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**STEENROD OPERATIONS ON ALGEBRAIC DE RHAM COHOMOLOGY, HODGE COHOMOLOGY, AND SPECTRAL SEQUENCES**

A dissertation submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

**Ryan Drury**

December 2019

The Dissertation of Ryan Drury
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Professor Junecue Suh, Chair

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## Abstract

Steenrod operations on algebraic De Rham cohomology, Hodge cohomology, and spectral sequences

by

Ryan Drury

Let $X$ be a topological space and $k$ a field of characteristic $p$. Let $A^{\cdot}$ be a bounded below complex of sheaves of differential graded commutative $\mathbb{F}_p$-algebras. We show that there exist Steenrod operations canonically defined on the sheaf hypercohomology groups, $\mathbf{H}^{\cdot}(X, A^{\cdot})$. These Steenrod operations satisfy most of their usual properties, including the Cartan formula and the Adem relations. Suppose further that $A^{\cdot}$ is equipped with a filtration $F^{\cdot}$, which is finite in each degree, and compatible with the product on $A^{\cdot}$. The filtration on $F^{\cdot}A^{\cdot}$ induces a spectral sequence that converges to $\mathbf{H}^{\cdot}(X, A^{\cdot})$, and we prove that the constructed Steenrod operations also have a compatible action on the $E_1$ and $E_\infty$ pages of this spectral sequence. When $X$ is a smooth projective variety over $k$, we obtain Steenrod operations on the algebraic De Rham cohomology groups, $H^{\cdot}_{\mathrm{DR}}(X/k)$, as well as the Hodge cohomology groups. The Steenrod operations on $H^{\cdot}_{\mathrm{DR}}(X/k)$ have a compatible action on the first and infinite pages of the Hodge to De Rham spectral sequence, as well as the spectral sequence from Katz and Oda related to the Gauss-Manin connection.

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# Chapter 1

# Introduction

Let $X$ be a topological space and $p$ a fixed prime. One has Steenrod operations defined on the singular cohomology groups of $X$ with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, as described in Section 4.L, page 487, of [3]. For $p > 2$, one has the Steenrod powers and Bockstein homomorphism:

$$P^i : H^n(X;\mathbb{F}_p) \to H^{n+2i(p-1)}(X;\mathbb{F}_p), \qquad \beta : H^n(X;\mathbb{F}_p) \to H^{n+1}(X;\mathbb{F}_p).$$

When $p = 2$, there is instead a single collection of maps, called the Steenrod squares:

$$\mathrm{Sq}^i : H^i(X;\mathbb{F}_2) \to H^{i+1}(X;\mathbb{F}_2).$$

In this case, we in fact have $\beta = \mathrm{Sq}^1$, and one may define $P^i = \mathrm{Sq}^{2i}$. One then has $\beta P^i = \mathrm{Sq}^{2i+1}$ (shown on page 496 of [3]). These operations satisfy a nice list of axioms.

For a complex of sheaves of abelian groups $A^\cdot$ on $X$, one defines the hypercohomology groups of $X$ with coefficients in $A^\cdot$ as the hyper right derived functors of the global section functor:

$$\mathbf{H}^\cdot(X, A^\cdot) = \mathbf{R}^\cdot T(A^\cdot) = H^\cdot(T(I^\cdot))$$

In the above, $A^\cdot \hookrightarrow I^\cdot$ is an injective resolution in the category of sheaves of abelian groups, $\mathrm{Sh}(X)$, and $T : \mathrm{Sh}(X) \to \mathrm{Ab}$ represents the global section functor. Now suppose $X$ is a smooth projective variety over a field $k$ of characteristic $p$. Then one has the De Rham complex

$\Omega^{\cdot}_{X/k}$, and under these conditions we may define the algebraic De Rham cohomology groups of $X$ over $k$ as the hypercohomology of $X$ with coefficients in $\Omega^{\cdot}_{X/k}$:

$$H^{\cdot}_{\mathrm{DR}}(X/k) = \mathbf{H}^{\cdot}(X, \Omega^{\cdot}_{X/k})$$

To define Steenrod operations on the algebraic De Rham cohomology groups, it suffices under these conditions to construct them for sheaf hypercohomology.

We make use of two papers that define Steenrod operations in a more general context. In [2], Epstein constructs Steenrod operations in a categorical setting. For an object $A$ in an abelian category with tensor product and a left exact functor $T$ satisfying certain properties, Steenrod operations are constructed on the right derived functors, $R^{\cdot}T(A)$. In [5], May constructs Steenrod operations on the (co)homology groups of a complex $K_{\cdot}$ with additional structure that satisfies certain axioms. In [2], the Steenrod operations constructed actually include the case of sheaf cohomology, and the construction is general enough to include étale cohomology, which is used for example on page 559 of [7]. However, in order to include cohomology with coefficients in a sheaf of complexes, we would need to generalize Epstein's construction to define Steenrod operations on the hyper right derived functors, $\mathbf{R}^{\cdot}T(A^{\cdot})$, of a complex $A^{\cdot}$. This is more or less what we do, but we limit ourselves to our category of interest.

Our approach was to use Epstein's techniques but fit into May's framework. Many of Epstein's lemmas are included in Chapter 3, but generalized when needed to the case of complexes. For a bounded below complex of sheaves of graded commutative $\mathbb{F}_p$-algebras on $X$, we choose an injective resolution $A^{\cdot} \hookrightarrow I^{\cdot}$ in $\mathrm{Sh}_{\mathbb{F}_p}(X)$, the category of sheaves of $\mathbb{F}_p$ vector spaces on $X$, and define $K^{\cdot} = T(I^{\cdot})$. We then show in Chapter 5 that the complex $T(I^{\cdot})$, equipped with structure induced by $A^{\cdot}$, satisfies the axioms required by May's machinery in [5], and we obtain Steenrod operations on the cohomology groups, $H^{\cdot}(T(I^{\cdot})) = \mathbf{H}^{\cdot}(X, A^{\cdot})$. The Steenrod operations are canonically determined by $A^{\cdot}$ and its graded product structure. This establishes Steenrod operations on $H^{\cdot}_{\mathrm{DR}}(X/k)$.

Now that we have Steenrod operations on sheaf hypercohomology, one can consider various spectral sequences associated with hypercohomology, and ask if it is possible to define Steenrod operations on the pages of these spectral sequences in a compatible way. The tools required for this are developed in Chapter 6, and are then applied in Chapter 7. If one assumes in addition that $A^{\cdot}$ has a filtration that is finite in each degree and whose filtrations is compatible with the product on $A^{\cdot}$, then it is shown in Chapter 7 that the Steenrod operations defined in Chapter 5 have a canonical and compatible action on the $E_1$ and $E_\infty$ pages of the spectral sequence converging to $\mathbf{H}^{\cdot}(X, A^{\cdot})$, induced by the filtration on $F^{\cdot}A^{\cdot}$. As a special case, we have that the Steenrod operations on $H^{\cdot}_{\mathrm{DR}}(X/k)$ have a compatible action on the Hodge to De Rham spectral sequence, where $E_1^{a,b} = H^b(X, \Omega^a_{X/k})$. Any filtration will work, as long as it is finite in each degree and compatible with the product, so similar results for other spectral sequences are certainly possible. It should be noted that the canonical filtration of the De Rham complex is not compatible with the wedge product, and thus, it appears we cannot apply these methods to this filtration and its associated spectral sequence.

# Chapter 2

# Steenrod Operations of May

In this chapter I will state the definitions and theorems used by May. The below is a simplified version of Definition 2.1 of [5], on page 160.

**Definition 2.0.1.** *Let $p$ be a prime. Let $\Lambda$ be a commutative ring, which we will later take to be $\mathbb{F}_p$, the finite field of order $p$. Let $\Sigma_p$ denote the symmetric group on $p$ elements. Let $\pi$ be a subgroup of $\Sigma_p$, which we will later take to be the cyclic subgroup of order $p$ generated by the $p$-cycle, $\alpha = (1\ 2\ \dots\ p)$. Let $W.$ be a $\pi$-free resolution of $\Lambda$. Let $V.$ be a $\Sigma_p$-free resolution of $\Lambda$. Let $j. : W. \to V.$ be a morphism of $\Lambda\pi$ complexes making the diagram below commute.*

$$\begin{tikzcd} W. \arrow[r, dashed] \arrow[d, two heads] & V. \arrow[d, two heads] \\ \Lambda \arrow[r, "1"] & \Lambda \end{tikzcd}$$

*Note $j.$ is a quasi-isomorphism. As specified in Definition 1.2, page 157 of [5], the resolution $W.$ can be constructed explicitly so that $W_i$ is a free $\Lambda\pi$ module of rank 1 for each $i \geq 0$, with generator $e_i$, and has the following differentials for all $i \geq 0$:*

$$d^W_{2i+1}(e_{2i+1}) = (\alpha - 1)e_{2i}$$
$$d^W_{2i+2}(e_{2i+2}) = (1 + \alpha + \cdots + \alpha^{p-1})e_{2i+1}$$

*Define $\mathcal{C}(\pi,\Lambda)$ to be the category whose objects are pairs $(K.,\theta.)$, where $K.$ is a $\Lambda$ chain complex with a graded product making it a homotopy associative differential $\Lambda$-algebra and*

$$\theta. : W. \otimes_\Lambda (K.)^{[p]} \to K.$$

*is a $\Lambda\pi$ chain map. In May's paper there is also an n, but for our purposes we just take $n = \infty$. $(K.)^{[p]}$ represents the complex $K.$ tensored with itself $p$ times over $\Lambda$. Unlabeled tensor products should be assumed to be over $\Lambda$. The group actions of $\Sigma_p$ and $\pi$ on $K.$ are trivial, while $\Sigma_p$ and $\pi$ act on $(K.)^{[p]}$ by permuting tensors with the appropriate sign, taking into account the grading on $K.$. We then let $\pi$ act diagonally on $W. \otimes_\Lambda (K.)^{[p]}$. We additionally require $\theta.$ to satisfy the following axioms:*

1. *The restriction of $\theta.$ to $e_0 \otimes (K.)^{[p]}$ is $\Lambda$ homotopic to the iterated product, $(K.)^{[p]} \to K.$ in some fixed order.*

2. *There is a $\Lambda\Sigma_p$ chain map $\phi.$ such that $\theta.$ is $\Lambda\pi$-homotopic to the following composition:*

$$W. \otimes_\Lambda (K.)^{[p]} \xrightarrow{j\otimes 1} V. \otimes_\Lambda (K.)^{[p]} \xrightarrow{\phi} K.$$

*A morphism $f. : (K.,\theta.) \to (K'.,\theta'.)$ in this category is a $\Lambda$ chain map $f. : K. \to K'.$ making the diagram below commute up to $\Lambda\pi$-homotopy:*

$$\begin{array}{ccc} W. \otimes (K.)^{[p]} & \xrightarrow{\theta.} & K. \\ \big\downarrow{\scriptstyle 1\otimes f.^{[p]}} & & \big\downarrow{\scriptstyle f.} \\ W. \otimes (K'.)^{[p]} & \xrightarrow{\theta'.} & K'. \end{array}$$

Let $\mathcal{C}(p)$ be an abbreviation for $\mathcal{C}(\pi,\mathbb{F}_p)$. An object $(K.,\theta.) \in \mathcal{C}(\pi,\mathbb{F}_p)$ is said to be reduced mod $p$ if it comes from the reduction mod $p$ of an object $(\tilde{K}.,\tilde{\theta}.) \in \mathcal{C}(\pi,\mathbb{Z})$, and $\tilde{K}.$ is a flat $\mathbb{Z}$-module. *Given two objects $(K.,\theta)$, $(L.,\theta')$ in $\mathcal{C}(\pi,\Lambda)$, one can define the tensor product, $(K. \otimes L.,\tilde{\theta}.)$, where $K. \otimes_\Lambda L.$ is the usual tensor product of chain complexes and $\tilde{\theta}.$ is given by the following composition:*

$$\begin{tikzcd} W_\cdot \otimes (K_\cdot \otimes L_\cdot)^{[p]} \arrow[r, "\psi_\cdot \otimes S_\cdot"] & W_\cdot \otimes W_\cdot \otimes K_\cdot^{[p]} \otimes L_\cdot^{[p]} \arrow[d, "1 \otimes U_\cdot \otimes 1"] & \\ & (W_\cdot \otimes K_\cdot^{[p]}) \otimes (W_\cdot \otimes L_\cdot^{[p]}) \arrow[r, "\theta \otimes \theta'"] & K_\cdot \otimes L_\cdot \end{tikzcd}$$

*In the above $\psi_\cdot : W_\cdot \to W_\cdot \otimes_\Lambda W_\cdot$ is a $\Lambda\pi$ chain map making the diagram below commute:*

$$\begin{tikzcd} W_\cdot \arrow[r, "\psi_\cdot"] \arrow[d, two heads] & W_\cdot \otimes_\Lambda W_\cdot \arrow[d, two heads] \\ \Lambda \arrow[r, "1"] & \Lambda \end{tikzcd}$$

*The chain map $S_\cdot : (K_\cdot \otimes L_\cdot)^{[p]} \to K_\cdot^{[p]} \otimes L_\cdot^{[p]}$ is the shuffling isomorphism, where for $k_i \in K_{\deg(k_i)}$ and $l_i \in L_{\deg(l_i)}$, $i = 1, \ldots, p$, one has:*

$$S((k_1 \otimes l_1) \otimes \cdots \otimes (k_p \otimes l_p)) = (-1)^s (k_1 \otimes \cdots \otimes k_p) \otimes (l_1 \otimes \cdots \otimes l_p)$$

*where $(-1)^s$ is the sign that is incurred from transposing the $k_i$ terms and the $l_i$ terms. We have:*

$$s = \sum_{1 \le i < i' \le p} \deg(l_i) \cdot \deg(k_{i'})$$

*In the diagram, $U : W_\cdot \otimes K_\cdot^{[p]} \to K_\cdot^{[p]} \otimes W_\cdot$ is given by $U(x \otimes y) = (-1)^{\deg x \deg y} y \otimes x$. Note that I'm using different names for these maps than those used by May because I will later use $T$ to denote a functor.* Given an object $(K_\cdot, \theta) \in \mathcal{C}(\pi, \Lambda)$, the graded product on $K_\cdot$ gives rise to a chain map, $m_\cdot : K_\cdot \otimes K_\cdot \to K_\cdot$ and $(K_\cdot, \theta)$ is called a Cartan object if $m_\cdot : (K_\cdot \otimes K_\cdot, \tilde{\theta}_\cdot) \to (K_\cdot, \theta)$ is a morphism in $\mathcal{C}(\pi, \Lambda)$. That is, if the diagram below commutes up to $\Lambda\pi$ homotopy:

$$\begin{tikzcd} W_\cdot \otimes (K_\cdot \otimes K_\cdot)^{[p]} \arrow[r, "\tilde{\theta}_\cdot"] \arrow[d, "1 \otimes m_\cdot^{[p]}"] & K_\cdot \otimes K_\cdot \arrow[d, "m_\cdot"] \\ W_\cdot \otimes K_\cdot^{[p]} \arrow[r, "\theta_\cdot"] & K_\cdot \end{tikzcd}$$

Given an object $(K., \theta.) \in \mathcal{C}(\pi, \mathbb{F}_p)$, one can define Steenrod operations on the homology groups $H.(K.)$. The following is Definition 2.2 from [5], on page 161:

**Definition 2.0.2.** *Let $(K., \theta.)$ be an object in $\mathcal{C}(\pi, \mathbb{F}_p)$. Let $[x] \in H_q(K.)$, and $i \geq 0$. Then $[e_i \otimes x^{[p]}]$ is a well defined element of $H_{pq+i}(W. \otimes_\pi K.^{[p]})$. Define $D_i(x) = \theta.([e_i \otimes x^{[p]}]) \in H_{pq+i}(K.)$. For $p = 2$ define the Steenrod squares on $H.(K.)$ as follows:*

$$P_s(x) = 0 \quad \text{if } s < q\text{ ;} \qquad P_s(x) = D_{s-q}(x) \quad \text{if } s \geq q$$

*For $p > 2$, define the two operations $P_s : H_q(K.) \to H_{q+2s(p-1)}(K.)$ and $\beta P_s : H_q(K.) \to H_{q+2s(p-1)-1}(K.)$, as follows:*

$$P_s(x) = \begin{cases} 0 & \text{if } 2s < q \\ (-1)^s \nu(q) D_{(2s-q)(p-1)}(x) & \text{if } 2s \geq q \end{cases}$$

$$\beta P_s(x) = \begin{cases} 0 & \text{if } 2s \leq q \\ (-1)^s \nu(q) D_{(2s-q)(p-1)-1}(x) & \text{if } 2s > q \end{cases}$$

*Where $\nu(2j+\varepsilon) = (-1)^j (m!)^\varepsilon$ for $\varepsilon = 0, 1$ $j \in \mathbb{Z}$, and $m = (p-1)/2$. Note that $\beta P_s$ is a single symbol that is not a priori related to the Bockstein operator.*

The following is part of Proposition 2.3 from page 162 of [5], and will be needed later to show that the constructed Steenrod operations are natural.

**Lemma 2.0.3.** *Let $(K., \theta.)$ be an object in $\mathcal{C}(p)$ and consider $D_i : H_q(K.) \to H_{pq+i}(K.)$. For every morphism, $f. : (K., \theta.) \to (K'., \theta'.)$ in $\mathcal{C}(p)$, and $i \geq 0$, one has:*

$$f_* \circ D_i = D_i \circ f_* : H_q(K.) \to H_{pq+i}(K'.)$$

*Proof.* This proven by May, but because it is important I give a quick explicit proof. Let $[x] \in H_q(K.)$.

$$f_*(D_i([x])) = [f(\theta(e_i \otimes_\pi x^{[p]}))] = [\theta'(e_i \otimes_\pi f(x)^{[p]})] = D_i(f_*([x]))$$

$\square$

Corollary 2.7 on page 165 of [5] asserts that if $(K_\cdot, \theta_\cdot)$ is a Cartan object, then the Steenrod operations for $H_\cdot(K_\cdot)$ will satisfy the Cartan formula.

**Corollary 2.0.4.** *Let $(K_\cdot, \theta_\cdot)$ and $(L_\cdot, \theta'_\cdot)$ be objects in $\mathcal{C}(p)$. Let $x \in H_q(K_\cdot)$, and $y \in H_r(L_\cdot)$. Then*

$$P_s(x \otimes y) = \sum_{i+j=s} P_i(x) \otimes P_j(y)$$

*and if $p > 2$,*

$$\beta P_{s+1}(x \otimes y) = \sum_{i+j=s} \beta P_{i+1}(x) \otimes P_j(y) + (-1)^q P_i(x) \otimes \beta P_{j+1}(y)$$

*The above is called the external Cartan formula. If $(K_\cdot, \theta_\cdot)$ is a Cartan object, then we have the internal Cartan formula below for $P_s$, and if $p > 2$, $\beta P_s$, on $H_\cdot(K_\cdot)$.*

$$P_s(xy) = \sum_{i+j=s} P_i(x) P_j(y)$$

$$\beta P_{s+1}(xy) = \sum_{i+j=s} \beta P_{i+1}(x) P_j(y) + (-1)^{\deg x} P_i(x) \beta P_{j+1}(y)$$

For the Adem relations to hold, an additional axiom must be satisfied by $(K_\cdot, \theta_\cdot)$. The following definition is paraphrased from the beginning of section 4 of [5], on page 172.

**Definition 2.0.5.** *Let $\Sigma_{p^2}$ act as permutations on the set $\{(i, j) \mid 1 \le i \le p,\ 1 \le j \le p\}$. Embed $\pi = \langle \alpha \rangle$ as a subgroup of $\Sigma_{p^2}$ by letting $\alpha(i, j) = (i+1, j)$. Define $\alpha_i \in \Sigma_{p^2}$ with $\alpha_i(i, j) = (i, j+1)$ and $\alpha_i(k, j) = (k, j)$ for $k \neq i$. Set $\beta = \alpha_1 \cdots \alpha_p$ so that $\beta(i, j) = (i, j+1)$. Then:*

$$\alpha\alpha_i = \alpha_{i+1}\alpha \qquad \alpha_i\alpha_j = \alpha_j\alpha_i \qquad \alpha\beta = \beta\alpha$$

*Let $\alpha_i$ generate $\pi_i$ and $\beta$ generate $\nu$, so that $\pi_i$ and $\nu$ are cyclic of order $p$. Set $\sigma = \pi\nu$ and let $\tau$ be generated by the $\alpha_i$ and $\alpha$. Then $\sigma \subset \tau$ and $\tau$ is a Sylow-$p$ subgroup of $\Sigma_{p^2}$, and $\tau$ is a split extension of $\pi_1 \cdots \pi_p$ by $\pi$. Let $W_1 = W$ and $W_2 = W$ be $\pi$-free and $\nu$-free resolutions of $\mathbb{F}_p$ respectively. Let $\nu$ operate trivially on $W_1$ and $\pi$ operate trivially on $W_2$. Let $\sigma$ operate diagonally on $W_1 \otimes_{\mathbb{F}_p} W_2$. Then $W_1 \otimes_{\mathbb{F}_p} W_2$ is a $\sigma$-free resolution of $\mathbb{F}_p$.*

*If $M$ is a $\nu$-module, let $\tau$ operate on $M^{[p]}$ by letting $\alpha$ operate by cyclic permutations of the tensors and by letting $\alpha_i$ act on the ith factor of $M^{[p]}$, as does $\beta$. Let $\alpha_i$ operate trivially on $W_1$. Then $\tau$ operates on $W_1$ and we let $\tau$ operate diagonally on $W_1 \otimes M^{[p]}$. In particular, $W_1 \otimes W_2^{[p]}$ is then a $\tau$-free resolution of $\mathbb{F}_p$.*

*Let $K$ be any $\mathbb{F}_p$-complex. We let $\Sigma_{p^2}$ act by permutations on the tensors of $K^{[p^2]}$, with the $(i,j)$th factor of $K^{[p^2]}$ being the jth factor of $K$ in the ith factor of $K^{[p]}$ in $K^{[p^2]} = (K^{[p]})^{[p]}$. Let $\nu$ operate on $W_2 \otimes K^{[p]}$, with $\beta$ acting by cyclic permutations of the tensors of $K^{[p]}$. Then $\tau$ has an action on $W_1 \otimes (W_2 \otimes K^{[p]})^{[p]}$.*

*Let $Y$ be a $\Sigma_{p^2}$-free resolution of $\mathbb{F}_p$, and let $w : W_1 \otimes W_2^{[p]} \to Y$ be a $\mathbb{F}_p\tau$ chain map over $\mathbb{F}_p$:*

$$\begin{tikzcd}
W_1 \otimes W_2^{[p]} \arrow[r, dashed, "w."] \arrow[d, two heads] & Y \arrow[d, two heads] \\
\mathbb{F}_p \arrow[r, "1"] & \mathbb{F}_p
\end{tikzcd}$$

*which exists because $W_1 \otimes W_2^{[p]}$ is a free $\tau$ resolution of $\mathbb{F}_p$ and $Y$ is acyclic.*

With all of the above established, we say an object $(K,\theta) \in \mathcal{C}(p)$ is an Adem object if there exists a $\Sigma_{p^2}$-morphism $\xi : Y \otimes K^{[p^2]} \to K$ such that the following diagram is commutative up to $\tau$-homotopy:

$$\begin{tikzcd}
(W_1 \otimes W_2^{[p]}) \otimes K^{[p^2]} \arrow[r, "w\otimes 1"] \arrow[d, "1\otimes S"] & Y \otimes K^{[p^2]} \arrow[dr, "\xi"] & \\
 & & K \\
W_1 \otimes (W_2 \otimes K^{[p]})^{[p]} \arrow[r, "1\otimes\theta^{[p]}"] & W_1 \otimes K^{[p]} \arrow[ur, "\theta"] &
\end{tikzcd}$$

*The map $S : W_2^{[p]} \otimes K^{[p^2]} \to (W_2 \otimes K^{[p]})^{[p]}$ shuffles tensors with sign, which is a $\tau$-morphism. In the above, $\Sigma_{p^2}$ acts trivially on $K$ and $\alpha_i$ acts trivially on $W_1 \otimes K^{[p]}$.*

It is then proven in Theorem 4.7, on page 178 of [5], that if $(K_\cdot, \theta_\cdot)$ is an Adem object, then the Adem relations hold for the operations on $H_\cdot(K_\cdot)$.

**Theorem 2.0.6.** *The following relations among $P_s$ and $\beta P_s$ are valid on all homology classes of all Adem objects in $\mathcal{C}(p)$.*

1. *If $p = 2$ and $a > 2b$:*

$$P_a P_b = \sum_i \binom{2i-a}{a-b-i-1} P_{a+b-i} P_i$$

2. *If $p > 2$ and $a > pb$:*

$$P_a P_b = \sum_i (-1)^{a+i} \binom{pi-a}{a-(p-1)b-i-1} P_{a+b-i} P_i$$
$$\beta P_a P_b = \sum_i (-1)^{a+i} \binom{pi-a}{a-(p-1)b-i-1} \beta P_{a+b-i} P_i$$

3. *If $p > 2$ and $a \geq pb$:*

$$\begin{aligned} P_a \beta P_b &= \sum_i (-1)^{a+i} \binom{pi-a}{a-(p-1)b-i} \beta P_{a+b-i} P_i \\ &\quad - \sum_i (-1)^{a+i} \binom{pi-a-1}{a-(p-1)b-i} P_{a+b-i} \beta P_i \\ \beta P_a \beta P_b &= -\sum_i (-1)^{a+i} \binom{pi-a-1}{a-(p-1)b-i} \beta P_{a+b-i} \beta P_i \end{aligned}$$

In Section 5 of [5], page 182, May restated his results with indices for cohomology instead of homology, using the convention $K_{-q} = K^q$.

**Definition 2.0.7.** *Let $(K^{\cdot}, \theta^{\cdot}) \in \mathcal{C}(p)$, and consider $W^{\cdot}$ as before but now graded with non-positive superscripts. Let $x \in H^q(K^{\cdot})$. Then we have $D_i(x) = \theta^{\cdot}(e^{-i} \otimes x^{[p]}) \in H^{pq-i}(K^{\cdot})$, for $i \geq 0$, and $D_i = 0$ for $i < 0$. We may define $P^s(x) = P_{-s}(x)$, and if $p > 2$, $\beta P^s(x) = \beta P_{-s}(x)$.*

We have the formulas in the following corollary:

**Corollary 2.0.8.** *The definitions and properties of the Steenrod operations with cohomological indices become the following. Let $(K^{\cdot}, \theta^{\cdot}) \in \mathcal{C}(\pi, \mathbb{F}_p)$, and $x \in H^q(K^{\cdot})$.*

1. *For $p = 2$:*

$$P^s(x) = D_{q-s}(x) \in H^{q+s}(K^{\cdot})$$

2. *For $p > 2$:*

$$P^s(x) = (-1)^s \nu(-q) D_{(q-2s)(p-1)}(x) \in H^{q+2s(p-1)}(K^{\cdot})$$
$$\beta P^s(x) = (-1)^s \nu(-q) D_{(q-2s)(p-1)-1}(x) \in H^{q+2s(p-1)+1}(K^{\cdot})$$

*where in the above, $D_i = 0$ for $i < 0$ and $\nu(-q) = (-1)^j (m!)^{\varepsilon}$, with $q = 2j - \varepsilon$, $\varepsilon = 0$ or 1, and $m = (p-1)/2$.*

*In the case $p = 2$, May notes it would be standard to use $Sq^s$ instead of $P^s$, but he uses $P^s$ so that the Cartan and Adem relations are still the same in both cases. We have the following properties:*

1. *For $p = 2$, $P^s(x) = 0$ when $s > q$ and $P^q(x) = x^2$.*
2. *For $p > 2$, $P^s(x) = 0$ when $2s > q$, $\beta P^s(x) = 0$ when $2s > q$, and $P^s(x) = x^p$ when $2s = q$.*

*The formulas $P^s(x) = 0$ when $s < 0$ and $P^0 = 1$ are not true in general. If $(K_{\cdot}, \theta_{\cdot})$ is reduced mod p, then:*

1. *$\beta P^{s-1} = sP^s$ if $p = 2$, and $\beta P^s$ is the composition of $P^s$ with the Bockstein $\beta$ if $p > 2$.*

*The external Cartan formula is now:*

$$
\begin{aligned}
P^s(x \otimes y) &= \sum_{i+j=s} P^i(x) \otimes P^j(y) \\
\beta P^{s+1}(x \otimes y) &= \sum_{i+j=s} \beta P^{i+1}(x) \otimes P^j(y) + (-1)^{\deg x} P^i(x) \otimes \beta P^{j+1}(y) \qquad \textit{for } p > 2.
\end{aligned}
$$

*So if $(K^{\cdot}, \theta^{\cdot}) \in \mathcal{C}(p)$ is a Cartan object, then we will have:*

$$
\begin{aligned}
P^s(xy) &= \sum_{i+j=s} P^i(x) P^j(y) \\
\beta P^{s+1}(xy) &= \sum_{i+j=s} \beta P^{i+1}(x) P^j(y) + (-1)^{\deg x} P^i(x) \beta P^{j+1}(y) \qquad \textit{for } p > 2.
\end{aligned}
$$

The Adem relations with cohomological indices are stated in Corollary 5.1, page 183 of [5], included below:

**Corollary 2.0.9.** *The following relations among the $P^s$ and $\beta P^s$ are valid on all cohomology classes of all Adem objects in $\mathcal{C}(p)$.*

1. *If $p \geq 2$, $a < pb$, and $\varepsilon = 0$ or $1$ if $p > 2$, $\varepsilon = 0$ if $p = 2$, then:*

$$\beta^{\varepsilon} P^a P^b = \sum_i (-1)^{a+i} \binom{a - pi}{(p-1)b - a + i - 1} \beta^{\varepsilon} P^{a+b-i} P^i$$

2. *If $p > 2$, $a \leq pb$, and $\varepsilon = 0$ or $1$, then*

$$
\begin{aligned}
\beta^{\varepsilon} P^a \beta P^b = {} & (1-\varepsilon) \sum_i (-1)^{a+i} \binom{a - pi}{(p-1)b - a + i - 1} \beta P^{a+b-i} P^i \\
& - \sum_i (-1)^{a+i} \binom{a - pi - 1}{(p-1)b - a + i} \beta^{\varepsilon} P^{a+b-i} \beta P^i
\end{aligned}
$$

Now that these results from [5] have been summarized here, I will construct Steenrod operations in a few contexts by constructing explicit objects $(K^{\cdot}, \theta^{\cdot})$, proving that they belong to the category $\mathcal{C}(p)$, and that the objects are both Cartan and Adem.

# Chapter 3

# Conventions and Tools

In this chapter I will establish some conventions, list some results of Epstein, and work out a few useful tools.

