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Stability of Compact Leaves of Foliations[†‡]

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Introduction

The classical case

Let $\mathscr{F}$ be a foliation of a manifold M. Suppose $\mathscr{F}$ has a compact leaf L; under what conditions is it certain that nearby foliations have nearby compact leaves?

Consider the classical case where $\mathscr{F}$ is the orbit foliation of a smooth flow $\varphi_t: M \to M (t \in \mathscr{R})$. The answer is phrased in terms of the *Poincaré map h* of the closed orbit L. Let $E \subset M$ be a local section of the flow at a point $x_0 \in L$. For each $x \in E$ sufficiently close to x_0, let $h(x) = \varphi_t(x)$, where t is the smallest positive number such that $\varphi_t(x) \in E$. Then h is a diffeomorphism between neighborhoods of x_0 in E with a fixed point at x_0. The classical result states that if the fixed point x_0 is stable, then any sufficiently small perturbation of the flow has a nearby closed orbit L' whose period is near that of L. Moreover, if 1 is not an eigenvalue of the derivatives of h at x_0, then L' is unique. There may be other nearby closed orbits, but they will wrap around L' several times and have much larger periods.

The theorems below generalize the classical result. In place of the Poincaré map of a closed orbit we use the holonomy homomorphism of a compact leaf.

Outline of the chapter

In Section 1, Theorem 1.1 is stated and an intuitive proof given; then variations and applications are presented. In Section 2, the theory of holonomy is developed and topologies for spaces of foliations are defined. The proofs of the theorems appear in Section 3. The last section contains various remarks and problems.

1 Statement of results

Notation

The following notation will be used consistently:

M is a C^∞ manifold of dimension $n + k$, without boundary, endowed with a metric $d(x, y)$ induced by a C^∞ Riemannian metric on M.

$\mathrm{Fol}_k{}^r(M)$ is the space of C^r foliations of M of codimension k; the topology is described in Section 2.

$\mathscr{F}$ denotes an element of $\mathrm{Fol}_k{}^r(M)$.

L is a compact leaf of $\mathscr{F}$ and $x_0 \in L$ is a base point for the fundamental group $\pi_1(L, x_0)$.

α denotes an element of $\pi_1(L, x_0)$. The holonomy of α for $\mathscr{F}$ is a C^r local diffeomorphism $H(\alpha)$ of $(R^k, 0)$. It is well defined up to C^r conjugacy. If $r \geq 1$, the derivative of $H(\alpha)$ at 0 is the linear holonomy $LH(\alpha) \in GL(k)$.

THEOREM 1.1 Assume:

(a) $r \geq 1$;

(b) α is in the center of $\pi_1(L, x_0)$;

(c) 1 is not an eigenvalue of $LH(\alpha)$.

Then there exists $\varepsilon_0 > 0$ with the following properties: If $0 < \varepsilon < \varepsilon_0$, there exists a neighborhood $\mathbf{N} \subset \mathrm{Fol}_k{}^r(M)$ of $\mathscr{F}$ such that for every $\mathscr{F}' \in \mathbf{N}$ there is a compact leaf L' of $\mathscr{F}'$ and a map $h: L \to L'$ satisfying $d(x, h(x)) < \varepsilon$. Moreover, L' is unique, and the map h is a homotopy equivalence.

This can be proved as follows: Let $\{D(x)\}_{x \in L}$ be a family of open k-disks transverse to L, $x \in D(x)$, giving a smooth fibering of a tubular neighborhood of L.

Given $A > 0$ there exists $\delta > 0$ and a neighborhood $\mathbf{N}$ of $\mathscr{F}$ with the following properties: If $\mathscr{F}' \in \mathbf{N}$, $x_0 \in L$, $y_0 \in D(x)$, $d(x_0, y_0) < \delta$, and $u: [0, 1] \to L$ is a smooth path of length $\leq A$ with $u(0) = x_0$, then there exists a unique path $u': [0, 1] \to M$ such that:

(1) u' lies in the $\mathscr{F}'$ leaf of y_0;

(2) $u'(0) = y_0$;

(3) $u'(t) \in D(u(t))$, $0 \leq t \leq 1$.

Moreover, $u'(1)$ is unchanged if u is replaced by a path v of length $\leq A$, homotopic to u rel$\{0, 1\}$.

We call the map $h(u, \mathscr{F}'): y_0 \mapsto u'(1)$ the $\mathscr{F}'$ holonomy of the path u. It is a C^r diffeomorphism from a neighborhood of x_0 in $D(x_0)$ to a neighborhood of x_1 in $D(x_1)$, which *depends continuously on $\mathscr{F}'$*.

Let w be a loop in L based at x_0 representing, α of Theorem 1.1. Then $h(w, \mathscr{F}')$ depends only on α, except for changes of domain. Hypothesis (c) of Theorem 1.1 guarantees that if $\mathscr{F}'$ is close enough to $\mathscr{F}$, then $h(w, \mathscr{F}')$ *will have a unique fixed point close to x_0*. Denote this fixed point by $g(x_0)$.

For every $x \in L$, let u_x be a smooth path in L from x to x_0; let u_{x_0} be the constant path. We may assume a uniform bound $b = $ diameter L for the length of u_x. Let w_x be the loop $u_x^{-1} w u_x$ based at x, in L. Then $h(w_x, \mathscr{F}')$ will have a unique fixed point $g(x) \in D(x)$ near x.

If $\mathscr{F}'$ *is sufficiently close to $\mathscr{F}$, the fixed point $g(x)$ is independent of the choice of u_x*. To see this, let v be another path in L from x to x_0, of length $\leq b$. Put $z_x = v^{-1} w v$. Then z_x is homotopic to

$$(v^{-1} u_x) w_x (v^{-1} u_x)^{-1}.$$

Since $\alpha \in$ center $\pi_1(L, x_0)$, it follows that w_x represents a central element of $\pi_1(L, x)$. Therefore, z_x is homotopic to w_x. Hence $h(z_x, \mathscr{F}')$ $= h(w_x, \mathscr{F}')$ on the intersection of their domains. Uniqueness of fixed points shows that they have the same fixed point.

