung Won Lee

Committee in charge:

Professor Nolan Wallach, Chair
Professor Kenneth Intriligator
Professor Ramamohan Paturi
Professor Cristian Popescu
Professor Hans Wenzl

2012

The dissertation of Seung Won Lee is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

____

____

____

____

____

Chair

University of California, San Diego

2012

DEDICATION

To my parents

TABLE OF CONTENTS

Signature Page iii

Dedication iv

Table of Contents v

Acknowledgements vii

Vita and Publications viii

Abstract of the Dissertation ix

Chapter 1 Introduction 1
1.1 Background 1
1.2 Main Results 2

Chapter 2 Principal Series and Intertwining Operators 7
2.1 Principal Series Representation and the Underlying $(\mathfrak{g}, K)$-module 7
2.2 Meromorphic Continuation of Intertwining Operators . . 9
2.3 Harmonics on $\mathfrak{p}$ and Kostant P^{γ} matrix 10
2.4 Relationship between the P^{γ} matrix and Intertwining Operators of Principal Series Representations 11
2.5 Product Formula of p_{γ} 12

Chapter 3 Small K-types 14
3.1 Small K-types and Principal Series Admitting Small K-types 14
3.2 Small K-types for connected, simply connected split real form of simple Lie types 15
3.3 Embedding of Metalinear group $\widetilde{GL(n, \mathbb{R})}$ or $\widetilde{SL(n, \mathbb{R})}$ into G 20

Chapter 4 The P^{ξ} matrix 25
4.1 Definition 25
4.2 Relationship between P^{ξ} matrix and Intertwining Operators of Principal Series Representations 26

Chapter 5 $\widetilde{SL(n, \mathbb{R})}$ and Metalinear Group $\widetilde{GL(n, \mathbb{R})}$ 28
5.1 The Group $\widetilde{SL(n, \mathbb{R})}$ and the Metalinear Group $\widetilde{GL(n, \mathbb{R})}$ 28
5.2 Irreducible Representations of $Pin(n)$ and Small $Pin(n)$ types relative to the Metalinear Group 30

5.3 Structure of irreducible K_α-modules in H_α and $H_\alpha \otimes V_\tau$. 31
5.4 Frobenius Reciprocity in our Context 34
5.5 $\mathfrak{t}_\alpha$-weights on certain vectors 36
5.6 Restriction of $Pin(n)$-modules to $Spin(n)$ and $\mathfrak{t}_\alpha$-weights 37

Chapter 6 Product Formula of p_ξ for $\widetilde{SL(n,\mathbb{R})}$ 39
6.1 Comparison of $\mathfrak{t}_\alpha$-weights 39
6.2 Divisibility 42
6.2.1 Definition of the P^ξ matrix revisited 42
6.2.4 Divisibility 43
6.3 Product Formula of p_ξ for the group $\widetilde{SL(n,\mathbb{R})}$ 47

Chapter 7 Product formula of p_ξ for the connected, simply connected $\mathbb{R}$-split Lie group of simple Lie type other than A_n and C_n 48
7.1 Comparision of $\mathfrak{t}_\alpha$-weights 50
7.2 Comparison of $\mathfrak{t}_\alpha$-weights for short roots of $Lie(\widetilde{SO(q+1,q)})$ 51
7.2.1 The Groups $\widetilde{Spin(p,q)}$, $\widetilde{Pin(p,q)}$ ($p \geq q \geq 3,\ p = q\ or\ p = q+1$) and their Small K-types 51
7.2.4 Restriction of $Pin(q)\times Pin(q)$ modules to $Spin(q)\times Spin(q)$ and $\mathfrak{t}_{\epsilon_1}$-weights 53
7.2.13 Comparison of $\mathfrak{t}_\alpha$-weights for short roots of $Lie(\widetilde{SO(q+1,q)})$ 59
7.3 Comparison of $\mathfrak{t}_\alpha$-weights for type F_4 and Comparison of $\mathfrak{t}_\alpha$-weights for short roots of $Lie(G_2)$ 66
7.4 Product formula of p_ξ for the connected, simply connected $\mathbb{R}$-split Lie group of simple Lie type other than A_n and C_n 69

Chapter 8 Computation of $p_\xi(\nu)$ and Determinants of Intertwining Operators 71
8.1 Computation in Rank One Case 71
8.2 Computation of $p_\xi(\nu)$ and Determinants of Intertwining Operators for $\widetilde{SL(n,\mathbb{R})}$ 73
8.2.1 $\widetilde{SL(3,\mathbb{R})}$ 73
8.2.2 $\widetilde{SL(4,\mathbb{R})}$ 75
8.2.5 General case of $\widetilde{SL(n,\mathbb{R})}$ 77
8.3 Application: Cyclicity of V_τ and Irreducibility of Unitary Principal Series 78

Bibliography 81

ACKNOWLEDGEMENTS

I would like to take this opportunity to thank my family, especially my parents, for their unconditional love and support. They have provided me with so many opportunities and have always been supportive of my decisions. I am truly grateful for their faith in me.

I also would like to thank my fellow graduate students for both mathematical and non-mathematical interactions that were crucial for the completion of my graduate study. I am grateful to Raul Gomez and Oded Yacobi for sharing their knowledge and insights. Numerous discussions with Jon Middleton have also been very helpful in the last couple years. I would like to give special thanks to Mike Gao, Andre Minor, and Ravi Shroff, who have been more of brothers than just friends ever since I came to San Diego.

My advisor, Professor Nolan Wallach, has provided more help than anyone else. It was his guidance and patience that made this thesis possible, which I did not believe would be possible. I would like to thank him for teaching me so many things, especially teaching me what passion towards mathematics is. A few years ago I asked Jeb Willenbring, a former student of Professor Wallach's, what he misses the most from San Diego. I expected his answer to be the great weather, beautiful beaches, etc..., but his answer was the conversations he used to have with Professor Wallach. I certainly understand Jeb now.

VITA

2005	B.A. in Mathematics, Cornell University
2009	M.A. in Mathematics, University of California, San Diego
2012	Ph.D. in Mathematics, University of California, San Diego

ABSTRACT OF THE DISSERTATION

Determinants of Intertwining Operators between Genuine Principal Series Representations of Nonlinear Real Split Groups

by

Seung Won Lee

Doctor of Philosophy in Mathematics

University of California, San Diego, 2012

Professor Nolan Wallach, Chair

Classification of "small K-types" for the connected, simply connected split real form of simple Lie type other than type C_n is obtained via Clifford algebras which completes the list of all small K-types of $dim > 1$ for the connected, simply connected split real form of simple Lie types. An analog, P^ξ, of Kostant's P^γ matrix is defined for a K-type V_ξ of principal series admitting a small K-type, and a product formula of the determinant of P^ξ over the rank one subgroups corresponding to the reduced restricted roots is proved. The product formula and the relationship between P^ξ and intertwining operator between the genuine principal series representations give a method to compute the shift factors of Vogan and Wallach's generalization of Leslie Cohn's determinant formula for the restriction of the intertwining operator to a K-isotypic component given in terms of ratios of classical gamma functions. The determinant of the intertwining operator between the genuine principal series representations of $\widetilde{SL(n,\mathbb{R})}$ $(n \geq 3)$ is obtained as a ratio of classical gamma functions.

Chapter 1

Introduction

1.1 Background

In their 1990 paper, David Vogan and Nolan Wallach proved a difference equation for intertwining operators for C^∞ principal series by tensoring principal series with a finite dimensional spherical representation of G. This difference equation was used to prove a meromorphic continuation of the intertwining operators for C^∞ principal series. It was also used to derive a generalization of Leslie Cohn's determinant formula for the restriction of the intertwining operator to a K-isotypic component. This determinant is given in terms of ratios of classical gamma functions with appropriate shifts that are yet unknown in general. In the same year, Chen-bo Zhu generalized Vogan and Wallach's work by showing that associated with each irreducible finite dimensional representation of G, there is a functional equation relating intertwining operators.

The intertwining operators are also related to Kostant's 1971 paper in which he proves irreducibility of spherical principal series and existence of complementary series. In more detail, let H be the the space of K-harmonics on $\mathfrak{p}_{\mathbb{C}}$ where $\mathfrak{p}$ is the -1 eigenspace of the Cartan involution on $Lie(G)$. It is a result of Kostant and Rallis that spherical principal series is isomorphic with H as a K module. For a K-type γ, let $E_\gamma = (V_\gamma^*)^{^0M}$, let $\epsilon_1, ..., \epsilon_{l(\gamma)}$ be a basis of E_γ, and let $v_1, ..., v_{l(\gamma)}$ be a basis of $V_\gamma^{^0M}$ where ${}^0M$ is the centralizer of $\mathfrak{a}$ in K. Let Q' be the projection map onto the first summand of $U(\mathfrak{g}) = U(\mathfrak{a}) \oplus \mathfrak{n}U(\mathfrak{g}) \oplus U(\mathfrak{g})\mathfrak{k}$. Kostant defines what he calls the P^γ matrix by $(P^\gamma)_{i,j} = (Q'(\epsilon_i(v_j)))$. The critical point of the paper is the explicit determination of the determinant of P^γ. Kostant achieves the determinant in split rank one case, and he derives the general formula for the determinant by proving a product formula over the rank one subgroups corresponding to the reduced restricted roots. In their 1977 paper, Kenneth Johnson and Nolan Wallach proved a formula of the intertwining operator $A_s(\nu)$ for spherical principal series in terms of Kostant's P^γ matrices that is $A_s(\nu)(\lambda\otimes v) = \lambda\circ P^\gamma(\nu)^{-1}P^\gamma(s(\nu-\rho)+\rho)\otimes v$ where $\lambda \in E_\gamma$, $v \in V_\gamma$, and $s \in W(A)$. In light of this formula and Kostant's product formula of the determinant of P^γ, one can obtain the appropriate shifts in the gamma functions that give the determinant of the intertwining operator on the γ isotypic component.

1.2 Main Results

Similar technique of Kostant's may be applied to principal series representations that admit so called "small K-types". In the second volume of his book Real Reductive Groups, Nolan Wallach defines small K-type V_τ to be an irreducible representation of K whose irreducibility is preserved under restriction to ${}^0M$, and gives examples for all real forms over $\mathbb{R}$ of all simple Lie types. Moreover, he proves that as K modules, $I_{P,\sigma,\nu}$ is isomorphic with $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^K} V_\tau$ where $I_{P,\sigma,\nu}$ is the underlying $(\mathfrak{g}, K)$-module of the principal series induced from the ${}^0M$ irreducible representation $\sigma = V_\tau|_{{}^0M}$ with minimal parabolic subgroup P, which also covers Kostant and Rallis' result because trivial representation is a small K-type.

Based on the results above, for a K-type V_ξ that occurs in $I_{P,\sigma,\nu}$, we define an analogue of Kostant's P^γ matrix, P^ξ, whose definition is as follows. Let $T_1^\xi, ..., T_{n(\xi)}^\xi$ be a basis of $Hom_{{}^0M}(V_\tau, V_\xi)$ and $\epsilon_1^\xi, ..., \epsilon_{n(\xi)}^\xi$ be a basis of $Hom_K(V_\xi, U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^K} V_\tau)$. Let $Q_\nu : U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^K} V_\tau \longrightarrow I_{P,\sigma,\nu}$ be the corresponding isomorphism as K-modules, and define $R_\nu : U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^K} V_\tau \longrightarrow V_\tau$ by $R_\nu(Z) := Q_\nu(Z)(e)$. Note every map defined above intertwines ${}^0M$ action, hence $R_\nu \circ \epsilon_i \circ T_j$ does also for all i and j. Define by $P^\xi(\nu)$ the $n(\xi)$ by $n(\xi)$ matrix such that $(P^\xi(\nu))_{i,j}$ is the polynomial in ν in which $R_\nu \circ \epsilon_i^\xi \circ T_j^\xi$ acts on V_τ, and define by P^ξ the $n(\xi)$ by

$n(\xi)$ matrix obtained from $P^\xi(\nu)$ by replacing the entries with the corresponding elements in $U(\mathfrak{a}_\mathbb{C}) \otimes End(V_\tau)$.

Let G be any of the connected, simply connected split real form of simple Lie type other than type C_n with maximal compact subgroup K. First, by a relationship between ${}^0M$ and Clifford algebra, we show that the examples of small K-types given by Nolan Wallach exhaust the list of all small K-types, which completes the list of all small K-types of $dim > 1$ for the connected, simply connected split real form of simple Lie types. If K is a product of two groups, denote by p_1 and p_2 the projection onto the first factor and the second factor respectively. Denote by s the $Spin$ representation of $Spin(n)$ for n odd, and either of the two half-$Spin$ representations of $Spin(n)$ for n even.

Theorem 3.3.2 Let G be any of the connected, simply connected split real form of simple Lie type other than type C_n with maximal compact subgroup K. The following is a complete list of all small K-types.

Type	K	Small K-type
A_n $(n \geq 2)$	$Spin(n+1)$	s
B_n $(n \geq 3)$	$Spin(n+1) \times Spin(n)$	$s \circ p_1$ or $s \circ p_2$ for n odd, $s \circ p_2$ for n even
D_n $(n \geq 3)$	$Spin(n) \times Spin(n)$	$s \circ p_1$ or $s \circ p_2$
E_6	$Sp(4)$	standard 8 dimensional representation
E_7	$SU(8)$	standard 8 dimensional representation or its dual representation
E_8	$Spin(16)$	standard 16 dimensional representation
F_4	$Sp(3) \times SU(2)$	standard 2 dimensional representation $\circ\, p_2$
G_2	$SU(2) \times SU(2)$	standard 2 dimensional representation $\circ\, p_1$ *or* p_2

Second, for a K-type V_ξ of principal series admitting a small K-type V_τ, a product formula of the determinant of P^ξ over the rank one subgroups corresponding to the reduced restricted roots is proved. In more detail, let ϕ be a positive root of $Lie(G)$, and let G_ϕ be the corresponding rank one subgroup. G_ϕ has its semisimple part the group generated by the metaplectic group $Mp_2(\mathbb{R})$ and ${}^0M$.

Let K_ϕ be the maximal compact subgroup of G_ϕ generated by a torus and ${}^0M$. Let $V_\xi = \oplus_{j=1}^{n(\xi)} V_{\tau_j} \oplus W$ where V_{τ_j} is an irreducible K_ϕ module such that $V_{\tau_j} \cong V_\tau$ as ${}^0M$ modules for all $j = 1, ..., n(\xi)$ and W is a K_ϕ submodule of V_ξ such that $dim Hom_{{}^0M}(V_\tau, W) = 0$. Let $p_\phi = p^\phi_{\tau_1} ... p^\phi_{\tau_{n(\xi)}}$ where $p^\phi_{\tau_j}$ is the determinant of P^{τ_j} matrix of the rank one case of G_ϕ with K_ϕ-type V_{τ_j}. Denote by $p_{(\phi)} = T_{\rho_\phi - \rho}(p_\phi)$ where $T_{\rho_\phi - \rho}$ is translation by $\rho_\phi - \rho$. The following is a product formula of p_ξ, the determinant of P^ξ, over the rank one subgroups corresponding to the reduced restricted roots for the connected, simply connected split real form of simple Lie type other than type C_n.

Theorem 6.3.1 & **Theorem 7.4.1** There exists a nonzero scalar c such that

$$p_\xi(\nu) = c\Pi_{\phi \in \Phi^+} p_{(\phi)}(\nu)$$

Third, Johnson and Wallach's formula of the intertwining operator for spherical principal series in terms of Kostant's P^γ matrix remains true. Let $v \in V_\tau$. We look at $P^\xi(\nu)$ as a map of $\bigoplus_{j=1}^{n(\xi)} T_j^\xi(V_\tau) \longrightarrow \bigoplus_{j=1}^{n(\xi)} T_j^\xi(V_\tau)$ by setting $P^\xi(\nu) T_i^\xi(v) = \Sigma_{j=1}^{n(\xi)} T_j^\xi(P_{ji}^\xi(\nu)v)$. If $V_\xi = \bigoplus_{j=1}^{n(\xi)} T_j^\xi(V_\tau) \oplus W$ is a decomposition of V_ξ as a ${}^0M$-module, we look at $P^\xi(\nu)$ as a map on V_ξ where $P^\xi(\nu)$ acts as above on $\bigoplus_{j=1}^{n(\xi)} T_j^\xi(V_\tau)$ and acts trivially on W. We now look at $P^\xi(\nu)$ as an operator on $Hom_{{}^0M}(V_\xi, V_\tau)$ where $P^\xi(\nu) \cdot \lambda = \lambda \circ P^\xi(\nu)$. With the new definition for P^ξ, the formula of the intertwining operator obtained by Kenneth Johnson and Nolan Wallach remains true for the underlying $(\mathfrak{g}, K)$ module $I_{P,\sigma,\nu}$ in general.

Theorem 4.2.3 Given $s \in W(A)$ the Weyl group of $\mathfrak{a}$, let $A_s(\nu) : I_{P,\sigma,\nu} \longrightarrow I_{P,\sigma,s(\nu-\rho)+\rho}$ be such that $A_s(\nu)\tau_\nu = \tau_{s(\nu-\rho)+\rho}$ and $A_s(\nu) \circ \pi_{\tau,\nu}(u) = \pi_{\tau,s(\nu-\rho)+\rho}(u) \circ A_s(\nu)$ for all $u \in U(\mathfrak{g})$. Then

$$A_s(\nu)(\lambda \otimes v) = \lambda \circ P^\xi(\nu)^{-1} P^\xi(s(\nu - \rho) + \rho) \otimes v$$

for $\lambda \in Hom_{{}^0M}(V_\xi, V_\tau)$ and $v \in V_\xi$, if $det\ P^\xi(\nu) \neq 0$ and $det\ P^\xi(s(\nu-\rho)+\rho) \neq 0$ for all $\xi \in \hat{K}$ that occurs in $I_{P,\sigma,\nu}$.

The determinant of the intertwining operator between the genuine principal series representations of $\widetilde{SL(n,\mathbb{R})}$ $(n \geq 3)$ is obtained as a ratio of classical gamma functions. $\widetilde{SL(n,\mathbb{R})}$ $(n \geq 3)$ is the connected, simply connected two-fold covering group of $SL(n,\mathbb{R})$ whose maximal compact subgroup K is $Spin(n)$. Let V_τ be the spin representation of $Spin(n)$ for n odd and either of the two half-spin representations of $Spin(n)$ for n even. Denote by η the nontrivial element of the covering homomorphism $p : Spin(n) \to SO(n)$, where we assume -1 action of η. Then, $\mathbb{C}[^0M]/ < \eta + 1 >$ is isomorphic with the subalgebra of $Cliff_n$ spanned by the even number of products of the generators. Thus, $\mathbb{C}[^0M]/ < \eta+1 >$ is isomorphic to the simple matrix algebra $M_{2^{\frac{n-1}{2}}}(\mathbb{C})$ for n odd and to a direct sum of two isomorphic copies of the simple matrix algebra $M_{2^{\frac{n-2}{2}}}(\mathbb{C})$ for n even. Based on this observation, we have that V_τ is a small K-type, and Weyl dimension formula implies that above examples of small K-types exhaust the list of all small K-types for the group $\widetilde{SL(n,\mathbb{R})}$.

Define $q_\nu : 2\mathbb{N} + 1 \longrightarrow \mathbb{C}[\nu]$ where $\mathbb{N} = \{0, 1, 2, 3, ...\}$ as follows.

$q_\nu(m) := \Pi_{l=0}^{\frac{m-1}{4}} \Pi_{j=0}^{l-1}(\nu + 2j + \frac{1}{2})(\nu + 2j + \frac{3}{2})$ if $4 \mid m - 1$
$q_\nu(m) := \Pi_{l=0}^{\frac{m-3}{4}} \Pi_{j=0}^{l-1}(\nu + 2j + \frac{1}{2})(\nu + 2j + \frac{3}{2}) \times \Pi_{k=0}^{\frac{m-3}{4}} (\nu + 2k + \frac{1}{2})$ if $4 \mid m - 3$

Define $\Gamma_\nu(m) : 2\mathbb{N} + 1 \longrightarrow M$ where M is the space of meromorphic functions in ν as follows.

$\Gamma_\nu(m) := \Pi_{l=0}^{\frac{m-1}{4}} \Pi_{j=0}^{l-1} \frac{\Gamma(\nu-2j+\frac{1}{2})}{\Gamma(\nu-2j-\frac{3}{2})} \frac{\Gamma(\nu+2j+\frac{1}{2})}{\Gamma(\nu+2j+\frac{5}{2})}$ if $4 \mid m - 1$

$\Gamma_\nu(m) := \Pi_{l=0}^{\frac{m-3}{4}} \Pi_{j=0}^{l-1} \frac{\Gamma(\nu-2j+\frac{1}{2})}{\Gamma(\nu-2j-\frac{3}{2})} \frac{\Gamma(\nu+2j+\frac{1}{2})}{\Gamma(\nu+2j+\frac{5}{2})} \times \Pi_{k=0}^{\frac{m-3}{4}} \frac{\Gamma(\nu-2k+\frac{1}{2})}{\Gamma(\nu-2k-\frac{3}{2})}$ if $4 \mid m - 3$

Now, given an irreducible $Spin(n)$-module $V_\xi \subseteq I_{P,\sigma,\nu}$ with highest weight $\xi = \xi_1\epsilon_1 + ... + \xi_n\epsilon_k$, branch down to $Spin(3)$ that occurs in the top left corner. Let $\{\frac{j_1}{2}, ..., \frac{j_{m_\xi}}{2}\}$ be the set of highest weights of $Spin(3)$-modules that occur in the branching counting multiplicity. Denote by $A(\nu)$ the intertwining operator $A_s(\nu)$ with s the longest element of the Weyl group. The following formulas are with ρ-shifts.

$$p_\xi(\nu) = \Pi_{\alpha\in\Phi^+} \Pi_{k=1}^{m_\xi} q_{(\nu,\alpha)}(j_k)$$

$$detA(\nu)|_{I_{P,\sigma,\nu}(\xi)} = (\frac{p_\xi(-\nu)}{p_\xi(\nu)})^{dim(V_\xi)} = ((\Pi_{\alpha\in\Phi^+}\Pi_{k=1}^{m_\xi}\Gamma_{(\nu,\alpha)}(j_k))^{\frac{2}{dim(V_\tau)}})^{dim(V_\xi)}$$

where if $n = 2k+1$, $dim(V_\xi) = \Pi_{1\leq i<j\leq k}\frac{(\xi_i+\rho_i)^2-(\xi_j+\rho_j)^2}{\rho_i^2-\rho_j^2}\Pi_{1\leq i\leq k}\frac{\xi_i+\rho_i}{\rho_i}$ with $\rho_i = k-i+\frac{1}{2}$, $dim(V_\tau) = 2^k$, and if $n = 2k$, $dim(V_\xi) = \Pi_{1\leq i<j\leq k}\frac{(\xi_i+\rho_i)^2-(\xi_j+\rho_j)^2}{\rho_i^2-\rho_j^2}$ with $\rho_i = k-i$, $dim(V_\tau) = 2^{k-1}$.

Last, cyclicity of a small K-type V_τ in $I_{P,\sigma,\nu}$ is determined by the determinant of P^ξ matrix. V_τ is cyclic in $I_{P,\sigma,\nu}$ if and only if $p_\xi(\nu) \neq 0$ for every K-type ξ that occurs in $I_{P,\sigma,\nu}$. By the product formula of $p_\xi(\nu)$, we obtain the following result and its corollary.

Theorem 8.3.1 Let G be any of the connected, simply connected split real form of simple Lie type other than type C_n with maximal compact subgroup K. Let V_τ be a small K-type and let $\sigma = V_\tau|_{{}^0M}$. If $Re(\nu,\alpha) \geq 0$ for every $\alpha \in \Phi^+$, i.e. in the closed Langlands chamber, $V_\tau \subseteq I_{P,\sigma,\nu}$ is cyclic.

Corollary 8.3.2 Let G be any of the connected, simply connected split real form of simple Lie type other than type C_n with maximal compact subgroup K. Let V_τ be a small K-type and let $\sigma = V_\tau|_{{}^0M}$. The unitary principal series $(\pi_{P,\sigma,\nu}, H^{P,\sigma,\nu})$ ($Re\ \nu = 0$) is irreducible.

Chapter 2

Principal Series and Intertwining Operators

2.1 Principal Series Representation and the Underlying $(\mathfrak{g}, K)$-module

Let G be a real reductive group with maximal compact subgroup K defined as the set of fixed elements of a Cartan involution θ. Let $\mathfrak{g}_\circ$ be the Lie algebra of G with complexification $\mathfrak{g}$. Let (P, A) be a standard p-pair with Langlands decomposition $P = {}^0MAN$ with Levi factor ${}^0MA$ and unipotent radical N.

Definition 2.1.1.

- A Hilbert representation of G on a topological vector space V over $\mathbb{C}$ is a homomorphism π of G to $GL(V)$ such that the map $G \times V \to V$ given by $(g, v) \mapsto \pi(g)v$ is continuous.

- A closed subspace W of V is invariant if $\pi(g)W \subseteq W$ for all $g \in G$. (π, V) is irreducible if the only invariant subspaces of V are 0 and V.

- A Hilbert representation (π, V) of G is unitary if $\pi(g)$ is a unitary operator for all $g \in G$.

Definition 2.1.2. Let V be a $\mathfrak{g}$-module and a K-module. V is a $(\mathfrak{g}, K)$-module if:

1. $k.X.v = Ad(k)X.k.v$ for all $v \in V,\ k \in K,\ X \in \mathfrak{g}$.
2. If $v \in V$, then $K.v$ spans a finite dimensional vector subspace of V, W_v, such that the action of K on W_v is continuous.
3. If $Y \in Lie(K)_{\mathbb{C}}$ and $v \in V$ then $\frac{d}{dt}|_{t=0} exp(tY)v = Y.v$.

Let (σ, H_σ) be an irreducible Hilbert representation of ${}^0M$ that is unitary when restricted to $K \cap {}^0M$, and let $\nu \in (Lie(A)_{\mathbb{C}})^*$. Define ${}^\infty H^{P,\sigma,\nu}$ as the space of all smooth functions $f : G \longrightarrow H_\sigma$ such that $f(mang) = \sigma(m)a^{\nu+\rho}f(g)$ for $m \in {}^0M$, $a \in A$, $n \in N$, and $g \in G$. Define for $f, g \in {}^\infty H^{P,\sigma,\nu}$

$$< f, g >= \int_K < f(k), g(k) > dk$$

Denote by $H^{P,\sigma,\nu}$ the Hilbert space completion of ${}^\infty H^{P,\sigma,\nu}$. From 1.5.3 of [RRG I], we know that the right regular action $\pi_{P,\sigma,\nu}(g)f(x) = f(xg)$ gives a Hilbert Representation $(\pi_{P,\sigma,\nu}, H^{P,\sigma,\nu})$ of G.

Definition 2.1.3. The representation $(\pi_{P,\sigma,\nu}, H^{P,\sigma,\nu})$ above is called a principal series representation of G.

If $X \in \mathfrak{g}$, then $X.f(g) = \frac{d}{dt}|_{t=0} f(g \cdot exp(tX))$ gives a natural action of $\mathfrak{g}$ on $H^{P,\sigma,\nu}$ induced from $\pi_{P,\sigma,\nu}$. We will also denote this action of $\mathfrak{g}$ by $\pi_{P,\sigma,\nu}$. For $\gamma \in \hat{K}$, denote by $H^{P,\sigma,\nu}(\gamma)$ the sum of all the K-invariant, finite dimensional subspaces of $H^{P,\sigma,\nu}$ that are in the class of γ. Denote by $I_{P,\sigma,\nu}$ the algebraic direct sum $\oplus_{\gamma \in \hat{K}} H^{P,\sigma,\nu}(\gamma) \cap {}^\infty H^{P,\sigma,\nu}$. The following is Lemma 3.3.5 of [RRG I].

Lemma 2.1.4. *$(\pi_{P,\sigma,\nu}, I_{P,\sigma,\nu})$ is a $(\mathfrak{g}, K)$-module.*

$(\pi_{P,\sigma,\nu}, I_{P,\sigma,\nu})$ is called the underlying $(\mathfrak{g}, K)$-module of the principal series representation $(\pi_{P,\sigma,\nu}, H^{P,\sigma,\nu})$. Consider the following Theorem of Harish-Chandra.

Theorem 2.1.5. *There is a bijection between the set of irreducible unitary representations of G and the set of irreducible $(\mathfrak{g}, K)$-modules admitting a positive definite $(\mathfrak{g}, K)$-invariant Hermitian form.*

R. Langlands has shown that every irreducible $(\mathfrak{g}, K)$-module can be realized as a quotient of an underlying $(\mathfrak{g}, K)$-module of some principal series representation.

Definition 2.1.6. The triple (P, σ, ν) is called a Langlands data if P is a parabolic subgroup of G, (σ, H_σ) is an irreducible unitary representation of ${}^0M$ such that $(H_\sigma)_{K\cap {}^0M}$ is tempered, i.e. the matrix coefficient $m \mapsto <\sigma(m)v, w>$ lies in $L^{2+\epsilon}({}^0M)$ for every $\epsilon > 0$ for all $v, w \in (H_\sigma)_{K\cap {}^0M}$, and $\nu \in (Lie(A)_\mathbb{C})^*$ such that $Re(\nu, \alpha) > 0$ for all $\Phi(P, A)$.

Definition 2.1.7. Define for $\nu \in (Lie(A)_\mathbb{C})^*$ the intertwining operator $J_{\overline{P}|P}(\nu)$: ${}^\infty H^{P,\sigma,\nu} \longrightarrow {}^\infty H^{P,\sigma,\nu}$ as $(J_{\overline{P}|P}(\nu)f)(k) = \int_{\overline{N}} f_\nu(\overline{n}k)d\overline{n}$.

Theorem 2.1.8. *(Langlands) Let V be an irreducible $(\mathfrak{g}, K)$-module. Then there exists a Langlands data (P, σ, ν) such that V is $(\mathfrak{g}, K)$-isomorphic with the unique irreducible quotient of $I_{P,\sigma,\nu}$, which is $(\mathfrak{g}, K)$ isomorphic to $J_{\overline{P}|P}(I_{P,\sigma,\nu})$.*

The theorem of Harish-Chandra suggests to classify irreducible unitary representations of G, it is enough to study irreducible $(\mathfrak{g}, K)$-modules. The theorem of Langlands realizes an irreducible $(\mathfrak{g}, K)$-module as a unique quotient of the underlying $(\mathfrak{g}, K)$-module of some principal series representation. Hence, the problem of Unitary Dual reduces down to finding among the underlying $(\mathfrak{g}, K)$-modules of principal series $I_{P,\sigma,\nu}$ with (P, σ, ν) a Langlands data the ones that admit a positive definite $(\mathfrak{g}, K)$-invariant Hermitian form.

2.2 Meromorphic Continuation of Intertwining Operators

In 1990, David Vogan and Nolan Wallach achieved a meromorphic continuation of the intertwining operators via the difference equation satisfied by the intertwining operators. The following is Theorem 2.2 of [VW].

Theorem 2.2.1. *There exist polynomials $b_{\sigma,\lambda}$ and $D_{\sigma,\lambda}$ in ν with values in $\mathbb{C}$ and $U(\mathfrak{g})^K$, respectively, with $b_{\sigma,\lambda} \neq 0$ s.t.*

$$b_{\sigma,\lambda}(\nu)J_{\overline{P}|P}(\nu)f = J_{\overline{P}|P}(\nu + \lambda)\pi_{P,\sigma,\nu+\lambda}(D_{\sigma,\lambda}(\nu))f$$

for $f \epsilon I_\sigma^\infty$ *and* $Re(\nu, \alpha) > c_\sigma$ *for all* $\alpha \epsilon \Phi(P, A)$.

Meromorphic continuation of the intertwining operators has been achieved in the past for the K-finite space $I_{P,\sigma,\nu}$. The novelty of the theorem stated above is that the authors were able to achieve the meromorphic continuation for I_σ^∞ the space of C^∞ vectors by tensoring with a finite dimensional G-module. In addition, using the two polynomials above, the authors were able to compute a determinant formula of the intertwining operator on each K-isotypic component that generalizes Leslie Cohn's determinant formula. The following is Theorem 4.6 of [VW].

Theorem 2.2.2.

$$det J_{\overline{P}|P(\nu)}|_{I_\sigma(\gamma)} = \prod_{\alpha\epsilon\Sigma} \frac{\prod_{i=1}^{r_\alpha(\sigma)} \Gamma(((\nu,\alpha)/4(\rho_\alpha,\alpha)) - a_{i,\alpha(\sigma)})^{(\gamma:\sigma)}}{\prod_{i=1}^{r_\alpha(\sigma)(\gamma:\sigma)} \Gamma(((\nu,\alpha)/4(\rho_\alpha,\alpha)) - b_{i,\alpha(\sigma,\gamma)})}$$

The determinant formula is important for numerous reasons. First, by Langlands' classification theorem, the determinant formula gives the reduction points of $I_{P,\sigma,\nu}$. Also, the determinant formula can be used to show existence of Complementary Series Representations, a subset of the unitary dual of G.

