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ABSTRACT OF THE DISSERTATION

On Kähler manifolds with certain curvature bounds

By

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Doctor of Philosophy in Mathematics

University of California, Irvine, 2019

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This dissertation discusses the Frankel conjecture and the Kähler-Ricci flow approach to it. Frankel conjecture (first proved by Mori and Siu-Yau independently) states that every compact Kähler manifold of positive bisectional curvature is biholomorphic to the complex projective space. On the other hand, the Ricci flow introduced by Hamilton was used by Bando and Mok to generalize Siu-Yau's theorem to nonnegative bisectional curvature case. It's natural to ask if there is a primarily flow proof of the original Frankel conjecture.

The convergence of Kähler-Ricci flow on compact manifolds with positive bisectional curvature would imply such a proof, however not yet been completed. The advances closest to this target might be a series of papers by Phong-Song-Sturm-Weinkove along with the improvements by Cao-Zhu and Zhang. We are going to survey their works in this thesis, and also cover some new result proved by the author.

This thesis is organized as follows. In chapter one, we first collect some fundamental facts of Kähler geometry, and then go over the convergence theory of Kähler-Ricci flow on Fano manifolds built on stability conditions. In chapter two, we review known results on bisectional curvature, and then relate the curvature to the former stability conditions. Finally we will state and prove our new result.

Introduction

Frankel conjecture, which states that every compact Kähler manifold of positive bisectional curvature is biholomorphic to the complex projective space, was first made by T. Frankel [15] in 1961. The dimension two case was settled by Andreotti-Frankel [15] and dimension three case by T. Mabuchi [25]. In 1980, Siu-Yau [34] solved Frankel conjecture in full generality, by using harmonic map from complex projective line into the manifold, and characterization of the complex projective space obtained earlier by Kobayashi-Ochiai [24]. An independent proof was given by S. Mori [27] in 1979, where he used algebraic method to prove the Hartshorne conjecture, which includes Frankel conjecture as its special case.

On the other hand, in 1982, R. Hamilton [20] introduced the Ricci flow as a new powerful tool in differential geometry. Its version on Kähler manifolds, the Kähler-Ricci flow was used right afterwards by S. Bando [2] to generalize Siu-Yau's theorem to nonnegative bisectional curvature case, in dimension three. The n-dimensional classification was done by N. Mok [26] combining flow method and Mori's algebraic method. Later H. Gu [19] simplified Mok's arguments. Since all their proofs rely on Kähler-Ricci flow, it's natural to ask if there is a primarily flow proof of the original Frankel conjecture.

The first attempt along this direction is the work of Chen-Tian [11][12]. With first assuming the Frankel conjecture, they proved that the Kähler-Ricci flow starting from any initial metric with positive bisectional curvature, would converge to the Fubini-Study metric. Later, G.

Perelman showed the convergence of the the flow with only assuming the existence of a Kähler-Einstein metric (detailed proof can be found in [5], chapter 6). In 2009, Chen-Sun-Tian [10] obtained a proof of Frankel conjecture by using Kähler-Ricci flow and soliton. They used induction on dimension, part of Siu-Yau's idea and some Morse theory. After a couple of years, He-Sun [22] gave another independent proof through so-called Sasaki-Ricci flow.

However, up to now, the convergence of Kähler-Ricci flow on compact manifolds with positive bisectional curvature, has not yet been completely proved, without a priori assuming the Frankel conjecture. The advances closest to this target might be a series of papers by Phong, Sturm, Song and Weinkove [31][29][30], along with the improvements by Cao-Zhu [8] and Z. Zhang [36]. We are going to survey their works in this note.

This note is organized as follows. In chapter one, we first collect some fundamental facts of Kähler geometry, and then go over the convergence theory of Kähler-Ricci flow on Fano manifolds built on stability conditions, due to [31][29] and [36]. In chapter two, we review Mok's result on bisectional curvature and the improvements by [10][8], and then relate the curvature to the former stability conditions as in [30]. Some new result by the author is also included. We try to make the note as self-contained as much, with assuming that the reader is familiar with Kähler-Ricci flow at the level of standard textbooks (such as [5], chapter 3 and 5).

Chapter 1

Kähler-Ricci Flow on Fano Manifolds

In this chapter, we first collect the basic definitions and results in Kähler geometry which will be used later, and then cover the convergence theory of Kähler-Ricci flow developed by Phong-Song-Sturm-Weinkove [29]. They reduced the convergence to some stability conditions, but we will adapt Z. Zhang's method [36] on some steps, since his approach seems more simple and transparent.

.1 Preliminaries

.1.1 Kähler manifolds

Let (X^n, g) be a compact complex manifold of complex dimension n with the Hermitian metric g. In local holomorphic coordinates $(z^1, \cdots, z^n)$, denote its Kähler form by

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j > 0.$$

By definition, g is Kähler means that its Kähler form ω is a closed real $(1,1)$-form, or equivalently,

$$\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}} \quad \text{and} \quad \partial_{\bar{k}} g_{i\bar{j}} = \partial_{\bar{j}} g_{i\bar{k}}$$

for all $i, j, k = 1, \cdots n$.

The cohomology class $[\omega]$ represented by ω in $H^2(X, \mathbb{R})$ is called the Kähler class of metric $g_{i\bar{j}}$. By the Hodge theory, two Kähler metrics $g_{i\bar{j}}$ and $\tilde{g}_{i\bar{j}}$ belong to the same Kähler class if and only if $g_{i\bar{j}} = \tilde{g}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi$, or equivalently,

$$\omega = \tilde{\omega} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \varphi$$

for some real-valued smooth function φ on X.

The volume of (X, g) is written as

$$\mathrm{Vol}\,(X, g) = \int_X \frac{\omega^n}{n!} = \int_X \det(g) \wedge_{k=1}^n (\frac{\sqrt{-1}}{2} dz^k \wedge d\bar{z}^k).$$

Clearly, by Stokes' theorem, $\mathrm{Vol}\,(X, g) = \mathrm{Vol}\,(X, \tilde{g})$ if g and $\tilde{g}$ are in the same Kähler class. We will just use V to denote $\mathrm{Vol}\,(X, g)$.

The Christoffel symbols of the metric $g_{i\bar{j}}$ are given by

$$\Gamma^k_{ij} = g^{\bar{\ell} k} \partial_i g_{j\bar{\ell}} \quad \text{and} \quad \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = g^{\bar{k}\ell} \partial_{\bar{i}} g_{\ell\bar{j}}$$

where $(g^{\bar{j}i}) = (g_{i\bar{j}})^{-1}$. Given any $T^{1,0}$-tensor v_i, its covariant derivatives are defined as

$$\nabla_j v_i = \partial_j v_i - \Gamma^k_{ij} v_k \quad \text{and} \quad \nabla_{\bar{j}} v_i = \partial_{\bar{j}} v_i.$$

The covariant derivatives of $T^{0,1}$-tensor are just defined as the conjugate of the above.

The curvature tensor of the metric $g_{i\bar{j}}$ is given by $R_{i\ k\bar{\ell}}^{\ j} = -\partial_{\bar{\ell}}\Gamma_{ik}^{j}$, or by lowering j to the second index:

$$R_{i\bar{j}k\bar{\ell}} = g_{p\bar{j}}R_{i\ k\bar{\ell}}^{\ p} = -\partial_k\partial_{\bar{\ell}}g_{i\bar{j}} + g^{\bar{q}p}\partial_k g_{i\bar{q}}\partial_{\bar{\ell}}g_{p\bar{j}}.$$

From the Kähler condition, it's not hard to see

$$R_{i\bar{j}k\bar{\ell}} = R_{k\bar{j}i\bar{\ell}} \quad \text{and} \quad R_{i\bar{j}k\bar{\ell}} = R_{i\bar{\ell}k\bar{j}}, \qquad \text{(1st Bianchi identity)}$$

$$\nabla_p R_{i\bar{j}k\bar{\ell}} = \nabla_k R_{i\bar{j}p\bar{\ell}} \quad \text{and} \quad \nabla_{\bar{q}} R_{i\bar{j}k\bar{\ell}} = \nabla_{\bar{\ell}} R_{i\bar{j}k\bar{q}}. \qquad \text{(2nd Bianchi identity)}$$

The commutation rules of covariant differentiations are as follows:

$$[\nabla_k, \nabla_j]v_i = 0, \qquad [\nabla_{\bar{k}}, \nabla_{\bar{j}}]v_i = 0,$$

$$[\nabla_k, \nabla_{\bar{j}}]v_i = -R_{k\bar{j}i\bar{\ell}}v^{\bar{\ell}}, \qquad [\nabla_k, \nabla_{\bar{j}}]w_{\bar{\ell}} = R_{k\bar{j}i\bar{\ell}}w^i.$$

We say that (X, g) has positive (holomorphic) bisectional curvature, or positive holomorphic sectional curvature at a point $x \in X$, if

$$R_{i\bar{j}k\bar{\ell}}v^i v^{\bar{j}} w^k w^{\bar{\ell}} > 0, \quad \text{or} \quad R_{i\bar{j}k\bar{\ell}}v^i v^{\bar{j}} v^k v^{\bar{\ell}} > 0$$

respectively, for all nonzero vectors v and w in the holomorphic tangent space $T_x^{1,0}X$.

The Ricci tensor of the metric $g_{i\bar{j}}$ is obtained by taking the trace of $R_{i\bar{j}k\bar{\ell}}$:

$$R_{i\bar{j}} = g^{\bar{\ell}k}R_{i\bar{j}k\bar{\ell}} = -\partial_i\partial_{\bar{j}}\log\det(g).$$

It is clear that the Ricci form

$$\text{Ric} = \frac{\sqrt{-1}}{2} \sum_{i,j} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

is real and closed. It is well known that the first Chern class $c_1(X) \in H^2(X, \mathbb{Z})$ of X is represented by the Ricci form:

$$[\text{Ric}\,] = \pi c_1(X).$$

A compact Kähler manifold is called Fano if its first Chern class is positive, i.e., contains a positive representative.

Finally, the scalar curvature of the metric $g_{i\bar{j}}$ is

$$R = g^{\bar{j}i} R_{i\bar{j}}.$$

Hence, the total scalar curvature

$$\int_X R \frac{\omega^n}{n!} = \int_X \text{Ric} \wedge \frac{\omega^{n-1}}{(n-1)!}$$

depends only on the Kähler class of ω and the first Chern class $c_1(X)$.

From now on, by abusing the notations, we will just write ω^n to denote the volume form $\omega^n/n!$.

.1.2 Kähler-Einstein metric and Futaki invariant

A Kähler metric $g_{i\bar{j}}$ is called Kähler-Einstein if

$$R_{i\bar{j}} = \lambda g_{i\bar{j}}$$

for some real number $\lambda \in \mathbb{R}$. A classical example is the complex projective space $\mathbb{CP}^n$ with Fubini-Study metric g_{FS}:

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log\left(1 + \sum_{k=1}^{n} |z_k|^2\right)$$

which satisfies

$$R_{i\bar{j}} = (n+1) g_{i\bar{j}}.$$

Clearly, if X admits a Kähler-Einstein metric g, then the first Chern class is necessarily definite, as

$$\pi c_1(X) = \lambda [\omega_g].$$

When $c_1(X) = 0$ it follows from S.-T. Yau's solution [35] to the Calabi conjecture that, in each Kähler class there exists a unique Calabi-Yau metric (i.e. Ricci-flat metric). Moreover, when $c_1(X) < 0$, T. Aubin [1] and Yau [35] independently proved the existence of a unique Kähler-Einstein metric in the class $-\pi c_1(X)$.

However, in the Fano case (i.e. $c_1(X) > 0$) Kähler-Einstein metric does not always exist. Among many other ones, in 1983 A. Futaki [16] introduced his famous obstruction, which now is called Futaki invariant, defined as follows:

Choose any Kähler metric g with $[\omega_g] = \pi c_1(X)$. Then its Kähler form and its Ricci form lie in the same cohomology class. Hence, by the Hodge theory, there exists a real-valued smooth function u, called the Ricci potential of metric g, such that

$$R_{i\bar{j}} + \partial_i \partial_{\bar{j}} u = g_{i\bar{j}}.$$

Let $\eta(X)$ denote the space of holomorphic vector fields on X, and W be any element in $\eta(X)$. Then the functional $F\colon \eta(X) \to \mathbb{C}$ defined by

$$F(W) = \int_X W(u) \omega^n = \int_X (W \cdot \nabla u) \omega^n$$

is called the Futaki invariant.

In [16], Futaki proved that $F(W)$ depends only on the class $\pi c_1(X)$, but not the special choice of metric g. Obviously, if a Fano manifold X admits a positive Kähler-Einstein metric, then the Ricci potential u must be constant, and the Futaki invariant F vanishes. F has strong relation to the notion called 'geometric stability', which plays a central role in the existence problem of Kähler-Einstein metric on Fano manifolds, in the general case.