It follows easily that if x and y are sufficiently close points of L, then $g(x)$ and $g(y)$ are in same leaf L' of $\mathscr{F}'$, and are close in the induced Riemannian metric on L'. Therefore, the map $g: L \to L'$ is continuous. This suffices to make L' compact. Uniqueness of L' follows from uniqueness of fixed points.

The argument just given can be made precise. The most serious difficulty is the necessity of finding an a priori bound on the lengths of all paths involved. In addition, it relies too much on uniqueness of fixed

points, making generalization difficult. A more rigorous proof is given in Section 3.

Further results

Just as with flows, there are stability theorems without uniqueness if no assumption is made on the linear holonomy. The theorems below require drastic restrictions on the foliation, which may turn out to be unnecessary.

An isolated fixed point of a local homeomorphism of $(R^k, 0)$ is *essential* if it has nonzero index.

THEOREM 1.2   Suppose:

(a) $k \leq 2$ and $r = 0$;
(b) $\alpha \in \text{center } \pi_1(L, x_0)$;
(c) $H(\alpha)$ has an essential isolated fixed point at x_0.

Let $U \subset M$ be a neighborhood of L. Then there exists a neighborhood $\mathbf{N} \subset \text{Fol}_k{}^0(M)$ of $\mathscr{F}$ with the following properties:

(d)   every $\mathscr{F}' \in \mathbf{N}$ has a compact leaf in U if $k = 1$;
(e)   every $\mathscr{F}' \in \mathbf{N}$ which is transversely real analytic has a compact leaf in U if $k = 2$.

For arbitrary codimension and C^0 perturbations, there is a much weaker conclusion:

THEOREM 1.3   Suppose:

(a) $r = 0$ and k arbitrary;
(b) $\alpha \in \text{center } \pi_1(L, x_0)$;
(c) $H(\alpha)$ has an essential isolated fixed point at x_0.

Let $U \subset M$ be a neighborhood of L. Then there exists a neighborhood $\mathbf{N} \subset \text{Fol}_k{}^0(M)$ such that every $\mathscr{F}' \in \mathbf{N}$ has a leaf entirely contained in U.

Assumptions about the Fundamental Group of L

The hypothesis that α is central can be weakened. An element α of a group G is *accessible* if there are subgroups

$$G_0 \subset \cdots \subset G_q \subset G,$$

such that α generates G_0, G_{i-1} is normal in G_i ($i = 1, \ldots, q$), and G_q

has finite index in G. If $G_q = G$, α is *directly accessible*. If G is nilpotent, every element is directly accessible.

THEOREM 1.4   Theorems 1.1 and 1.2(d) are valid if (b) is replaced by: α is accessible. In particular, they are valid if (b) is replaced by: α belongs to a nilpotent subgroup of finite index.

For Theorems 1.2(e) and 1.3, there is a weaker improvement.

THEOREM 1.5   Theorems 1.2(e) and 1.3 are valid if (b) is replaced by: α belongs to the center of a subgroup of finite index.

Proof   Apply Theorem 1.2(e) or 1.3 to the covering space of a tubular neighborhood of L corresponding to the subgroup of finite index.

By passing to covering spaces and replacing α by a power we obtain:

THEOREM 1.6   The conclusions of Theorem 1.1 are valid under the following assumptions:

(a) $r \geq 1$;
(b) $\pi_1(L, x_0)$ has a nilpotent subgroup of finite index p;
(c)   the spectrum of $LH(\alpha)$ does not contain any mth root of unity, $1 \leq m \leq p$.

THEOREM 1.7   The conclusion of Theorem 1.2(d) is valid under the following assumptions:

(a) $k = 1$ and $r = 0$;
(b) $\pi_1(L, x_0)$ has a nilpotent subgroup of finite index p;
(c) $H(\alpha^m)$ has an essential isolated fixed point at x_0, $1 \leq m \leq p$.

Similar variations of Theorems 1.2(e) and 1.3 are true. They are left to the reader, as are the proofs of Theorems 1.6 and 1.7.

The stability theorem of Reeb

The following result, a part of Reeb's theorem [8, B, II, 21], can be proved by the same techniques used for the preceding results.

THEOREM 1.9   Let r and k be arbitrary and $\pi_1(L)$ finite. Given a neighborhood $U \subset M$ of L, there exist neighborhoods $V \subset M$ of L, and $\mathbf{N} \subset \mathrm{Fol}_k{}^r(M)$ of $\mathscr{F}$, such that if $\mathscr{F}' \in \mathbf{N}$, then every leaf of $\mathscr{F}'$ meeting V is compact and simply connected and is contained in U.

The foliations of Reeb and Lawson

Theorem 1.2 can be applied to the celebrated Reeb foliation of the 3-sphere S^3 (see [8, p. 122]). This foliation $\mathscr{R} \in \mathrm{Fol}_1^\infty(S^3)$ has a unique compact leaf $T^2 \subset S^3$, diffeomorphic to the 2-torus embedded in the usual way. There are generators α_1, $\alpha_2 \in \pi_1(T^2)$ represented by loops bounding disks, one in each of the components of $S^3 - T^2$. The holonomy $H(\alpha_i)$ is represented by a C^∞ local diffeomorphism f_i of $(R^1, 0)$ such that

$$f_1(x) \begin{cases} = x, & \text{for} \quad x \leq 0, \\ < x & \text{for} \quad x > 0, \end{cases}$$

and

$$f_2(x) \begin{cases} = x & \text{for} \quad x \geq 0, \\ > x & \text{for} \quad x < 0. \end{cases}$$

Let $\alpha = \alpha_1 + \alpha_2 \in \pi_1(T^2)$. Then $H(\alpha)$ is represented by $f = f_1 \circ f_2$ and $|f(x)| < |x|$, for $x \neq 0$. Thus $H(\alpha)$ has an essential isolated fixed point at 0.

THEOREM 1.10   Let $N \subset S^3$ be an open neighborhood of T^2 and let $\mathscr{R}_N$ denote the foliation of N induced by the Reeb foliation $\mathscr{R}$. There is a neighborhood $\mathbf{N} \subset \mathrm{Fol}_1^0(N)$ of $\mathscr{R}_N$ such that every foliation in $\mathbf{N}$ has a compact leaf.

Proof   Apply Theorem 1.2 with $M = U = N$.