2.3 Harmonics on $\mathfrak{p}$ and Kostant P^γ matrix

Let G be a connected semisimple Lie group with maximal compact subgroup K defined as the set of fixed elements of a Cartan involution θ. Denote by $\mathfrak{g}_\circ$ the lie algebra of G and let $\mathfrak{g}_\circ = \mathfrak{k}_\circ \oplus \mathfrak{p}_\circ$ be its Cartan decomposition where $\mathfrak{k}_\circ$ is $+1$ eigenspace and $\mathfrak{p}_\circ$ is -1 eigenspace of the Cartan involution θ of $\mathfrak{g}_\circ$. Let $\mathfrak{a}_\circ$ be a maximal abelian subalgebra of $\mathfrak{p}_\circ$, and let ${}^0\mathfrak{m}_\circ$ be the centralizer of $\mathfrak{a}_\circ$ in $\mathfrak{k}_\circ$. We drop the subscript $\circ$ to denote the complexifications of the subspaces of $\mathfrak{g}_\circ$ introduced above. Let $\mathfrak{g} = \mathfrak{a} \oplus {}^0\mathfrak{m} \oplus \Sigma_{\phi\in\Phi(\mathfrak{g},\mathfrak{a})}\mathfrak{g}^\phi$ be the root space decomposition, and let $\mathfrak{n} = \Sigma_{\phi\in\Phi^+(\mathfrak{g},\mathfrak{a})}\mathfrak{g}^\phi$.

Let $S(\mathfrak{p})$ be the space of symmetric polynomials on $\mathfrak{p}$ and denote by $S^j(\mathfrak{p})$ the space of homogeneous polynomials on $\mathfrak{p}$ of degree j. K acts on $P(\mathfrak{p})$ as K acts on $\mathfrak{p}$, and K acts on $S^j(\mathfrak{p})$ for all $j \in \mathbb{Z}_{\geq 0}$. Let $S(\mathfrak{p})^K$ the space of K invariants on $S(\mathfrak{p})$. $S(\mathfrak{p})^K$ is graded by degree. Denote by $S(\mathfrak{p})^K_+$ the subspace of K invariants of $S(\mathfrak{p})$ of degree strictly greater than 0. Note the subspace $S(\mathfrak{p})^j \cap (S(\mathfrak{p})S(\mathfrak{p})^K_+)$ is K-invariant, and hence there is a K-invariant subspace H^j of $S^j(\mathfrak{p})$ such that $S^j(\mathfrak{p}) = H^j \oplus \{S(\mathfrak{p})^j \cap (S(\mathfrak{p})S(\mathfrak{p})^K_+)\}$.

Definition 2.3.1. $H = \bigoplus_{j \geq 0} H^j$ is the space of harmonics on $\mathfrak{p}$.

Theorem 2.3.2. *(Kostant-Rallis [KR]) The map $h \otimes f \mapsto hf$ from $H \otimes S(\mathfrak{p})^K$ to $S(\mathfrak{p})$ is a linear bijection, and $H \cong Ind^K_{^0M}(1)$ as K modules.*

Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and consider the decomposition $U(\mathfrak{g}) = U(\mathfrak{a}) \oplus U(\mathfrak{g})\mathfrak{k} \oplus \mathfrak{n}U(\mathfrak{g})$.

Definition 2.3.3. Let $Q' : U(\mathfrak{g}) \longrightarrow U(\mathfrak{a}) \oplus U(\mathfrak{g})\mathfrak{k} \oplus \mathfrak{n}U(\mathfrak{g})$ be the projection onto the first summand.

Denote by $symm : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ the symmetrization map. Let V_γ be an irreducible K module that occurs in H hence in $symm(H)$. Denote by $V_\gamma^{^0M}$ the subspace of ^0M invariant elements where ^0M is the centralizer of $\mathfrak{a}_\circ$ in K. $Lie(^0M) = {^0\mathfrak{m}_\circ}$. Let $dim V_\gamma^{^0M} = l(\gamma)$. Let $\epsilon_1, ..., \epsilon_{l(\gamma)}$ be a basis of $Hom_K(V_\gamma, symm(H))$, $v_1, ..., v_{l(\gamma)}$ be a basis of $V_\gamma^{^0M}$.

Definition 2.3.4.

- P^γ is an $l(\gamma)$ by $l(\gamma)$ matrix with $P^\gamma_{ij} = Q'(\epsilon_i(v_j))$.
- $p_\gamma = det\ P^\gamma$.

2.4 Relationship between the P^γ matrix and Intertwining Operators of Principal Series Representations

Let G be a connected semisimple Lie group with maximal compact subgroup K. Let V_γ be a K-type that occurs in H the space of harmonics on $\mathfrak{p}$ of G.

Definition 2.4.1. For $\nu \in \mathfrak{a}^*$, $P^\gamma(\nu) = (P^\gamma_{ij}(\nu))$.

Let $\epsilon_1, ..., \epsilon_{l(\gamma)}$ be a basis of $Hom_K(V_\gamma, symm(H))$, $v_1, ..., v_{l(\gamma)}$ be a basis of $V_\gamma^{^0M}$. $P^\gamma(\nu)$ is a map of $V_\gamma^{^0M} \to V_\gamma^{^0M}$ by $P^\gamma(\nu)v_i = \Sigma P^\gamma_{ji}(\nu)v_j$. Consider the K-module isomorphism $Hom_K(V_\gamma, symm(H)) \otimes V_\gamma \to I_{P,triv,\nu}(\gamma)$ given by $(\lambda \otimes v)(k) = \lambda(k \cdot v)$ with $\lambda \in Hom_K(V_\gamma, symm(H))$ and $v \in V_\gamma$ where $(\pi_\nu, I_{P,triv,\nu})$ is the underlying $(\mathfrak{g}, K)$-module of the principal series representation $(\pi_{P,triv,\nu}, H^{P,triv,\nu})$. As $Hom_K(V_\gamma, symm(H)) \cong (V_\gamma^*)^{^0M}$, we can consider the above map as a K-module isomorphism $(V_\gamma^*)^{^0M} \otimes V_\gamma \to I_{P,triv,\nu}(\gamma)$. For $s \in W(A)$ the Weyl group of $\mathfrak{a}$, let $A_s(\nu) : I_{P,triv,\nu} \to I_{P,triv,s(\nu-\rho)+\rho}$ be the map such that $A_s(\nu) \circ \pi_\nu(u) = \pi_{s(\nu-\rho)+\rho}(u) \circ A_s(\nu)$ for $u \in U(\mathfrak{g})$ and $A_s(\nu) \cdot 1_\nu = 1_{s(\nu-\rho)+\rho}$. The following is Lemma 7.5 of [JW].

Theorem 2.4.2. *Let* $\lambda \in (V_\gamma^*)^{^0M}$, $v \in V_\gamma$.

$$A_s(\nu)(\lambda \otimes v) = \lambda \circ P^\gamma(\nu)^{-1} P^\gamma(s(\nu - \rho) + \rho) \otimes v$$

if det $P^\gamma(\nu) \neq 0$ *and det* $P^\gamma(s(\nu-\rho)+\rho) \neq 0$ *for all* $\gamma \in \hat{K}$ *that occurs in* H.

Let $A(\nu) = A_s(\nu)$ with s the longest element of the Weyl group. Then, for a minimal parabolic subgroup P of G, we have

$$J_{\overline{P}|P}(\nu)f = (c(\nu)A(\nu + \rho)f) \circ k^*$$

where $c(\nu)$ is Harish-Chandra c-function on the trivial K-type and k^* is a representative of $s \in W(A) = N_K(A)/Z_K(A)$. Therefore, determinant of $P^\gamma(\nu)$ gives the shift factors in the classical gamma functions in Theorem 2.2.2 modulo those from Harish-Chandra c-function on the trivial K-type.

2.5 Product Formula of p_γ

Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}_\circ$. Denote by Φ_1^+ the set of reduced restricted roots of $\mathfrak{g}_\circ$. If $\alpha \in \Phi_1^+$, denote by $\mathfrak{g}_{\alpha_\circ} = \mathfrak{a}_\circ \oplus {}^0\mathfrak{m}_\circ + \Sigma_{j=-2}^2 \mathfrak{g}_\circ^{j\alpha}$ and denote by G_α the analytic subgroup of G with Lie algebra $\mathfrak{g}_{\alpha_\circ}$. Let K_α, ${}^0M_\alpha$ play the roles of K, ${}^0M$ for the case $\mathfrak{g}_{\alpha_\circ} = \mathfrak{k}_{\alpha_\circ} \oplus \mathfrak{p}_{\alpha_\circ}$, and denote by H_α the space of harmonics on $\mathfrak{p}_\alpha$. Let V_γ be a K-type that occurs in the space of harmonics H of G. For $\alpha \in \Phi_1^+$ of $\mathfrak{g}_\circ$, let $Span\ K_\alpha \cdot V_\gamma^{0M} = \oplus_{j=1}^{l(\gamma)} V_{\gamma_j^\alpha}$ be a decomposition into irreducible K_α modules. For $f \in \mathfrak{a}^*$, let $T_f : U(\mathfrak{a}) \to U(\mathfrak{a})$ be defined by $(T_f p)(g) = p(g-f)$. Let $\rho = \frac{1}{2}\Sigma_{\phi\in\Phi^+}\phi$ and let ρ_α play the role of ρ for $\mathfrak{g}_{\alpha_\circ}$.

Definition 2.5.1. Let $\alpha \in \Phi_1^+$.

- $p_\alpha = p_{\gamma_1^\alpha} \ldots p_{\gamma_{l(\gamma)}^\alpha}$.
- $p_{(\alpha)} = T_{\rho_\alpha - \rho} p_\alpha$.

The following is Theorem 2.4.6 of [Kos].

Theorem 2.5.2. *There exists a nonzero scalar c such that*

$$p_\gamma = c\ \Pi_{\alpha\in\Phi_1^+}\ p_{(\alpha)}$$

Chapter 3

Small K-types

3.1 Small K-types and Principal Series Admitting Small K-types

Let G be a real reductive Lie group with maximal compact subgroup $K = \{g \in G | \Theta(g) = g\}$, the subgroup of fixed elements of a Cartan involution θ. Let (P, A) be a minimal p-pair such that $P = {}^0MAN$ with ${}^0M$ the centralizer of A in K and unipotent radical N.

Definition 3.1.1. An irreducible representation (τ, V_τ) of K is a small K-type if irreducibility is preserved under restriction of K to ${}^0M$.

Let $\sigma = \tau|_{{}^0M}$. If $I_{P,\sigma,\nu}$ is the underlying $(\mathfrak{g}, K)$ module of a principal series that admits a small K-type (τ, V_τ), one can describe the set of K-types that occur in $I_{P,\sigma,\nu}$ and their multiplicities using Frobenius Reciprocity.

Lemma 3.1.2. *Let V_ξ be an irreducible representation of K. V_ξ occurs in $I_{P,\sigma,\nu}$ if and only if $V_\xi|_{{}^0M}$ contains a copy of $V_\tau|_{{}^0M}$, with multiplicity dim $Hom_{{}^0M}(V_\xi, V_\tau)$, the number of copies of $V_\tau|_{{}^0M}$ within $V_\xi|_{{}^0M}$.*

Proof. We have $Hom_K(V_\xi, I_{P,\sigma,\nu}) \cong Hom_{{}^0M}(V_\xi, V_\tau)$ by Frobenius Reciprocity. Since V_τ is a small K-type, $V_\tau|_{{}^0M}$ is irreducible. Thus by Schur's Lemma, V_ξ will occur in $I_{P,\sigma,\nu}$ if $V_\xi|_{{}^0M}$ contains a copy of $V_\tau|_{{}^0M}$ with multiplicity of dim $Hom_{{}^0M}(V_\xi, V_\tau)$. □

Consider the decomposition $U(\mathfrak{g}) = U(\mathfrak{a})U(\mathfrak{k}) \oplus \mathfrak{n}U(\mathfrak{g})$. Let Q be the projection onto the first summand. If η is an automorphism of $U(\mathfrak{a})$ given by $\eta(H) = H + \rho(H)$ for $H \in \mathfrak{a}$, let $\eta \otimes \tau : U(\mathfrak{a}) \otimes U(\mathfrak{k}) \longrightarrow U(\mathfrak{a}) \otimes End(V_\tau)$ be defined by $\eta \otimes \tau(a \otimes k) = \eta(a) \otimes \tau(k)$. By Lemma 11.3.2 of [RRG II], there is a homomorphism $\gamma_\tau : U(\mathfrak{g})^K \longrightarrow U(\mathfrak{a})$ such that $(\eta \otimes \tau)(Q(g)) = \gamma_\tau(g) \otimes I$, which will give a natural action on V_τ, i.e. the action of $U(\mathfrak{g})^K$ on V_τ considered as a subspace of $I_{P,\sigma,\nu}$.

The following is a theorem in 11.3.6 of [RRG II] that gives another realization of $I_{P,\sigma,\nu}$ as a K-module.

Theorem 3.1.3. *$U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^{\mathfrak{k}}} V_\tau \cong I_{P,\sigma,\nu}$ as K-modules and the two modules have equivalent semi-simplifications.*

3.2 Small K-types for connected, simply connected split real form of simple Lie types

Let $\mathfrak{g}_\mathbb{R}$ be a semisimple Lie algebra over $\mathbb{R}$, and let $\mathfrak{g}$ be its complexification. Denote by $G_\mathbb{C}$ the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ and by $G_\mathbb{R}$ the connected subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{g}_\mathbb{R}$. Let G be a covering group of $G_\mathbb{R}$ with covering homomorphism p where we denote the kernel by Z. Fix a maximal compact subgroup K of G and let U be a compact real form of $G_\mathbb{C}$ such that $G_\mathbb{R} \cap U = K_\mathbb{R} = p(K)$. Let (P, A) be a minimal p-pair and $P = {}^0MAN$ be the Langlands decomposition as before.

The following theorem is from 11.A.2.1 of [RRG II] whose proof is a case by case argument that gives examples of small K-types for all real forms over $\mathbb{R}$ of all simple Lie types.

Theorem 3.2.1. *Let $\chi \in \hat{Z}$. There exists an irreducible representation (τ, V) of K such that $\tau|_Z = \chi I$ and $\tau|_{{}^0M}$ is irreducible.*

We also have the following theorem from 11.A.2.11 of [RRG II].

Theorem 3.2.2. *If $\mathfrak{g}_\mathbb{R}$ is split over $\mathbb{R}$, then $G_\mathbb{R}$ always has a two-fold covering group.*

We consider the case $\mathfrak{g}_{\mathbb{R}}$ split over $\mathbb{R}$. If G is simple and not of type C I, it is also simply connected. Second, the rank one subgroups of G corresponding to the reduced restricted roots have their semisimple part as the group generated by the metaplectic group $Mp_2(\mathbb{R})$ and ${}^0M$, which simplify the product formula of p_ξ and the computation. Hence, we assume from now on G is a two-fold covering group of a split real simple Lie group $G_{\mathbb{R}}$. Let η be the nontrivial element of $Z = \mu_2$. If $\chi(\eta) = Id$, there is no difference between the representations of G and $G_{\mathbb{R}}$. Therefore, we assume $\chi(\eta) = -Id$.

The following are split real simple Lie groups and their examples of small K-types from chapter 11 of [RRG II].

- Type A I. $G_{\mathbb{R}} = SL(n, \mathbb{R})$, $n \geq 3$

  The universal covering group $\widetilde{SL(n,\mathbb{R})}$ of $SL(n,\mathbb{R})$ $(n \geq 3)$ is a central μ_2-extension with maximal compact subgroup $K = Spin(n)$. ${}^0M_{SL(n,\mathbb{R})}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1}$, and ${}^0M_{\widetilde{SL(n,\mathbb{R})}}$ is a nonabelian group of order 2^n.

$$
\begin{array}{ccccc}
1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow \\
\mu_2 & & \mu_2 & & \mu_2 \\
\downarrow & & \downarrow & & \downarrow \\
{}^0M_{\widetilde{SL(n,\mathbb{R})}} & \hookrightarrow & Spin(n) & \hookrightarrow & \widetilde{SL(n,\mathbb{R})} \\
\downarrow & & \downarrow & & \downarrow \\
{}^0M_{SL(n,\mathbb{R})} & \hookrightarrow & SO(n) & \hookrightarrow & SL(n,\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1
\end{array}
$$

  Let η be the nontrivial element in μ_2. As discussed above, we assume η action of -1. $\mathbb{C}[{}^0M_{\widetilde{SL(n,\mathbb{R})}}]/ < \eta + 1 >$ is isomorphic to the subalgebra of $Cliff_n$ spanned by the even number of products of the generators, thus $\mathbb{C}[{}^0M_{\widetilde{SL(n,\mathbb{R})}}]/ < \eta + 1 >$ is a simple matrix algebra for n odd and a direct sum of two simple matrix algebras for n even. If we choose for τ the Spin representation of $Spin(n)$ for n-odd and either of the half-Spin representations of $Spin(n)$ for n-even, τ is a small K-type as restriction to ${}^0M_{\widetilde{SL(n,\mathbb{R})}}$ preserves irreducibility.

- Type BD I. $G_{\mathbb{R}} = Spin(p,q)$ with $p = q$ or $p = q+1$, $q \geq 3$

  $K_{\mathbb{R}} = (Spin(p)\times Spin(q))/\{1,(-1,-1)\}$ and $K = Spin(p)\times Spin(q)$. ${}^0M_{\widetilde{Spin(p,q)}}$ is isomorphic to ${}^0M_{\widetilde{SL(q,\mathbb{R})}} \times \mu_2$ where ${}^0M_{\widetilde{SL(q,\mathbb{R})}}$ sits inside K diagonally, and μ_2 can be either of $(\pm 1, 1)$ or $(1, \pm 1)$ where either is a subgroup in the center of K. In the case $p = q+1$, ${}^0M_{\widetilde{Spin(p,q)}} = \{(\pm diag(g,1), g) \mid g \in {}^0M_{\widetilde{SL(q,\mathbb{R})}},\ 1 \in Spin(p-q) = Spin(1)\}$. $Z = \{1,(-1,-1)\} \leq K = Spin(p)\times Spin(q)$. If χ is nontrivial, choose σ to be the $Spin$ representation and either of the two half-$Spin$ representations of $Spin(q)$ for q odd and even respectively. Also denote by σ either of the two half-$Spin$ representations of $Spin(p) = Spin(q+1)$ for q odd. Let p_1 denote the projection of K onto $Spin(p)$, and let p_2 denote the projection of K onto $Spin(q)$. If q is odd and $\tau = \sigma \circ p_1$ or $\tau = \sigma \circ p_2$, τ is a small K-type as in the example of type A I. If q is even and $\tau = \sigma \circ p_2$, τ is a small K-type as in the example of type A I.

- Type C I. $G_{\mathbb{R}} = Sp(n,\mathbb{R})$

  $K_{\mathbb{R}} = U(n)$. The universal covering group of $G_{\mathbb{R}} = Sp(n,\mathbb{R})$ is a central extension and we have

  $$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{Sp(n,\mathbb{R})} \longrightarrow Sp(n,\mathbb{R}) \longrightarrow 1$$

  The two fold cover of $G_{\mathbb{R}} = Sp(n,\mathbb{R})$ is a central μ_2 extension. A character of $U(n)$ extends to a character of the two-fold covering group of $U(n)$ that gives us half-integrals. For τ we may choose extension of characters of $U(n)$ to its two-fold cover. τ is a small K-type as the representation is 1 dimensional.

- Type E V.

  $K_{\mathbb{R}} = SU(8)/\pm 1$ and $K = SU(8)$. From the extended dynkin diagram in chapter 6 of [Bou], there is a nontrivial homomorphism δ from $SL(8,\mathbb{R})$ to $G_{\mathbb{R}}$. δ is easily seen to be injective by going to the complexification. Hence we may assume ${}^0M_{\mathbb{R}}$ is contained in the image of δ and $\delta(SO(8)) \subseteq K_{\mathbb{R}}$. Let $\tilde{\delta}$ be the lift of δ to a homomorphism of $Spin(8)$ into $SU(8)$. The corresponding representation of $Spin(8)$ cannot factor through $SO(8)$, hence it must be one of the two half-spin representations. Because $\tilde{\delta}$ is a homomorphism of the simply connected covering group of $SL(8,\mathbb{R})$ into G, $\tilde{\delta}$ is injective on ${}^0M$ for $\widetilde{SL(8,\mathbb{R})}$. If we choose τ the standard 8-dimensional representation of $SU(8)$ or its dual representation, τ is a small K-type as in the example for A I.

- Type E I.

  $K = Sp(4)$. We embed $G_{\mathbb{R}}$ into that of the case of E V as the identity component of ${}^0M_Q$ where Q is a parabolic subgroup of $G_{\mathbb{R}}$ of case E V. This homomorphism δ maps $K_{\mathbb{R}}$ into $SU(8)/\{\pm 1\}$. The lift to K must be the standard eight dimensional representation. Therefore, $Z = \{\pm 1\}$. Let η be the nontrivial element of Z. $\mathbb{C}[{}^0M_{\widetilde{E_6}}]/ < \eta+1 > \cong Cliff_6$ using the case of E V. Therefore, if we choose τ to be the standard 8-dimensional representation of $Sp(4)$, irreducibility is preserved under restriction of $K = Sp(4)$ to ${}^0M_{\widetilde{E_6}}$ as in the example for A I.

- E VIII.

  $K_{\mathbb{R}} = SO(16)$ and $K = Spin(16)$. The highest weight of K action on $\mathfrak{p}$ is $-\alpha_1$. Therefore this action is one of the two half-spin representations, say s_+, and $Z = kers_+$. There is a nontrivial homomorphism δ from $SL(9,\mathbb{R})$ to $G_{\mathbb{R}}$ from the extended dynkin diagram in chapter 6 of [Bou]. δ is injective as $SL(9,\mathbb{R})$ has trivial center. We may assume again ${}^0M_{\mathbb{R}}$ is contained in the image of δ and $\delta(SO(9)) \subseteq K_{\mathbb{R}}$. Let $\tilde{\delta}$ be the lift of δ to a homomorphism of $Spin(9)$ into $Spin(16)$. Let π be a 16 dimensional representation of $Spin(16)$ by using the covering homomorphism $p : Spin(16) \to SO(16)$ where the $ker(p)$ is the diagonal μ_2 in $\mu_2 \times \mu_2$ the center of $Spin(16)$. $\mu = \pi \circ \tilde{\delta}$ is a 16 dimensional representation of $Spin(9)$. Weyl dimension formula suggests that there are exactly three irreducible representations of $Spin(9)$ with dimension at most 16. They are the trivial representation, σ the 9-dimensional representation corresponding to the covering of $SO(9)$, and the 16-dimensional spin representation. Since μ is nontrivial, either $\mu = 7 \cdot 1 \oplus \sigma$ or $\mu =$ the spin representation. In the first case, $\tilde{\delta}$ is the standard embedding of $Spin(9)$ into $Spin(16)$. But it must push down to $SO(9)$ hence it is not possible. Thus μ must be the spin representation, and ${}^0M$ is isomorphic with that of $\widetilde{SL(9,\mathbb{R})}$. We choose $\tau = \mu$, then the result for $\widetilde{SL(9,\mathbb{R})}$ implies τ is a small K-type.

- F I.

  $K = Sp(3) \times SU(2)$, and the highest weight of K action on $\mathfrak{p}$ is $-\alpha_1$. $(1,-1)$ and $(-1,1)$ both act on $\mathfrak{p}$ by $-I$, and hence $Z = \{Id, (-1,-1)\}$. ${}^0M_{\mathbb{R}} \cong (\mathbb{Z}/2\mathbb{Z})^4$. There is a nontrivial homomorphism δ from $Spin(5,4)$ into $G_{\mathbb{R}}$. The adjoint representation of $\mathfrak{g}$ restricted to $Spin(5,4)$ splits into the adjoint representation of $Spin(5,4)$ and the spin representation, hence δ is injective. We can choose δ so that δ maps $(Spin(5) \times Spin(4))/\{Id, (-1,-1)\}$ into $(Sp(3) \times SU(2))/\{Id, (-1,-1)\}$. $Spin(5) \cong Sp(2)$ and $Spin(4) \cong SU(2) \times SU(2)$. The lift $\tilde{\delta}$ of δ to $(Spin(5) \times Spin(4))$ must be given by the obvious map of $Sp(2) \times Sp(1)$ into $Sp(3)$. The image of $Spin(5,4)$ contains the split Cartan subgroup of $G_{\mathbb{R}}$. Therefore ${}^0M$ is isomorphic to that of $\widetilde{Spin(5,4)}$, hence ${}^0M \cong {}^0M_{\widetilde{SL(4,\mathbb{R})}} \times \mu_2$ from the case of BD I. Let p_2 be the projection of $Sp(3) \times SU(2)$ onto the second factor, and σ the standard 2-dimensional representation of $SU(2)$. If $\tau = \sigma p_2$, it is a small K-type as it is for $Spin(5,4)$.

- G I.

  $K = SU(2) \times SU(2)$ with $G_{\mathbb{R}}$ the split adjoint group of G_2. The action of K is the tensor proudct of 2-dimensional representation with 4-dimensional representation. $K_{\mathbb{R}} = (SU(2) \times SU(2))/\{Id, (-1,-1)\}$, and ${}^0M_{\mathbb{R}} \cong (\mathbb{Z}/2\mathbb{Z})^2$. From the extended dynkin diagram in chapter 6 of [Bou], there is a nontrivial homomorphism δ of $SL(3,\mathbb{R})$ to $G_{\mathbb{R}}$. δ is injective as $SL(3,\mathbb{R})$ has trivial center. Hence we may assume ${}^0M_{\mathbb{R}}$ is contained in the image of δ. $\delta(SO(3))$ is the diagonal $SO(3)$ in $K_{\mathbb{R}}$ as it is the only possibility. Hence the image of the lift of δ to $\widetilde{SL(3,\mathbb{R})}$ contains ${}^0M$. Let σ be the standard 2-dimensional representation of $SU(2)$. Let p_1 be the projection of $K = SU(2)$ x $SU(2)$ onto the first factor and p_2 be the projection of $K = SU(2)$ x $SU(2)$ onto the second factor. Let $\tau = \sigma p_1$ or $\tau = \sigma p_2$. Then, as in type A I., τ is a small K-type.

3.3 Embedding of Metalinear group $\widetilde{GL(n,\mathbb{R})}$ or $\widetilde{SL(n,\mathbb{R})}$ into G

For purposes of product formula of p_ξ, we introduce certain embedded subgroup G_0 of G where G is any of the connected, simply connected split real form of simple Lie type other than type C_n, and G_0 is isomorphic to either the metalinear group $\widetilde{GL(n,\mathbb{R})}$ or $\widetilde{SL(n,\mathbb{R})}$ for appropriate n.

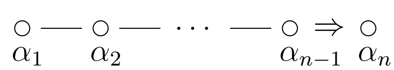
We first introduce an embbeded subgroup of $G_\mathbb{R}$ isomorphic to G_0/μ_2 using dynkin diagram or extended dynkin diagram from chapter 6 of [Bou] where μ_2 is the kernel of both of the covering homomorphisms $G \to G_\mathbb{R}$ and $G_0 \to G_0/\mu_2$.

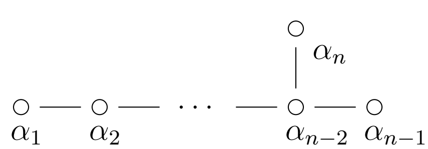
- A I.

$G_0/\mu_2 \cong GL(n,\mathbb{R})$.

Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are $\alpha_1, ..., \alpha_{n-1}$. L is isomorphic with $GL(n,\mathbb{R})$. Note the $\mu_2 \leq {}^0M_{G_\mathbb{R}}$ from the node α_n is contained in L.

- BD I.

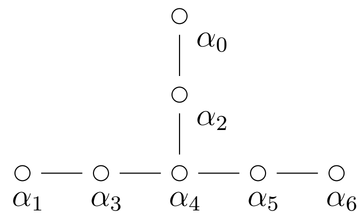




$G_0/\mu_2 \cong SL(n,\mathbb{R})$.

Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are $\alpha_1, ..., \alpha_{n-1}$. The identity component of L is isomorphic with $SL(n,\mathbb{R})$.

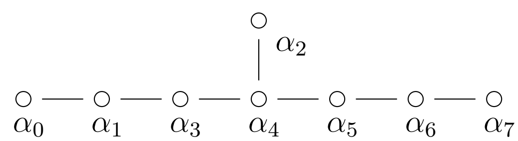
- E I.



$G_0/\mu_2 \cong GL(6, \mathbb{R})$.

Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$. The identity component of L is isomorphic with $SL(6, \mathbb{R})$. The embedded subgroup isomorphic to $GL(6, \mathbb{R})$ is generated by the $SL(6, \mathbb{R})$, μ_2 from the node α_2, and $\mathbb{R}_{>0}$ from the node α_0.

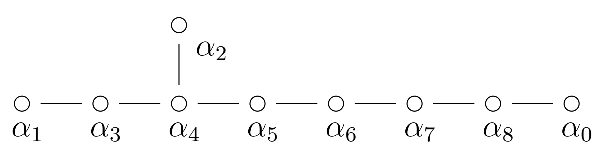
- E V.



$G_0/\mu_2 \cong GL(7, \mathbb{R})$.

Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are $\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$. The identity component of L is isomorphic with $SL(8, \mathbb{R})$, and $GL(7, \mathbb{R}) \hookrightarrow SL(8, \mathbb{R}) \hookrightarrow G_{\mathbb{R}}$ is obtained as in the case of A I.

- E VIII.



$G_0/\mu_2 \cong GL(8, \mathbb{R})$.

Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_0$. The identity component of L is isomorphic with $SL(9, \mathbb{R})$, and $GL(8, \mathbb{R}) \hookrightarrow SL(9, \mathbb{R}) \hookrightarrow G_{\mathbb{R}}$ is obtained as in the case of A I.

- F I.

$$\begin{matrix} \circ & - & \circ & - & \circ & \Rightarrow & \circ & - & \circ \\ \alpha_0 & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 \end{matrix}$$

  $G_0/\mu_2 \cong Spin(5,4)$.

  Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are $\alpha_0, \alpha_1, \alpha_2, \alpha_3$. The identity component of L is isomorphic with $Spin(5,4)$.

- G I.

$$\begin{matrix} \circ & \Lleftarrow & \circ & - & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_0 \end{matrix}$$

  $G_0/\mu_2 \cong SL(3,\mathbb{R})$.

  Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are α_2, α_0. The identity component of L is isomorphic with $SL(3,\mathbb{R})$. From the discussion of small $K = SU(2) \times SU(2)$-type in 11.A.2.8 of [RRG II], the embedding lifts to an embedding of $\widetilde{SL(3,\mathbb{R})}$ into G such that the maximal compact subgroup $SU(2)$ of $\widetilde{SL(3,\mathbb{R})}$ embeds into the maximal compact subgroup $SU(2) \times SU(2)$ of G diagonally.

Denote by $p : G \to G_{\mathbb{R}}$ the covering homomorphism and let $i : G_0/\mu_2 \hookrightarrow G_{\mathbb{R}}$ be the embedding described above. Denote by G_0 the embedded subgroup of G given by $p^{-1}(i(G_0/\mu_2))$ and denote by K_0 the maximal compact subgroup of G_0. G_0 is isomorphic to $\widetilde{GL(n,\mathbb{R})}$, $\widetilde{SL(n,\mathbb{R})}$, $\widetilde{GL(6,\mathbb{R})}$, $\widetilde{GL(7,\mathbb{R})}$, $\widetilde{GL(8,\mathbb{R})}$, $\widetilde{Spin(5,4)}$, $\widetilde{SL(3,\mathbb{R})}$ for G of type $A_n, BD\ I,\ E_6,\ E_7,\ E_8,\ F_4$, and G_2 respectively.

Lemma 3.3.1. *Let H be the space of harmonics on $\mathfrak{p}$ of G, V_τ a small K-type, V_ξ a K-type that occurs in $H \otimes V_\tau$. The restriction of K to K_0 preserve ${}^0M_G$-invariants of H and the decomposition $V_\xi|_{{}^0M_G}$. Moreover, ${}^0M_G$ is isomorphic to ${}^0M_{\widetilde{GL(n,\mathbb{R})}}$ or ${}^0M_{\widetilde{SL(n,\mathbb{R})}} \times \mu_2$ for appropriate n.*

Proof. For G of type BD I., ${}^0M_G \cong {}^0M_{G_0} \times \mu_2$ as discussed in the example of small K-types for G. The μ_2 can either be $(\pm 1, 1)$ or $(1, \pm 1) \leq K = Spin(p) \times$

$Spin(q)$. Any choice of μ_2 acts trivially on H as both are central in K. We can also make a choice of μ_2 that will act trivially on V_τ as the small K-type is the $Spin$ representation or either of the two half $Spin$ representations after projection onto the first or the second factor of K. Thus we have the statement of the lemma for G of type BD I.

Now consider G of type A_n, E_6, E_7, E_8, F_4, and G_2. As ${}^0M_{G_\mathbb{R}}$ is generated by μ_2s from each node of the dynkin diagram, we may assume that $i({}^0M_{G_0/\mu_2}) = {}^0M_{G_\mathbb{R}}$. Let $p : G \to G_\mathbb{R}$ be the covering homomorphism. We have ${}^0M_{G_0} = p^{-1}(i({}^0M_{G_0/\mu_2}))$ and ${}^0M_G = p^{-1}({}^0M_{G_\mathbb{R}})$, hence ${}^0M_G = {}^0M_{G_0}$ and we have the statement of the lemma for G of type A_n, E_6, E_7, E_8, F_4, and G_2.

$\square$

Theorem 3.3.2. *Let G be any of the connected, simply connected $\mathbb{R}$-split Lie group of simple Lie type other than type C_n with maximal compact subgroup K. The above examples of small K-types exhaust the list of all small K-types.*

Proof. By Lemma 3.3.1, ${}^0M_G$ is isomorphic to ${}^0M_{\widetilde{GL(n,\mathbb{R})}}$ or ${}^0M_{\widetilde{SL(n,\mathbb{R})}} \times \mu_2$ for appropriate n. Suppose ${}^0M_G \cong {}^0M_{\widetilde{GL(n,\mathbb{R})}}$. If $n = 2k+1$, a small K-type must have dimension 2^k. If $n = 2k$, a small K-type must have dimension 2^k. Suppose ${}^0M_G \cong {}^0M_{\widetilde{SL(n,\mathbb{R})}} \times \mu_2$. If $n = 2k+1$, a small K-type must have dimension 2^k. If $n = 2k$, a small K-type must have dimension 2^{k-1}.