## 3.1 Group Actions and Adjoint Isomorphisms

**Definition 3.1.1.** *Let $G$ be a group, $\Lambda$ a commutative ring, and let $A, B, C$ be $\Lambda G$ modules. We let $G$ act diagonally on the tensor product $A \otimes_\Lambda B$. That is, for $a \otimes b \in A \otimes B$ and $g \in G$:*

$$g \cdot (a \otimes b) = ga \otimes gb$$

*We also let $G$ act diagonally on $Hom_\Lambda(A, B)$. So for $a \in A$, $f \in Hom_\Lambda(A, B)$, and $g \in G$:*

$$(g \cdot f)(a) = g(f(g^{-1}a))$$

*In the case that $G$ acts trivially on $B$, we have the identity:*

$$(g \cdot f)(a) = f(g^{-1}a)$$

**Lemma 3.1.2.** *Let $A, B, C$ be $\Lambda G$ modules. Let $G$ act diagonally on $Hom_\Lambda(B, C)$ and $A \otimes_\Lambda B$. Then there is an isomorphism of abelian groups:*

$$\Phi : Hom_{\Lambda G}(A, Hom_\Lambda(B, C)) \to Hom_{\Lambda G}(A \otimes_\Lambda B, C)$$

*where for $f \in Hom_{\Lambda G}(A, Hom_{\Lambda}(B,C))$ and $a \otimes b \in A \otimes B$, one sets:*

$$\Phi(f)(a \otimes b) = f(a)(b)$$

*Proof.* Since $\Lambda$ is a commutative ring, we have the following isomorphism, by the more classical version of this lemma:

$$\Phi : \mathrm{Hom}_{\Lambda}(A, \mathrm{Hom}_{\Lambda}(B,C)) \to \mathrm{Hom}_{\Lambda}(A \otimes_{\Lambda} B, C)$$

where $\Phi(f)(a \otimes b) = f(a)(b)$, and $\Phi^{-1}(h)(a)(b) = h(a \otimes b)$, for $f \in \mathrm{Hom}_{\Lambda}(A, \mathrm{Hom}_{\Lambda}(B,C))$ and $h \in \mathrm{Hom}_{\Lambda}(A \otimes_{\Lambda} B, C)$. Now all we have to prove is that when $f$ is a $\Lambda G$ morphism, $\Phi(f)$ is a $\Lambda G$ morphism, and when $h$ is a $\Lambda G$ morphism, $\Phi^{-1}(h)$ is a $\Lambda G$ morphism. Suppose $f \in \mathrm{Hom}_{\Lambda G}(A, \mathrm{Hom}_{\Lambda}(B,C))$ and $g \in G$:

$$\begin{aligned}
\Phi(f)(g \cdot (a \otimes b)) &= \Phi(f)(ga \otimes gb) \\
&= (f)(ga)(gb) \\
&= [g \cdot (f(a))](gb) && f \text{ is a } \Lambda G \text{ morphism.} \\
&= g[f(a)(g^{-1} \cdot gb)] && \text{Diagonal action on } f(a) \in \mathrm{Hom}_{\Lambda}(B,C) \\
&= g[f(a)(b)] \\
&= g[\Phi(f)(a \otimes b)]
\end{aligned}$$

So we see $\Phi(f) \in \mathrm{Hom}_{\Lambda G}(A \otimes_{\Lambda} B, C)$ when $f \in \mathrm{Hom}_{\Lambda G}(A, \mathrm{Hom}_{\Lambda}(B,C))$. Suppose $h \in \mathrm{Hom}_{\Lambda G}(A \otimes_{\Lambda} B, C)$. Then:

$$\begin{aligned}
\Phi^{-1}(h)(ga)(b) &= h(ga \otimes b) \\
&= h(g \cdot (a \otimes g^{-1}b)) \\
&= g[h(a \otimes g^{-1}b)] && h \text{ is a } \Lambda G \text{ morphism.} \\
&= g[\Phi^{-1}(h)(a)(g^{-1}b)] \\
&= [g \cdot (\Phi^{-1}(h)(a))](b) && \text{Diagonal action on } \Phi^{-1}(h)(a) \in \mathrm{Hom}_{\Lambda}(B,C).
\end{aligned}$$

This shows $\Phi^{-1}(h)(ga) = g \cdot \Phi^{-1}(h)(a)$, so $\Phi^{-1}(h) \in \mathrm{Hom}_{\Lambda G}(A, \mathrm{Hom}_{\Lambda}(B,C))$ when $h \in \mathrm{Hom}_{\Lambda G}(A \otimes_{\Lambda} B, C)$. Now we can conclude that $\Phi$ restricts to give us a natural isomorphism:

$$\mathrm{Hom}_{\Lambda G}(A, \mathrm{Hom}_{\Lambda}(B,C)) \to \mathrm{Hom}_{\Lambda G}(A \otimes_{\Lambda} B, C)$$

□

**Lemma 3.1.3.** *Suppose $A, B, C$ are as in Lemma 3.1.2. Suppose futhermore that the action of $G$ on $C$ is trivial. Then the following abelian groups are equal:*

$$Hom_{\Lambda G}(A \otimes_{\Lambda} B, C) = Hom_{\Lambda G}(A \otimes_{\Lambda G} B, C)$$

*Proof.* The canonical surjection $\pi : A \otimes_{\Lambda} B \to A \otimes_{\Lambda G} B$ induces the following inclusion by precomposition:

$$\pi^* : \mathrm{Hom}_{\Lambda G}(A \otimes_{\Lambda G} B, C) \hookrightarrow \mathrm{Hom}_{\Lambda G}(A \otimes_{\Lambda} B, C)$$

Now suppose that $f \in \mathrm{Hom}_{\Lambda G}(A \otimes_{\Lambda} B, C)$ and the action of $G$ on $C$ is trivial. Then I claim that $f$ is well defined in $\mathrm{Hom}_{\Lambda G}(A \otimes_{\Lambda G} A, C)$. Let $a \in A$, $b \in B$, and $g \in G$. Note in $A \otimes_{\Lambda G} B$, one has $(ga) \otimes_{\Lambda G} b = (ag^{-1}) \otimes_{\Lambda G} b = a \otimes_{\Lambda G} (g^{-1}b)$. I must show $f(ga \otimes_{\Lambda} b) = f(a \otimes_{\Lambda} g^{-1}b)$.

$$\begin{aligned} f(ga \otimes_{\Lambda} b) &= f(g \cdot (a \otimes_{\Lambda} g^{-1}b)) & & \\ &= g \cdot f(a \otimes_{\Lambda} g^{-1}b) & & f \text{ is a } G \text{ morphism.} \\ &= f(a \otimes_{\Lambda} g^{-1}b) & & g \text{ acts trivially on } C. \end{aligned}$$

Thus $f$ is well defined, and this shows $\mathrm{Hom}_{\Lambda G}(A \otimes_{\Lambda} B, C) = \mathrm{Hom}_{\Lambda G}(A \otimes_{\Lambda G} B, C)$.

□

**Definition 3.1.4.** *Let $A^{\cdot}$ and $B^{\cdot}$ be complexes in an abelian category with tensor product, countable direct sums, and countable direct products. Then we define the total complex of the tensor product $(A^{\cdot} \otimes B^{\cdot})^{\cdot}$, also denoted $Tot^{\cdot}(A^{\cdot} \otimes B^{\cdot})$.*

$$(A^\cdot \otimes B^\cdot)^n = \bigoplus_{i+j=n} A^i \otimes B^j$$

$$d^n_{(A^\cdot \otimes B^\cdot)^\cdot} = \bigoplus_{i+j=n} \left( d^i_A \otimes 1^j_B + (-1)^i (1^i_A \otimes d^j_B) \right)$$

*Similarly, we define the total Hom complex $Hom^\cdot(A^\cdot, B^\cdot)$ as follows:*

$$Hom^n(A^\cdot, B^\cdot) = \prod_{i \in \mathbb{Z}} Hom(A^i, B^{i+n})$$

$$d^n_{Hom^\cdot(A^\cdot,B^\cdot)}(f^\cdot) = \prod_{i \in \mathbb{Z}} \left( d^{i+n}_B \circ f^i + (-1)^{n+1} f^{i+1} \circ d^i_A \right)$$

*The choice of sign in $d^\cdot_{Hom^\cdot(A^\cdot,B^\cdot)}$ is determined by the choice of sign in $d^\cdot_{(A^\cdot \otimes B^\cdot)^\cdot}$, if one wants the map $\Phi^\cdot$ in Lemma 3.1.5 to induce a chain map without requiring an additional sign.*

**Lemma 3.1.5.** *Let $A^\cdot$, $B^\cdot$, and $C^\cdot$ be complexes of $\Lambda G$ modules. Let $G$ act diagonally on $Hom^\cdot_\Lambda(B^\cdot, C^\cdot)$ and $(A^\cdot \otimes B^\cdot)^\cdot$. Then there is an isomorphism of complexes of abelian groups:*

$$\Phi^\cdot : Hom^\cdot_{\Lambda G}(A^\cdot, Hom^\cdot_\Lambda(B^\cdot, C^\cdot)) \to Hom^\cdot_{\Lambda G}((A^\cdot \otimes_\Lambda B^\cdot)^\cdot, C^\cdot)$$

*induced by the map $\Phi$ from Lemma 3.1.2.*

*Proof.* For now we define $\Phi^\cdot : \mathrm{Hom}^\cdot_\Lambda(A^\cdot, \mathrm{Hom}^\cdot_\Lambda(B^\cdot, C^\cdot)) \to \mathrm{Hom}^\cdot_\Lambda((A^\cdot \otimes_\Lambda B^\cdot)^\cdot, C^\cdot)$, and like in Lemma 3.1.2, we will show that $\Phi^\cdot$ restricts to an isomorphism $\mathrm{Hom}^\cdot_{\Lambda G}(A^\cdot, \mathrm{Hom}^\cdot_\Lambda(B^\cdot, C^\cdot)) \to \mathrm{Hom}^\cdot_{\Lambda G}((A^\cdot \otimes_\Lambda B^\cdot)^\cdot, C^\cdot)$. To ease notation, let $D^\cdot_R = \mathrm{Hom}^\cdot_R(A^\cdot, \mathrm{Hom}^\cdot_\Lambda(B^\cdot, C^\cdot))$ and $E^\cdot_R = \mathrm{Hom}^\cdot_R((A^\cdot \otimes_\Lambda B^\cdot)^\cdot, C^\cdot)$, for $R = \Lambda$ and $R = \Lambda G$. Let $n, k, l \in \mathbb{Z}$, and $f^\cdot \in D^n_\Lambda$. Let $a \in A^k$ and $b \in B^l$. We have $f^k(a) \in \mathrm{Hom}^{n+k}_\Lambda(B^\cdot, C^\cdot)$. So $f^k(a)^l(b) \in C^{n+k+l}$. Define $\Phi^n(f^\cdot) \in E^n_\Lambda$ by:

$$\Phi^n(f^\cdot)^{k+l}(a \otimes_\Lambda b) = f^k(a)^l(b) \in C^{n+k+l}$$

Denote the inverse map, $\beta^\cdot : E^\cdot_\Lambda \to D^\cdot_\Lambda$, where for $h \in E^n_\Lambda$, $a \in A^k$, $b \in B^l$, we define:

$$\beta^n(h^\cdot)^k(a)^l(b) = h^{k+l}(a \otimes b) \in C^{n+k+l}$$

So we have $\beta^n(h^\cdot)^k(a) \in \mathrm{Hom}^{n+k}_\Lambda(B^\cdot, C^\cdot)$, and that $\beta^n(h^\cdot) \in E^n_\Lambda$. We have shown that $\Phi^\cdot$ and $\beta^\cdot$ are degree 0 maps between chain complexes, and it is clear that $\Phi^\cdot$ and $\beta^\cdot$ are inverse maps in each degree. Now I show that $\Phi^{\cdot}$ commutes with the differential. Let $f \in D^n_\Lambda$, $a \in A^k$, and $b \in B^l$.

$$\begin{aligned}
&\Phi^{n+1}(d^n_{D_\Lambda}(f^{\cdot}))^{k+l}(a \otimes b) \\
&\quad = [(d^n_{D_\Lambda}(f^{\cdot}))^k(a)]^l(b) \\
&\quad = [d^{n+k}_{\mathrm{Hom}_\Lambda(B^{\cdot},C^{\cdot})}(f^k(a)) + (-1)^{n+1} f^{k+1}(d^k_A a)]^l(b) \\
&\quad = d^{n+k}_{\mathrm{Hom}_\Lambda(B^{\cdot},C^{\cdot})}(f^k(a))^l(b) + (-1)^{n+1} f^{k+1}(d^k_A a)^l(b) \\
&\quad = d^{n+k+l}_C(f^k(a)^l(b)) + (-1)^{n+k+1} f^k(a)^{l+1}(d^l_B b) + (-1)^{n+1} f^{k+1}(d^k_A a)^l(b) \\
&\quad = d^{n+k+l}_C(\Phi^n(f^{\cdot})^{k+l}(a \otimes b)) \\
&\qquad\quad + (-1)^{n+k+1}(\Phi^n(f^{\cdot})^{k+l+1}(a \otimes d^l_B b)) + (-1)^{n+1}(\Phi^n(f^{\cdot})^{k+l+1}(d^k_A a \otimes b)) \\
&\quad = d^{n+k+l}_C(\Phi^n(f^{\cdot})^{k+l}(a \otimes b)) + (-1)^{n+1}\Phi^n(f^{\cdot})^{k+l+1}(d^k_A a \otimes b + (-1)^k a \otimes d^l_B b) \\
&\quad = d^{n+k+l}_C(\Phi^n(f^{\cdot})^{k+l}(a \otimes b)) + (-1)^{n+1}\Phi^n(f^{\cdot})^{k+l+1}(d^{k+l}_{A^{\cdot}\otimes B^{\cdot}}(a \otimes b)) \\
&\quad = (d^n_{E_\Lambda}(\Phi^n(f^{\cdot})))^{k+l}(a \otimes b)
\end{aligned}$$

Thus we have $\Phi^{n+1}(d^n_{D_\Lambda}(f^{\cdot})) = d^n_{E_\Lambda}(\Phi^n(f^{\cdot}))$, showing $\Phi^{\cdot}$ is a chain map. This implies the inverse map $\beta^{\cdot}$ is also a chain map, so $\Phi^{\cdot}$ is an isomorphism of chain complexes.

Finally, the arguments in Lemma 3.1.2 carry through to this case as well, and $\Phi^{\cdot}$ restricts to the desired isomorphism $D^{\cdot}_{\Lambda G} \to E^{\cdot}_{\Lambda G}$.

$\square$

**Lemma 3.1.6.** *Let $A^{\cdot}, B^{\cdot}, C^{\cdot}$ be complexes of $\Lambda G$-modules. Suppose $f^{\cdot}, g^{\cdot}$ are two homotopic $\Lambda G$ chain maps:*

$$f^{\cdot}, g^{\cdot} : A^{\cdot} \to Hom^{\cdot}_\Lambda(B^{\cdot}, C^{\cdot})$$

*with homotopy $h^{\cdot} : A^{\cdot} \to Hom^{\cdot}_\Lambda(B^{\cdot}, C^{\cdot})[-1]$. Because $f^{\cdot}$ and $g^{\cdot}$ are chain maps, we may regard them as cycles in the complex of abelian groups:*

$$f^{\cdot}, g^{\cdot} \in Z^0(Hom^{\cdot}_{\Lambda G}(A^{\cdot}, Hom^{\cdot}_\Lambda(B^{\cdot}, C^{\cdot})))$$

*We also have* $h^{\cdot} \in Hom^{-1}_{\Lambda G}(A^{\cdot}, Hom^{\cdot}_{\Lambda}(B^{\cdot}, C^{\cdot}))$*, with* $d^{-1}_{Hom_{\Lambda G}(A^{\cdot},\ Hom^{\cdot}_{\Lambda}(B^{\cdot}, C^{\cdot}))}(h^{\cdot}) = f^{\cdot} - g^{\cdot}$*. Recall the isomorphism from Lemma 3.1.5:*

$$\Phi^{\cdot} : Hom^{\cdot}_{\Lambda G}(A^{\cdot}, Hom^{\cdot}_{\Lambda}(B^{\cdot}, C^{\cdot})) \to Hom^{\cdot}_{\Lambda G}((A^{\cdot} \otimes B^{\cdot})^{\cdot}, C^{\cdot})$$

*Then* $\Phi^0(f^{\cdot})$ *and* $\Phi^0(g^{\cdot})$ *are* $\Lambda G$ *chain maps, and are homotopic by homotopy* $\Phi^{-1}(h^{\cdot})$*.*

*Proof.* Because $\Phi^{\cdot}$ is an isomorphism of chain complexes, all the relations satisfied by $f^{\cdot}$, $g^{\cdot}$, and $h^{\cdot}$ will be satisfied by $\Phi^0(f^{\cdot})$, $\Phi^0(g^{\cdot})$ and $\Phi^{-1}(h^{\cdot})$ respectively. Thus, we have:

$$\Phi^0(f^{\cdot}), \Phi^0(g^{\cdot}) \in Z^0(\mathrm{Hom}^{\cdot}_{\Lambda G}((A^{\cdot} \otimes B^{\cdot})^{\cdot}, C^{\cdot}))$$

which implies $\Phi^0(f^{\cdot})$ and $\Phi^0(g^{\cdot})$ are chainmaps, and we also have:

$$\Phi^0(f^{\cdot}) - \Phi^0(g^{\cdot}) = d^{-1}_{\mathrm{Hom}_{\Lambda G}((A^{\cdot} \otimes B^{\cdot})^{\cdot}, C^{\cdot})}(\Phi^{-1}(h^{\cdot}))$$

which implies $\Phi^0(f^{\cdot})$ and $\Phi^0(g^{\cdot})$ are homotopic by homotopy $\Phi^{-1}(h^{\cdot})$.

$\square$

**Lemma 3.1.7.** *Let* $\Lambda$ *be a commutative ring,* $\pi$ *a group, and let* $A^{\cdot}, B^{\cdot}, C^{\cdot}, D^{\cdot}$ *be complexes of* $\Lambda\pi$ *modules, with* $\Lambda\pi$ *morphisms:*

$$f^{\cdot} : A^{\cdot} \to Hom^{\cdot}_{\Lambda}(B^{\cdot}, C^{\cdot}), \qquad g^{\cdot} : C^{\cdot} \to D^{\cdot}$$

*Note* $\Phi^0(f^{\cdot}) : Tot^{\cdot}(A^{\cdot} \otimes_{\Lambda} B^{\cdot}) \to C^{\cdot}$*. We have the* $\Lambda\pi$ *morphism,* $(g^{\cdot})_* : Hom_{\Lambda}(B^{\cdot}, C^{\cdot}) \to Hom_{\Lambda}(B^{\cdot}, D^{\cdot})$ *induced by post-composition with* $g^{\cdot}$*. We have the composition:*

$$(g^{\cdot})_* \circ f^{\cdot} : A^{\cdot} \to Hom^{\cdot}_{\mathbb{F}_p}(B^{\cdot}, D^{\cdot})$$

*And the adjoint map:*

$$\Phi^0((g^{\cdot})_* \circ f^{\cdot}) : Tot^{\cdot}(A^{\cdot} \otimes B^{\cdot}) \to D^{\cdot}$$

*Then we have the identity,* $\Phi^0((g^{\cdot})_* \circ f^{\cdot}) = g^{\cdot} \circ \Phi^0(f^{\cdot})$*. That is, if we have the diagram:*

$$A^{\cdot} \xrightarrow{f^{\cdot}} Hom_{\mathbb{F}_p}(B^{\cdot}, C^{\cdot}) \xrightarrow{(g^{\cdot})_*} Hom_{\mathbb{F}_p}(B^{\cdot}, D^{\cdot}),$$

*then applying $\Phi^0$ yields the diagram below:*

$$Tot^{\cdot}(A^{\cdot} \otimes B^{\cdot}) \xrightarrow{\Phi^0(f^{\cdot})} C^{\cdot} \xrightarrow{g^{\cdot}} D^{\cdot}$$

*Proof.* Let $a \in A^i$ and $b \in B^j$. We have:

$$\begin{aligned}
\Phi^0((g^{\cdot})_* \circ f^{\cdot})^{i+j}(a \otimes b) &= g^{\cdot}_*(f^i(a))^j(b) \\
&= g^{i+j}(f^i(a)^j(b)) \\
&= g^{i+j}(\Phi^0(f^{\cdot})^{i+j}(a \otimes b)) \\
&= (g^{\cdot} \circ (\Phi^0(f^{\cdot})))^{i+j}(a \otimes b)
\end{aligned}$$

Thus, we have the equality $\Phi^0((g^{\cdot})_* \circ f^{\cdot}) = g^{\cdot} \circ \Phi^0(f^{\cdot})$.

□

**Lemma 3.1.8.** *Let $\Lambda$ be a commutative ring, $\pi$ a group, and $A^{\cdot}, B^{\cdot}, C^{\cdot}, D^{\cdot}$ complexes of $\Lambda\pi$ modules. Let $f^{\cdot}$ and $g^{\cdot}$ be $\Lambda\pi$ morphisms:*

$$f^{\cdot} : A^{\cdot} \to Hom^{\cdot}_{\Lambda}(B^{\cdot}, C^{\cdot}), \qquad g^{\cdot} : D^{\cdot} \to B^{\cdot}$$

*There is a $\Lambda\pi$ morphism $(g^{\cdot})^* : Hom^{\cdot}_{\mathbb{F}_p}(B^{\cdot}, C^{\cdot}) \to Hom^{\cdot}_{\mathbb{F}_p}(D^{\cdot}, C^{\cdot})$ induced by precomposition with $g^{\cdot}$. We have the composite $\Lambda\pi$ morphism:*

$$(g^{\cdot})^* \circ f^{\cdot} : A^{\cdot} \to Hom^{\cdot}_{\mathbb{F}_p}(D^{\cdot}, C^{\cdot})$$

*Note $\Phi^0((g^{\cdot})^* \circ f^{\cdot}) : Tot^{\cdot}(A^{\cdot} \otimes_{\Lambda} D^{\cdot}) \to C^{\cdot}$. We claim we have the identity, $\Phi^0((g^{\cdot})^* \circ f^{\cdot}) = \Phi^0(f^{\cdot}) \circ (1^{\cdot}_A \otimes g^{\cdot})$. That is, if we have the diagram below:*

$$A^{\cdot} \xrightarrow{f^{\cdot}} Hom^{\cdot}_{\mathbb{F}_p}(B^{\cdot}, C^{\cdot}) \xrightarrow{(g^{\cdot})^*} Hom^{\cdot}_{\mathbb{F}_p}(D^{\cdot}, C^{\cdot})$$

*Then applying $\Phi^0$ yields the diagram:*

$$Tot^{\cdot}(A^{\cdot} \otimes_{\Lambda} D^{\cdot}) \xrightarrow{1^{\cdot}_A \otimes g^{\cdot}} Tot^{\cdot}(A^{\cdot} \otimes_{\Lambda} B^{\cdot}) \xrightarrow{\Phi^0(f^{\cdot})} C^{\cdot}$$

*Proof.* Let $x \in A^i$ and $y \in D^j$. We have:

$$\begin{aligned}
\Phi^0((g^{\cdot})^* \circ f^{\cdot})^{i+j}(x \otimes y) &= (g^{\cdot})^*(f^i(x))^j(y) \\
&= f^i(x)^j(g^j(y)) \\
&= \Phi^0(f^{\cdot})^{i+j}(x \otimes g^j(y)) \\
&= (\Phi^0(f^{\cdot}) \circ (1^{\cdot}_A \otimes g^{\cdot}))^{i+j}(x \otimes y)
\end{aligned}$$

This shows $\Phi^0((g^{\cdot})^* \circ f^{\cdot}) = \Phi^0(f^{\cdot}) \circ (1^{\cdot}_A \otimes g^{\cdot})$.

□

**Lemma 3.1.9.** *Let $\Lambda$ be a commutative ring, $\pi$ a group, and $A^{\cdot}, B^{\cdot}, C^{\cdot}, D^{\cdot}$ complexes of $\Lambda\pi$ modules. Suppose there are $\Lambda\pi$ morphisms $f^{\cdot}, g^{\cdot}$:*

$$f^{\cdot} : A^{\cdot} \to Hom^{\cdot}_{\Lambda}(B^{\cdot}, C^{\cdot}) \qquad\qquad g^{\cdot} : D^{\cdot} \to A^{\cdot}$$

*We have the composite $\Lambda\pi$ morphism:*

$$f^{\cdot} \circ g^{\cdot} : D^{\cdot} \to Hom^{\cdot}_{\Lambda}(B^{\cdot}, C^{\cdot})$$

*and $\Phi^0(f^{\cdot} \circ g^{\cdot}) : Tot^{\cdot}(D^{\cdot} \otimes_{\Lambda} B^{\cdot}) \to C^{\cdot}$. We claim we have the identity $\Phi^0(f^{\cdot} \circ g^{\cdot}) = \Phi^0(f^{\cdot}) \circ (g^{\cdot} \otimes 1^{\cdot}_B)$. That is, if we have the diagram below:*

$$D^{\cdot} \xrightarrow{g^{\cdot}} A^{\cdot} \xrightarrow{f^{\cdot}} Hom^{\cdot}_{\Lambda}(B^{\cdot}, C^{\cdot})$$

*Then applying $\Phi^0$ gives the diagram below:*

$$Tot^{\cdot}(D^{\cdot} \otimes_{\Lambda} B^{\cdot}) \xrightarrow{g^{\cdot} \otimes 1^{\cdot}_{B}} Tot^{\cdot}(A^{\cdot} \otimes_{\Lambda} B^{\cdot}) \xrightarrow{\Phi^{0}(f^{\cdot})} C^{\cdot}$$

*Proof.* Let $x \in D^{i}$ and $y \in B^{j}$.

$$\begin{aligned}
\Phi^{0}(f^{\cdot} \circ g^{\cdot})^{i+j}(x \otimes y) &= f^{i}(g^{i}(x))^{j}(y) \\
&= \Phi^{0}(f^{\cdot})^{i+j}(g^{i}(x) \otimes y) \\
&= (\Phi^{0}(f^{\cdot}) \circ (g^{\cdot} \otimes 1^{\cdot}_{B}))^{i+j}(x \otimes y)
\end{aligned}$$

This shows $\Phi^{0}(f^{\cdot} \circ g^{\cdot}) = \Phi^{0}(f^{\cdot}) \circ (g^{\cdot} \otimes 1^{\cdot}_{B})$.

□

**Lemma 3.1.10.** *Let $\Lambda$ be a commutative ring, $\pi$ a group, and $B^{\cdot}$, $C^{\cdot}$, $E^{\cdot}$, and $F^{\cdot}$ be complexes of $\Lambda\pi$ modules. There is a natural $\Lambda\pi$ chain map:*

$$\rho^{\cdot} : Tot^{\cdot}(Hom^{\cdot}_{\Lambda}(B^{\cdot}, C^{\cdot}) \otimes_{\Lambda} Hom^{\cdot}_{\Lambda}(E^{\cdot}, F^{\cdot})) \to Hom^{\cdot}_{\Lambda}(Tot^{\cdot}(B^{\cdot} \otimes_{\Lambda} E^{\cdot}), Tot^{\cdot}(C^{\cdot} \otimes_{\Lambda} F^{\cdot}))$$

*where for $h^{\cdot}_{1} \in Hom^{i}_{\Lambda}(B^{\cdot}, C^{\cdot})$, $h^{\cdot}_{2} \in Hom^{j}_{\Lambda}(E^{\cdot}, F^{\cdot})$, $b \in B^{l}$ and $e \in E^{k}$, we define:*

$$\rho^{i+j}(h^{\cdot}_{1} \otimes h^{\cdot}_{2})^{l+k}(b \otimes e) = (-1)^{lj}(h^{l}_{1}(b) \otimes h^{k}_{2}(e))$$

*Proof.* First I show that $\rho$ is a $\pi$ morphism. Degrees are omitted in the below. Let $g \in \pi$:

$$\begin{aligned}
\rho(g \cdot (h_{1} \otimes h_{2}))(b \otimes e) &= \rho(g \cdot h_{1} \otimes g \cdot h_{2})(b \otimes e) \\
&= (-1)^{lj}((g \cdot h_{1})(b) \otimes (g \cdot h_{2})(e)) \\
&= (-1)^{lj}(gh_{1}(g^{-1}b) \otimes gh_{2}(g^{-1}e)) \\
&= g \cdot (-1)^{lj}(h_{1}(g^{-1}b) \otimes h_{2}(g^{-1}e)) \\
&= g \cdot \rho(h_{1} \otimes h_{2})(g^{-1}b \otimes g^{-1}e) \\
&= g \cdot \rho(h_{1} \otimes h_{2})(g^{-1} \cdot (b \otimes e))
\end{aligned}$$

$$= (g \cdot \rho(h_1 \otimes h_2))(b \otimes e)$$

Thus $\rho(g \cdot (h_1 \otimes h_2)) = g \cdot \rho(h_1 \otimes h_2)$, so we have that $\rho$ is a $\pi$ morphism. Now I will show that it is a chain map. This is where the choice of sign is important.

$$\begin{aligned}
&\rho(d(h_1 \otimes h_2))(b \otimes e) \\
&\quad = \rho((dh_1 \otimes h_2) + (-1)^i (h_1 \otimes dh_2))(b \otimes e) \\
&\quad = (-1)^{lj}((dh_1)(b) \otimes h_2(e)) + (-1)^{i+l(j+1)}(h_1(b) \otimes (dh_2)(e)) \\
&\quad = (-1)^{lj}\big((d(h_1(b)) + (-1)^{i+1} h_1(db)) \otimes h_2(e)\big) \\
&\qquad\quad + (-1)^{i+l(j+1)}\big(h_1(b) \otimes (d(h_2(e)) + (-1)^{j+1} h_2(de))\big) \\
&\quad = (-1)^{lj}(d(h_1(b)) \otimes h_2(e)) + (-1)^{i+1+lj}(h_1(db) \otimes h_2(e)) \\
&\qquad\quad + (-1)^{i+l(j+1)}(h_1(b) \otimes d(h_2(e))) + (-1)^{i+l(j+1)+j+1}(h_1(b) \otimes h_2(de))
\end{aligned}$$

$$\begin{aligned}
&(d\rho(h_1 \otimes h_2))(b \otimes e) \\
&\quad = d(\rho(h_1 \otimes h_2)(b \otimes e)) + (-1)^{i+j+1}\rho(h_1 \otimes h_2)(d(b \otimes e)) \\
&\quad = d((-1)^{lj}(h_1(b) \otimes h_2(e))) + (-1)^{i+j+1}\rho(h_1 \otimes h_2)((db \otimes e) + (-1)^l (b \otimes de)) \\
&\quad = (-1)^{lj} d(h_1(b)) \otimes h_2(e) + (-1)^{lj+l+i} h_1(b) \otimes d(h_2(e)) \\
&\qquad\quad (-1)^{i+j+1+j(l+1)} h_1(db) \otimes h_2(e) + (-1)^{i+j+1+l+lj} h_1(b) \otimes h_2(de)
\end{aligned}$$

Matching up the terms in the above, we just need to check that the signs are the same. The sign on the $d(h_1(b)) \otimes h_2(e)$ is $(-1)^{lj}$ in both equations. On the $h_1(db) \otimes h_2(e)$ term, we have: $(-1)^{i+1+lj} = (-1)^{i+j+1+j(l+1)}$, since the $j$'s cancel. On the $h_1(b) \otimes d(h_2(e))$ term, we have: $(-1)^{i+l(j+1)} = (-1)^{lj+l+1}$. On the $h_1(b) \otimes h_2(de)$ term we have: $(-1)^{i+l(j+1)+j+1} = (-1)^{i+j+1+l+lj}$. Thus the expressions are equal, and we have shown $\rho \circ d = d\rho$.