Similar reasoning applies to the Lawson foliation $\mathscr{L} \in \mathrm{Fol}_1^\infty(S^5)$ (see Lawson [6]). The fundamental group of the unique compact leaf lies in an exact sequence

$$0 \to Z \times Z \times Z \to \pi_1(L) \to Z_3 \to 0,$$

and there exists $\alpha \in \pi_1(L)$ such that $H(\alpha)$ has a contracting fixed point at 0. We may choose such an element in $Z \times Z \times Z$, replacing α by α^3, if necessary. Therefore α is accessible and we obtain:

THEOREM 1.11   The conclusion of Theorem 1.10 is valid if S^3, T^2, and $\mathscr{R}$ are replaced by S^5, L, and $\mathscr{L}$.

Lawson has also constructed a C^∞ codimension 1 foliation $\mathscr{L}(n)$ of S^n for $n = 2^p + 3$, $p = 2, 3, \ldots$. Each of these has a unique compact

leaf L diffeomorphic to

$$S^1 + SO((n+1)/2)/SO((n-1)/2),$$

which has fundamental group Z. The holonomy of the generator $\alpha \in \pi_1(L)$ has a contracting fixed point at 0. Therefore Theorem 1.10 applies to $\mathscr{L}(n)$.

Commuting diffeomorphisms

The following result comes out of the proof of Theorem 1.2.

THEOREM 1.12   A family of real analytic diffeomorphisms of an open or closed 2-disk has a common periodic point, provided one of them, f_0, has a nonempty compact fixed point set and is not the identity. In fact, there is a common periodic point which is a fixed point of f_0.

2   Holonomy

The topology of $\mathrm{Fol}_k{}^r(M)$

Let V, W be C^r p-manifolds. A C^r *local diffeomorphism* $f: V \to W$ means a C^r diffeomorphism

$$f: D(f) \approx R(f),$$

where $D(f) \subset V$ and $R(f) \subset W$ are open sets. $D(f)$ and $R(f)$ are called the *domain* and *range* of f, respectively. Any two local diffeomorphisms can be composed: the composition $f'f$ of f with $f': V' \to W'$ is defined to be the map

$$D(f'f) = f^{-1}(R(f) \cap D(f')) \to f'(R(f) \cap D(f')) = R(f'f),$$
$$x \mapsto f'(f(x)).$$

The empty map $\varnothing$, $D(\varnothing) = R(\varnothing) = \varnothing$, is allowed. The set of all C^r local diffeomorphisms from V to W is denoted by $\mathrm{Loc}^r(V, W)$, or $\mathrm{Loc}^r(V)$ if $W = V$.

Let $I^k(t) = \{x \in R^k \mid -t < x_i < t,\ i = 1, \ldots, k\}$ and put $I^k(1) = I^k$ = open unit cube in R^k.

Given a C^r local diffeomorphism $f \in \mathrm{Loc}^r(I^k)$, let $\|f\|_r$ denote the usual C^r norm of f, i.e., the supremum of all the numbers $|g(x)|$, where x ranges over $D(f)$ and g ranges over the coordinate functions of f and their partial derivatives of order $\leq r$. If $f = \varnothing$, put $\|\varnothing\| = 1$.

Define a pseudometric on $\text{Loc}^r(I^k)$ as follows: $d(f, g)$ is the maximum of the four numbers

$$\text{Haus}(D(f), D(g)), \qquad \text{Haus}(R(f), R(g)),$$
$$\min\{\| (f - g) \mid D(f) \cap D(g) \|_r, 1\},$$
$$\min\{\| (f^{-1} - g^{-1}) \mid R(f) \cap R(g) \|_r, 1\}.$$

Here $\text{Haus}(X, Y)$ denotes the *Hausdorff distance* between sets X, $Y \subset I^k$:

$$\text{Haus}(X, Y) = \begin{cases} \max\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\} \\ \qquad \text{if} \quad X \neq \varnothing \neq Y, \\ 0 \quad \text{if} \quad X = Y = \varnothing, \\ 1 \quad \text{otherwise.} \end{cases}$$

Observe that $d(f, g) = 0$ means that $D(f) \cap D(g)$ is dense in $D(f)$ and in $D(g)$, $R(f) \cap R(g)$ is dense in $R(f)$ and in $R(g)$, and $f = g$ on $D(f) \cap D(g)$. If $D(f)$ is the interior of its closure, and also $D(g)$ is the interior of its closure, then $d(f, g) = 0$ implies $f = g$.

A *chart* (Ω, f) on the C^r manifold M is a C^r diffeomorphism $f: \Omega \approx I^{n+k}$, where $\Omega \subset M$ is an open set. The distance between charts is defined as

$$d((\Omega, f), (\Omega_1, f_1)) = d(I, f_1 f^{-1}),$$

where I denotes the identity map of I^k. The d on the right is the pseudo-metric in $\text{Loc}^r(I^k)$. With this distance function, the set of charts on M is a complete pseudometric space.

A C^r *foliation* of codimension k on M is decomposition $\mathscr{F}$ of M into disjoint connected sets called *leaves*, having the following property: There is a covering of M by charts (Ω, f), $f: \Omega \approx I^n \times I^k$, such that $f^{-1}(I^n \times y)$ is a connected component of the intersection of Ω with a leaf, for all $y \in I^k$. The set $f^{-1}(I^n \times y)$ is an Ω *plaque* of $\mathscr{F}$, Ω is an $\mathscr{F}$ *domain*, and (Ω, f) is an $\mathscr{F}$ *chart*.

An $\mathscr{F}$ chart (Ω, f) is *regular* if there exists an $\mathscr{F}$ chart (Λ, g) such that $\bar{\Omega} \subset \Lambda$ and $\Omega = g^{-1}(I^{n+k}(\frac{1}{2}))$. This implies $\Omega = \text{int } \bar{\Omega}$.

A plaque of a regular $\mathscr{F}$ chart is a *regular* plaque. The closure of a regular plaque is a compact subset of the leaf containing it and is contained in another plaque.

A covering of M by $\mathscr{F}$ charts can be shrunk to a covering by regular $\mathscr{F}$ charts.