For type A_{n-1}, a small $K = Spin(n)$-type must have dimension 2^k if $n = 2k+1$ and 2^{k-1} if $n = 2k$. Weyl dimension formula implies that a small K-type must be the $Spin$ representation for n odd and either of the two half-$Spin$ representation for n even.

For type B_n, $K = Spin(n+1) \times Spin(n)$ and an irreducible representation of K is an outer tensor product of irreducible representations of $Spin(n+1)$ and $Spin(n)$. A small $K = Spin(n+1) \times Spin(n)$-type must have dimension 2^k if $n = 2k+1$ and 2^{k-1} if $n = 2k$. Weyl dimension formula implies the following. If n is odd, a small K-type must be either the $Spin$ representation after projection onto $Spin(n)$ or either of the two half $Spin$ representations after projection onto $Spin(n+1)$. If n is even, a small K-type must be either of the two half $Spin$ representations after projection onto $Spin(n)$.

For type D_n, $K = Spin(n) \times Spin(n)$ and an irreducible representation of K is an outer tensor product of irreducible representations of each of the two $Spin(n)$s. A small $K = Spin(n) \times Spin(n)$-type must have dimension 2^k if $n = 2k+1$ and 2^{k-1} if $n = 2k$. Weyl dimension formula implies that for n odd, a small K-type must be the $Spin$ representation after projection onto either of the two $Spin(n)$s and for n even, a small K-type must be either of the two half-$Spin$ representations after projection onto either of the two $Spin(n)$s.

For type E_6, a small $K = Sp(4)$-type must have dimension 8. By Weyl dimension formula, the standard 8 dimensional representation of K is the only irreducible representation of K of dimension 8.

For type E_7, a small $K = SU(8)$-type must have dimension 8. By Weyl dimension formula, the standard 8 dimensional representation of K and its dual representation are the only irreducible representations of K of dimension 8.

For type E_8, a small $K = Spin(16)$-type must have dimension 16. By Weyl dimension formula, the standard 16 dimensional representation of K after projection onto $SO(16)$ is the only irreducible representation of K of dimension 16.

For type F_4, a small $K = Sp(3) \times SU(2)$-type must have dimension 2. An irreducible representation of K is an outer tensor product of irreducible representations of $Sp(3)$ and $SU(2)$. By Weyl dimension formula, the standard 6 dimensional representation of $Sp(3)$ is the smallest-dimensional nontrivial irreducible representation of $Sp(3)$. Therefore, the 2 dimensional representation after projection onto $SU(2)$ is the only choice.

For type G_2, a small $K = SU(2) \times SU(2)$-type must have dimension 2. An irreducible representation of K is an outer tensor product of irreducible representations of each of the two copies of $SU(2)$. Therefore, a small K-type must be the 2 dimensional representation after projection onto either of the two copies of $SU(2)$.

□

Chapter 4

The P^ξ matrix

4.1 Definition

Let G be any of the connected, simply connected split real form of simple Lie type, and denote by K a maximal compact subgroup. Let V_ξ be a K-type that occurs in $symm(H) \otimes V_\tau$ with a small K-type V_τ and let $\sigma = V_\tau|_{{}^0M}$. Recall $n(\xi)$ is the number of copies of V_τ in V_ξ restricted to ${}^0M$. By Frobenius reciprocity, V_ξ has multiplicity $n(\xi)$ in $symm(H) \otimes V_\tau$. Let $T_1^\xi, ..., T_{n(\xi)}^\xi$ be a basis of $Hom_{{}^0M}(V_\tau, V_\xi)$ and $\epsilon_1^\xi, ..., \epsilon_{n(\xi)}^\xi$ be a basis of $Hom_K(V_\xi, symm(H) \otimes V_\tau)$. Let $Q_\nu : symm(H) \otimes V_\tau \longrightarrow I_{P,\sigma,\nu}$ be the corresponding isomorphism as K-modules, and define $R_\nu : symm(H) \otimes V_\tau \longrightarrow V_\tau$ by $R_\nu(Z) := Q_\nu(Z)(e)$. Every map defined in this paragraph intertwines ${}^0M$ action, hence $R_\nu \circ \epsilon_i \circ T_j$ also for all i and j.

Definition 4.1.1.

- Define by $P^\xi(\nu)$ the $n(\xi)$ by $n(\xi)$ matrix where $(P^\xi(\nu))_{i,j}$ is the polynomial in ν in which $R_\nu \circ \epsilon_i^\xi \circ T_j^\xi$ acts on V_τ, without ρ-shift.

- Define by P^ξ the $n(\xi)$ by $n(\xi)$ matrix obtained from $P^\xi(\nu)$ by replacing entries with the corresponding elements in $U(\mathfrak{a})$.

- Denote by p_ξ and $p_\xi(\nu)$ the determinants of P^ξ and $P^\xi(\nu)$ respectively.

4.2 Relationship between P^ξ matrix and Intertwining Operators of Principal Series Representations

$P^\xi(\nu)$ as a map of $\bigoplus_{j=1}^{n(\xi)} T_j^\xi(V_\tau) \longrightarrow \bigoplus_{j=1}^{n(\xi)} T_j^\xi(V_\tau)$ by setting $P^\xi(\nu)T_i^\xi(v) = \Sigma_{j=1}^{n(\xi)} T_j^\xi(P_{ji}^\xi(\nu)v)$ for $v \in V_\tau$. If $V_\xi = \bigoplus_{j=1}^{n(\xi)} T_j^\xi(V_\tau) \oplus W$ is a decomposition of V_ξ as a ${}^0M$-module, we look at $P^\xi(\nu)$ as a map on V_ξ where $P^\xi(\nu)$ acts as above on $\bigoplus_{j=1}^{n(\xi)} T_j^\xi(V_\tau)$ and acts trivially on W. Thus, we may consider $P^\xi(\nu)$ as an operator on $Hom_{{}^0M}(V_\xi, V_\tau)$ where $P^\xi(\nu).\lambda = \lambda \circ P^\xi(\nu)$.

Definition 4.2.1.

- If $\lambda \in Hom_{{}^0M}(V_\xi, V_\tau)$ and $v \in V_\xi$, define $(\lambda \otimes v)(\mathrm{k}) = \lambda(\rho_\xi(k)v)$.
- Define for $a \in Hom_K(V_\xi, symm(H)\otimes V_\tau)$ and $v \in V_\xi$, $B_\nu^\xi(a)(v) = \pi_{\tau,\nu}(a(v))(e)$ where $\pi_{\tau,\nu}(a(v))$ denotes the first factor action on the second factor with $\pi_{\tau,\nu}$ action by an abuse of notation. Then $B_\nu^\xi : Hom_K(V_\xi, symm(H) \otimes V_\tau) \longrightarrow Hom_{{}^0M}(V_\xi, V_\tau)$.
- Let $T_\nu : symm(H)\otimes V_\tau \longrightarrow I_{P,\sigma,\nu}$ be defined by $T_\nu(\Sigma(u_j \otimes v_j)) = \Sigma(\pi_{\tau,\nu}(u_j)v_j)$.

We have a K-module isomorphism $I_{P,\sigma,\nu}(\xi) \cong Hom_{{}^0M}(V_\xi, V_\tau) \otimes V_\xi$ using above.

There exists $\nu_0 \in \mathfrak{a}^*$ such that T_{ν_0} is a bijection from 11.3.6 of [RRG II].

Lemma 4.2.2. *$T_\nu \circ T_{\nu_0}^{-1}(\lambda \otimes v) = \lambda \circ P^\xi(\nu) \otimes v$ for $\lambda \in Hom_{{}^0M}(V_\xi, V_\tau)$ and $v \in V_\xi$.*

Proof. This proof is almost word for word as the proof of Lemma 7.3 of [JW].

Let $\hat{\delta}_\xi : Hom_K(V_\xi, symm(H) \otimes V_\tau) \longrightarrow Hom_{{}^0M}(V_\xi, V_\tau)$ be defined so that $T_{\nu_0}(a(v)) = \hat{\delta}_\xi(a) \otimes v$ for $a \in Hom_K(V_\xi, symm(H) \otimes V_\tau)$ and $v \in V_\xi$. By the above $B_{\nu_0}^\xi(a(v)) = \hat{\delta}_\xi(a)$. Now $T_\nu \circ T_{\nu_0}^{-1}(B_{\nu_0}^\xi(a) \otimes v) = B_\nu^\xi(a) \otimes v$. But $B_\nu^\xi(a_i)(T_j^\xi(V_\tau)) = P_{ij}^\xi(\nu)$ where $\{a_i\}$ is a basis of $Hom_{{}^0M}(V_\xi, V_\tau)$ and $T_j^\xi(V_\tau)$ is say for block diagonal P^ξ, or even an identity for $P^\xi(\nu_0)$ because $P^\xi(\nu_0)$ is invertible. Thus $B_\nu^\xi(a) = B_{\nu_0}^\xi(a) \circ P^\xi(\nu)$. Hence if $B_{\nu_0}^\xi(a) = \lambda$, then $T_\nu \circ T_{\nu_0}^{-1}(\lambda \otimes v) = \lambda \circ P^\xi(\nu) \otimes v$. $\square$

Theorem 4.2.3. *Given* $s \in W(A)$ *the Weyl group of* $\mathfrak{a}$*, let* $A_s(\nu) : I_{P,\sigma,\nu} \longrightarrow I_{P,s\sigma,s(\nu-\rho)+\rho}$ *be such that* $A_s(\nu)\tau_\nu = \tau_{s(\nu-\rho)+\rho}$ *and* $A_s(\nu) \circ \pi_{\tau,\nu}(u) = \pi_{\tau,s(\nu-\rho)+\rho}(u) \circ A_s(\nu)$ *for all* $u \in U(\mathfrak{g})$*. Then*

$$A_s(\nu)(\lambda \otimes v) = \lambda \circ P^\xi(\nu)^{-1} P^\xi(s(\nu-\rho)+\rho) \otimes v$$

for $\lambda \in Hom_{{}^0M}(V_\xi, V_\tau)$ *and* $v \in V_\xi$*, if det* $P^\xi(\nu) \neq 0$ *and det* $P^\xi(s(\nu-\rho)+\rho) \neq 0$ *for all* $\xi \in \hat{K}$ *that occurs in* $I_{P,\sigma,\nu}$*.*

Proof. This proof is almost word for word as the proof of Lemma 7.5 of [JW]. If $u \otimes w \in \beta(H) \otimes V_\tau$ is a simple tensor, then

$$\begin{aligned} A_s(\nu)T_\nu(u \otimes w) &= A_s(\nu)\pi_{\tau,\nu}(u)w_\nu && (4.2.1)\\ &= \pi_{\tau,s(\nu-\rho)+\rho}(u)A_s(\nu)w_\nu && (4.2.2)\\ &= \pi_{\tau,s(\nu-\rho)+\rho}(u)w_{s(\nu-\rho)+\rho} && (4.2.3)\\ &= T_{s(\nu-\rho)+\rho}(u \otimes v) && (4.2.4) \end{aligned}$$

Hence $A_s(\nu)T_\nu \circ T_{\nu_0}^{-1}(\lambda \otimes v) = T_{s(\nu-\rho)+\rho} \circ T_{\nu_0}^{-1}(\lambda \otimes v)$. Thus,

$$\begin{aligned} A_s(\nu)(\lambda \otimes v) &= (T_{s(\nu-\rho)+\rho} \circ T_{\nu_0}^{-1}) \circ (T_\nu \circ T_{\nu_0}^{-1})^{-1}(\lambda \otimes v) && (4.2.5)\\ &= \lambda \circ P^\xi(\nu)^{-1} P^\xi(s(\nu-\rho)+\rho) \otimes v && (4.2.6) \end{aligned}$$

by Lemma 4.2.2. □

Let $A(\nu) = A_s(\nu)$ with s the longest element of the Weyl group. Then, for a minimal parabolic subgroup P of G, we have

$$J_{\overline{P}|P}(\nu)f = (c_\tau(\nu)A(\nu+\rho)f) \circ k^*$$

where $c_\tau(\nu)$ is Harish-Chandra c-function on the small K-type V_τ and k^* is a representative of $s \in W(A) = N_K(A)/Z_K(A)$. Therefore, determinant of $P^\xi(\nu)$ gives the shift factors in the classical gamma functions in Theorem 2.2.2 modulo those from Harish-Chandra c-function on the small K-type V_τ.

Chapter 5

$\widetilde{SL(n,\mathbb{R})}$ and Metalinear Group $\widetilde{GL(n,\mathbb{R})}$

5.1 The Group $\widetilde{SL(n,\mathbb{R})}$ and the Metalinear Group $\widetilde{GL(n,\mathbb{R})}$

Let $\widetilde{SL(n,\mathbb{R})}$ be the connected, simply connected covering group of $SL(n,\mathbb{R})$ for $n \geq 3$. If θ is the Cartan involution of $SL(n,\mathbb{R})$ defined by $\theta(g) = (g^{-1})^t$, the set of fixed points of θ is the maximal compact subgroup $SO(n)$ of $SL(n,\mathbb{R})$, and the set of fixed points of the lift of θ is the maximal compact subgroup $K = Spin(n)$ of $\widetilde{SL(n,\mathbb{R})}$.

Let $\mathfrak{g}_\circ = Lie(\widetilde{SL(n,\mathbb{R})}) = Lie(SL(n,\mathbb{R})) = \mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{a}_\circ \subseteq \mathfrak{g}_\circ$ be the subalgebra of diagonal matrices and $\mathfrak{n}_\circ \subseteq \mathfrak{g}_\circ$ the subalgebra of strictly upper triangular matrices. Let $P = {}^0MAN$ be a minimal parabolic subgroup of $\widetilde{SL(n,\mathbb{R})}$ with ${}^0M = Z_K(A)$, $A = exp(\mathfrak{a}_\circ)$, and $N = exp(\mathfrak{n}_\circ)$.

For $n \geq 3$, $\widetilde{SL(n,\mathbb{R})}$ is a two-fold covering group of $SL(n,\mathbb{R})$, and the group $\widetilde{SL(n,\mathbb{R})}$ is a central μ_2-extension of the group $SL(n,\mathbb{R})$. Denote by η the nontrivial element of the group μ_2. Then, $\eta \in {}^0M \leq K$ as η is central and the center is in K by Theorem 7.2.5 of [HAHS]. Hence, ${}^0M$ is a central μ_2-extension of the group ${}^0M_{SL(n,\mathbb{R})}$ of the diagonal elements of $SO(n)$.

$$\begin{array}{ccccc}
1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow \\
\mu_2 & & \mu_2 & & \mu_2 \\
\downarrow & & \downarrow & & \downarrow \\
{}^0M & \hookrightarrow & Spin(n) & \hookrightarrow & \widetilde{SL(n,\mathbb{R})} \\
\downarrow & & \downarrow & & \downarrow \\
{}^0M_{SL(n,\mathbb{R})} & \hookrightarrow & SO(n) & \hookrightarrow & SL(n,\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1
\end{array}$$

Similarly for $n \geq 3$, consider the Metalinear group $\widetilde{GL(n,\mathbb{R})}$, the double cover of $GL(n,\mathbb{R})$. In this case, $O(n)$ is the maximal compact subgroup of $GL(n,\mathbb{R})$, and $Pin(n)$ is the maximal compact subgroup of $\widetilde{GL(n,\mathbb{R})}$.

$$\begin{array}{ccccc}
1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow \\
\mu_2 & & \mu_2 & & \mu_2 \\
\downarrow & & \downarrow & & \downarrow \\
{}^0M & \hookrightarrow & Pin(n) & \hookrightarrow & \widetilde{GL(n,\mathbb{R})} \\
\downarrow & & \downarrow & & \downarrow \\
{}^0M_{GL(n,\mathbb{R})} & \hookrightarrow & O(n) & \hookrightarrow & GL(n,\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1
\end{array}$$

We now discuss in more detail the embedding described in chapter 3. Consider the standard embedding of $i : GL(n-1,\mathbb{R}) \hookrightarrow SL(n,\mathbb{R})$ by

$$i(g) = \begin{matrix} g & 0 \\ 0 & det(g)^{-1} \end{matrix}$$

Let $p : \widetilde{SL(n,\mathbb{R})} \to SL(n,\mathbb{R})$ be the covering homomorphism. Since $p^{-1}(i(GL(n-1,\mathbb{R}))) \cong \widetilde{GL(n-1,\mathbb{R})}$, there is a natural inclusion $\widetilde{i} : \widetilde{GL(n-1,\mathbb{R})} \hookrightarrow \widetilde{SL(n,\mathbb{R})}$. Under this inclusion, $Pin(n-1) \hookrightarrow Spin(n)$.

Lemma 5.1.1. *Let $n \geq 4$. Given the embedding $\widetilde{i}: \widetilde{GL(n-1,\mathbb{R})} \hookrightarrow \widetilde{SL(n,\mathbb{R})}$, we have $\widetilde{i}({}^0M_{\widetilde{GL(n-1,\mathbb{R})}}) = {}^0M_{\widetilde{SL(n,\mathbb{R})}}$.*

Proof. ${}^0M_{SL(n,\mathbb{R})} = \{\begin{pmatrix} g & \\ & deg(g)^{-1} \end{pmatrix} \mid g \in O(n-1) \text{ diagonal}\}$, the image of ${}^0M_{GL(n-1,\mathbb{R})}$ under the inclusion map i. Hence $i({}^0M_{GL(n-1,\mathbb{R})}) = {}^0M_{SL(n,\mathbb{R})}$. Consider the maximal parabolic subgroup $P_{max} = \{\begin{pmatrix} g & x \\ 0 & det(g)^{-1} \end{pmatrix} \mid g \in GL(n-1,\mathbb{R}),\ x \in \mathbb{R}^{n-1}\}$ of $\widetilde{SL(n,\mathbb{R})}$ with Levi factor $M_{P_{max}} = i(GL(n-1,\mathbb{R}))$. $P \leq P_{max}$ where P is the minimal parabolic subgroup of $SL(n,\mathbb{R})$ consisting of upper triangular matrices. Let $p: \widetilde{SL(n,\mathbb{R})} \to SL(n,\mathbb{R})$ be the covering homomorphism for $n \geq 4$, and let $\widetilde{P} = p^{-1}(P)$. Then, $\widetilde{P} = {}^0M_{\widetilde{SL(n,\mathbb{R})}}AN$ and $M_{\widetilde{P_{max}}} = p^{-1}(M_{P_{max}}) = p^{-1}(i(GL(n-1,\mathbb{R})))$. Thus ${}^0M_{\widetilde{SL(n,\mathbb{R})}} \subseteq p^{-1}(i(GL(n-1,\mathbb{R}))) = \widetilde{i}(\widetilde{GL(n-1,\mathbb{R})})$. Therefore, $\widetilde{i}({}^0M_{\widetilde{GL(n-1,\mathbb{R})}}) = {}^0M_{\widetilde{SL(n,\mathbb{R})}}$.

□

5.2 Irreducible Representations of $Pin(n)$ and Small $Pin(n)$ types relative to the Metalinear Group

We recall two theorems that can be found in section 5.5.5 of [GW] that describe irreducible regular representations of $Pin(n)$. The theorems are stated in terms of Orthogonal and Special Orthogonal groups in [GW]; however, the same statements are true for the pair Pin and Spin groups.

Definition 5.2.1.

- If $n = 2k+1$ is odd, let $g_0 = -I \in O(2k+1)$. If $n = 2k$ is even, let $g_0 \in O(2k)$ be the diagonal matrix whose entries are all 1 except for last $g_{0\,2k,2k} = -1$. Let $p: Pin(n) \to O(n)$ be the covering homomorphism and let ζ be any choice of $p^{-1}(g_0)$.

- Let (π_λ, V_λ) be the irreducible representation of $Spin(n)$ with highest weight λ and let $(\rho_\lambda, V_\lambda)$ be the induced representation $Ind_{Spin(2k)}^{Pin(2k)}(\pi_\lambda)$.

Theorem 5.2.2. *The irreducible regular representations of $Pin(2k+1)$ are of the form $(\pi_\lambda^\epsilon, V_\lambda^\epsilon)$, where $(\pi_\lambda^\epsilon, V_\lambda^\epsilon)$ restricted to $Spin(2k+1)$ is the highest weight representation (π_λ, V_λ), and ζ acts on V_λ^ϵ by ϵI where $\epsilon = \pm$.*

Theorem 5.2.3. *Let $k \geq 2$. The irreducible representation (σ, W) of $Pin(2k)$ is one of the following two types.*

- *Suppose $dim W^{\mathfrak{n}^+} = 1$ and $\mathfrak{h}$ acts by the weight λ on $W^{\mathfrak{n}^+}$. $(\sigma, W) \cong (\pi_\lambda^\epsilon, V_\lambda^\epsilon)$ where $(\pi_\lambda^\epsilon, V_\lambda^\epsilon)$ restricted to $Spin(2k)$ is the highest weight representation (π_λ, V_λ), and ζ acts on $W^{\mathfrak{n}^+}$ by ϵI where $\epsilon = \pm$.*
- *Suppose $dim W^{\mathfrak{n}^+} = 2$. Then $\mathfrak{h}$ has two distinct weights λ and $\zeta \cdot \lambda$ on $W^{\mathfrak{n}^+}$, and $(\sigma, W) \cong (\rho_\lambda, V_\lambda)$.*

We list the small $K = Pin(n)$-types for the Metalinear Group $\widetilde{GL(n, \mathbb{R})}$ by Theorem 5.2.2 and 5.2.3.

Definition 5.2.4.

- If n is even, let V_{T} be the Pin-representation.
- If n is odd, let V_{T} be either of the two Pin-representations.

5.3 Structure of irreducible K_α-modules in H_α and $H_\alpha \otimes V_\tau$

Recall that for the group $\widetilde{SL(n, \mathbb{R})}$ and the maximal compact subgroup $K = Spin(n)$ we have as the small K-type (τ, V_τ) the spin representation for n odd and either of the two half-spin representations for n even. Let $\alpha = \epsilon_i - \epsilon_j$ be a positive root of $Lie(\widetilde{SL(n, \mathbb{R})})$ and denote by E_α an n by n matrix with entry 1 in the position (i,j) and 0 elsewhere.

Definition 5.3.1. $\mathfrak{t}_\alpha = i(E_\alpha + \theta(E_\alpha))$.

Lemma 5.3.2. $Ad(^0M)|_{\mathfrak{t}_\alpha} = \{\pm 1\}$.

Proof. ${}^0M = Z_K(A)$, thus ${}^0M$ acts by a character on $\mathfrak{n}_\alpha$ because it is 1-dimensional. Since the square of any character of ${}^0M$ is equal to 1 by 2.2.2 of [RRG I], any element of ${}^0M$ must act on E_α by ± 1. Therefore, ${}^0M$ will act by ± 1 on $E_\alpha + \theta(E_\alpha)$, hence on $\mathfrak{t}_\alpha$. If α is a simple root, $exp(\pi i \mathfrak{t}_\alpha) \in {}^0M$ will act by $+1$ on $\mathfrak{t}_\alpha$ and if β is a simple root connected to α in the dynkin diagram, $exp(\pi i \mathfrak{t}_\beta) \in {}^0M$ will act by -1 on $\mathfrak{t}_\alpha$. Since all positive roots of $Lie(\widetilde{SL(n,\mathbb{R})})$ are conjugates by elements of the Weyl group $N_K(A)/Z_K(A)$ and $N_K(A)$ acts on ${}^0M$, for any positive root α there is an element of ${}^0M$ that will act by $+1$ and an element of ${}^0M$ that will act by -1. □

Definition 5.3.3.

- Let ${}^0M^+$ (*resp.* ${}^0M^-$) be the set of elements of ${}^0M$ that act on $\mathfrak{t}_\alpha$ by $+1$ (*resp.* -1).
- Let V_τ^+ (*resp.* V_τ^-) be the subspace of V_τ consisting of $\mathfrak{t}_\alpha$ weight vectors of weights $+\frac{1}{2}$ (*resp.* $-\frac{1}{2}$).

For a positive root $\alpha = \epsilon_i - \epsilon_j$ of $Lie(\widetilde{SL(n,\mathbb{R})})$, let $X_\alpha = E_{ii} - E_{jj}$, $Y_\alpha = E_\alpha - \theta(E_\alpha)$, $Z_\alpha = X_\alpha + iY_\alpha$ where E_{kk} is a diagonal matrix with entry 1 in the position (k,k) and 0 elsewhere and E_α is a matrix with entry 1 in the position (i,j) and 0 elsewhere. Then, $E_\alpha \in \mathfrak{n}_\alpha$, $\theta(E_\alpha) \in \mathfrak{n}_{-\alpha}$ and $\mathfrak{t}_\alpha \in \mathfrak{k} = \mathfrak{so}(n,\mathbb{C})$ with $[\mathfrak{t}_\alpha, \overline{Z_\alpha^l}] = 2l\overline{Z_\alpha^l}$ and $[\mathfrak{t}_\alpha, Z_\alpha^l] = -2lZ_\alpha^l$.

Definition 5.3.4.

- Let $\mathfrak{g}_{\alpha_\circ} = \mathfrak{a}_\circ \oplus \mathfrak{n}_{\alpha_\circ} \oplus \overline{\mathfrak{n}_{\alpha_\circ}}$ and G_α be the rank one subgroup of $\widetilde{SL(n,\mathbb{R})}$ generated by $exp(\mathfrak{g}_{\alpha_\circ})$ and ${}^0M$. G_α is the group generated by $Mp(2,\mathbb{R})_\alpha$ the semisimple part of the group generated by $exp(\mathfrak{g}_{\alpha_\circ})$, ${}^0M$, and $exp(\mathfrak{a}_\circ^\alpha)$ where $\mathfrak{a}_\circ^\alpha = \bigoplus_{\beta\in\Phi^+-\{\alpha\}} \mathbb{R}X_\beta$.
- Let K_α be the subgroup of K generated by $exp(i\mathbb{R}\mathfrak{t}_\alpha)$ and ${}^0M$, the maximal compact subgroup of G_α.
- Let H_α be the space of harmonics on $\mathfrak{p}_\alpha = \mathfrak{a} \oplus \mathbb{C}Y_\alpha$ for the group G_α. H_α as a space is $\bigoplus_{l\geq 0}(\mathbb{C}Z_\alpha^l \oplus \mathbb{C}\overline{Z_\alpha}^l)$.

- Let $\mathfrak{g}_{\alpha,\mathfrak{gl}(n,\mathbb{R})_\circ} = \mathbb{R}I_n \oplus \mathfrak{g}_{\alpha_\circ}$. Let $G_{\alpha,\widetilde{GL(n,\mathbb{R})}}$ be the rank one subgroup of the metalinear group $\widetilde{GL(n,\mathbb{R})}$ generated by $exp(\mathfrak{g}_{\alpha,\mathfrak{gl}(n,\mathbb{R})_\circ})$ and ${}^0M_{\widetilde{GL(n,\mathbb{R})}}$. $G_{\alpha,\widetilde{GL(n,\mathbb{R})}}$ is the group generated by $SL(2,\mathbb{R})_\alpha$ the semisimple part of the group generated by $exp(\mathfrak{g}_{\alpha,\mathfrak{gl}(n,\mathbb{R})_\circ})$, ${}^0M_{\widetilde{GL(n,\mathbb{R})}}$, and $exp(\mathbb{R}I_n \oplus \mathfrak{a}^\alpha_\circ)$ where $\mathfrak{a}^\alpha_\circ = \bigoplus_{\beta\in\Phi^+-\{\alpha\}} \mathbb{R}X_\beta$.

- Let $K_{\alpha,\widetilde{GL(n,\mathbb{R})}}$ be the subgroup of $Pin(n)$ generated by $exp(i\mathbb{R}\mathfrak{t}_\alpha)$ and ${}^0M_{\widetilde{GL(n,\mathbb{R})}}$, the maximal compact subgroup of $G_{\alpha,\widetilde{GL(n,\mathbb{R})}}$.

- Let $H_{\alpha,\mathfrak{gl}(n,\mathbb{R})}$ be the space of harmonics on $\mathfrak{p}_{\alpha,\mathfrak{gl}(n,\mathbb{R})} = \mathbb{C}I_n \oplus \mathfrak{a} \oplus \mathbb{C}Y_\alpha$ for the group $G_{\alpha,\widetilde{GL(n,\mathbb{R})}}$. $H_{\alpha,\mathfrak{gl}(n,\mathbb{R})}$ as a space is $\bigoplus_{l\geq 0}(\mathbb{C}Z^l_\alpha \oplus \mathbb{C}\overline{Z_\alpha}^l)$.

By Lemma 5.1.1, ${}^0M_{\widetilde{SL(n,\mathbb{R})}} \cong {}^0M_{\widetilde{GL(n-1,\mathbb{R})}}$. Therefore, the semisimple parts of G_α and $G_{\alpha,\widetilde{GL(n-1,\mathbb{R})}}$ are isomorphic and $K_\alpha \cong K_{\alpha,\widetilde{GL(n-1,\mathbb{R})}}$ for α a positive root of $Lie(\widetilde{GL(n-1,\mathbb{R})}) \subseteq Lie(\widetilde{SL(n,\mathbb{R})})$. Thus the space of harmonics for G_α and $G_{\alpha,\widetilde{GL(n-1,\mathbb{R})}}$ is the same with the same $K_\alpha \cong K_{\alpha,\widetilde{GL(n-1,\mathbb{R})}}$ action, and the following Lemmas and Theorems are true for both G_α and $G_{\alpha,\widetilde{GL(n-1,\mathbb{R})}}$.

Lemma 5.3.5. *Let V_{γ_α} be an irreducible nontrivial K_α-module that occurs in H_α. There exist exactly two $\mathfrak{t}_\alpha$-weights on V_{γ_α}, and they are dual representations.*

Proof. Start with the $\mathfrak{t}_\alpha$ weight vector of weight $2l$, $\overline{Z^l_\alpha}$. ${}^0M$ is a group of finite order that centralizes $\mathfrak{a}$ and ${}^0M^\pm$ act by ± 1 on $\mathfrak{t}_\alpha$. Therefore, ${}^0M^+$ will fix $\overline{Z^l_\alpha}$ and ${}^0M^-$ will move $\overline{Z^l_\alpha}$ to Z^l_α. Hence the weights are 2l and -2l. □

Lemma 5.3.6. *Let V_{ξ_α} be an irreducible K_α-module that occurs in $H_\alpha \otimes V_\tau$. There exist exactly two $\mathfrak{t}_\alpha$-weights on V_{ξ_α}, and they are dual.*

Proof. Let $v \in V_{\xi_\alpha}$ be a $\mathfrak{t}_\alpha$ weight vector of weight c, which is nonzero as it is in the form of $2j \pm \frac{1}{2}$, because the only weights of $\mathfrak{t}_\alpha$ acting on V_τ are $\pm\frac{1}{2}$. If $m \in {}^0M$, $\mathfrak{t}_\alpha.m.v = m.m^{-1}.\mathfrak{t}_\alpha.m.m^{-1}.m.v = \pm m.\mathfrak{t}_\alpha.v = \pm c * m.v$. Since ${}^0M$ acts irreducibly on V_{ξ_α} the result follows. □

Theorem 5.3.7. *Let V_{γ_α} be an irreducible K_α-module that occurs in H_α. V_{γ_α} as a space is the span of $\{\overline{Z^l_\alpha}, Z^l_\alpha\}$ for some l. Moreover, ${}^0M$-invariant elements are $\mathbb{C}(\overline{Z^l_\alpha} + Z^l_\alpha)$.*

Proof. This follows from Lemma 5.3.5. □

Theorem 5.3.8. *Let V_{ξ_α} be an irreducible K_α-module that occurs in $H_\alpha \otimes V_\tau$. V_{ξ_α} as a space is either $(\overline{Z^l_\alpha} \otimes V^+_\tau) \oplus (Z^l_\alpha \otimes V^-_\tau)$ or $(\overline{Z^l_\alpha} \otimes V^-_\tau) \oplus (Z^l_\alpha \otimes V^+_\tau)$ for some l.*

Proof. Consider $\overline{Z^l_\alpha} \otimes v$ for some l and $v \in V^+_\tau$. By Lemma 5.3.6, K_α module generated by this element is contained in $(\overline{Z^l_\alpha} \otimes V^+_\tau) \oplus (Z^l_\alpha \otimes V^-_\tau)$. Since the dimensions of the two spaces are equal the inclusion is an equality. We argue similarly if we start with a vector in V^-_τ. □

5.4 Frobenius Reciprocity in our Context

Recall H the space of harmonics on $\mathfrak{p}$ for the group $\widetilde{SL(n,\mathbb{R})}$, $H \cong I_{P,triv,\nu}$ and $H \otimes V_\tau \cong I_{P,\tau,\nu}$ as $Spin(n)$-modules where V_τ is the Spin representation for n odd and either of the two half-Spin representations for n even. The space of harmonics H for the metalinear group $\widetilde{GL(n,\mathbb{R})}$ is the same as that of the group $\widetilde{SL(n,\mathbb{R})}$, and $H \otimes V_{\mathrm{T}} \cong I_{P,\mathrm{T},\nu}$ as $Pin(n)$-modules where V_{T} is either of the two Pin representations for n odd and the Pin representation for n even.

Lemma 5.4.1.

- *If V_ξ is an irreducible $Spin(n)$ module that occurs in $H \otimes V_\tau$,*

$$V_\xi|_{{}^0M_{\widetilde{SL(n,\mathbb{R})}}} = \bigoplus_{j=1}^{dim(V_\xi)/dim(V_\tau)} V_{\tau_j}$$

  where $V_{\tau_j} \cong V_\tau$ as ${}^0M_{\widetilde{SL(n,\mathbb{R})}}$-modules for all j.