□

**Corollary 3.1.11.** *Let $\Lambda$ be a commutative ring, $\pi$ a group, and $B^{\cdot}$, $C^{\cdot}$, be complexes of $\Lambda\pi$ modules. Let $r \geq 1$. There is a natural $\Lambda\pi$ chain map:*

$$\rho_r^{\cdot} : Tot^{\cdot}((Hom_{\Lambda}^{\cdot}(B^{\cdot}, C^{\cdot}))^{[r]}) \to Hom_{\Lambda}^{\cdot}(Tot^{\cdot}((B^{\cdot})^{[r]}), Tot^{\cdot}((C^{\cdot})^{[r]}))$$

*defined by the r-fold iteration of $\rho^{\cdot}$ from Lemma 3.1.10.*

*Proof.* The statement that $\rho_p^{\cdot}$ is a $\Lambda\pi$ chain map follows from the fact that $\rho^{\cdot}$ is a $\Lambda\pi$ chain map, from Lemma 3.1.10. □

**Lemma 3.1.12.** *Let $\Lambda$ be a commutative ring, $\pi$ a group, and $A^{\cdot}$, $B^{\cdot}$, $C^{\cdot}$, $D^{\cdot}$, $E^{\cdot}$, and $F^{\cdot}$ complexes of $\Lambda\pi$ modules. Suppose we have $\Lambda\pi$ chain maps:*

$$f^{\cdot} : A^{\cdot} \to Hom_{\Lambda}^{\cdot}(B^{\cdot}, C^{\cdot}) \qquad g^{\cdot} : D^{\cdot} \to Hom_{\Lambda}^{\cdot}(E^{\cdot}, F^{\cdot})$$

*Consider the composition:*

$$\begin{array}{ccc} Tot^{\cdot}(A^{\cdot} \otimes_{\Lambda} D^{\cdot}) & \xrightarrow{f^{\cdot} \otimes g^{\cdot}} & Tot^{\cdot}(Hom_{\Lambda}^{\cdot}(B^{\cdot}, C^{\cdot}) \otimes_{\Lambda} Hom_{\Lambda}^{\cdot}(E^{\cdot}, F^{\cdot})) \\ & & \downarrow{\scriptstyle \rho^{\cdot}} \\ & & Hom_{\Lambda}^{\cdot}(Tot^{\cdot}(B^{\cdot} \otimes_{\Lambda} E^{\cdot}), Tot^{\cdot}(C^{\cdot} \otimes_{\Lambda} F^{\cdot})) \end{array}$$

*where $\rho^{\cdot}$ is the map from Lemma 3.1.10. We have:*

$$\Phi^0(\rho^{\cdot} \circ (f^{\cdot} \otimes g^{\cdot})) : Tot^{\cdot}(Tot^{\cdot}(A^{\cdot} \otimes_{\Lambda} D^{\cdot}) \otimes_{\Lambda} Tot^{\cdot}(B^{\cdot} \otimes_{\Lambda} E^{\cdot})) \to Tot^{\cdot}(C^{\cdot} \otimes_{\Lambda} F^{\cdot})$$

$$\Phi^0(f^{\cdot}) : Tot^{\cdot}(A^{\cdot} \otimes_{\Lambda} B^{\cdot}) \to C^{\cdot} \qquad \Phi^0(g^{\cdot}) : Tot^{\cdot}(D^{\cdot} \otimes_{\Lambda} E^{\cdot}) \to F^{\cdot}$$

*We claim $\Phi^0(\rho^{\cdot} \circ (f^{\cdot} \otimes g^{\cdot}))$ is equal to the following composition:*

$$\begin{array}{ccc} Tot^{\cdot}(Tot^{\cdot}(A^{\cdot} \otimes_{\Lambda} D^{\cdot}) \otimes_{\Lambda} Tot^{\cdot}(B^{\cdot} \otimes_{\Lambda} E^{\cdot})) & \xrightarrow{1_A^{\cdot} \otimes U^{\cdot} \otimes 1_E^{\cdot}} & Tot^{\cdot}(Tot^{\cdot}(A^{\cdot} \otimes_{\Lambda} B^{\cdot}) \otimes_{\Lambda} Tot^{\cdot}(C^{\cdot} \otimes_{\Lambda} E^{\cdot})) \\ & & \downarrow{\scriptstyle \Phi^0(f^{\cdot}) \otimes \Phi^0(g^{\cdot})} \\ & & Tot^{\cdot}(C^{\cdot} \otimes_{\Lambda} F^{\cdot}) \end{array}$$

*where $U^{\cdot} : Tot^{\cdot}(D^{\cdot} \otimes_{\Lambda} B^{\cdot}) \to Tot^{\cdot}(B^{\cdot} \otimes_{\Lambda} D^{\cdot})$ swaps tensors with sign.*

*Proof.* Let $a \in A^i$, $b \in B^j$, $d \in D^k$, and $e \in E^l$. We have:

$$\begin{aligned}
&\Phi^0(\rho^\cdot \circ (f^\cdot \otimes g^\cdot))^{i+j+k+l}((a\otimes b)\otimes(d\otimes e)) \\
&\quad = (\rho^\cdot \circ (f^\cdot \otimes g^\cdot))^{i+j}(a\otimes b)^{k+l}(d\otimes e) \\
&\quad = \rho^{i+j}(f^i(a)\otimes g^j(b))^{k+l}(d\otimes e) \\
&\quad = (-1)^{kj}(f^i(a)^k(d)\otimes g^j(b)^l(e)) \\
&\quad = (-1)^{kj}[(\Phi^0(f^\cdot)^{i+k}(a\otimes d))\otimes(\Phi^0(g^\cdot)^{j+l}(b\otimes e))] \\
&\quad = (-1)^{kj}(\Phi^0(f^\cdot)\otimes\Phi^0(g^\cdot))((a\otimes d)\otimes(b\otimes e)) \\
&\quad = (\Phi^0(f^\cdot)\otimes\Phi^0(g^\cdot))(a\otimes U^{k+j}(b\otimes d)\otimes e) \\
&\quad = (\Phi^0(f^\cdot)\otimes\Phi^0(g^\cdot))(1^\cdot_A\otimes U^\cdot\otimes 1^\cdot_E)^{i+j+k+l}((a\otimes b)\otimes(d\otimes e))
\end{aligned}$$

Thus, we have shown:

$$\Phi^0(\rho^\cdot \circ (f^\cdot \otimes g^\cdot)) = (\Phi^0(f^\cdot)\otimes\Phi^0(g^\cdot))\circ(1^\cdot_A\otimes U^\cdot\otimes 1^\cdot_E)$$

□

**Lemma 3.1.13.** *Let $\Lambda$ be a commutative ring, $\pi$ a group, $A^\cdot$, $B^\cdot$, and $C^\cdot$ be complexes of $\Lambda\pi$ modules, and let $r \geq 1$. Let $f^\cdot : A^\cdot \to \mathbf{Hom}^\cdot_\Lambda(B^\cdot, C^\cdot)$ be a $\Lambda\pi$ chain map. We have the composition:*

$$Tot^\cdot((A^\cdot)^{[r]}) \xrightarrow{(f^\cdot)^{[r]}} (Hom^\cdot_\Lambda(B^\cdot,C^\cdot))^{[r]} \xrightarrow{\rho^\cdot_r} Hom^\cdot_\Lambda(Tot^\cdot((B^\cdot)^{[r]}), Tot^\cdot((C^\cdot)^{[r]}))$$

*where $\rho^\cdot_r$ is the natural $\Lambda\pi$ chain map from Corollary 3.1.11. Let $S^\cdot : Tot^\cdot((A^\cdot)^{[r]} \otimes (B^\cdot)^{[r]}) \to Tot^\cdot((A^\cdot \otimes B^\cdot)^{[r]})$ be the shuffling isomorphism. Then we have the identity $\Phi^0(\rho^\cdot_r \circ (f^\cdot)^{[p]}) = \Phi^0(f^\cdot)^{[r]} \circ S^\cdot$. That is, applying $\Phi^0$ to the diagram above yields the diagram below:*

$$Tot^\cdot((A^\cdot)^{[r]} \otimes_\Lambda (B^\cdot)^{[r]}) \xrightarrow{S^\cdot} Tot^\cdot((A^\cdot \otimes_\Lambda B^\cdot)^{[r]}) \xrightarrow{(f^\cdot)^{[r]}} Tot^\cdot((C^\cdot)^{[r]})$$

*Proof.* Let $a_i \in A^{j_i}$ and $b_i \in B^{k_i}$ for $i = 1, \ldots, r$. Define $s = \sum_{i>i'} j_i \cdot k_{i'}$, $n_1 = \sum_i j_i$, $n_2 = \sum_i k_i$, and $n = n_1 + n_2$. We have, $S^n(a \otimes b) = (-1)^s c$.

$$\begin{aligned}
&\Phi^0(\rho^\cdot_r \circ (f^\cdot)^{[r]})^n((a_1 \otimes \cdots \otimes a_r) \otimes (b_1 \otimes \cdots \otimes b_r)) \\
&\quad = (\rho^\cdot_r \circ (f^\cdot)^{[r]})^{n_1}(a_1 \otimes \cdots \otimes a_r)^{n_2}(b_1 \otimes \cdots \otimes b_r) \\
&\quad = \rho^{n_1}_r(f^{j_1}(a_1) \otimes \cdots \otimes f^{j_r}(a_r))^{n_2}(b_1 \otimes \cdots \otimes b_r) \\
&\quad = (-1)^s(f^{j_1}(a_1)^{k_1}(b_1) \otimes \cdots \otimes f^{j_r}(a_r)^{k_r}(b_r)) \\
&\quad = (-1)^s(\Phi^0(f^\cdot)^{j_1+k_1}(a_1 \otimes b_1) \otimes \cdots \otimes \Phi^0(f^\cdot)^{j_r+k_r}(a_r \otimes b_r)) \\
&\quad = (-1)^s(\Phi^0(f^\cdot)^{[r]})^n((a_1 \otimes b_1) \otimes \cdots \otimes (a_r \otimes b_r)) \\
&\quad = (\Phi^0(f^\cdot)^{[r]})^n(S^n((a_1 \otimes \cdots \otimes a_r) \otimes (b_1 \otimes \cdots \otimes b_r)))
\end{aligned}$$

This shows the relation $\Phi^0(\rho^\cdot_r \circ (f^\cdot)^{[r]}) = \Phi^0(f^\cdot)^{[r]} \circ S^\cdot$.

□

**Lemma 3.1.14.** *Let $\Lambda$ be a commutative ring, $\pi$ a group, and $A^\cdot$, $B^\cdot$, $C^\cdot$, $D^\cdot$, complexes of $\Lambda\pi$ modules. Let $\Phi^\cdot_1$, $\Phi^\cdot_2$, and $\Phi^\cdot_3$ denote the $\Lambda\pi$ chain map isomorphisms from Lemma 3.1.5:*

$$\begin{aligned}
&\Phi^\cdot_1 : Hom^\cdot_\Lambda(B^\cdot, Hom^\cdot_\Lambda(C^\cdot, D^\cdot)) \to Hom^\cdot_\Lambda(B^\cdot \otimes_\Lambda C^\cdot, D^\cdot) \\
&\Phi^\cdot_2 : Hom^\cdot_{\Lambda\pi}(A^\cdot, Hom^\cdot_\Lambda(B^\cdot, Hom_\Lambda(C^\cdot, D^\cdot))) \to Hom^\cdot_{\Lambda\pi}(A^\cdot \otimes_\Lambda B^\cdot, Hom^\cdot_\Lambda(C^\cdot, D^\cdot)) \\
&\Phi^\cdot_3 : Hom^\cdot_{\Lambda\pi}(A^\cdot \otimes_\Lambda B^\cdot, Hom^\cdot_\Lambda(C^\cdot, D^\cdot)) \to Hom^\cdot_{\Lambda\pi}(A^\cdot \otimes_\Lambda B^\cdot \otimes_\Lambda C^\cdot, D^\cdot) \\
&\Phi^\cdot_4 : Hom^\cdot_{\Lambda\pi}(A^\cdot, Hom^\cdot_\Lambda(B^\cdot \otimes_\Lambda C^\cdot, D^\cdot)) \to Hom^\cdot_{\Lambda\pi}(A^\cdot \otimes_\Lambda B^\cdot \otimes_\Lambda C^\cdot, D^\cdot)
\end{aligned}$$

*and suppose $f^\cdot$ is a $\Lambda\pi$ chain map in the composition below:*

$$A^\cdot \xrightarrow{f^\cdot} Hom^\cdot_\Lambda(B^\cdot, Hom^\cdot_\Lambda(C^\cdot, D^\cdot)) \xrightarrow{\Phi^\cdot_1} Hom^\cdot_\Lambda(B^\cdot \otimes C^\cdot, D^\cdot)$$

*We have, $\Phi^0_2(f^\cdot) : Tot^\cdot(A^\cdot \otimes_\Lambda B^\cdot) \to Hom^\cdot_\Lambda(C^\cdot, D^\cdot)$. I claim we have the identity:*

$$\Phi^0_4(\Phi^\cdot_1 \circ f^\cdot) = \Phi^0_3(\Phi^0_2(f^\cdot))$$

*Proof.* Let $a \in A^i$, $b \in B^j$, and $c \in C^k$. We have:

$$\begin{aligned}
\Phi_4^0(\Phi_1^{\cdot} \circ f^{\cdot})^{i+j+k}(a \otimes b \otimes c) &= \Phi_1^i(f^i(a))^{j+k}(b \otimes c) \\
&= f^i(a)^j(b)^k(c) \\
&= \Phi_2^0(f^{\cdot})^{i+j}(a \otimes b)^k(c) \\
&= \Phi_3^0(\Phi_2^0(f^{\cdot}))^{i+j+k}(a \otimes b \otimes c)
\end{aligned}$$

Thus we have the strange looking identity, $\Phi_4^0(\Phi_1^{\cdot} \circ f^{\cdot}) = \Phi_3^0(\Phi_2^0(f^{\cdot}))$. □

**Lemma 3.1.15.** *Let $\mathcal{A}$ be an abelian category, let $A^{\cdot}$, $B^{\cdot}$, and $C^{\cdot}$ be complexes in $\mathcal{A}$, let $g_1^{\cdot}, g_2^{\cdot} : A^{\cdot} \to B^{\cdot}$ be chain maps that are homotopic by homotopy $h^{\cdot} : A^{\cdot} \to B[-1]^{\cdot}$. Let $f^{\cdot} : B^{\cdot} \to C^{\cdot}$ be a chain map. Then $f^{\cdot}g_1^{\cdot}$ and $f^{\cdot}g_2^{\cdot}$ are homotopic by homotopy $f^{\cdot}h^{\cdot}$.*

*Proof.* Let $n \in \mathbb{Z}$. We have:

$$\begin{aligned}
f^n g_1^n - f^n g_2^n &= f^n(g_1^n - g_2^n) \\
&= f^n(d_B^{n-1}h^n + h^{n+1}d_A^n) \\
&= d_C^{n-1}(f^{n-1}h^n) + (f^n h^{n+1})d_A^n
\end{aligned}$$

□

**Lemma 3.1.16.** *Let $\mathcal{A}$ be an abelian category, let $A^{\cdot}$, $B^{\cdot}$, and $C^{\cdot}$ be complexes in $A$, let $g_1^{\cdot}, g_2^{\cdot} : A^{\cdot} \to B^{\cdot}$ be chain maps homotopic by homotopy $h^{\cdot} : A^{\cdot} \to B[-1]^{\cdot}$. Let $f^{\cdot} : C^{\cdot} \to A^{\cdot}$ be a chain map. Then $g_1^{\cdot}f^{\cdot}$ and $g_2^{\cdot}f^{\cdot}$ are homotopic by homotopy $h^{\cdot}f^{\cdot}$.*

*Proof.* Let $n \in \mathbb{Z}$. We have:

$$\begin{aligned}
g_1^n f^n - g_2^n f^n &= (g_1^n - g_2^n)f^n \\
&= (d_B^{n-1}h^n + h^{n+1}d_A^n)f^n
\end{aligned}$$

$$= d_B^{n-1}(h^n f^n) + (h^{n+1} f^{n+1}) d_C^n$$

□

## 3.2 Sheaves with Group Action and Resolutions

**Definition 3.2.1.** *Let $\Lambda$ be a commutative ring, $G$ be a finite group. and $F$ be a sheaf of $\Lambda$ modules on a topological space $X$. Let $M$ be a $\Lambda G$ module. Define $\mathrm{Pr}\,\mathbf{Hom}_\Lambda(M,F)$ to be the following presheaf of $\Lambda G$-modules on $X$. For an open set $U \subset X$, define:*

$$\mathrm{Pr}\,\mathbf{Hom}_\Lambda(M,F)(U) = Hom_\Lambda(M,F(U))$$

*and let $\mathbf{Hom}_\Lambda(M,F)$ be the sheafification. The action of $G$ is diagonal on Hom, and trivial on $F$. That is, if $f \in \mathrm{Pr}\,\mathbf{Hom}_\Lambda(M,F)(U)$, $m \in M$, $g \in G$, then:*

$$(g \cdot f)(m) = f(g^{-1}m)$$

Note that in [2], $M$ in the above was required to be finitely generated. We can get away without this restriction in our context, but in this paper's applications $M$ will always be finitely generated.

**Lemma 3.2.2.** *Let $\Lambda$, $G$, $X$, $F$, and $M$ be given as in Definition 3.2.1, but assume in addition that $M$ is a free $\Lambda$ module. Then the presheaf $\mathrm{Pr}\,\mathbf{Hom}_\Lambda(M,F)$ is actually a sheaf.*

*Proof.* Let $\{x_i\}_{i\in I}$ be a $\Lambda$ basis of $M$. For $N$ a $\Lambda$ module, we have the natural isomorphism:

$$\mathrm{Hom}_\Lambda(M,N) \cong \prod_{i\in I} N$$

where a $f \in \mathrm{Hom}_\Lambda(M,N)$ is sent to the element $(f(x_i))_{i\in I} \in \prod_{i\in I} N$. Thus, for every open set $U \subseteq X$, we have the natural identification:

$$\mathrm{Pr}\,\mathbf{Hom}_\Lambda(M,F)(U) = \mathrm{Hom}_\Lambda(M,F(U)) \cong \prod_{i\in I} F(U)$$

And since $\Pr \mathbf{Hom}_\Lambda(M,F)$ is naturally isomorphic to the product of sheaves, $\prod_{i\in I} F$, we have $\Pr \mathbf{Hom}_\Lambda(M,F)$ is a sheaf.

□

The next three lemmas are motivated by the proof of Theorem 2.2.1 from [2], on page 157.

**Lemma 3.2.3.** *Let $\Lambda$, $G$, $X$, and $F$ be as in Definition 3.2.1. The contravariant functor $\mathbf{Hom}_\Lambda(-,F)$ : ${}_{\Lambda G}Mod \to Sh_{\Lambda G}$ is left exact.*

*Proof.* Let

$$N_0 \xrightarrow{f} N_1 \xrightarrow{g} N_2 \to 0$$

be exact in ${}_{\Lambda G}$Mod. We have the induced sequence:

$$0 \to \mathbf{Hom}_\Lambda(N_2,F) \xrightarrow{\tilde{g}^*} \mathbf{Hom}_\Lambda(N_1,F) \xrightarrow{\tilde{f}^*} \mathbf{Hom}_\Lambda(N_0,F)$$

with $\tilde{g}^*$ and $\tilde{f}^*$ being induced by precomposition with $g$ and $f$ respectively. For every $x \in X$, the sequence on the stalks becomes:

$$0 \to \mathrm{Hom}_\Lambda(N_2,F_x) \xrightarrow{g^*} \mathrm{Hom}_\Lambda(N_1,F_x) \xrightarrow{f^*} \mathrm{Hom}_\Lambda(N_0,F_x)$$

Now the maps $g^*$ and $f^*$ really are precomposition with $g$ and $f$, so the sequence is left exact by the left exactness of $\mathrm{Hom}_\Lambda(-,F_x) : {}_{\Lambda G}\mathrm{Mod} \to {}_{\Lambda G}\mathrm{Mod}$. Since this sequence is left exact for all $x \in X$, the sequence of sheaves is left exact. □

**Lemma 3.2.4.** *Let $\Lambda$, $G$, $X$, $F$, and $M$ be as in Definition 3.2.1, and suppose $M$ be a free $\Lambda G$ module. Then for every sheaf $A$ in $Sh_\Lambda(X)$, we have the natural isomorphism of abelian groups:*

$$Hom_{\Lambda G}(M, Hom_\Lambda(A,F)) \cong Hom_{\Lambda G}(A, \mathbf{Hom}_\Lambda(M,F))$$

*Proof.* Because $M$ is a free $\Lambda G$ module, $M$ is also a free $\Lambda$ module, so by Lemma 3.2.2 we have the equality $\mathbf{Hom}_\Lambda(M,F)(U) = \mathrm{Hom}_\Lambda(M,F(U))$ for all open $U \subseteq X$. I first claim we have the natural isomorphism:

$$\rho : \mathrm{Hom}_\Lambda(M, \mathrm{Hom}_\Lambda(A,F)) \to \mathrm{Hom}_\Lambda(A, \mathbf{Hom}_\Lambda(M,F))$$

where $\rho$ is defined as follows, for all $f \in \mathrm{Hom}_\Lambda(M, \mathrm{Hom}_\Lambda(A,F))$, $m \in M$, $U \subseteq X$, and $a \in A(U)$:

$$\rho(f)_U(a)(m) = f(m)_U(a)$$

We may similarly define:

$$\beta : \mathrm{Hom}_\Lambda(A, \mathbf{Hom}_\Lambda(M,F)) \to \mathrm{Hom}_\Lambda(M, \mathrm{Hom}_\Lambda(A,F))$$

where for $h \in \mathrm{Hom}_\Lambda(A, \mathbf{Hom}_\Lambda(M,F))$, $m \in M$, $U \subseteq X$, and $a \in A(U)$:

$$\beta(h)(m)_U(a) = h_U(a)(m)$$

We have $\rho$ and $\beta$ are inverse to one another. Let $f \in \mathrm{Hom}_\Lambda(M, \mathrm{Hom}_\Lambda(A,F))$, $m \in M$, $U \subseteq X$, and $a \in A(U)$.

$$\beta(\rho(f))(m)_U(a) = \rho(f)_U(a)(m) = f(m)_U(a)$$

So $\beta \circ \rho = 1$. Now let $h \in \mathrm{Hom}_\Lambda(A, \mathbf{Hom}_\Lambda(M,F))$.

$$\begin{aligned} \rho(\beta(h))_U(a)(m) &= \beta(h)(m)_U(a) \\ &= h_U(a)(m) \end{aligned}$$

So $\rho \circ \beta = 1$. Thus $\rho$ is an isomorphism. It is clear that $\rho$ is natural in $A$. Finally, I must show that $\rho$ is a $G$ morphism. Let $g \in G$, $f \in \mathrm{Hom}_\Lambda(M, \mathrm{Hom}_\Lambda(A,F))$, $m \in M$, $U \subseteq X$, and $a \in A(U)$. We have:

$$(g \cdot \rho(f))_U(a)(m) = (g \cdot \rho(f)_U(a))(m)$$

$$
\begin{aligned}
&= \rho(f)_U(a)(g^{-1}\cdot m) \\
&= f(g^{-1}\cdot m)_U(a) \\
&= (g\cdot f)(m)_U(a) \\
&= \rho(g\cdot f)_U(a)(m)
\end{aligned}
$$

Thus $\rho$ is a $G$ morphism. Therefore $\rho$ induces an isomorphism of the $G$ equivariant part of its domain and range, which are precisely the $\Lambda G$ morphisms:

$$\rho_G : \mathrm{Hom}_{\Lambda G}(M, \mathrm{Hom}_{\Lambda}(A,F)) \to \mathrm{Hom}_{\Lambda G}(A, \mathbf{Hom}_{\Lambda}(M,F))$$

$\square$

**Lemma 3.2.5.** *For $\Lambda$, $G$, $X$, $F$, and $M$ given in Definition 3.2.1, the object $\mathbf{Hom}_{\Lambda}(M,F)$ is an object representing the contravariant functor $Hom_{\Lambda G}(M, Hom_{\Lambda}(-,F)) : Sh_{\Lambda G}(X) \to Ab$. That is, we have the natural isomorphism of abelian groups for all sheaves A in the category $Sh_{\Lambda G}(X)$:*

$$Hom_{\Lambda G}(M, Hom_{\Lambda}(A,F)) \cong Hom_{\Lambda G}(A, \mathbf{Hom}_{\Lambda}(M,F))$$

*Proof.* In [2], Theorem 2.2.1, on page 157, it is proven that the functor $\mathrm{Hom}_{\Lambda G}(M, \mathrm{Hom}_{\Lambda}(-,F))$ is representable, with the additional condition that $M$ is finitely generated. Here I will prove that the sheaf defined in Definition 3.2.1 is an object representing this functor.

We may choose a partial free resolution of $M$:

$$N_1 \to N_0 \to M \to 0$$

where $N_j$ is a free $\Lambda G$ module and the sequence is exact at $N_0$ and $M$. We then have the diagram below:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\Lambda G}(M, \mathrm{Hom}_\Lambda(A,F)) & \dashrightarrow & \mathrm{Hom}_{\Lambda G}(A, \mathbf{Hom}_\Lambda(M,F)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\Lambda G}(N_0, \mathrm{Hom}_\Lambda(A,F)) & \xrightarrow{\cong} & \mathrm{Hom}_{\Lambda G}(A, \mathbf{Hom}_\Lambda(N_0,F)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\Lambda G}(N_1, \mathrm{Hom}_\Lambda(A,F)) & \xrightarrow{\cong} & \mathrm{Hom}_{\Lambda G}(A, \mathbf{Hom}_\Lambda(N_1,F))
\end{array}$$

where the right column is due to Lemma 3.2.3 and the bottom two isomorphisms are from Lemma 3.2.4. This diagram allows us to induce the isomorphism:

$$\mathrm{Hom}_{\Lambda G}(M, \mathrm{Hom}_\Lambda(A,F)) \cong \mathrm{Hom}_{\Lambda G}(A, \mathbf{Hom}_\Lambda(M,F))$$

And this induced isomorphism is natural because the isomorphisms along the two bottom rows are.

□

**Lemma 3.2.6.** *Suppose $W$ is a projective $\Lambda G$ module and $I$ is an injective object in the category of sheaves of $\Lambda$ modules on a topological space $X$. Then $\mathbf{Hom}_\Lambda(W,I)$ is an injective object in the category of sheaves of $\Lambda G$ modules on $X$.*

*Proof.* This is proved in [2], Corollary 2.3.3, on page 158 in an abstract setting. I include a direct proof below.

It suffices to prove that the functor $\mathrm{Hom}_{\Lambda G}(-, \mathbf{Hom}_\Lambda(W,I)) : \mathrm{Sh}_{\Lambda G}(X) \to \mathrm{Ab}$ is right exact. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathrm{Sh}_{\Lambda G}(X)$. Because $I$ is injective in $\mathrm{Sh}_\Lambda(X)$, and $0 \to A \to B \to C \to 0$ is exact in $\mathrm{Sh}_\Lambda(X)$, we have:

$$0 \to \mathrm{Hom}_\Lambda(C,I) \to \mathrm{Hom}_\Lambda(B,I) \to \mathrm{Hom}_\Lambda(A,I) \to 0$$

is exact in ${}_{\Lambda G}\mathrm{Mod}$, where $G$ acts diagonally on Hom and trivially on $I$. Then because $W$ is a projective object in ${}_{\Lambda G}\mathrm{Mod}$, we have the exact sequence of abelian groups:

$$0 \to \mathrm{Hom}_{\Lambda G}(W, \mathrm{Hom}_{\Lambda}(C,I)) \to \mathrm{Hom}_{\Lambda G}(W, \mathrm{Hom}_{\Lambda}(B,I)) \to \mathrm{Hom}_{\Lambda G}(W, \mathrm{Hom}_{\Lambda}(A,I)) \to 0$$

Now by Lemma 3.2.5, the exact sequence above is naturally isomorphic to the following:

$$0 \to \mathrm{Hom}_{\Lambda G}(C, \mathbf{Hom}_{\Lambda}(W,I)) \to \mathrm{Hom}_{\Lambda G}(B, \mathbf{Hom}_{\Lambda}(W,I)) \to \mathrm{Hom}_{\Lambda G}(A, \mathbf{Hom}_{\Lambda}(W,I)) \to 0$$

This shows the functor $\mathrm{Hom}_{\Lambda G}(-, \mathbf{Hom}_{\Lambda}(W,I))$ is right exact, and thus, $\mathbf{Hom}_{\Lambda}(W,I)$ is an injective object in the category $\mathrm{Sh}_{\Lambda G}(X)$.

□

**Definition 3.2.7.** *Given a complex of $\Lambda G$ modules $M.$ and a complex of sheaves of $\Lambda$ modules $F^{\cdot}$, we can define the total complex $\mathbf{Hom}^{\cdot}_{\Lambda}(M., F^{\cdot})$, in a similar way to the total Hom complex of Definition 3.1.4.*

**Lemma 3.2.8.** *Let $W.$ be a projective resolution of $\Lambda$ in the cateogry of $\Lambda G$ modules, and let $I^{\cdot}$ be an injective resolution of $A$ in the category of sheaves of $\Lambda$ modules on a topological space $X$. Then $\mathbf{Hom}^{\cdot}_{\Lambda}(W., I^{\cdot})$ is an injective resolution of $A$ in the category of sheaves of $\Lambda G$ modules on $X$.*

*Proof.* This is a special case of Theorem 2.4.6 of [2], on page 161. □

**Lemma 3.2.9.** *Let $X$ be a topological space, $k$ a field, $G$ a finite group, and $M$ a $kG$ module. Then the functor $\mathbf{Hom}_k(M,-) : Sh_k(X) \to Sh_{kG}(X)$ is exact.*

*Proof.* Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence in $\mathrm{Sh}_k(X)$. I claim $0 \to \mathbf{Hom}_k(M,A) \xrightarrow{f_*} \mathbf{Hom}_k(M,B) \xrightarrow{g_*} \mathbf{Hom}_k(M,C) \to 0$ is exact. Let $x \in X$. Since $0 \to A_x \to B_x \to C_x \to 0$ is exact in $\mathrm{Vect}(k)$ and $\mathrm{Hom}_k(M,-) : \mathrm{Vect}(k) \to {}_{kG}\mathrm{Mod}$ is exact, we have exactness of:

$$0 \to \mathrm{Hom}_k(M,A_x) \xrightarrow{(f_x)_*} \mathrm{Hom}_k(M,B_x) \xrightarrow{(g_x)_*} \mathrm{Hom}_k(M,C_x) \to 0$$

And the above sequence is the following:

$$0 \to \mathbf{Hom}_k(M,A)_x \xrightarrow{(f_*)_x} \mathbf{Hom}_k(M,B)_x \xrightarrow{(g_*)_x} \mathbf{Hom}_k(M,C)_x \to 0$$

This shows the sequence $0 \to \mathbf{Hom}_k(M,A) \to \mathbf{Hom}_k(M,B) \to \mathbf{Hom}_k(M,C) \to 0$ is exact on all stalks, and hence is exact.

□

**Lemma 3.2.10.** *For $k$ any field, in the category of sheaves of $k$ vector spaces on $X$, the tensor product is an exact bifunctor.*

*Proof.* We have that tensor product is an exact bifunctor in the category of $k$ vector spaces. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathrm{Sh}_k(X)$, and $D$ an object in $\mathrm{Sh}_k(X)$. The sequence $0 \to D \otimes_k A \to D \otimes_k B \to D \otimes_k C \to 0$ is exact if and only if it is exact at all stalks of $X$. Taking the stalk at a point $x \in X$ gives the sequence:

$$0 \to D_x \otimes_k A_x \to D_x \otimes_k B_x \to D_x \otimes_k C_x \to 0$$

which is exact because $0 \to A_x \to B_x \to C_x \to 0$ is exact in $\mathrm{Vect}(k)$ and $D_x \otimes_k -$ is exact in $\mathrm{Vect}(k)$. This shows $D \otimes_k - : \mathrm{Sh}_k(X) \to \mathrm{Sh}_k(X)$ is an exact functor. The argument for $- \otimes_k D$ is symmetric. Thus tensor product is an exact bifunctor.

□

**Lemma 3.2.11.** *Let $X$ be a topological space and $k$ a field. Suppose $A$ and $B$ are sheaves of $k$ vector spaces on $X$, and $J^{\cdot}$ and $K^{\cdot}$ are resolutions of $A$ and $B$ respectively in the category $Sh_k(X)$, with embeddings $\varepsilon : A \to J^0$ and $\gamma : B \to K^0$. Then $Tot^{\cdot}(J^{\cdot} \otimes_k K^{\cdot})$ is a resolution of $A \otimes_k B$, with embedding $\varepsilon \otimes_k \gamma$.*

*Proof.* Special case of Lemma 3.2.16 in which $A^{\cdot}$ and $B^{\cdot}$ are concentrated in degree 0.

□

**Corollary 3.2.12.** *Let $A$, $J^{\cdot}$, and $X$ be as in Lemma 3.2.11, and let $r > 0$. Then $Tot^{\cdot}((J^{\cdot})^{[r]})$ is a resolution of $A^{[r]}$.*

*Proof.* Inductive application of Lemma 3.2.11 on $r$. □

The next two lemmas are from [6]. Their numbers are subject to change but their tags are permanent.