Let $\mathscr{F}$, $\mathscr{F}' \in \text{Fol}_k^r(M)$. Let S be a finite set of $\mathscr{F}$ charts, and suppose $\varepsilon > 0$. We say $\mathscr{F}'$ is an (S, ε) *perturbation* of $\mathscr{F}$ if there exists a

set S' of $\mathscr{F}'$ charts and a bijection $S \to S'$, denoted by $(\Omega, f) \mapsto (\Omega', f')$, such that

$$d((\Omega, f), (\Omega', f')) < \varepsilon \qquad \text{for all} \quad (\Omega, f) \in S.$$

This is indicated by

$$d(\mathscr{F}, S; \mathscr{F}', S') < \varepsilon.$$

If the charts in S' are regular, $\mathscr{F}'$ is a *regular* (S, ε) perturbation.

The topology on $\mathrm{Fol}_k{}^r(M)$ is generated by sets of the form $\mathbf{N}(\mathscr{F}, S, \varepsilon)$ = the set of all regular (S, ε) perturbations of $\mathscr{F}$, where S is a finite set of regular $\mathscr{F}$ charts. If $r > 0$, this topology can also be defined by uniform C^{r-1} convergence on compact subsets of M of tangent planes of leaves.

The topology on $\mathrm{Fol}_k{}^r(M)$ is Hausdorff. In fact, it is not hard to prove:

THEOREM 2.1   The space $\mathrm{Fol}_k{}^r(M)$ has a complete metric.

Coherent sets of charts

A set S of $\mathscr{F}$ domains is *coherent* if whenever $\Omega_1, \Omega_2, \Omega_3 \in S$ and $\Omega_1 \cup \Omega_2 \cup \Omega_3$ is connected, there exists an $\mathscr{F}$ domain Ω, not necessarily in S, such that $\Omega_1 \cup \Omega_2 \cup \Omega_3 \subset \Omega$ and $P \cap \Omega_i$ is an Ω_i plaque for every Ω plaque P ($i = 1, 2, 3$).

A set of $\mathscr{F}$ charts is *coherent* if the corresponding set of $\mathscr{F}$ domains is coherent.

LEMMA 2.2   (a)   Let $K \subset M$ be a compact set and S_0 a set of $\mathscr{F}$ domains whose union contains K. Then there is a finite coherent set of $\mathscr{F}$ domains which refines S_0 and whose union contains K.

(b)   Let S be a finite coherent set of $\mathscr{F}$ charts. There exists $\varepsilon > 0$ such that if $d(\mathscr{F}, S; \mathscr{F}', S') < \varepsilon$ then S' is a coherent set of $\mathscr{F}'$ charts.

Proof   Compare Reeb [8: B,II,20].

Let S be a set of $\mathscr{F}$ charts. A *chain in S* means a sequence of charts $\omega = ((\Omega_0 f_0), \ldots, (\Omega_m, f_m))$ such that each $(\Omega_1, f_i) \in S$ and $\Omega_i \cap \Omega_{i-1} \neq \varnothing$, $i = 1, \ldots, m$. A point $x \in \Omega_0$ *admits* ω if there are Ω_i plaques P_i, $i = 0, \ldots, m$, such that $x \in P_0$ and $P_{i-1} \cap P_i \neq \varnothing$, $i = 1, \ldots, m$. We also say P_0 admits ω. We call $(P_0, \ldots, P_m)$ a *plaque chain contained in* ω.

LEMMA 2.3   Let S be a coherent set of $\mathscr{F}$ charts and ω an S chain. If P_0 admits ω, then P_m is unique.

Proof By induction on m; the inductive step reduces to the case $m = 1$. Since $P_0 \cap P_1 \neq \varnothing$, there is a plaque P of some $\mathscr{F}$ domain such that $P \cap \Omega_i = P_i$. Any other Ω_1 plaque P_1' meeting P_0 is contained in the same leaf as P_0; hence $P \cap \Omega_1 = P_1' = P_1$.

LEMMA 2.4   Let x admit ω. There exists $\varepsilon > 0$ such that if $d(\mathscr{F}, \omega;$ $\mathscr{F}', \omega') < \varepsilon$, and $d(x, x') < \varepsilon$, then x' admits ω'.

Holonomy

Let (Ω, f) be an $\mathscr{F}$ chart. The set $\Omega/\mathscr{F}$ of Ω plaques is identified with I^k by the map

$$f_* : \Omega/\mathscr{F} \to I^k, \qquad P \mapsto \pi f(P),$$

where $\pi : I^n \times I^k \to I^k$ is the projection. We give $\Omega/\mathscr{F}$ the C^r differential structure that makes f_* a C^r diffeomorphism.

For every coherent chain $\omega = ((\Omega_0, f_0), \ldots, (\Omega_m, f_m))$ of $\mathscr{F}$ we define a C^r local diffeomorphism

$$G(\omega) : \Omega_0/\mathscr{F} \to \Omega_m/\mathscr{F},$$

called the *geometric holonomy* of ω. Let $DG(\omega)$ be, the set of Ω_0 plaques P_0 admitting ω; define $G(\omega)P_0 = P_m$, where $(P_0, \ldots, P_m)$ is the unique plaque chain contained in ω beginning with P_0. Note that $G(\omega)$ depends only on $\Omega_0, \ldots, \Omega_m$, not on $f_0, \ldots, f_m$.

The *algebraic holonomy* of ω is the C^r local diffeomorphism

$$H(\omega) = (f_{m*})G(\omega)(f_{0*})^{-1} : I^k \to I^k.$$

Note that $H(\omega)$ is independent of $f_1, \ldots, f_{m-1}$.

Equivalently, $H(\omega)$ is the composition $u_m \cdots u_1$ of the C^r local diffeomorphism

$$u_i : I^k \approx I^k \times 0 \subset I^n \times I^k \xrightarrow{f_i f_{i-1}^{-1}} I^n \times I^k \xrightarrow{\pi} I^k.$$

This is well defined because ω is coherent.

LEMMA 2.5   Let S be a finite coherent set of regular $\mathscr{F}$ charts and T a finite set of chains in S. Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(\mathscr{F}, S; \mathscr{F}', S') < \delta$, then

$$d(H(\omega), H(\omega')) < \varepsilon,$$

for every $\omega \in T$ (where ω' is the chain in S' corresponding to ω).