- *If V_Ξ is an irreducible $Pin(n)$ module that occurs in $H \otimes V_{\mathrm{T}}$,*

$$V_\Xi|_{{}^0M_{\widetilde{GL(n,\mathbb{R})}}} = \bigoplus_{j=1}^{dim(V_\Xi)/dim(V_{\mathrm{T}})} V_{\mathrm{T}_j}$$

  where $V_{\mathrm{T}_j} \cong V_{\mathrm{T}}$ as ${}^0M_{\widetilde{GL(n,\mathbb{R})}}$-modules for all j.

Proof. First consider the statement for the group $\widetilde{SL(n,\mathbb{R})}$. $\mathbb{C}[{}^0M_{\widetilde{SL(n,\mathbb{R})}}]/< \eta+1 >$ is the subalgebra of $Cliff_n$ spanned by the even number of elements of the usual basis elements from 11.A.2.8 of [RRG II]. Hence we have the result if n is odd. If n is even, V_{τ_j} can be either of the half spin representations restricted to ${}^0M_{\widetilde{SL(n,\mathbb{R})}}$. I claim that only $V_\tau|_{{}^0M_{\widetilde{SL(n,\mathbb{R})}}}$ is allowed. Let V_τ and $\overline{V_\tau}$ be the two half spin representation of $Spin(n)$ for n even, and let ω be a choice of $p^{-1}(-Id)$ where $p : Spin(n) \to SO(n)$. ω distinguishes the two representations as ${}^0M_{\widetilde{SL(n-1,\mathbb{R})}}$ do not. ω acts trivially on H as $-Id \in {}^0M_{SL(n,\mathbb{R})}$ and is central in $SO(n)$. Therefore, $(H \otimes V_\tau)|_{{}^0M_{\widetilde{SL(n,\mathbb{R})}}} = \bigoplus V_{\tau_j}$ with $V_{\tau_j} \cong V_\tau$ as ${}^0M_{\widetilde{SL(n,\mathbb{R})}}$-modules, and $(H \otimes \overline{V_\tau})|_{{}^0M_{\widetilde{SL(n,\mathbb{R})}}} = \bigoplus V_{\tau_j}$ with $V_{\tau_j} \cong \overline{V_\tau}$ as ${}^0M_{\widetilde{SL(n,\mathbb{R})}}$-modules.

Consider the statement for the metalinear group $\widetilde{GL(n,\mathbb{R})}$. By Lemma 5.1.1, ${}^0M_{\widetilde{GL(n-1,\mathbb{R})}} \cong {}^0M_{\widetilde{SL(n,\mathbb{R})}}$. Therefore, we have the result for n even similarly as above. If n is odd, we argue similarly as above using ζ defined in 5.2.1. $\square$

Lemma 5.4.2.

- *Let V_ξ be an irreducible $Spin(n)$ module that occurs in $H \otimes V_\tau$, and let $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct $Spin(n)$-types that occur in H, with $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$ for all j. If $l(\gamma_j) = dim V_{\gamma_j}^{{}^0M_{\widetilde{SL(n,\mathbb{R})}}}$, then $dim(V_\xi)/dim(V_\tau) = dim\, Hom_{Spin(n)}(V_\xi, H \otimes V_\tau) = \Sigma_{j=1}^N l(\gamma_j)$.*

- *Let V_Ξ be an irreducible $Pin(n)$ module that occurs in $H \otimes V_\mathrm{T}$, and let $V_{\Gamma_1}, ..., V_{\Gamma_L}$ be distinct $Pin(n)$-types that occur in H, with $V_\Xi \subseteq V_{\Gamma_j} \otimes V_\mathrm{T}$ for all j. If $l(\Gamma_j) = dim V_{\Gamma_j}^{{}^0M_{\widetilde{GL(n,\mathbb{R})}}}$, then $dim(V_\Xi)/dim(V_\mathrm{T}) = dim\, Hom_{Pin(n)}(V_\Xi, H \otimes V_\mathrm{T}) = \Sigma_{j=1}^L l(\Gamma_j)$.*

Proof. Since V_τ is multiplicity free, by Corollary 3.4 of [Ku], we know V_ξ occurs in $V_\gamma \otimes V_\tau$ exactly once if it does. Now each of V_{γ_j} occurs in H exactly $l(\gamma_j)$ many times by Frobenius Reciprocity, hence the multiplicity of V_ξ in $H \otimes V_\tau$ is exactly $\Sigma_{j=1}^N l(\gamma_j)$ which is $dim(V_\xi)/dim(V_\tau)$ by Lemma 5.4.1 and Frobenius Reciprocity. We argue similarly for the statement of the metalinear group $\widetilde{GL(n,\mathbb{R})}$. $\square$

5.5 $\mathfrak{t}_\alpha$-weights on certain vectors

Recall the definition of $\mathfrak{t}_\alpha$ from 5.3.1. Let V_γ be an irreducible $Spin(n)$-module that occurs in the harmonics H in $\mathfrak{p}$ of $\widetilde{SL(n,\mathbb{R})}$. Let Span $K_\alpha.V_\gamma^{{}^0M_{\widetilde{SL(n,\mathbb{R})}}} = \bigoplus_{j=1}^{l(\gamma)} W_j$ be a decomposition into irreducible K_α-modules. Let V_Γ be an irreducible $Pin(n)$-module that occurs in the harmonics on $\mathfrak{p}$ of $\widetilde{GL(n,\mathbb{R})}$. Let Span $K_{\alpha,\widetilde{GL(n,\mathbb{R})}}\cdot V_\Gamma^{{}^0M_{\widetilde{GL(n,\mathbb{R})}}} = \bigoplus_{j=1}^{l(\Gamma)} X_j$ a decomposition into irreducible $K_{\alpha,\widetilde{GL(n,\mathbb{R})}}$-modules where $K_{\alpha,\widetilde{GL(n,\mathbb{R})}}$ is the group generated by the torus and ${}^0M_{\widetilde{GL(n,\mathbb{R})}}$.

Definition 5.5.1.

- Let $\delta^\gamma_{\alpha,j}$ be the dominant $\mathfrak{t}_\alpha$ weight on W_j for $j = 1, ..., l(\gamma)$ given by Lemma 5.3.5.

- Let $\delta^\Gamma_{\alpha,j}$ be the dominant $\mathfrak{t}_\alpha$ weight on X_j for $j = 1, ..., l(\Gamma)$ by the remark in section 5.3.

Let V_ξ be an irreducible $Spin(n)$-module that occurs in $H \otimes V_\tau$. Let $V_\xi = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j}$ be a decomposition into irreducible K_α-modules where $V_{\tau_j} \cong V_\tau$ as ${}^0M_{\widetilde{SL(n,\mathbb{R})}}$-modules for all j by Lemma 5.4.1.

Let V_Ξ be an irreducible $Pin(n)$-module that occurs in $H \otimes V_{\mathrm{T}}$. Let $V_\Xi = \bigoplus_{j=1}^{n(\Xi)} V_{\mathrm{T}_j}$ be a decomposition into irreducible $K_{\alpha,\widetilde{GL(n,\mathbb{R})}}$-modules where $V_{\mathrm{T}_j} \cong V_{\mathrm{T}}$ as ${}^0M_{\widetilde{GL(n,\mathbb{R})}}$-modules for all j by the remark after Lemma 5.4.1.

Definition 5.5.2.

- Let $\delta^\xi_{\alpha,j}$ be the dominant $\mathfrak{t}_\alpha$ weight on V_{τ_j} for $j = 1, ..., n(\xi)$ given by Lemma 5.3.6.

- Let $\delta^\Xi_{\alpha,j}$ be the dominant $\mathfrak{t}_\alpha$ weight on V_{T_j} for $j = 1, ..., n(\Xi)$ by the remark in section 5.3.

5.6 Restriction of $Pin(n)$-modules to $Spin(n)$ and $\mathfrak{t}_\alpha$-weights

Let V_γ be an irreducible $Spin(n)$-module that occurs in the harmonics on $\mathfrak{p}$ of $\widetilde{SL(n,\mathbb{R})}$, and V_Γ be an irreducible $Pin(n)$-module that occurs in the harmonics on $\mathfrak{p}$ of $\widetilde{GL(n,\mathbb{R})}$.

Theorem 5.6.1. *Let n=2k, $(m_1,...,m_k)$ be the highest weight of V_γ, and assume V_γ occurs in V_Γ if we restrict to Spin(n) from Pin(n).*

- *If $m_k \neq 0$, then dim $V_\gamma^{{}^0M_{\widetilde{SL(n,\mathbb{R})}}}$ = dim $V_\Gamma^{{}^0M_{\widetilde{GL(n,\mathbb{R})}}}$, $l(\gamma) = l(\Gamma)$ and $\delta^\gamma_{\alpha,j} = \delta^\Gamma_{\alpha,j}$ for all j after reordering.*
- *If $m_k = 0$, let $V_\Gamma = V_\gamma^\epsilon$ where $\epsilon = \pm$ is the signature of g_0 on the highest weight vector given to us by Theorem 5.2.3. Then, $V_\gamma^{{}^0M_{\widetilde{SL(n,\mathbb{R})}}} = V_\gamma^{+\,{}^0M_{\widetilde{GL(n,\mathbb{R})}}} \oplus V_\gamma^{-\,{}^0M_{\widetilde{GL(n,\mathbb{R})}}} = V_\Gamma^{+\,{}^0M_{\widetilde{GL(n,\mathbb{R})}}} \oplus V_\Gamma^{-\,{}^0M_{\widetilde{GL(n,\mathbb{R})}}}$, and hence $\{\delta^\gamma_{\alpha,1},...,\delta^\gamma_{\alpha,l(\gamma)}\}$ is a disjoint union of that of $V_\gamma^{+\,{}^0M_{\widetilde{GL(n,\mathbb{R})}}}$ and $V_\gamma^{-\,{}^0M_{\widetilde{GL(n,\mathbb{R})}}}$.*

Proof. Since we are working with submodules of the harmonics, η acts trivially, hence we can ignore the tilde and work with ${}^0M_{SL(n,\mathbb{R})}$-invariants of $SO(n)$-modules and ${}^0M_{GL(n,\mathbb{R})}$-invariants of $O(n)$-modules. Also, remember $g_0 \in {}^0M_{GL(n,\mathbb{R})}$.

Let us assume first $m_k \neq 0$. Then, g_0 swaps the two $SO(2k)$ highest weight modules of highest weights $(m_1,...,m_k)$ and $(m_1,...,-m_k)$. Since g_0 commutes with ${}^0M_{SL(n,\mathbb{R})}$, g_0 will give us a bijection of ${}^0M_{SL(n,\mathbb{R})}$-invariants in the $SO(2k)$ highest weight representation of highest weight $(m_1,...,m_k)$ with ${}^0M_{SL(n,\mathbb{R})}$-invariants in the $SO(2k)$ highest weight representation of highest weight $(m_1,...,-m_k)$. Hence, it is now clear that dim $V_\gamma^{{}^0M_{\widetilde{SL(n,\mathbb{R})}}}$ = dim $V_\Gamma^{{}^0M_{\widetilde{GL(n,\mathbb{R})}}}$ as ${}^0M_{\widetilde{GL(n,\mathbb{R})}}$ is generated by ${}^0M_{\widetilde{SL(n,\mathbb{R})}}$ and ζ, a choice of $p^{-1}(g_0)$. Since g_0 leaves invariant $\mathfrak{t}^2_\alpha$, the statement of the $\mathfrak{t}_\alpha$-weights is now also clear with the help of Lemma 5.3.5.

Let us now assume $m_k = 0$, $v(m_1,...,m_k)$ the highest weight vector, and $v_1,...,v_{l(\gamma)}$ a basis of $V_\gamma^{{}^0M_{\widetilde{SL(n,\mathbb{R})}}}$ such that g_0 acts on v_j by ±1 for all j, which is possible since $g_0^2 = Id$ and g_0 commutes with ${}^0M_{SL(n,\mathbb{R})}$. Denote by $v_1^+,...,v^+_{l(\gamma)}$ and $v_1^-,...,v^-_{l(\gamma)}$ above basis thought of being in V_γ^+ and V_γ^- respectively. Now the only difference between V_γ^+ and V_γ^- is the action of g_0. If we denote by ϵ the action of g_0 on $v_1^+, ..., v_{l(\gamma)}^+$, I claim that g_0 will act by $-\epsilon$ on $v_1^-, ..., v_{l(\gamma)}^-$. Indeed, $v_j = X_j.v(m_1, ..., m_k)$ where $X_j \in U(\overline{\mathfrak{n}})$. Since g_0 acts by different signatures on $v(m_1, ..., m_k)$ for V_γ^+ and V_γ^-, the statement is now clear. □

Theorem 5.6.2. *Let n=2k+1 and assume V_γ occurs in V_Γ if we restrict to $Spin(n)$ from $Pin(n)$. Then, $g_0 = -Id$ will act trivially, and there is no difference between V_γ and V_Γ.*

Proof. We have $-Id \in Z(Pin(n))$ and $-Id \in {}^0M_{GL(n,\mathbb{R})}$, hence in order for V_Γ to be in the harmonics, it must act trivially. Now we have the result by Theorem 5.2.2. □

Let V_Ξ be an irreducible $Pin(n)$-module whose restriction to $Spin(n)$ contains a copy of V_ξ where $V_\xi \subseteq H \otimes V_\tau$. Let $V_\xi = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j}$ be a decomposition into irreducible K_α-modules. Recall the definition of ζ from 5.2.1.

Theorem 5.6.3. *If n=2k, then $V_\Xi|_{{}^0M_{\widetilde{GL(n,\mathbb{R})}}} = \bigoplus_{j=1}^{n(\xi)} (V_{\tau_j} \oplus \zeta.V_{\tau_j})$ and if n=2k+1, then $V_\Xi|_{{}^0M_{\widetilde{GL(n,\mathbb{R})}}} = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j}$. In either case, $\delta_{\alpha,j}^\xi = \delta_{\alpha,j}^\Xi$ for all j after reordering.*

Proof. Since ${}^0M_{\widetilde{GL(n,\mathbb{R})}}$ is generated by ${}^0M_{\widetilde{SL(n,\mathbb{R})}}$ and ζ, the statement is clear now along with the help of theorems 5.2.2 and 5.2.3. □

Chapter 6

Product Formula of p_ξ for $\widetilde{SL(n,\mathbb{R})}$

6.1 Comparison of $\mathfrak{t}_\alpha$-weights

Consider for the group $\widetilde{SL(n,\mathbb{R})}$ the small $K = Spin(n)$-type V_τ the spin representation for n odd and either of the half spin representations for n even. Let V_ξ be an irreducible $Spin(n)$-module that occurs in $H \otimes V_\tau$ and $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct $K = Spin(n)$-types that occur in H such that $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$ for all j. Let α be a positive root of $\mathfrak{sl}(n,\mathbb{R})$, $\{\delta^\xi_{\alpha,1}, ..., \delta^\xi_{\alpha,n(\xi)}\}$ be the set of $\mathfrak{t}_\alpha$ weights on V_ξ defined in 5.5.2, and let $\{\delta_{\alpha,1}, ..., \delta_{\alpha,\Sigma_{j=1}^N l(\gamma_j)}\}$ be the ones from $V_{\gamma_1}, ..., V_{\gamma_N}$ defined in 5.5.1 where $n(\xi) = dim\ Hom_K(V_\xi, H \otimes V_\tau)$ and $l(\gamma_j) = dim\ V_{\gamma_j}^{0M}$.

Theorem 6.1.1. *$n(\xi) = \Sigma_{j=1}^N l(\gamma_j)$ and we can reorder the set $\{\delta^\xi_{\alpha,1}, ..., \delta^\xi_{\alpha,n(\xi)}\}$ so that $\delta_{\alpha,j} = \delta^\xi_{\alpha,j} \pm \frac{1}{2}$ for all j.*

We first state and prove a lemma for the theorem.

Lemma 6.1.2. *Assume the statement of $\mathfrak{t}_\alpha$-weights in Theorem 6.1.1 for the modules of the group $Spin(n)$. Then the statement of $\mathfrak{t}_\alpha$-weights in Theorem 6.1.1 for the modules of the group $Pin(n)$ is also true.*

Proof. Assume first n is odd. The space of Harmonics on $\mathfrak{p}$ are the same for both $\widetilde{SL(n,\mathbb{R})}$ and $\widetilde{GL(n,\mathbb{R})}$, and the small $Pin(n)$-type V_{T} is the small $Spin(n)$-type V_τ if we restrict from $Pin(n)$ to $Spin(n)$. By Theorem 5.6.2, the restriction of

$Pin(n)$ to $Spin(n)$ will not change the assumptions of the modules in Theorem 6.1.1.

Given an irreducible $Pin(n)$-module that occurs in the Harmonics, the set of $\mathfrak{t}_\alpha$ weights of interest do not change restricted to $Spin(n)$ by Theorem 5.6.2. Given an irreducible $Pin(n)$-module that occurs in $H \otimes V_{\mathrm{T}}$, the set of $\mathfrak{t}_\alpha$ weights of interest do not change restricted to $Spin(n)$ by Theorem 5.6.3. Therefore, once we restrict $Pin(n)$ to $Spin(n)$, the comparison of $\mathfrak{t}_\alpha$-weights for the group $Pin(n)$ is that of $\mathfrak{t}_\alpha$-weights for the group $Spin(n)$.

Assume now n is even. The space of Harmonics on $\mathfrak{p}$ are the same for both $\widetilde{SL(n,\mathbb{R})}$ and $\widetilde{GL(n,\mathbb{R})}$, and the small $Pin(n)$-type V_{T} is a direct sum of the two half-spin representations V_τ and $\overline{V_\tau}$ if we restrict from $Pin(n)$ to $Spin(n)$. Let $V_{\Gamma_1}, ..., V_{\Gamma_M}$ be distinct $Pin(n)$ types that occur in H such that $V_\Xi \subseteq V_{\Gamma_j} \otimes V_{\mathrm{T}}$. Let us restrict V_{Γ_j} to $Spin(n)$. By Theorem 5.2.3, if $dim\ V_{\Gamma_j}^{\mathfrak{n}^+} = 2$, V_{Γ_j} is a direct sum of two irreducible $Spin(n)$ modules with last entries of the highest weights nonzero and negatives of each other, and if $dim\ V_{\Gamma_j}^{\mathfrak{n}^+} = 1$, V_{Γ_j} is irreducible as a $Spin(n)$ module. For each j, let V_{γ_j} be the choice of the irreducible $Spin(n)$-module that occurs in $V_{\Gamma_j}|_{Spin(n)}$ with last entry of the highest weight nonnegative, and reorder so that $V_{\gamma_1}, ..., V_{\gamma_N}$ are distinct $Spin(n)$ modules. $N \leq M$ as there may be j such that V_{γ_j} occurs twice with different g_0 signature on $V_{\gamma_j}^{\mathfrak{n}^+}$. Let V_ξ be the choice of the irreducible $Spin(n)$-module that occurs in $V_\Xi|_{Spin(n)}$ with last entry of the highest weight positive. Without loss of generality, assume $V_\xi \subseteq H \otimes V_\tau$. $V_{\gamma_1}, ..., V_{\gamma_N}$ are distinct $Spin(n)$ modules that occur in H such that $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$. Therefore, we can assume the statement of the $\mathfrak{t}_\alpha$ weights on these $Spin(n)$ modules. But by Theorem 5.6.1 and Theorem 5.6.3, comparison of $\mathfrak{t}_\alpha$ weights for the modules of the group $Pin(n)$ is that of $V_\xi, V_{\gamma_1}, ..., V_{\gamma_N}$ of $Spin(n)$. Therefore, we have the result for n even.

□

Proof. (Theorem 6.1.1)

$n(\xi) = \Sigma_{j=1}^{N} l(\gamma_j)$ by Lemma 5.4.2.

The Weyl group $W(A)$ of $\widetilde{SL(n,\mathbb{R})}$ is the symmetric group on n elements that permute the roots of $Lie(\widetilde{SL(n,\mathbb{R})})$, hence the positive roots of $Lie(\widetilde{SL(n,\mathbb{R})})$ are permuted by elements of $W(A)$. Since $\widetilde{SL(n,\mathbb{R})}$ is split, the Weyl group $W(A)$ is isomorphic to $N_K(A)/Z_K(A)$. Therefore, all positive roots of $Lie(\widetilde{SL(n,\mathbb{R})})$ are conjugate to each other by elements of $K = Spin(n)$, hence t_αs also. Thus the set of $\mathfrak{t}_\alpha$-weights of interest is independent of the choice of α.

Let n=3. Denote by λ the highest weight of V_ξ. Let $\lambda = \frac{p}{2}$ with p odd. If $p = 1$, there exists only one $V_\gamma \subseteq H$ with $V_\xi \subseteq V_\gamma \otimes V_\tau$ the trivial representation, and the claim is true. If $p = 3$, there exists only one $V_\gamma \subseteq H$ with $V_\xi \subseteq V_\gamma \otimes V_\tau$ the representation with highest weight 2. In this case, the weights are $\frac{1}{2}$ and $\frac{3}{2}$ for V_ξ and 0 and 2 for V_γ, hence the claim is also true. Suppose $p > 3$. Then, there exist exactly two such representations, call them $V_{\gamma_{p_1}}$ and $V_{\gamma_{p_2}}$ with highest weights $\frac{p-1}{2}$ and $\frac{p+1}{2}$. The weights of interest on V_ξ are $\frac{1}{2}, \frac{3}{2}, ..., \frac{p}{2}$. Now, the weights of interest on $V_{\gamma_{p_1}}$ and $V_{\gamma_{p_2}}$ are $0, 2, 4, ..., \frac{p-1}{2}$ and $2, 4, ..., \frac{p-1}{2}$ respectively if $\frac{p-1}{2}$ is even, and $2, 4, ..., \frac{p-3}{2}$ and $0, 2, ..., \frac{p+1}{2}$ respectively if $\frac{p+1}{2}$ is even.

The set of the weights of interest on V_ξ are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}..., \frac{p-2}{2}, \frac{p}{2}$. In the first case where $\frac{p-1}{2}$ is even, consider $(\frac{1}{2}-\frac{1}{2}), (\frac{3}{2}+\frac{1}{2}), (\frac{5}{2}-\frac{1}{2}), (\frac{7}{2}+\frac{1}{2}), ..., (\frac{p-2}{2}+\frac{1}{2}), (\frac{p}{2}-\frac{1}{2})$, which is $0, 2, 2, 4, 4, ..., \frac{p-1}{2}, \frac{p-1}{2}$. This is exactly the union of the weights on $V_{\gamma_{p_1}}$ and $V_{\gamma_{p_2}}$. In the second case where $\frac{p+1}{2}$ is even, consider $(\frac{1}{2} - \frac{1}{2}), (\frac{3}{2} + \frac{1}{2}), (\frac{5}{2} - \frac{1}{2}), (\frac{7}{2} + \frac{1}{2}), ..., (\frac{p-2}{2} - \frac{1}{2}), (\frac{p}{2} + \frac{1}{2})$, which is $0, 2, 2, 4, 4, ..., \frac{p-3}{2}, \frac{p-3}{2}, \frac{p+1}{2}$. This is again exactly the union of the weights on $V_{\gamma_{p_1}}$ and $V_{\gamma_{p_2}}$. Therefore the statement of the theorem is true for the case n=3.

We now proceed by induction. Assume the statement of the theorem for $\widetilde{SL(n,\mathbb{R})}$, and hence for $\widetilde{GL(n,\mathbb{R})}$ by Lemma 6.1.2. We prove the statement of the theorem for $\widetilde{SL(n+1,\mathbb{R})}$. Consider the embedding $\widetilde{GL(n,\mathbb{R})} \hookrightarrow \widetilde{SL(n+1,\mathbb{R})}$ with ${}^0M$ the same by Lemma 5.1.1.

We can restate the condition $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$ with $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$ where V_τ^* is the contragradient representation. Note the statement of the theorem is true with the restated condition for $\widetilde{GL(n,\mathbb{R})}$ by the induction hypothesis.

Let $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct irreducible $Spin(n+1)$-modules that occur in H such that $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$, and let $\bigoplus_{j=1}^N Span\ Pin(n).V_{\gamma_j}^{{}^0M} = \bigoplus_k W_k$ where each W_k is an irreducible $Pin(n)$-module. As the nontrivial element $\eta \in Z = \mu_2 \leq {}^0M$

acts by -1, $V_\xi|_{Pin(n)} = \bigoplus_j V_{\xi_j}$ where each of V_{ξ_j} occurs in $H \otimes V_\tau$ by Lemma 5.4.1, with H that of $\widetilde{GL(n,\mathbb{R})}$. We have $\bigoplus_k W_k \subseteq \bigoplus_j V_{\xi_j} \otimes V_\tau^*$ where each of W_k occurs in H of $\widetilde{GL(n,\mathbb{R})}$ as $\tilde{i}({}^0M_{\widetilde{GL(n-1,\mathbb{R})}}) = {}^0M_{\widetilde{SL(n,\mathbb{R})}}$ by Lemma 5.1.1 where $\tilde{i} : \widetilde{GL(n-1,\mathbb{R})} \hookrightarrow \widetilde{SL(n,\mathbb{R})}$ is the inclusion map.

Since the statement of the theorem is true for $\widetilde{GL(n,\mathbb{R})}$ with the restated condition and as the set of $\mathfrak{t}_\alpha$ weights of interest are the same after branching down to $Pin(n)$ as $\tilde{i} : \widetilde{GL(n-1,\mathbb{R})} \hookrightarrow \widetilde{SL(n,\mathbb{R})}$ by Lemma 5.1.1, we have the statement of the theorem for all positive roots α of $Lie(\widetilde{GL(n,\mathbb{R})}) \subseteq Lie(\widetilde{SL(n+1,\mathbb{R})})$ once we realize that $n(\xi) = \Sigma_{j=1}^N l(\gamma_j)$. Note $V_{\xi_j} \otimes V_\tau^*$ decomposes into distinct $Pin(n)$-modules by Corollary 3.4 of [Ku] as V_τ^* is multiplicity free. Therefore, if $W_k \cong W_l$ with $k \neq l$, then W_k and W_l cannot be contained in a single $V_{\xi_j} \otimes V_\tau^*$, important as the statement of the theorem for $\widetilde{GL(n,\mathbb{R})}$ also assumes distinct V_Γs. As the set of $\mathfrak{t}_\alpha$-weights of interest is independent of the choice of α of $Lie(\widetilde{SL(n+1,\mathbb{R})})$, we have the statement of the theorem. □

6.2 Divisibility

6.2.1 Definition of the P^ξ matrix revisited

Let $\mathfrak{g}_\circ = Lie(\widetilde{SL(n,\mathbb{R})}) = Lie(SL(n,\mathbb{R})) = \mathfrak{sl}(n,\mathbb{R})$ and recall the definitions of $\mathfrak{g}, \mathfrak{k}$, and $\mathfrak{p}$ from Chapter 2. H is the space of harmonics on $\mathfrak{p}$, J the subspace of K invariants in $S(\mathfrak{p})$, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Let $symm : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ be the symmetrization map.

The following is Lemma 1.4.2 of [Kos].

Lemma 6.2.2. $U(\mathfrak{g}) = symm(H)symm(J) \oplus U(\mathfrak{g})\mathfrak{k}$

The following is a Theorem from 11.3.6 of [RRG II].

Theorem 6.2.3. *$U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^{\mathfrak{k}}} V_\tau \cong I_{P,\tau,\nu}$ as K-modules.*

$symm(S(\mathfrak{p})) = symm(H)symm(J)$ from Lemma 6.2.2. Thus, $symm(H) \otimes V_\tau \cong I_{P,\tau,\nu}$ as K-modules from Theorem 6.2.3. Let V_ξ be a K-type that occurs in

$symm(H) \otimes V_\tau \cong I_{P,\tau,\nu}$. From Lemma 5.4.1,

$$V_\xi|_{^0M} = \bigoplus_{j=1}^{dim(V_\xi)/dim(V_\tau)} V_{\tau_j}$$

where each $V_{\tau_j} \cong V_\tau$ as ${}^0M$-modules.

Recall the definition of P^ξ matrix in section 4.1. To cut notations a little, set $V_{\tau_j} := T_j(V_\tau)$. Then, $P^\xi(\nu)$ is an $n(\xi)$ by $n(\xi)$ matrix with

$$(P^\xi(\nu))_{i,j} = (\epsilon_i(V_{\tau_j})(e))$$

Each entry of P^ξ is an element of $U(\mathfrak{a})$, or really an element of $U(\mathfrak{a}) \otimes End(V_\tau)$. Note $P^\xi(\nu)$ is without ρ-shift.

We are interested in $p_\xi(\nu)$ and p_ξ, determinants of $P^\xi(\nu)$ and P^ξ respectively, which will not depend on the choice of the bases up to a nonzero scalar multiple.

6.2.4 Divisibility

Let again $\mathfrak{a} \subseteq \mathfrak{g}$ be the subalgebra of diagonal elements and $\mathfrak{n} \subseteq \mathfrak{g}$ the subalgebra of strictly upper triangular elements. If α is a positive root of $\mathfrak{g}$, i.e. $\alpha = \epsilon_i - \epsilon_j$ for $1 \leq i < j \leq n$, recall E_α is an n by n matrix with entry 1 in the position (i,j) and 0 elsewhere. Let $\mathfrak{g}_\alpha$ be the Lie subalgebra of $\mathfrak{g}$ generated by E_α, $\theta(E_\alpha)$, and $\mathfrak{a}$. Then, $\mathfrak{g}_\alpha = \mathbb{C}\theta(E_\alpha) \oplus \mathfrak{a} \oplus \mathbb{C}E_\alpha = \theta\mathfrak{n}_\alpha \oplus \mathfrak{a} \oplus \mathfrak{n}_\alpha$ is the triangular decomposition, and $\mathfrak{g}_\alpha = \mathbb{C}(E_\alpha + \theta(E_\alpha)) \oplus (\mathbb{C}(E_\alpha - \theta(E_\alpha)) \oplus \mathfrak{a}) = \mathfrak{k}_\alpha \oplus \mathfrak{p}_\alpha$ is the Cartan decomposition. H_α is the space of harmonics on $\mathfrak{p}_\alpha$ as discussed in 5.3, and let $J_\alpha = S(\mathfrak{p}_\alpha)^{\mathfrak{k}_\alpha}$.

For $\alpha \in \Delta^+$ simple, let $\mathfrak{n}^\alpha = \bigoplus_{\psi \in \Phi^+ - \{\alpha\}} \mathfrak{g}^\psi$. Let $\mathfrak{k}^\alpha \subseteq \mathfrak{k}$ be spanned by $E_\psi + \theta E_\psi$ with $\psi \in \Phi^+ - \{\alpha\}$ so that $\mathfrak{k} = \mathfrak{k}_\alpha \oplus \mathfrak{k}^\alpha$. Then, $\mathfrak{g} = \mathfrak{n}^\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{k}^\alpha$. Note $\mathfrak{n}^\alpha$ is a Lie subalgebra of $\mathfrak{g}$ as α is simple.

Lemma 6.2.5. *For $\alpha \in \Delta^+$ simple,*

$$U(\mathfrak{g}) = symm(H_\alpha)symm(J_\alpha)U(\mathfrak{k}) \oplus \mathfrak{n}^\alpha U(\mathfrak{n}^\alpha)symm(H_\alpha)symm(J_\alpha)U(\mathfrak{k})$$

Proof. From Proposition 2.4.1 of [Kos], we have

$$U(\mathfrak{g}) = U(\mathfrak{n}^{\alpha})symm(H_{\alpha})symm(J_{\alpha}) \oplus U(\mathfrak{g})\mathfrak{k}$$

Hence, we have

$$U(\mathfrak{g}) = U(\mathfrak{n}^{\alpha})symm(H_{\alpha})symm(J_{\alpha})U(\mathfrak{k}) \tag{6.2.1}$$

$$= symm(H_{\alpha})symm(J_{\alpha})U(\mathfrak{k}) \oplus \mathfrak{n}^{\alpha}U(\mathfrak{n}^{\alpha})symm(H_{\alpha})symm(J_{\alpha})U(\mathfrak{k}) \tag{6.2.2}$$

□

Hence now by Theorem 6.2.3 and Lemma 6.2.5, we have the following K-module isomorphisms

$$I_{P,\tau,\nu} \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^{\mathfrak{k}}} V_{\tau} \tag{6.2.3}$$

$$\cong symm(H_{\alpha})symm(J_{\alpha}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^{\mathfrak{k}}} V_{\tau} \tag{6.2.4}$$

$$\oplus \;\; \mathfrak{n}^{\alpha}U(\mathfrak{n}^{\alpha})symm(H_{\alpha})symm(J_{\alpha}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^{\mathfrak{k}}} V_{\tau} \tag{6.2.5}$$

Let $\mathbb{C}[{}^0M]$ be the group algebra generated by ${}^0M$, and denote by $U(\mathfrak{k}_{\alpha})\Pi\mathbb{C}[{}^0M]$ the smash product of $U(\mathfrak{k}_{\alpha})$ with $\mathbb{C}[{}^0M]$, i.e. $U(\mathfrak{k}_{\alpha})\Pi\mathbb{C}[{}^0M]$ has a $(U(\mathfrak{k}_{\alpha}), \mathbb{C}[{}^0M])$-action on $symm(H_{\alpha}) \otimes V_{\tau}$ that is an analog of a $(\mathfrak{g}, K)$-action. If $I_{\tau} = U(\mathfrak{k}) \cap ker\tau$, $U(\mathfrak{k})/I_{\tau} \cong End(V_{\tau}) \cong (U(\mathfrak{k}_{\alpha})\Pi\mathbb{C}[{}^0M])/(ker\tau \cap (U(\mathfrak{k}_{\alpha})\Pi\mathbb{C}[{}^0M]))$ as ${}^0M$ acts irreducibly on V_{τ}.