.1.3 (Normalized) Kähler-Ricci flow

Now assume that we have a compact Fano Kähler manifold (X^n, g_0) such that $[\omega_{g_0}] = \pi c_1(X)$. The normalized Kähler-Ricci flow is

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} + g_{i\bar{j}}, \quad g(0) = g_0 \tag{.1.1}$$

or equivalently

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}\,(\omega) + \omega, \quad \omega(0) = \omega_0 (= \omega_{g_0}).$$

From the second equation, the evolution of the Kähler class shows

$$\frac{\partial}{\partial t}[\omega] = -\pi c_1(X) + [\omega] \quad \Rightarrow \quad [\omega] \equiv \pi c_1(X).$$

So the normalized Kähler-Ricci flow preserves the Kähler class, and hence the total volume of the manifold. H. Cao [6] proved that the solution to this flow exists for all $t > 0$.

With the metric evolving along the time, the Ricci potential u defined by

$$R_{i\bar{j}} + \partial_i \partial_{\bar{j}} u = g_{i\bar{j}} \tag{.1.2}$$

also evolves along the time. If we normalize u by constraint

$$\frac{1}{V}\int_X e^{-u}\omega^n = 1, \tag{.1.3}$$

then u is well-known to satisfy

$$\frac{\partial}{\partial t}u = \Delta u + u - a, \quad \text{where} \quad a := \frac{1}{V}\int_X ue^{-u}\omega^n. \tag{.1.4}$$

Since the solution ω_t to (.1.1) always lies in the same cohomology class as ω_0, there exists smooth real-valued function $\phi(t)$ on $X \times [0, +\infty)$ such that $\omega_t = \omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi$. The evolution equation of ϕ is

$$\frac{\partial}{\partial t}\phi = \log\frac{\omega_t^n}{\omega_0^n} + \phi + u(0), \quad \phi(0) = \phi_0 \tag{.1.5}$$

which is equivalent to (.1.1). Compare (.1.1) and (.1.2), we can tell that u and $\partial_t\phi$ are identical up to at most a time-dependent constant.

By straightforward computations, we can find that the evolutions of the volume form and curvatures are as follows:

$$\frac{\partial}{\partial t}\omega^n = (n - R)\omega^n; \tag{.1.6}$$

$$\frac{\partial}{\partial t}R = \Delta R + R_{i\bar{j}}R^{\bar{j}i} - R; \tag{.1.7}$$

$$\frac{\partial}{\partial t} R_{i\bar{j}} = \Delta R_{i\bar{j}} + R_{i\bar{j}a\bar{b}} R^{\bar{b}a} - R_{i\bar{k}} R^{\bar{k}}{}_{\bar{j}}; \tag{.1.8}$$

$$\begin{aligned} \frac{\partial}{\partial t} R_{i\bar{j}k\bar{\ell}} =& \Delta R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}a\bar{b}} R^{\bar{b}a}{}_{k\bar{\ell}} + R_{i\bar{b}a\bar{\ell}} R^{\bar{b}}{}_{\bar{j}k}{}^{a} - R_{i\bar{a}k\bar{b}} R^{\bar{a}}{}_{\bar{j}}{}^{\bar{b}}{}_{\bar{\ell}} \\ & - \frac{1}{2} \left(R_i{}^a R_{a\bar{j}k\bar{\ell}} + R^{\bar{a}}{}_{\bar{j}} R_{i\bar{a}k\bar{\ell}} + R_k{}^a R_{i\bar{j}a\bar{\ell}} + R^{\bar{a}}{}_{\bar{\ell}} R_{i\bar{j}k\bar{a}} \right). \end{aligned} \tag{.1.9}$$

Note here the Laplacian operator Δ is defined as $\frac{1}{2} g^{\bar{j}i} (\nabla_i \nabla_{\bar{j}} + \nabla_{\bar{j}} \nabla_i)$.

One of the deepest results in the theory of Kähler-Ricci flow is the following estimate proved by G. Perelman for a solution of (.1.1) (see [32] or chapter 5 in [5] for a detailed exposition). The first part is bound for the Ricci potential $u = u(t)$ defined by (.1.2) and (.1.3) and the second part is a non-collapsing theorem:

(i) There exists a constant C depending only on $g_{i\bar{j}}(0)$ such that

$$\|u\|_{C^0} + \|\nabla u\|_{C^0} + \|R\|_{C^0} \leq C. \tag{.1.10}$$

Note that taking trace of (.1.2) would yield $\Delta u = n - R$.

(ii) Let $\rho > 0$ be given. Then there exists $c > 0$ depending only on $g_{i\bar{j}}(0)$ and ρ such that for all points $x \in X$, all times $t \geq 0$ and all r with $0 < r \leq \rho$, we have

$$\int_{B_r(x)} \omega^n > c\, r^{2n}, \tag{.1.11}$$

where $B_r(x)$ is the geodesic ball of radius r centered at x with respect to the metric

$g = g(t)$.

If the solution $g_{i\bar{j}}(t)$ of the normalized Kähler-Ricci flow (.1.1) converges smoothly to a limit metric g_∞, then we must have $\lim_{t\to\infty} \frac{\partial}{\partial t}\phi = 0$. Equation (.1.5) tells that the limit metric satisfies

$$0 = \log \frac{\omega_\infty^n}{\omega_0^n} + \phi_\infty + u(0); \tag{.1.12}$$

making $\partial\bar{\partial}$ acts on both sides of (.1.12), we get

$$\begin{aligned} 0 &= -\mathrm{Ric}\,(\omega_\infty) + \mathrm{Ric}\,(\omega_0) + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi_\infty + \frac{\sqrt{-1}}{2}\partial\bar{\partial}u(0) \\ &= -\mathrm{Ric}\,(\omega_\infty) + \mathrm{Ric}\,(\omega_0) + \omega_\infty - \omega_0 + \omega_0 - \mathrm{Ric}\,(\omega_0) \\ &= -\mathrm{Ric}\,(\omega_\infty) + \omega_\infty, \end{aligned}$$

which shows that the limit metric must be Kähler-Einstein:

$$\mathrm{Ric}\,(\omega_\infty) = \omega_\infty. \tag{.1.13}$$

From now on, we will write NKRF as brief notation of normalized Kähler-Ricci flow, and K-E metric as brief notation of Kähler-Einstein metric. For convenience, we will also use repeated indices to represent summations, as well as the upper-lower index summations.

.2 Convergence of the Flow

.2.1 The C^0-estimate

In order to get the convergence of NKRF on compact Kähler manifold $(X^n, g(t))$, the standard step is to derive the so-called C^0-estimates, i.e. uniform C^0 bounds of the Kähler potential ϕ and its time derivative $\partial_t \phi$; then all higher order estimates can be obtained for free and so is the C^∞-convergence of the flow ([35], [6], and [5], chapter 3 and chapter 6).

The boundness of $\|\partial_t \phi\|_{C^0}$ is already included in Perelman's result (.1.10), so we just need to figure out $\|\phi\|_{C^0}$. The following proposition, which observes that that the integrability of $\|R - n\|_{C^0}$ over $t \in [0, \infty)$ implies a uniform bound for $\|\phi\|_{C^0}$, is due to Phong-Song-Sturm-Weinkove [29]:

Proposition 1.1. *Let $(X^n, g(t))$ be a compact Fano Kähler manifold with $g(t)$ as the solution to NKRF. Assume that the scalar curvature $R(t)$ along the flow satisfies*

$$\int_0^\infty \|R(t) - n\|_{C^0} \, dt \;<\; \infty, \tag{.2.1}$$

then $\|\phi\|_{C^0}$ is uniformly bounded along the NKRF.

Proof of Proposition 1.1. Let's recall the flow equation (.1.5)

$$\frac{\partial}{\partial t}\phi = \log \frac{\omega_t^n}{\omega_0^n} + \phi + u(0), \qquad \phi(0) = \phi_0; \tag{.2.2}$$

remember that Perelman's estimate for u implies $\|\partial_t \phi\|_{C^0} \leq C$.

Take derivative

$$\frac{d}{dt}\,(\log\frac{\omega_t^n}{\omega_0^n}) = g^{\bar{j}i}\partial_t g_{i\bar{j}} = -(R-n),$$

thus for any $t \in (0,\infty)$,

$$\left|\log\frac{\omega_t^n}{\omega_0^n}\right| = \left|\int_0^t (R-n)\,dt\right| \le \int_0^\infty \|R-n\|_{C^0} dt < \infty.$$

On the other hand, (.2.2) can be rewritten as

$$\phi = -\log\frac{\omega_t^n}{\omega_0^n} + \partial_t\phi - u(0),$$

then the uniform bound for $\|\phi\|_{C^0}$ follows from the uniform bound for $|\log(\omega_t^n/\omega_0^n)|$ and Perelman's uniform estimate for $\|\partial_t\phi\|_{C^0}$. □

From Proposition 1.1, we easily see that the exponential decay of $\|R(t)-n\|_{C^0}$ to 0 would give us the C^0-estimate, hence the convergence of the NKRF.

.2.2 A smoothing lemma

The following smoothing lemma is due to Bando [3] and refined in [29], which shows that the C^0-norm of u could control the C^0-norms of ∇u and Δu at later time:

Lemma 1.2. *There exist positive constants δ and K depending only on n with the following property. For any ε with $0 < \varepsilon \le \delta$ and any $t_0 \ge 0$, if*

$$\|u(t_0)\|_{C^0} \le \varepsilon,$$

then

$$\|\nabla u(t_0+2)\|_{C^0} + \|R(t_0+2) - n\|_{C^0} \leq K\varepsilon.$$

The proof consists of some delicate maximum-principle arguments.

Proof of Lemma 1.2. Without loss of generality, we can assume $t_0 = 0$ by making a translation in time. Recall the evolution equation (.1.4) of u:

$$\frac{\partial}{\partial t}u = \Delta u + u - a, \quad \text{where} \quad a = \frac{1}{V}\int_X ue^{-u}\omega^n.$$

It is convenient to define a new constant $c = c(t)$ for $t \geq 0$ by $\dot{c} = a + c$, $c(0) = 0$; then set $\hat{u}(t) = -u(t) - c(t)$. We have $\|\hat{u}(0)\|_{C^0} = \|u(0)\|_{C^0} \leq \varepsilon$ and $\hat{u}$ evolves by

$$\frac{\partial}{\partial t}\hat{u} = \Delta\hat{u} + \hat{u}.$$

Following [3], we calculate

$$\frac{\partial}{\partial t}\hat{u}^2 = \Delta\hat{u}^2 - 2|\nabla\hat{u}|^2 + 2\hat{u}^2, \tag{.2.3}$$

$$\frac{\partial}{\partial t}|\nabla\hat{u}|^2 = \Delta|\nabla\hat{u}|^2 - |\nabla\overline{\nabla}\hat{u}|^2 - |\nabla\nabla\hat{u}|^2 + |\nabla\hat{u}|^2, \tag{.2.4}$$

$$\frac{\partial}{\partial t}\Delta\hat{u} = \Delta(\Delta\hat{u}) + \Delta\hat{u} + |\nabla\overline{\nabla}\hat{u}|^2. \tag{.2.5}$$

From (.2.3) we have

$$\frac{\partial}{\partial t}\left(e^{-2t}(\hat{u}^2)\right) \leq \Delta\left(e^{-2t}(\hat{u}^2)\right), \tag{.2.6}$$

which gives $\|\hat{u}(t)\|_{C^0} \leq e^2\varepsilon$ for $t \in [0,2]$.

From (.2.4) we have

$$\frac{\partial}{\partial t}\left(e^{-2t}(\hat{u}^2 + t|\nabla\hat{u}|^2)\right) \leq \Delta\left(e^{-2t}(\hat{u}^2 + t|\nabla\hat{u}|^2)\right) \quad \text{when } t \geq 1, \tag{.2.7}$$

giving $\|\nabla\hat{u}\|_{C^0}(t) \leq e^2\varepsilon$ for $t \in [1,2]$.

We shall now prove a lower bound for $\Delta\hat{u}$. Set

$$H = e^{-(t-1)}(|\nabla\hat{u}|^2 - \varepsilon n^{-1}(t-1)\Delta\hat{u})$$

and compute using (.2.4) and (.2.5),

$$\frac{\partial}{\partial t}H = \Delta H - e^{-(t-1)}\left(\varepsilon n^{-1}\Delta\hat{u} + (1+\varepsilon n^{-1}(t-1))|\nabla\overline{\nabla}\hat{u}|^2 + |\nabla\nabla\hat{u}|^2\right).$$

For $t \in [1,2]$, using the inequality $(\Delta\hat{u})^2 \leq n|\nabla\overline{\nabla}\hat{u}|^2$ we obtain

$$\frac{\partial}{\partial t}H \quad \leq \quad \Delta H + e^{-(t-1)}n^{-1}(-\Delta\hat{u})(\varepsilon + \Delta\hat{u}). \tag{.2.8}$$

We claim that $H < 2e^4\varepsilon^2$ for $t \in [1,2]$. Otherwise, at the point $(x', t') \in X \times (1,2]$ when this inequality first fails we have $-\Delta\hat{u} \geq e^4\varepsilon$. But since $(\frac{\partial}{\partial t} - \Delta)H \geq 0$ at this point, we also have $\varepsilon + \Delta\hat{u} \geq 0$, which gives a contradiction. Hence at $t = 2$ we have $H < 2e^4\varepsilon^2$ and

$$\Delta\hat{u} > -2ne^5\varepsilon$$

on X.