Homotopy

Let S be a coherent set of $\mathscr{F}$ charts. An *elementary expansion* (ω, ω') in S is a pair of S chains such that ω' is obtained from $\omega = ((\Omega_0, f_0), \ldots,$ $(\Omega_m, f_m))$ by inserting a chart $(\Omega, f) \in S$ between (Ω_{j-1}, f_{j-1}) and (Ω_j, f_j), for some $j \in \{1, \ldots, m\}$, such that $\Omega_{j-1} \cap \Omega \cap \Omega_j \neq \varnothing$. We call ω and ω' *contiguous* if either (ω, ω') or (ω', ω) is an elementary expansion.

A *homotopy* in S is a sequence of S chains $\omega_0, \ldots, w_q$ such that ω_i and ω_{i-1} are contiguous, $i = 1, \ldots, q$. We call ω_0 and ω_q *homotopic* in S. Note that ω_q and ω_0 are coterminous.

From the coherence of S it follows that if ω and ω' are contiguous then

$$G(\omega) = G(\omega') \quad \text{on} \quad DG(\omega) \cap DG(\omega'),$$

and similarly for $H(\omega)$ and $H(\omega')$. More generally,

LEMMA 2.6   Let S be a finite coherent set of regular $\mathscr{F}$ charts and $\omega_0, \ldots, \omega_q$ a homotopy in S. Let B be a compact set contained in

$$DH(\omega_0) \cap \cdots \cap DH(\omega_q).$$

There exists $\delta > 0$ such that if $d(\mathscr{F}, S; \mathscr{F}', S') < \delta$ then:

(a) $\omega_0', \ldots, \omega_q'$ is a homotopy in S';
(b) $B \subset DH(\omega_0') \cap \cdots \cap DH(\omega_q')$;
(c) $H(\omega_0') = H(\omega_q')$ on B.

The compactness criterion

Let S be a set of $\mathscr{F}$ charts and $(\Omega_0, f_0) \in S$ a particular chart considered as a base point. A *loop* in S (at (Ω_0, f_0)) is an S chain ω that starts and ends at (Ω_0, f_0).

Let $\bigcup S$ denote the union of the domains of charts in S. Suppose S is coherent. A set of plaques $\Gamma \subset \Omega_0 / \mathscr{F}$ is *invariant under* ω if $\Gamma \subset DG(\omega)$ and $G(\omega)(\Gamma) = \Gamma$. The notion of a set $\Gamma \subset I^k$ being invariant under $H(\omega)$ is similarly defined.

LEMMA 2.7   Let S be a coherent set of regular $\mathscr{F}$ charts having finite cardinality m. Let $(\Omega_0, f_0) \in S$ be the base point, and $\Gamma \subset \Omega_0 / \mathscr{F}$ a set of plaques. Suppose:

(a) Γ is invariant under every $2m$ loop in S;
(b)   If P is a plaque of a chart in S then $\bar{P} \subset \bigcup S$.

Then every plaque P in Γ admits every S chain and the leaf L containing P lies entirely in $\bigcup S$. If Γ is finite, L is closed in M. If Γ is finite and $\bigcup S$ has compact closure in M, L is compact.

Proof By (a), every plaque in Γ admits every m-chain in S.

Let $P_0 \in \Gamma$ and L be the leaf containing P. By (b), for every $x \in L$ there is a chain of plaques $P_0, \ldots, P_q$ contained in an S chain, with $x \in P_q$ and $P_{i-1} \cap P_i \neq \varnothing, i = 1, \ldots, q$. Let $q(x)$ be the minimal such q. To prove $L \subset \bigcup S$ it suffices to prove that $q(x) \leq m$ for all x. It is enough to prove that if $q(x) \leq m + 1$, then $q(x) \leq m$: for then the first $m + 1$ elements of any S chain of length $>m$ can be replaced by a shorter chain, and this process can be repeated until a chain of length m is obtained.

Suppose then that $P_0, \ldots, P_{m+1}$ is a plaque chain contained in an S chain $(\Omega_0, \ldots, \Omega_{m+1})$, and $x \in P_{m+1}$. (We suppress the f_i, since they play no role.) Since S has only m elements there must exist $0 \leq i < j \leq m + 1$ with $\Omega_i = \Omega_j$.

Consider the S loop $\Omega_0, \ldots, \Omega_j, \Omega_{i-1}, \ldots, \Omega_0$, of length $i + j \leq 2m$. By (a) it contains a plaque chain

$$P_0, \ldots, P_j, P'_{i-1}, \ldots, P_0',$$

and $P_0' \in \Gamma$. This shows that P_j is connected to P_0' by the $i - 1$ chain $P_0', \ldots, P'_{i-1}$. Therefore, $P_0' \in \Gamma$ is connected to P_{m+1} by the plaque chain

$$P_0', \ldots, P'_{i-1}, P_j, \ldots, P_{m+1},$$

which is contained in the S chain $\Omega_0, \ldots, \Omega_{i-1}, \Omega_j, \ldots, \Omega_{m+1}$ whose length is $m + i - j + 1 \leq m$.

This shows $L \subset \bigcup S$. It also shows that if Γ is finite then L is contained in a finite union of plaques of S. Since a leaf contains the closure of every regular plaque in it, this shows that L is closed in M. If also $\bigcup S$ has compact closure, L is compact.

Charts for a compact leaf

LEMMA 2.8 Let L be a compact leaf of $\mathscr{F}$ and $U \subset M$ a neighborhood of L. Then there exists a finite coherent set S of regular $\mathscr{F}$ charts such that:

(a) $L \subset \bigcup S \subset U$;
(b) if $\Omega_1, \Omega_2, \Omega_3 \in S$ and $\Omega_1 \cap \Omega_2 \cap \Omega_3 \neq \varnothing$, then $\Omega_1 \cap \Omega_2 \cap \Omega_3 \cap L \neq \varnothing$;
(c) $\Omega \cap L$ is a plaque of Ω for all $\Omega \in S$.

Compare Reeb [8: B,I,1].

LEMMA 2.9 Let S be as in Lemma 2.8. There exists $\delta > 0$ such that if $d(\mathscr{F}, S; \mathscr{F}', S') < \delta$ then Lemma 2.8(a) and (b) are valid with S' replacing S.