Definition 6.2.6.

- For α simple, let

$$L_{\alpha} : U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^{\mathfrak{k}}} V_{\tau} \to symm(H_{\alpha})symm(J_{\alpha}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^{\mathfrak{k}}} V_{\tau}$$

$$\oplus \; \mathfrak{n}^{\alpha}U(\mathfrak{n}^{\alpha})symm(H_{\alpha})symm(J_{\alpha}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^{\mathfrak{k}}} V_{\tau}$$

  be the projection onto the first summand.

- Denote by Q the projection onto the first summand in $U(\mathfrak{g}) = U(\mathfrak{a})U(\mathfrak{k}) \oplus \mathfrak{n}U(\mathfrak{g})$ followed by the projection onto $U(\mathfrak{a}) \otimes (U(\mathfrak{k})/I_{\tau}) = U(\mathfrak{a}) \otimes End(V_{\tau})$.

Theorem 6.2.7. *Let $\alpha \in \Phi^+$ be a simple root, $\epsilon_1, ..., \epsilon_{n(\xi)}$ be a basis of $Hom_K(V_\xi, symm(H) \otimes V_\tau)$, $V_\xi = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j}$ with each V_{τ_j} an irreducible K_α-module. If $p_{\tau_j^\alpha}$ denotes the determinant of $P^{\tau_j^\alpha}$ matrix of the rank one case of G_α with K_α-type V_{τ_j}, then $p_{\tau_j^\alpha}$ divides p_ξ.*

Proof. This proof follows the method of the proof of Proposition 2.4.3 of [Kos] very closely.

Let $\alpha \in \Phi^+$ be a simple root, $\epsilon_1, ..., \epsilon_{n(\xi)}$ be a basis of

$Hom_K(V_\xi, symm(H) \otimes V_\tau)$, and let $V_\xi = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j}$ where each V_{τ_j} is an irreducible K_α-module. $(P^\xi)_{ij}$ is the action of $\epsilon_i(V_{\tau_j})(e)$ followed by replacement of elements in $\mathbb{C}[\nu]$ with corresponding elements in $S(\mathfrak{a})$, which is same as the action of $L_\alpha(\epsilon_i(V_{\tau_j}))(e)$ as L_α is a K_α-map since $[\mathfrak{g}_\alpha, \mathfrak{n}^\alpha] \subseteq \mathfrak{n}^\alpha$ for α simple, and elements in $\mathfrak{n}^\alpha$ will not contribute.

Recall V_τ^+ is the subspace of V_τ that consist of $\mathfrak{t}_\alpha$-weights of $\frac{1}{2}$, and denote by $V_{\tau_j}^+$ the subspace of V_{τ_j} that correspond to positive $\mathfrak{t}_\alpha$-weight space. Without loss of generality, assume $L_\alpha(\epsilon_i(V_{\tau_j}^+)) = \overline{Z_\alpha}^{l_j} R_{i,j}^\alpha \otimes V_\tau^+$ with $R_{i,j}^\alpha \in symm(J_\alpha)$. This is possible because of Theorem 5.3.8 which characterizes the irreducible K_α-modules in $H_\alpha \otimes V_\tau$ and L_α is a K_α-map. We have $symm(J_\alpha) \subseteq U(\mathfrak{g}_\alpha)^{\mathfrak{k}_\alpha}$ with $U(\mathfrak{g}_\alpha)^{\mathfrak{k}_\alpha}$ the subalgebra generated by $\mathfrak{t}_\alpha$, center of $\mathfrak{g}_\alpha$, and the Casimir element.

The action of $\overline{Z_\alpha}^{l_j} R_{i,j}^\alpha$ on V_τ^+ at the identity is $Q(\overline{Z_\alpha}^{l_j} R_{i,j}^\alpha)$, where by 3.5.6 of [RRG I], we have $Q(\overline{Z_\alpha}^{l_j} R_{i,j}^\alpha) = Q(R_{i,j}^\alpha) Q(\overline{Z_\alpha}^{l_j}) = r_{i,j}^\alpha Q(\overline{Z_\alpha}^{l_j})$ with $r_{i,j}^\alpha$ invariant under $\tilde{x_\alpha}$ the translated Weyl group element of simple reflection as $U(\mathfrak{g}_\alpha)^{\mathfrak{k}_\alpha}$ is the subalgebra generated by $\mathfrak{t}_\alpha$, center of $\mathfrak{g}_\alpha$, and the Casimir element. From the observation before, we have

$$Q(\overline{Z_\alpha}^{l_j}) \in U(\mathfrak{a}) \otimes (U(\mathfrak{k})/I_\tau) = U(\mathfrak{a}) \otimes End(V_\tau) \tag{6.2.6}$$

$$= U(\mathfrak{a}) \otimes (U(\mathfrak{k}_\alpha)\Pi\mathbb{C}[{}^0M])/(ker\tau \cap (U(\mathfrak{k}_\alpha)\Pi\mathbb{C}[{}^0M])) \tag{6.2.7}$$

We now see that action of $Q(\overline{Z_\alpha}^{l_j})$ on V_τ^+ is the determinant $p_{\tau_j}^\alpha$ of P^{τ_j} matrix for the rank one subgroup G_α with K_α type τ_j, and $p_{\tau_j}^\alpha$ divides p_ξ.

□

For $\phi \in \Phi^+$, define $p_\phi = p_{\tau_1^\phi} ... p_{\tau_{n(\xi)}^\phi}$ where $p_{\tau_j^\phi}$ corresponds to the determinant of $P^{\tau_j^\phi}$ matrix of the rank one case of G_ϕ with K_ϕ-type V_{τ_j}. Define now $p_{(\phi)} = T_{\rho_\phi - \rho} p_\phi$, where $T_{\rho_\phi - \rho}$ is the translation by $\rho_\phi - \rho$. Note each $p_{\tau_j^\phi}$ is a polynomial in $w_\phi = X_\phi \in \mathfrak{a}$ defined before. Hence p_ϕ and $p_{(\phi)}$ are also. Also, it is clear that $T_{\rho_\phi - \rho}(w_\phi) = w_\phi$ for ϕ simple, hence we have $p_\phi = p_{(\phi)}$ for ϕ simple.

Theorem 6.2.8. *For any $\phi \in \Phi^+$, $p_{(\phi)}$ divides p_ξ.*

Proof. This proof is almost word for word as in Proposition 2.4.5 of [Kos].

For $\phi \in \Phi^+$, define $O(\phi) = \Sigma m_i$, if $\phi = \Sigma_{\alpha \in \Delta^+} m_i \alpha_i$. If $O(\phi) = 1$, then $\phi \in \Delta^+$ hence the claim is true by Theorem 6.2.7 and above observation. We proceed by induction on $O(\phi)$. Assume $O(\phi) > 1$ and that the claim is true for all $\psi \in \Phi^+$ with $O(\psi) < O(\phi)$. We use the fact that for some $\alpha \in \Delta^+$, $< \phi, w_\alpha >$ is strictly greater than 0 and find a root $\psi \in \Phi^+$ such that $O(\psi) < O(\phi)$ and $\phi = x_\alpha \psi$ for some $\alpha \in \Delta^+$ where x_α is the Weyl group element of simple reflection. Note $\psi \neq \alpha$. But we have that $p_\xi = r_\alpha p_\alpha$, where r_α is invariant under the action of $\tilde{x_\alpha}$ by Theorem 6.2.7. Also, by induction hypothesis, $p_{(\psi)}$ divides $r_\alpha p_\alpha$. Since p_α is a polynomial in w_α whereas $p_{(\psi)}$ is a polynomial in w_ψ, and since $w_\alpha \neq w_\psi$, p_α and $p_{(\psi)}$ are mutually prime. Hence $p_{(\psi)}$ divides r_α, hence $\tilde{x_\alpha} p_{(\psi)}$ does also.

We now assert $\tilde{x_\alpha} p_{(\psi)} = p_{(\phi)}$ up to a nonzero scalar. Since $x_\alpha \psi = \phi$, we have $x_\alpha \mathfrak{g}_\psi = \mathfrak{g}_\phi$, $x_\alpha \mathfrak{k}_\psi = \mathfrak{k}_\phi$, and $x_\alpha \mathfrak{p}_\psi = \mathfrak{p}_\phi$. Moreover, $x_\alpha \mathfrak{a} = \mathfrak{a}$ and $x_\alpha \mathfrak{n}_\psi = \mathfrak{n}_\phi$. Therefore, for $u \in U(\mathfrak{g}_\psi)$, $x_\alpha Q(u) = Q(x_\alpha u)$. Also, $x_\alpha K_\psi x_\alpha^{-1} = K_\phi$. Furthermore, if $V_\xi = \oplus V_{\tau_j^\psi}$ is a decomposition into K_ψ-irreducibles and if $V_{\tau_j^\phi} = x_\alpha V_{\tau_j^\psi}$, then we know $V_\xi = \oplus V_{\tau_j^\phi}$ is a decomposition into K_ϕ-irreducibles. Hence it is now clear that $x_\alpha p_\psi = p_\phi$ up to a nonzero scalar.

But, $\tilde{x_\alpha} p_{(\psi)} = T_{-\rho} x_\alpha T_\rho T_{\rho_\psi - \rho} p_\psi = T_{-\rho} x_\alpha T_{\rho_\psi} p_\psi = T_{-\rho} x_\alpha T_{\rho_\psi} x_\alpha^{-1} x_\alpha p_\psi = T_{-\rho} T_{x_\alpha \rho_\psi} x_\alpha p_\psi = T_{\rho_\phi - \rho} p_\phi = p_{(\phi)}$, and this completes the assertion and $p_{(\phi)}$ divides p_ξ.

□

6.3 Product Formula of p_ξ for the group $\widetilde{SL(n,\mathbb{R})}$

Theorem 6.3.1. *There exists a non-zero scalar c such that*

$$p_\xi(\nu) = c\Pi_{\phi\in\Phi^+}p_{(\phi)}(\nu)$$

Proof. The right hand side divides the left hand side by Theorem 6.2.8. I claim the degrees of the two polynomials are the same.

Since our definition of p_ξ was independent of the basis up to a nonzero scalar, we may assume $\epsilon_i(V_{\tau_j}) \subseteq symm(H)_{d(i)} \otimes V_\tau$, i.e. we use a homogeneous basis. Therefore, the deg of the left hand side, $d(\xi)$, is at most $\Sigma d(i)$.

If $V_{\gamma_1}, ..., V_{\gamma_N}$ are distinct K-types in $symm(H)$ with $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$, $\Sigma d(i) = \Sigma_{j=1}^N d(\gamma_j)$ where $d(\gamma_j)$ is the sum of degrees in which V_{γ_j} occur in $symm(H)$. But, we have $d(\gamma_j) = \Sigma_{\phi\in\Phi^+} n_{\gamma_j}^\phi$ where $n_{\gamma_j}^\phi$ is the sum of degrees of which the irreducible K_ϕ-modules in Span$K_\phi.V_{\gamma_j}^{0M}$ occur in $symm(H_\phi)$ by Proposition 2.3.12 and Theorem 2.3.14 of [Kos]. But $n_{\gamma_j}^\phi$ only depends on the t_ϕ-weights in our case, hence $\Sigma d(i) = \Sigma_{j=1}^N d(\gamma_j) = \Sigma_{j=1}^N \Sigma_{\phi\in\Phi^+} n_{\gamma_j}^\phi = \Sigma_{\phi\in\Phi^+} deg(p_{(\phi)}(\nu))$ by Theorem 6.1.1 and the fact that $deg(Q(Z_\phi^l)) = l$ by Theorem 7.6 of [JW]. Therefore the degree of left hand side $\leq$ degree of right hand side, and with divisibility we have the statement of the theorem.

□

Chapter 7

Product formula of p_ξ for the connected, simply connected $\mathbb{R}$-split Lie group of simple Lie type other than A_n and C_n

In this chapter we will apply our results for $\widetilde{SL(n,\mathbb{R})}$ and $\widetilde{GL(n,\mathbb{R})}$ to derive a product formula for the groups of the title. Let G be any of the connected, simply connected $\mathbb{R}$-split Lie group of simple Lie type other than A_n and C_n with maximal compact subgroup K defined as the set of fixed elements of a Cartan involution θ. Denote by θ the corresponding Cartan involution of $Lie(G)$. As G is split, for a positive root α of $Lie(G)$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = \mathfrak{sl}_2$. Let $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] \cap \mathfrak{a}$ be such that $\alpha(h_\alpha) = 2$, and let $e_\alpha \in \mathfrak{n}_\alpha$ be such that $[e_\alpha, -\theta(e_\alpha)] = h_\alpha$. $(h_\alpha, e_\alpha, -\theta(e_\alpha))$ is an S-triple.

Definition 7.0.2. $\mathfrak{t}_\alpha = i * (e_\alpha + \theta(e_\alpha))$.

Let H be the space of harmonics on $\mathfrak{p}$, and V_τ be a choice of a small K-type from chapter 3.

Lemma 7.0.3. *Let G be as above for all types but B_n, E_7, and F_4. Let V_ξ be an irreducible K-module that occurs in $H \otimes V_\tau$. Then, $V_\xi|_{{}^0M} = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j}$ with $V_{\tau_j} \cong V_\tau$ as ${}^0M$ modules for all $j = 1, ..., n(\xi)$.*

Proof. For D_n, $K = Spin(n) \times Spin(n)$ where ${}^0M_{\widetilde{Spin(n,n)}}$ is isomorphic to ${}^0M_{\widetilde{SL(n,\mathbb{R})}} \times \mu_2$ where ${}^0M_{\widetilde{SL(n,\mathbb{R})}}$ sits diagonally in K. V_τ is the $Spin$ representation or either of the two half $Spin$ representations after projection onto either the first or the second factor of $K = Spin(n) \times Spin(n)$ depending on the parity of n. By Lemma 3.3.1, $(H \otimes V_\tau)|_{{}^0M_{\widetilde{Spin(n,n)}}}$ decomposes in the same way as $(H \otimes V_\tau)|_{{}^0M_{\widetilde{SL(n,\mathbb{R})}}}$ where ${}^0M_{\widetilde{SL(n,\mathbb{R})}}$ is the one sitting diagonally in $K = Spin(n) \times Spin(n)$. If n is odd, there is only one possible V_τ as in the case of $\widetilde{SL(n,\mathbb{R})}$. If n is even, choose an element ω of the set $p^{-1}(-Id \times -Id)$ where p is the covering homomorphism $p : Spin(n) \times Spin(n) \to SO(n) \times SO(n)$. ω distinguishes the two half $Spin$ representations as ${}^0M_{\widetilde{SL(n-1,\mathbb{R})}}$ do not. ω acts trivially on H as it is central and is an element of ${}^0M_{\widetilde{Spin(n,n)}}$. Therefore ω acts on the entire space $H \otimes V_\tau$ as it does on V_τ, and we have the statement of the lemma for D_n.

For E_6, E_8, and G_2, $\widetilde{i} : \widetilde{GL(6,\mathbb{R})} \hookrightarrow \widetilde{E_6}$, $\widetilde{i} : \widetilde{GL(8,\mathbb{R})} \hookrightarrow \widetilde{E_8}$, and $\widetilde{i} : \widetilde{SL(3,\mathbb{R})} \hookrightarrow \widetilde{G_2}$ with $\widetilde{i}({}^0M_{\widetilde{GL(6,\mathbb{R})}}) = {}^0M_{\widetilde{E_6}}$, $\widetilde{i}({}^0M_{\widetilde{GL(8,\mathbb{R})}}) = {}^0M_{\widetilde{E_8}}$, $\widetilde{i}({}^0M_{\widetilde{SL(3,\mathbb{R})}}) = {}^0M_{\widetilde{G_2}}$ by the proof of Lemma 3.3.1. Therefore, there can only be one V_τ, $Spin$-rep$|_{{}^0M}$. □

Remark For the connected, simply connected $\mathbb{R}$-split Lie groups of type B_n, E_7, and F_4, if $V_\xi \subseteq H \otimes V_\tau$, $V_\xi|_{K_\alpha} = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j} \oplus \bigoplus_k V_k$ where $V_{\tau_j} \cong V_\tau$ as ${}^0M$-modules, and $V_k \cong \overline{V_{\tau_k}}$ as ${}^0M$-modules where $\overline{V_{\tau_k}}$ is the other half $Spin$ representation or the other Pin representation restricted to ${}^0M$. Hence the weights of interest are just those of $\bigoplus_{j=1}^{n(\xi)} V_{\tau_j}$ from the definition of P^ξ matrix in 4.1.

Lemma 7.0.4. *Let G be any of the connected, simply connected $\mathbb{R}$-split Lie group of simple Lie type other than C_n with maximal compact subgroup K. Let $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct K-types that occur in H such that $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$. If $l(\gamma_j) = dim V_{\gamma_j}^{{}^0M}$, then $n(\xi) = \Sigma_{j=1}^N l(\gamma_j)$.*

Proof. Since V_τ is multiplicity free, by Corollary 3.4 of [Ku], we know V_ξ occurs in $V_\gamma \otimes V_\tau$ exactly once if it does. Now each of V_{γ_j} occurs in H exactly $l(\gamma_j)$ many times by Frobenius Reciprocity, hence the multiplicity of V_ξ in $H \otimes V_\tau$ is exactly $\Sigma_{j=1}^N l(\gamma_j)$ which is $n(\xi)$ defined in the remark above, by Frobenius Reciprocity. □

7.1 Comparision of t_α-weights

Recall the assumptions of Theorem 6.1.1.

Theorem 7.1.1. *Let G be as above other than type F_4. Let $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct K-types that occur in H such that $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$. $n(\xi) = \Sigma_{j=1}^N l(\gamma_j)$ and if α is a positive root of $Lie(G)$ but not short in the case of B_n and not short in the case of G_2 , then after reordering, $\delta_{\alpha,j}^\xi = \delta_{\alpha,j} \pm \frac{1}{2}$ for each $j = 1, ..., n(\xi)$.*

Proof. $n(\xi) = \Sigma_{j=1}^N l(\gamma_j)$ is by Lemma 7.0.4.

Recall from chapter 3 the subgroup $G_0 \leq G$ with maximal compact subgroup K_0 where G_0 is isomorphic to $\widetilde{SL(n, \mathbb{R})}$ or the metalinear group $\widetilde{GL(n, \mathbb{R})}$ for appropriate n. We have ${}^0M_{G_0} \leq {}^0M_G$ where the restriction of K to K_0 preserve ${}^0M_G$-invariants of H and the decomposition $V_\xi|_{{}^0M_G}$ by Lemma 3.3.1.

We can restate the condition $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$ as $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$. Note the statement of the theorem is true with the restated condition for G_0 by Theorem 6.1.1 and Lemma 6.1.2.

Let $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct irreducible K-modules that occur in H such that $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$, and let $\bigoplus_{j=1}^N Span\ K_0.V_{\gamma_j}^{{}^0M_G} = \bigoplus_k W_k$ where each W_k is an irreducible K_0-module. The nontrivial element $\eta \in \mu_2 \leq {}^0M_G$ acts by -1 where μ_2 is the kernel of the covering homomorphism $p : G_0 \to G_0/\mu_2$ which is also the kernel of the covering homomorphism $p : G \to G_\mathbb{R}$. Hence $V_\xi|_{K_0} = \bigoplus_j V_{\xi_j}$ where each of V_{ξ_j} occurs in $H \otimes V_\tau$ or $H \otimes \overline{V_\tau}$ with H that of G_0 and $\overline{V_\tau}$ the other half *Spin* representation or *Pin* representation. We have $\bigoplus_k W_k \subseteq \bigoplus_j V_{\xi_j} \otimes V_\tau^*$ where each of W_k occurs in H of G_0 by Lemma 3.3.1.

We assert that if $W_k \subseteq V_{\xi_j} \otimes V_\tau^*$, then $V_{\xi_j}|_{{}^0M_{G_0}}$ is equivalent to a multiple of V_τ and $V_{\xi_j} \subseteq H \otimes V_\tau$. Indeed, if $W_k \subseteq V_{\xi_j} \otimes V_\tau^*$, then $V_{\xi_j} \subseteq W_k \otimes V_\tau$, hence claim is true by Lemma 5.4.1. This observation is important because of the following. First, recall from remark in the beginning of the chapter the decomposition $V_\xi|_{K_\alpha} = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j} \oplus \bigoplus_k V_k$ where $V_{\tau_j} \cong V_\tau$ as ${}^0M_G$-modules, and $V_k \cong \overline{V_\tau}$ as ${}^0M_G$-modules where $\overline{V_\tau}$ is the other half *Spin* representation or the other *Pin* representation restricted to ${}^0M_G$. As $V_\xi \subseteq H \otimes V_\tau$, we only consider $V_{\tau_1}, ..., V_{\tau_{n(\xi)}}$ in the definition of P^ξ matrix.

Since the statement of the theorem is true for G_0 with the restated condition and as the set of $\mathfrak{t}_\alpha$ weights of interest are the same after branching down to K_0 by Lemma 3.3.1, we have the statement of the theorem for all positive roots α of $Lie(G_0) \subseteq Lie(G)$. Note $V_{\xi_j} \otimes V_\tau^*$ decomposes into distinct K_0-modules by Corollary 3.4 of [Ku] as V_τ^* is multiplicity free. Therefore, if $W_k \cong W_l$ with $k \neq l$, then W_k and W_l cannot be contained in a single $V_{\xi_j} \otimes V_\tau^*$, important as the statement of the theorem for G_0 also assumes distinct V_γs.

By Proposition 6.11 of [Bou], any positive root $\beta \in Lie(G)$ must be conjugate to some simple root α via an element of the Weyl group $W(A) = N_K(A)/Z_K(A)$, hence $\mathfrak{t}_\beta$ must be conjugate to $\mathfrak{t}_\alpha$ for some simple root α via an element of K. Therefore, the set of $\mathfrak{t}_\alpha$-weights of interest is the same for all positive root αs of same length and we have the statement of the theorem. $\square$

7.2 Comparison of $\mathfrak{t}_\alpha$-weights for short roots of $Lie(\widetilde{SO(q+1,q)})$

7.2.1 The Groups $\widetilde{Spin(p,q)}$, $\widetilde{Pin(p,q)}$ ($p \geq q \geq 3,\ p = q$ *or* $p = q+1$) and their Small K-types

Denote by $\widetilde{Spin(p,q)}$ the connected, simply connected $\mathbb{R}$-split Lie group of type B_q ($p = q+1,\ q \geq 3$) or D_q ($p = q \geq 3$) with maximal compact subgroup $K = Spin(p) \times Spin(q)$. ${}^0M_{\widetilde{Spin(p,q)}}$ is isomorphic to ${}^0M_{\widetilde{SL(q,\mathbb{R})}} \times \mu_2$ where ${}^0M_{\widetilde{SL(q,\mathbb{R})}}$ sits diagonally in K and the μ_2 can either be $(\pm 1, 1)$ or $(1, \pm 1) \leq K$.

$\widetilde{Spin(q,q)}$ has small K-type V_τ the $Spin$-representation or either of the two half $Spin$-representations of $Spin(q)$ depending on the parity of q, after projection onto either the first or the second factor of $K = Spin(q) \times Spin(q)$.

If q is odd, $\widetilde{Spin(q+1,q)}$ has small K-type V_τ the $Spin$-representation after projection onto the second factor of $K = Spin(q+1) \times Spin(q)$ or either of the two half $Spin$-representations of $Spin(q+1)$ after projection onto the first factor of $K = Spin(q+1) \times Spin(q)$. If q is even, $\widetilde{Spin(q+1,q)}$ has small K-type V_τ either of the two half $Spin$-representations of $Spin(q)$ after projection onto the second factor of $K = Spin(q+1) \times Spin(q)$.

Denote by $\widetilde{Pin(p,q)}$ the corresponding covering group of $Pin(p,q)$ with maximal compact subgroup $Pin(p) \times Pin(q)$. ${}^0M_{\widetilde{Pin(p,q)}}$ is isomorphic to ${}^0M_{\widetilde{GL(q,\mathbb{R})}} \times \mu_2$ where ${}^0M_{\widetilde{GL(q,\mathbb{R})}}$ sits diagonally in $Pin(p) \times Pin(q)$ and the μ_2 can either be $(\pm 1, 1)$ or $(1, \pm 1) \in Pin(p) \times Pin(q)$. The following are small $K = Pin(p) \times Pin(q)$-types for $\widetilde{Pin(p,q)}$.

Definition 7.2.2.

- $\widetilde{Pin(q,q)}$ has small K-type V_{T} the Pin representation or either of the two Pin representations of $Pin(q)$ depending on the parity of q, after projection onto either the first or the second factor of $K = Pin(q) \times Pin(q)$.

- If q is odd, $\widetilde{Pin(q+1,q)}$ has small K-type V_{T} either of the two Pin representations of $Pin(q)$ after projection onto the second factor of $K = Pin(q+1) \times Pin(q)$. If q is even, $\widetilde{Pin(q+1,q)}$ has small K-type V_{T} the Pin representation of $Pin(q)$ after projection onto the second factor of $K = Pin(q+1) \times Pin(q)$ or either of the two either of the two Pin representations of $Pin(q+1)$ after projection onto the first factor of $K = Pin(q+1) \times Pin(q)$.

Remark 7.2.2 ${}^0M_{\widetilde{Spin(p,q)}} \cong {}^0M_{\widetilde{SL(q,\mathbb{R})}} \times \mu_2$ and ${}^0M_{\widetilde{Pin(p,q)}} \cong {}^0M_{\widetilde{GL(q,\mathbb{R})}} \times \mu_2$. In either case the μ_2 acts trivially on H the space of harmonics as it is central in K and the μ_2 can be chosen to act trivially on the small K-type described above. Therefore, in the case of $\widetilde{Spin(p,q)}$, $(H \otimes V_\tau)|_{{}^0M_{\widetilde{Spin(p,q)}}}$ decomposes in the same way as $(H \otimes V_\tau)|_{{}^0M_{\widetilde{SL(q,\mathbb{R})}}}$ where ${}^0M_{\widetilde{SL(q,\mathbb{R})}}$ is the one sitting diagonally in $K = Spin(p) \times Spin(q)$. In the case of $\widetilde{Pin(p,q)}$, $(H \otimes V_{\mathrm{T}})|_{{}^0M_{\widetilde{Pin(p,q)}}}$ decomposes in the same way as $(H \otimes V_{\mathrm{T}})|_{{}^0M_{\widetilde{GL(q,\mathbb{R})}}}$ where ${}^0M_{\widetilde{GL(q,\mathbb{R})}}$ is the one sitting diagonally in $K = Pin(p) \times Pin(q)$.

Consider the embedding $i : O(q,q) \hookrightarrow SO(q+1,q+1)_\circ$ where the image of the maximal compact subgroup $O(q) \times O(q)$ of $O(q,q)$ under i is contained in the maximal compact subgroup $SO(q+1) \times SO(q+1)$ of $SO(q+1,q+1)_\circ$ such that if $(g,h) \in O(q) \times O(q)$,

$$i((g,h)) = \left(\begin{pmatrix} g & 0 \\ 0 & det(g)^{-1} \end{pmatrix}, \begin{pmatrix} h & 0 \\ 0 & det(h)^{-1} \end{pmatrix}\right) \in SO(q+1) \times SO(q+1)$$

Let $p : \widetilde{Spin(q+1}, q+1) \to SO(q+1,q+1)_\circ$ be the covering homomorphism. We have that $p^{-1}(i(O(q,q)))$ is a Lie subgroup isomorphic to $\widetilde{Pin(q,q)}$, hence we have an embedding $\widetilde{i} : \widetilde{Pin(q,q)} \hookrightarrow \widetilde{Spin(q+1}, q+1)$.

Lemma 7.2.3. *Consider the embedding* $\widetilde{i} : \widetilde{Pin(q,q)} \hookrightarrow \widetilde{Spin(q+1}, q+1)$ *described above. We have* $\widetilde{i}({}^0M_{\widetilde{Pin(q,q)}}) = {}^0M_{\widetilde{Spin(q+1},q+1)}$.

Proof. We have

$${}^0M_{SO(q+1,q+1)_\circ} = \{\left(\begin{pmatrix} g & 0 \\ 0 & det(g)^{-1} \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & det(g)^{-1} \end{pmatrix}\right) | g \in O(q) \ diagonal\}$$

, the image of ${}^0M_{O(q,q)}$ under the map i. Hence $i({}^0M_{O(q,q)}) = {}^0M_{SO(q+1,q+1)_\circ}$. As $\widetilde{i}({}^0M_{\widetilde{Pin(q,q)}}) = p^{-1}(i({}^0M_{O(q,q)}))$ and ${}^0M_{\widetilde{Spin(q+1},q+1)} = p^{-1}({}^0M_{SO(q+1,q+1)_\circ})$, we have the statement of the lemma. □

Let $p : \widetilde{Pin(q,q)} \to O(q,q)$ be the covering homomorphism. $Pin(q) \times Pin(q)$ is generated by $Spin(q) \times Spin(q)$, $p^{-1}(g_0, Id)$, and $p^{-1}(Id, g_0)$ where g_0 is defined in 5.2.1. ${}^0M_{\widetilde{Pin(q,q)}}$ is generated by ${}^0M_{\widetilde{Spin(q,q)}}$ and $p^{-1}(g_0, g_0)$.

7.2.4 Restriction of $Pin(q) \times Pin(q)$ modules to $Spin(q) \times Spin(q)$ and $\mathfrak{t}_{\epsilon_1}$-weights

Irreducible representations of $Spin(q) \times Spin(q)$ and $Pin(q) \times Pin(q)$ are outer tensor products of irreducible representations of $Spin(q)$ and $Pin(q)$ respectively, discussed in section 5.2. Recall the definition of ζ from 5.2.1.

Definition 7.2.5.

- Let $q = 2k$, and V_Γ be an irreducible representation of $Pin(q)$. Let V_γ be an irreducible representation of $Spin(q)$ that occurs in the restriction of V_Γ to $Spin(q)$ with highest weight $(\lambda_1, ..., \lambda_k)$. If $\lambda_k \neq 0$, denote by $\overline{V_\gamma} = \zeta \cdot V_\gamma$ so that $V_\Gamma = V_\gamma \oplus \overline{V_\gamma}$. If $\lambda_k = 0$, denote by V_γ^+ (*resp.* V_γ^-) the $Pin(q)$ module whose restriction to $Spin(q)$ is V_γ with action of ζ on the highest weight vector by $+Id$ (*resp.* $-Id$).

- Let $q = 2k+1$, and V_Γ be an irreducible representation of $Pin(q)$. Let V_γ be an irreducible representation of $Spin(q)$ that occurs in the restriction of V_Γ to $Spin(q)$. Denote by V_γ^+ (*resp.* V_γ^-) the $Pin(q)$ module whose restriction to $Spin(q)$ is V_γ with action of ζ by $+Id$ (*resp.* $-Id$).

Let V_γ and V_Γ be an irreducible $Spin(q)\times Spin(q)$-module and an irreducible $Pin(q) \times Pin(q)$-module respectively. Let $V_\gamma = V_{\gamma_1} \otimes V_{\gamma_2}$ and $V_\Gamma = V_{\Gamma_1} \otimes V_{\Gamma_2}$ where V_{γ_1}, V_{γ_2} are irreducible $Spin(q)$ modules and V_{Γ_1}, V_{Γ_2} are irreducible $Pin(q)$ modules.

Lemma 7.2.6. *Let $q = 2k$, $(m_1, ..., m_k)$ the highest weight of V_{γ_1}, $(n_1, ..., n_k)$ the highest weight of V_{γ_2}, and assume V_γ occurs in V_Γ if we restrict to $Spin(q)\times Spin(q)$ from $Pin(q) \times Pin(q)$.*

- *If $m_k \neq 0$ and $n_k \neq 0$, $V_\Gamma|_{Spin(q)\times Spin(q)} = \bigoplus_{\epsilon_1=\pm,\epsilon_2=\pm} V_{(m_1,...,m_{k-1},\epsilon_1 m_k)} \otimes V_{(n_1,...,n_{k-1},\epsilon_2 n_k)}$ where $(\zeta, Id) \in Pin(q) \times Pin(q)$ swaps the two highest weight modules $V_{(m_1,...,m_{k-1},\epsilon_1 m_k)}\otimes V_{(n_1,...,n_{k-1},\epsilon_2 n_k)}$ and $V_{(m_1,...,m_{k-1},-\epsilon_1 m_k)}\otimes V_{(n_1,...,n_{k-1},\epsilon_2 n_k)}$, (Id, ζ) swaps the two highest weight modules $V_{(m_1,...,m_{k-1},\epsilon_1 m_k)}\otimes V_{(n_1,...,n_{k-1},\epsilon_2 n_k)}$ and $V_{(m_1,...,m_{k-1},\epsilon_1 m_k)} \otimes V_{(n_1,...,n_{k-1},-\epsilon_2 n_k)}$.*

- *If $m_k \neq 0$ and $n_k = 0$, $V_\Gamma|_{Spin(q)\times Spin(q)} = \bigoplus_{\epsilon_1=\pm} V_{(m_1,...,m_{k-1},\epsilon_1 m_k)}\otimes V_{(n_1,...,n_{k-1},0)}$ where (ζ, Id) swaps the two highest weight modules $V_{(m_1,...,m_{k-1},\epsilon_1 m_k)}\otimes V_{(n_1,...,n_{k-1},0)}$ and $V_{(m_1,...,m_{k-1},-\epsilon_1 m_k)} \otimes V_{(n_1,...,n_{k-1},0)}$ and (Id, ζ) acts on the highest weight vector of $V_{(m_1,...,m_{k-1},\epsilon_1 m_k)} \otimes V_{(n_1,...,n_{k-1},0)}$ by $\pm Id$.*

- *If $m_k = 0$ and $n_k \neq 0$, $V_\Gamma|_{Spin(q)\times Spin(q)} = \bigoplus_{\epsilon_2=\pm} V_{(m_1,...,m_{k-1},0)}\otimes V_{(n_1,...,n_{k-1},\epsilon_2 n_k)}$ where (Id, ζ) swaps the two highest weight modules $V_{(m_1,...,m_{k-1},0)}\otimes V_{(n_1,...,n_{k-1},\epsilon_2 n_k)}$ and $V_{(m_1,...,m_{k-1},0)} \otimes V_{(n_1,...,n_{k-1},-\epsilon_2 n_k)}$ and (ζ, Id) acts on the highest weight vector of $V_{(m_1,...,m_{k-1},0)} \otimes V_{(n_1,...,n_{k-1},\epsilon_2 n_k)}$ by $\pm Id$.*

- *If* $m_k = 0$ *and* $n_k = 0$, $V_\Gamma|_{Spin(q)\times Spin(q)} = V_{(m_1,...,m_{k-1},0)} \otimes V_{(n_1,...,n_{k-1},0)}$ *where* (ζ, Id) *and* (Id, ζ) *act by* $\pm Id$ *on the highest weight vector of* $V_{(m_1,...,m_{k-1},0)} \otimes V_{(n_1,...,n_{k-1},0)}$.