**Lemma 3.2.13.** *The Stacks Project: Tag 013P* [1] *(Lemma 13.18.6)*

*Let $\mathcal{A}$ be an abelian category. Consider a solid diagram:*

$$\begin{tikzcd} K^\cdot \arrow[r, "\alpha^\cdot"] \arrow[d, "\gamma"] & L^\cdot \arrow[dl, dashed, "\beta^\cdot"] \\ I^\cdot & \end{tikzcd}$$

*where $I^\cdot$ is bounded below and consists of injective objects, and $\alpha^\cdot$ is a quasi-isomorphism.*

1. *There exists a map of complexes $\beta^\cdot$ making the diagram commute up to homotopy.*

2. *If $\alpha^\cdot$ is injective in every degree then we can find a $\beta^\cdot$ which makes the diagram commute.*

**Lemma 3.2.14.** *The Stacks Project: Tag 013S* [2] *(Lemma 13.18.7)*

*Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{tikzcd} K^\cdot \arrow[r, "\alpha^\cdot"] \arrow[d, "\gamma"] & L^\cdot \arrow[dl, dashed, "\beta_i^\cdot"] \\ I^\cdot & \end{tikzcd}$$

*where $I^\cdot$ is bounded below and consists of injective objects, and $\alpha^\cdot$ is a quasi-isomorphism. Any two morphisms $\beta_1^\cdot$, $\beta_2^\cdot$ making the diagram commute up to homotopy are homotopic.*

**Definition 3.2.15.** *In an abelian category, a chain map $\varepsilon^\cdot : A^\cdot \to B^\cdot$ will be called a resolution if $\varepsilon^\cdot$ is injective in each degree and a quasi-isomorphism. If $B^\cdot$ is bounded from below and injective in each degree, then $\varepsilon^\cdot$ is called an injective resolution.*

The following generalizes Lemma 3.2.11 to the case where $A$ and $B$ are complexes.

**Lemma 3.2.16.** *Let $X$ be a topological space and $k$ a field. Suppose $A^\cdot$, $B^\cdot$, $J^\cdot$, $K^\cdot$ are complexes in $Comp(Sh_k(X))$, with chain maps $\varepsilon^\cdot : A^\cdot \to J^\cdot$ and $\gamma : B^\cdot \to K^\cdot$.*

1. *If $\varepsilon^\cdot$ and $\gamma$ are injective in each degree, then $\varepsilon^\cdot \otimes \gamma$ is injective in each degree.*

[1] https://stacks.math.columbia.edu/tag/013P

[2] https://stacks.math.columbia.edu/tag/013S

2. *If* $\varepsilon^{\cdot}$ *and* $\gamma^{\cdot}$ *are both quasi-isomorphisms then* $\varepsilon^{\cdot} \otimes \gamma^{\cdot}$ *is a quasi-isomorphism.*

*Proof.* Suppose $\varepsilon^{\cdot}$ and $\gamma^{\cdot}$ are injective in each degree. Let $x \in X$. Then $\varepsilon^{\cdot}_x$ and $\gamma^{\cdot}_x$ are injective maps of vector spaces in each degree. Hence, we have $\varepsilon^k_x \otimes_k \gamma^{n-k}_x : A^k \otimes B^{n-k} \to J^k \otimes K^{n-k}$ is injective for all $n$ and $k$. Then we have:

$$(\varepsilon_x \otimes_k \gamma_x)^n : \mathrm{Tot}^n(A^{\cdot} \otimes_k B^{\cdot}) \to \mathrm{Tot}^n(J^{\cdot} \otimes_k K^{\cdot})$$

is a direct sum of injective maps, and hence, is injective. Finally, the injectivity of $(\varepsilon^{\cdot} \otimes_k \gamma^{\cdot})^n_x = (\varepsilon^{\cdot}_x \otimes_k \gamma^{\cdot}_x)^n$ for every $x \in X$ implies $(\varepsilon^{\cdot} \otimes_k \gamma^{\cdot})^n$ is an injective map of sheaves. Thus $(\varepsilon^{\cdot} \otimes_k \gamma^{\cdot})^{\cdot}$ is injective in every degree when $\varepsilon^{\cdot}$ and $\gamma^{\cdot}$ are.

Now suppose $\varepsilon^{\cdot}$ and $\gamma^{\cdot}$ are quasi-isomorphisms. Let $x \in X$. We have $\varepsilon^{\cdot}_x$ and $\gamma^{\cdot}_x$ are quasi-isomorphisms. Using page 113 of [1], we have the exact sequences of $\mathbb{F}_p$ vector spaces:

$$0 \to H^{\cdot}(A^{\cdot}_x) \otimes H^{\cdot}(B^{\cdot}_x) \to H^{\cdot}(A^{\cdot}_x \otimes B^{\cdot}_x) \to \mathrm{Tor}_1(H^{\cdot}(A^{\cdot}_x), H^{\cdot}(B^{\cdot}_x))[1] \to 0$$

$$0 \to H^{\cdot}(K^{\cdot}_x) \otimes H^{\cdot}(J^{\cdot}_x) \to H^{\cdot}(K^{\cdot}_x \otimes J^{\cdot}_x) \to \mathrm{Tor}_1(H(K^{\cdot}_x), H(J^{\cdot}_x))[1] \to 0$$

The sequences exists because we are in the category of $\mathbb{F}_p$ vector spaces, so all the following Tor groups are zero, since all objects are flat:

$$\mathrm{Tor}_1(B^{\cdot}(A^{\cdot}_x), B^{\cdot}(B^{\cdot}_x)) = 0 = \mathrm{Tor}_1(H^{\cdot}(A^{\cdot}_x), B^{\cdot}(B^{\cdot}_x))$$
$$\mathrm{Tor}_1(B^{\cdot}(A^{\cdot}_x), Z^{\cdot}(B^{\cdot}_x)) = 0 = \mathrm{Tor}_1(H^{\cdot}(A^{\cdot}_x), Z^{\cdot}(B^{\cdot}_x))$$

$$\mathrm{Tor}_1(B^{\cdot}(K^{\cdot}_x), B^{\cdot}(J^{\cdot}_x)) = 0 = \mathrm{Tor}_1(H^{\cdot}(K^{\cdot}_x), B^{\cdot}(J^{\cdot}_x))$$
$$\mathrm{Tor}_1(B^{\cdot}(K^{\cdot}_x), Z^{\cdot}(J^{\cdot}_x)) = 0 = \mathrm{Tor}_1(H^{\cdot}(K^{\cdot}_x), Z^{\cdot}(J^{\cdot}_x))$$

Furthermore, the Tor groups at the end of the sequence are zero, so the Künneth map gives isomorphisms:

$$H^{\cdot}(A^{\cdot}_x)\otimes_k H^{\cdot}(B^{\cdot}_x) \to H^{\cdot}(A^{\cdot}_x \otimes B^{\cdot}_x)$$
$$H^{\cdot}(K^{\cdot}_x)\otimes_k H^{\cdot}(J^{\cdot}_x) \to H^{\cdot}(K^{\cdot}_x \otimes J^{\cdot}_x)$$

where the Künneth map is given by $[a]\otimes[b]\mapsto[a\otimes b]$. We now have the commutative diagram:

$$\begin{array}{ccc} H^{\cdot}(A^{\cdot}_x)\otimes H^{\cdot}(B^{\cdot}_x) & \longrightarrow & H^{\cdot}(A^{\cdot}_x\otimes B^{\cdot}_x) \\ \Big\downarrow{\scriptstyle \overline{\varepsilon^{\cdot}_x}\otimes\overline{\gamma_x}} & & \Big\downarrow{\scriptstyle \overline{\varepsilon^{\cdot}_x\otimes\gamma_x}} \\ H^{\cdot}(K^{\cdot}_x)\otimes H^{\cdot}(J^{\cdot}_x) & \longrightarrow & H^{\cdot}(K^{\cdot}_x\otimes J^{\cdot}_x) \end{array}$$

where the horizontal maps are the Künneth maps. Because the diagram commutes and every other map is an isomorphism, we have $\overline{\varepsilon^{\cdot}_x\otimes\gamma_x}$ is an isomorphism. Hence, $\varepsilon^{\cdot}_x\otimes_k\gamma_x$ is a quasi-isomorphism, and since $\varepsilon^{\cdot}\otimes_k\gamma$ is a quasi-isomorphism on all stalks, we have $\varepsilon^{\cdot}\otimes_k\gamma$ is a quasi-isomorphism.

□

**Corollary 3.2.17.** *Let $X$ be a topological space, $k$ a field, $A^{\cdot}$ a complex in $Sh_k(X)$, and $\varepsilon^{\cdot}: A^{\cdot}\to J^{\cdot}$ a resolution of $A^{\cdot}$, and $r\geq 1$. Then the following is a resolution.*

$$(\varepsilon^{\cdot})^{[r]}:(A^{\cdot})^{[r]}\to(J^{\cdot})^{[r]}$$

*Proof.* Use induction on $r$ and apply Lemma 3.2.16. □

**Lemma 3.2.18.** *Let $\mathcal{A}$ be an abelian category and suppose $A^{\cdot}$ and $B^{\cdot}$ are complexes in $\mathcal{A}$ with $A^{\cdot}$ bounded above and $B^{\cdot}$ bounded below. Then $Hom^{\cdot}(A^{\cdot},B^{\cdot})$ is bounded below. Furthermore, for each $n\in\mathbb{Z}$ only finitely many terms in the product below are non-zero:*

$$Hom^n(A^{\cdot},B^{\cdot})=\prod_{i\in\mathbb{Z}}Hom(A^i,B^{i+n})$$

*Proof.* Let $k,l\in\mathbb{Z}$ such that $A^n=0$ for all $n\geq k$ and $B^n=0$ for all $n\leq l$. I claim $\mathrm{Hom}^{\cdot}(A^{\cdot},B^{\cdot})$ is bounded below by $l-k$ and for all $n\in\mathbb{Z}$, there are only finitely many $i\in\mathbb{Z}$ such that

$\mathrm{Hom}(A^i, B^{i+n}) \neq 0$. Suppose $A^i \neq 0$ and $B^{i+n} \neq 0$. Then we must have $i < k$ and $i + n > l$. Thus, for every fixed $n$, we have $i$ contained within the finite range of integers, $l - n < i < k$. Thus, for every $n$, there are only finitely many $i$ such that $\mathrm{Hom}(A^i, B^{i+n}) \neq 0$. In particular, when $l - n \geq k$, there are no values for $i$ in which $\mathrm{Hom}(A^i, B^{i+n}) \neq 0$. This implies $\mathrm{Hom}^n(A^\cdot, B^\cdot) = 0$ for all $n \leq l - k$. Thus the complex $\mathrm{Hom}^\cdot(A^\cdot, B^\cdot)$ is bounded below at $l - k$.

□

**Corollary 3.2.19.** *Let $\mathcal{A}$ be an abelian category and suppose $A_\cdot$ and $B^\cdot$ are complexes in $\mathcal{A}$ with $B^\cdot$ bounded below and $A_\cdot$ (lowered index) bounded below. Then $Hom^\cdot(A_\cdot, B^\cdot)$ is bounded below, and for every $n \in \mathbb{Z}$ only finitely many terms in the product are non-zero:*

$$Hom^n(A_\cdot, B^\cdot) = \prod_{i \in \mathbb{Z}} Hom(A_i, B^{n-i})$$

*Proof.* When one raises the index on $A_\cdot$ using the convention $A_n = A^{-n}$, we have $A^\cdot$ is bounded above and the result follows from Lemma 3.2.18. □

**Corollary 3.2.20.** *Let $X$ be a topological space, $\Lambda$ a commutative ring, $G$ a finite group, $M_\cdot$ a (lowered index) bounded below complex of finitely generated $\Lambda G$ modules, and $A^\cdot$ a bounded below complex in $Sh_\Lambda(X)$. Then $\mathbf{Hom}^\cdot_\Lambda(M_\cdot, A^\cdot)$ is a bounded below complex in $Sh_{\Lambda G}(X)$, and for every $n \in \mathbb{Z}$, only finitely many terms in the product below are non-zero:*

$$\mathbf{Hom}^n_\Lambda(M_\cdot, A^\cdot) = \prod_{i \in \mathbb{Z}} \mathbf{Hom}_\Lambda(M_i, A^{n-i})$$

*Proof.* Use $\mathbf{Hom}_\Lambda$ in place of Hom of Lemma 3.2.18, and follow with Corollary 3.2.19. □

**Lemma 3.2.21.** *Let $X$ be a topological space, $k$ a field, $G$ a finite group, $A^\cdot$ a complex of sheaves of $k$ vector spaces, and $\varepsilon^\cdot : A^\cdot \to I^\cdot$ an injective resolution of $A^\cdot$ in the category $Sh_k(X)$. That is, $I^\cdot$ is injective in each degree, bounded from below, and $\varepsilon^\cdot$ is an injective quasi-isomorphism. Let $V_\cdot$ be a $G$ projective resolution of $k$, finitely generated in each degree, with augmentation map $\pi_\cdot : V_\cdot \to k[0]_\cdot$. Then the following is an injective resolution of $A^\cdot$ in the category $Sh_{kG}(X)$:*

$$\nu^\cdot : A^\cdot = \mathbf{Hom}^\cdot_k(k, A^\cdot) \to \mathbf{Hom}^\cdot_k(V_\cdot, I^\cdot)$$

*where $\nu^{\cdot}$ is induced by pre-composition by $\pi_{.}$ and post-composition by $\varepsilon^{\cdot}$.*

*Proof.* This is more or less a generalization of Theorem 2.4.6 of [2], on page 161, in which $A^{\cdot}$ is concentrated in degree 0, although we only work in the context of sheaves of vector spaces.

Since $V_{.}$ and $I^{\cdot}$ are bounded below, $\mathbf{Hom}_k^{\cdot}(V_{.}, I^{\cdot})$ is bounded from below by Corollary 3.2.20. For a fixed degree $n$ we have:

$$\mathbf{Hom}_k^n(V_{.}, I^{\cdot}) = \prod_{j\in\mathbb{Z}} \mathbf{Hom}_k(V_j, I^{n-j})$$

By Lemma 3.2.6, each $\mathbf{Hom}_k^n(V_j, I^{n-j})$ is an injective object in $\mathrm{Sh}_{kG}(X)$. Then because products of injective objects are injective, we have $\mathbf{Hom}_k^n(V_{.}, I^{\cdot})$ is an injective object in $\mathrm{Sh}_{kG}(X)$. Thus $\mathbf{Hom}_k^{\cdot}(V_{.}, I^{\cdot})$ is a bounded below complex of injective objects in $\mathrm{Sh}_{kG}(X)$. Now I must show $\nu^{\cdot}$ is injective in each degree, and a quasi-isomorphism.

To show that $\nu^{\cdot}$ is injective in each degree, observe that $\nu^{\cdot}$ is induced by post-composition with the monomorphism $\varepsilon^{\cdot} : A^{\cdot} \to I^{\cdot}$, and pre-composition by the epimorphism, $\pi : V_0 \to k$, both of which are injective operations on Hom sets.

To show that $\nu^{\cdot}$ is a quasi-isomorphism, it suffices to check on the stalks. Let $x \in X$. I claim $\nu_x^{\cdot}$ is a quasi-isomorphism. After taking the stalk, we are in the category of $k$ vector spaces, and we have the second Künneth exact sequences from [1], page 114:

$$0 \to \mathrm{Ext}^1(H_{.}(k[0]_{.}), H^{\cdot}(A_x^{\cdot}))[1] \to H^{\cdot}(\mathrm{Hom}_k^{\cdot}(k[0]_{.}, A_x^{\cdot})) \to \mathrm{Hom}_k^{\cdot}(k[0]_{.}, A_x^{\cdot}) \to 0$$
$$0 \to \mathrm{Ext}^1(H_{.}(V_{.}), H^{\cdot}(I_x^{\cdot}))[1] \to H^{\cdot}(\mathrm{Hom}_k^{\cdot}(V_{.}, I_x^{\cdot})) \to \mathrm{Hom}_k^{\cdot}(V_{.}, I_x^{\cdot}) \to 0$$

We have these sequences because we are in the category of $k$ vector spaces, so all objects are projective and injective, so the following $\mathrm{Ext}^1$ groups are zero:

$$\mathrm{Ext}^1(B_{.}(k[0]_{.}), B^{\cdot}(A_x^{\cdot})) = 0 = \mathrm{Ext}^1(B_{.}(k[0]_{.}), H^{\cdot}(A_x^{\cdot}))$$
$$\mathrm{Ext}^1(Z_{.}(k[0]_{.}), B^{\cdot}(A_x^{\cdot})) = 0 = \mathrm{Ext}^1(Z_{.}(k[0]_{.}), H^{\cdot}(A_x^{\cdot}))$$

$$\operatorname{Ext}^1(B.(V.), B^{\cdot}(I^{\cdot}_x)) = 0 = \operatorname{Ext}^1(B.(V.), H^{\cdot}(I^{\cdot}_x))$$
$$\operatorname{Ext}^1(Z.(V.), B^{\cdot}(I^{\cdot}_x)) = 0 = \operatorname{Ext}^1(Z.(V.), H^{\cdot}(I^{\cdot}_x))$$

We also have $\operatorname{Ext}^1(H.(k[0].), H^{\cdot}(A^{\cdot}_x))[1] = \operatorname{Ext}^1(H.(V.), H^{\cdot}(I^{\cdot}_x))[1] = 0$, so the Künneth maps are isomorphisms:

$$H^{\cdot}(\operatorname{Hom}^{\cdot}_k(k[0].,A^{\cdot}_x)) \to \operatorname{Hom}^{\cdot}_k(H.(k[0].), H^{\cdot}(A^{\cdot}_x))$$
$$H^{\cdot}(\operatorname{Hom}^{\cdot}_k(V.,I^{\cdot}_x)) \to \operatorname{Hom}^{\cdot}_k(H.(V.), H^{\cdot}(I^{\cdot}_x))$$

Because of how the Künneth maps are defined, we have the following commutative square:

$$\begin{tikzcd}
H^{\cdot}(\operatorname{Hom}^{\cdot}_k(k[0].,A^{\cdot}_x)) \arrow[r] \arrow[d, "\overline{\nu^{\cdot}_x}"] & \operatorname{Hom}^{\cdot}_k(H.(k[0].), H^{\cdot}(A^{\cdot}_x)) \arrow[d, "\beta"] \\
H^{\cdot}(\operatorname{Hom}^{\cdot}_k(V.,I^{\cdot}_x)) \arrow[r] & \operatorname{Hom}^{\cdot}_k(H.(V.), H^{\cdot}(I^{\cdot}_x))
\end{tikzcd}$$

The horizontal maps are the Künneth isomorphisms, and $\beta$ is precomposition by $\overline{\pi.}$ and postcomposition by $\overline{\varepsilon^{\cdot}_x}$. Since both $\overline{\pi.}$ and $\overline{\varepsilon^{\cdot}_x}$ are isomorphisms, $\beta$ is an isomorphism. Then since every other edge in the square is an isomorphism and the square commutes, $\overline{\nu^{\cdot}_x}$ is an isomorphism. So we have shown $\nu^{\cdot}_x$ is a quasi-isomorphism for all $x \in X$, and this implies $\nu^{\cdot}$ is a quasi-isomorphism. Now all the desired properties have been shown.

$\square$

# Chapter 4

# Steenrod Operations on Sheaf Cohomology

In this chapter we give a construction of Steenrod operations on sheaf cohomology using May's framework. These operations are known to exist and were constructed by Epstein in [2]. Here I will just express Epstein's construction in terms of May's, citing many of his results along the way. The next section generalizes this section to the case of sheaf hypercohomology, making the results of this section a special case. Because of this, I will actually omit all proofs in this chapter, and just state which results are special cases of those in the next chapter.

Let $X$ be a topological space. Let $A$ be a sheaf of commutative $\mathbb{F}_p$-algebras on $X$. Choose an injective resolution $\iota : A \hookrightarrow I^{\cdot}$ in the category of sheaves of $\mathbb{F}_p$ vector spaces. Let $T$ denote the global section functor, $\Gamma(X,-)$. Define $K^{\cdot} = T(I^{\cdot})$. Then the cohomology groups of $K^{\cdot}$ compute the sheaf cohomology groups of $X$ with coefficients in $A$:

$$H^n(K^{\cdot}) = H^n(X,A)$$

We will construct a homotopy associative product on $K^{\cdot}$, which defines an associative cup product on $H^{\cdot}(X,A)$. We will then define a map $\theta^{\cdot} : W_{\cdot} \otimes (K^{\cdot})^{[p]} \to K^{\cdot}$, which satisfies the axioms needed for $(K^{\cdot},\theta^{\cdot})$ to be an object in category $\mathcal{C}(p)$ defined by May. We will then show $(K^{\cdot},\theta^{\cdot})$ is a Cartan and Adem object, which will show the Steenrod operations on $H^{\cdot}(X,A)$ satisfy the Cartan formula and Adem relations.

## 4.1 The Product on $K^{\cdot}$

In this section we will construct the product that makes $K^{\cdot}$ into a homotopy associative graded $\mathbb{F}_p$ algebra. Let $m : A \otimes A \to A$ denote the multiplication map on $A$.

**Definition 4.1.1.** *Consider the diagram below:*

$$\begin{tikzcd} Tot^{\cdot}(I^{\cdot} \otimes I^{\cdot}) \arrow[r, dashed, "\tilde{m}^{\cdot}"] & I^{\cdot} \\ A \otimes A \arrow[u, hook, "\iota\otimes\iota"] \arrow[r, "m"] & A \arrow[u, hook, "\iota"'] \end{tikzcd}$$

*By Lemma 3.2.11, $Tot^{\cdot}(I^{\cdot} \otimes I^{\cdot})$ is a resolution of $A \otimes A$. Because of this and the fact that $I^{\cdot}$ is an injective resolution of $A$, there exists a chain map $\tilde{m}^{\cdot}$ unique up to homotopy making the diagram commute. We then apply the global section functor $T$ and precompose by the natural map $\gamma$ to obtain the product $M^{\cdot} : Tot^{\cdot}(K^{\cdot} \otimes K^{\cdot}) \to K^{\cdot}$:*

$$M^{\cdot} : Tot^{\cdot}(T(I^{\cdot}) \otimes T(I^{\cdot})) \xrightarrow{\gamma} T(Tot^{\cdot}(I^{\cdot} \otimes I^{\cdot})) \xrightarrow{T(\tilde{m}^{\cdot})} T(I^{\cdot})$$

*Note this definition only defines $M^{\cdot}$ uniquely up to homotopy.*

**Lemma 4.1.2.** *The product map $M^{\cdot}$ makes $K^{\cdot}$ a homotopy associative $\mathbb{F}_p$ algebra. The cup product induced on the cohomology groups $H^{\cdot}(K^{\cdot})$ is associative.*

*Proof.* Special case of Lemma 5.1.3, where here $A^{\cdot} = A[0]^{\cdot}$ is concentrated in degree 0. $\square$

**Definition 4.1.3.** *In a similar fashion, we may define a $p$ iterated product, using the diagram below:*

$$\begin{tikzcd} Tot^{\cdot}((I^{\cdot})^{[p]}) \arrow[r, dashed, "\tilde{m}^{\cdot}_p"] & I^{\cdot} \\ A^{[p]} \arrow[u, hook, "\iota^{[p]}"] \arrow[r, "m_p"] & A \arrow[u, hook, "\iota"'] \end{tikzcd}$$

*where $m_p$ denotes the $p$-fold product on $A$. By Corollary 3.2.12, $Tot^{\cdot}((I^{\cdot})^{[p]})$ is a resolution of $A^{\cdot}$. Because $I^{\cdot}$ is an injective resolution of $A$, there is a chain map $\tilde{m}^{\cdot}_p$ unique up to* *homotopy making the diagram commute. We can then define a p-fold product, $M_p : (K^\cdot)^{[p]} \to K^\cdot$ by applying the global section functor $T$ and precomposing with the natural map $\gamma_p$:*

$$M^\cdot_p : T(I^\cdot)^{[p]} \xrightarrow{\gamma_p} T((I^\cdot)^{[p]}) \xrightarrow{T(m^\cdot_p)} T(I^\cdot)$$

**Lemma 4.1.4.** *The p-fold product of $M^\cdot$ is $\mathbb{F}_p$ homotopic to $M^\cdot_p$ as chain maps, $Tot^\cdot((K^\cdot)^{[p]}) \to K^\cdot$.*

*Proof.* Special case of Lemma 5.1.5 where $A^\cdot$ is concentrated in degree 0. □

## 4.2 Construction of $\theta^\cdot$

In this section we construct the map $\theta^\cdot$ associated to the homotopy associative $\mathbb{F}_p$-DGA, $K^\cdot = T(I^\cdot)$. Recall the standard free resolution $W.$ of $\mathbb{F}_p$ in the category of $\mathbb{F}_p\pi$ modules. Because $I^\cdot$ is an injective resolution of $A$ in the category $\mathrm{Sh}_{\mathbb{F}_p}(X)$, by Lemma 3.2.8, $\mathbf{Hom}^\cdot_{\mathbb{F}_p}(W., I^\cdot)$ is an injective resolution of $A$ in the category $\mathrm{Sh}_{\mathbb{F}_p\pi}(X)$. By Corollary 3.2.12, $\mathrm{Tot}^\cdot((I^\cdot)^{[p]})$ is a resolution of $A^{[p]}$ in the category $\mathrm{Sh}_{\mathbb{F}_p}(X)$. We now let $\pi$ act on $A^{[p]}$ and $\mathrm{Tot}^\cdot((I^\cdot)^{[p]})$ by cyclic permutations of the tensors, with sign change determined by the grading on $I^\cdot$. Because multiplication in $A$ is commutative, $m_p : A^{[p]} \to A$ is a $\pi$ morphism. Hence, we have the solid diagram below in which all solid arrows are $\mathbb{F}_p\pi$ morphisms:

$$\begin{tikzcd}
\mathrm{Tot}^\cdot((I^\cdot)^{[p]}) \arrow[r, dashed, "\beta^\cdot"] & \mathbf{Hom}^\cdot_{\mathbb{F}_p}(W., I^\cdot) \\
A^{[p]} \arrow[u, hook, "\iota^{[p]}"] \arrow[r, "m_p"] & A \arrow[u, hook, "\nu^\cdot"']
\end{tikzcd}$$

By Corollary 3.2.12, $\mathrm{Tot}^\cdot((I^\cdot)^{[p]})$ is a resolution of $A^{[p]}$. Because all morphisms are $\mathbb{F}_p\pi$ morphisms and because $\mathbf{Hom}^\cdot_{\mathbb{F}_p}(W., I^\cdot)$ is an injective resolution of $A$ in $\mathrm{Sh}_{\mathbb{F}_p\pi}(X)$, there exists a $\mathbb{F}_p\pi$ chain map $\beta^\cdot$ unique up to homotopy making the diagram commute. We then define the map $\hat{\theta}^\cdot : \mathrm{Tot}^\cdot((K^\cdot)^{[p]}) \to \mathrm{Hom}^\cdot_{\mathbb{F}_p}(W., K^\cdot)$ by the following composition:

$$\hat{\theta}^{\cdot} : \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}) \xrightarrow{\gamma_p} T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]})) \xrightarrow{T(\beta^{\cdot})} T(\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W.,I^{\cdot})) = \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W.,T(I^{\cdot}))$$

We have $\gamma_p$ is a $\mathbb{F}_p\pi$ morphism, as well as $T(\beta^{\cdot})$, so $\hat{\theta}^{\cdot}$ is a $\mathbb{F}_p\pi$ morphism. By applying the adjoint isomorphism $\Phi^{\cdot}$ from Lemma 3.1.5, we obtain:

$$\Phi^0(\hat{\theta}^{\cdot}) : \mathrm{Tot}^{\cdot}((K^{\cdot})^{[p]} \otimes_{\mathbb{F}_p} W.) \to K^{\cdot}$$

We then define $\theta^{\cdot}$ by precomposing with the swapping isomorphism $U^{\cdot} : \mathrm{Tot}^{\cdot}(W. \otimes_{\mathbb{F}_p} (K^{\cdot})^{[p]}) \to \mathrm{Tot}^{\cdot}((K^{\cdot})^{[p]} \otimes_{\mathbb{F}_p} W.)$:

$$\theta^{\cdot} = \Phi^0(\hat{\theta}) \circ U^{\cdot}$$

## 4.3 Verification of Axioms

In order to show the object $(T(I^{\cdot}),\theta^{\cdot})$ constructed in this section belongs to May's category $\mathcal{C}(p)$, the two following lemmas must be proven:

**Lemma 4.3.1.** *Let $K^{\cdot} = T(I^{\cdot})$ and $\theta^{\cdot}$ be as in the previous sections. Then the restriction of $\theta^{\cdot}$ to $e_0 \otimes (K^{\cdot})^{[p]}$ is $\mathbb{F}_p$ homotopic to the iterated product $M_p : Tot^{\cdot}((K^{\cdot})^{[p]}) \to K^{\cdot}$.*

*Proof.* Special case of Lemma 5.3.1, where $A^{\cdot} = A[0]^{\cdot}$ is concentrated in degree 0. □

**Lemma 4.3.2.** *Let $K^{\cdot} = T(I^{\cdot})$ and $\theta^{\cdot}$ be as in the previous sections. Then there exists a $\Sigma_p$ chain map $\phi^{\cdot} : Tot^{\cdot}(V. \otimes (K^{\cdot})^{[p]}) \to K^{\cdot}$ such that $\theta^{\cdot}$ is $\mathbb{F}_p\pi$ homotopic to the composition:*

$$Tot^{\cdot}(W. \otimes (K^{\cdot})^{[p]}) \xrightarrow{j\otimes 1} Tot^{\cdot}(V. \otimes (K^{\cdot})^{[p]}) \xrightarrow{\phi} K^{\cdot}$$

*Proof.* Special case of Lemma 5.3.2, in which we have $A^{\cdot}$ is concentrated in degree 0. □

**Corollary 4.3.3.** *The object $(K^{\cdot},\theta^{\cdot})$ belongs to the category $\mathcal{C}(p)$.*

*Proof.* The required properties are shown in Lemma 4.3.1 and Lemma 4.3.2. □

**Lemma 4.3.4.** *The object $(K^{\cdot}, \theta^{\cdot})$ is a Cartan object. That is, given $\hat{\theta}^{\cdot}$ as defined in the second half of Definition 2.0.1, the following diagram commutes up to $\mathbb{F}_p\pi$ homotopy:*

$$\begin{array}{ccc} Tot^{\cdot}(W_{\cdot} \otimes (K^{\cdot} \otimes K^{\cdot})^{[p]}) & \xrightarrow{\tilde{\theta}^{\cdot}} & Tot^{\cdot}(K^{\cdot} \otimes K^{\cdot}) \\ \downarrow{\scriptstyle 1\otimes M^{[p]}} & & \downarrow{\scriptstyle M} \\ Tot^{\cdot}(W_{\cdot} \otimes (K^{\cdot})^{[p]}) & \xrightarrow{\theta^{\cdot}} & K^{\cdot} \end{array}$$

*Proof.* Special case of Lemma 5.3.3, where here $A^{\cdot}$ is concentrated in degree 0. □

**Lemma 4.3.5.** *The object $(K^{\cdot}, \theta^{\cdot})$ is an Adem object. That is, there is a $\Sigma_{p^2}$ chain map $\xi^{\cdot}$ : $Y_{\cdot} \otimes_{\mathbb{F}_p} (K^{\cdot})^{[p]} \to K^{\cdot}$ such that the following diagram commutes up to $\tau$-homotopy.*

$$\begin{tikzcd} (W_1 \otimes W_2^{[p]}) \otimes K^{[p^2]} \arrow[r, "w\otimes 1"] \arrow[dd, "1\otimes S"] & Y \otimes K^{[p^2]} \arrow[dr, "\xi"] & \\ & & K \\ W_1 \otimes (W_2 \otimes K^{[p]})^{[p]} \arrow[r, "1\otimes\theta^{[p]}"] & W_1 \otimes K^{[p]} \arrow[ur, "\theta"] & \end{tikzcd}$$

*Recall the definitions of $W_1, W_2, Y$ and the group actions involved from Definition 2.0.5.*

*Proof.* Special case of Lemma 5.3.4 where here we have $A^{\cdot}$ is concentrated in degree 0. □

At this point we may construct Steenrod operations on $H^{\cdot}(K^{\cdot})$ using the $\theta^{\cdot}$ map of this section by applying Corollaries 2.0.8 and 2.0.9.

## 4.4 Further Properties

The following is shown by Epstein in [2], in part 6 of section 10, page 204.

**Lemma 4.4.1.** *For the Steenrod operations constructed on sheaf cohomology, we have $P^i = 0$ and $\beta P^i = 0$ for all $i < 0$.*

The following is shown by Epstein in [2], in part 7 of section 11.1, on page 205.