Paths and chains

A path $u: [0, 1] \to M$ is *contained* in a chain $(\Omega_0, \ldots, \Omega_p)$ if there is a subdivision

$$0 = t_0 < \cdots < t_{p+1} = 1,$$

such that $u([t_i, t_{i+1}]) \subset \Omega_i$, $i = 0, \ldots, p$.

LEMMA 2.10 Let S be a coherent set of $\mathscr{F}$ domains.

(a) If $\omega = (\Omega_0, \ldots, \Omega_p)$ is a chain in S then ω contains a path. If $x \in \Omega_0$ admits ω, then ω contains a path in the leaf of x, starting from x;
(b) Let ω_0, ω_1 be coterminous S chains. Let u_0, u_1 be coterminous paths contained in ω_0, ω_1, respectively. Then u_0 and u_1 are homotopic in $\bigcup S$ (rel end points) if and only if ω_0 and ω_q are homotopic in S;
(c) Let S, L be as in Lemma 2.8. If in (b), u_0 and u_1 lie in L and ω_0 and ω_1 are homotopic in S, then u_0 and u_1 are homotopic in L.

Outline of proof (a) is apparent. If u_0 and u_1 of (b) are homotopic in $\bigcup S$, the homotopy can be replaced by a succession of small homotopies, each taking place inside a single domain of S. The succession of domains leads to a homotopy from ω_0 to ω_1. If ω_0 and ω_1 are contiguous, coherence of S is used to prove u_0 and u_1 homotopic, from which (b) follows. Likewise for (c).

The set $\pi_1(S, (\Omega_0, f_0))$ of all S loops based at (Ω_0, f_0) is a group under the obvious composition.

LEMMA 2.11 Let L, S be as in Lemma 2.8. Let $x_0 \in L$ and $(\Omega_0, f_0) \in S$ be base points. The function assigning to every loop in L a loop in S containing it induces an isomorphism from the fundamental group $\pi_1(L, x_0)$ to the group $\pi_1(S, (\Omega_0 f_0))$ of homotopy classes of S loops.

Proof Follows from Lemma 2.10.

3 Proofs of theorems

Let L be a compact leaf of $\mathscr{F}$. Let S be a finite coherent set of regular $\mathscr{F}$ charts of cardinality m as in Lemma 2.8, such that $f(\Omega \cap L) = 0 \in I^k$ for all $(\Omega, f) \in S$. Let (f_0, Ω_0) be the base point. Let $(\mathscr{F}', S')$ denote a

regular (S, δ) perturbation of $(\mathscr{F}, S)$ for some $\delta > 0$. The proofs are based on the following principle:

LEMMA 3.1   There exists $\delta_0 > 0$ with the following properties: If $0 < \delta \leq \delta_0$ and there is a point $y_0 \in I^k$ fixed under $H(\omega)$ for every S' loop ω of length $2m$, then the $\Omega_0{}'$ plaque $\Gamma = (f_{0*}')^{-1}(y_0)$ is contained in a compact $\mathscr{F}\,'$ leaf. If there is a subset $Z \subset I^k$ invariant under every S' loop ω of length $2m$, then the $\mathscr{F}\,'$ leaves containing $(f_{0*}')^{-1}(Z)$ are contained in $\bigcup S'$.

Proof   Follows from Lemma 2.7, since (b) of Lemma 2.7 is preserved by small (S, δ) perturbations of $\mathscr{F}$.

Proof of Theorem 1.1 as amended in Theorem 1.4

By passing to a suitable finite covering space of a neighborhood of L we may assume $\alpha \in \pi_1(L, x_0)$ is directly accessible. Let

$$G_0 \subset \cdots \subset G_\nu = \pi_1(L, x_0)$$

be subgroups, each normal in the next, with G_0 generated by α.

Let $\varphi : \pi_1(L, x_0) \to \pi_1(S, (\Omega_0, f_0))$ be the isomorphism described in Lemma 2.11.

Let Λ_j be the set of all loops in S that represent elements of $\varphi(G_j)$, $j = 0, \ldots, \nu$.

Let G_j' be the set of S' chains corresponding to G_j.

Let λ be an S loop representing $\varphi(\alpha)$. Since 1 is not an eigenvalue of $LH(\lambda)$, there is a compact neighborhood B of 0 in I^k such that $B \subset DH(\lambda) \cap RH(\lambda)$ and 0 is the only fixed point of $H(\lambda)$ in B. Given any neighborhood $B_0 \subset B$ of 0 we can make δ so small that:

(1) $B \subset DH(\lambda') \cap RH(\lambda')$;

(2) $H(\lambda')$ has a unique fixed point $y_0 \in B$;

(3) $y_0 \in B_0$.

The following statement will be proved by recursion on $j = 0, \ldots, \nu$:

$(*)_j$:   Let $T_j \subset \Lambda_j$ be a finite subset. If δ is small enough then y_0 is fixed under ω for all $\omega \in T_j'$. T_j' is the set of S' loops corresponding to T_j under the bijective map $S \to S'$.

Once $(*)_\nu$ is proved, the existence of a compact $\mathscr{F}\,'$ leaf follows from Lemma 3.1, taking T_ν to be the set of all $2m$-loops in S.

For the case $j = 0$, let $T_0 \subset \Lambda_0$ be a finite set of S loops. We may assume T_0 closed under inversion of loops. For each $\tau \in T_0$ choose a

homotopy $h(\tau)$ in S from λ to $\tau^{-1}\tau$:

$$h(\tau) = \omega_0(\tau), \ldots, \omega_q(\tau) \qquad q = q(\tau);$$

each $\omega_i(\tau)$ is an S loop; $\omega_0(\tau) = \lambda$ and $\omega_q(\tau) = \tau^{-1}\lambda\tau$.

Put $E = \{\omega_i(\tau) \mid \tau \in T_0, \; 0 \le i \le q(\tau)\}$. Then E is a finite set of S loops. If $x \in L$ and ω is any S loop, then x admits ω, by Lemma 2.8. Hence by Lemma 2.4, if δ is small enough there is a compact neighborhood B_0 of 0 in I^k such that if $x \in \Omega_0$ and $\pi f_0(x) \in B_0$, then x admits every loop in E. Moreover, since $H(\omega) = 0$ for every S loop ω, we may take δ so small and B_0 so small that for all S' loops $\omega \in E'$ and $\tau \in T_0'$:

(4) $B_0 \subset DH(\omega) \cap DH(\tau^{-1}\lambda'\tau) \cap DH(\lambda')$;
(5) $H(\tau^{-1}\lambda'\tau)b_0 \cup H(\lambda')B_0 \subset B$.