Proof. As irreducible representations of $Pin(q) \times Pin(q)$ are outer tensor products of irreducible representations of each of the two $Pin(q)$s, we have the statements of the lemma by Theorem 5.2.3. □

Lemma 7.2.7. *Let* $q = 2k+1$, $(m_1, ..., m_k)$ *the highest weight of* V_{γ_1}, $(n_1, ..., n_k)$ *the highest weight of* V_{γ_2}, *and assume* V_γ *occurs in* V_Γ *if we restrict to* $Spin(q)\times Spin(q)$ *from* $Pin(q)\times Pin(q)$. *Then,* $V_\Gamma|_{Spin(q)\times Spin(q)} = V_{\gamma_1} \otimes V_{\gamma_2}$ *where* (ζ, Id) *and* (Id, ζ) *act by* $\pm Id$ *on* $V_{\gamma_1} \otimes V_{\gamma_2}$.

Proof. As irreducible representations of $Pin(q) \times Pin(q)$ are outer tensor products of irreducible representations of each of the two $Pin(q)$s, we have the statements of the lemma by Theorem 5.2.2. □

Consider the short root ϵ_1 of $Lie(\widetilde{Spin(q+1,q)})$. $\epsilon_1 = \frac{1}{2}(\alpha_1 + \alpha_2)$ where $\alpha_1 = \epsilon_1 + \epsilon_2$, $\alpha_2 = \epsilon_1 - \epsilon_2$ are positive roots of $Lie(\widetilde{Spin(q,q)})$. Recall the definition of $\mathfrak{t}_{\epsilon_1}$ from 7.0.2. We have $\mathfrak{t}_{\epsilon_1} = \frac{1}{2}(\mathfrak{t}_{\alpha_1} + \mathfrak{t}_{\alpha_2})$. Note $t_{\epsilon_1} \in Lie(Spin(q) \times Spin(q))_{\mathbb{C}}$ for all $q \geq 3$.

Recall the definition of $\mathfrak{t}_{\epsilon_1}$ weights $\delta^\gamma_{\epsilon_1,j}$ and $\delta^\Gamma_{\epsilon_1,j}$ from 5.5.1.

Theorem 7.2.8. *Let* $q = 2k$, *and assume* V_Γ *occurs in the harmonics* H *of* $\widetilde{Pin(q,q)}$. *Let* $(m_1, ..., m_k)$ *be the highest weight of* V_{γ_1}, $(n_1, ..., n_k)$ *be the highest weight of* V_{γ_2}, *and assume* $V_\gamma = V_{\gamma_1} \otimes V_{\gamma_2}$ *occurs in* V_Γ *if we restrict to* $Spin(q) \times Spin(q)$ *from* $Pin(q) \times Pin(q)$.

- *If* $m_k \neq 0$ *and* $n_k \neq 0$, *then dim* $V_\Gamma^{{}^0M_{\widetilde{Pin(q,q)}}}$ = *dim* $(V_{\gamma_1} \otimes V_{\gamma_2})^{{}^0M_{\widetilde{Spin(q,q)}}}$ + *dim* $(V_{\gamma_1} \otimes \overline{V_{\gamma_2}})^{{}^0M_{\widetilde{Spin(q,q)}}}$, *and* $\{\delta^\Gamma_{\epsilon_1,1}, ..., \delta^\Gamma_{\epsilon_1,l(\Gamma)}\}$ *is the disjoint union of those from* $(V_{\gamma_1} \otimes V_{\gamma_2})^{{}^0M_{\widetilde{Spin(q,q)}}}$ *and* $(V_{\gamma_1} \otimes \overline{V_{\gamma_2}})^{{}^0M_{\widetilde{Spin(q,q)}}}$.

- *If* $m_k \neq 0$ *and* $n_k = 0$ *or if* $m_k = 0$ *and* $n_k \neq 0$, *then dim* $V_\Gamma^{{}^0M_{\widetilde{Pin(q,q)}}}$ = *dim* $V_\gamma^{{}^0M_{\widetilde{Spin(q,q)}}}$, *and* $\{\delta^\Gamma_{\epsilon_1,1}, ..., \delta^\Gamma_{\epsilon_1,l(\Gamma)}\}$ *is the same as the set on* $V_\gamma^{{}^0M_{\widetilde{Spin(q,q)}}}$.

- *If* $m_k = 0$ *and* $n_k = 0$, *then dim* $V_\gamma^{{}^0M_{\widetilde{Spin(q,q)}}}$ $=$ *dim* $(V_{\gamma1}^+ \otimes V_{\gamma2}^+)^{{}^0M_{\widetilde{Pin(q,q)}}}$ $+$ *dim* $(V_{\gamma1}^+ \otimes V_{\gamma2}^-)^{{}^0M_{\widetilde{Pin(q,q)}}}$ $=$ *dim* $(V_{\gamma1}^- \otimes V_{\gamma2}^-)^{{}^0M_{\widetilde{Pin(q,q)}}}$ $+$ *dim* $(V_{\gamma1}^- \otimes V_{\gamma2}^+)^{{}^0M_{\widetilde{Pin(q,q)}}}$, *and* $\{\delta^\gamma_{\epsilon_1,1}, ..., \delta^\gamma_{\epsilon_1,l(\gamma)}\}$ *is the disjoint union of those from* $(V_{\gamma1}^+ \otimes V_{\gamma2}^+)^{{}^0M_{\widetilde{Pin(q,q)}}}$ *and* $(V_{\gamma1}^+ \otimes V_{\gamma2}^-)^{{}^0M_{\widetilde{Pin(q,q)}}}$ *or the disjoint union of those from* $(V_{\gamma1}^- \otimes V_{\gamma2}^-)^{{}^0M_{\widetilde{Pin(q,q)}}}$ *and* $(V_{\gamma1}^- \otimes V_{\gamma2}^+)^{{}^0M_{\widetilde{Pin(q,q)}}}$.

Proof. As we are working with submodules of the harmonics, we can ignore the tilde and consider ${}^0M_{O(q,q)}$ and ${}^0M_{SO(q,q)_\circ}$ invariants. Recall that ${}^0M_{O(q,q)}$ is generated by ${}^0M_{SO(q,q)_\circ}$ and (g_0, g_0).

First consider the case $m_k \neq 0$ and $n_k \neq 0$. From Lemma 7.2.6, we have $V_\Gamma|_{SO(q)\times SO(q)} = (V_{\gamma_1} \otimes V_{\gamma_2}) \oplus (\overline{V_{\gamma_1}} \otimes \overline{V_{\gamma_2}}) \oplus (\overline{V_{\gamma_1}} \otimes V_{\gamma_2}) \oplus (V_{\gamma_1} \otimes \overline{V_{\gamma_2}})$ where $(g_0, g_0) \in O(q) \times O(q)$ swaps the two highest weight modules $V_{\gamma_1} \otimes V_{\gamma_2}$ and $\overline{V_{\gamma_1}} \otimes \overline{V_{\gamma_2}}$, and the two highest weight modules $\overline{V_{\gamma_1}} \otimes V_{\gamma_2}$ and $V_{\gamma_1} \otimes \overline{V_{\gamma_2}}$. As (g_0, g_0) commutes with ${}^0M_{SO(q,q)_\circ}$, (g_0, g_0) gives us a bijection of ${}^0M_{SO(q,q)_\circ}$ invariants of $V_{\gamma_1} \otimes V_{\gamma_2}$ with those of $\overline{V_{\gamma_1}} \otimes \overline{V_{\gamma_2}}$, and a bijection of ${}^0M_{SO(q,q)_\circ}$ invariants of $\overline{V_{\gamma_1}} \otimes V_{\gamma_2}$ with those of $V_{\gamma_1} \otimes \overline{V_{\gamma_2}}$. As ${}^0M_{O(q,q)}$ is generated by ${}^0M_{SO(q,q)_\circ}$ and (g_0, g_0), and as (g_0, g_0) leaves invariant $t^2_{\epsilon_1}$, we have the first statement of the lemma.

Consider the case $m_k \neq 0$ and $n_k = 0$ or if $m_k = 0$ and $n_k \neq 0$. Without loss of generality, assume $m_k \neq 0$ and $n_k = 0$. From Lemma 7.2.6, we have $V_\Gamma|_{SO(q)\times SO(q)} = (V_{\gamma_1} \otimes V_{\gamma_2}) \oplus (\overline{V_{\gamma_1}} \otimes V_{\gamma_2})$ where (g_0, g_0) swaps the two highest weight modules $V_{\gamma_1} \otimes V_{\gamma_2}$ and $\overline{V_{\gamma_1}} \otimes V_{\gamma_2}$. As (g_0, g_0) commutes with ${}^0M_{SO(q,q)_\circ}$, (g_0, g_0) gives us a bijection of ${}^0M_{SO(q,q)_\circ}$ invariants of $V_{\gamma_1} \otimes V_{\gamma_2}$ with those of $\overline{V_{\gamma_1}} \otimes V_{\gamma_2}$. As ${}^0M_{O(q,q)}$ is generated by ${}^0M_{SO(q,q)_\circ}$ and (g_0, g_0), and as (g_0, g_0) leaves invariant $t^2_{\epsilon_1}$, we have the second statement of the lemma.

Now consider the case $m_k = 0$ and $n_k = 0$. Let $v_1, ..., v_{l(\gamma)}$ a basis of $V_\gamma^{{}^0M_{SO(q,q)_\circ}}$ such that (g_0, g_0) acts on v_j by $\pm Id$ for all j, which is possible since $(g_0, g_0)^2 = (Id, Id)$ and (g_0, g_0) commutes with ${}^0M_{SO(q,q)_\circ}$. Denote by $v_1^+, ..., v_{l(\gamma)}^+$ and $v_1^-, ..., v_{l(\gamma)}^-$ above basis thought of being in $(V_{\gamma1}^+ \otimes V_{\gamma2}^+)$ and $(V_{\gamma1}^+ \otimes V_{\gamma2}^-)$ respectively. If we denote by ϵ the action of (g_0, g_0) on $v_1^+, ..., v_{l(\gamma)}^+$, we assert that (g_0, g_0) will act by $-\epsilon$ on $v_1^-, ..., v_{l(\gamma)}^-$. Indeed, $v_j = X_j.v$ where $X_j \in U(\overline{\mathfrak{n}})$ and v is the highest weight vector of V_γ. Since (g_0, g_0) acts by different signatures on v for $(V_{\gamma1}^+ \otimes V_{\gamma2}^+)$ and $(V_{\gamma1}^+ \otimes V_{\gamma2}^-)$, the statement is now clear. We argue exactly the same way for the modules $(V^-_{\gamma_1} \otimes V^-_{\gamma_2})$ and $(V^-_{\gamma_1} \otimes V^+_{\gamma_2})$.

□

Theorem 7.2.9. *Let* $q = 2k+1$ *and assume* V_Γ *occurs in the harmonics* H *of* $\widetilde{Pin(q,q)}$. *Assume* V_γ *occurs in* V_Γ *if we restrict to* $Spin(q) \times Spin(q)$ *from* $Pin(q) \times Pin(q)$. *Then dim* $V_\Gamma^{{}^0M_{\widetilde{Pin(q,q)}}}$ = *dim* $V_\gamma^{{}^0M_{\widetilde{Spin(q,q)}}}$, *and* $\{\delta^\Gamma_{\epsilon_1,1}, ..., \delta^\Gamma_{\epsilon_1,l(\Gamma)}\}$ *is exactly those of* $V_\gamma^{{}^0M_{\widetilde{Spin(q,q)}}}$.

Proof. As we are working with submodules of the harmonics, we can ignore the tilde and consider ${}^0M_{O(q,q)}$ and ${}^0M_{SO(q,q)_\circ}$ invariants. Recall that ${}^0M_{O(q,q)}$ is generated by ${}^0M_{SO(q,q)_\circ}$ and (g_0, g_0).

First, $V_\Gamma = V_\gamma$ as a space by Lemma 7.2.7. Since V_Γ occurs in the harmonics and $(g_0, g_0) \in {}^0M_{O(q,q)}$ is central, it must act by identity. Therefore, (g_0, g_0) acts by either $(+,+)$ or $(-,-)$. In any case, there is no difference between $V_\Gamma^{{}^0M_{\widetilde{Pin(q,q)}}}$ and $V_\gamma^{{}^0M_{\widetilde{Spin(q,q)}}}$. □

Lemma 7.2.10. *Let* V_Ξ *be an irreducible* $Pin(q) \times Pin(q)$*-module that occurs in* $H \otimes V_{\mathrm{T}}$. *Then,* $V_\Xi|_{{}^0M_{\widetilde{Pin(q,q)}}} = \bigoplus_{j=1}^{n(\Xi)} V_{\mathrm{T}_j}$ *where* $V_{\mathrm{T}_j} \cong V_{\mathrm{T}}$ *as* ${}^0M_{\widetilde{Pin(q,q)}}$*-modules for all* $j = 1, ..., n(\Xi)$.

Proof. First recall from Remark 7.2.2 that $(H \otimes V_{\mathrm{T}})|_{{}^0M_{\widetilde{Pin(q,q)}}}$ decomposes in the same way as $(H \otimes V_{\mathrm{T}})|_{{}^0M_{\widetilde{GL(q,\mathbb{R})}}}$ where ${}^0M_{\widetilde{GL(q,\mathbb{R})}}$ is the one sitting diagonally in $K = Pin(q) \times Pin(q)$. If q is even, there is only one choice of $V_{\mathrm{T}}|_{{}^0M_{\widetilde{Pin(q,q)}}}$. If q is odd, $(\zeta,\zeta) \in {}^0M_{\widetilde{Pin(q,q)}}$ distinguishes the two small $Pin(q) \times Pin(q)$ types defined in 7.2.2 as ${}^0M_{\widetilde{Spin(q,q)}}$ does not. But, (ζ,ζ) is also central, hence it must act trivially on H. Therefore, (ζ,ζ) acts by a single sign on $H \otimes V_{\mathrm{T}}$ and we have the statement of the lemma. □

Recall the definition of t_{ϵ_1} weights $\delta^\xi_{\epsilon_1,j}$ and $\delta^\Xi_{\epsilon_1,j}$ from 5.5.2. Denote by K_{ϵ_1} the group generated by $exp(i * \mathfrak{t}_{\epsilon_1})$ and ${}^0M_{\widetilde{Spin(q,q)}}$, and denote by $K_{\epsilon_1,\widetilde{Pin(q,q)}}$ the group generated by $exp(i * \mathfrak{t}_{\epsilon_1})$ and ${}^0M_{\widetilde{Pin(q,q)}}$.

Theorem 7.2.11. *Let* $q = 2k$ *and let* V_Ξ *be an irreducible* $Pin(q) \times Pin(q)$*-module that occurs in* $H \otimes V_{\mathrm{T}}$ *so that its restriction to* $Spin(q) \times Spin(q)$ *contains a copy* *of* V_ξ *where* $V_\xi \subseteq H_{\widetilde{Spin(q,q)}} \otimes V_\tau$ *or* $V_\xi \subseteq H_{\widetilde{Spin(q,q)}} \otimes \overline{V_\tau}$. *Let* $V_\xi = V_{\xi_1} \otimes V_{\xi_2}$ *with* $(m_1, ..., m_k)$ *the highest weight of* V_{ξ_1} *and* $(n_1, ..., n_k)$ *the highest weight of* V_{ξ_2}.

- *Let* V_{T} *be the* Pin *representation of* $Pin(q)$ *after projection onto the first factor of* $Pin(q) \times Pin(q)$. *If* $n_k \neq 0$, *then* $\{\delta^{\Xi}_{\epsilon_1,1}, ..., \delta^{\Xi}_{\epsilon_1,n(\Xi)}\}$ *is the disjoint union of those from the* $Spin(q) \times Spin(q)$ *modules* $V_{\xi_1} \otimes V_{\xi_2}$ *and* $V_{\xi_1} \otimes \overline{V_{\xi_2}}$. *If* $n_k = 0$, *then* $\{\delta^{\Xi}_{\epsilon_1,1}, ..., \delta^{\Xi}_{\epsilon_1,n(\Xi)}\}$ *is exactly same as those from the* $Spin(q) \times Spin(q)$ *module* $V_{\xi_1} \otimes V_{\xi_2}$.

- *Let* V_{T} *be the* Pin *representation of* $Pin(q)$ *after projection onto the second factor of* $Pin(q) \times Pin(q)$. *If* $m_k \neq 0$, *then* $\{\delta^{\Xi}_{\epsilon_1,1}, ..., \delta^{\Xi}_{\epsilon_1,n(\Xi)}\}$ *is the disjoint union of those from the* $Spin(q) \times Spin(q)$ *modules* $V_{\xi_1} \otimes V_{\xi_2}$ *and* $V_{\xi_1} \otimes \overline{V_{\xi_2}}$. *If* $m_k = 0$, *then* $\{\delta^{\Xi}_{\epsilon_1,1}, ..., \delta^{\Xi}_{\epsilon_1,n(\Xi)}\}$ *is exactly same as those from the* $Spin(q) \times Spin(q)$ *module* $V_{\xi_1} \otimes V_{\xi_2}$.

Proof. Assume V_{T} is the Pin representation of $Pin(q)$ after projection onto the first factor of $Pin(q) \times Pin(q)$. $m_k \neq 0$ as highest weight of V_{ξ_1} must be half-integral.

Consider the case $n_k \neq 0$. From Lemma 7.2.6, we have $V_\Xi|_{Spin(q)\times Spin(q)} = (V_{\xi_1} \otimes V_{\xi_2}) \oplus (\overline{V_{\xi_1}} \otimes \overline{V_{\xi_2}}) \oplus (\overline{V_{\xi_1}} \otimes V_{\xi_2}) \oplus (V_{\xi_1} \otimes \overline{V_{\xi_2}})$ where (ζ, ζ) swaps $V_{\xi_1} \otimes V_{\xi_2}$ and $\overline{V_{\xi_1}} \otimes \overline{V_{\xi_2}}$, and (ζ, ζ) swaps $\overline{V_{\xi_1}} \otimes V_{\xi_2}$ and $V_{\xi_1} \otimes \overline{V_{\xi_2}}$.

Let $(V_{\xi_1} \otimes V_{\xi_2})|_{K_{\epsilon_1}} = \oplus_{j=1}^{n(\xi_1,\xi_2)} V_j$ where $V_j \cong V_\tau$ as ${}^0M_{\widetilde{Spin(q,q)}}$-modules for all j or $V_j \cong \overline{V_\tau}$ as ${}^0M_{\widetilde{Spin(q,q)}}$-modules for all j with V_τ and $\overline{V_\tau}$ the two small $Spin(q) \times Spin(q)$ types by Lemma 7.0.3. Let $(\overline{V_{\xi_1}} \otimes V_{\xi_2})|_{K_{\epsilon_1}} = \oplus_{k=1}^{n(\overline{\xi_1},\xi_2)} W_k$ where $W_k \cong V_\tau$ as ${}^0M_{\widetilde{Spin(q,q)}}$-modules for all k or $W_k \cong \overline{V_\tau}$ as ${}^0M_{\widetilde{Spin(q,q)}}$-modules for all k with V_τ and $\overline{V_\tau}$ the two small $Spin(q) \times Spin(q)$ types by Lemma 7.0.3.

We have $V_\Xi|_{K_{\epsilon_1,\widetilde{Pin(q,q)}}} = \bigoplus_{j=1}^{n(\xi_1,\xi_2)} (V_j \oplus (\zeta,\zeta)\cdot V_j) \oplus \bigoplus_{k=1}^{n(\overline{\xi_1},\xi_2)} (W_k \oplus (\zeta,\zeta)\cdot W_k)$ by Lemma 7.2.10. Therefore we have the first statement of the first case of the lemma as (ζ, ζ) commutes with $\mathfrak{t}^2_{\epsilon_1}$.

If $n_k = 0$, we have by Lemma 7.2.6 $V_\Xi|_{Spin(q)\times Spin(q)} = (V_{\xi_1} \otimes V_{\xi_2}) \oplus (\overline{V_{\xi_1}} \otimes V_{\xi_2})$ where (ζ, ζ) swaps $V_{\xi_1} \otimes V_{\xi_2}$ and $\overline{V_{\xi_1}} \otimes V_{\xi_2}$. Let $(V_{\xi_1} \otimes V_{\xi_2})|_{K_{\epsilon_1}} = \oplus_{j=1}^{n(\xi_1,\xi_2)} V_j$ where $V_j \cong V_\tau$ as ${}^0M_{\widetilde{Spin(q,q)}}$-modules for all j or $V_j \cong \overline{V_\tau}$ as ${}^0M_{\widetilde{Spin(q,q)}}$-modules for all j with V_τ and $\overline{V_\tau}$ the two small $Spin(q) \times Spin(q)$ types by Lemma 7.0.3.

We have $V_\Xi|_{K_{\epsilon_1,\widetilde{Pin(q,q)}}} = \bigoplus_{j=1}^{n(\xi_1,\xi_2)}(V_j \oplus (\zeta,\zeta)\cdot V_j)$ by Lemma 7.2.10. Therefore we have the second statement of the first case of the lemma as (ζ,ζ) commutes with $\mathfrak{t}_{\epsilon_1}^2$.

The second case can be shown similarly as in the first case. □

Theorem 7.2.12. *Let $q = 2k+1$ and let V_Ξ be an irreducible $Pin(q) \times Pin(q)$-module that occurs in $H \otimes V_\mathrm{T}$ so that its restriction to $Spin(q) \times Spin(q)$ is an irreducible module V_ξ where $V_\xi \subseteq H_{\widetilde{Spin(q,q)}} \otimes V_\tau$. Let $V_\Xi|_{K_{\epsilon_1,\widetilde{Pin(q,q)}}} = \bigoplus_{j=1}^{n(\Xi)} V_{\mathrm{T}_j}$, and $V_\xi|_{K_{\epsilon_1,\widetilde{Spin(q,q)}}} = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j}$. Then, $n(\Xi) = n(\xi)$, and $\delta_{\epsilon_1,j}^{\Xi} = \delta_{\epsilon_1,j}^{\xi}$ for all j after reordering.*

Proof. ${}^0M_{\widetilde{Pin(q,q)}}$ is generated by ${}^0M_{\widetilde{Spin(q,q)}}$ and (ζ,ζ) where (ζ,ζ) acts by a single sign on the entire space $H \otimes V_\mathrm{T}$. Therefore, it is clear that $n(\Xi) = n(\xi)$ by Lemma 7.2.7, and hence the statement of the weights is also clear. □

7.2.13 Comparison of $\mathfrak{t}_\alpha$-weights for short roots of $Lie(\widetilde{SO(q+1,q)})$

Recall the assumptions of Theorem 6.1.1.

Theorem 7.2.14. *Let $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct $K = Spin(q) \times Spin(q)$-types that occur in $H_{\widetilde{Spin(q,q)}}$ such that $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$. $\Sigma_{j=1}^N l(\gamma_j) = n(\xi)$. If V_τ is the Spin representation or either of the two half-Spin representations of $Spin(q)$ after projection onto the first factor of K, after reordering, $\delta_{\epsilon_1,j}^{\xi} = \delta_{\epsilon_1,j} \pm \frac{1}{2}$ for each $j = 1, ..., n(\xi)$. If V_τ is the Spin representation or either of the two half-Spin representations of $Spin(q)$ after projection onto the second factor of K, after reordering, $\delta_{\epsilon_1,j}^{\xi} = \delta_{\epsilon_1,j}$ for each $j = 1, ..., n(\xi)$.*

First we state a lemma for the Theorem.

Lemma 7.2.15. *Assume the statement of $\mathfrak{t}_{\epsilon_1}$-weights in Theorem 7.2.14 for the modules of the group $Spin(q) \times Spin(q)$. Then the statement of $\mathfrak{t}_{\epsilon_1}$-weights in Theorem 7.2.14 for the modules of the group $Pin(q) \times Pin(q)$ is also true.*

Proof. Assume V_T is Pin representation or either of the two Pin representations of $Pin(q)$ after projection onto the second factor of $Pin(q) \times Pin(q)$.

Assume first q is odd hence V_{T} is either of the two Pin representations of $Pin(q)$ after projection onto the second factor of $Pin(q) \times Pin(q)$. Let $V_{\Xi} \subseteq H_{\widetilde{Pin(q,q)}} \otimes V_{\mathrm{T}}$ and let $V_{\Gamma_1}, ..., V_{\Gamma_N}$ be distinct $Pin(q) \times Pin(q)$ modules that occur in $H_{\widetilde{Pin(q,q)}}$ such that $V_{\Xi} \subseteq V_{\Gamma_j} \otimes V_{\mathrm{T}}$ for all j. Without loss of generality, assume (ζ, ζ) acts on V_{T} by Id, i.e. (Id, ζ) acts on V_{T} by Id as V_{T} is the Pin representation of $Pin(q)$ after projection of $Pin(q) \times Pin(q)$ onto the second factor. Without loss of generality, assume (Id, ζ) acts on V_{Ξ} by Id. As $V_{\Xi} \subseteq V_{\Gamma_j} \otimes V_{\mathrm{T}}$ for all j, (Id, ζ) must act by Id on V_{Γ_j} for all j. As $(\zeta, \zeta) \in {}^0M_{\widetilde{Pin(q,q)}}$ is central, (ζ, ζ) must act by Id on V_{Γ_j} for all j. Therefore, we have that the action of (ζ, ζ) on V_{Γ_j} must be $(+, +)$ for all j. This observation gives us the following. If $V_{\Gamma_j}|_{(Spin(q) \times Spin(q))} = V_{\gamma_j}$ where $V_{\gamma_1}, ..., V_{\gamma_N}$ are irreducible $Spin(q) \times Spin(q)$ modules by Lemma 7.2.7, $V_{\gamma_1}, ..., V_{\gamma_N}$ are distinct.

Now let $V_{\Xi}|_{(Spin(q) \times Spin(q))} = V_{\xi}$ where V_{ξ} is irreducible by Lemma 7.2.7. We have $V_{\xi} \subseteq H_{\widetilde{Spin(q,q)}} \otimes V_{\tau}$, and $V_{\gamma_1}, ..., V_{\gamma_N}$ are distinct irreducible modules that occur in $H_{\widetilde{Spin(q,q)}}$ by Theorem 7.2.9, and $V_{\xi} \subseteq V_{\gamma_j} \otimes V_{\tau}$ for all $j = 1, ..., N$. By Theorem 7.2.9 and Theorem 7.2.12, this restriction of $Pin(q) \times Pin(q)$ to $Spin(q) \times Spin(q)$ do not change the set of $\mathfrak{t}_{\epsilon_1}$ weights of interest. Therefore, by the assumption of the lemma, we have the result for q odd.

Assume now $q = 2k$ is even hence V_{T} is Pin representation after projection onto the second factor of $Pin(q) \times Pin(q)$. Let $V_{\Xi} \subseteq H_{\widetilde{Pin(q,q)}} \otimes V_{\mathrm{T}}$ and let $V_{\Gamma_1}, ..., V_{\Gamma_M}$ be distinct $Pin(q) \times Pin(q)$ modules that occur in $H_{\widetilde{Pin(q,q)}}$ such that $V_{\Xi} \subseteq V_{\Gamma_j} \otimes V_{\mathrm{T}}$ for all j. By Lemma 7.2.6, let $V_{\xi_1} \otimes V_{\xi_2}$ be a choice of an irreducible $Spin(q) \times Spin(q)$ module that occurs in $V_{\Xi}|_{(Spin(q) \times Spin(q))}$ such that $(m_1, ..., m_k)$ is the highest weight of V_{ξ_1} and $(n_1, ..., n_k)$ is the highest weight of V_{ξ_2} with m_k, $n_k \geq 0$.

Assume first $m_k = 0$. For each $j = 1, ..., M$, let $V_{\gamma_{j,1}} \otimes V_{\gamma_{j,2}}$ be a choice of an irreducible $Spin(q) \times Spin(q)$ module that occurs in $V_{\Gamma_j}|_{(Spin(q) \times Spin(q))}$ with last entries of the highest weights of $V_{\gamma_{j,1}}$ and $V_{\gamma_{j,2}}$ nonnegative. In fact, $V_{\gamma_{j,1}} = V_{\xi_1}$. Now, reorder so that $V_{\gamma_{1,1}} \otimes V_{\gamma_{1,2}}, ..., V_{\gamma_{N,1}} \otimes V_{\gamma_{N,2}}$ are distinct $Spin(q) \times Spin(q)$ modules. $N \leq M$ as there may be j such that $V_{\gamma_{j,1}} \otimes V_{\gamma_{j,2}}$ occurs twice with different (Id, ζ) signature on the highest weight vector. $V_{\xi_1} \otimes V_{\xi_2} \subseteq H_{\widetilde{Spin(q,q)}} \otimes V_{\tau}$

or $V_{\xi_1} \otimes V_{\xi_2} \subseteq H_{\widetilde{Spin(q,q)}} \otimes \overline{V_\tau}$ but not both by Lemma 7.0.3. Without loss of generality, assume $V_{\xi_1} \otimes V_{\xi_2} \subseteq H_{\widetilde{Spin(q,q)}} \otimes V_\tau$. $V_{\gamma_{1,1}} \otimes V_{\gamma_{1,2}}, ..., V_{\gamma_{N,1}} \otimes V_{\gamma_{N,2}}$ are distinct $Spin(q) \times Spin(q)$ modules that occur in $H_{\widetilde{Spin(q,q)}}$ by Theorem 7.2.8, and we have $V_{\xi_1} \otimes V_{\xi_2} \subseteq (V_{\gamma_{j,1}} \otimes V_{\gamma_{j,2}}) \otimes V_\tau$ for all $j = 1, ..., N$. Therefore, we can assume the statement of the $\mathfrak{t}_{\epsilon_1}$ weights on these $Spin(q) \times Spin(q)$ modules by the assumption of the lemma. But by Theorem 7.2.8 and Theorem 7.2.11, comparison of $\mathfrak{t}_{\epsilon_1}$ weights for the modules of the group $Pin(q) \times Pin(q)$ is that of $V_{\xi_1} \otimes V_{\xi_2}, V_{\gamma_{1,1}} \otimes V_{\gamma_{1,2}}, ..., V_{\gamma_{N,1}} \otimes V_{\gamma_{N,2}}$ of $Spin(q) \times Spin(q)$. Therefore, we have the result for $q = 2k$ where $m_k = 0$.

Assume now $m_k \neq 0$. For each $j = 1, ..., M$, let $V_{\gamma_{j,1}} \otimes V_{\gamma_{j,2}}$ be a choice of an irreducible $Spin(q) \times Spin(q)$ module that occurs in $V_{\Gamma_j}|_{(Spin(q) \times Spin(q))}$ with last entry of the highest weight of $V_{\gamma_{j,2}}$ nonnegative. We have $V_{\gamma_{j,1}} = V_{\xi_1}$ for all j. Now, reorder so that $V_{\gamma_{1,1}} \otimes V_{\gamma_{1,2}}, ..., V_{\gamma_{N,1}} \otimes V_{\gamma_{N,2}}$ are distinct $Spin(q) \times Spin(q)$ modules. $N \leq M$ as there may be j such that $V_{\gamma_{j,1}} \otimes V_{\gamma_{j,2}}$ occurs twice with different (Id, ζ) signature on the highest weight vector. $V_{\xi_1} \otimes V_{\xi_2} \subseteq H_{\widetilde{Spin(q,q)}} \otimes V_\tau$ or $V_{\xi_1} \otimes V_{\xi_2} \subseteq H_{\widetilde{Spin(q,q)}} \otimes \overline{V_\tau}$ but not both by Lemma 7.0.3. Without loss of generality, assume $V_{\xi_1} \otimes V_{\xi_2} \subseteq H_{\widetilde{Spin(q,q)}} \otimes V_\tau$. We have $V_{\xi_1} \otimes V_{\xi_2} \subseteq (V_{\gamma_{j,1}} \otimes V_{\gamma_{j,2}}) \otimes V_\tau$ for all $j = 1, ..., N$. Therefore, we can assume the statement of the $\mathfrak{t}_{\epsilon_1}$ weights on these $Spin(q) \times Spin(q)$ modules by the assumption of the lemma. Note for some j, $dim\ (V_{\gamma_{j,1}} \otimes V_{\gamma_{j,2}})^{{}^0M_{\widetilde{Spin(q,q)}}}$ may be zero because of the first statement of Theorem 7.2.8. In this case, we just ignore $(V_{\gamma_{j,1}} \otimes V_{\gamma_{j,2}})$ in the comparison of $\mathfrak{t}_{\epsilon_1}$ weights.