By considering the quantity

$$K = e^{-(t-1)}(|\nabla \hat{u}|^2 + \varepsilon n^{-1}(t-1)\Delta \hat{u}),$$

we can similarly prove that $\Delta \hat{u} < 2ne^5\varepsilon$ at $t = 2$. Since $\nabla u = -\nabla \hat{u}$, $\Delta u = -\Delta \hat{u} = n - R$, this completes the proof of the lemma. □

Remark 1.3. *In the statement of the lemma,* $(t_0 + 2)$ *could be replaced by* $(t_0 + \zeta)$ *for any positive constant* ζ*, at the expense of allowing the constants* δ *and* K *to depend on* ζ*.*

By combining Lemma 1.2 and Proposition 1.1, we know the exponential decay of $\|u(t)\|_{C^0}$ to 0 would be enough to make the NKRF converge.

.2.3 From C^0-norm to L^2-norm

To further refine the condition for NKRF to converge, the authors of [29] proved the following proposition, by making use of Perelman's non-collapsing theorem:

Proposition 1.4. *The Ricci potential* $u(t)$ *and its average* $a(t)$ *satisfy the following inequalities, where the constant* C *depends only on* $g_{i\bar{j}}(0)$*:*

(i) $0 \leq -a \leq \|u - a\|_{C^0}$*;*

(ii) $\|u - a\|_{C^0}^{n+1} \leq C\, \|u - a\|_{L^2}$*.*

Proof of Proposition 1.4. First, as a consequence of Jensen's inequality and the convexity of exponential function,

$$a = \frac{1}{V}\int_X u\, e^{-u}\omega^n \leq \log\,(\frac{1}{V}\int_X e^u e^{-u}\omega^n) = 0. \tag{.2.9}$$

On the other hand, from (.1.3) we know e^{-u} has average 1 with respect to the measure ω^n, and thus $\max_X(u) \geq 0$. Hence $-a \leq \max_X(u-a)$, and (i) is proved.

Next, let $A = \|u-a\|_{C^0} = |u-a|(x_0)$. Then $|u-a| \geq \frac{A}{2}$ on the ball $B_r(x_0)$ of radius $r = \frac{A}{2\|\nabla u\|_{C^0}}$ centered at x_0. If $r < \rho$, where ρ is some fixed uniform radius in Perelman's non-collapsing result (.1.11), then

$$\int_X (u-a)^2\omega^n \geq \int_{B_r(x_0)} \frac{A^2}{4}\omega^n \;\geq\; c\frac{A^2}{4}\left(\frac{A}{2\|\nabla u\|_{C^0}}\right)^{2n} \tag{.2.10}$$

and thus

$$\|u-a\|_{C^0}^{n+1} \;\leq\; C_1\|\nabla u\|_{C^0}^n\|u-a\|_{L^2} \;\leq\; C\,\|u-a\|_{L^2}. \tag{.2.11}$$

On the other hand, if $r > \rho$, then integrating over the ball $B_\rho(x_0)$ gives

$$\int_X (u-a)^2\omega^n \geq \int_{B_\rho(x_0)} \frac{A^2}{4}\omega^n \;=\; \frac{A^2}{4}\int_{B_\rho(x_0)}\omega^n \tag{.2.12}$$

and hence $\|u-a\|_{C^0} \leq C_2\,\|u-a\|_{L^2}$, which turns out to be a stronger estimate than (ii). □

With Proposition 1.4 in hand, for the convergence of NKRF to a K-E metric, we just need to prove that $\|u-a\|_{L^2}$ converges exponentially fast to 0 along the flow. The next step is to relate this target with some 'stability conditions'. [29] first raised an approach to achieve this goal, but we adapt the more simplified one by Z. Zhang [36] here. First we shall do some preparations. The following observation is made in [29] and N. Pali [28]:

Lemma 1.5. *The time derivative of $a(t)$ equals to*

$$\frac{1}{V}\int_X |\nabla u|^2 e^{-u}\omega^n - \frac{1}{V}\int_X (u-a)^2 e^{-u}\omega^n. \tag{.2.13}$$

Proof of Lemma 1.5. Compute using (.1.4) and (.1.6):

$$\begin{aligned}
\frac{d}{dt}a =& \frac{1}{V}\frac{d}{dt}\int_X ue^{-u}\omega^n \\
=& \frac{1}{V}\int_X (\Delta u + u - a)e^{-u}\omega^n - \frac{1}{V}\int_X ue^{-u}(\Delta u + u - a)\omega^n \\
&+ \frac{1}{V}\int_X ue^{-u}\Delta u\,\omega^n \\
=& \frac{1}{V}\int_X \Delta u e^{-u}\omega^n - \frac{1}{V}\int_X u(u-a)e^{-u}\omega^n \\
=& \frac{1}{V}\int_X |\nabla u|^2 e^{-u}\omega^n - \frac{1}{V}\int_X (u-a)^2 e^{-u}\omega^n,
\end{aligned}$$

where in the last line we used the equality

$$\int_X (\Delta u)e^{-u}\omega^n = \int_X |\nabla u|^2 e^{-u}\omega^n.$$

□

We also need the following Poincaré-type inequality on Fano manifolds (see for example, [17], Theorem 2.4.3):

Lemma 1.6. *Let u satisfy (.1.2). Then the following inequality*

$$\frac{1}{V}\int_X f^2 e^{-u}\omega^n \leq \frac{1}{V}\int_X |\nabla f|^2 e^{-u}\omega^n + (\frac{1}{V}\int_X fe^{-u}\omega^n)^2 \tag{.2.14}$$

holds for all $f \in C^\infty(X)$.

Proof of Lemma 1.6. The desired inequality is equivalent to the fact that the lowest positive eigenvalue λ of the following operator

$$L(f) := -g^{\bar{j}i}\nabla_i\nabla_{\bar{j}}f + g^{\bar{j}i}\nabla_i u \cdot \nabla_{\bar{j}}f = \lambda f, \tag{.2.15}$$

with eigenfunction f satisfies $\lambda \geq 1$. Note that this operator is self-adjoint with respect to the inner product

$$\frac{1}{V}\int_X (\quad \cdot \quad) e^{-u}\omega^n,$$

and that its kernel consists of constants.

Applying $\nabla_{\bar{\ell}}$ and using commutation rule for covariant derivatives in the first term gives

$$-g^{\bar{j}i}\nabla_i\nabla_{\bar{j}}\nabla_{\bar{\ell}}f + R^{\bar{p}}{}_{\bar{\ell}}\nabla_{\bar{p}}f + g^{\bar{j}i}\nabla_{\bar{\ell}}\nabla_i u \cdot \nabla_{\bar{j}}f + g^{\bar{j}i}\nabla_{\bar{\ell}}\nabla_{\bar{j}}f \cdot \nabla_i u = \lambda\nabla_{\bar{\ell}}f.$$

Now integrate with respect to $g^{\bar{\ell}k}\nabla_k f\, e^{-u}\omega^n$ and integrate by parts. In view of the fact that

$$R_{i\bar{\ell}} + \partial_i\partial_{\bar{\ell}}u = g_{i\bar{\ell}},$$

we obtain

$$\int_X |\overline{\nabla\nabla} f|^2 e^{-u}\omega^n + \int_X |\overline{\nabla} f|^2 e^{-u}\omega^n = \lambda\int_X |\overline{\nabla} f|^2 e^{-u}\omega^n, \tag{.2.16}$$

from which the desired inequality $\lambda \geq 1$ follows at once. □

We are ready to go over Zhang's result in next subsection.

.2.4 'Strong' Poincaré-type inequality

Z. Zhang [36] used a smart argument to get the convergence of NKRF. His first theorem is as follows:

Theorem 1.7. *Let $u(t)$ and $a(t)$ be defined as in (.1.2) and (.1.4). If along the NKRF,*

$$\int_X |\nabla u|^2 e^{-u}\omega^n \geq (1+\delta)\int_X (u-a)^2 e^{-u}\omega^n \tag{.2.17}$$

holds for a uniform constant $\delta > 0$ independent of t, then $\|u-a\|_{L^2}$ converges exponentially fast to 0; i.e. NKRF converges in the C^∞ sense to a K-E metric.

We need to divide the proof into a couple of lemmas.

For convenience, let's introduce notations

$$Y = \frac{1}{V}\int_X (u-a)^2 e^{-u}\omega^n, \quad Z = \frac{1}{V}\int_X (|\nabla u|^2 - (u-a)^2)e^{-u}\omega^n$$

at each time t.

Lemma 1.8. *Along the NKRF we have $Z(t) \to 0$ as $t \to \infty$.*

Proof of Lemma 1.8. From Perelman's result (.1.10), $u(t)$ is uniformly bounded for all $t > 0$, then so is its average $a(t)$. By Lemma 1.5 and Lemma 1.6, $Z(t) = da/dt \geq 0$ for any t. Then observe that

$$\int_0^\infty Z(t)dt = \lim_{t\to\infty} a(t) - a(0) < \infty.$$

To show $Z(t) \to 0$, it suffices to prove that dZ/dt is uniformly bounded. Recall the evolution equations (.2.3) (.2.4):

$$\frac{\partial}{\partial t}u^2 = \Delta u^2 - 2|\nabla u|^2 + 2u(u-a),$$
$$\frac{\partial}{\partial t}|\nabla u|^2 = \Delta|\nabla u|^2 - |\nabla\nabla u|^2 - |\nabla\overline{\nabla} u|^2 + |\nabla u|^2;$$

then by direct calculation:

$$\begin{aligned}\frac{dZ}{dt} &= \frac{d}{dt}\frac{1}{V}\int_X (|\nabla u|^2 - (u-a)^2)e^{-u}\omega^n \\ &= \frac{1}{V}\int_X \big[\Delta|\nabla u|^2 - |\nabla\nabla u|^2 - |\nabla\overline{\nabla} u|^2 + 3|\nabla u|^2 - \Delta(u-a)^2 \\ &\quad -2(u-a)^2 - (|\nabla u|^2 - (u-a)^2)(u-a)\big]e^{-u}\omega^n \\ &= \frac{1}{V}\int_X \big[-|\nabla\nabla u|^2 - |\nabla\overline{\nabla} u|^2 + 3|\nabla u|^2 - 2(u-a)^2 \\ &\quad +(|\nabla u|^2 - (u-a)^2)(-\Delta u + |\nabla u|^2 - u + a)\big]e^{-u}\omega^n\end{aligned}$$

is uniformly bounded by Perelman's estimate (.1.10). Note that in the second equality, we used

$$\frac{\partial}{\partial t}(e^{-u}\omega^n) = (-\Delta u - u + a + n - R)e^{-u}\omega^n = -(u-a)e^{-u}\omega^n.$$

□

Lemma 1.9. *Assume as in Theorem 1.7, then $Y(t) \to 0$ as $t \to \infty$.*

Proof of Lemma 1.9. Lemma 1.5 and condition (.2.17) imply

$$Z = \frac{da}{dt} \geq \frac{\delta}{V}\int_X (u-a)^2 e^{-u}\omega^n,$$

then use Lemma 1.8. □

Again by Perelman's estimate (.1.10), $\|u-a\|_{L^2}^2 = \int_X (u-a)^2\omega^n$ and $Y = \frac{1}{V}\int_X (u-a)^2 e^{-u}\omega^n$ are uniformly equivalent. To get the exponential decay of $\|u-a\|_{L^2}$, we just need to prove the same thing for Y.

Lemma 1.10. *Assume as in Theorem 1.7, then there exist positive constants γ and B depending only on $g_{i\bar{j}}(0)$ and δ such that*

$$Y(t) \leq Be^{-\gamma t}, \quad \forall t \in [0,\infty).$$

Proof of Lemma 1.10. By Proposition 1.4 (ii) and Lemma 1.9, $\|u-a\|_{C^0} \to 0$ as $t \to \infty$. Thus,

$$\begin{aligned}
\frac{d}{dt}Y &= \frac{1}{V}\int_X \left[2(u-a)(\Delta u + u - a - \frac{da}{dt}) - (u-a)^3\right]e^{-u}\omega^n \\
&= \frac{1}{V}\int_X \left[2(u-a)|\nabla u|^2 - 2|\nabla u|^2 + 2(u-a)^2 - (u-a)^3\right]e^{-u}\omega^n \\
&\leq \frac{1}{V}\int_X \left[(-2+2\|u-a\|_{C^0})|\nabla u|^2 + (2+\|u-a\|_{C^0})(u-a)^2\right]e^{-u}\omega^n \\
&\leq \big((-2+2\|u-a\|_{C^0})(1+\delta) + (2+\|u-a\|_{C^0})\big)Y \\
&\leq -\delta \cdot Y
\end{aligned}$$

whenever t is large enough. Note here we used the condition (.2.17) in the fourth line. This suffices to complete the proof of the lemma, as well as Theorem 1.7. □

Our next job is to relate the convergence of NKRF with some stability conditions on Fano manifolds. This idea was first explored by Phong-Sturm [31] and Phong-Song-Sturm-

Weinkove [29], where they indeed got some nice theorems. Zhang basically recovered and improved their results by using Theorem 1.7, which is more natural to be connected with stability conditions. We will finish this part in next section.