**Lemma 4.4.2.** *For the Steenrod operations constructed on sheaf cohomology, we have $P^0$ : $H^n(K^{\cdot}) \to H^n(K^{\cdot})$ is induced by the Frobenius map, $fr : A \to A$, on the sheaf of commutative $\mathbb{F}_p$-algebras, $A$.*

The following shows that the Steenrod operations are natural.

**Lemma 4.4.3.** *Let $X$ be a topological space. Let $A$ and $B$ be two sheaves of commutative $\mathbb{F}_p$ algebras on $X$, with $f : A \to B$ a morphism of sheaves of $\mathbb{F}_p$ algebras. There are induced morphisms for each $n \in \mathbb{Z}$:*

$$H^n(X, f) : H^n(X,A) \to H^n(X,B)$$

*We have $H^{\cdot}(X, f)$ commutes with $D_i$. As a result, $H^{\cdot}(X, f)$ commutes with the Steenrod operations constructed on $H^{\cdot}(X,A)$ and $H^{\cdot}(X,B)$.*

*Proof.* Special case of Lemma 5.4.1 in which $A^{\cdot}$ and $B^{\cdot}$ are both concentrated in degree zero. □

# Chapter 5

# Steenrod Operations on Sheaf Hypercohomology

In this chapter I will show the Steenrod operations described in Corollary 2.0.8 can be constructed on the algebraic De Rham cohomology groups and the Hodge cohomology groups of a smooth projective variety $X$ over a field $k$ of characteristic $p$. I will establish both the Cartan formula and the Adem relations. Steenrod operations on algebraic De Rham cohomology can be obtained by constructing Steenrod operations on crystalline cohomology using the framework of Epstein, and then reducing mod $p$. However, Steenrod operations on Hodge cohomology does not appear to have a prior construction. The approach of this section is to generalize Epstein's machinery to complexes and then apply May's framework. In this chapter, unlabeled tensor products are to be over $\mathbb{F}_p$.

**Definition 5.0.1.** *Let $X$ be a smooth projective scheme over a field $k$ of characteristic $p$. Let $\Omega^{\cdot}_{X/k}$ denote the De Rham complex of $X$. Then the algebraic De Rham cohomology groups of $X$ may be computed as the hypercohomology of $X$ with coefficients in $\Omega^{\cdot}_{X/k}$. That is, given an injective resolution $\iota^{\cdot} : \Omega^{\cdot}_{X/k} \to I^{\cdot}$ in $Sh_{\mathbb{F}_p}(X)$, and global section functor $T(-) = \Gamma(X,-)$, one has:*

$$H^{\cdot}_{DR}(X/k) = \mathbf{H}^{\cdot}(X, \Omega^{\cdot}_{X/k}) = H^{\cdot}(T(I^{\cdot}))$$

For the rest of this section I will let $X$ denote an arbitrary topological space and $A^{\cdot}$ a bounded below complex of sheaves of differential graded commutative $\mathbb{F}_p$ algebras on $X$. I will construct Steenrod operations on the hypercohomology groups $\mathbf{H}^{\cdot}(X, A^{\cdot})$. Choose an injective resolution $\iota^{\cdot} : A^{\cdot} \to I^{\cdot}$ in the category of sheaves of $\mathbb{F}_p$ vector spaces on $X$. Such a resolution can be obtained by taking $I^{\cdot}$ to be the total complex of a Cartan Eilenberg resolution of $A^{\cdot}$. Define $K^{\cdot} = T(I^{\cdot})$. The product on $A^{\cdot}$ induces a product on $K^{\cdot}$, unique up to homotopy, making $K^{\cdot}$ into a homotopy associative differential graded $\mathbb{F}_p$ algebra. I will construct a $\mathbb{F}_p\pi$ chain map $\theta^{\cdot} : \mathrm{Tot}^{\cdot}(W_{\cdot} \otimes (K^{\cdot})^{[p]}) \to K^{\cdot}$ and show that $(K^{\cdot}, \theta^{\cdot})$ belongs to May's category $\mathcal{C}(p)$. I will also show $(K^{\cdot}, \theta^{\cdot})$ is both a Cartan and Adem object.

## 5.1 The Product on $K^{\cdot}$

In this section we construct the homotopy associative graded product on $K^{\cdot}$.

**Definition 5.1.1.** *Let $m^{\cdot} : Tot^{\cdot}(A^{\cdot} \otimes A^{\cdot}) \to A^{\cdot}$ denote the graded commutative product on $A^{\cdot}$. Consider the solid diagram below in the category $Sh_{\mathbb{F}_p}(X)$:*

$$\begin{tikzcd}
Tot^{\cdot}(I^{\cdot} \otimes_{\mathbb{F}_p} I^{\cdot}) \arrow[r, dashed, "\tilde{m}^{\cdot}"] & I^{\cdot} \\
Tot^{\cdot}(A^{\cdot} \otimes_{\mathbb{F}_p} A^{\cdot}) \arrow[u, hook, "\iota^{\cdot}\otimes\iota^{\cdot}"] \arrow[r, "m^{\cdot}"] & A^{\cdot} \arrow[u, hook, "\iota^{\cdot}"']
\end{tikzcd}$$

*By Lemma 3.2.16, $\iota^{\cdot} \otimes \iota^{\cdot} : Tot^{\cdot}(A^{\cdot} \otimes A^{\cdot}) \to Tot^{\cdot}(I^{\cdot} \otimes I^{\cdot})$ is a resolution. Because $\iota^{\cdot} : A^{\cdot} \to I^{\cdot}$ is an injective resolution, by Lemma 3.2.13, there exists a chain map $\tilde{m}^{\cdot}$ making the diagram commute. By Lemma 3.2.14, $\tilde{m}^{\cdot}$ is unique up to homotopy. We then obtain the product $M^{\cdot} : Tot^{\cdot}(K^{\cdot} \otimes K^{\cdot}) \to K^{\cdot}$ by the following composition:*

$$M^{\cdot} : Tot^{\cdot}(T(I^{\cdot}) \otimes_{\mathbb{F}_p} T(I^{\cdot})) \xrightarrow{\gamma} T(Tot^{\cdot}(I^{\cdot} \otimes_{\mathbb{F}_p} I^{\cdot})) \xrightarrow{T(\tilde{m}^{\cdot})} T(I^{\cdot})$$

*where $\gamma$ is the natural map. Note this definition only defines $M^{\cdot}$ uniquely up to homotopy.*

**Definition 5.1.2.** *For all $n, m \in \mathbb{Z}$, define the cup product:*

$$\cup^{n,m} : H^n(K^{\cdot}) \otimes H^m(K^{\cdot}) \to H^{n+m}(K^{\cdot})$$

*by the composition:*

$$\cup^{n,m} : H^n(K^\cdot)\otimes H^m(K^\cdot) \xhookrightarrow{\psi} H^{n+m}(Tot^\cdot(K^\cdot\otimes K^\cdot)) \xrightarrow{H^{n+m}(M^\cdot)} H^{n+m}$$

*where $\psi$ is an injection induced by the Künneth isomorphism $H^\cdot(K^\cdot)\otimes H^\cdot(K^\cdot) \cong H^\cdot(K^\cdot \otimes K^\cdot)$, which we have because we are in the category Vect($\mathbb{F}_p$). That is, $\psi$ is the map:*

$$[x]\otimes[y] \mapsto [x\otimes y]$$

*for $x \in Z^n(K^\cdot)$ and $y \in Z^m(K^\cdot)$. The uniqueness of $H^\cdot(M^\cdot)$ implies $\cup^{\cdot,\cdot}$ is uniquely defined.*

**Lemma 5.1.3.** *The product $M^\cdot : Tot^\cdot(K^\cdot\otimes K^\cdot) \to K^\cdot$ makes $K^\cdot$ a homotopy associative differential graded $\mathbb{F}_p$ algebra. The induced cup product $\cup^{\cdot,\cdot}$ on $H^\cdot(K^\cdot)$ is associative.*

*Proof.* We have the diagram below:

$$\begin{tikzcd}
\mathrm{Tot}^\cdot(I^\cdot\otimes I^\cdot\otimes I^\cdot) \arrow[r, shift left, "1\otimes\tilde{m}^\cdot"] \arrow[r, shift right, "\tilde{m}^\cdot\otimes 1"'] & \mathrm{Tot}^\cdot(I^\cdot\otimes I^\cdot) \arrow[r, "\tilde{m}^\cdot"] & I^\cdot \\
\mathrm{Tot}^\cdot(A^\cdot\otimes A^\cdot\otimes A^\cdot) \arrow[u, hook, "(\iota^\cdot)^{[3]}"] \arrow[r, shift left, "1\otimes m^\cdot"] \arrow[r, shift right, "m^\cdot\otimes 1"'] & \mathrm{Tot}^\cdot(A^\cdot\otimes A^\cdot) \arrow[u, hook, "(\iota^\cdot)^{[2]}"] \arrow[r, "m^\cdot"] & A^\cdot \arrow[u, hook, "\iota^\cdot"]
\end{tikzcd}$$

Because $A^\cdot$ is an associative differential graded $\mathbb{F}_p$ algebra, the two composites along the bottom row are equal: $m^\cdot\circ(1\otimes m^\cdot) = m^\cdot\circ(m^\cdot\otimes 1)$. Define $f_1^\cdot = \tilde{m}^\cdot\circ(1\otimes\tilde{m}^\cdot)$ and $f_2^\cdot = \tilde{m}^\cdot\circ(\tilde{m}^\cdot\otimes 1)$, the two composite maps along the top row of the diagram. By Corollary 3.2.17, $(\iota^\cdot)^{[3]} : \mathrm{Tot}^\cdot((A^\cdot)^{[3]}) \to \mathrm{Tot}^\cdot((I^\cdot)^{[3]})$ is a resolution, and we still have $\iota^\cdot : A^\cdot \to I^\cdot$ is an injective resolution in $\mathrm{Sh}_{\mathbb{F}_p}(X)$. Because $f_1^\cdot, f_2^\cdot : \mathrm{Tot}^\cdot((I^\cdot)^{[3]}) \to I^\cdot$ are two morphisms over the triple iterated product, $\mathrm{Tot}^\cdot((A^\cdot)^{[3]}) \to A^\cdot$, we can invoke Lemma 3.2.14 to obtain a $\mathbb{F}_p$ homotopy $h^\cdot$ between $f_1^\cdot$ and $f_2^\cdot$. We have $h^\cdot : \mathrm{Tot}^\cdot((I^\cdot)^{[3]}) \to I^\cdot[-1]$, and $h^\cdot$ satisfies the relation for all $n\in\mathbb{Z}$,

$$f_1^n - f_2^n = d_I^{n-1}\circ h^n + h^{n+1}\circ d^n_{\mathrm{Tot}((I^\cdot)^{[3]})}$$

Because $T$ is an additive functor, we have $T(f_1^\cdot)$ and $T(f_2^\cdot)$ are homotopic by homotopy $T(h^\cdot)$. That is:

$$T(f_1^n) - T(f_2^n) = d_{T(I)}^{n-1} \circ T(h^n) + T(h^{n+1}) \circ d_{T(\mathrm{Tot}((I^\cdot)^{[3]}))}^n$$

Let $\gamma_3 : \mathrm{Tot}^\cdot(T(I^\cdot)^{[3]}) \to T(\mathrm{Tot}^\cdot((I^\cdot)^{[3]}))$ be the natural map. By precomposing the above with $\gamma_3$ we obtain:

$$T(f_1^n) \circ \gamma_3^n - T(f_2^n) \circ \gamma_3^n = d_{T(I)}^{n-1} \circ (T(h^n) \circ \gamma_3^n) + (T(h^{n+1}) \circ \gamma_3^{n+1}) \circ d_{\mathrm{Tot}(T(I^\cdot)^{[3]})}^n$$

Thus $T(f_1^\cdot) \circ \gamma_3$ and $T(f_2^\cdot) \circ \gamma_3$ are homotopic by homotopy $T(h^\cdot) \circ \gamma_3$. But $T(f_1^\cdot) \circ \gamma_3 = M^\cdot \circ (1 \otimes M^\cdot)$ and $T(f_2^\cdot) \circ \gamma_3 = M^\cdot \circ (M^\cdot \otimes 1)$, and the fact that these two maps are homotopic shows the product $M^\cdot$ on $K^\cdot = T(I^\cdot)$ is homotopy associative. This then implies the induced product on the cohomology groups $H^\cdot(K^\cdot)$ is associative.

□

**Definition 5.1.4.** *We can define a p iterated product, $M_p^\cdot : Tot^\cdot((K^\cdot)^{[p]}) \to K^\cdot$, as follows. Consider the solid diagram below:*

$$\begin{tikzcd} Tot^\cdot((I^\cdot)^{[p]}) \arrow[r, dashed, "\tilde{m}_p^\cdot"] & I^\cdot \\ Tot^\cdot((A^\cdot)^{[p]}) \arrow[u, "(\iota^\cdot)^{[p]}"] \arrow[r, "m_p^\cdot"] & A^\cdot \arrow[u, hook, "\iota^\cdot"'] \end{tikzcd}$$

*By Corollary 3.2.17, $(\iota^\cdot)^{[p]} : Tot^\cdot((A^\cdot)^{[p]}) \to Tot^\cdot((I^\cdot)^{[p]})$ is a resolution. Because $\iota^\cdot : A^\cdot \to I^\cdot$ is an injective resolution, by Lemma 3.2.13, there exists a chain map $\tilde{m}_p^\cdot$ making the diagram commute. By Lemma 3.2.14, $\tilde{m}_p^\cdot$ is unique up to homotopy. We can then define a p iterated product, $M_p^\cdot : Tot^\cdot((K^\cdot)^{[p]}) \to K^\cdot$ by the following composition:*

$$M_p^\cdot : Tot^\cdot(T(I^\cdot)^{[p]}) \xrightarrow{\gamma_p} T(Tot^\cdot((I^\cdot)^{[p]})) \xrightarrow{T(\tilde{m}_p^\cdot)} T(I^\cdot)$$

*It should be noted that $M_p^\cdot$ is only a $\mathbb{F}_p$ chain map, not a $\mathbb{F}_p\pi$ chain map. Although each solid arrow is a $\mathbb{F}_p\pi$ chain map, the objects of $I^\cdot$ will almost always fail to be injective in the category $Sh_{\mathbb{F}_p\pi}(X)$.*

**Lemma 5.1.5.** *The chain map $M_p^\cdot$ is $\mathbb{F}_p$ homotopic to a p-fold product of $M^\cdot$, in some order.*

*Proof.* Define $f^{\cdot} = \tilde{m}^{\cdot} \circ (1 \otimes \tilde{m}^{\cdot}) \circ \cdots \circ (1^{[p-1]} \otimes \tilde{m}^{\cdot}) : \mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}) \to I^{\cdot}$. $f^{\cdot}$ is a $p$-fold product of $\tilde{m}^{\cdot}$. We have $m_p^{\cdot} = m^{\cdot} \circ (1 \otimes m^{\cdot}) \circ \cdots \circ (1^{[p-1]} \otimes m^{\cdot})$ because the product $m^{\cdot}$ on $A^{\cdot}$ is associative. Thus, $f^{\cdot}$ and $\tilde{m}_p^{\cdot}$ are two $\mathbb{F}_p$ chain maps over $m_p^{\cdot}$, where $(\iota^{\cdot})^{[p]} : \mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]}) \to \mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]})$ is a resolution by Corollary 3.2.17, and $\iota^{\cdot} : A^{\cdot} \to I^{\cdot}$ is an injective resolution. By Lemma 3.2.14, there is a $\mathbb{F}_p$ homotopy $h^{\cdot}$ between $\tilde{m}_p^{\cdot}$ and $f^{\cdot}$. That is, $h^{\cdot} : \mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}) \to I[-1]^{\cdot}$, and $h^{\cdot}$ satisfies for all $n \in \mathbb{Z}$:

$$\tilde{m}_p^n - f^n = d_I^{n-1} \circ h^n + h^{n+1} \circ d^n_{\mathrm{Tot}((I^{\cdot})^{[p]})}$$

By applying the additive global section functor $T$, we obtain a homotopy $T(h^{\cdot})$ between $M_p^{\cdot} = T(\tilde{m}_p^{\cdot})$ and $T(f^{\cdot})$.

$$T(\tilde{m}_p^n) - T(f^n) = d^{n-1}_{T(I)} \circ T(h^n) + T(h^{n+1}) \circ d^n_{T(\mathrm{Tot}((I^{\cdot})^{[p]}))}$$

By precomposing the above with $\gamma_p$, we obtain:

$$T(\tilde{m}_p^n) \circ \gamma_p^n - T(f^n) \circ \gamma_p^n = d^{n-1}_{T(I)} \circ (T(h^n) \circ \gamma_p^n) + (T(h^{n+1}) \circ \gamma_p^{n+1}) \circ d^n_{\mathrm{Tot}(T(I^{\cdot})^{[p]})}$$

So $T(h^{\cdot}) \circ \gamma_p^n$ is a homotopy between $M_p^{\cdot} = T(\tilde{m}_p^{\cdot}) \circ \gamma_p$ and $T(f^{\cdot}) \circ \gamma_p$. But we have $T(f^{\cdot}) \circ \gamma_p = M^{\cdot} \circ (1 \otimes M^{\cdot}) \circ \cdots \circ (1^{[p-1]} \otimes M^{\cdot})$, so the result is shown.

□

## 5.2 Construction of $\theta^{\cdot}$

Now that the homotopy associative differential graded product on $K^{\cdot}$ has been defined, I will construct the map $\theta^{\cdot} : \mathrm{Tot}^{\cdot}(W_{\cdot} \otimes_{\mathbb{F}_p} (K^{\cdot})^{[p]}) \to K^{\cdot}$. Consider the solid diagram below:

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}) \arrow[r, dashed, "\beta^{\cdot}"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, I^{\cdot}) \\
\mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]}) \arrow[u, "(\iota^{\cdot})^{[p]}"] \arrow[r, "m_p^{\cdot}"] & A \arrow[u, "\nu^{\cdot}"']
\end{tikzcd}$$

By Corollary 3.2.17, $(\iota^{\cdot})^{[p]} : \mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]}) \to \mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]})$ is a resolution. We let $\Sigma_p$ act on $\mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]})$ and $\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]})$ by permutation of tensors with sign change based upon the grading, and trivially on $A^\cdot$. We see that $(\iota^\cdot)^{[p]}$ is a $\mathbb{F}_p\Sigma_p$ morphism, as well as a $\mathbb{F}_p\pi$ morphism. Because the product on $A^\cdot$ is graded commutative, $m_p^\cdot$ is a $\mathbb{F}_p\Sigma_p$ and $\mathbb{F}_p\pi$ morphism. By Lemma 3.2.21, $\nu^\cdot : A^\cdot \to \mathbf{Hom}^\cdot_{\mathbb{F}_p}(W_\cdot, I^\cdot)$ is an injective resolution of $A^\cdot$ in the category $\mathrm{Sh}_{\mathbb{F}_p\pi}(X)$. Now we may invoke Lemma 3.2.13, with the abelian category $\mathcal{A} = \mathrm{Sh}_{\mathbb{F}_p\pi}(X)$ to obtain a $\mathbb{F}_p\pi$ chain map $\beta^\cdot$ making the diagram commute. By Lemma 3.2.14, $\beta^\cdot$ is unique up to homotopy. We can now define $\hat{\theta}^\cdot : \mathrm{Tot}^\cdot((K^\cdot)^{[p]}) \to \mathrm{Hom}^\cdot_{\mathbb{F}_p}(W_\cdot, K^\cdot)$ by the following composition:

$$\hat{\theta}^\cdot : \mathrm{Tot}^\cdot(T(I^\cdot)^{[p]}) \xrightarrow{\gamma_p} T(\mathrm{Tot}^\cdot((I^\cdot)^{[p]})) \xrightarrow{T(\beta^\cdot)} T(\mathbf{Hom}^\cdot_{\mathbb{F}_p}(W_\cdot, I^\cdot)) = \mathrm{Hom}^\cdot_{\mathbb{F}_p}(W_\cdot, T(I^\cdot))$$

Because the above maps are in $\mathrm{Comp}(_{\mathbb{F}_p\pi}\mathrm{Mod})$, $\hat{\theta}^\cdot$ is a $\mathbb{F}_p\pi$ morphism. By applying the adjoint isomorphism $\Phi^\cdot$ from Lemma 3.1.5, we obtain the $\mathbb{F}_p\pi$ morphism:

$$\Phi^0(\hat{\theta}^\cdot) : \mathrm{Tot}^\cdot((K^\cdot)^{[p]} \otimes_{\mathbb{F}_p} W_\cdot) \to K^\cdot$$

We now define $\theta^\cdot : \mathrm{Tot}^\cdot(W_\cdot \otimes_{\mathbb{F}_p} (K^\cdot)^{[p]}) \to K^\cdot$ with the composition:

$$\theta^\cdot = \Phi^0(\hat{\theta}^\cdot) \circ U^\cdot$$

where $U^\cdot : \mathrm{Tot}^\cdot(W_\cdot \otimes_{\mathbb{F}_p} (K^\cdot)^{[p]}) \to \mathrm{Tot}^\cdot((K^\cdot)^{[p]} \otimes_{\mathbb{F}_p} W_\cdot)$ swaps tensors with sign change based upon the grading.

## 5.3 Verification of Axioms

Now that $(K^\cdot, \theta^\cdot)$ has been defined, I will show that $(K^\cdot, \theta^\cdot)$ belongs to May's category $\mathcal{C}(p)$, and that $(K^\cdot, \theta^\cdot)$ is both Cartan and Adem.

**Lemma 5.3.1.** *The restriction of* $\theta^\cdot$ *to* $e_0 \otimes (K^\cdot)^{[p]}$ *is* $\mathbb{F}_p$ *homotopic to a p-fold product on* $K^\cdot$.

*Proof.* We have the solid diagram below:

$$\begin{tikzcd}
\mathbb{F}_p[0]_\cdot \arrow[r, dashed, "\iota_\cdot"] \arrow[d, two heads] & W_\cdot \arrow[d, two heads] \\
\mathbb{F}_p \arrow[r, "1"] & \mathbb{F}_p
\end{tikzcd}$$

Because $W_\cdot$ is a resolution of $\mathbb{F}_p$ in $\mathrm{Vect}(\mathbb{F}_p)$, and $\mathbb{F}_p[0]_\cdot$ is a projective resolution of $\mathbb{F}_p$ in $\mathrm{Vect}(\mathbb{F}_p)$, there is a $\mathbb{F}_p$ chain map $l_\cdot$ making the square commute, and $l_\cdot$ is unique up to homotopy. In fact we can define $l_\cdot$ explicitly with $l_0(1) = e_0 \in W_0$. Now we have the diagram below:

$$\begin{tikzcd}
\mathrm{Tot}^\cdot((I^\cdot)^{[p]}) \arrow[r, "\beta^\cdot"] \arrow[rr, bend left, "\tilde{m}^\cdot_p"] & \mathbf{Hom}^\cdot_{\mathbb{F}_p}(W_\cdot, I^\cdot) \arrow[r, "(l_\cdot)^*"] & \mathbf{Hom}^\cdot_{\mathbb{F}_p}(\mathbb{F}_p[0]_\cdot, I^\cdot) = I^\cdot \\
\mathrm{Tot}^\cdot((A^\cdot)^{[p]}) \arrow[r, "m^\cdot_p"] \arrow[u, hook, "(\iota^\cdot)^{[p]}"] & A^\cdot \arrow[r, "1^\cdot_A"] \arrow[u, hook, "\nu^\cdot"] & A^\cdot \arrow[u, hook, "\iota^\cdot"]
\end{tikzcd}$$

Note that $\tilde{m}^\cdot_p$ is only a $\mathbb{F}_p$ morphism, while $\beta^\cdot$ is a $\mathbb{F}_p\pi$ morphism, and $(l_\cdot)^*$ is a $\mathbb{F}_p$ morphism. We have that $\tilde{m}^\cdot_p$ and $(l_\cdot)^* \circ \beta^\cdot$ are both $\mathbb{F}_p$ chain maps extending the iterated multiplication map $m^\cdot_p$ in $A^\cdot$. By Lemma 3.2.14, there is a $\mathbb{F}_p$ homotopy $h^\cdot$ from $\tilde{m}^\cdot_p$ to $(l_\cdot)^* \circ \beta^\cdot$. We then apply the global section functor to obtain a homotopy $T(h^\cdot)$ from $(l_\cdot)^* \circ T(\beta^\cdot)$ to $T(\tilde{m}^\cdot_p)$, so the square in the diagram below commutes up to homotopy.

$$\begin{tikzcd}
\mathrm{Tot}^\cdot(T(I^\cdot)^{[p]}) \arrow[r, "\gamma_p"] & T(\mathrm{Tot}^\cdot((I^\cdot)^{[p]})) \arrow[r, "T(\beta^\cdot)"] \arrow[d, "T(\tilde{m}^\cdot_p)"'] & T(\mathbf{Hom}^\cdot_{\mathbb{F}_p}(W_\cdot, I^\cdot)) = \mathrm{Hom}^\cdot_{\mathbb{F}_p}(W_\cdot, T(I^\cdot)) \arrow[d, "(l_\cdot)^*"] \\
 & T(I^\cdot) \arrow[r, "\lambda"] & T(\mathbf{Hom}^\cdot_{\mathbb{F}_p}(\mathbb{F}_p[0]_\cdot, I^\cdot)) = \mathrm{Hom}^\cdot_{\mathbb{F}_p}(\mathbb{F}_p[0]_\cdot, T(I^\cdot))
\end{tikzcd}$$

$\lambda$ is the natural isomorphism. Recall $M^\cdot_p = T(\tilde{m}^\cdot_p) \circ \gamma_p$, and $\hat{\theta}^\cdot = T(\beta^\cdot) \circ \gamma_p$. By precomposing with $\gamma_p$, we get that the maps:

$$\lambda \circ M^\cdot_p,\ (l_\cdot)^* \circ \hat{\theta}^\cdot\ : \mathrm{Tot}^\cdot(T((I^\cdot)^{[p]})) \to \mathrm{Hom}^\cdot_{\mathbb{F}_p}(\mathbb{F}_p[0]_\cdot, T(I^\cdot))$$

are homotopic by homotopy $T(h^\cdot) \circ \gamma_p$. By Lemma 3.1.6, the chain maps:

$$\Phi^0(\lambda \circ M^\cdot_p), \Phi^0((l_\cdot)^* \circ \hat{\theta}^\cdot) : \mathrm{Tot}^\cdot(T(I^\cdot)^{[p]} \otimes \mathbb{F}_p[0]_\cdot) \to T(I^\cdot)$$

are homotopic by homotopy $\Phi^{-1}(T(h^{\cdot}) \circ \gamma_p)$. Let $\alpha \in \mathrm{Tot}^n(T(I^{\cdot})^{[p]})$ and $x \in \mathbb{F}_p[0]_0$. Then we have:

$$\begin{aligned}
\Phi^0(\lambda \circ M_p^n)(\alpha \otimes x) &= \lambda(M_p^n(\alpha))(x) \\
&= x \cdot M_p^n(\alpha) \\
&= M_p^n(x \cdot \alpha)
\end{aligned}$$

Which is the $p$ iterated product on $T(I^{\cdot})$ with a scalar multiplication. By Lemma 3.1.8, we have $\Phi^0((l.)^* \otimes \hat{\theta}^{\cdot}) = \Phi^0(\hat{\theta}^{\cdot}) \circ (1 \otimes l.)$. We now have the homotopy commutativity of the right square below:

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}(W. \otimes T(I^{\cdot})^{[p]}) \arrow[r, "U^{\cdot}"] \arrow[rr, bend left, "\theta^{\cdot}"] & \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]} \otimes W.) \arrow[r, "\Phi^0(\hat{\theta}^{\cdot})"] & T(I^{\cdot}) \\
\mathrm{Tot}^{\cdot}(\mathbb{F}_p[0]. \otimes T(I^{\cdot})^{[p]}) \arrow[u, "l.\otimes 1"] \arrow[r, "U^{\cdot}"] & \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]} \otimes \mathbb{F}_p[0].) \arrow[u, "1\otimes l."] \arrow[r] & \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}) \arrow[u, "M_p^{\cdot}"]
\end{tikzcd}$$

Since $U^{\cdot}$ just swaps tensors with sign, the left square commutes. Since the bottom row consists of isomorphisms, we can walk from $\mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]})$ to $T(I^{\cdot})$ in two paths, one of which is directly with $M_p^{\cdot}$ which is the $p$ iterated product on $T(I^{\cdot})$, and the other is to walk around the perimeter, which is the restriction of $\theta^{\cdot}$ to $l_0(1) \otimes T(I^{\cdot})^{[p]}$, and since $l_0(1) = e_0 \in W_0$, the fact that these two maps are homotopic proves the result.