Now, $V_{\gamma_{1,1}} \otimes \overline{V_{\gamma_{1,2}}}, ..., V_{\gamma_{N,1}} \otimes \overline{V_{\gamma_{N,2}}}$ are distinct $Spin(q) \times Spin(q)$ modules such that $V_{\xi_1} \otimes \overline{V_{\xi_2}} \subseteq (V_{\gamma_{j,1}} \otimes \overline{V_{\gamma_{j,2}}}) \otimes \overline{V_\tau}$ for all $j = 1, ..., N$. Therefore, we can also assume the statement of the $\mathfrak{t}_{\epsilon_1}$ weights on these $Spin(q) \times Spin(q)$ modules by the assumption of the lemma. Note for some j, $dim\ (V_{\gamma_{j,1}} \otimes \overline{V_{\gamma_{j,2}}})^{{}^0M_{\widetilde{Spin(q,q)}}}$ may be zero because of the first statement of Theorem 7.2.8. In this case, we just ignore $(V_{\gamma_{j,1}} \otimes \overline{V_{\gamma_{j,2}}})$ in the comparison of $\mathfrak{t}_{\epsilon_1}$ weights.

By Theorem 7.2.8 and Theorem 7.2.11, comparison of $\mathfrak{t}_{\epsilon_1}$ weights for the modules of the group $Pin(q) \times Pin(q)$ is that of $V_{\xi_1} \otimes V_{\xi_2}, V_{\gamma_{1,1}} \otimes V_{\gamma_{1,2}}, ..., V_{\gamma_{N,1}} \otimes V_{\gamma_{N,2}}$ of $Spin(q) \times Spin(q)$ and $V_{\xi_1} \otimes \overline{V_{\xi_2}}, V_{\gamma_{1,1}} \otimes \overline{V_{\gamma_{1,2}}}, ..., V_{\gamma_{N,1}} \otimes \overline{V_{\gamma_{N,2}}}$ of $Spin(q) \times Spin(q)$. Therefore, we have the result for $q = 2k$ where $m_k \neq 0$.

The case where V_{T} is Pin representation or either of the two Pin representations of $Pin(q)$ after projection onto the first factor of $Pin(q) \times Pin(q)$ can be shown similarly as above.

□

Proof. (Theorem 7.2.14)

$\Sigma_{j=1}^{N} l(\gamma_j) = n(\xi)$ by Lemma 7.0.4.

We first prove the statement of the theorem for $q = 3$. In this case, $K = Spin(3) \times Spin(3) \cong SU(2) \times SU(2)$ where ${}^0M$ is isomorphic to ${}^0M_{\widetilde{SL(3,\mathbb{R})}} \times \mu_2$ with ${}^0M_{\widetilde{SL(3,\mathbb{R})}}$ sitting in K diagonally and μ_2 central in K. Again, by Remark 7.2.2, this μ_2 acts trivially on H and V_τ. We have $\mathfrak{t}_{\epsilon_1} = (\mathfrak{t}_\beta, 0) \in \mathfrak{su}_2 \oplus \mathfrak{su}_2$ where β is that of $Lie(\widetilde{SL(3,\mathbb{R})})$. An irreducible representation of $K = SU(2) \times SU(2)$ is an outer tensor product of that of each of the two $SU(2)$s. Let V_r and W_s be irreducible representations of each of $SU(2)$ with highest weights r and s respectively. Note the weights of V_r are $-r, -r+1, ..., r-1, r$ and similarly for W_s. Denote these weight vectors by $v_{-r}, v_{-r+1}, ..., v_{r-1}, v_r$ and similarly for W_s.

Assume first V_τ is the $Spin$ representation of $Spin(3)$ after projection onto the second factor of $K = Spin(3) \times Spin(3)$.

Let $V_\xi \subseteq H \otimes V_\tau$ with $V_\xi \cong V_r \otimes W_s$. There is at most two $V_\gamma \subseteq H$ such that $V_\xi \subseteq V_\gamma \otimes V_\tau$, $V_{\gamma_1} = V_r \otimes W_{s_1}$ and $V_{\gamma_2} = V_r \otimes W_{s_2}$ with $s_1 = s + \frac{1}{2}$ and $s_2 = s - \frac{1}{2}$. Consider $v_i \otimes w_j \in V_r \otimes W_{s_1}$ and $v_i \otimes w_j \in V_r \otimes W_{s_2}$. In order for them to be candidates for dominant $\mathfrak{t}_{\epsilon_1}$-weight vectors from ${}^0M$ invariant vectors, $i + j$ must be even, and $i \geq 0$.

First, if $i + s_1$ is even and $i \geq 0$, then cover $v_i \otimes w_{s_1}$ with $v_i \otimes w_s$ and $v_i \otimes w_{-s_1}$ with $v_i \otimes w_{-s}$. Now assume $i + j$ is even with $i \geq 0$ and $j \neq s_1$. We use $v_i \otimes w_{j \pm \frac{1}{2}} \in V_r \otimes W_s$ to cover the two $v_i \otimes w_j \in V_r \otimes W_{s_1}$ and $v_i \otimes w_j \in V_r \otimes W_{s_2}$.

The only ambiguity is when $i = j = 0$, since we can't use both $v_0 \otimes w_{\pm \frac{1}{2}} \in V_r \otimes W_s$ as the two come from a single V_τ. But exactly one of the two sets $\{r, s_1\}$ and $\{r, s_2\}$ must consist of two numbers that are of different parity. Without loss of generality assume r and s_1 are of different parity. Then $v_0 \otimes w_0 \in V_r \otimes W_{s_1}$ is not ${}^0M$ invariant, hence the ambiguity is now cleared.

Now assume V_τ is the *Spin* representation of $Spin(3)$ after projection onto the first factor of $K = Spin(3) \times Spin(3)$.

Let $V_\xi \subseteq H \otimes V_\tau$ with $V_\xi \cong V_r \otimes W_s$. There is at most two $V_\gamma \subseteq H$ such that $V_\xi \subseteq V_\gamma \otimes V_\tau$, $V_{\gamma_1} = V_{r_1} \otimes W_s$ and $V_{\gamma_2} = V_{r_2} \otimes W_s$ with $r_1 = r + \frac{1}{2}$ and $r_2 = r - \frac{1}{2}$. Consider $v_i \otimes w_j \in V_{r_1} \otimes W_s$ and $v_i \otimes w_j \in V_{r_2} \otimes W_s$. In order for them to be candidates for dominant $\mathfrak{t}_{\epsilon_1}$-weight vectors from ${}^0M$ invariant vectors, $i + j$ must be even, and $i \geq 0$.

First, if $r_1 + j$ is even, then cover $v_{r_1} \otimes w_j$ with $v_r \otimes w_j$ and $v_{r_1} \otimes w_{-j}$ with $v_r \otimes w_{-j}$. Now assume $i+j$ is even with $i \geq 0$ and $i \neq r_1$. We use $v_{i \pm \frac{1}{2}} \otimes w_j \in V_r \otimes W_s$ to cover the two $v_i \otimes w_j \in V_{r_1} \otimes W_s$ and $v_i \otimes w_j \in V_{r_2} \otimes W_s$.

The only ambiguity is when $i = j = 0$, since we can't use both $v_{\pm\frac{1}{2}} \otimes w_0 \in V_r \otimes W_s$ as $v_{-\frac{1}{2}} \otimes w_0$ is not a dominant $\mathfrak{t}_{\epsilon_1}$-weight vector. But exactly one of the two sets $\{r_1, s\}$ and $\{r_2, s\}$ must consist of two numbers that are of different parity. Without loss of generality assume r_1 and s are of different parity. Then $v_0 \otimes w_0 \in V_{r_1} \otimes W_s$ is not ${}^0M$ invariant, hence the ambiguity is now cleared.

We now proceed with induction. Assume the statement of the theorem for $\widetilde{Spin(q,q)}$, hence the statement of the theorem for $\widetilde{Pin(q,q)}$ with maximal compact subgroup $Pin(q) \times Pin(q)$ by Lemma 7.2.15. We prove the statement of the theorem for $\widetilde{Spin(q+1,q+1)}$. Note $\widetilde{i}({}^0M_{\widetilde{Pin(q,q)}}) = {}^0M_{\widetilde{Spin(q+1,q+1)}}$ by Lemma 7.2.3 where $\widetilde{i}$ is the embedding $\widetilde{i} : \widetilde{Pin(q,q)} \hookrightarrow \widetilde{Spin(q+1,q+1)}$.

The condition $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$ can be restated as $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$. Note the statement of the theorem is true for $\widetilde{Pin(q,q)}$ with the restated condition. Let $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct irreducible $Spin(q+1) \times Spin(q+1)$-modules that occur in H such that $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$, and let $\bigoplus_{j=1}^N Span\ (Pin(q) \times Pin(q)).V_{\gamma_j}^{{}^0M_{\widetilde{Spin(q+1,q+1)}}} = \bigoplus_k W_k$ where each W_k is an irreducible $Pin(q) \times Pin(q)$-module. As the nontrivial element $\eta \in \mu_2 \leq {}^0M_{\widetilde{Spin(q+1,q+1)}}$ acts by -1 where μ_2 is the kernel of the covering homomorphism $p : \widetilde{Spin(q+1,q+1)} \to Spin(q+1,q+1)$, $V_\xi|_{Pin(q)\times Pin(q)} = \bigoplus_j V_{\xi_j}$ where each of V_{ξ_j} occurs in $H_{\widetilde{Pin(q,q)}} \otimes V_\tau$ by Lemma 7.0.3. We have $\bigoplus_k W_k \subseteq \bigoplus_j V_{\xi_j} \otimes V_\tau^*$ where we also know each of W_k occurs in $H_{\widetilde{Pin(q,q)}}$ since $\widetilde{i}({}^0M_{\widetilde{Pin(q,q)}}) = {}^0M_{\widetilde{Spin(q+1,q+1)}}$ by Lemma 7.2.3 where $\widetilde{i} : \widetilde{Pin(q,q)} \hookrightarrow \widetilde{Spin(q+1,q+1)}$ is the embeding.

Since the statement of the theorem is true for $\widetilde{Pin(q,q)}$ with the restated condition and as the set of $\mathfrak{t}_{\epsilon_1}$ weights of interest are the same after branching down to $Pin(q) \times Pin(q)$ because $\widetilde{i}({}^0M_{\widetilde{Pin(q,q)}}) = {}^0M_{\widetilde{Spin(q+1,q+1)}}$ by Lemma 7.2.3, we have the statement of the theorem for $\mathfrak{t}_{\epsilon_1}$. Note $V_{\xi_j} \otimes V_\tau^*$ decomposes into distinct $Pin(q) \times Pin(q)$-modules by Corollary 3.4 of [Ku] as V_τ^* is multiplicity free. Therefore, if $W_k \cong W_l$ with $k \neq l$, then W_k and W_l cannot be contained in a single $V_{\xi_j} \otimes V_\tau^*$, important as the statement of the theorem for $\widetilde{Pin(q,q)}$ also assumes distinct V_Γs. □

Remark For the connected, simply connected $\mathbb{R}$-split Lie groups of type B_q, if $V_\xi \subseteq H \otimes V_\tau$, $V_\xi|_{{}^0M} = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j} \oplus \bigoplus_k V_k$ where $V_{\tau_j} \cong V_\tau$ as ${}^0M$-modules, and $V_k \cong \overline{V_\tau}$ as ${}^0M$-modules. Hence the weights of interest are just that of $\bigoplus_{j=1}^{n(\xi)} V_{\tau_j}$ from the definition of P^ξ matrix in 4.1.

Theorem 7.2.16. *Let α be a short root of $Lie(\widetilde{Spin(q+1,q)})$. Let $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct $K = Spin(q+1) \times Spin(q)$-types that occur in H of $\widetilde{Spin(q+1,q)}$ such that $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$. $\Sigma_{j=1}^N l(\gamma_j) = n(\xi)$. If V_τ is a small K-type after projection onto the first factor of $K = Spin(q+1) \times Spin(q)$, after reordering, $\delta_{\alpha,j}^\xi = \delta_{\alpha,j} \pm \frac{1}{2}$ for each $j = 1, ..., n(\xi)$. If V_τ is a small K-type after projection onto the second factor of $K = Spin(q+1) \times Spin(q)$, after reordering, $\delta_{\alpha,j}^\xi = \delta_{\alpha,j}$ for each $j = 1, ..., n(\xi)$.*

We first need a lemma for the theorem. Consider the embedding $i : SO(q,q)_\circ \hookrightarrow SO(q+1,q)_\circ$ where the image of the maximal compact subgroup $SO(q) \times SO(q)$ of $SO(q,q)_\circ$ under i is contained in the maximal compact subgroup $SO(q+1) \times SO(q)$ of $SO(q+1,q)_\circ$ such that if $(g,h) \in SO(q) \times SO(q)$,

$$i((g,h)) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, h \in SO(q+1) \times SO(q)$$

Let $p : \widetilde{Spin(q+1,q)} \to SO(q+1,q)_\circ$ be the covering homomorphism. We have that $p^{-1}(i(SO(q,q)_\circ))$ is a Lie subgroup isomorphic to $\widetilde{Spin(q,q)}$, hence we have an embedding $\widetilde{i} : \widetilde{Spin(q,q)} \hookrightarrow \widetilde{Spin(q+1,q)}$.

Lemma 7.2.17. *Consider the embedding $\widetilde{i} : \widetilde{Spin(q,q)} \hookrightarrow \widetilde{Spin(q+1,q)}$ described above. We have $\widetilde{i}({}^0M_{\widetilde{Spin(q,q)}}) = {}^0M_{\widetilde{Spin(q+1,q)}}$.*

Proof. We have

$$ {}^0M_{SO(q+1,q)_\circ} = \{ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, g \mid g \in {}^0M_{SL(q,\mathbb{R})} \} $$

, the image of ${}^0M_{SO(q,q)_\circ}$ under the map i. Hence $i({}^0M_{SO(q,q)_\circ}) = {}^0M_{SO(q+1,q)_\circ}$. As $\widetilde{i}({}^0M_{\widetilde{Spin(q,q)}}) = p^{-1}(i({}^0M_{SO(q,q)_\circ}))$ and ${}^0M_{\widetilde{Spin(q+1,q)}} = p^{-1}({}^0M_{SO(q+1,q)_\circ})$, we have the statement of the lemma. □

Proof. (Theorem 7.2.16)

$\Sigma_{j=1}^N l(\gamma_j) = n(\xi)$ is Lemma 7.0.4.

We first show the statement for $\alpha = \epsilon_1$.

Consider the embedding $\widetilde{i} : \widetilde{Spin(q,q)} \hookrightarrow \widetilde{Spin(q+1,q)}$ where $\widetilde{i}({}^0M_{\widetilde{Spin(q,q)}}) = {}^0M_{\widetilde{Spin(q+1,q)}}$ by Lemma 7.2.17.

We can restate the condition $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$ as $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$. Note the statement of the theorem is true with the restated condition for $\widetilde{Spin(q,q)}$ by Theorem 7.2.14.

Let $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct irreducible K-modules that occur in H such that $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$, and let $\bigoplus_{j=1}^N Span\ (Spin(q) \times Spin(q)).V_{\gamma_j}^{{}^0M_{\widetilde{Spin(q+1,q)}}} = \bigoplus_k W_k$ where each W_k is an irreducible $Spin(q) \times Spin(q)$-module. As the nontrivial element $\eta \in \mu_2 \leq {}^0M_{\widetilde{Spin(q+1,q)}}$ acts by -1 where μ_2 is the kernel of the covering homomorphism $p : \widetilde{Spin(q+1,q)} \to Spin(q+1,q)$, $V_\xi|_{(Spin(q)\times Spin(q))} = \bigoplus_j V_{\xi_j}$ where each of V_{ξ_j} occurs in $H \otimes V_\tau$ or $H \otimes \overline{V_\tau}$ with H that of $\widetilde{Spin(q,q)}$ and $\overline{V_\tau}$ the other half *Spin* representation. We have $\bigoplus_k W_k \subseteq \bigoplus_j V_{\xi_j} \otimes V_\tau^*$ where each of W_k occurs in H of $\widetilde{Spin(q,q)}$ as $\widetilde{i}({}^0M_{\widetilde{Spin(q,q)}}) = {}^0M_{\widetilde{Spin(q+1,q)}}$ by Lemma 7.2.17 where $\widetilde{i}$ is the embedding $\widetilde{i} : \widetilde{Spin(q,q)} \hookrightarrow \widetilde{Spin(q+1,q)}$.

We assert that if $W_k \subseteq V_{\xi_j} \otimes V_\tau^*$, then $V_{\xi_j}|_{{}^0M_{\widetilde{Spin(q,q)}}}$ is equivalent to a multiple of V_τ and $V_{\xi_j} \subseteq H_{\widetilde{Spin(q,q)}} \otimes V_\tau$. Indeed, if $W_k \subseteq V_{\xi_j} \otimes V_\tau^*$, then $V_{\xi_j} \subseteq W_k \otimes V_\tau$, hence claim is true by Lemma 7.0.3. This observation is important because of the following. First, recall from remark in the beginning of the chapter the decomposition $V_\xi|_{K_{\epsilon_1}} = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j} \oplus \bigoplus_k V_k$ where $V_{\tau_j} \cong V_\tau$ as ${}^0M_{\widetilde{Spin(q+1,q)}}$-modules, and $V_k \cong \overline{V_\tau}$ as ${}^0M_{\widetilde{Spin(q+1,q)}}$-modules where $\overline{V_\tau}$ is the other half *Spin* representation. As $V_\xi \subseteq H \otimes V_\tau$, we only consider $V_{\tau_1}, ..., V_{\tau_{n(\xi)}}$ in the definition of P^ξ matrix.

Since the statement of the theorem is true for $\widetilde{Spin(q,q)}$ with the restated condition and as the set of $\mathfrak{t}_{\epsilon_1}$ weights of interest are the same after branching down to $Spin(q) \times Spin(q)$ because $\widetilde{i}({}^0M_{\widetilde{Spin(q,q)}}) = {}^0M_{\widetilde{Spin(q+1,q)}}$ by Lemma 7.2.17, we have the statement of the theorem for $\alpha = \epsilon_1$. Note $V_{\xi_j} \otimes V_\tau^*$ decomposes into distinct $Spin(q) \times Spin(q)$-modules by Corollary 3.4 of [Ku] as V_τ^* is multiplicity free. Therefore, if $W_k \cong W_l$ with $k \neq l$, then W_k and W_l cannot be contained in a single $V_{\xi_j} \otimes V_\tau^*$, important as the statement of the theorem for $\widetilde{Spin(q,q)}$ also assumes distinct V_γs.

By Proposition 6.11 of [Bou], any positive short root α of $Lie(\widetilde{Spin(q+1,q)})$ must be conjugate to ϵ_1 via an element of the Weyl group $W(A) = N_K(A)/Z_K(A)$ Therefore, the set of $\mathfrak{t}_\alpha$-weights of interest is the same as that of $\mathfrak{t}_{\epsilon_1}$ and we have the statement of the theorem.

□

7.3 Comparison of $\mathfrak{t}_\alpha$-weights for type F_4 and Comparison of $\mathfrak{t}_\alpha$-weights for short roots of $Lie(G_2)$

Recall the assumptions of Theorem 6.1.1.

Theorem 7.3.1. *Let G be the connected, simply connected $\mathbb{R}$-split Lie group of type F_4. Let $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct $K = Sp(3) \times SU(2)$-types that occur in H of G such that $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$. $\Sigma_{j=1}^N l(\gamma_j) = n(\xi)$. If α is a short root of Lie(G), after reordering, $\delta^\xi_{\alpha,j} = \delta_{\alpha,j}$ for each $j = 1, ..., n(\xi)$. If α is a long root of Lie(G), after reordering, $\delta^\xi_{\alpha,j} = \delta_{\alpha,j} \pm \frac{1}{2}$ for each $j = 1, ..., n(\xi)$.*

Proof. $\Sigma_{j=1}^N l(\gamma_j) = n(\xi)$ by Lemma 7.0.4.

Recall from chapter 3 the embedded subgroup $\widetilde{i} : \widetilde{Spin(5,4)} \hookrightarrow G$ with maximal compact subgroup $Spin(5) \times Spin(4)$. We have $\widetilde{i}({}^0M_{\widetilde{Spin(5,4)}}) = {}^0M_G$ where the restriction of $K = Sp(3) \times SU(2)$ to $Spin(5) \times Spin(4)$ preserve ${}^0M_G$-invariants of H and the decomposition $V_\xi|_{{}^0M_G}$ by Lemma 3.3.1. But we also have the embedding $\widetilde{i} : \widetilde{Spin(4,4)} \hookrightarrow \widetilde{Spin(5,4)}$ with $\widetilde{i}({}^0M_{\widetilde{Spin(4,4)}}) = {}^0M_{\widetilde{Spin(5,4)}}$ by Lemma 7.2.17. Hence we also have the embedding $\widetilde{i} : \widetilde{Spin(4,4)} \hookrightarrow G$ where

$\widetilde{i}({}^0M_{\widetilde{Spin(4,4)}}) = {}^0M_G$ and restriction of $K = Sp(3) \times SU(2)$ to the maximal compact subgroup $Spin(4) \times Spin(4)$ of $\widetilde{Spin(4,4)}$ preserve ${}^0M_G$-invariants of H and the decomposition $V_\xi|_{{}^0M_G}$.

If α is a positive long root of $Lie(G)$, by Proposition 6.11 of [Bou], $\mathfrak{t}_\alpha$ must be conjugate to $\mathfrak{t}_\beta$ via an element of $K = Sp(3) \times SU(2)$ where β is a positive root of $Lie(\widetilde{Spin(4,4)}) \subseteq Lie(G)$. If α is a positive short root of $Lie(G)$, $\mathfrak{t}_\alpha$ must be conjugate to $\mathfrak{t}_{\epsilon_1}$ where ϵ_1 is a positive short root of $Lie(\widetilde{Spin(5,4)}) \subseteq Lie(G)$. We have $\mathfrak{t}_{\epsilon_1} \in Lie(Spin(4) \times Spin(4))_{\mathbb{C}}$. Therefore, it will be enough to show the statement for α a positive root of $Lie(\widetilde{Spin(4,4)}) \subseteq Lie(G)$ and ϵ_1 a positive root of $Lie(\widetilde{Spin(5,4)}) \subseteq Lie(G)$ where $\mathfrak{t}_{\epsilon_1} \in Lie(Spin(4) \times Spin(4))_{\mathbb{C}}$.

We can restate the condition $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$ as $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$. Note the statement of the theorem is true with the restated condition for $\widetilde{Spin(4,4)}$ by Theorem 7.1.1 and Theorem 7.2.14.

Let $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct irreducible K-modules that occur in H such that $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$, and let $\bigoplus_{j=1}^N Span\ (Spin(4) \times Spin(4)).V_{\gamma_j}^{{}^0M_G} = \bigoplus_k W_k$ where each W_k is an irreducible $Spin(4) \times Spin(4)$-module. The nontrivial element $\eta \in \mu_2 \leq {}^0M_G$ acts by -1 where μ_2 is the kernel of the covering homomorphism $p : \widetilde{Spin(4,4)} \to Spin(4,4)$ which is also the kernel of the covering homomorphism $p : G \to G_{\mathbb{R}}$. Hence $V_\xi|_{Spin(4)\times Spin(4)} = \bigoplus_j V_{\xi_j}$ where each of V_{ξ_j} occurs in $H \otimes V_\tau$ or $H \otimes \overline{V_\tau}$ with H that of $\widetilde{Spin(4,4)}$ and $\overline{V_\tau}$ the other half $Spin$ representation after projection onto the second factor of $Spin(4) \times Spin(4)$. We have $\bigoplus_k W_k \subseteq \bigoplus_j V_{\xi_j} \otimes V_\tau^*$ where each of W_k occurs in H of $\widetilde{Spin(4,4)}$ as $\widetilde{i}({}^0M_{\widetilde{Spin(4,4)}}) = {}^0M_G$.

We assert that if $W_k \subseteq V_{\xi_j} \otimes V_\tau^*$, then $V_{\xi_j}|_{{}^0M_{\widetilde{Spin(4,4)}}}$ is equivalent to a multiple of V_τ and $V_{\xi_j} \subseteq H_{\widetilde{Spin(4,4)}} \otimes V_\tau$. Indeed, if $W_k \subseteq V_{\xi_j} \otimes V_\tau^*$, then $V_{\xi_j} \subseteq W_k \otimes V_\tau$, hence claim is true by Lemma 7.0.3. This observation is important because of the following. First, recall from remark in the beginning of the chapter the decomposition $V_\xi|_{K_\alpha} = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j} \oplus \bigoplus_k V_k$ where $V_{\tau_j} \cong V_\tau$ as ${}^0M_G$-modules, and $V_k \cong \overline{V_\tau}$ as ${}^0M_G$-modules where $\overline{V_\tau}$ is the other half $Spin$ representation restricted to ${}^0M_G$. As $V_\xi \subseteq H \otimes V_\tau$, we only consider $V_{\tau_1}, ..., V_{\tau_{n(\xi)}}$ in the definition of P^ξ matrix.

Since the statement of the theorem is true for $\widetilde{Spin(4,4)}$ with the restated condition by Theorem 7.1.1 and Theorem 7.2.14 and as the set of $\mathfrak{t}_\alpha$ weights of interest and the set of $\mathfrak{t}_{\epsilon_1}$ weights of interest remain the same after branching down to $Spin(4) \times Spin(4)$ because $\widetilde{i}({}^0M_{\widetilde{Spin(4,4)}}) = {}^0M_{\widetilde{Spin(5,4)}}$, we have the statement of the theorem for all positive roots α of $Lie(\widetilde{Spin(4,4)}) \subseteq Lie(G)$ and $\mathfrak{t}_{\epsilon_1} \in Lie(Sp(3) \times SU(2))_{\mathbb{C}}$. Note $V_{\xi_j} \otimes V_\tau^*$ decomposes into distinct $Spin(4) \times Spin(4)$-modules by Corollary 3.4 of [Ku] as V_τ^* is multiplicity free. Therefore, if $W_k \cong W_l$ with $k \neq l$, then W_k and W_l cannot be contained in a single $V_{\xi_j} \otimes V_\tau^*$, important as the statement of the theorem for $\widetilde{Spin(4,4)}$ also assumes distinct V_γs. □

Theorem 7.3.2. *Let G be the connected, simply connected $\mathbb{R}$-split Lie group of type G_2. Let $V_{\gamma_1}, ..., V_{\gamma_N}$ be distinct $K = SU(2) \times SU(2)$-types that occur in H of G such that $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$. $\Sigma_{j=1}^N l(\gamma_j) = n(\xi)$. Let α be a short root of $Lie(G)$. If V_τ is the standard 2 dimensional representation of $SU(2)$ after projection onto the first factor of K, after reordering, $\delta^\xi_{\alpha,j} = \delta_{\alpha,j}$ for each $j = 1, ..., n(\xi)$. If V_τ is the standard 2 dimensional representation of $SU(2)$ after projection onto the second factor of K, after reordering, $\delta^\xi_{\alpha,j} = \delta_{\alpha,j} \pm \frac{1}{2}$ for each $j = 1, ..., n(\xi)$.*

Proof. $\Sigma_{j=1}^N l(\gamma_j) = n(\xi)$ by Lemma 7.0.4.

The maximal compact subgroup K of G is $SU(2) \times SU(2)$ where the first $SU(2)$ comes from α_0 the long root in the extended dynkin diagram of G_2 in section 3.3 and the second $SU(2)$ comes from α_1 the short simple root. By the definition of $\mathfrak{t}_{\alpha_1}$ from 7.0.2, we see that $\mathfrak{t}_{\alpha_1}$ must be $(0, \mathfrak{t}_\beta) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ where β is a positive root of $Lie(SL(3, \mathbb{R}))$. We have that ${}^0M_G \cong {}^0M_{\widetilde{SL(3,\mathbb{R})}}$ sitting in $SU(2) \times SU(2)$ diagonally.

V_τ is the standard 2 dimensional representation of $SU(2)$ after projection onto either the first factor or the second factor of K. Therefore, our situation is exactly that of the two comparisons of $\mathfrak{t}_{\epsilon_1}$ weights for $\widetilde{Spin(3,3)}$ in the proof of Theorem 7.2.14, hence we have the statement of the theorem for α_1. By Proposition 6.11 of [Bou], any positive short root $\alpha \in Lie(G)$ must be conjugate to α_1 via an element of the Weyl group $W(A) = N_K(A)/Z_K(A)$, hence $\mathfrak{t}_\alpha$ must be conjugate to $\mathfrak{t}_{\alpha_1}$ via an element of K. Therefore, the set of $\mathfrak{t}_\alpha$-weights of interest is the same as that of $\mathfrak{t}_{\alpha_1}$ and we have the statement of the theorem.

□

7.4 Product formula of p_ξ for the connected, simply connected $\mathbb{R}$-split Lie group of simple Lie type other than A_n and C_n

Recall the notations from chapter 6. The following is Theorem 6.3.1 for the connected, simply connected $\mathbb{R}$-split Lie type other than A_n and C_n.

Theorem 7.4.1. *There exists a non-zero scalar c such that*

$$p_\xi(\nu) = c\Pi_{\phi\in\Phi^+} p_{(\phi)}(\nu)$$

Proof. The proof of Theorem 6.3.1 was completed with divisibility and degree argument. First consider α a positive root of $Lie(G)$ other than the short roots of type B_n, F_4, and G_2. The semisimple part of G_α is the group generated by $Mp(2,\mathbb{R})$ and ${}^0M$, and the K_α module $H_\alpha \otimes V_\tau$ is exactly that of $\widetilde{SL(n,\mathbb{R})}$ case from section 5.3. Therefore, divisibility argument is exactly that of $\widetilde{SL(n,\mathbb{R})}$.

Now consider α a positive short root of type B_n, F_4, and G_2.

If the small K-type is after projection onto the first factor $Spin(n+1)$ of $K = Spin(n+1) \times Spin(n)$ for type B_n or if the small K-type is after projection onto the second factor $SU(2)$ of $K = SU(2) \times SU(2)$ for type G_2, the situation is exactly the same as above. Now assume otherwise. The semisimple part of G_α is the group generated by $Mp(2,\mathbb{R})$ and ${}^0M$. Let V_{ξ_α} be an irreducible K_α module that occurs in $H_\alpha \otimes V_\tau$. The weights of $\mathfrak{t}_\alpha$ on V_{ξ_α} are even integers as $\mathfrak{t}_\alpha$ acts trivially on V_τ. V_{ξ_α} is isomorphic as a K_α module to an irreducible K_α submodule of $(\overline{Z^l_\alpha} \otimes V_\tau) \oplus (Z^l_\alpha \otimes V_\tau)$ for some l.

Let $V_\xi \subseteq H \otimes V_\tau$, $\epsilon_1, ..., \epsilon_{n(\xi)}$ be a basis of $Hom_K(V_\xi, H \otimes V_\tau)$. Let $V_\xi|_{K_\alpha} = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j} \oplus W$ where V_{τ_j} is an irreducible K_α module isomorphic to V_τ as ${}^0M$ modules for all j, and W is a multiple of $\overline{V_\tau}|_{{}^0M}$.

Without loss of generality, let $v_j \in V_{\tau_j}$ be a dominant $\mathfrak{t}_\alpha$ weight vector. $L_\alpha(\epsilon_i(v_j)) \in \overline{Z^l_\alpha} symm(J_\alpha) \otimes V_\tau$ for some $l \in \mathbb{Z}_{\geq 0}$ where L_α is defined in 6.2.6 and $symm(J_\alpha) \subseteq U(\mathfrak{g}_\alpha)^{\mathfrak{k}_\alpha}$ with $U(\mathfrak{g}_\alpha)^{\mathfrak{k}_\alpha}$ the subalgebra generated by $\mathfrak{t}_\alpha$, center of $\mathfrak{g}_\alpha$, and the Casimir element. Recall the projection map $Q : U(\mathfrak{g}) \to U(\mathfrak{a})U(\mathfrak{k}) \oplus \mathfrak{n}U(\mathfrak{g})$ onto the first factor. As $\mathfrak{t}_\alpha$ acts trivially on V_τ, we still have that the action of

$\overline{Z^l_\alpha} symm(J_\alpha)$ on V_τ at the identity is given by $Q(\overline{Z^l_\alpha} symm(J_\alpha))$, and hence the rest of the argument is exactly that of $\widetilde{SL(n,\mathbb{R})}$.

The degree argument is exactly that of the proof of Theorem 6.3.1, using Theorem 7.1.1, Theorem 7.2.16, Theorem 7.3.1, and Theorem 7.3.2. □

Chapter 8

Computation of $p_\xi(\nu)$ and Determinants of Intertwining Operators

In this chapter we derive a general formula of $p_\xi(\nu)$ for the group $\widetilde{SL(n,\mathbb{R})}$ ($n \geq 3$) as a product over those of rank one subgroups corresponding to the positive roots. In addition, for G any of the connected, simply connected split real form of simple Lie type other than type C_n, we prove cyclicity of a small K-type $V_\tau \subseteq I_{P,\sigma,\nu}$ in the closed Langlands chamber, and use this to prove irreducibility of unitary principal series admitting a small K-type.