.3 Reduction to Stability Conditions

.3.1 Stability conditions

Let's first introduce the stability conditions we are going to use later. As before, we always assume $(X^n, g(t))$ evolves by NKRF.

One condition is the vanishing of the Futaki invariant, on the certain Kähler class $\pi c_1(X)$:

$$F(W) = \int_X W(u)\omega^n = \int_X (W \cdot \nabla u)\omega^n = 0 \tag{.3.1}$$

for $\forall W \in \eta(X)$ and $\omega \in \pi c_1(X)$.

Others are lower bounds for some second-order differential operators. Recall the operator in (.2.15) acting on smooth functions:

$$L(f) = -g^{\bar{j}i}\nabla_i\nabla_{\bar{j}} f + g^{\bar{j}i}\nabla_i u \cdot \nabla_{\bar{j}} f; \tag{.3.2}$$

by Lemma 1.6, the lowest positive eigenvalue of L is 1. We will post condition on the second positive eigenvalue of L, which we denote by ν, as in Zhang [36]. Namely,

$$\nu(t) \geq 1 + b \quad \text{for a uniform constant} \quad b > 0. \tag{.3.3}$$

This condition is closely related to the lower bounds of second-order differential operators on the space of smooth sections of $T^{1,0}X$. Introduce as in Phong-Song-Sturm-Weinkove [30] two inner products on the space of smooth $T^{1,0}$-vector fields on X:

$$\langle U, W\rangle_0 = \frac{1}{V}\int_X g_{i\bar{j}} U^i \overline{W^j}\omega^n, \quad \langle U, W\rangle_u = \frac{1}{V}\int_X g_{i\bar{j}} U^i \overline{W^j} e^{-u}\omega^n.$$

Define two operators on the same space:

$$-g^{\bar{j}i}\nabla_i\nabla_{\bar{j}}, \quad -g^{\bar{j}i}\nabla_i\nabla_{\bar{j}} + g^{\bar{j}i}\nabla_i u \cdot \nabla_{\bar{j}}. \tag{.3.4}$$

It's clear that the first one is self-adjoint with respect to $\langle \ , \ \rangle_0$ and second one to $\langle \ , \ \rangle_u$. Obviously both operators have 0 as their smallest eigenvalue and $\eta(X)$ as the corresponding eigenspace. Let $\mu(t)$ and $\tilde{\mu}(t)$ be the lowest positive (second smallest) eigenvalue of $-g^{\bar{j}i}\nabla_i\nabla_{\bar{j}}$ and $-g^{\bar{j}i}\nabla_i\nabla_{\bar{j}} + g^{\bar{j}i}\nabla_i u \cdot \nabla_{\bar{j}}$ respectively, then we can post the conditions:

$$\mu(t) \geq c \quad \text{for a uniform constant} \quad c > 0, \tag{.3.5}$$

or

$$\tilde{\mu}(t) \geq \tilde{c} \quad \text{for a uniform constant} \quad \tilde{c} > 0. \tag{.3.6}$$

Now we can summarize all stability conditions we need here:

(A) X has vanished Futaki invariant on $\pi c_1(X)$;

(B) $\nu(t) \geq 1 + b$ for a uniform constant $b > 0$;

(C) $\mu(t) \geq c$ for a uniform constant $c > 0$;

(C') $\tilde{\mu}(t) \geq \tilde{c}$ for a uniform constant $\tilde{c} > 0$.

.3.2 More on stability conditions

Here we provide four lemmas on these conditions. Denote by π_0 and π_u the orthogonal projections of $T^{1,0}$-vector fields onto $\eta(X)$ with respect to $\langle\ ,\ \rangle_0$ and $\langle\ ,\ \rangle_u$. Let $\nabla u = \nabla^i u \frac{\partial}{\partial z^i} = g^{\bar{j}i}\partial_{\bar{j}} u \frac{\partial}{\partial z^i}$ be the complex gradient field of u. The first lemma is observed by Phong-Sturm [31]:

Lemma 1.11. *If condition (A) holds, then* $\pi_0(\nabla u) \equiv 0$.

Proof of Lemma 1.11. From (.3.1), the vanishing of Futaki invariant means $F(W) = \int_X (W \cdot \nabla u)\omega^n = 0$ for $\forall W \in \eta(X)$. Choose W to be $\pi_0(\nabla u)$, then

$$0 = \frac{1}{V}\int_X (W \cdot \nabla u)\omega^n = \langle \pi_0(\nabla u), \nabla u\rangle_0 = \langle \pi_0(\nabla u), \pi_0(\nabla u)\rangle_0,$$

which implies $\pi_0(\nabla u) \equiv 0$. □

The second lemma is proved by Zhang [36]:

Lemma 1.12. *Let* $\nabla u = \pi_u(\nabla u) + U$ *be the orthogonal decomposition with respect to* $\langle\ ,\ \rangle_u$. *Then*

$$\langle \pi_u(\nabla u), \pi_u(\nabla u)\rangle_0 \leq \langle U, U\rangle_0, \tag{.3.7}$$

if condition (A) holds.

Proof of Lemma 1.12. Similarly, choosing W to be $\pi_u(\nabla u)$ yields

$$0 = \langle \pi_u(\nabla u), \nabla u\rangle_0 = \langle \pi_u(\nabla u), \pi_u(\nabla u)\rangle_0 + \langle \pi_u(\nabla u), U\rangle_0,$$

then the conclusion follows from the Cauchy-Schwarz inequality:

$$\langle \pi_u(\nabla u), \pi_u(\nabla u)\rangle_0 = -\langle \pi_u(\nabla u), U\rangle_0 \leq \langle \pi_u(\nabla u), \pi_u(\nabla u)\rangle_0^{1/2} \cdot \langle U, U\rangle_0^{1/2}.$$

□

Now we turn to explore the properties of ν, μ and $\tilde{\mu}$.

We know that 0, having $\eta(X)$ as its eigenspace, is the smallest eigenvalue of operators $-g^{\bar{j}i}\nabla_i\nabla_{\bar{j}}$ and $-g^{\bar{j}i}\nabla_i\nabla_{\bar{j}} + g^{\bar{j}i}\nabla_i u \cdot \nabla_{\bar{j}}$. Then μ and $\tilde{\mu}$ as the second smallest eigenvalue(s), can be determined as the largest numbers such that:

$$\begin{aligned} \int_X |\overline{\nabla} W|^2 \omega^n &\geq \mu \int_X |W|^2 \omega^n, \quad \forall \langle W, \eta(X)\rangle_0 = 0; \\ \int_X |\overline{\nabla} W|^2 e^{-u}\omega^n &\geq \tilde{\mu} \int_X |W|^2 e^{-u}\omega^n, \quad \forall \langle W, \eta(X)\rangle_u = 0. \end{aligned}$$

We also define $\operatorname{osc}_X(u) := \max_X(u) - \min_X(u)$, which is bounded for all time $t \geq 0$ due to Perelman's uniform estimate (.1.10).

The following lemma is due to Phong-Song-Sturm-Weinkove [30] and refined in Zhang [36]:

Lemma 1.13. *The eigenvalues μ and $\tilde{\mu}$ satisfy*

$$e^{-\operatorname{osc}_X(u)}\mu \leq \tilde{\mu} \leq e^{\operatorname{osc}_X(u)}\mu. \tag{.3.8}$$

Or in other words, μ and $\tilde{\mu}$ are uniformly equivalent to each other.

Proof of Lemma 1.13. Let U be a smooth $T^{1,0}$-vector field such that $\langle U, \eta(X)\rangle_u = 0$ and decompose it with respect to $\langle \quad , \quad \rangle_0$ as $U = W + \xi$ with $\xi \in \eta(X)$ and $\langle W, \xi \rangle_0 = 0$. Then,

$$0 = \langle U, \xi \rangle_u = \langle \xi, \xi \rangle_u + \langle W, \xi \rangle_u$$

yields $\langle \xi, \xi \rangle_u = -\langle W, \xi \rangle_u$. Hence,

$$\langle U, U \rangle_u = \langle U, W \rangle_u = \langle W, W \rangle_u + \langle \xi, W \rangle_u = \langle W, W \rangle_u - \langle \xi, \xi \rangle_u \leq \langle W, W \rangle_u.$$

Now, since $\xi \in \eta(X)$,

$$\begin{aligned}\frac{1}{V}\int_X |\overline{\nabla} U|^2 e^{-u}\omega^n \geq e^{-\max_X(u)}\frac{1}{V}\int_X |\overline{\nabla} U|^2 \omega^n &= e^{-\max_X(u)}\frac{1}{V}\int_X |\overline{\nabla} W|^2 \omega^n \\ &\geq \mu e^{-\max_X(u)}\langle W, W\rangle_0 \geq \mu e^{-\operatorname{osc}_X(u)}\langle W, W\rangle_u \\ &\geq \mu e^{-\operatorname{osc}_X(u)}\langle U, U\rangle_u.\end{aligned}$$

In particular, $\tilde{\mu} \geq \mu e^{-\operatorname{osc}_X(u)}$. The other inequality follows similarly. □

Our last lemma is observed also by Zhang [36]:

Lemma 1.14. *The eigenvalues ν and $\tilde{\mu}$ satisfy*

$$\nu \geq 1 + \tilde{\mu}. \tag{.3.9}$$

Combined with Lemma 1.13, we have $\nu \geq 1 + e^{-\operatorname{osc}_X(u)}\mu$.

Proof of Lemma 1.14. Recall the equation (.2.16):

$$\int_X |\overline{\nabla}\overline{\nabla} f|^2 e^{-u}\omega^n + \int_X |\overline{\nabla} f|^2 e^{-u}\omega^n = \lambda \int_X |\overline{\nabla} f|^2 e^{-u}\omega^n,$$

we can see that the eigenfunctions of L with $\lambda = 1$ are the ones whose complex gradient fields being holomorphic. Let ψ be an eigenfunction of $\lambda = \nu$, then $\langle \nabla\psi, \eta(X)\rangle_u = 0$. We immediately get

$$(\nu - 1)\int_X |\nabla\psi|^2 e^{-u}\omega^n = \int_X |\nabla_{\bar{i}}\nabla_{\bar{j}}\psi|^2 e^{-u}\omega^n \geq \tilde{\mu}\int_X |\nabla\psi|^2 e^{-u}\omega^n.$$

□

Lemma 1.13 and Lemma 1.14 basically tell us that, condition (C) and (C') are equivalent, and either of them implies condition (B).

.3.3 Stability and convergence

Now we can prove the following main theorem due to Zhang [36]:

Theorem 1.15. *Suppose on $(X, g(t))$, conditions (A) and (B) hold. Then along the NKRF,*

$$\int_X |\nabla u|^2 e^{-u}\omega^n \geq (1+\delta)\int_X (u-a)^2 e^{-u}\omega^n \tag{.3.10}$$

holds for a uniform constant $\delta > 0$ depending only on the constant b in condition (B) and the upper bound of $\mathrm{osc}_X(u)$. By Theorem 1.7, we know NKRF converges in the C^∞ sense to a K-E metric.

From Lemma 1.13 and Lemma 1.14, we immediately have the following corollary:

Corollary 1.16. *Suppose on $(X, g(t))$, conditions (A) and (C) (or conditions (A) and (C')) hold. Then NKRF converges in the C^∞ sense to a K-E metric.*

Remark 1.17. *Some versions of Corollary 1.16, with slightly stronger assumptions, appeared first in Phong-Song-Sturm-Weinkove* [29] *and* [30].

Proof of Theorem 1.15. Denote by $L^2(X, e^{-u}\omega^n)$ the space of L^2 functions on X with respect to the inner product

$$\frac{1}{V}\int_X (\quad \cdot \quad) e^{-u}\omega^n. \tag{.3.11}$$

Let

$$\lambda_0 = 0 < \lambda_1 = 1 < \lambda_2 = \nu < \lambda_3 \cdots$$

be the sequence of eigenvalues of the operator $L = -g^{\bar{j}i}\nabla_i\nabla_{\bar{j}} + g^{\bar{j}i}\nabla_i u \cdot \nabla_{\bar{j}}$ acting on function space $L^2(X, e^{-u}\omega^n)$.