□

**Lemma 5.3.2.** *There exists a $\Sigma_p$ chain map $\phi^{\cdot} : Tot^{\cdot}(V. \otimes_{\mathbb{F}_p} (K^{\cdot})^{[p]}) \to K^{\cdot}$ such that $\theta^{\cdot}$ is $\mathbb{F}_p\pi$ homotopic to the composition:*

$$Tot^{\cdot}(W. \otimes_{\mathbb{F}_p} (K^{\cdot})^{[p]}) \xrightarrow{j^{\cdot}\otimes 1} Tot^{\cdot}(V. \otimes_{\mathbb{F}_p} (K^{\cdot})^{[p]}) \xrightarrow{\phi^{\cdot}} K^{\cdot}$$

*Proof.* Recall $j.$ is a $\mathbb{F}_p\pi$ morphism making the diagram below commute.

$$\begin{tikzcd} W. \arrow[r, dashed, "j."] \arrow[d, two heads] & V. \arrow[d, two heads] \\ \mathbb{F}_p \arrow[r, "1"] & \mathbb{F}_p \end{tikzcd}$$

In the above $W.$ is a free $\mathbb{F}_p\pi$ resolution of $\mathbb{F}_p$ and $V.$ is a free $\mathbb{F}_p\Sigma_p$ resolution of $\mathbb{F}_p$. Because $W.$ is a projective $\mathbb{F}_p\pi$ resolution of $\mathbb{F}_p$ and $V.$ is a resolution of $\mathbb{F}_p$, such a $\mathbb{F}_p\pi$ morphism $j.$ exists, and is unique up to homotopy. Consider the diagram below:

$$\begin{tikzcd} \mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}) \arrow[r, dashed, "\alpha^{\cdot}"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(V.,I^{\cdot}) \\ \mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]}) \arrow[r, "m^{\cdot}_p"] \arrow[u, hook, "(\iota^{\cdot})^{[p]}"] & A^{\cdot} \arrow[u, hook, "\eta^{\cdot}"'] \end{tikzcd}$$

Because $V.$ is a projective resolution of $\mathbb{F}_p$ in ${}_{\mathbb{F}_p\Sigma_p}\mathrm{Mod}$, and $I^{\cdot}$ is an injective resolution of $A^{\cdot}$ in $\mathrm{Sh}_{\mathbb{F}_p}(X)$, by Lemma 3.2.21, $\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(V.,I^{\cdot})$ is an injective resolution of $A^{\cdot}$ in the category $\mathrm{Sh}_{\mathbb{F}_p\Sigma_p}(X)$. By Corollary 3.2.17, $\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]})$ is a resolution of $\mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]})$. Because the product on $A^{\cdot}$ is graded commutative, the iterated product $m^{\cdot}_p$ is a $\mathbb{F}_p\Sigma_p$ morphism. The vertical arrows are $\mathbb{F}_p\Sigma_p$ morphisms as well. Now by Lemma 3.2.13, there exists a $\mathbb{F}_p\Sigma_p$ chain map $\alpha^{\cdot}$ making the diagram commute, and by Lemma 3.2.14, $\alpha^{\cdot}$ is unique up to homotopy. Define $\hat{\phi}^{\cdot}$ by the composition, $\hat{\phi}^{\cdot} = T(\alpha^{\cdot})\circ\gamma_p$:

$$\hat{\phi}^{\cdot} : \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}) \xrightarrow{\gamma_p} T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]})) \xrightarrow{T(\alpha^{\cdot})} T(\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(V.,I^{\cdot})) = \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(V.,T(I^{\cdot}))$$

With $\Phi^{\cdot}$ the adjoint map from Lemma 3.1.5, we have $\Phi^0(\hat{\phi}^{\cdot}) : \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}\otimes V.) \to T(I^{\cdot})$. We now define $\phi^{\cdot} = \Phi^0(\hat{\phi}^{\cdot})\circ U^{\cdot}$, where $U^{\cdot} : \mathrm{Tot}^{\cdot}(V.\otimes T(I^{\cdot})^{[p]}) \to \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}\otimes V.)$ swaps tensors with sign. Consider the diagram below:

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}) \arrow[r, "\alpha^{\cdot}"] \arrow[rr, bend left, "\beta^{\cdot}"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(V_{\cdot}, I^{\cdot}) \arrow[r, "(j_{\cdot})^*"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, I^{\cdot}) \\
\mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]}) \arrow[u, "(\iota^{\cdot})^{[p]}"] \arrow[r, "m^{\cdot}_p"] & \mathrm{Tot}^{\cdot}(A^{\cdot}) \arrow[u, "\eta^{\cdot}"] \arrow[r, "1^{\cdot}_A"] & \mathrm{Tot}^{\cdot}(A^{\cdot}) \arrow[u, "\nu^{\cdot}"]
\end{tikzcd}$$

We have $(j_{\cdot})^*$ makes the square on the right commute and $\alpha^{\cdot}$ makes the square on the left commute, so $(j_{\cdot})^* \circ \alpha^{\cdot}$ makes the rectangle commute. We also have $\beta^{\cdot}$ makes the perimeter commute by its construction. Because $\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, I^{\cdot})$ is an injective resolution of $A^{\cdot}$ in $\mathrm{Sh}_{\mathbb{F}_p\pi}(X)$, $\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]})$ is a resolution of $\mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]})$, and both $(j_{\cdot})^* \circ \alpha^{\cdot}$ and $\beta^{\cdot}$ are $\mathbb{F}_p\pi$ morphisms making the rectangle commute, by Lemma 3.2.14, there is a $\mathbb{F}_p\pi$ homotopy $h^{\cdot}$ from $\beta^{\cdot}$ to $(j_{\cdot})^* \circ \alpha^{\cdot}$. Because $T$ is an additive functor, $T(\beta^{\cdot})$ and $T((j_{\cdot})^* \circ \alpha^{\cdot}) = (j_{\cdot})^* \circ T(\alpha^{\cdot})$ are homotopic by homotopy $T(h^{\cdot})$. Then precomposing by $\gamma_p$ shows that $\hat{\phi}^{\cdot} = T(\alpha^{\cdot}) \circ \gamma_p$ is homotopic to $(j_{\cdot})^* \circ \hat{\theta}^{\cdot} = (j_{\cdot})^* \circ T(\beta^{\cdot}) \circ \gamma_p$ by homotopy $T(h^{\cdot}) \circ \gamma_p$. These two maps are $\mathbb{F}_p\pi$ chain maps:

$$\hat{\phi}^{\cdot},\ (j_{\cdot})^* \circ \gamma_p :\ \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}) \to \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, T(I^{\cdot}))$$

Using the adjoint isomorphism from Lemma 3.1.5 we obtain $\mathbb{F}_p\pi$ chain maps:

$$\Phi^0(\hat{\phi}^{\cdot}),\ \Phi^0((j_{\cdot})^* \circ \hat{\theta}^{\cdot}) :\ \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]} \otimes V_{\cdot}) \to T(I^{\cdot})$$

and by Lemma 3.1.6 these are homotopic by homotopy $\Phi^{-1}(T(h^{\cdot}) \circ \gamma_p)$. By Lemma 3.1.8, we have $\Phi^0((j_{\cdot})^* \circ \hat{\theta}^{\cdot}) = \Phi^0(\hat{\theta}^{\cdot}) \circ (1 \otimes j_{\cdot})$. We have the diagram below:

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}(V_{\cdot} \otimes T(I^{\cdot})^{[p]}) \arrow[r, "U^{\cdot}"] & \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]} \otimes V_{\cdot}) \arrow[r, "\Phi^0(\hat{\phi}^{\cdot})"] & T(I^{\cdot}) \\
\mathrm{Tot}^{\cdot}(W_{\cdot} \otimes T(I^{\cdot})^{[p]}) \arrow[u, "j_{\cdot} \otimes 1"] \arrow[r, "U^{\cdot}"] & \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]} \otimes W_{\cdot}) \arrow[u, "1 \otimes j_{\cdot}"] \arrow[ur, "\Phi^0(\hat{\theta}^{\cdot})"'] &
\end{tikzcd}$$

We have shown that the triangle in the above diagram commutes up to $\mathbb{F}_p\pi$ homotopy, and we have that the square on the left commutes. This implies the two compositions,

$$\phi^{\cdot}\circ(j.\otimes 1)=\Phi^0(\hat{\phi}^{\cdot})\circ U^{\cdot}\circ(j.\otimes 1),\qquad \theta^{\cdot}=\Phi^0(\hat{\theta}^{\cdot})\circ U^{\cdot}:\qquad \mathrm{Tot}^{\cdot}(W.\otimes T(I^{\cdot})^{[p]})\to T(I^{\cdot})$$

are $\mathbb{F}_p\pi$ homotopic, which was the result to be shown.

□

**Lemma 5.3.3.** *The object $(K^{\cdot},\theta^{\cdot})$ is a Cartan object. That is, given $\tilde{\theta}^{\cdot}$ as defined in Definition 2.0.1, the following diagram commutes up to $\mathbb{F}_p\pi$ homotopy:*

$$\begin{array}{ccc}
Tot^{\cdot}(W.\otimes_{\mathbb{F}_p}(K^{\cdot}\otimes_{\mathbb{F}_p}K^{\cdot})^{[p]}) & \xrightarrow{\tilde{\theta}^{\cdot}} & Tot^{\cdot}(K^{\cdot}\otimes_{\mathbb{F}_p}K^{\cdot}) \\
\big\downarrow{\scriptstyle 1\otimes(M^{\cdot})^{[p]}} & & \big\downarrow{\scriptstyle M^{\cdot}} \\
Tot^{\cdot}(W.\otimes_{\mathbb{F}_p}(K^{\cdot})^{[p]}) & \xrightarrow{\theta^{\cdot}} & K^{\cdot}
\end{array}$$

*Proof.* Given $\beta^{\cdot}$ as constructed previously in this section, define the $\mathbb{F}_p\pi$ morphism $\tilde{\beta}^{\cdot}:\mathrm{Tot}^{\cdot}((I^{\cdot}\otimes I^{\cdot})^{[p]})\to \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W.,\mathrm{Tot}^{\cdot}(I^{\cdot}\otimes I^{\cdot}))$ by the following composition:

$$\begin{array}{ccc}
\tilde{\beta}^{\cdot}:\mathrm{Tot}^{\cdot}((I^{\cdot}\otimes I^{\cdot})^{[p]}) & \xrightarrow{S^{\cdot}} & \mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}\otimes(I^{\cdot})^{[p]}) \\
& \swarrow{\scriptstyle \beta^{\cdot}\otimes\beta^{\cdot}} & \\
\mathrm{Tot}^{\cdot}(\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W.,I^{\cdot})\otimes\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W.,I^{\cdot})) & \xrightarrow{\rho^{\cdot}} & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}.(W.\otimes W.),\mathrm{Tot}^{\cdot}(I^{\cdot}\otimes I^{\cdot})) \\
& & \big\downarrow{\scriptstyle (\psi.)^*} \\
& & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W.,I^{\cdot}\otimes I^{\cdot})
\end{array}$$

In the above, $\Sigma_p$ acts on $\mathrm{Tot}^{\cdot}((I^{\cdot}\otimes I^{\cdot})^{[p]})$ by permuting two-tensors, and $\Sigma_p$ has diagonal action on $\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}\otimes(I^{\cdot})^{[p]})$. On each open set, the natural map $S^{\cdot}$ sends $(a_1\otimes b_1)\otimes\cdots\otimes(a_p\otimes b_p)$ to $(a_1\otimes\cdots\otimes a_p)\otimes(b_1\otimes\cdots\otimes b_p)$, with sign change based upon degree. Recall $\psi.:\mathrm{Tot}.(W.\otimes W.)\to W.$ is a $\mathbb{F}_p\pi$ morphism over $\mathbb{F}_p$, and $\pi$ acts diagonally on $\mathrm{Tot}.(W.\otimes W.)$. Recall the product, $\tilde{m}^{\cdot}:\mathrm{Tot}^{\cdot}(I^{\cdot}\otimes I^{\cdot})\to I^{\cdot}$. We have two compositions, $\beta\circ(\tilde{m}^{\cdot})^{[p]}$, and $(\tilde{m}^{\cdot})_*\circ\tilde{\beta}$:

$$\begin{array}{ccc}
\mathrm{Tot}^{\cdot}((I^{\cdot}\otimes I^{\cdot})^{[p]}) & \xrightarrow{(\tilde{m}^{\cdot})^{[p]}} & \mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}) \\
\big\downarrow{\scriptstyle \tilde{\beta}^{\cdot}} & & \big\downarrow{\scriptstyle \beta^{\cdot}} \\
\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W.,\mathrm{Tot}^{\cdot}(I^{\cdot}\otimes I^{\cdot})) & \xrightarrow{(\tilde{m}^{\cdot})_*} & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W.,I^{\cdot})
\end{array}$$

I claim the square above commutes up to $\mathbb{F}_p\pi$ homotopy. We have the diagram below:

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}((I^{\cdot}\otimes I^{\cdot})^{[p]}) \arrow[r, "(\tilde{m}^{\cdot})^{[p]}"] & \mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}) \arrow[r, "\beta^{\cdot}"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot},I^{\cdot}) \\
\mathrm{Tot}^{\cdot}((A^{\cdot}\otimes A^{\cdot})^{[p]}) \arrow[u, "(\iota^{\cdot}\otimes\iota^{\cdot})^{[p]}"] \arrow[r, "(m^{\cdot})^{[p]}"] & \mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]}) \arrow[u, "(\iota^{\cdot})^{[p]}"] \arrow[r, "m^{\cdot}_p"] & A^{\cdot} \arrow[u, "\nu^{\cdot}"]
\end{tikzcd}$$

with all squares commutative. Thus $\beta^{\cdot}\circ(\tilde{m}^{\cdot})^{[p]}$ is a chain map over $m^{\cdot}_{2p} = m^{\cdot}_p\circ(m^{\cdot})^{[p]}$, the $2p$-iterated product on $A^{\cdot}$. I will show $(m^{\cdot})_*\circ\tilde{\beta}$ is also a chain map over $m^{\cdot}_{2p}$. Let $\varepsilon_{\cdot} : W_{\cdot} \to F_p[0]_{\cdot}$ denote the $\pi$ projective resolution of $\mathbb{F}_p$, and recall $\nu^{\cdot} = (\varepsilon_{\cdot})^*\circ(\iota^{\cdot})_*$ from Lemma 3.2.21. We have the diagram:

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}((A^{\cdot}\otimes A^{\cdot})^{[p]}) \arrow[r, hook, "(\iota^{\cdot}\otimes\iota^{\cdot})^{[p]}"] \arrow[d, "S^{\cdot}"] & \mathrm{Tot}^{\cdot}((I^{\cdot}\otimes I^{\cdot})^{[p]}) \arrow[d, "S^{\cdot}"] \arrow[dddd, bend left=60, "\tilde{\beta}^{\cdot}"] \\
\mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]}\otimes(A^{\cdot})^{[p]}) \arrow[r, hook, "(\iota^{\cdot})^{[p]}\otimes(\iota^{\cdot})^{[p]}"] \arrow[d, "m^{\cdot}_p\otimes m^{\cdot}_p"] & \mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}\otimes(I^{\cdot})^{[p]}) \arrow[d, "\beta^{\cdot}\otimes\beta^{\cdot}"] \\
\mathrm{Tot}^{\cdot}(A^{\cdot}\otimes A^{\cdot}) \arrow[r, hook, "(\varepsilon^*_{\cdot}\circ\iota^{\cdot}_*)\otimes(\varepsilon^*_{\cdot}\circ\iota^{\cdot}_*)"] \arrow[d, "1"] & \mathrm{Tot}^{\cdot}(\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot},I^{\cdot})\otimes\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot},I^{\cdot})) \arrow[d, "\rho^{\cdot}"] \\
\mathrm{Tot}^{\cdot}(A^{\cdot}\otimes A^{\cdot}) \arrow[r, hook, "(\varepsilon_{\cdot}\otimes\varepsilon_{\cdot})^*\circ(\iota^{\cdot}\otimes\iota^{\cdot})_*"] \arrow[d, "1"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}_{\cdot}(W_{\cdot}\otimes W_{\cdot}),\mathrm{Tot}^{\cdot}(I^{\cdot}\otimes I^{\cdot})) \arrow[d, "(\psi_{\cdot})^*"] \\
\mathrm{Tot}^{\cdot}(A^{\cdot}\otimes A^{\cdot}) \arrow[r, hook, "(\varepsilon_{\cdot})^*\circ(\iota^{\cdot}\otimes\iota^{\cdot})_*"] \arrow[d, "m^{\cdot}"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot},\mathrm{Tot}^{\cdot}(I^{\cdot}\otimes I^{\cdot})) \arrow[d, "(\tilde{m}^{\cdot})_*"] \\
A^{\cdot} \arrow[r, hook, "\varepsilon^*_{\cdot}\circ\iota^{\cdot}_*"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot},I^{\cdot})
\end{tikzcd}$$

Along the right column, the composition of the top 4 morphisms forms $\tilde{\beta}^{\cdot}$. The composition along the left column is the $2p$-iterated product on $A^{\cdot}$, $m^{\cdot}_{2p}$. Note that because $A^{\cdot}$ is graded commutative, the shuffling isomorphism $S^{\cdot}$ does not affect the result. I claim that each square in the diagram above commutes. The top square commutes by the naturality of $S^{\cdot}$. The second square commutes by the construction of $\beta^{\cdot}$. The third square commutes by the naturality of $\rho^{\cdot}$. The fourth square commutes by the construction of $\psi_{\cdot}$. And finally the bottom square commutes by the construction of $\tilde{m}^{\cdot}$. Thus, we have shown $(\tilde{m}^{\cdot})_* \circ \tilde{\beta}^{\cdot}$ is also a $\mathbb{F}_p\pi$ morphism over $m^{\cdot}_{2p}$. Since $\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W.,I^{\cdot})$ is an injective resolution of $A^{\cdot}$ in $\mathrm{Sh}_{\mathbb{F}_p\pi}(X)$ by Lemma 3.2.21, and $\mathrm{Tot}^{\cdot}((I^{\cdot}\otimes I^{\cdot})^{[p]})$ is a resolution of $\mathrm{Tot}^{\cdot}((A^{\cdot}\otimes A^{\cdot})^{[p]})$ by Corollary 3.2.17, we can invoke Lemma 3.2.14 to obtain a $\mathbb{F}_p\pi$ homotopy $h^{\cdot}$ from $\beta^{\cdot}\circ(\tilde{m}^{\cdot})^{[p]}$ to $(\tilde{m}^{\cdot})_*\circ\tilde{\beta}^{\cdot}$. Since $T$ is an additive functor, we have $T(h^{\cdot})$ is a homotopy from $T(\beta^{\cdot}\circ(\tilde{m}^{\cdot})^{[p]})$ to $T((\tilde{m}^{\cdot})_*\circ\tilde{\beta}^{\cdot})$. This shows the bottom square in the diagram below commutes up to $\mathbb{F}_p\pi$ homotopy.

$$\begin{array}{ccc}
\mathrm{Tot}^{\cdot}((T(I^{\cdot})\otimes T(I^{\cdot}))^{[p]}) & \xrightarrow{(M^{\cdot})^{[p]}} & \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}) \\
\big\downarrow{\scriptstyle\gamma_{2p}} & & \big\downarrow{\scriptstyle\gamma_p} \\
T(\mathrm{Tot}^{\cdot}((I^{\cdot}\otimes I^{\cdot})^{[p]})) & \xrightarrow{T((\tilde{m}^{\cdot})^{[p]})} & T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]})) \\
\big\downarrow{\scriptstyle T(\tilde{\beta}^{\cdot})} & & \big\downarrow{\scriptstyle T(\beta^{\cdot})} \\
\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W.,T(\mathrm{Tot}^{\cdot}(I^{\cdot}\otimes I^{\cdot}))) & \xrightarrow{T(\tilde{m}^{\cdot})_*} & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W.,T(I^{\cdot}))
\end{array} \tag{5.1}$$

Since the top square commutes, the perimeter commutes up to $\mathbb{F}_p\pi$ homotopy. Going around the top is the composition $T(\beta^{\cdot})\circ\gamma_p\circ(M^{\cdot})^{[p]} = \hat{\theta}^{\cdot}\circ(M^{\cdot})^{[p]}$, while the bottom route is $T(\tilde{m}^{\cdot})\circ T(\beta^{\cdot})\circ\gamma_{2p}$. I claim the bottom composition, $T(\tilde{m}^{\cdot})\circ T(\beta^{\cdot})\circ\gamma_{2p}$, is equal to $\hat{\hat{\theta}}^{\cdot}\circ M^{\cdot}$, for a map:

$$\hat{\hat{\theta}}^{\cdot} : \mathrm{Tot}^{\cdot}((T(I^{\cdot})\otimes T(I^{\cdot}))^{[p]}) \to \mathrm{Hom}_{\mathbb{F}_p}(W.,\mathrm{Tot}^{\cdot}(T(I^{\cdot})\otimes T(I^{\cdot})))$$

where $\hat{\hat{\theta}}^{\cdot}$ is related to $\tilde{\theta}^{\cdot}$ by $\tilde{\theta}^{\cdot} = \Phi^0(\hat{\hat{\theta}}^{\cdot})\circ U^{\cdot}_{W.\otimes(T(I^{\cdot})\otimes T(I^{\cdot}))^{[p]}}$. Here I use

$$U^{\cdot}_{A^{\cdot}\otimes B^{\cdot}} : \mathrm{Tot}^{\cdot}(A^{\cdot}\otimes B^{\cdot}) \to \mathrm{Tot}^{\cdot}(B^{\cdot}\otimes A^{\cdot})$$

to denote the isomorphism that swaps tensors with sign based on degree. Since $\tilde{\theta}^{\cdot}$ is given and $\Phi^0$ and $U^{\cdot}$ are isomorphisms, we can define $\hat{\hat{\theta}}^{\cdot} = (\Phi^0)^{-1}(\tilde{\theta}^{\cdot}\circ(U^{\cdot}_{W.\otimes(T(I^{\cdot})\otimes T(I^{\cdot}))^{[p]}})^{-1})$. Recall $\tilde{\theta}^{\cdot}$ is defined by the following composition from Definition 2.0.1:

$$
\begin{tikzcd}
\tilde{\theta}^{\cdot} : \mathrm{Tot}^{\cdot}(W_{\cdot} \otimes (T(I^{\cdot}) \otimes T(I^{\cdot}))^{[p]}) \arrow[r, "\psi_{\cdot} \otimes S^{\cdot}"] & \mathrm{Tot}^{\cdot}(\mathrm{Tot}_{\cdot}(W_{\cdot} \otimes W_{\cdot}) \otimes \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]} \otimes T(I^{\cdot})^{[p]})) \arrow[d, "1 \otimes U^{\cdot}_{W_{\cdot} \otimes T(I^{\cdot})^{[p]}} \otimes 1"] \\
& \mathrm{Tot}^{\cdot}(W_{\cdot} \otimes T(I^{\cdot})^{[p]} \otimes W_{\cdot} \otimes T(I^{\cdot})^{[p]}) \arrow[d, "\theta^{\cdot} \otimes \theta^{\cdot}"] \\
& \mathrm{Tot}^{\cdot}(T(I^{\cdot}) \otimes T(I^{\cdot}))
\end{tikzcd}
$$

I claim that $\hat{\tilde{\theta}}^{\cdot}$ is given by the following composition:

$$
\begin{tikzcd}
\hat{\tilde{\theta}}^{\cdot} : \mathrm{Tot}^{\cdot}((T(I^{\cdot}) \otimes T(I^{\cdot}))^{[p]}) \arrow[r, "S^{\cdot}"] & \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]} \otimes T(I^{\cdot})^{[p]}) \arrow[d, "\hat{\theta}^{\cdot} \otimes \hat{\theta}^{\cdot}"] \\
& \mathrm{Tot}^{\cdot}(\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, T(I^{\cdot})) \otimes \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, T(I^{\cdot}))) \arrow[d, "\rho^{\cdot}"] \\
& \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}_{\cdot}(W_{\cdot} \otimes W_{\cdot}), \mathrm{Tot}^{\cdot}(T(I^{\cdot}) \otimes T(I^{\cdot}))) \arrow[d, "(\psi_{\cdot})^*"] \\
& \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, \mathrm{Tot}^{\cdot}(T(I^{\cdot}) \otimes T(I^{\cdot})))
\end{tikzcd}
$$

We can derive this algebraically using the rules of Lemmas 3.1.7, 3.1.8, 3.1.9, and 3.1.12.

$$
\begin{aligned}
&\Phi^0((\psi_{\cdot})^* \circ \rho^{\cdot} \circ (\hat{\theta}^{\cdot} \otimes \hat{\theta}^{\cdot}) \circ S^{\cdot}) \circ U^{\cdot}_{W_{\cdot} \otimes (T(I^{\cdot}) \otimes T(I^{\cdot}))^{[p]}} \\
&\quad = \Phi^0(\rho^{\cdot} \circ (\hat{\theta}^{\cdot} \otimes \hat{\theta}^{\cdot}) \circ S^{\cdot}) \circ (1 \otimes \psi_{\cdot}) \circ U^{\cdot}_{W_{\cdot} \otimes (T(I^{\cdot}) \otimes T(I^{\cdot}))^{[p]}} \\
&\quad = \Phi^0(\rho^{\cdot} \circ (\hat{\theta}^{\cdot} \otimes \hat{\theta}^{\cdot})) \circ (S^{\cdot} \otimes 1) \circ (1 \otimes \psi_{\cdot}) \circ U^{\cdot}_{W_{\cdot} \otimes (T(I^{\cdot}) \otimes T(I^{\cdot}))^{[p]}} \\
&\quad = (\Phi^0(\hat{\theta}^{\cdot}) \otimes \Phi^0(\hat{\theta}^{\cdot})) \circ (1 \otimes U^{\cdot}_{T(I^{\cdot})^{[p]} \otimes W_{\cdot}} \otimes 1) \circ (S^{\cdot} \otimes \psi_{\cdot}) \circ U^{\cdot}_{W_{\cdot} \otimes (T(I^{\cdot}) \otimes T(I^{\cdot}))^{[p]}} \\
&\quad = (\Phi^0(\hat{\theta}^{\cdot}) \otimes \Phi^0(\hat{\theta}^{\cdot})) \circ (1 \otimes U^{\cdot}_{T(I^{\cdot})^{[p]} \otimes W_{\cdot}} \otimes 1) \circ U^{\cdot}_{(W_{\cdot} \otimes W_{\cdot}) \otimes (T(I^{\cdot})^{[p]} \otimes T(I^{\cdot})^{[p]})} \circ (\psi_{\cdot} \otimes S^{\cdot}) \\
&\quad = (\Phi^0(\hat{\theta}^{\cdot}) \otimes \Phi^0(\hat{\theta}^{\cdot})) \circ (U^{\cdot}_{W_{\cdot} \otimes T(I^{\cdot})^{[p]}} \otimes U^{\cdot}_{W_{\cdot} \otimes T(I^{\cdot})^{[p]}}) \circ (1 \otimes U^{\cdot}_{T(I^{\cdot})^{[p]} \otimes W_{\cdot}} \otimes 1) \circ (\psi_{\cdot} \otimes S^{\cdot}) \\
&\quad = ((\Phi^0(\hat{\theta}^{\cdot}) \circ U^{\cdot}_{W_{\cdot} \otimes T(I^{\cdot})^{[p]}}) \otimes (\Phi^0(\hat{\theta}^{\cdot}) \circ U^{\cdot}_{W_{\cdot} \otimes T(I^{\cdot})^{[p]}})) \circ (1 \otimes U^{\cdot}_{T(I^{\cdot})^{[p]} \otimes W_{\cdot}} \otimes 1) \circ (\psi_{\cdot} \otimes S^{\cdot})
\end{aligned}
$$

$$= (\theta^{\cdot} \otimes \theta^{\cdot}) \circ (1 \otimes U^{\cdot}_{T(I^{\cdot})^{[p]} \otimes W_{\cdot}} \otimes 1) \circ (\psi_{\cdot} \otimes S^{\cdot})$$
$$= \tilde{\theta}$$

And the above equation implies:

$$\tilde{\hat{\theta}}^{\cdot} = (\psi_{\cdot})^{*} \circ \rho^{\cdot} \circ (\hat{\theta}^{\cdot} \otimes \hat{\theta}^{\cdot}) \circ S^{\cdot}$$

Now I claim the following rectangle commutes, where the composition along the left column is $\tilde{\hat{\theta}}^{\cdot}$, and the upper path is from the lower path of the square in diagram 5.1.

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}((T(I^{\cdot}) \otimes T(I^{\cdot}))^{[p]}) \arrow[r, "\gamma_{2p}"] \arrow[d, "S^{\cdot}"] & T(\mathrm{Tot}^{\cdot}((I^{\cdot} \otimes I^{\cdot}))^{[p]}) \arrow[d, "T(S^{\cdot})"] \\
\mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]} \otimes T(I^{\cdot})^{[p]}) \arrow[r, "\gamma_{2p}"] \arrow[d, "\hat{\theta}^{\cdot} \otimes \hat{\theta}^{\cdot}"] & T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]} \otimes (I^{\cdot})^{[p]})) \arrow[d, "T(\beta^{\cdot} \otimes \beta^{\cdot})"] \\
\mathrm{Tot}^{\cdot}(\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, T(I^{\cdot})) \otimes \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, T(I^{\cdot}))) \arrow[r, "\gamma"] \arrow[d, "\rho^{\cdot}"] & T(\mathrm{Tot}^{\cdot}(\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, I^{\cdot}) \otimes \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, I^{\cdot}))) \arrow[d, "T(\rho^{\cdot})"] \\
\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}_{\cdot}(W_{\cdot} \otimes W_{\cdot}), \mathrm{Tot}^{\cdot}(T(I^{\cdot}) \otimes T(I^{\cdot}))) \arrow[r, "(\gamma^{\cdot})_*"] \arrow[d, "(\psi_{\cdot})^*"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}^{\cdot}(W_{\cdot} \otimes W_{\cdot}), T(\mathrm{Tot}^{\cdot}(I^{\cdot} \otimes I^{\cdot}))) \arrow[d, "(\psi_{\cdot})^*"] \\
\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, \mathrm{Tot}^{\cdot}(T(I^{\cdot}) \otimes T(I^{\cdot}))) \arrow[r, "(\gamma^{\cdot})_*"] \arrow[dr, "(M^{\cdot})_*"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, T(\mathrm{Tot}^{\cdot}(I^{\cdot} \otimes I^{\cdot}))) \arrow[d, "T(\tilde{m}^{\cdot})_*"] \\
 & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, T(I^{\cdot}))
\end{tikzcd}$$

Recall $\hat{\theta}^{\cdot} = T(\beta^{\cdot}) \circ \gamma_p$ and $M^{\cdot} = T(\tilde{m}^{\cdot}) \circ \gamma$. These two facts and the naturality of the $\gamma$ maps imply each of the faces in the above diagram commute. Since the composition along the lower path is $(M^{\cdot})_* \circ \tilde{\hat{\theta}}^{\cdot}$, and the composition along the upper path is $T(\tilde{m}^{\cdot}) \circ T(\tilde{\beta}^{\cdot}) \circ \gamma_{2p}$, it has been shown these two maps are equal. And since the later map has been shown to be $\mathbb{F}_p\pi$ homotopic to $\hat{\theta}^{\cdot} \circ (M^{\cdot})^{[p]}$, we now have the square below commutes up to $\mathbb{F}_p\pi$ homotopy:

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}((T(I^{\cdot})\otimes T(I^{\cdot}))^{[p]}) \arrow[r, "(M^{\cdot})^{[p]}"] \arrow[d, "\hat{\hat{\theta}}^{\cdot}"] & \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}) \arrow[d, "\hat{\theta}^{\cdot}"] \\
\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot},\mathrm{Tot}^{\cdot}(T(I^{\cdot})\otimes T(I^{\cdot}))) \arrow[r, "(M^{\cdot})_*"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot},T(I^{\cdot}))
\end{tikzcd}$$

By Lemma 3.1.6, this implies the following chain maps are $\mathbb{F}_p\pi$ homotopic:

$$\Phi^0(\hat{\theta}^{\cdot}\circ(M^{\cdot})^{[p]}),\ \Phi^0((M^{\cdot})_*\circ\hat{\hat{\theta}}^{\cdot}):\ \mathrm{Tot}^{\cdot}((T(I^{\cdot})\otimes T(I^{\cdot}))^{[p]}\otimes W_{\cdot})\to T(I^{\cdot})$$

By Lemma 3.1.9, we have $\Phi^0(\hat{\theta}\circ(M^{\cdot})^{[p]})=\Phi^0(\hat{\theta})\circ((M^{\cdot})^{[p]}\otimes 1)$. By Lemma 3.1.7, we have $\Phi^0((M^{\cdot})_*\circ\hat{\hat{\theta}}^{\cdot})=M^{\cdot}\circ\Phi^0(\hat{\hat{\theta}}^{\cdot})$. We have now shown the right square in the diagram below commutes up to $\mathbb{F}_p\pi$ homotopy:

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}(W_{\cdot}\otimes(T(I^{\cdot})\otimes T(I^{\cdot}))^{[p]}) \arrow[r, "U^{\cdot}"] \arrow[d, "1\otimes(M^{\cdot})^{[p]}"] & \mathrm{Tot}^{\cdot}((T(I^{\cdot})\otimes T(I^{\cdot}))^{[p]}\otimes W_{\cdot}) \arrow[r, "\Phi^0(\hat{\hat{\theta}}^{\cdot})"] \arrow[d, "(M^{\cdot})^{[p]}\otimes 1"] & \mathrm{Tot}^{\cdot}(T(I^{\cdot})\otimes T(I^{\cdot}))) \arrow[d, "M^{\cdot}"] \\
\mathrm{Tot}^{\cdot}(W_{\cdot}\otimes T(I^{\cdot})^{[p]}) \arrow[r, "U^{\cdot}"] & \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}\otimes W_{\cdot}) \arrow[r, "\Phi^0(\hat{\theta}^{\cdot})"] & T(I^{\cdot})
\end{tikzcd}$$

while the square on the left commutes by the naturality of $U^{\cdot}$. This shows the rectangle commutes up to $\mathbb{F}_p\pi$ homotopy. The composition along the upper path is $M^{\cdot}\circ\tilde{\theta}^{\cdot}$ and the composition along the lower path is $\theta^{\cdot}\circ(1\otimes(M^{\cdot})^{[p]})$. The fact that these two compositions are $\mathbb{F}_p\pi$ homotopic was the result to be shown.