Moreover by Lemma 2.6 we can assume, for all $\tau \in T_0'$:

(6) $H(\tau^{-1}\lambda'\tau) \mid B_0 = H(\lambda') \mid B_0$.

Therefore by (3), $H(\tau^{-1}\lambda'\tau)y_0 = y_0$, and so, using (1),

$$H(\lambda')H(\tau)y_0 = H(\tau)y_0 \qquad \text{for all} \quad \tau \in T_0'.$$

By (2) $H(\tau)y_0 = y_0$. We have proved $(*)_0$.

Now suppose $j > 0$ and $(*)_{j-1}$ is true. Given a finite set of S loops $T_j \subset \Lambda_j$, define

$$T_{j-1} = \{\tau^{-1}\lambda\tau \mid \tau \in T_j\}.$$

Since G_{j-1} is normal in G_j, we have $T_{j-1} \subset \Lambda_{j-1}$. Choose δ so small that, for all $\tau \in T_j'$,

(7) $y_0 \in B \cap DH(\tau)$;
(8) $H(\tau)y_0 \in B$;

and so small that by $(*)_{j-1}$: $H(\tau^{-1}\lambda'\tau)y_0 = y_0$. Since $y_0 \in DH(\tau)$, this implies $H(\lambda')H(\tau)y_0 = H(\tau)y_0$. By (8) and (2),

$$H(\tau)y_0 = y_0.$$

Therefore $(*)_j$ is true for $j = 0, \ldots, \nu$, and Theorem 1.1 is proved.

To prove the uniqueness statement of Theorem 1.1., we show that if $h: L \to L'$ maps L onto a leaf of $\mathscr{F}'$ and h is close enough to the inclusion, then $\Gamma \subset \Omega_0' \cap L'$, where Γ is the plaque of Lemma 3.1. We may assume that for each $\mathscr{F}$ domain Ω of S, $h(\Omega \cap L)$ intersects a single Ω' plaque $P(\Omega) \subset L'$. The map of plaques $\Omega \cap L \mapsto P(\Omega)$

takes plaque chains of S into plaque chains of S'. Let $\lambda = (\Omega_0, \ldots, \Omega_p = \Omega_0)$. Then the S' loop $\lambda' = (\Omega_0', \ldots, \Omega_p' = \Omega_0')$ contains the S' loop of plaques

$$(P(\Omega_0), \ldots, P(\Omega_p) = P(\Omega_0)).$$

This means that the plaque $P(\Omega_0) \subset L' \cap \Omega_0'$ is fixed under $G(\lambda')$. Since $G(\lambda')$ has a unique fixed point, $P(\Omega_0) = \Gamma$.

It remains to prove the existence of an ε-map $h: L \to L'$ which is also a homotopy equivalence. This is easy in the C^r case, $r \geq 1$, using the fibers $\{D_x\}_{x \in L}$ of a smooth tubular neighborhood of L: define $h(x) = D_x \cap L'$. In this case, h is actually a C^r diffeomorphism.

When $r = 0$, let N (respectively, N') be the simplicial complex which is the nerve of the family of domains of S (respectively, S'). The correspondence $S \to S'$ induces a simplicial isomorphism $g: N \to N'$ if δ is small enough. There are maps $u: L \to N$ and $u': L' \to N'$ such that $u(x)$ belongs to the simplex $(\Omega_0, \ldots, \Omega_s)$ whenever $x \in \Omega_0 \cap \cdots \cap \Omega_s$, and similarly for u'. There are also maps $v: N \to L$, $v': N' \to L'$ such that $vu: L \to L$ takes each simplex $(\Omega_0, \ldots, \Omega_s)$ into $\Omega_0 \cup \cdots \cup \Omega_s$ and similarly for $v'u'$ (see Milnor [7], p. 279). The maps vu and $v'u'$ are homotopic to identity maps.

The composite map

$$h: L \xrightarrow{\;u\;} N \xrightarrow{\;g\;} N' \xrightarrow{\;v'\;} L'$$

will be an ε-map if the domains of S are sufficiently small.

The map

$$h': L' \xrightarrow{\;u'\;} N' \xrightarrow{\;g^{-1}\;} N \xrightarrow{\;v\;} L$$

will be a homotopy inverse to h. To see this we must prove hh' and $h'h$ homotopic to identity maps. We do this for $h'h$, the other case being analogous. It suffices to prove $uh'h: L \to N$ homotopic to u, since $vu \simeq 1_L$. But

$$uh'h = uvg^{-1}u'v'gu \simeq uvg^{-1}gu = uvu \simeq u.$$

This completes the proof of Theorem 1.1 as amended in Theorem 1.4.

Proof of Theorem 1.2 as amended in Theorem 1.4

First suppose $k = 1$. Passing to a double covering if necessary, we assume $\mathscr{F}$ and $\mathscr{F}'$ transversely oriented.

Let $F \subset I^1$ be the nonempty compact set of fixed points of $H(\lambda')$. Since 0 is an isolated and essential fixed point of $H(\lambda)$, by taking δ small enough we may assume $F \subset B_0$, where (1) above is satisfied.

Put inf $F = y_0 \in F$, sup $F = y_1 \in F$. We may assume $[y_0, y_1] \subset B_0$.

We prove inductively that $(*)_j$ above is valid. The proof for $j = 0$ is similar to the previous one; the point is that since $H(\lambda') = H(\tau^{-1}\lambda'\tau)$ on B_0, it must be true that $H(\tau)y_0 \in F \subset [y_0, y_1]$, for all $\tau \in T_0'$. But $H(\tau)y_0 > y_0$ is impossible, for since $\mathcal{F}'$ is transversely oriented, $H(\tau^{-1})$ preserves order, so we would have $y_0 > H(\tau^{-1})y_0$. But $H(\tau^{-1})y_0 \in F$, since we assumed T_0 closed under inversion. Hence $H(\tau)y_0 = y_0$. The inductive step is similar. The rest of the proof of Theorem 1.2, $k = 1$, is similar to the proof of Theorem 1.1.