We know the eigenspace of $\lambda_0 = 0$ is just the kernel of L, which consists of constants. In view of equation (.2.16), eigenfunctions of $\lambda_1 = 1$ are the ones whose complex gradient fields being holomorphic. Let E_k denote the eigenspace of λ_k, then we can write

$$u = u_0 + u_1 + u_2 + \cdots$$

as the unique orthogonal decomposition with respect to the inner product (.3.11), where $u_k \in E_k$ for each k. Note that $u_0 \equiv a$ and $\nabla u_1 \in \eta(X)$.

For any $k \geq 2$ we have $\lambda_k(t) \geq \nu(t) \geq 1 + b$ by condition (B). Thus, from integration by parts

$$\begin{aligned}
\int_X (u-a)^2 e^{-u}\omega^n &= \sum_{k=1}^{\infty}\int_X |u_k|^2 e^{-u}\omega^n = \sum_{k=1}^{\infty}\lambda_k^{-1}\int_X |\nabla u_k|^2 e^{-u}\omega^n \\
&\leq \int_X |\nabla u_1|^2 e^{-u}\omega^n + \sum_{k=2}^{\infty}\frac{1}{1+b}\int_X |\nabla u_k|^2 e^{-u}\omega^n \\
&= \int_X (|\nabla u_1|^2 + \frac{1}{1+b}|U|^2)e^{-u}\omega^n,
\end{aligned}$$

here U is defined by $\nabla u = \pi_u(\nabla u) + U$, where

$$\pi_u(\nabla u) = \nabla u_1 \quad \text{and} \quad U = \sum_{k=2}^{\infty}\nabla u_k.$$

By Lemma 1.12, condition (A) implies

$$\begin{aligned}
\frac{1}{V}\int_X |\nabla u_1|^2 e^{-u}\omega^n &\leq e^{-\min_X(u)}\langle \nabla u_1, \nabla u_1\rangle_0 \\
&\leq e^{-\min_X(u)}\langle U, U\rangle_0 \leq e^{\operatorname{osc}_X(u)}\frac{1}{V}\int_X |U|^2 e^{-u}\omega^n.
\end{aligned}$$

Hence, by direct calculation,

$$\begin{aligned}
\int_X (u-a)^2 e^{-u}\omega^n &\leq \int_X (|\nabla u_1|^2 + \frac{1}{1+b}|U|^2)e^{-u}\omega^n \\
&= \int_X (\frac{b}{1+b}|\nabla u_1|^2 + \frac{1}{1+b}|\nabla u|^2)e^{-u}\omega^n \\
&\leq \int_X (\frac{b}{1+b}\frac{e^{\operatorname{osc}_X(u)}}{1+e^{\operatorname{osc}_X(u)}}|\nabla u|^2 + \frac{1}{1+b}|\nabla u|^2)e^{-u}\omega^n \\
&= \frac{be^{\operatorname{osc}_X(u)} + e^{\operatorname{osc}_X(u)} + 1}{(1+b)(1+e^{\operatorname{osc}_X(u)})}\int_X |\nabla u|^2 e^{-u}\omega^n,
\end{aligned}$$

then we can choose

$$\delta = \frac{b}{be^{\operatorname{osc}_X(u)} + e^{\operatorname{osc}_X(u)} + 1} > 0$$

and proof of the theorem is complete. □

Up to now, we have finished the convergence theory of NKRF on Fano manifolds built on stability conditions. We haven't yet touched bisectional curvature or Frankel conjecture; those would form the topics of next chapter.

Chapter 2

The Role of Bisectional Curvature

In this chapter, we first prove that the Kähler-Ricci flow preserves the positivity of bisectional curvature, due to S. Bando [2] and N. Mok [26] (this indeed provides a Kähler-Ricci flow approach to the Frankel conjecture, thanks to Goldberg-Kobayashi [18]); then go over the improved curvature pinching estimates obtained by Chen-Sun-Tian [10] and Cao-Zhu [8]. Later, we shall confirm that the curvature condition indeed implies the stability condition, as in Phong-Song-Sturm-Weinkove [30]. At last, we state and prove some new result due to the author.

.1 Bisectional Curvature along the Flow

.1.1 Preserving positive bisectional curvature

Let's start with the following version of Hamilton's strong tensor maximum principle proved by Bando ([2], Proposition 1):

Proposition 2.1. *Let (X^n, g) be an n-dimensional Kähler manifold with the metric g pos-*

sibly changes with time t. Consider a tensor h which has the same type and symmetric properties as the curvature tensor, satisfying the following equation:

$$\frac{\partial}{\partial t}h = \Delta h + H(h).$$

Suppose the smooth function H has the following property:

$(*)$ If $h \geq 0$ and there exist two nonzero vectors $v, w \in T^{1,0}_x X$ such that $h_{v\bar{v}w\bar{w}}(x) = 0$, then $H(h)_{v\bar{v}w\bar{w}}(x) \geq 0$.

If h is nonnegative at $t = 0$, then it remains so. Moreover, if at $t = 0$, h is positive at one point, then it's positive everywhere for all $t > 0$.

We omit the proof of Proposition 2.1 since it's a purely partial-differential-equation argument.

Based on Proposition 2.1, Bando [2] (in dimension 3) and Mok [26] (in all dimensions) proved the theorem as follows:

Theorem 2.2. *Let $(X^n, g(t))$ be a compact Kähler manifold with $g(t)$ as the solution to Kähler-Ricci flow (normalized or not normalized). Suppose $g(0)$ has nonnegative bisectional curvature, then so does $g(t)$ for all $t > 0$; furthermore, if g(0) also has bisectional curvature being positive at one point, then $g(t)$ has positive bisectional curvature at $\forall x \in X$ for all $t > 0$.*

We follow the simplified proof by H. Cao ([5], chapter 5):

First recall the evolution equation of the curvature tensor (for convenience, we lower all the indices here):

$$\frac{\partial}{\partial t}R_{i\bar{j}k\bar{\ell}} =\Delta R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}a\bar{b}}R_{b\bar{a}k\bar{\ell}} + R_{i\bar{\ell}a\bar{b}}R_{b\bar{a}k\bar{j}} - R_{i\bar{a}k\bar{b}}R_{a\bar{j}b\bar{\ell}}$$
$$- \frac{1}{2}(R_{i\bar{a}}R_{a\bar{j}k\bar{\ell}} + R_{a\bar{j}}R_{i\bar{a}k\bar{\ell}} + R_{k\bar{a}}R_{i\bar{j}a\bar{\ell}} + R_{a\bar{\ell}}R_{i\bar{j}k\bar{a}}). \qquad (.1.1)$$

Note that the normalization of the flow only adds the second term $R_{i\bar{j}k\bar{\ell}}$ on right hand side and doesn't affect the argument here.

Let us denote by

$$H(Rm)_{i\bar{j}k\bar{\ell}} :=R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}a\bar{b}}R_{b\bar{a}k\bar{\ell}} + R_{i\bar{\ell}a\bar{b}}R_{b\bar{a}k\bar{j}} - R_{i\bar{a}k\bar{b}}R_{a\bar{j}b\bar{\ell}}$$
$$- \frac{1}{2}(R_{i\bar{a}}R_{a\bar{j}k\bar{\ell}} + R_{a\bar{j}}R_{i\bar{a}k\bar{\ell}} + R_{k\bar{a}}R_{i\bar{j}a\bar{\ell}} + R_{a\bar{\ell}}R_{i\bar{j}k\bar{a}}), \qquad (.1.2)$$

so that

$$\frac{\partial}{\partial t}R_{i\bar{j}k\bar{\ell}} = \Delta R_{i\bar{j}k\bar{\ell}} + H_{i\bar{j}k\bar{\ell}}. \qquad (.1.3)$$

Then by Proposition 2.1, it suffices to show that the property $(*)$ holds: for any $T^{1,0}$-vectors $V = (v^i)$ and $W = (w^i)$, we have

$$(*) \quad H_{i\bar{j}k\bar{\ell}}v^i v^{\bar{j}} w^k w^{\bar{\ell}} \geq 0 \quad \text{whenever} \quad R_{i\bar{j}k\bar{\ell}}v^i v^{\bar{j}} w^k w^{\bar{\ell}} = 0,$$

or simply,

$$H_{V\overline{V}W\overline{W}} := H(V,\overline{V},W,\overline{W}) \geq 0 \quad \text{whenever} \quad R_{V\overline{V}W\overline{W}} := \mathrm{Rm}(V,\overline{V},W,\overline{W}) = 0.$$

We divide the proof into a couple of lemmas:

Lemma 2.3. *Assume as in Theorem 2.2. If* $R_{V\overline{V}W\overline{W}} = 0$*, then we have*

$$R_{V\overline{Z}W\overline{W}} = R_{V\overline{V}W\overline{Z}} = 0$$

for any $T^{1,0}$*-vector* Z*.*

Proof of Lemma 2.3. For real parameter $s \in \mathbb{R}$, consider

$$G(s) = \mathrm{Rm}(V + sZ, \overline{V} + s\overline{Z}, W, \overline{W}).$$

Since the bisectional curvature is nonnegative and $R_{V\overline{V}W\overline{W}} = 0$, it follows that $G'(0) = 0$ which implies

$$\mathrm{Re}\ (R_{V\overline{Z}W\overline{W}}) = 0.$$

Suppose $R_{V\overline{Z}W\overline{W}} \neq 0$, and let $R_{V\overline{Z}W\overline{W}} = |R_{V\overline{Z}W\overline{W}}|e^{\sqrt{-1}\theta}$. Then, replacing Z by $e^{-\sqrt{-1}\theta}Z$ in the above yields

$$0 = \mathrm{Re}\ (e^{-\sqrt{-1}\theta}R_{V\overline{Z}W\overline{W}}) = |R_{V\overline{Z}W\overline{W}}|,$$

a contradiction. Thus, we must have

$$R_{V\overline{Z}W\overline{W}} = 0.$$

Similarly, we also have $R_{V\overline{V}W\overline{Z}} = 0$.

□

By Lemma 2.3, we see that if $R_{V\overline{V}W\overline{W}} = 0$ then

$$H_{V\overline{V}W\overline{W}} = R_{V\overline{V}Y\overline{Z}}R_{Z\overline{Y}W\overline{W}} + |R_{V\overline{W}Y\overline{Z}}|^2 - |R_{V\overline{Y}W\overline{Z}}|^2.$$

Therefore, property $(*)$ follows immediately from the next lemma:

Lemma 2.4. *Assume as in Theorem 2.2. Then, for any $T^{1,0}$-vectors Y and Z,*

$$R_{V\overline{V}Y\overline{Z}}R_{Z\overline{Y}W\overline{W}} \geq |R_{V\overline{W}Y\overline{Z}}|^2 + |R_{V\overline{Y}W\overline{Z}}|^2$$

if $R_{V\overline{V}W\overline{W}} = 0$.

Proof of Lemma 2.4. Consider

$$\begin{aligned} I(s) =& \mathrm{Rm}(V + sY, \overline{V} + s\overline{Y}, W + sZ, \overline{W} + s\overline{Z}) \\ =& s^2 \left(R_{V\overline{V}Z\overline{Z}} + R_{Y\overline{Y}W\overline{W}} + R_{V\overline{Y}W\overline{Z}} + R_{Y\overline{V}Z\overline{W}} + R_{V\overline{Y}Z\overline{W}} + R_{Y\overline{V}W\overline{Z}} \right) \\ & + O(s^3). \end{aligned}$$

Here we have used Lemma 2.3.