□

**Lemma 5.3.4.** *The object $(K^{\cdot},\theta^{\cdot})$ is an Adem object. That is, there is a $\Sigma_{p^2}$ chain map $\xi^{\cdot}$ : $Y_{\cdot}\otimes_{\mathbb{F}_p}(K^{\cdot})^{[p^2]}\to K^{\cdot}$ such that the following diagram commutes up to $\tau$-homotopy (dropping the $\mathrm{Tot}^{\cdot}$ and dots notation below, and tensors are over $\mathbb{F}_p$).*

$$\begin{tikzcd}
(W_1 \otimes W_2^{[p]}) \otimes K^{[p^2]} \arrow[r, "w\otimes 1"] \arrow[dd, "1\otimes S"] & Y \otimes K^{[p^2]} \arrow[dr, "\xi"] & \\
 & & K \\
W_1 \otimes (W_2 \otimes K^{[p]})^{[p]} \arrow[r, "1\otimes\theta^{[p]}"] & W_1 \otimes K^{[p]} \arrow[ur, "\theta"] &
\end{tikzcd}$$

*Recall the definitions of $W_1, W_2, Y$ and the group actions involved from Definition 2.0.5. In the above I use $S^{\cdot}$ instead of $U^{\cdot}$ for the graded tensor shuffling isomorphism, $S^{\cdot}$ : $Tot^{\cdot}(W_{2,\cdot}^{[p]} \otimes ((K^{\cdot})^{[p]})^{[p]}) \to Tot^{\cdot}((W_{2,\cdot} \otimes K^{[p]})^{[p]})$ because I'm already using $U^{\cdot}$ to denote the graded tensor product swapping isomorphism $Tot^{\cdot}(A^{\cdot} \otimes B^{\cdot}) \to Tot^{\cdot}(B^{\cdot} \otimes A^{\cdot})$ for various $A^{\cdot}$, $B^{\cdot}$.*

*Proof.* The map $m^{\cdot}_{p^2}$ is the $p^2$-iterated product in $A^{\cdot}$, which is a $\Sigma_{p^2}$ morphism because the product is graded commutative. We have the $\mathbb{F}_p\pi$ projective resolution $\varepsilon_{1,\cdot} : W_{1,\cdot} \twoheadrightarrow \mathbb{F}_p$, the $\mathbb{F}_p\nu$ projective resolution $\varepsilon_{2,\cdot} : W_{2,\cdot} \twoheadrightarrow \mathbb{F}_p$, and the $\mathbb{F}_p\Sigma_{p^2}$ projective resolution $\kappa_{\cdot} : Y_{\cdot} \twoheadrightarrow \mathbb{F}_p$. We still have the injective resolution in $\mathrm{Sh}_{\mathbb{F}_p}(X)$, $\iota^{\cdot} : A^{\cdot} \hookrightarrow I^{\cdot}$. By Lemma 3.2.21, we have the injective resolution in $\mathrm{Sh}_{\mathbb{F}_p\Sigma_{p^2}}(X)$:

$$\kappa^*_{\cdot} \circ \iota^{\cdot}_* : A^{\cdot} = \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(\mathbb{F}_p, A^{\cdot}) \hookrightarrow \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(Y_{\cdot}, I^{\cdot})$$

By Corollary 3.2.17, the following is a resolution:

$$(\iota^{\cdot})^{[p^2]} : \mathrm{Tot}^{\cdot}((A^{\cdot})^{[p^2]}) \to \mathrm{Tot}^{\cdot}((I^{\cdot})^{[p^2]})$$

We have the solid diagram below:

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p^2]}) \arrow[r, dashed, "\zeta^{\cdot}"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(Y_{\cdot}, I^{\cdot}) \\
\mathrm{Tot}^{\cdot}((A^{\cdot})^{[p^2]}) \arrow[u, hook, "(\iota^{\cdot})^{[p^2]}"] \arrow[r, "m^{\cdot}_{p^2}"] & A^{\cdot} \arrow[u, hook, "\kappa^*_{\cdot}\circ\iota^{\cdot}_*"']
\end{tikzcd} \tag{5.2}$$

Note all the solid arrows in the diagram are $\mathbb{F}_p\Sigma_{p^2}$ morphisms. By Lemma 3.2.13, there is a $\mathbb{F}_p\Sigma_{p^2}$ morphism $\zeta^{\cdot}$ making the diagram commute, and by Lemma 3.2.14, $\zeta^{\cdot}$ is unique up to homotopy. Recall the $\mathbb{F}_p\tau$ chain map $w_\cdot$ from Definition 2.0.5 is defined to make the following diagram commute:

$$\begin{tikzcd}
\mathrm{Tot}_\cdot(W_{1,\cdot} \otimes (W_{2,\cdot}^{[p]})) \arrow[r, "w_\cdot"] \arrow[d, two heads, "\varepsilon_{1,\cdot}\otimes\varepsilon_{2,\cdot}^{[p]}"] & Y_\cdot \arrow[d, two heads, "\kappa_\cdot"] \\
\mathbb{F}_p \arrow[r, "1"] & \mathbb{F}_p
\end{tikzcd}$$

We have the following diagram with commutative squares:

$$\begin{tikzcd}
\mathrm{Tot}^\cdot((I^\cdot)^{[p^2]}) \arrow[r, "\zeta^\cdot"] & \mathbf{Hom}^\cdot_{\mathbb{F}_p}(Y_\cdot, I^\cdot) \arrow[r, "(w_\cdot)^*"] & \mathbf{Hom}^\cdot_{\mathbb{F}_p}(\mathrm{Tot}_\cdot(W_{1,\cdot} \otimes (W_{2,\cdot})^{[p]}), I^\cdot) \\
\mathrm{Tot}^\cdot((A^\cdot)^{[p^2]}) \arrow[u, hook, "(\iota^\cdot)^{[p^2]}"] \arrow[r, "m^\cdot_{p^2}"] & A^\cdot \arrow[u, hook, "\kappa_\cdot^*\circ\iota^\cdot_*"'] \arrow[r, "1"] & A^\cdot \arrow[u, hook, "(\varepsilon_{1,\cdot}\otimes\varepsilon_{2,\cdot}^{[p]})^*\circ\iota^\cdot_*"']
\end{tikzcd}$$

Because $\mathrm{Tot}_\cdot(W_{1,\cdot} \otimes (W_{2,\cdot})^{[p]})$ is a free resolution of $\mathbb{F}_p$ in ${}_{\mathbb{F}_p\tau}\mathrm{Mod}$, by Lemma 3.2.21, $\mathbf{Hom}^\cdot_{\mathbb{F}_p}(\mathrm{Tot}_\cdot(W_{1,\cdot} \otimes (W_{2,\cdot})^{[p]}), I^\cdot)$ is an injective resolution of $A^\cdot$ in $\mathrm{Sh}_{\mathbb{F}_p\tau}(X)$. By Lemma 3.2.14, any other such $\mathbb{F}_p\tau$ chain map over $m^\cdot_{p^2}$ will be $\mathbb{F}_p\tau$ homotopic to $(w_\cdot)^* \circ \zeta^\cdot$. Define $\beta^\cdot_1 = \beta^\cdot$ to be the $\mathbb{F}_p\pi$ chain map constructed in Section 5.2, which made the following diagram of $\mathbb{F}_p\pi$ chain maps commute:

$$\begin{tikzcd}
\mathrm{Tot}^\cdot((I^\cdot)^{[p]}) \arrow[r, "\beta^\cdot_1"] & \mathbf{Hom}^\cdot_{\mathbb{F}_p}(W_{1,\cdot}, I^\cdot) \\
\mathrm{Tot}^\cdot((A^\cdot)^{[p]}) \arrow[u, "(\iota^\cdot)^{[p]}"] \arrow[r, "m^\cdot_p"] & A^\cdot \arrow[u, "\varepsilon^*_{1,\cdot}\circ\iota^\cdot_*"']
\end{tikzcd} \tag{5.3}$$

Define $\beta^\cdot_2$ uniquely up to $\mathbb{F}_p\nu$ homotopy by the diagram below:

$$\begin{tikzcd}
\mathrm{Tot}^\cdot((I^\cdot)^{[p]}) \arrow[r, dashed, "\beta^\cdot_2"] & \mathbf{Hom}^\cdot_{\mathbb{F}_p}(W_{2,\cdot}, I^\cdot) \\
\mathrm{Tot}^\cdot((A^\cdot)^{[p]}) \arrow[u, "(\iota^\cdot)^{[p]}"] \arrow[r, "m^\cdot_p"] & A^\cdot \arrow[u, "\varepsilon^*_{2,\cdot}\circ\iota^\cdot_*"']
\end{tikzcd} \tag{5.4}$$

Set $\theta^{\cdot}_1 = \theta^{\cdot}$, $\hat{\theta}^{\cdot}_1 = \hat{\theta}^{\cdot}$, and define $\hat{\theta}^{\cdot}_2 = T(\beta^{\cdot}_2) \circ \gamma_p$ and $\theta^{\cdot}_2 = \Phi^0(\hat{\theta}^{\cdot}_2) \circ U^{\cdot}_{W_2}$, where $U^{\cdot}_{W_i} : W_{i,\cdot} \otimes T(I^{\cdot})^{[p]} \to T(I^{\cdot})^{[p]} \otimes W_{i,\cdot}$ is the swapping isomorphism for $i = 1,2$. Let $U_1 : W_{1,\cdot} \otimes W^{[p]}_{2,\cdot} \to W^{[p]}_{2,\cdot} \otimes W_{1,\cdot}$. We have the diagram below:

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}((A^{\cdot})^{[p^2]}) \arrow[r, hook, "(\iota^{\cdot})^{[p^2]}"] \arrow[d, "(m^{\cdot}_p)^{[p]}"] & \mathrm{Tot}^{\cdot}((I^{\cdot})^{[p^2]}) \arrow[d, "(\beta^{\cdot}_2)^{[p]}"] \\
\mathrm{Tot}^{\cdot}((A^{\cdot})^{p}) \arrow[r, hook, "(\varepsilon^*_{2,\cdot} \circ \iota^{\cdot}_*)^{[p]}"] \arrow[d, "1"] & \mathrm{Tot}^{\cdot}((\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{2,\cdot}, I^{\cdot}))^{[p]}) \arrow[d, "\rho^{\cdot}_p"] \\
\mathrm{Tot}^{\cdot}((A^{\cdot})^{p}) \arrow[r, hook, "(\varepsilon^{[p]}_{2,\cdot})^* \circ ((\iota^{\cdot})^{[p]})_*"] \arrow[d, "m^{\cdot}_p"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}_{\cdot}((W_{2,\cdot})^{[p]}), \mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]})) \arrow[d, "(\beta^{\cdot}_1)_*"] \\
A^{\cdot} \arrow[r, hook, "(\varepsilon^{[p]}_{2,\cdot})^* \circ (\varepsilon^*_{1,\cdot} \circ \iota^{\cdot}_*)_*"] \arrow[d, "1"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}_{\cdot}((W_{2,\cdot})^{[p]}), \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{1,\cdot}, I^{\cdot})) \arrow[d, "\Phi^{\cdot}"] \\
A^{\cdot} \arrow[r, hook, "(\varepsilon^{[p]}_{2,\cdot} \otimes \varepsilon_{1,\cdot})^* \circ \iota^{\cdot}_*"] \arrow[d, "1"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}^{\cdot}((W_{2,\cdot})^{[p]} \otimes W_{1,\cdot}), I^{\cdot}) \arrow[d, "(U^{\cdot}_1)^*"] \\
A^{\cdot} \arrow[r, hook, "(\varepsilon_{1,\cdot} \otimes \varepsilon^{[p]}_{2,\cdot})^* \circ \iota^{\cdot}_*"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}^{\cdot}(W_{1,\cdot} \otimes (W_{2,\cdot})^{[p]}, I^{\cdot})
\end{tikzcd}$$

In the above, $\rho^{\cdot}_p$ is the natural $\mathbb{F}_p\tau$ chain map from Corollary 3.1.11, and $\Phi^{\cdot}$ is the natural $\mathbb{F}_p\tau$ isomorphism of complexes from Lemma 3.1.5. The top square commutes because diagram 5.4 commutes. The second square commutes by the naturality of $\rho^{\cdot}_p$. The third square commutes because diagram 5.3 commutes. The fourth square commutes by the naturality of $\Phi^{\cdot}$. And the fifth square commutes by the naturality of $U^{\cdot}_1$. The composition along the left column is $m^{\cdot}_{p^2}$. By Lemma 3.2.14, we get that $(w_{\cdot})^* \circ \zeta^{\cdot}$ and the composition along the right column of the above diagram are $\mathbb{F}_p\tau$ homotopic. That is, the diagram below commutes up to $\mathbb{F}_p\tau$ homotopy.

$$
\begin{tikzcd}
\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p^2]}) \arrow[r, "\zeta^{\cdot}"] \arrow[d, "(\beta_2^{\cdot})^{[p]}"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(Y.,I^{\cdot}) \arrow[d, "(w.)^*"] \\
\mathrm{Tot}^{\cdot}((\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{2,\cdot},I^{\cdot}))^{[p]}) \arrow[d, "\rho_p^{\cdot}"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot.}(W_{1,\cdot}\otimes(W_{2,\cdot})^{[p]}),I^{\cdot}) \\
\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot.}((W_{2,\cdot})^{[p]}),\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]})) \arrow[d, "(\beta_1^{\cdot})_*"] & \\
\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot.}((W_{2,\cdot})^{[p]}),\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{1,\cdot},I^{\cdot})) \arrow[r, "\Phi^{\cdot}"] & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot.}((W_{2,\cdot})^{[p]}\otimes W_{1,\cdot}),I^{\cdot}) \arrow[uu, "(U_1^{\cdot})^*"']
\end{tikzcd}
$$

After applying the additive global section functor $T$, we get the following commutes up to $\mathbb{F}_p\tau$ homotopy.

$$
\begin{tikzcd}
T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p^2]})) \arrow[r, "T(\zeta^{\cdot})"] \arrow[d, "T((\beta_2^{\cdot})^{[p]})"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(Y.,T(I^{\cdot})) \arrow[d, "(w.)^*"] \\
T(\mathrm{Tot}^{\cdot}((\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{2,\cdot},I^{\cdot}))^{[p]})) \arrow[d, "T(\rho_p^{\cdot})"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot.}(W_{1,\cdot}\otimes(W_{2,\cdot})^{[p]}),T(I^{\cdot})) \\
\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot.}((W_{2,\cdot})^{[p]}),T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}))) \arrow[d, "T(\beta_1^{\cdot})_*"] & \\
\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot.}((W_{2,\cdot})^{[p]}),\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{1,\cdot},T(I^{\cdot}))) \arrow[r, "\Phi^{\cdot}"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot.}((W_{2,\cdot})^{[p]}\otimes W_{1,\cdot}),T(I^{\cdot})) \arrow[uu, "(U_1^{\cdot})^*"']
\end{tikzcd}
$$

(5.5)

Let $\gamma_{p^2} : \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}) \to T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}))$ be the natural map. Define the $\mathbb{F}_p\Sigma_{p^2}$ morphism $\hat{\xi} : \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p^2]}) \to \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(Y.,T(I^{\cdot}))$ by the composition $\hat{\xi} = T(\zeta^{\cdot})\circ\gamma_{p^2}$. Set $\xi = \Phi^0(\hat{\xi})\circ U_Y^{\cdot}$, where $U_Y : Y.\otimes T(I^{\cdot})^{[p^2]} \to T(I^{\cdot})^{[p^2]}\otimes Y.$. The following diagram has commuting squares:

$$
\begin{tikzcd}
\mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p^2]}) \arrow[r, "\gamma_{p^2}"] \arrow[d, "(\hat{\theta}_2^{\cdot})^{[p]}"] & T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p^2]})) \arrow[d, "T((\beta_2^{\cdot})^{[p]})"] \\
\mathrm{Tot}^{\cdot}((\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{2,\cdot}, T(I^{\cdot})))^{[p]}) \arrow[r, "\gamma_p"] \arrow[d, "\rho^{\cdot}_p"] & T(\mathrm{Tot}^{\cdot}((\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{2,\cdot}, I^{\cdot}))^{[p]})) \arrow[d, "T(\rho^{\cdot}_p)"] \\
\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}^{\cdot}((W_{2,\cdot})^{[p]}), \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]})) \arrow[r, "(\gamma_p)_*"] \arrow[d, "(\hat{\theta}_1^{\cdot})_*"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}_{\cdot}((W_{2,\cdot})^{[p]}), T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}))) \arrow[d, "T(\beta_1^{\cdot})_*"] \\
\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}_{\cdot}((W_{2,\cdot})^{[p]}), \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{1,\cdot}, T(I^{\cdot}))) \arrow[r, "1"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}_{\cdot}((W_{2,\cdot})^{[p]}), \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{1,\cdot}, T(I^{\cdot})))
\end{tikzcd}
\tag{5.6}
$$

By precomposing the left column of diagram 5.5 with diagram 5.6, we get the following commutes up to $\mathbb{F}_p\tau$ homotopy:

$$
\begin{tikzcd}
\mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p^2]}) \arrow[r, "\hat{\xi}^{\cdot}"] \arrow[d, "(\hat{\theta}_2^{\cdot})^{[p]}"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(Y_{\cdot}, T(I^{\cdot})) \arrow[d, "(w_{\cdot})^*"] \\
\mathrm{Tot}^{\cdot}((\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{2,\cdot}, T(I)^{\cdot}))^{[p]}) \arrow[d, "\rho^{\cdot}_p"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}_{\cdot}(W_{1,\cdot} \otimes (W_{2,\cdot})^{[p]}), T(I^{\cdot})) \\
\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}_{\cdot}((W_{2,\cdot})^{[p]}), \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]})) \arrow[d, "(\hat{\theta}_1^{\cdot})_*"] & \\
\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}_{\cdot}((W_{2,\cdot})^{[p]}), \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{1,\cdot}, T(I^{\cdot}))) \arrow[r, "\Phi^{\cdot}"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(\mathrm{Tot}_{\cdot}((W_{2,\cdot})^{[p]} \otimes W_{1,\cdot}), T(I^{\cdot})) \arrow[uu, "(U_1^{\cdot})^*"']
\end{tikzcd}
$$

By Lemma 3.1.6, we get that the diagram we obtain after applying $\Phi^0$ commutes up to $\mathbb{F}_p\tau$ homotopy. By using the rules of Lemmas 3.1.7, 3.1.8, and 3.1.9, we get for the top path:

$$
\Phi^0((w_{\cdot})^* \circ \hat{\xi}^{\cdot}) = \Phi^0(\hat{\xi}^{\cdot}) \circ (1 \otimes w_{\cdot})
$$

Along the bottom path, applying $\Phi^0$ gives:

$$
\begin{aligned}
&\Phi^0((U_1^{\cdot})^* \circ \Phi^{\cdot} \circ (\hat{\theta}_1^{\cdot})_* \circ \rho_p^{\cdot} \circ (\hat{\theta}_2^{\cdot})^{[p]}) \\
&\quad = \Phi^0(\Phi^{\cdot} \circ (\hat{\theta}_1^{\cdot})_* \circ \rho_p^{\cdot} \circ (\hat{\theta}_2^{\cdot})^{[p]}) \circ (1 \otimes U_1^{\cdot}) && \text{By Lemma 3.1.8} \\
&\quad = \Phi^0(\Phi^0((\hat{\theta}_1^{\cdot})_* \circ \rho_p^{\cdot} \circ (\hat{\theta}_2^{\cdot})^{[p]})) \circ (1 \otimes U_1^{\cdot}) && \text{By Lemma 3.1.14} \\
&\quad = \Phi^0(\hat{\theta}_1^{\cdot} \circ \Phi^0(\rho_p^{\cdot} \circ (\hat{\theta}_2^{\cdot})^{[p]})) \circ (1 \otimes U_1^{\cdot}) && \text{By Lemma 3.1.7} \\
&\quad = \Phi^0(\hat{\theta}_1^{\cdot}) \circ (\Phi^0(\rho_p^{\cdot} \circ (\hat{\theta}_2^{\cdot})^{[p]}) \otimes 1) \circ (1 \otimes U_1^{\cdot}) && \text{By Lemma 3.1.9} \\
&\quad = \Phi^0(\hat{\theta}_1^{\cdot}) \circ ((\Phi^0(\hat{\theta}_2^{\cdot})^{[p]} \circ \tilde{S}^{\cdot}) \otimes 1) \circ (1 \otimes U_1^{\cdot}) && \text{By Lemma 3.1.13} \\
&\quad = \Phi^0(\hat{\theta}_1^{\cdot}) \circ (\Phi^0(\hat{\theta}_2^{\cdot})^{[p]} \otimes 1) \circ (\tilde{S}^{\cdot} \otimes 1) \circ (1 \otimes U_1^{\cdot})
\end{aligned}
$$

where $\tilde{S}^{\cdot} : (T(I^{\cdot})^{[p^2]} \otimes W_{2,\cdot}^{[p]}) \to (T(I^{\cdot})^{[p]} \otimes W_{2,\cdot})^{[p]}$ is the shuffling isomorphism. The last line in the computation above is the composition below:

$$
\begin{tikzcd}
T(I^{\cdot})^{[p^2]} \otimes (W_{1,\cdot} \otimes W_{2,\cdot}^{[p]}) \arrow[r, "1\otimes U_1^{\cdot}"] & T(I^{\cdot})^{[p^2]} \otimes (W_{2,\cdot}^{[p]} \otimes W_{1,\cdot}) \arrow[d, "\tilde{S}^{\cdot}\otimes 1"] \\
& (T(I^{\cdot})^{[p]} \otimes W_{2,\cdot})^{[p]} \otimes W_{1,\cdot} \arrow[d, "\Phi^0(\hat{\theta}_2^{\cdot})^{[p]}\otimes 1"] \\
& T(I^{\cdot})^{[p]} \otimes W_{1,\cdot} \arrow[d, "\Phi^0(\hat{\theta}_1^{\cdot})"] \\
& T(I^{\cdot})
\end{tikzcd}
$$

Since the composition above is $\mathbb{F}_p\tau$ homotopic to $\Phi^0(\hat{\zeta}^{\cdot}) \circ (1 \otimes w_{\cdot})$, we can can precompose both by the following tensor swapping isomorphism,

$$U_2^{\cdot} : (W_{1,\cdot} \otimes W_{2,\cdot}^{[p]}) \otimes T(I^{\cdot})^{[p^2]} \to T(I^{\cdot})^{[p^2]} \otimes (W_{1,\cdot} \otimes W_{2,\cdot}^{[p]})$$

and by Lemma 3.1.16, the results will be $\mathbb{F}_p\tau$ homotopic. For the top path we have:

$$\Phi^0(\hat{\xi}^{\cdot}) \circ (1 \otimes w_{\cdot}) \circ (U_2^{\cdot}) = \Phi^0(\hat{\xi}^{\cdot}) \circ U_Y^{\cdot} \circ (w_{\cdot} \otimes 1) = \xi^{\cdot} \circ (w_{\cdot} \otimes 1)$$

Denote:

$$
\begin{aligned}
U_3^{\cdot} &: (W_{2,\cdot}^{[p]} \otimes W_{1,\cdot}) \otimes T(I^{\cdot})^{[p^2]} \to T(I^{\cdot})^{[p^2]} \otimes (W_{2,\cdot}^{[p]} \otimes W_{1,\cdot}) \\
U_4^{\cdot} &: W_{1,\cdot} \otimes (W_{2,\cdot} \otimes T(I^{\cdot})^{[p]})^{[p]} \to (W_{2,\cdot} \otimes T(I^{\cdot})^{[p]})^{[p]} \otimes W_{1,\cdot}
\end{aligned}
$$

For the lower path, we have:

$$
\begin{aligned}
&\Phi^0(\hat{\theta}_1^{\cdot}) \circ (\Phi^0(\hat{\theta}_2^{\cdot})^{[p]} \otimes 1) \circ (\tilde{S}^{\cdot} \otimes 1) \circ (1 \otimes U_1^{\cdot}) \circ U_2^{\cdot} \\
&\quad = \Phi^0(\hat{\theta}_1^{\cdot}) \circ (\Phi^0(\hat{\theta}_2^{\cdot})^{[p]} \otimes 1) \circ (\tilde{S}^{\cdot} \otimes 1) \circ U_3^{\cdot} \circ (U_1^{\cdot} \otimes 1) \\
&\quad = \Phi^0(\hat{\theta}_1^{\cdot}) \circ (\Phi^0(\hat{\theta}_2^{\cdot})^{[p]} \otimes 1) \circ U_4^{\cdot} \circ (1 \otimes (U_{W_2}^{\cdot})^{[p]}) \circ (1 \otimes S^{\cdot}) \\
&\quad = \Phi^0(\hat{\theta}_1^{\cdot}) \circ U_{W_1}^{\cdot} \circ (1 \otimes \Phi^0(\hat{\theta}_2^{\cdot})^{[p]}) \circ (1 \otimes (U_{W_2}^{\cdot})^{[p]}) \circ (1 \otimes S^{\cdot}) \\
&\quad = \theta_1^{\cdot} \circ (1 \otimes (\theta_2^{\cdot})^{[p]}) \circ (1 \otimes S^{\cdot})
\end{aligned}
$$

Thus we have shown the diagram below commutes up to $\mathbb{F}_p\tau$ homotopy:

$$
\begin{tikzcd}
\mathrm{Tot}^{\cdot}(W_{1,\cdot} \otimes W_{2,\cdot}^{[p]} \otimes T(I^{\cdot})^{[p^2]}) \arrow[r, "w.\otimes 1"] \arrow[d, "1\otimes S^{\cdot}"] & \mathrm{Tot}^{\cdot}(Y. \otimes T(I^{\cdot})^{[p^2]}) \arrow[dr, "\xi^{\cdot}"] & \\
 & & K \\
\mathrm{Tot}^{\cdot}(W_{1,\cdot} \otimes (W_{2,\cdot} \otimes T(I^{\cdot})^{[p]})^{[p]}) \arrow[r, "1\otimes(\theta_2^{\cdot})^{[p]}"] & \mathrm{Tot}^{\cdot}(W_{1,\cdot} \otimes T(I^{\cdot})^{[p]}) \arrow[ur, "\theta_1^{\cdot}"'] &
\end{tikzcd}
$$

And this was the result to be shown.

$\square$

**Theorem 5.3.5.** *Given a topological space $X$, and a bounded below complex $A^{\cdot}$ of sheaves of differential graded commutative $\mathbb{F}_p$ algebras on $X$, there exist canonically defined Steenrod* *operations on the sheaf hypercohomology groups* $\mathbf{H}^{\cdot}(X, A^{\cdot})$*. These Steenrod operations satisfy the formulas in Corollaries 2.0.8 and 2.0.9, Cartan formula and Adem relations included.*

*Proof.* We have $(K^{\cdot}, \theta^{\cdot}) \in \mathcal{C}$, due to the existence of the homotopy associative product established in Lemma 5.1.3, and the axioms required of $\theta^{\cdot}$ are proven in Lemmas 5.3.1 and 5.3.2. This allows us to define Steenrod operations on $H^{\cdot}(K^{\cdot}) = \mathbf{H}^{\cdot}(X, A^{\cdot})$. Because of Lemmas 5.3.3 and 5.3.4, the Cartan formula and Adem relations are valid.

□

## 5.4 Naturality

In this section I will show that the Steenrod operations constructed on sheaf hypercohomology are natural.

**Lemma 5.4.1.** *Let $X$ be a topological space and suppose $A^{\cdot}$ and $B^{\cdot}$ are two bounded below complexes of differential graded commutative $\mathbb{F}_p$ algebras on $X$. Suppose $f^{\cdot} : A^{\cdot} \to B^{\cdot}$ is a differential graded $\mathbb{F}_p$ algebra homomorphism. Then there are induced maps for all $n \in \mathbb{Z}$:*

$$\mathbf{H}^n(X, f^{\cdot}) : \mathbf{H}^n(X, A^{\cdot}) \to \mathbf{H}^n(X, B^{\cdot})$$

*We have that $\mathbf{H}^{\cdot}(X, f^{\cdot})$ commutes with $D_i$ for all $i \geq 0$. As a consequence, $\mathbf{H}^{\cdot}(X, f^{\cdot})$ commutes with the Steenrod operations constructed on $\mathbf{H}^{\cdot}(X, A^{\cdot})$ and $\mathbf{H}^{\cdot}(X, B^{\cdot})$ respectively.*

*Proof.* We first define the morphisms, $\mathbf{H}^n(X, f^{\cdot})$. Let $\iota^{\cdot} : A^{\cdot} \to I^{\cdot}$ and $\kappa^{\cdot} : B^{\cdot} \to J^{\cdot}$ be injective resolutions in $\mathrm{Sh}_{\mathbb{F}_p}(X)$. We have the diagram below:

$$\begin{tikzcd} I^{\cdot} \arrow[r, dashed, "\tilde{f}^{\cdot}"] & J^{\cdot} \\ A^{\cdot} \arrow[u, hook, "\iota^{\cdot}"] \arrow[r, "f^{\cdot}"] & B^{\cdot} \arrow[u, hook, "\kappa^{\cdot}"'] \end{tikzcd}$$

Because $J^{\cdot}$ is injective in each degree and bounded below, and because $\iota^{\cdot}$ is an injective quasi-isomorphism, we have by Lemma 3.2.13 that there is a $\mathbb{F}_p$ chain map $\tilde{f}^{\cdot}$ that makes the diagram commute. By Lemma 3.2.14, $\tilde{f}^{\cdot}$ is unique up to homotopy. Because $T$ is an additive functor, we have $T(\tilde{f}^{\cdot})$ is unique up to homotopy. Thus, we get well defined morphisms:

$$H^n(T(\tilde{f}^\cdot)) : H^n(T(I^\cdot)) \to H^n(T(J^\cdot))$$

and the above is precisely $\mathbf{H}^n(X, f^\cdot)$. To show that $\mathbf{H}^\cdot(X, f^\cdot)$ commutes with $D_i$, it suffices by Lemma 2.0.3 to show that $T(\tilde{f}^\cdot)$ is a morphism in the category $\mathcal{C}(p)$. Let the following denote the $\theta$ maps for $A^\cdot$ and $B^\cdot$ respectively.

$$\theta_A^\cdot : W. \otimes T(I^\cdot)^{[p]} \to T(I^\cdot)$$
$$\theta_B^\cdot : W. \otimes T(J^\cdot)^{[p]} \to T(J^\cdot)$$

We must show that $T(\tilde{f}^\cdot)$ is a morphism between the objects $(T(I^\cdot), \theta_A^\cdot)$ and $(T(J^\cdot), \theta_B^\cdot)$. That is, I must show the following diagram commutes up to $\mathbb{F}_p\pi$ homotopy:

$$\begin{array}{ccc} W. \otimes T(I^\cdot)^{[p]} & \xrightarrow{\theta_A^\cdot} & T(I^\cdot) \\ \downarrow{\scriptstyle 1\otimes T(\tilde{f}^\cdot)^{[p]}} & & \downarrow{\scriptstyle T(\tilde{f}^\cdot)} \\ W. \otimes T(J^\cdot)^{[p]} & \xrightarrow{\theta_B^\cdot} & T(J^\cdot) \end{array}$$

Let $\beta_A^\cdot$ and $\beta_B^\cdot$ denote the $\mathrm{Sh}_{\mathbb{F}_p\pi}(X)$ chain maps making the diagrams below commute, each unique up to homotopy:

$$\begin{array}{ccc} \mathrm{Tot}^\cdot(I^\cdot)^{[p]} & \xrightarrow{\beta_A^\cdot} & \mathbf{Hom}^\cdot_{\mathbb{F}_p}(W., I^\cdot) \\ {\scriptstyle (\iota^\cdot)^{[p]}}\uparrow & & \uparrow{\scriptstyle \varepsilon_\cdot^* \circ \iota_*^\cdot} \\ \mathrm{Tot}^\cdot(A^\cdot)^{[p]} & \xrightarrow{m_{A,p}^\cdot} & A^\cdot \end{array} \qquad \begin{array}{ccc} \mathrm{Tot}^\cdot(J^\cdot)^{[p]} & \xrightarrow{\beta_B^\cdot} & \mathbf{Hom}^\cdot_{\mathbb{F}_p}(W., J^\cdot) \\ {\scriptstyle (\kappa^\cdot)^{[p]}}\uparrow & & \uparrow{\scriptstyle \varepsilon_\cdot^* \circ \kappa_*^\cdot} \\ \mathrm{Tot}^\cdot(B^\cdot)^{[p]} & \xrightarrow{m_{B,p}^\cdot} & B^\cdot \end{array}$$

In the above $m_{A,p}^\cdot$ and $m_{B,p}^\cdot$ denote the $p$ iterated products on $A^\cdot$ and $B^\cdot$ respectively. Because $f^\cdot$ is a differential graded $\mathbb{F}_p$ algebra homomorphism, we have the commutative square:

$$\begin{array}{ccc}
\mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]}) & \xrightarrow{m^{\cdot}_{A,p}} & A^{\cdot} \\
\Big\downarrow{\scriptstyle (f^{\cdot})^{[p]}} & & \Big\downarrow{\scriptstyle f^{\cdot}} \\
\mathrm{Tot}^{\cdot}((B^{\cdot})^{[p]}) & \xrightarrow{m^{\cdot}_{B,p}} & B^{\cdot}
\end{array}$$