Theorem 1.2, $k = 2$ is a consequence of Lemma 3.1 and the following result.

LEMMA 3.2   Let $D \subset R^2$ be an open disk. Let $\{h_i\}$ be a family of real analytic diffeomorphisms of D onto open sets $U_i \subset R^2$. Suppose there is an h_0 in the family such that, for all i, j:

(a)   The fixed point set F of h_0 is compact and nonempty;

(b) $h_i(F) \subset D$;

(c) $h_i h_j(x) = h_j h_i(x)$ if $x \in h_i^{-1}D \cap h_j^{-1}D$.

Then there is a common periodic point for all the h_i.

Proof Since F is a compact real analytic variety, it is a finite simplicial complex [4, 5] without interior. By (b) and (c), $h_1, \ldots, h_q$ leave F invariant. Since the h_i leave invariant the finite set of vertices that belong to more than two simplices, the lemma must be true if F is not a finite disjoint union of circles.

Suppose F is a finite disjoint union of circles; let C be an innermost one. Replacing each h_i by an iterate if necessary, we may assume they leave invariant the closed disk B bounded by C.

By Brouwer's fixed point theorem, each h_i has a fixed point in B. Let $F_i \subset B$ denote the fixed point set of $h_i \mid B$. Unless $h_i =$ identity, F_i has empty interior. If $C \cup F_i$ is not a finite disjoint union of circles it contains a common periodic point. But if $C \cup F_i$ is such a union, we shall show $F_i = C$. For otherwise, the union K of the components of F_i in $B - C$ is a compact set invariant under h_0, and therefore the nonwandering set of $h_0 \mid K$ is not empty. The "plane translation" theorem of Brouwer [1, 3] implies that a homeomorphism of an open disk which preserves orientation and has a nonwandering point, has a fixed point.

Since h_0 has no fixed points in $B - C$, $F_i \subset G$ and so $F_i = G$. Therefore all h_i leave C pointwise fixed. This proves Lemma 3.1.

Proof of Theorem 1.3

If δ is small enough, this is similar to the proof of Theorem 1.2, $k = 1$. $H(\lambda')$ will have a nonempty fixed point set $Z \subset I^k$ as near 0 as desired and invariant under every $2m$ loop of S'. Now apply Lemma 3.1.

Proof of Theorem 1.9

By passing to a finite covering space of a neighborhood of L, we may assume $\pi_1(L, x_0) = 0$. Then every S loop is null homotopic, i.e., homotopic to the loop $\omega_0 = ((\Omega_0, f_0), (\Omega_0, f_0))$. Observe that $H(\omega_0)$ is the identity map of I^k. If δ is small enough, it follows from Lemma 2.6 that there is a neighborhood $B_0 \subset I^k$ of 0 such that $B_0 \subset DH(\omega) \cap RH(\omega)$ and $H(\omega) \mid B_0 = $ identity, for every $2m$ loop ω in S'. By Lemma 3.1 every leaf of $\mathscr{F}'$ corresponding under f_0' to a point of B_0 is compact. This is sufficient to prove Theorem 1.9.

4   Remarks and questions

4.1   Theorem 1.2 can be proved for C^r foliations under the apparently weaker hypothesis of a "C^r essential" isolated fixed point. The reader may formulate this precisely, the idea being that any C^r small perturbation has a nearby fixed point. But maybe C^r essential implies essential.

4.2   Is Lemma 3.1 true without assuming real analytic? It can be proved for a finite set $h_0, \ldots, h_q$ satisfying:

(a)   the fixed point set of h_0 is a finite simplicial complex;

(b) $h_1, \ldots, h_{q-1}$ are C^2 diffeomorphisms, if $q > 1$.

If $q = 1$, this can be improved; the existence of a common periodic point follows from the Cartwright–Littlewood fixed point theorem and the assumption that the fixed point set of h_0 has finitely many components (all compact).

Can Lemma 3.1 be improved to obtain a common fixed point? Do two commuting homeomorphisms of a compact 2-disk have a common fixed point? This is true in the complex analytic case (see Shields [10]).

4.3   Can some analogue of Theorem 1.2 be proved for higher codimension? There are counter examples to Lemma 3.1 for R^3, but perhaps a generalization to R^3 can be proved using the added hypothesis that the index of the fixed point set of h_0 is nonzero.

4.4   Simple examples show that Theorems 1.1 and 2.2 are no longer true if condition (b) is dropped entirely, that is, if no assumption is made on the position of the holonomy element α in the fundamental group.

References

[1] L. E. J. Brouwer, Beweis des ebenen Translationenssatzes, *Math. Ann.* **72** (1912), 37–54.

[2] A. Haefliger, Variétés feuilletées, *Ann. Scuola Norm. Sup. Pisa* [3], **16** (1962), 367–996.

[3] B. von Kerékjártó, "Vorlesungen über Topologie," Vol. I, p. 95. Springer-Verlag, Berlin and New York, 1923.

[4] B. C. Koopman and A. B. Brown, On the covering of analytic loci by complexes, *Trans. Amer. Math. Soc.* **34** (1932), 231–251.

[5] S. Lojasiewicz, Triangulation of semi-analytic sets, *Ann. Scuola Norm. Sup. Pisa* [3], **18** (1964), 449–474.

[6] H. B. Lawson, Codimension-one foliations of spheres, *Bull. Amer. Math. Soc.* **77** (1971), 437–438.

[7] J. Milnor, On spaces having the homotopy type of a CW-complex, *Trans. Amer. Math. Soc.* **90** (1959), 272–280.

[8] G. Reeb, Sur certains propriétés topologiques des variétés feuilletées, *Actualités Sci. Indust.* 1183 (1952).

[9] G. Reeb, Sur les structures feuilletées de co-dimension un et sur un théorème de M. A. Denjoy, *Ann. Inst. Fourier (Grenoble)* **11** (1961), 185–200.

[10] A. Shields, On fixed points of commuting analytic functions, *Proc. Amer. Math. Soc.* **15** (1964), 703–706.