Since $I(s) \geq 0$ and $I(0) = 0$, we have $I''(0) \geq 0$. Hence, by taking $Y = \zeta^k e_k$ and $Z = \eta^\ell e_\ell$ with respect to any basis $\{e_1, \cdots e_n\}$, we obtain a real, semi-positive definite bilinear form $Q(Y, Z)$:

$$\begin{aligned} Q(Y, Z) :=& R_{V\overline{V}Z\overline{Z}} + R_{Y\overline{Y}W\overline{W}} + R_{V\overline{Y}W\overline{Z}} + R_{Y\overline{V}Z\overline{W}} + R_{V\overline{Y}Z\overline{W}} + R_{Y\overline{V}W\overline{Z}} \\ =& R_{V\overline{V}k\bar{\ell}}\eta^k\eta^{\bar{\ell}} + R_{k\bar{\ell}W\overline{W}}\zeta^k\zeta^{\bar{\ell}} + R_{V\bar{k}W\bar{\ell}}\zeta^{\bar{k}}\eta^{\bar{\ell}} + R_{k\overline{V}\ell\overline{W}}\zeta^k\eta^\ell \\ & + R_{V\bar{k}\ell\overline{W}}\zeta^{\bar{k}}\eta^\ell + R_{k\overline{V}W\bar{\ell}}\zeta^k\eta^{\bar{\ell}} \geq 0. \end{aligned}$$

Next, we need a useful linear algebra fact :

Proposition 2.5. *Let M and N be two $m \times m$ real symmetric semi-positive definite matrices,* *and let K be a real $m \times m$ matrix such that the $2m \times 2m$ real symmetric matrix*

$$P_1 = \begin{pmatrix} M & K \\ K^T & N \end{pmatrix}$$

is semi-positive definite. Then, we have

$$\mathrm{Tr}(MN) \geq \mathrm{Tr}(K^T K) = |K|^2.$$

Proof of Proposition 2.5. Consider the associated matrix

$$P_2 = \begin{pmatrix} N & -K \\ -K^T & M \end{pmatrix},$$

it is clear that P_2 is also symmetric and semi-positive definite. Thus, we get

$$\mathrm{Tr}(P_1 P_2) \geq 0.$$

However,

$$P_1 P_2 = \begin{pmatrix} MN - KK^T & KM - MK \\ K^T N - NK^T & NM - K^T K \end{pmatrix};$$

therefore,

$$\mathrm{Tr}(MN) - |K|^2 = \frac{1}{2}\mathrm{Tr}(P_1 P_2) \geq 0.$$

□

As a special case, by taking

$$P_1 = \begin{pmatrix} ReA & ImA & Re(B+D) & Im(B-D) \\ -ImA & ReA & -Im(B+D) & Re(B-D) \\ Re(B+D)^T & -Im(B+D)^T & ReC & ImC \\ Im(B-D)^T & Re(B-D)^T & -ImC & ReC \end{pmatrix},$$

we immediately obtain the following :

Corollary 2.6. *Let A, B, C, D be complex matrices with A and C being Hermitian. Suppose that the (real) quadratic form*

$$A_{k\bar{\ell}}\eta^k\overline{\eta^\ell} + C_{k\bar{\ell}}\zeta^k\overline{\zeta^\ell} + 2Re(B_{k\bar{\ell}}\eta^k\overline{\zeta^\ell}) + 2Re(D_{k\ell}\eta^k\zeta^\ell), \quad \eta, \zeta \in \mathbb{C}^n,$$

is semi-positive definite. If we write everthing out in real coordinates and use Proposition 2.5, we would have

$$Tr(AC) \geq |B|^2 + |D|^2,$$

i.e.

$$\sum_{k,\ell} A_{k\bar{\ell}}C_{\ell\bar{k}} \geq \sum_{k,\ell} |B_{k\bar{\ell}}|^2 + |D_{k\ell}|^2.$$

Now, by applying Corollary 2.6 to the above real semi-positive definite bilinear form Q, we get

$$R_{V\overline{V}Y\overline{Z}}R_{Z\overline{Y}W\overline{W}} \geq |R_{V\overline{W}Y\overline{Z}}|^2 + |R_{V\overline{Y}W\overline{Z}}|^2.$$

We have thus proved property $(*)$, which concludes the proof of Theorem 2.2. $\square$

By Theorem 2.2, if the NKRF starting from any Kähler metric with positive bisectional curvature CONVERGES, then the limit metric would be a K-E metric with positive bisectional curvature. From a classical result of Goldberg-Kobayashi ([18], Theorem 5), such metric must be globally isometric to the Fubini-Study metric. Thus, the underlying manifold is biholomorphic to complex projective space. In other words, the convergence of NKRF with positive bisectional curvature implies the Frankel conjecture.

To prove the convergence, we wish to show that the conditions (A) and (C) (in Chapter 1) hold under positive bisectional curvature, due to Corollary 1.16. Before trying to do that, we first strengthen our curvature condition, as in the following two subsections.

.1.2 On the lower bound of bisectional curvature

In the last subsection we already know the positivity of bisectional curvature is preserved under NKRF. It will be interesting to study how the lower bound of bisectional curvature behaves along NKRF, if the initial metric has bisectional curvature bounded below from 0. The following theorem is due to Chen-Sun-Tian [10]:

Theorem 2.7. *Let $(X^n, g(t))$ be a compact Kähler manifold with $g(t)$ as the solution to NKRF. Suppose that along the flow $g(t)$ has positive bisectional curvature, and the Ricci curvature of $g(t)$ satisfies $Ric(g(t)) \geq Cg(t)$ for a uniform constant $C > 0$. Then the bisectional curvature of $g(t)$ has a uniform positive lower bound.*

Remark 2.8. *To study the lower bound of bisectional curvature under NKRF, it's natural to think of computing the evolution equation of tensor $R_{i\bar{j}k\bar{\ell}} - c(t)(g_{i\bar{j}}g_{k\bar{\ell}} + g_{i\bar{\ell}}g_{k\bar{j}})$; in fact, this idea was carried out by X. Chen [9], where he obtained some pinching estimates for* *holomorphic sectional and bisectional curvatures. In our case, Chen-Sun-Tian considered a different tensor to deal with and got the above result.*

Proof of Theorem 2.7. First recall the NKRF equation and evolutions of curvatures:

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} + g_{i\bar{j}};$$

$$\frac{\partial}{\partial t} R = \Delta R + |Ric|^2 - R;$$

$$\frac{\partial}{\partial t} Ric = \Delta Ric + Ric \cdot Rm - Ric^2;$$

$$\begin{aligned}\frac{\partial}{\partial t} R_{i\bar{j}k\bar{\ell}} =& \Delta R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}a\bar{b}} R_{b\bar{a}k\bar{\ell}} + R_{i\bar{\ell}a\bar{b}} R_{b\bar{a}k\bar{j}} - R_{i\bar{a}k\bar{b}} R_{a\bar{j}b\bar{\ell}} \\ & - \frac{1}{2}(R_{i\bar{a}} R_{a\bar{j}k\bar{\ell}} + R_{a\bar{j}} R_{i\bar{a}k\bar{\ell}} + R_{k\bar{a}} R_{i\bar{j}a\bar{\ell}} + R_{a\bar{\ell}} R_{i\bar{j}k\bar{a}}).\end{aligned}$$

Here we define

$$(Ric \cdot Rm)_{i\bar{j}} := R_{b\bar{a}} R_{i\bar{j}a\bar{b}}, \tag{.1.4}$$

and

$$(Ric^2)_{i\bar{j}} := R_{i\bar{k}} R_{k\bar{j}}. \tag{.1.5}$$

Now we put $S \coloneqq Rm - c(g * Ric)$, where c is a function of t, and

$$(g * Ric)_{i\bar{j}k\bar{\ell}} \coloneqq g_{i\bar{j}}R_{k\bar{\ell}} + g_{k\bar{\ell}}R_{i\bar{j}} + g_{i\bar{\ell}}R_{k\bar{j}} + g_{k\bar{j}}R_{i\bar{\ell}}. \tag{.1.6}$$

We shall compute the evolution of tensor $S_{i\bar{j}k\bar{\ell}}$ and use maximum principle.

Taking trace on tensor S yields

$$S_{k\bar{\ell}} = (1 - (n+2)c)R_{k\bar{\ell}} - cR \cdot g_{k\bar{\ell}},$$

if we denote by $Sic_{k\bar{\ell}} \coloneqq g^{\bar{j}i}S_{i\bar{j}k\bar{\ell}}$, then we can write the above equation as

$$Sic = (1 - (n+2)c)Ric - cR \cdot g. \tag{.1.7}$$

Therefore, by a straightforward calculation, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}R_{i\bar{j}k\bar{\ell}} =& \Delta S_{i\bar{j}k\bar{\ell}} + H(S)_{i\bar{j}k\bar{\ell}} \\
&+ c(g * \Delta Ric)_{i\bar{j}k\bar{\ell}} + c(g * Ric)_{i\bar{j}k\bar{\ell}} \\
&+ c[(g * (Ric \cdot S))_{i\bar{j}k\bar{\ell}} + (Ric * Sic)_{i\bar{j}k\bar{\ell}}] + I \\
&- c[(Ric * Ric)_{i\bar{j}k\bar{\ell}} + (Ric^2 * g)_{i\bar{j}k\bar{\ell}}],
\end{aligned} \tag{.1.8}$$

where $H(S)_{i\bar{j}k\bar{\ell}}$ is defined as (.1.2) with $R_{i\bar{j}k\bar{\ell}}$ replaced by $S_{i\bar{j}k\bar{\ell}}$:

$$\begin{aligned}
H(S)_{i\bar{j}k\bar{\ell}} \coloneqq& S_{i\bar{j}k\bar{\ell}} + S_{i\bar{j}a\bar{b}}S_{b\bar{a}k\bar{\ell}} + S_{i\bar{\ell}a\bar{b}}S_{b\bar{a}k\bar{j}} - S_{i\bar{a}k\bar{b}}S_{a\bar{j}b\bar{\ell}} \\
&- \frac{1}{2}(R_{i\bar{a}}S_{a\bar{j}k\bar{\ell}} + R_{a\bar{j}}S_{i\bar{a}k\bar{\ell}} + R_{k\bar{a}}S_{i\bar{j}a\bar{\ell}} + R_{a\bar{\ell}}S_{i\bar{j}k\bar{a}}),
\end{aligned}$$

and

$$
\begin{aligned}
I \coloneqq c^2[&(g * Ric)_{i\bar{j}a\bar{b}}(g * Ric)_{b\bar{a}k\bar{\ell}} + (g * Ric)_{i\bar{\ell}a\bar{b}}(g * Ric)_{b\bar{a}k\bar{j}} \\
&- (g * Ric)_{i\bar{a}k\bar{b}}(g * Ric)_{a\bar{j}b\bar{\ell}}]. \qquad (.1.9)
\end{aligned}
$$

We also calculate:

$$
\begin{aligned}
\frac{\partial}{\partial t}(-c(g * Ric)_{i\bar{j}k\bar{\ell}}) =& - c'(g * Ric)_{i\bar{j}k\bar{\ell}} - c(\frac{\partial g}{\partial t} * Ric)_{i\bar{j}k\bar{\ell}} - c(g * \frac{\partial Ric}{\partial t})_{i\bar{j}k\bar{\ell}} \\
=& - c'(g * Ric)_{i\bar{j}k\bar{\ell}} - c(g * Ric)_{i\bar{j}k\bar{\ell}} + c(Ric * Ric)_{i\bar{j}k\bar{\ell}} \\
& - c(g * (\Delta Ric))_{i\bar{j}k\bar{\ell}} - c(g * (Ric \cdot Rm))_{i\bar{j}k\bar{\ell}} \\
& + c(g * (Ric^2))_{i\bar{j}k\bar{\ell}}. \qquad (.1.10)
\end{aligned}
$$

It follows from (.1.8)∼(.1.10) that

$$\begin{aligned}\frac{\partial}{\partial t}S_{i\bar{j}k\bar{\ell}} =&\Delta S_{i\bar{j}k\bar{\ell}} + H(S)_{i\bar{j}k\bar{\ell}} + c(g * \Delta Ric)_{i\bar{j}k\bar{\ell}} + c(g * Ric)_{i\bar{j}k\bar{\ell}} \\ &+ c[(g * (Ric \cdot S))_{i\bar{j}k\bar{\ell}} + (Ric * Sic)_{i\bar{j}k\bar{\ell}}] + I \\ &- c[(Ric * Ric)_{i\bar{j}k\bar{\ell}} + (Ric^2 * g)_{i\bar{j}k\bar{\ell}}] \\ &- c'(g * Ric)_{i\bar{j}k\bar{\ell}} - c(g * Ric)_{i\bar{j}k\bar{\ell}} + c(Ric * Ric)_{i\bar{j}k\bar{\ell}} \\ &- c(g * (\Delta Ric))_{i\bar{j}k\bar{\ell}} - c(g * (Ric \cdot Rm))_{i\bar{j}k\bar{\ell}} + c(g * (Ric^2))_{i\bar{j}k\bar{\ell}} \\ =&\Delta S_{i\bar{j}k\bar{\ell}} + H(S)_{i\bar{j}k\bar{\ell}} + c^2(g * (Ric \cdot (g * Ric))_{i\bar{j}k\bar{\ell}} \\ &+ c(Ric * Sic)_{i\bar{j}k\bar{\ell}} + I - c'(g * Ric)_{i\bar{j}k\bar{\ell}} \\ =&\Delta S_{i\bar{j}k\bar{\ell}} + H(S)_{i\bar{j}k\bar{\ell}} + c^2(g * (Ric \cdot (g * Ric)))_{i\bar{j}k\bar{\ell}} \\ &+ c(Ric * Ric)_{i\bar{j}k\bar{\ell}} - c^2(Ric * ((n+2)Ric + R \cdot g))_{i\bar{j}k\bar{\ell}} \\ &+ I - c'(g * Ric)_{i\bar{j}k\bar{\ell}}.\end{aligned}$$

Note that $I = O(c^2)$. Since $Ric(g(t)) \geq Cg(t)$ for $t \geq 0$, if $c(t) \equiv c > 0$ is sufficiently small, we have

$$\frac{\partial}{\partial t}S_{i\bar{j}k\bar{\ell}} \geq \Delta S_{i\bar{j}k\bar{\ell}} + H(S)_{i\bar{j}k\bar{\ell}}. \tag{.1.11}$$