8.1 Computation in Rank One Case

Let G be any of the connected, simply connected split real form of simple Lie type other than type C_n. For any positive root α, $Lie(G_\alpha) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus Z(\mathfrak{g}_\alpha)$. Recall for G_α,

$$H = \bigoplus_{l \geq 0} Z^l \oplus \bigoplus_{l > 0} \overline{Z}^l$$

where $Z = X + iY$ is discussed in chapter 5, and we drop the notation α. Recall the projection map $Q : U(\mathfrak{g}) \longrightarrow U(\mathfrak{a})U(\mathfrak{k}) \oplus \mathfrak{n}U(\mathfrak{g})$ onto the first summand. To compute $p_\xi(\nu)$ for a K-type V_ξ that occurs in $H \otimes V_\tau$, we compute $Q(Z^l)$ and $Q(\overline{Z}^l)$

with $\mathfrak{t}$ weight $\pm\frac{1}{2}$ or 0. To do this, we use $Q'(Z^l)$ and $Q'(\overline{Z}^l)$ already computed in [JW] where $Q' : U(\mathfrak{g}) \longrightarrow U(\mathfrak{a}) \oplus \mathfrak{n}U(\mathfrak{g}) \oplus U(\mathfrak{g})\mathfrak{k}$ is the projection onto the first summand. This is because $Q(Z^l)$ and $Q(\overline{Z}^l)$ can be written as a sum of two different parts, one in $U(\mathfrak{a})$, and the other in $U(\mathfrak{a})U(\mathfrak{k})\mathfrak{k}$, where the former is exactly $Q'(Z^l)$ and $Q'(\overline{Z}^l)$ respectively.

Theorem 8.1.1. *$Q(Z^l) = \Pi_{j=0}^{l-1}(X+2j-t)$ and $Q(\overline{Z}^l) = \Pi_{j=0}^{l-1}(X+2j+t)$, where t is a weight of $\mathfrak{t}$.*

Proof. We prove the first formula.

From Theorem 7.6 of [JW], we have $Q'(Z^l) = \Pi_{j=0}^{l-1}(X+2j)$. We wish to find the shift from $U(\mathfrak{a})U(\mathfrak{k})\mathfrak{k}$ part as it is the only difference between Q and Q'. If $l = 1$, then $Z = X + iY = X + i(2E + i\mathfrak{t})$. Hence the statement is true. Now we proceed with the method of induction. Assume the statement for $l-1$ and we show the statement for l. We have $Z^l = ZZ^{l-1} = (X + 2iE - \mathfrak{t})Z^{l-1}$. After dropping the $\mathfrak{n}$ part E, we have $(X - \mathfrak{t})Z^{l-1}$ left. There will be exactly two $U(\mathfrak{a})U(\mathfrak{k})\mathfrak{k}$ shifts, one from $X(U(\mathfrak{a})U(\mathfrak{k})\mathfrak{k}$ part of $Z^{l-1})$ and the other from $-\mathfrak{t}(Z^{l-1}) = -Z^{l-1}(\mathfrak{t} - 2(l-1))$ by the commutation relation. Hence, the overall shift is $(X + 2(l-1))(Q(Z^{l-1}) - Q'(Z^{l-1})) - tQ(Z^{l-1})$. But, since $Q'(Z^l) = (X + 2(l-1))Q'(Z^{l-1})$, we have

$$Q(Z^l) = Q'(Z^l) + shift = (X + 2(l-1))Q'(Z^{l-1}) \tag{8.1.1}$$

$$+ (X + 2(l-1))(Q(Z^{l-1}) - Q'(Z^{l-1})) - tQ(Z^{l-1}) \tag{8.1.2}$$

$$= (X + 2(l-1))Q(Z^{l-1}) - tQ(Z^{l-1}) \tag{8.1.3}$$

$$= (X + 2(l-1) - t)Q(Z^{l-1}) \tag{8.1.4}$$

Hence we have the first formula, and second can be shown similarly. □

If $\mathfrak{t}$ acts nontrivially on V_τ, we have the following.

For the ξ-type $\overline{Z}^l \otimes V_\tau^+ \oplus Z^l \otimes V_\tau^-$,

$$p_\xi(\nu) = \Pi_{j=0}^{l-1}(\nu + 2j + \frac{1}{2})$$

and for the ξ-type $\overline{Z}^l \otimes V_\tau^- \oplus Z^l \otimes V_\tau^+$,

$$p_\xi(\nu) = \Pi_{j=0}^{l-1}(\nu + 2j - \frac{1}{2})$$

If $\mathfrak{t}$ acts trivially on V_τ, for any of the ξ-type that occurs in $(\overline{Z}^l \otimes V_\tau) \oplus (Z^l \otimes V_\tau)$,

$$p_\xi(\nu) = \Pi_{j=0}^{l-1}(\nu + 2j)$$

8.2 Computation of $p_\xi(\nu)$ and Determinants of Intertwining Operators for $\widetilde{SL(n,\mathbb{R})}$

Computations are with ρ-shifts.

8.2.1 $\widetilde{SL(3,\mathbb{R})}$

For $\widetilde{SL(3,\mathbb{R})}$, the set of positive roots of Lie$(\widetilde{SL(3,\mathbb{R})}) \otimes \mathbb{C} = \mathfrak{sl}_3$ consists of $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, and $\alpha_3 = \epsilon_1 - \epsilon_3$. As α_js are conjugates by elements in $K = Spin(3)$ for all $j \in \{1, 2, 3\}$, the set of dominant weights that determine $p_\xi(\nu)$ will be exactly the same for all three $\mathfrak{t}_{\alpha_1}$, $\mathfrak{t}_{\alpha_2}$, and $\mathfrak{t}_{\alpha_3}$.

Consider an irreducible representation of $K = Spin(3) \cong SU(2)$ with highest weight $\frac{p}{2}$ with p odd. The dominant $\mathfrak{t}_\alpha$-weights of interest are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ..., \frac{p}{2}$, all with multiplicity one. We have the following using the product formula and the computation in rank one case from section 8.1.

Let $\xi = \frac{p}{2}$ with $(p-1)$ divisible by 4. Then, up to a nonzero scalar,

$$p_\xi(\nu) = \Pi_{l=0}^{\frac{p-1}{4}} \Pi_{j=0}^{l-1}(\nu_1 + 2j + \frac{1}{2})(\nu_1 + 2j + \frac{3}{2})(\nu_2 + 2j + \frac{1}{2})$$

$$\times\ (\nu_2 + 2j + \frac{3}{2})(\nu_1 + \nu_2 + 2j + \frac{1}{2})(\nu_1 + \nu_2 + 2j + \frac{3}{2})$$

Let $\xi = \frac{p}{2}$ with $(p-3)$ divisible by 4. Then, up to a nonzero scalar,

$$p_\xi(\nu) = \Pi_{l=0}^{\frac{p-3}{4}} \Pi_{j=0}^{l-1}(\nu_1 + 2j + \frac{1}{2})(\nu_1 + 2j + \frac{3}{2})(\nu_2 + 2j + \frac{1}{2})$$

$$\times\ (\nu_2 + 2j + \frac{3}{2})(\nu_1 + \nu_2 + 2j + \frac{1}{2})(\nu_1 + \nu_2 + 2j + \frac{3}{2})$$

$$\times \Pi_{k=0}^{\frac{p-3}{4}}(\nu_1+2k+\frac{1}{2})(\nu_2+2k+\frac{1}{2})(\nu_1+\nu_2+2k+\frac{1}{2})$$

Let $\sigma = V_\tau|_{^0M}$. By Theorem 4.2.3, the determinant of the intertwiner $A(\nu) : I_{P,\sigma,\nu} \longrightarrow I_{\overline{P},\sigma,\nu}$ on ξ-type is the following.

Let $\xi = \frac{p}{2}$ with $(p-1)$ divisible by 4. Then, determinant of $A(\nu)|_{(I_{P,\sigma,\nu}(\xi))}$ up to a nonzero scalar is

$$\Pi_{l=0}^{\frac{p-1}{4}}\Pi_{j=0}^{l-1}(\frac{(\nu_1-2j-\frac{1}{2})}{(\nu_1+2j+\frac{1}{2})}\frac{(\nu_1-2j-\frac{3}{2})}{(\nu_1+2j+\frac{3}{2})}\frac{(\nu_2-2j-\frac{1}{2})}{(\nu_2+2j+\frac{1}{2})})^{p+1}$$

$$\times (\frac{(\nu_2-2j-\frac{3}{2})}{(\nu_2+2j+\frac{3}{2})}\frac{(\nu_1+\nu_2-2j-\frac{1}{2})}{(\nu_1+\nu_2+2j+\frac{1}{2})}\frac{(\nu_1+\nu_2-2j-\frac{3}{2})}{(\nu_1+\nu_2+2j+\frac{3}{2})})^{p+1}$$

In terms of gamma functions, this is

$$\Pi_{l=0}^{\frac{p-1}{4}}\Pi_{j=0}^{l-1}(\frac{\Gamma(\nu_1-2j+\frac{1}{2})}{\Gamma(\nu_1-2j-\frac{3}{2})}\frac{\Gamma(\nu_1+2j+\frac{1}{2})}{\Gamma(\nu_1+2j+\frac{5}{2})}\frac{\Gamma(\nu_2-2j+\frac{1}{2})}{\Gamma(\nu_2-2j-\frac{3}{2})}\frac{\Gamma(\nu_2+2j+\frac{1}{2})}{\Gamma(\nu_2+2j+\frac{5}{2})})^{p+1}$$

$$\times (\frac{\Gamma(\nu_1+\nu_2-2j+\frac{1}{2})^2}{\Gamma(\nu_1+\nu_2-2j-\frac{3}{2})}\frac{\Gamma(\nu_1+\nu_2+2j+\frac{1}{2})}{\Gamma(\nu_1+\nu_2+2j+\frac{5}{2})})^{p+1}$$

Let $\xi = \frac{p}{2}$ with $(p-3)$ divisible by 4. Then, determinant of $A(\nu)|_{(I_{P,\sigma,\nu}(\xi))}$ up to a nonzero scalar is

$$\Pi_{l=0}^{\frac{p-3}{4}}\Pi_{j=0}^{l-1}(\frac{(\nu_1-2j-\frac{1}{2})}{(\nu_1+2j+\frac{1}{2})}\frac{(\nu_1-2j-\frac{3}{2})}{(\nu_1+2j+\frac{3}{2})}\frac{(\nu_2-2j-\frac{1}{2})}{(\nu_2+2j+\frac{1}{2})})^{p+1}$$

$$\times(\frac{(\nu_2-2j-\frac{3}{2})}{(\nu_2+2j+\frac{3}{2})}\frac{(\nu_1+\nu_2-2j-\frac{1}{2})}{(\nu_1+\nu_2+2j+\frac{1}{2})}\frac{(\nu_1+\nu_2-2j-\frac{3}{2})}{(\nu_1+\nu_2+2j+\frac{3}{2})})^{p+1}$$

$$\times \Pi_{k=0}^{\frac{p-3}{4}}(\frac{(\nu_1-2k-\frac{1}{2})}{(\nu_1+2k+\frac{1}{2})}\frac{(\nu_2-2k-\frac{1}{2})}{(\nu_2+2k+\frac{1}{2})}\frac{(\nu_1+\nu_2-2k-\frac{1}{2})}{(\nu_1+\nu_2+2k+\frac{1}{2})})^{p+1}$$

In terms of gamma functions, this is

$$\Pi_{l=0}^{\frac{p-3}{4}}\Pi_{j=0}^{l-1}(\frac{\Gamma(\nu_1-2j+\frac{1}{2})}{\Gamma(\nu_1-2j-\frac{3}{2})}\frac{\Gamma(\nu_1+2j+\frac{1}{2})}{\Gamma(\nu_1+2j+\frac{5}{2})}\frac{\Gamma(\nu_2-2j+\frac{1}{2})}{\Gamma(\nu_2-2j-\frac{3}{2})}\frac{\Gamma(\nu_2+2j+\frac{1}{2})}{\Gamma(\nu_2+2j+\frac{5}{2})})^{p+1}$$

$$\times (\frac{\Gamma(\nu_1+\nu_2-2j+\frac{1}{2})}{\Gamma(\nu_1+\nu_2-2j-\frac{3}{2})}\frac{\Gamma(\nu_1+\nu_2+2j+\frac{1}{2})}{\Gamma(\nu_1+\nu_2+2j+\frac{5}{2})})^{p+1}$$

$$\times \Pi_{k=0}^{\frac{p-3}{4}}(\frac{\Gamma(\nu_1-2k+\frac{1}{2})}{\Gamma(\nu_1-2k-\frac{3}{2})}\frac{\Gamma(\nu_2-2k+\frac{1}{2})}{\Gamma(\nu_2-2k-\frac{3}{2})}\frac{\Gamma(\nu_1+\nu_2-2k+\frac{1}{2})}{\Gamma(\nu_1+\nu_2-2k-\frac{3}{2})})^{p+1}$$

8.2.2 $\widetilde{SL(4,\mathbb{R})}$

For $\widetilde{SL(4,\mathbb{R})}$, the set of positive roots of Lie$(\widetilde{SL(4,\mathbb{R})}) \otimes \mathbb{C} = \mathfrak{sl}_4$ consists of $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, $\alpha_3 = \epsilon_3 - \epsilon_4$, $\alpha_4 = \epsilon_1 - \epsilon_3$, $\alpha_5 = \epsilon_1 - \epsilon_4$, $\alpha_6 = \epsilon_2 - \epsilon_4$.

$K = Spin(4) \cong SU(2) \times SU(2)$, hence Lie$(K) \otimes \mathbb{C} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. In this isomorphism, we have the following

$$\mathfrak{t}_{\alpha_1} \longrightarrow \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \oplus \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$\mathfrak{t}_{\alpha_2} \longrightarrow \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$\mathfrak{t}_{\alpha_3} \longrightarrow \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \oplus \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$\mathfrak{t}_{\alpha_4} \longrightarrow \begin{bmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{bmatrix}$$

$$\mathfrak{t}_{\alpha_5} \longrightarrow \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$\mathfrak{t}_{\alpha_4} \longrightarrow \begin{bmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -\frac{1}{2}i \\ \frac{1}{2}i & 0 \end{bmatrix}$$

$\mathfrak{t}_\alpha s$ are all conjugates by elements in $K = Spin(4) \cong SU(2) \times SU(2)$ for all $\alpha \in \{\alpha_1, ..., \alpha_6\}$, hence given an irreducible representation of K, the set of dominant $\mathfrak{t}_\alpha$-weights counting multiplicity will be independent of $\alpha \in \{\alpha_1, ..., \alpha_6\}$.

We have the following correspondence of highest weights.

$$(\epsilon_1, \epsilon_2) \ of \ Spin(4) \longrightarrow (\frac{\epsilon_1 - \epsilon_2}{2}, \frac{\epsilon_1 + \epsilon_2}{2}) \ of \ SU(2) \times SU(2)$$

$$(r + s, s - r) \ of \ Spin(4) \longleftarrow (r, s) \ of \ SU(2) \times SU(2)$$

Let $\tau_1 = \mathbb{C}^2 \otimes triv$ and $\tau_2 = triv \otimes \mathbb{C}^2$ be the two half spin representations of $K = Spin(4) \cong SU(2) \times SU(2)$. Let $\sigma_1 = V_{\tau_1}|_{{}^0M}$ and $\sigma_2 = V_{\tau_2}|_{{}^0M}$. If (r, s) is a highest weight of $SU(2)\times SU(2)$ with r half-integral and s integral, $(r,s)\subseteq I_{P,\sigma_1,\nu}$. If (r,s) is a highest weight of $SU(2)\times SU(2)$ with r integral and s half-integral, $(r,s)\subseteq I_{P,\sigma_2,\nu}$.

Consider the highest weight module of $SU(2)\times SU(2)$ of highest weight (r,s) that occurs in either $I_{P,\sigma_1,\nu}$ or $I_{P,\sigma_2,\nu}$. Assume without loss of generality $r>s$. Note $r\neq s$ because one of $r,\ s$ has to be half integral and the other has to be an integral. The set of dominant $\mathfrak{t}_\alpha$-weights that occur in the highest weight module of highest weight (r,s) counting multiplicity is the following.

$$\{\frac{1}{2},\frac{3}{2},...,r+s,\frac{1}{2},\frac{3}{2},...,r+s-1,\frac{1}{2},\frac{3}{2},...,r+s-2,....,\frac{1}{2},\frac{3}{2},...,r-s\}$$

Definition 8.2.3. Define $q_\nu^{\pm}:2\mathbb{N}+1\longrightarrow\mathbb{C}[\nu]$ as follows.

$$q_\nu^+(m):=\Pi_{l=0}^{\frac{m-1}{4}}\Pi_{j=0}^{l-1}(\nu+2j+\tfrac{1}{2})(\nu+2j+\tfrac{3}{2})\text{ if }4\mid m-1$$
$$q_\nu^+(m):=\Pi_{l=0}^{\frac{m-3}{4}}\Pi_{j=0}^{l-1}(\nu+2j+\tfrac{1}{2})(\nu+2j+\tfrac{3}{2})\times\ \Pi_{k=0}^{\frac{m-3}{4}}(\nu+2k+\tfrac{1}{2})\text{ if }4\mid m-3$$
$$q_\nu^-(m):=\Pi_{l=0}^{\frac{m-1}{4}}\Pi_{j=0}^{l-1}(\nu-2j-\tfrac{1}{2})(\nu-2j-\tfrac{3}{2})\text{ if }4\mid m-1$$
$$q_\nu^-(m):=\Pi_{l=0}^{\frac{m-3}{4}}\Pi_{j=0}^{l-1}(\nu-2j-\tfrac{1}{2})(\nu-2j-\tfrac{3}{2})\times\ \Pi_{k=0}^{\frac{m-3}{4}}(\nu-2k-\tfrac{1}{2})\text{ if }4\mid m-3$$

Definition 8.2.4. Define $G_\nu(m):2\mathbb{N}+1\longrightarrow M$ where M is the space of meromorphic functions, as follows.

$$\Gamma_\nu(m):=\Pi_{l=0}^{\frac{m-1}{4}}\Pi_{j=0}^{l-1}\frac{\Gamma(\nu-2j+\frac{1}{2})}{\Gamma(\nu-2j-\frac{3}{2})}\frac{\Gamma(\nu+2j+\frac{1}{2})}{\Gamma(\nu+2j+\frac{5}{2})}\text{ if }4\mid m-1$$
$$\Gamma_\nu(m):=\Pi_{l=0}^{\frac{m-3}{4}}\Pi_{j=0}^{l-1}\frac{\Gamma(\nu-2j+\frac{1}{2})}{\Gamma(\nu-2j-\frac{3}{2})}\frac{\Gamma(\nu+2j+\frac{1}{2})}{\Gamma(\nu+2j+\frac{5}{2})}\times\ \Pi_{k=0}^{\frac{m-3}{4}}\frac{\Gamma(\nu-2k+\frac{1}{2})}{\Gamma(\nu-2k-\frac{3}{2})}\text{ if }4\mid m-3$$

From the above analysis of the dominant $\mathfrak{t}_\alpha$-weights and the above functions we have the following. In the products below, indices of Π_{r-s}^{r+s} are $r-s,r-s+1,r-s+2,...,r+s$, all half integrals. Let V_τ be any of V_{τ_1} or V_{τ_2}, and let $\sigma=V_\tau|_{{}^0M}$.

$$p_\xi(\nu)=p_{(r,s)}=\Pi_{m=r-s}^{r+s}q_{\nu_1}^+(2m)q_{\nu_2}^+(2m)q_{\nu_3}^+(2m)q_{\nu_1+\nu_2}^+(2m)q_{\nu_2+\nu_3}^+(2m)q_{\nu_1+\nu_2+\nu_3}^+(2m)$$

$$detA(\nu)|_{(I_{P,\sigma,\nu}(\xi))}=detA(\nu)|_{(I_{P,\sigma,\nu}((r,s)))}$$
$$=\Pi_{m=r-s}^{r+s}\left(\frac{q_{\nu_1}^-(2m)}{q_{\nu_1}^+(2m)}\frac{q_{\nu_2}^-(2m)}{q_{\nu_2}^+(2m)}\frac{q_{\nu_3}^-(2m)}{q_{\nu_3}^+(2m)}\frac{q_{\nu_1+\nu_2}^-(2m)}{q_{\nu_1+\nu_2}^+(2m)}\frac{q_{\nu_2+\nu_3}^-(2m)}{q_{\nu_2+\nu_3}^+(2m)}\frac{q_{\nu_1+\nu2+\nu_3}^-(2m)}{q_{\nu_1+\nu_2+\nu_3}^+(2m)}\right)^{dim(V_\xi)}$$

In terms of Gamma Functions,

$$detA(\nu)|_{(I_{P,\sigma,\nu}(\xi))} = detA(\nu)|_{(I_{P,\sigma,\nu}((r,s)))}$$

$$= \Pi_{m=r-s}^{r+s}(\Gamma_{\nu_1}(2m)\Gamma_{\nu_2}(2m)\Gamma_{\nu_3}(2m)\Gamma_{\nu_1+\nu_2}(2m)\Gamma_{\nu_2+\nu_3}(2m)\Gamma_{\nu_1+\nu_2+\nu_3}(2m))^{dim(V_\xi)}$$

If $r < s$, the only difference is that the parameters for the products will start from $s - r$ instead of $r - s$. The formula of $dim(V_\xi)$ is given in the next subsection.

8.2.5 General case of $\widetilde{SL(n,\mathbb{R})}$

In this subsection, we give a formula of $p_\xi(\nu)$ and the determinant of $A(\nu)$ the intertwiner for $\widetilde{SL(n,\mathbb{R})}$. First, consider the following lemma.

Lemma 8.2.6. *Let Φ^+ be the set of positive roots of $Lie(\widetilde{SL(n,\mathbb{R})})$. If α_1, $\alpha_2 \in \Phi^+$, then $\mathfrak{t}_{\alpha_1}$ and $\mathfrak{t}_{\alpha_2}$ are conjugates of each other by an element in $Spin(n)$.*

Proof. $Lie(\widetilde{SL(n,\mathbb{R})})$ is simply laced. Therefore, by Proposition 6.11 [Bou], all positive roots are conjugates by an element of the Weyl group, $N_K(A)/Z_K(A)$. Therefore, $\mathfrak{t}_{\alpha_1}$ and $\mathfrak{t}_{\alpha_2}$ are conjugates of each other by an element in $Spin(n)$. □

By the lemma, given an irreducible representation V_ξ that occurs in $I_{P,\sigma,\nu}$, the set of dominant $\mathfrak{t}_\alpha$-weights of V_ξ counting multiplicity is independent of $\alpha \in \Phi^+$. Hence, we just need the set of dominant $\mathfrak{t}_\alpha$-weights of V_ξ counting multiplicity for some α. We choose $\alpha = \epsilon_1 - \epsilon_2$.

Recall that given $V_\xi = \bigoplus_{j=1}^{dim(V_\xi)/dim(V_\tau)} V_{\tau_j}$ with each V_{τ_j} an irreducible K_α-module, there is a unique dominant $\mathfrak{t}_\alpha$-weight on each V_{τ_j}. This fact along with the formula in rank one case given in section 8.1 allows us to compute the factors of $p_\xi(\nu)$ coming from $\alpha = \epsilon_1 - \epsilon_2$ by branching V_ξ down to $\widetilde{SO(2)} \subseteq Spin(n)$ where $\widetilde{SO(2)}$ denotes double cover of $SO(2)$ and $SO(2)$ is that occurring in the top left corner such that $\mathfrak{t}_{\epsilon_1-\epsilon_2} \subseteq Lie(\widetilde{SO(2)})$. However, we will branch down to $Spin(3)$ that occurs in the top left corner instead of going a step further down to $\widetilde{SO(2)}$ to simplify notations.

Given an irreducible $Spin(n)$-module $V_\xi \subseteq I_{P,\sigma,\nu}$ with highest weight $\xi = \xi_1\epsilon_1 + ... + \xi_k\epsilon_k$, branch down to $Spin(3)$ where $Spin(3)$ is as in above. Let $\{\frac{j_1}{2}, ..., \frac{j_{m_\xi}}{2}\}$ be the set of highest weights of $Spin(3)$-modules that occur in the branching counting multiplicity. We have

$$p_\xi(\nu) = \Pi_{\alpha\in\Phi^+}\Pi_{k=1}^{m_\xi} q^+_{(\nu,\alpha)}(j_k)$$

$$detA(\nu)|_{I_{P,\tau,\nu}(\xi)} = (\frac{p_\xi(-\nu)}{p_\xi(\nu)})^{dim(V_\xi)} = ((\Pi_{\alpha\in\Phi^+}\Pi_{k=1}^{m_\xi}\Gamma_{(\nu,\alpha)}(j_k))^{\frac{2}{dim(V_\tau)}})^{dim(V_\xi)}$$

where if $n = 2k+1$, $dim(V_\xi) = \Pi_{1\le i<j\le k}\frac{(\xi_i+\rho_i)^2-(\xi_j+\rho_j)^2}{\rho_i^2-\rho_j^2}\Pi_{1\le i\le k}\frac{\xi_i+\rho_i}{\rho_i}$ with $\rho_i = k-i+\frac{1}{2}$, $dim(V_\tau) = 2^k$, and if $n = 2k$, $dim(V_\xi) = \Pi_{1\le i<j\le k}\frac{(\xi_i+\rho_i)^2-(\xi_j+\rho_j)^2}{\rho_i^2-\rho_j^2}$ with $\rho_i = k-i$, $dim(V_\tau) = 2^{k-1}$.

8.3 Application: Cyclicity of V_τ and Irreducibility of Unitary Principal Series

Recall the computation of $p_\xi(\nu)$ in rank one case, now with ρ-shift. If $\mathfrak{t}$ acts nontrivially on V_τ, we have the following.

For the ξ-type $\overline{Z}^l \otimes V_\tau^+ \oplus Z^l \otimes V_\tau^-$,

$$p_\xi(\nu) = \Pi_{j=0}^{l-1}(\nu + 2j + \frac{3}{2})$$

and for the ξ-type $\overline{Z}^l \otimes V_\tau^- \oplus Z^l \otimes V_\tau^+$,

$$p_\xi(\nu) = \Pi_{j=0}^{l-1}(\nu + 2j + \frac{1}{2})$$

If $\mathfrak{t}$ acts trivially on V_τ, for any of the ξ-type that occurs in $(\overline{Z}^l \otimes V_\tau) \oplus (Z^l \otimes V_\tau)$,

$$p_\xi(\nu) = \Pi_{j=0}^{l-1}(\nu + 2j + 1)$$

ρ-shift simplifies determinant formula of P^ξ for the following reason. Let α be a simple root, and β be a non-simple root of $\mathfrak{sl}_n$, τ_α and τ_β be K_α and K_β types respectively such that τ_α and τ_β are the same types. In general, $p_{\delta_\alpha}(\nu) \neq p_{\delta_\beta}(\nu)$

without the ρ-shift. However, with ρ-shift, we have $p_{\delta_\alpha}(\nu) = p_{\delta_\beta}(\nu)$, which is built into the proof of Theorem 6.2.8. Consider the following example of the case $\widetilde{SL(3,\mathbb{R})}$. Let ξ be the 4-dimensional $K = Spin(3)$-type. We have $p_\xi(\nu) = (\nu_1 - \frac{1}{2})(\nu_2 - \frac{1}{2})(\nu_1 + \nu_2 - \frac{3}{2})$ before ρ-shift and $p_\xi(\nu) = (\nu_1 + \frac{1}{2})(\nu_2 + \frac{1}{2})(\nu_1 + \nu_2 + \frac{1}{2})$ after ρ-shift.

The following discussion is from 11.3.6 of [RRG II].

Let G be a real semisimple Lie group with maximal compact subgroup K. Let V_τ be a small K-type and let $\sigma = V_\tau|_{{}^0M}$. From chapter 10 of [RRG II], we know there exists $c \geq 0$ such that if $Re(\nu, \alpha) \geq c$ for all $\alpha \in \Phi^+$, then det $J_{\overline{P}|P}(\nu)|_{I_{P,\sigma,\nu}(\tau)} \neq 0$. Hence we have $\pi_{P,\sigma,\nu}(U(\mathfrak{g}))I_\sigma(\tau) = I_{P,\sigma,\nu}$ for these νs. This induces a surjective $(\mathfrak{g}, K)$-module homomorphism

$$\mu_{\tau,\nu} : U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^{\mathfrak{k}}U(\mathfrak{k})} V_{\tau,\nu} \longrightarrow I_{P,\sigma,\nu}$$

where the map is the action of the first factor on the second, which gives cyclicity of V_τ for above νs.

Moreover, it is also shown in 11.3.6 of [RRG II] that $U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^{\mathfrak{k}}U(\mathfrak{k})} V_{\tau,\nu} \cong I_{P,\sigma,\nu}$ as K-modules independent of ν. This result implies that the K-isotypic components of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^{\mathfrak{k}}U(\mathfrak{k})} V_{\tau,\nu}$ and $I_{P,\sigma,\nu}$ are exactly the same. Hence if V_τ is cyclic in $I_{P,\sigma,\nu}$, then $\mu_{\tau,\nu}$ is a $(\mathfrak{g}, K)$-module isomorphism.

Theorem 8.3.1. *Let G be any of the connected, simply connected split real form of simple Lie type other than type C_n with maximal compact subgroup K. Let V_τ be a small K-type and let $\sigma = V_\tau|_{{}^0M}$. If $Re(\nu, \alpha) \geq 0$ for every $\alpha \in \Phi^+$, i.e. in the closed Langlands chamber, $V_\tau \subseteq I_{P,\sigma,\nu}$ is cyclic.*

Proof. By the definition of $P^\xi(\nu)$ and above discussion, V_τ is cyclic if and only if $p_\xi(\nu) \neq 0$ for every K-type ξ that occurs in $I_{P,\sigma,\nu}$. By Theorem 6.3.1 and Theorem 7.4.1, $p_\xi(\nu)$ is a product of those of rank one subgroups G_α of G where $\alpha \in \Phi^+$. As G is split, the semisimple part of $Lie(G_\alpha)$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Let $n(\xi) = dim\ Hom_K(V_\tau, V_\xi)$. The product formula of p_ξ and the formulas in the rank one case given in the beginning of the section suggest that for a K-type ξ that occurs in $I_{P,\sigma,\nu}$, $\alpha \in \Phi^+$, and $j = 1, ..., n(\xi)$, there exist $l^\xi_{\alpha,j}$, $m^\xi_{\alpha,j}$, $n^\xi_{\alpha,j} \in \mathbb{Z}_{\geq 0}$

such that $p_\xi(\nu)$ is equal to

$$\Pi_{\alpha\in\Phi^+}\Pi_{j=1}^{n(\xi)}\Pi_{p=0}^{l^\xi_{\alpha,j}-1}\Pi_{q=0}^{m^\xi_{\alpha,j}-1}\Pi_{r=0}^{n^\xi_{\alpha,j}-1}(\frac{2(\nu,\alpha)}{(\alpha,\alpha)}+2p+\frac{1}{2})(\frac{2(\nu,\alpha)}{(\alpha,\alpha)}+2q+1)(\frac{2(\nu,\alpha)}{(\alpha,\alpha)}+2r+\frac{3}{2})$$

up to a nonzero scalar multiple. Hence, if $Re(\nu,\alpha)\geq 0$ for every $\alpha\in\Phi^+$, $p_\xi(\nu)\neq 0$ for every K-type ξ that occurs in $I_{P,\sigma,\nu}$ and $V_\tau\subseteq I_{P,\sigma,\nu}$ is cyclic. □

Corollary 8.3.2. *Let G be any of the connected, simply connected split real form of simple Lie type other than type C_n with maximal compact subgroup K. Let V_τ be a small K-type and let $\sigma = V_\tau|_{{}^0M}$. The unitary principal series $(\pi_{P,\sigma,\nu}, H^{P,\sigma,\nu})$ $(Re\ \nu = 0)$ is irreducible.*

Proof. Suppose $H^{P,\sigma,\nu}$ with $Re\ \nu = 0$ is reducible. By Theorem 3.4.11 of [RRG I], the underlying $(\mathfrak{g},K)$-module $I_{P,\sigma,\nu}$ is reducible. Therefore, there is a proper, nontrivial, closed $(\mathfrak{g},K)$-invariant subspace W of $I_{P,\sigma,\nu}$ that does not contain V_τ. Unitarity implies that the orthogonal complement of W, $W^\perp$, is a nontrivial, closed $(\mathfrak{g},K)$-invariant subspace that contains V_τ, which is a contradiction as V_τ is cyclic by Theorem 8.3.1.

□

Bibliography

[Bou] Bourbaki, N. *Lie Groups and Lie Algebras Chapters 4-6,* Springer, 1968.

[GW] Goodman, R. and Wallach, N., *Symmetry, Representations, and Invariants*, Graduate Texts in Mathematics, Springer, 2009.

[HAHS] Wallach, N. R., *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, New York, 1973.

[JW] Johnson, K. and Wallach, N., *Composition Series and Intertwining Operators for the Spherical Principal Series I*, Transactions of the American Mathematical Society, Vol. 229, (May, 1977), pp.137-173.

[Kos] Kostant, B., *On the existence and irreducibility of certain series of representations*, Lie Groups and their Representations ed. by I. M. Gelfand, Halsted, New York, 1975, pp.231-329.

[KR] Kostant, B. and Rallis, S., *Orbits and Lie group representations associated to symmetric spaces*, Amer. J. Math. 93 (1971), pp. 753-809.

[Ku] Kumar, S., *Tensor Product Decomposition, Proceedings of the International Congress of Mathematicians*, Hyderabad, India, 2010.

[RRG I] Wallach, N. R., *Real Reductive Groups I*, Academic Press, San Diego, 1988.

[RRG II] Wallach, N. R., *Real Reductive Groups II*, Academic Press, San Diego, 1992.

[VW] Vogan, D. and Wallach, N. R., *Intertwining Operators for Real Reductive Groups*, Advances in Mathematics 82 (1990), pp.203-243.