Now apply the same proof of Theorem 2.2 with $R_{i\bar{j}k\bar{\ell}}$ replaced by $S_{i\bar{j}k\bar{\ell}}$, we see $H(S)_{i\bar{i}j\bar{j}} \geq 0$ whenever $S_{i\bar{i}j\bar{j}} = 0$. Then by Proposition 2.1, $S_{i\bar{i}j\bar{j}} \geq 0$ for all $t \geq 0$, i.e. $R_{i\bar{i}j\bar{j}} \geq c(g{*}Ric)_{i\bar{i}j\bar{j}} \geq c \cdot C(g * g)_{i\bar{i}j\bar{j}}$. □

.1.3 On the lower bound of Ricci curvature

Theorem 2.7 tells us that a positive lower bound of bisectional curvature is preserved under NKRF, provided that the Ricci curvature is uniformly bounded below from 0. Of course we wish to remove this extra condition. It was done by Cao-Zhu [8]:

Theorem 2.9. *Let $(X^n, g(t))$ be a compact Kähler manifold with $g(t)$ as the solution to NKRF. Suppose that along the flow $g(t)$ has positive bisectional curvature. Then the Ricci curvature of $g(t)$ satisfies $Ric(g(t)) \geq Cg(t)$ for a uniform constant $C > 0$.*

Putting Theorem 2.2, Theorem 2.7 and Theorem 2.9 together, we get a satisfactory curvature pinching estimate:

Corollary 2.10. *Let $(X^n, g(t))$ be a compact Kähler manifold with $g(t)$ as the solution to NKRF. Suppose that $g(0)$ has positive bisectional curvature. Then the bisectional curvature of $g(t)$ has a uniform positive lower bound along the flow.*

To prove Theorem 2.9, we need first invoke the so-called 'Hamilton's compactness theorem' ([21], Main Theorem 1.2):

Theorem 2.11. *Let $\{X_k\} = \{(X_k, g_k, x_k, E^k)\}$ be a sequence of evolving complete marked Riemannian manifolds which are solutions to the Ricci flow. Here X is the underlying manifold, g is the Riemannian metric, x is a marked point on X and E is an orthonormal frame at x at $t = 0$ with respect to $g(0)$.*

Suppose that:

- *(i) The Riemann curvature tensors of X_k are uniformly bounded by a constant A for all k and all $t \geq 0$;*
- *(ii) The injectivity radii of X_k at the x_k at time $t = 0$ are uniformly bounded below by a constant $\delta > 0$ for all k.*

Then there exists a subsequence which converges to an evolving complete marked Riemannian manifold $\tilde{X} = (X, \tilde{g}, \tilde{x}, \tilde{E})$ *which is also a solution of the Ricci flow, with its Riemann curvature tensors bounded above by* A *and its injectivity radius at* x *at time* $t = 0$ *bounded below by* δ.

Proof of Theorem 2.9. We argue by contradiction. Suppose the conclusion is not true. Then we can find a sequence of positive numbers $\epsilon_k \to 0$, and a sequence of points $\{(x_k, t_k)\}_{k=1}^{\infty}$ in space-time with $x_k \in X$ and $t_k \to \infty$ as $k \to \infty$, such that

$$\min_{1 \le i,j \le n} R_{i\bar{j}}(x_k, t_k) \le \epsilon_k.$$

Now we can choose a unitary frame $E^k = \{e_1^k, \cdots, e_n^k\}$ at the point x_k and the time t_k so that

$$R_{1\bar{1}}(x_k, t_k) = \min_{1 \le i \le n} R_{i\bar{i}}(x_k, t_k).$$

By Perelman's result (or by Cao-Chen-Zhu [7], see also chapter 5 in [5]), the scalar curvature of $g(t)$ is uniformly bounded. Since $g(t)$ has positive bisectional curvature, its bisectional curvatures are also uniformly bounded, and then so are all its curvature tensors. The condition (i) in Theorem 2.11 is satisfied. As for condition (ii), the injectivity radii are uniformly bounded by Perelman's non-collapsing theorem.

Thus we can apply Theorem 2.11, with adding the complex structure J into the data of X. Namely, for a sequence of compact marked solutions $\{(X, J, g(t_k + t), x_k, E^k)\}$ to NKRF with positive bisectional curvature, there exists a subsequence converges to a compact marked solution $\tilde{X} = (X, \tilde{J}, \tilde{g}(t), \tilde{x}, \tilde{E})$ to NKRF with nonnegative bisectional curvature $\tilde{R}_{i\bar{i}j\bar{j}} \ge 0$

and

$$\tilde{R}_{1\bar{1}}(\tilde{x}, 0) = \lim_{k\to\infty} R_{1\bar{1}}(x_k, t_k) = 0. \qquad (.1.12)$$

Here $\tilde{E}$ is a unitary frame at the marked point $\tilde{x}$ at $t = 0$, and $\tilde{J}$ is a complex structure on X, possibly different from J.

Now we use the following result of H. Gu ([19], Theorem 1.2):

Proposition 2.12. *Given any Kähler metric $h_{i\bar{j}}$ with nonnegative bisectional curvature on a compact, irreducible, simply connected Kähler manifold M^n. Then, under the NKRF, either the bisectional curvature becomes positive everywhere after a short time, or (M^n, h) is isometrically biholomorphic to a Hermitian symmetric space of rank ≥ 2.*

We know that any compact Fano Kähler manifold is simply connected, by Yau's solution to Calabi conjecture [35] and a result of Kobayashi [23]. Also, by a theorem of Bishop-Goldberg [4] (see also [18], Theorem 4), the second betti number of X is 1. Hence, X is irreducible.

On the other hand, since $(X, \tilde{g})$ has nonnegative bisectional curvature, (.1.12) tells us that for $\forall t \in [0, \infty)$, the bisectional curvature $\tilde{R}_{i\bar{i}j\bar{j}}(t)$ vanishes along some direction at some point on X at t. Therefore, Proposition 2.12 implies that $(X, \tilde{J}, \tilde{g})$ is isometrically biholomorphic to a Hermitian symmetric space of rank ≥ 2.

This would lead to a contradiction, because $\tilde{g}$ is Kähler-Einstein, i.e., $\tilde{R}_{i\bar{j}} = \tilde{g}_{i\bar{j}}$, in turn implies that

$$||R_{i\bar{j}} - g_{i\bar{j}}||_{C^0}(t_k) \to 0,$$

contradicting (.1.12). This finishes the proof of Theorem 2.9. □

Now, remember that we already have Corollary 1.16 and Corollary 2.10 in hand. Therefore, in order to prove Frankel conjecture by Kähler-Ricci flow, we just need to show that a positive lower bound of bisectional curvature implies:

(A) The vanishing of Futaki invariant on $\pi c_1(X)$;

and

(C) A positive lower bound of the lowest positive eigenvalue μ of operator $-g^{\bar{j}i}\nabla_i\nabla_{\bar{j}}$ on smooth $T^{1,0}$-vector fields.

We will prove the (C) part in next section, following Phong-Song-Sturm-Weinkove [30]; however, the (A) part is still open up to today. It remains the only missing step of this incomplete proof.

.2 Curvature and Stability Conditions

.2.1 The lower bound of μ

We follow Phong-Song-Sturm-Weinkove [30] here. To connect bisectional curvature and the lower bound of μ, we need two steps: the first is to derive the curvature condition which implies a lower bound of μ, by some standard Bochner-type technique; the second is to relate the bisectional curvature with that curvature condition.

We introduce some notions of positivity. A tensor $T_{i\bar{j}k\bar{\ell}}$ is called *Griffiths* nonnegative if

$$T_{i\bar{j}k\bar{\ell}} v^i v^{\bar{j}} w^k w^{\bar{\ell}} \geq 0 \tag{.2.1}$$

for all $T^{1,0}$-vectors v, w. For brevity we write $T_{i\bar{j}k\bar{\ell}} \geq_{Gr} 0$. It's obvious that $R_{i\bar{j}k\bar{\ell}} \geq_{Gr} 0$ means the condition of nonnegative bisectional curvature.

We say that a tensor $T_{i\bar{j}k\bar{\ell}}$ is *Nakano* nonnegative if

$$T_{i\bar{j}k\bar{\ell}} h^{ik} h^{\overline{j\ell}} \geq 0 \tag{.2.2}$$

for any $T^{1,0} \otimes T^{1,0}$-tensor h, and we write $T_{i\bar{j}k\bar{\ell}} \geq_{Na} 0$ for short. Clearly *Nakano* nonnegativity is stronger than *Griffiths* nonnegativity.

Next lemma describes the curvature condition which guarantees a positive lower bound for μ:

Lemma 2.13. *Suppose that a Kähler metric g satisfies*

$$R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}} g_{k\bar{\ell}} - c\, g_{i\bar{j}} g_{k\bar{\ell}} \geq_{Na} 0 \tag{.2.3}$$

for some constant $c > 0$. Then $\mu \geq c$.

Proof of Lemma 2.13. Recall the commutation rules:

$$(\nabla_i \nabla_{\bar{\ell}} - \nabla_{\bar{\ell}} \nabla_i) V^k = g^{\bar{n}k} R_{i\bar{\ell}m\bar{n}} V^m,$$
$$(\nabla_i \nabla_{\bar{\ell}} - \nabla_{\bar{\ell}} \nabla_i) w_{\bar{j}} = g^{\bar{q}p} R_{i\bar{\ell}p\bar{j}} w_{\bar{q}}.$$

Let V be an eigenvector of the operator $-g^{\bar{j}i}\nabla_i \nabla_{\bar{j}}$ with eigenvalue μ; then

$$-g^{\bar{j}i} \nabla_i \nabla_{\bar{j}} V^k = \mu V^k. \tag{.2.4}$$

Applying $\nabla_{\bar{\ell}}$ and using the commutation rules yield

$$-g^{\bar{j}i} \nabla_i \nabla_{\bar{\ell}} \nabla_{\bar{j}} V^k + g^{\bar{j}i} g^{\bar{n}k} R_{i\bar{\ell}m\bar{n}} \nabla_{\bar{j}} V^m + g^{\bar{j}i} g^{\bar{q}p} R_{i\bar{\ell}p\bar{j}} \nabla_{\bar{q}} V^k = \mu \nabla_{\bar{\ell}} V^k. \tag{.2.5}$$

Multiply by $g^{\bar{\ell}r} g_{k\bar{s}} \nabla_r \overline{V^s}$ to obtain

$$\begin{aligned} &- g^{\bar{\ell}r} g_{k\bar{s}} g^{\bar{j}i} \nabla_r \overline{V^s} \nabla_i \nabla_{\bar{\ell}} \nabla_{\bar{j}} V^k + g^{\bar{\ell}r} g^{\bar{j}i} R_{i\bar{\ell}m\bar{s}} \nabla_r \overline{V^s} \nabla_{\bar{j}} V^m \\ &+ g^{\bar{\ell}r} g_{k\bar{s}} g^{\bar{q}p} R_{p\bar{\ell}} \nabla_r \overline{V^s} \nabla_{\bar{q}} V^k = \mu\, g^{\bar{\ell}r} g_{k\bar{s}} \nabla_r \overline{V^s} \nabla_{\bar{\ell}} V^k. \end{aligned} \tag{.2.6}$$

By (.2.3), after integrating by parts:

$$\mu \int_X |\nabla_{\bar{\ell}} V^k|^2 \omega^n \geq c \int_X |\nabla_{\bar{\ell}} V^k|^2 \omega^n + \int_X |\nabla_{\bar{\ell}} \nabla_{\bar{j}} V^k|^2 \omega^n, \tag{.2.7}$$

and hence $\mu \geq c$. □

We have a version for the lowest positive eigenvalue $\tilde{\mu}$ of operator $-g^{\bar{j}i}\nabla_i \nabla_{\bar{j}} + g^{\bar{j}i} \nabla_i u \cdot \nabla_{\bar{j}}$ on smooth $T^{1,0}$-vector fields as well (though we won't use it later):

Lemma 2.14. *Suppose that a Kähler metric g satisfies*

$$R_{i\bar{j}k\bar{\ell}} + (1-\tilde{c})g_{i\bar{j}}g_{k\bar{\ell}} \geq_{Na} 0 \tag{.2.8}$$

for some constant $\tilde{c} > 0$. Then $\tilde{\mu} \geq \tilde{c}$.

Proof of Lemma 2.14. Recall that u is the Ricci potential. Let V be an eigenvector of $-g^{\bar{j}i}\nabla_i\nabla_{\bar{j}} + g^{\bar{j}i}\nabla_i u \cdot \nabla_{\bar{j}}$ with eigenvalue $\tilde{\mu}$. Then

$$-g^{\bar{j}i}\nabla_i\nabla_{\bar{j}}V^k + g^{\bar{j}i}\nabla_i u \cdot \nabla_{\bar{j}}V^k = \tilde{\mu}V^k. \tag{.2.9}$$

Applying $\nabla_{\bar{\ell}}$ as before, using the commutation rules and the definition of u we have

$$\begin{aligned}&-g^{\bar{j}i}\nabla_i\nabla_{\bar{\ell}}\nabla_{\bar{j}}V^k + g^{\bar{j}i}g^{\bar{n}k}R_{i\bar{\ell}m\bar{n}}\nabla_{\bar{j}}V^m + g^{\bar{j}i}g^{\bar{q}p}R_{i\bar{\ell}p\bar{j}}\nabla_{\bar{q}}V^k \\ &+g^{\bar{j}i}\nabla_{\bar{\ell}}\nabla_{\bar{j}}V^k\nabla_i u + \nabla_{\bar{\ell}}V^k - g^{\bar{j}i}R_{i\bar{\ell}}\nabla_{\bar{j}}V^k = \tilde{\mu}\nabla_{\bar{\ell}}V^k.\end{aligned} \tag{.2.10}$$

Multiply by $g^{\bar{\ell}r}g_{k\bar{s}}\nabla_r\overline{V^s}$ to obtain

$$\begin{aligned}&-g^{\bar{\ell}r}g_{k\bar{s}}g^{\bar{j}i}\nabla_r\overline{V^s}\nabla_i\nabla_{\bar{\ell}}\nabla_{\bar{j}}V^k + (R_{i\bar{j}k\bar{\ell}} + g_{i\bar{j}}g_{k\bar{\ell}})\nabla^{\bar{j}}\overline{V^\ell}\nabla^i V^k \\ &+g^{\bar{\ell}r}g_{k\bar{s}}g^{\bar{j}i}\nabla_r\overline{V^s}\nabla_{\bar{\ell}}\nabla_{\bar{j}}V^k\nabla_i u = \tilde{\mu}\, g^{\bar{\ell}r}g_{k\bar{s}}\nabla_r\overline{V^s}\nabla_{\bar{\ell}}V^k.\end{aligned} \tag{.2.11}$$

Using (.2.8) and integrating with respect to $e^{-u}\omega^n$ yield

$$\tilde{\mu}\int_X |\nabla_{\bar{\ell}}V^k|^2 e^{-u}\omega^n \geq \tilde{c}\int_X |\nabla_{\bar{\ell}}V^k|^2 e^{-u}\omega^n + \int_X |\nabla_{\bar{\ell}}\nabla_{\bar{j}}V^k|^2 e^{-u}\omega^n, \tag{.2.12}$$

and hence $\tilde{\mu} \geq \tilde{c}$. □

We also need the following proposition:

Proposition 2.15. *Let (X^n, g) be a compact Kähler manifold. Suppose that the curvature of metric g satisfies*

$$R_{i\bar{j}k\bar{\ell}} - c\, g_{i\bar{j}} g_{k\bar{\ell}} \geq_{Gr} 0 \tag{.2.13}$$

for some constant $c > 0$. Then

$$R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}} g_{k\bar{\ell}} - n\, c\, g_{i\bar{j}} g_{k\bar{\ell}} \geq_{Na} 0. \tag{.2.14}$$

Proposition 2.15 along with Lemma 2.13 would fulfill our purpose:

Theorem 2.16. *Let (X^n, g) be a compact Kähler manifold. Suppose that the bisectional curvature of metric g has a positive lower bound:*

$$R_{i\bar{i}j\bar{j}} \geq c\, g_{i\bar{i}} g_{j\bar{j}}$$

for some constant $c > 0$. Then μ has a positive lower bound:

$$\mu \geq n\, c.$$

Proposition 2.15 is the analogy of a result by Demailly-Skoda [14] (see also [13], Proposition 10.14). It requires the following lemma ([13], Lemma 10.15):

Lemma 2.17. *Let $q \geq 3$ be an integer and let x^λ, y^λ for $1 \leq \lambda \leq n$ be complex numbers. Let U_q^n be the set of n-tuples of qth roots of unity and define complex numbers*

$$x'_{(\sigma)} = \sum_{\lambda=1}^{n} x^\lambda \overline{\sigma_\lambda}, \quad y'_{(\sigma)} = \sum_{\lambda=1}^{n} y^\lambda \overline{\sigma_\lambda}, \quad \textit{for each } \sigma = (\sigma_1, \ldots, \sigma_n) \in U_q^n.$$

Then for every pair (α, β) with $1 \le \alpha, \beta \le n$, the following holds:

$$q^{-n} \sum_{\sigma \in U_q^n} x'_{(\sigma)} \overline{y'_{(\sigma)}} \sigma_\alpha \overline{\sigma_\beta} = \begin{cases} x^\alpha \overline{y^\beta}, & \text{if } \alpha \neq \beta; \\ \sum_{\lambda=1}^n x^\lambda \overline{y^\lambda}, & \text{if } \alpha = \beta. \end{cases} \tag{.2.15}$$

Proof of Lemma 2.17. We only require the following elementary claim: the coefficient of $x^\gamma \overline{y^\delta}$ in the left hand side of (.2.15) is $q^{-n} \sum_{\sigma \in U_q^n} \sigma_\alpha \overline{\sigma_\beta}\, \overline{\sigma_\gamma} \sigma_\delta$, and this is equal to 1 if $\{\alpha, \delta\} = \{\beta, \gamma\}$ and 0 otherwise. Indeed, for the second alternative, assume without loss of generality that $\alpha \notin \{\beta, \gamma\}$ and then observe that

$$\sum_{\sigma \in U_q^n} \sigma_\alpha \overline{\sigma_\beta}\, \overline{\sigma_\gamma} \sigma_\delta = \begin{cases} e^{2\pi i/q} \sum_{\sigma \in U_q^n} \sigma_\alpha \overline{\sigma_\beta}\, \overline{\sigma_\gamma} \sigma_\delta, & \alpha \neq \delta; \\ e^{4\pi i/q} \sum_{\sigma \in U_q^n} \sigma_\alpha \overline{\sigma_\beta}\, \overline{\sigma_\gamma} \sigma_\delta, & \alpha = \delta. \end{cases} \tag{.2.16}$$

To get (.2.16), replace σ by the element of U_q^n obtained by multiplying the α component of σ by $e^{2\pi i/q}$. □

Proof of Proposition 2.15. For convenience, assume that we are calculating at a point where $g_{i\bar{j}} = \delta_{ij}$. Fix a $T^{1,0} \otimes T^{1,0}$-tensor h, we want to show

$$(R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}} g_{k\bar{\ell}} - n\, c\, g_{i\bar{j}} g_{k\bar{\ell}}) h^{ik} \overline{h^{j\ell}} \ge 0. \tag{.2.17}$$

Define vectors

$$V_{(\sigma)} = V^i_{(\sigma)} \frac{\partial}{\partial z^i} \quad \text{with components} \quad V^i_{(\sigma)} = \sum_{\lambda=1}^n h^{i\lambda} \overline{\sigma_\lambda} \in \mathbb{C};$$

$$W_{(\sigma)} = W^k_{(\sigma)} \frac{\partial}{\partial z^k} \quad \text{with components} \quad W^k_{(\sigma)} = \sigma_k \in \mathbb{C}.$$

Then, by assumption (.2.13),

$$
\begin{aligned}
0 \le & \sum_{i,j,k,\ell} (R_{i\bar{j}k\bar{\ell}} - c\, g_{i\bar{j}} g_{k\bar{\ell}}) q^{-n} \sum_{\sigma \in U_q^n} V^i_{(\sigma)} \overline{V^j_{(\sigma)}} W^k_{(\sigma)} \overline{W^\ell_{(\sigma)}} \\
= & \sum_{i,j} \sum_{k \neq \ell} R_{i\bar{j}k\bar{\ell}} q^{-n} \sum_{\sigma \in U_q^n} V^i_{(\sigma)} \overline{V^j_{(\sigma)}} \sigma_k \overline{\sigma_\ell} \\
& + \sum_{i,j} \sum_{k=\ell} (R_{i\bar{j}k\bar{\ell}} - c\, g_{i\bar{j}} g_{k\bar{\ell}}) q^{-n} \sum_{\sigma \in U_q^n} V^i_{(\sigma)} \overline{V^j_{(\sigma)}} \sigma_k \overline{\sigma_\ell} \\
= & \sum_{i,j} \sum_{k \neq \ell} R_{i\bar{j}k\bar{\ell}} h^{ik} h^{\overline{j\ell}} + \sum_{i,j,k} (R_{i\bar{j}} - n\, c\, g_{i\bar{j}}) h^{ik} h^{\overline{jk}},
\end{aligned}
$$

where we have made use of Lemma 2.17 on the second equality. Hence

$$
\begin{aligned}
& (R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}} g_{k\bar{\ell}} - n\, c\, g_{i\bar{j}} g_{k\bar{\ell}}) h^{ik} h^{\overline{j\ell}} \\
= & \sum_{i,j} \sum_{k} R_{i\bar{j}k\bar{k}} h^{ik} h^{\overline{jk}} + \sum_{i,j} \sum_{k \neq \ell} R_{i\bar{j}k\bar{\ell}} h^{ik} h^{\overline{j\ell}} + \sum_{i,j,k} (R_{i\bar{j}} - n\, c\, g_{i\bar{j}}) h^{ik} h^{\overline{jk}} \ge 0,
\end{aligned}
$$

since the first term on right hand side is nonnegative by the assumption of nonnegative bisectional curvature. □

.2.2 The vanishing of Futaki invariant

Unfortunately, up to now we still don't know how to directly prove that the positivity of bisectional curvature implies the vanishing of Futaki invariant.

Putting Corollary 1.16, Corollary 2.10 and Theorem 2.16 together gives us the following conclusion (Phong-Song-Sturm-Weinkove [30], Cao-Zhu [8]):

Theorem 2.18. *Let (X^n, g) be a compact Kähler manifold with positive bisectional curvature and vanished Futaki invariant on $\pi c_1(X)$. If we run the NKRF on X, it converges to the Fubini-Study metric. Thus X is biholomorphic to $\mathbb{CP}^n$.*

.3 An Alternative Attempt

It's hard to find the direct relation of bisectional curvature and holomorphic invariant. So we want to try something different. The following result is proved by the author, as the first step of an alternative attempt to a Kähler-Ricci flow proof of Frankel conjecture:

Proposition 2.19. *Let $(X^n, g(t))$ be a compact Fano Kähler manifold with $g(t)$ as the solution to NKRF with positive bisectional curvature. Assume that $\|R(t) - n\|_{L^2}$ converges exponentially fast to 0 along the flow. Then the NKRF converges to the Fubini-Study metric.*

Proof of Proposition 2.19. From Proposition 1.1, the exponential decay of $\|R(t)-n\|_{C^0}$ to 0 implies the convergence of NKRF. Then we just need to show that $\|R(t)-n\|_{C^0}$ is controlled by $\|R(t) - n\|_{L^2}$. The argument resembles the proof of Proposition 1.4 (ii).

We can take $A = \|R - n\|_{C^0} = |R - n|(x_1)$. Then $|R - n| \geq \frac{A}{2}$ on the ball $B_r(x_1)$ of radius $r = \frac{A}{2\|\nabla R\|_{C^0}}$ centered at x_1. If $r < \rho$, where ρ is some fixed uniform radius in Perelman's non-collapsing theorem, then

$$\int_X (R-n)^2\omega^n \geq \int_{B_r(x_1)} \frac{A^2}{4}\omega^n \ \geq \ c\frac{A^2}{4}\left(\frac{A}{2\|\nabla R\|_{C^0}}\right)^{2n} \tag{.3.1}$$

and thus

$$\|R-n\|_{C^0}^{n+1} \ \leq \ C_1\|\nabla R\|_{C^0}^n\|R-n\|_{L^2}. \tag{.3.2}$$

By Cao-Chen-Zhu [7], all curvature tensors are uniformly bounded along NKRF with non-negative bisectional curvature. Due to W. Shi's estimate ([33], Theorem 1.1), the derivatives of curvature tensors are also uniformly bounded. Hence

$$\|R-n\|_{C^0}^{n+1} \ \leq \ C_1\|\nabla R\|_{C^0}^n\|R-n\|_{L^2} \ \leq \ C\,\|R-n\|_{L^2}. \tag{.3.3}$$

On the other hand, if $r > \rho$, then integrating over the ball $B_\rho(x_1)$ gives

$$\int_X (R-n)^2 \omega^n \geq \int_{B_\rho(x_1)} \frac{A^2}{4} \omega^n = \frac{A^2}{4} \int_{B_\rho(x_1)} \omega^n \tag{.3.4}$$

and hence $\|R-n\|_{C^0} \leq C_2 \|R-n\|_{L^2}$.

In either way, the exponential decay of $\|R(t)-n\|_{L^2}$ implies the exponential decay of $\|R(t)-n\|_{C^0}$.

Remark 2.20. *In the above proposition, we actually only used the uniform boundness, but not the positivity of bisectional curvature. More work needs to be done on connecting the condition in Proposition 2.19 and the positivity of bisectional curvature.*

□

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