This shows that the top rows in the two diagrams below are both chain maps in $\mathrm{Sh}_{\mathbb{F}_p\pi}(X)$ over $f^{\cdot} \circ m^{\cdot}_{A,p} = m^{\cdot}_{B,p} \circ (f^{\cdot})^{[p]}$:

$$\begin{array}{ccccc}
\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}) & \xrightarrow{\beta^{\cdot}_A} & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, I^{\cdot}) & \xrightarrow{\tilde{f}^{\cdot}_*} & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, J^{\cdot}) \\
{\scriptstyle (\iota^{\cdot})^{[p]}}\Big\uparrow & & {\scriptstyle \varepsilon^*_{\cdot} \circ \iota^{\cdot}_*}\Big\uparrow & & {\scriptstyle \varepsilon^*_{\cdot} \circ \kappa^{\cdot}_*}\Big\uparrow \\
\mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]}) & \xrightarrow{m^{\cdot}_{A,p}} & A^{\cdot} & \xrightarrow{f^{\cdot}} & B^{\cdot}
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}) & \xrightarrow{(\tilde{f}^{\cdot})^{[p]}} & \mathrm{Tot}^{\cdot}((J^{\cdot})^{[p]}) & \xrightarrow{\beta^{\cdot}_B} & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, J^{\cdot}) \\
{\scriptstyle (\iota^{\cdot})^{[p]}}\Big\uparrow & & {\scriptstyle (\kappa^{\cdot})^{[p]}}\Big\uparrow & & {\scriptstyle \varepsilon^*_{\cdot} \circ \kappa^{\cdot}_*}\Big\uparrow \\
\mathrm{Tot}^{\cdot}((A^{\cdot})^{[p]}) & \xrightarrow{(f^{\cdot})^{[p]}} & \mathrm{Tot}^{\cdot}((B^{\cdot})^{[p]}) & \xrightarrow{m^{\cdot}_{B,p}} & B^{\cdot}
\end{array}$$

Note that every square in the above commutes by construction. We have the bottom rows are equal and all maps are $\mathbb{F}_p\pi$ chain maps. We also have $\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, J^{\cdot})$ is bounded below and is $\mathrm{Sh}_{\mathbb{F}_p\pi}(X)$-injective in each degree. And finally, because $(\iota^{\cdot})^{[p]}$ is an injective quasi-isomorphism, we can invoke Lemma 3.2.14 to obtain that the two top rows, $\tilde{f}^{\cdot}_* \circ \beta^{\cdot}_A$ and $\beta^{\cdot}_B \circ (\tilde{f}^{\cdot})^{[p]}$, are $\mathbb{F}_p\pi$ homotopic. That is, the diagram below commutes up to $\mathbb{F}_p\pi$ homotopy:

$$\begin{array}{ccc}
\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}) & \xrightarrow{\beta^{\cdot}_A} & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, I^{\cdot}) \\
\Big\downarrow{\scriptstyle (\tilde{f}^{\cdot})^{[p]}} & & \Big\downarrow{\scriptstyle \tilde{f}^{\cdot}_*} \\
\mathrm{Tot}^{\cdot}((J^{\cdot})^{[p]}) & \xrightarrow{\beta^{\cdot}_B} & \mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, J^{\cdot})
\end{array}$$

Because $T$ is an additive functor, we have the square on the right in the diagram below commutes up to $\mathbb{F}_p\pi$ homotopy:

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}) \arrow[r, "\gamma_p"] \arrow[d, "T(\tilde{f}^{\cdot})^{[p]}"] & T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]})) \arrow[r, "T(\beta^{\cdot}_A)"] \arrow[d, "T((\tilde{f}^{\cdot})^{[p]})"] & T(\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W.,I^{\cdot})) = \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W.,T(I^{\cdot})) \arrow[d, "T(\tilde{f}^{\cdot})_*"] \\
\mathrm{Tot}^{\cdot}(T(J^{\cdot})^{[p]}) \arrow[r, "\gamma_p"] & T(\mathrm{Tot}^{\cdot}((J^{\cdot})^{[p]})) \arrow[r, "T(\beta^{\cdot}_B)"] & T(\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W.,J^{\cdot})) = \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W.,T(J^{\cdot}))
\end{tikzcd}$$

while the square on the left commutes by the naturality of $\gamma_p$. Thus the perimeter commutes up to $\mathbb{F}_p\pi$ homotopy. We have $\hat{\theta}^{\cdot}_A = T(\beta^{\cdot}_A)\circ\gamma_p$ and $\hat{\theta}^{\cdot}_B = T(\beta^{\cdot}_B)\circ\gamma_p$. By composing the horizontal arrows along the top and bottom rows, we get the diagram below commutes up to $\mathbb{F}_p\pi$ homotopy:

$$\begin{tikzcd}
\mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}) \arrow[r, "\hat{\theta}^{\cdot}_A"] \arrow[d, "T(\tilde{f}^{\cdot})^{[p]}"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W.,T(I^{\cdot})) \arrow[d, "T(\tilde{f}^{\cdot})_*"] \\
\mathrm{Tot}^{\cdot}(T(J^{\cdot})^{[p]}) \arrow[r, "\hat{\theta}^{\cdot}_B"] & \mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W.,T(J^{\cdot}))
\end{tikzcd}$$

That is, $T(\tilde{f}^{\cdot})_*\circ\hat{\theta}^{\cdot}_A$ and $\hat{\theta}^{\cdot}_B\circ T(\tilde{f}^{\cdot})^{[p]}$ are $\mathbb{F}_p\pi$ homotopic. By Lemma 3.1.6, $\Phi^0(T(\tilde{f}^{\cdot})_*\circ\hat{\theta}^{\cdot}_A)$ and $\Phi^0(\hat{\theta}^{\cdot}_B\circ T(\tilde{f}^{\cdot})^{[p]})$ are $\mathbb{F}_p\pi$ homotopic, where $\Phi^{\cdot}$ is the adjoint isomorphism of Lemma 3.1.5. By Lemma 3.1.7, we have:

$$\Phi^0(T(\tilde{f}^{\cdot})_*\circ\hat{\theta}^{\cdot}_A) = T(\tilde{f}^{\cdot})\circ\Phi^0(\hat{\theta}^{\cdot}_A)$$

By Lemma 3.1.9, we have:

$$\Phi^0(\hat{\theta}^{\cdot}_B\circ T(\tilde{f}^{\cdot})^{[p]}) = \Phi^0(\hat{\theta}^{\cdot}_B)\circ(T(\tilde{f}^{\cdot})^{[p]}\otimes 1)$$

Thus, we have shown the square on the right in the diagram below commutes up to $\mathbb{F}_p\pi$ homotopy:

$$\begin{tikzcd}
W.\otimes T(I^{\cdot})^{[p]} \arrow[r, "U^{\cdot}"] \arrow[d, "1\otimes T(\tilde{f}^{\cdot})^{[p]}"] & T(I^{\cdot})^{[p]}\otimes W. \arrow[r, "\Phi^0(\hat{\theta}^{\cdot}_A)"] \arrow[d, "T(\tilde{f}^{\cdot})^{[p]}\otimes 1"] & T(I^{\cdot}) \arrow[d, "T(\tilde{f}^{\cdot})"] \\
W.\otimes T(J^{\cdot})^{[p]} \arrow[r, "U^{\cdot}"] & T(J^{\cdot})^{[p]}\otimes W. \arrow[r, "\Phi^0(\hat{\theta}^{\cdot}_B)"] & T(J^{\cdot})
\end{tikzcd}$$

while the square on the left commutes by the naturality of $U^{\cdot}$. Thus the perimeter commutes up to $\mathbb{F}_p\pi$ homotopy. Because $\theta_A^{\cdot} = \Phi^0(\hat{\theta}_A^{\cdot}) \circ U^{\cdot}$ and $\theta_B^{\cdot} = \Phi^0(\hat{\theta}_B^{\cdot}) \circ U^{\cdot}$, we have shown the diagram below commutes up to $\mathbb{F}_p$ homotopy:

$$\begin{tikzcd} W. \otimes T(I^{\cdot})^{[p]} \arrow[r, "\theta_A^{\cdot}"] \arrow[d, "1\otimes T(\tilde{f}^{\cdot})^{[p]}"] & T(I^{\cdot}) \arrow[d, "T(\tilde{f}^{\cdot})"] \\ W. \otimes T(J^{\cdot})^{[p]} \arrow[r, "\theta_B^{\cdot}"] & T(J^{\cdot}) \end{tikzcd}$$

This shows $T(\tilde{f}^{\cdot}) : (T(I^{\cdot}), \theta_A^{\cdot}) \to (T(J^{\cdot}), \theta_B^{\cdot})$ is a morphism in May's category $\mathcal{C}(p)$. Now by Lemma 2.0.3, we have that $H^{\cdot}(T(\tilde{f}^{\cdot}))$ commutes with $D_i$, and as a consequence, with the Steenrod operations on $H^{\cdot}(T(I^{\cdot}))$ and $H^{\cdot}(T(J^{\cdot}))$. So by their respective definitions, we have $\mathbf{H}^{\cdot}(X, f^{\cdot})$ commutes with the Steenrod operations on $\mathbf{H}^{\cdot}(X, A^{\cdot})$ and $\mathbf{H}^{\cdot}(X, B^{\cdot})$.

□

We now develop some lemmas that will allow us to apply Lemma 5.4.1 to algebraic De Rham cohomology and Hodge cohomology later.

**Lemma 5.4.2.** *Let $X$ and $Y$ will denote smooth projective varieties over a field of characteristic $p$, and $f : X \to Y$ will be a morphism of schemes over $k$. Let $A$ be a sheaf of $O_X$ modules on $X$ and $B$ a sheaf of $O_Y$ modules on $Y$.*

## 5.5 Further Questions

In the case that $A^{\cdot}$ is concentrated in degree 0, there are Lemmas 4.4.1 and 4.4.2, which are proven by Epstein in [2]. It is natural to ask if these lemmas also hold when $A^{\cdot}$ is not concentrated in degree 0.

**Question 5.5.1.** *For the Steenrod operations $P^{\cdot}$ and $\beta P^{\cdot}$ constructed in this section, do we have $P^i = 0$ and $\beta P^i = 0$ for all $i < 0$?*

**Question 5.5.2.** *When $A^{\cdot}$ is concentrated in degree zero, we have for the Steenrod operations $P^{\cdot}$, $\beta P^{\cdot}$ constructed in this section, that $P^0 : H^n(K^{\cdot}) \to H^n(K^{\cdot})$ is induced by the Frobenius map, $fr : A \to A$, on the sheaf of $\mathbb{F}_p$-algebras, $A$, by Lemma 4.4.2. Is there a similar result that holds when $A^{\cdot}$ is not concentrated in degree 0, and instead just bounded below?*

## 5.6 Applications

In this section I will apply Theorem 5.3.5 to a few different sheaves of bounded below complexes of differential graded commutative $\mathbb{F}_p$ algebras, $A^{\cdot}$.

### 5.6.1 Algebraic De Rham Cohomology

Let $X$ be a smooth projective variety over a field $k$ of characteristic $p$. Let $A^{\cdot} = \Omega^{\cdot}_{X/k}$ be the De Rham complex of $X$ over $k$. We have $\Omega^{\cdot}_{X/k}$ is concentrated in non-negative degree, and the wedge product on $\Omega^{\cdot}_{X/k}$ makes it a sheaf of differential graded commutative $\mathbb{F}_p$ algebras. The sheaf hypercohomology of $X$ with coefficients in $\Omega^{\cdot}_{X/k}$ computes the algebraic De Rham cohomology groups of $X$:

$$H^n_{\mathrm{DR}}(X/k) = \mathbf{H}^n(X, \Omega^{\cdot}_{X/k})$$

Thus under these conditions, the Steenrod operations from Theorem 5.3.5 are defined on $H^{\cdot}_{\mathrm{DR}}(X/k)$.

### 5.6.2 Hodge Cohomology

Let $X$, $k$ be as in the previous section. Let the Hodge complex $\tilde{\Omega}^{\cdot}_{X/k}$ be the De Rham complex $\Omega^{\cdot}_{X/k}$ but with zero differential. Like before, we still have $\tilde{\Omega}^{\cdot}_{X/k}$ equipped with the wedge product is a bounded below complex of sheaves of differential graded commutative $\mathbb{F}_p$ algebras on $X$. Under these conditions one may compute the Hodge cohomology of $X$ over $k$ as the hypercohomology groups, $\mathbf{H}^n(X, \tilde{\Omega}^{\cdot}_{X/k})$. Thus we can apply Theorem 5.3.5 to obtain Steenrod operations on the Hodge cohomology groups of $X$ as well.

# Chapter 6

# Filtrations

In this chapter I will develop some lemmas for filtered complexes. I expect most results to hold in a general abelian category, but I only prove them in the category of sheaves of abelian groups for simplicity. Other results are specific to the category of sheaves of $k$ vector spaces, where $k$ is any field. Sections 6.2 and 6.3 do not contain new results, and may be skipped. They are only included to make the arguments used in sections 6.4 and 6.5 easier to follow.

## 6.1 Definitions

**Definition 6.1.1.** *Let $\mathcal{A}$ be an abelian category. A filtered object $F^{\cdot}A$ of $\mathcal{A}$ is an object $A$ of $\mathcal{A}$ and a collection of subobjects $F^mA$ of $A$ for every $m \in \mathbb{Z}$, such that $F^{m+1}A$ is a subobject of $F^mA$ for all $m \in \mathbb{Z}$. If $F^{\cdot}A$ and $F^{\cdot}B$ are filtered objects of $\mathcal{A}$, a filtered morphism $F^{\cdot}f : F^{\cdot}A \to F^{\cdot}B$ is a morphism $f : A \to B$ in $\mathcal{A}$ such that $f$ restricts to a morphism $F^mf : F^mA \to F^mB$ for all $m \in \mathbb{Z}$. Let $Fil(\mathcal{A})$ denote the category of filtered objects of $\mathcal{A}$ with filtered morphisms. Let $Fil^f(\mathcal{A})$ denote the full subcategory of $Fil(\mathcal{A})$ whose objects are finitely filtered, as in Definition 6.1.2 (3). Define $gr^mA = F^mA/F^{m+1}A$. For a filtered morphism $F^{\cdot}f : F^{\cdot}A \to F^{\cdot}B$, one has the well defined morphisms $gr^mf : gr^mA \to gr^mB$ in $\mathcal{A}$.*

**Definition 6.1.2.** *Let $\mathcal{A}$ be an abelian category and let $F^{\cdot}A$ be a filtered object of $\mathcal{A}$.*

1. *The filtration on $F^{\cdot}A$ is said to* terminate *if there is a $m \in \mathbb{Z}$ such that $F^mA = 0$.*

2. *The filtration on $F^{\cdot}A$ is said to* begin *if there is a $m \in \mathbb{Z}$ such that $F^m A = A$.*

3. *The filtration is* finite *if it begins and terminates, ie, there is a $m_1 \in \mathbb{Z}$ and a $m_2 \in \mathbb{Z}$ such that $F^{m_1}A = A$ and $F^{m_2}A = 0$.*

4. *The filtration is called* exhaustive *if $A = \cup_{m\in\mathbb{Z}} F^m A$.*

5. *The filtration is called* separated *if $\cap_{m\in\mathbb{Z}} F^m A = 0$.*

6. *We say $F^{\cdot}A$ is* inductively filtered *if the filtration terminates and is exhaustive.*

**Definition 6.1.3.** *Let $\mathcal{A}$ be an abelian category. A filtered complex $F^{\cdot}A^{\cdot}$ in $\mathcal{A}$ is an object of $Fil(Comp(\mathcal{A}))$. That is, $A^{\cdot}$ is a complex in $\mathcal{A}$ and each $F^m A^{\cdot}$ is a subcomplex of $A^{\cdot}$, with $F^{m+1}A^{\cdot}$ being a subcomplex of $F^m A^{\cdot}$ for all $m \in \mathbb{Z}$. A filtered chain map $F^{\cdot} f^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ is a morphism in $Fil(Comp(\mathcal{A}))$. That is, $f^{\cdot} : A^{\cdot} \to B^{\cdot}$ is a chain map and $f^{\cdot}$ restricts to a chain map $F^m f^{\cdot} : F^m A^{\cdot} \to F^m B^{\cdot}$ for all $m \in \mathbb{Z}$. For $m \in \mathbb{Z}$, one defines the chain complex $gr^m A^{\cdot} = F^m A^{\cdot}/F^{m+1}A^{\cdot}$ in $\mathcal{A}$. One has the well defined chain map $gr^m f^{\cdot} : gr^m A^{\cdot} \to gr^m B^{\cdot}$.*

**Definition 6.1.4.** *Let $\mathcal{A}$ be an abelian category and suppose $F^{\cdot}A^{\cdot}$ is a filtered complex in $\mathcal{A}$.*

1. *We say the filtration on $F^{\cdot}A^{\cdot}$* terminates in each degree *if for all $n \in \mathbb{Z}$, the filtration on $F^{\cdot}A^n$ terminates. That is, for all $n \in \mathbb{Z}$, there is a $m_n \in \mathbb{Z}$ such that $F^{m_n}A^n = 0$.*

2. *The filtration on $F^{\cdot}A^{\cdot}$ is said to* begin in each degree *if the filtration on $F^{\cdot}A^n$ begins for all $n \in \mathbb{Z}$. That is, for all $n \in Z$, there is a $m_n \in \mathbb{Z}$ such that $F^{m_n}A^n = A^n$.*

3. *The filtration on $F^{\cdot}A^{\cdot}$ is called* finite in each degree *if $F^{\cdot}A^n$ is finitely filtered for each $n \in \mathbb{Z}$. That is, for every $n \in \mathbb{Z}$, there is a $m_n \in \mathbb{Z}$ and $m'_n \in \mathbb{Z}$ such that $F^{m_n}A^n = A^n$ and $F^{m'_n}A^n = 0$.*

4. *We say the filtration on $F^{\cdot}A^{\cdot}$ is* exhaustive *if the notion of exhaustive in Definition 6.1.2 (4) applies to $F^{\cdot}A^{\cdot}$ where one regards $A^{\cdot}$ as a filtered object in the category $Comp(\mathcal{A})$. This means for all $n \in \mathbb{Z}$, $A^n = \cup_{m\in\mathbb{Z}} F^m A^n$.*

5. *The filtration on $F^{\cdot}A^{\cdot}$ is* separated *if the notion of separated from Definition 6.1.2 (5) applies to $A^{\cdot}$ as a filtered object in the category $Comp(\mathcal{A})$. That is, for all $n \in \mathbb{Z}$, one has $\cap_{m\in\mathbb{Z}} F^m A^n = 0$.*

6. *The filtration on $F^{\cdot}A^{\cdot}$ is called* inductive in each degree *if it terminates in each degree and the filtration is exhaustive.*

7. *The filtration on $F^{\cdot}A^{\cdot}$ is said to* terminate uniformly *if the filtration on $F^{\cdot}A^{\cdot}$ terminates when one regards $A^{\cdot}$ as a filtered object of $Comp(\mathcal{A})$. That is, there is a $m \in \mathbb{Z}$ such that $F^m A^{\cdot} = 0^{\cdot}$.*

8. *The filtration on $F^{\cdot}A^{\cdot}$ is said to* begin uniformly *if the filtration on $F^{\cdot}A^{\cdot}$ begins where one sees $A^{\cdot}$ as an object of $Comp(\mathcal{A})$. That is, there is a $m \in \mathbb{Z}$, such that $F^m A^{\cdot} = A^{\cdot}$.*

9. *The filtration on $F^{\cdot}A^{\cdot}$ is said to be* uniformly finite *if it both begins uniformly and terminates uniformly. That is, there is a $m \in \mathbb{Z}$ and $m' \in \mathbb{Z}$ such that $F^m A^{\cdot} = A^{\cdot}$ and $F^{m'} A^{\cdot} = 0^{\cdot}$.*

**Lemma 6.1.5.** *Let $\mathcal{A}$ be an abelian category and suppose $F^{\cdot}f : F^{\cdot}A \to F^{\cdot}B$ is a filtered morphism of filtered objects in $\mathcal{A}$. If $F^m f$ is injective (respectively surjective) for all $m \in \mathbb{Z}$, then $gr^m f$ is injective (respectively surjective) for all $m \in \mathbb{Z}$.*

*Proof.* Let $m \in \mathbb{Z}$. We have the diagram:

$$\begin{tikzcd}
0 \arrow[r] & F^{m+1}A \arrow[r, hook] \arrow[d, "F^{m+1}f"] & F^m A \arrow[r, two heads] \arrow[d, "F^m f"] & \mathrm{gr}^m A \arrow[r] \arrow[d, "\mathrm{gr}^m f"] & 0 \\
0 \arrow[r] & F^{m+1}B \arrow[r, hook] & F^m B \arrow[r, two heads] & \mathrm{gr}^m B \arrow[r] & 0
\end{tikzcd}$$

If we assume $F^{m+1}f$ and $F^m f$ are injective, it follows from the Five Lemma that $\mathrm{gr}^m f$ is injective. Similarly, if $F^{m+1}f$ and $F^m f$ are surjective, it also follows from the Five Lemma that $\mathrm{gr}^m f$ is surjective.

□

**Corollary 6.1.6.** *Let $\mathcal{A}$ be an abelian category, suppose $F^{\cdot}f^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ is a filtered chain map. Suppose for all $m \in \mathbb{Z}$ $F^m f^{\cdot}$ is injective (respectively surjective) in each degree. Then $gr^m f^{\cdot}$ is injective (respectively surjective) in each degree.*

*Proof.* Let $n \in \mathbb{Z}$. We have $F^m f^n$ is injective (respectively surjective) for all $m \in \mathbb{Z}$. By Lemma 6.1.5, $\mathrm{gr}^m f^n$ is injective (respectively surjective) for all $m \in \mathbb{Z}$. □

**Definition 6.1.7.** *Let $\mathcal{A}$ be an abelian category, and let $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ be filtered complexes in $\mathcal{A}$. A filtered chain map $F^{\cdot}f^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ is called a* filtered quasi-isomorphism *if for all $m \in \mathbb{Z}$, $gr^m f^{\cdot} : gr^m A^{\cdot} \to gr^m B^{\cdot}$ is a quasi-isomorphism.*

**Definition 6.1.8.** *Let $\mathcal{A}$ be an abelian category, and let $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ be filtered complexes in $\mathcal{A}$. A filtered chain map $F^{\cdot}f^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ is called a* strong filtered quasi-isomorphism *if for all $m \in \mathbb{Z}$, $F^m f^{\cdot} : F^m A^{\cdot} \to F^m B^{\cdot}$ is a quasi-isomorphism.*

**Lemma 6.1.9.** *Let $\mathcal{A}$ be an abelian category, let $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ be filtered complexes in $\mathcal{A}$, and suppose $F^{\cdot}f^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ is a strong filtered quasi-isomorphism. Then $F^{\cdot}f^{\cdot}$ is also a filtered quasi-isomorphism in the sense of Definition 6.1.7.*

*Proof.* Let $m \in \mathbb{Z}$. We have the diagram of complexes in $\mathcal{A}$ with exact rows below:

$$\begin{tikzcd}
0 \arrow[r] & F^{m+1}A^{\cdot} \arrow[r, hook] \arrow[d, "F^{m+1}f^{\cdot}"] & F^m A^{\cdot} \arrow[r, two heads] \arrow[d, "F^m f^{\cdot}"] & \mathrm{gr}^m A^{\cdot} \arrow[r] \arrow[d, "\mathrm{gr}^m f^{\cdot}"] & 0 \\
0 \arrow[r] & F^{m+1}B^{\cdot} \arrow[r, hook] & F^m B^{\cdot} \arrow[r, two heads] & \mathrm{gr}^m B^{\cdot} \arrow[r] & 0
\end{tikzcd}$$

Since $F^{m+1}f^{\cdot}$ and $F^m f^{\cdot}$ are quasi-isomorphisms, it follows that $\mathrm{gr}^m f^{\cdot}$ is a quasi-isomorphism, as one can take the long exact sequence of cohomology groups and apply the Five Lemma. Since $\mathrm{gr}^m f^{\cdot}$ is a quasi-isomorphism for all $m \in \mathbb{Z}$, we have $F^{\cdot}f^{\cdot}$ is a filtered quasi-isomorphism in the sense of Definition 6.1.7. □

**Lemma 6.1.10.** *Let $\mathcal{A}$ be an abelian category and suppose $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ are filtered complexes in $\mathcal{A}$ whose filtrations begin in each degree, as in Definition 6.1.4 (2). Let $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ be a strong quasi-isomorphism. Then $\varepsilon^{\cdot} : A^{\cdot} \to B^{\cdot}$ is a quasi-isomorphism.*

*Proof.* Let $n \in \mathbb{Z}$. Because the filtrations of $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ both begin in each degree, we can find a $m_n \in \mathbb{Z}$ such that $F^{m_n}A^n = A^n$, $F^{m_n}A^{n-1} = A^{n-1}$, $F^{m_n}B^n = B^n$, and $F^{m_n}B^{n-1} = B^{n-1}$. Since $F^{\cdot}\varepsilon^{\cdot}$ is a strong quasi-isomorphism, $F^{m_n}\varepsilon^{\cdot}$ is a quasi-isomorphism, and we have $H^n(F^{m_n}\varepsilon^{\cdot})$ is an isomorphism. But we have $H^n(F^{m_n}\varepsilon^{\cdot}) = H^n(\varepsilon^{\cdot})$ because $H^n(A^{\cdot}) = \ker(d_A^n)/\mathrm{im}(d_A^{n-1}) = \ker(F^{m_n}d_A^n)/\mathrm{im}(F^{m_n}d_A^{n-1}) = H^n(F^{m_n}A^{\cdot})$, and similarly $H^n(B^{\cdot}) = H^n(F^{m_n}B^{\cdot})$. Since $H^n(\varepsilon^{\cdot})$ is an isomorphism for all $n \in \mathbb{Z}$, $\varepsilon^{\cdot}$ is a quasi-isomorphism.

□

**Lemma 6.1.11.** *Let $\mathcal{A}$ be an abelian category and suppose $F^\cdot A^\cdot$ and $F^\cdot B^\cdot$ are two filtered complexes in $\mathcal{A}$, with $F^\cdot\varepsilon^\cdot : F^\cdot A^\cdot \to F^\cdot B^\cdot$ a filtered quasi-isomorphism. Then for every $m \in \mathbb{Z}$ and $i \in \mathbb{N}$, we have $F^\cdot\varepsilon^\cdot$ induces a quasi-isomorphism:*

$$(F^m/F^{m+i})\varepsilon^\cdot : F^mA^\cdot/F^{m+i}A^\cdot \to F^mB^\cdot/F^{m+i}B^\cdot$$

*Proof.* Proceed by induction on $i$. In the case $i = 1$, we have for all $m \in \mathbb{Z}$, $(F^m/F^{m+1})\varepsilon^\cdot = \mathrm{gr}^m\varepsilon^\cdot$, which is a quasi-isomorphism because $F^\cdot\varepsilon^\cdot$ is a filtered quasi-isomorphism. Now suppose there is an $i \in \mathbb{N}$ such that, for all $m \in \mathbb{Z}$, the map $(F^m/F^{m+j})\varepsilon^\cdot : F^mA^\cdot/F^{m+j}A^\cdot \to F^mB^\cdot/F^{m+j}B^\cdot$ is a quasi-isomorphism, for all $1 \leq j < i$. Let $m \in \mathbb{Z}$. I must show $(F^m/F^{m+i})\varepsilon^\cdot : F^mA^\cdot/F^{m+i}A^\cdot \to F^mB^\cdot/F^{m+i}B^\cdot$ is a quasi-isomorphism. We have the diagram below:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^{m+1}A^\cdot/F^{m+i}A^\cdot & \longrightarrow & F^mA^\cdot/F^{m+i}A^\cdot & \longrightarrow & \mathrm{gr}^mA^\cdot & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (F^{m+1}/F^{m+i})\varepsilon^\cdot} & & \Big\downarrow{\scriptstyle (F^m/F^{m+i})\varepsilon^\cdot} & & \Big\downarrow{\scriptstyle \mathrm{gr}^m\varepsilon^\cdot} & & \\
0 & \longrightarrow & F^{m+1}B^\cdot/F^{m+i}B^\cdot & \longrightarrow & F^mB^\cdot/F^{m+i}B^\cdot & \longrightarrow & \mathrm{gr}^mB^\cdot & \longrightarrow & 0
\end{array}$$

We have $\mathrm{gr}^m\varepsilon^\cdot$ is a quasi-isomorphism by hypothesis and $(F^{m+1}/F^{m+i})\varepsilon^\cdot$ is a quasi-isomorphism by induction. Hence by the Five Lemma we get that $(F^m/F^{m+i})\varepsilon^\cdot$ is a quasi-isomorphism as well.

□

**Lemma 6.1.12.** *Let $\mathcal{A}$ be an abelian category and suppose $F^\cdot A^\cdot$ and $F^\cdot B^\cdot$ are two filtered complexes in $\mathcal{A}$ whose filtrations terminate in each degree. Let $F^\cdot\varepsilon^\cdot : F^\cdot A^\cdot \to F^\cdot B^\cdot$ be a filtered quasi-isomorphism. Then $F^\cdot\varepsilon^\cdot$ is a strong filtered quasi-isomorphism.*

*Proof.* Let $m, n \in \mathbb{Z}$. I must show $H^n(F^m\varepsilon^\cdot) : H^n(F^mA^\cdot) \to H^n(F^mB^\cdot)$ is an isomorphism. Because $F^\cdot A^\cdot$ and $F^\cdot B^\cdot$ have filtrations that terminate in each degree, we can find a $m_n \in \mathbb{Z}$ such that:

$$F^{m_n}A^n = F^{m_n}A^{n-1} = F^{m_n}B^n = F^{m_n}B^{n-1} = 0$$

If $m \geq m_n$ then of course we are done. Otherwise we can set $i = m_n - m \in \mathbb{N}$ and we have by Lemma 6.1.11 that:

$$(F^m/F^{m_n}\varepsilon^\cdot) : F^mA^\cdot/F^{m_n}A^\cdot \to F^mB^\cdot/F^{m_n}B^\cdot$$

is a quasi-isomorphism. In degrees $n$ and $n-1$ we have $F^mA^n/F^{m_n}A^n = F^mA^n$, $F^mA^{n-1}/F^{m_n}A^{n-1} = F^mA^{n-1}$, and similarly for $F^\cdot B^\cdot$. This implies that the isomorphism:

$$H^n((F^m/F^{m_n})\varepsilon^\cdot) : H^n(F^mA^\cdot/F^{m_n}A^\cdot) \to H^n(F^mB^\cdot/F^{m_n}B^\cdot)$$

is equal to the map:

$$H^n(F^m\varepsilon^\cdot) : H^n(F^mA^\cdot) \to H^n(F^mB^\cdot)$$

Thus, $H^n(F^m\varepsilon^\cdot)$ is an isomorphism. Since $m,n \in \mathbb{Z}$ were arbitrary, we have $F^\cdot\varepsilon^\cdot$ is a strong filtered quasi-isomorphism.

□

**Corollary 6.1.13.** *Let $\mathcal{A}$ be an abelian category and let $F^\cdot A^\cdot$ and $F^\cdot B^\cdot$ be filtered complexes in $\mathcal{A}$ that are finitely filtered in each degree. Let $F^\cdot\varepsilon^\cdot : F^\cdot A^\cdot \to F^\cdot B^\cdot$ be a filtered quasi-isomorphism. Then $F^\cdot\varepsilon^\cdot$ is a strong quasi-isomorphism and $\varepsilon^\cdot : A^\cdot \to B^\cdot$ is a quasi-isomorphism.*

*Proof.* By Lemma 6.1.12, $F^\cdot\varepsilon^\cdot$ is a strong quasi-isomorphism. Then by Lemma 6.1.10, $\varepsilon^\cdot$ is a quasi-isomorphism. □

**Definition 6.1.14.** *Let $\mathcal{A}$ be an abelian category and let $F^\cdot A^\cdot$ be a filtered complex in $\mathcal{A}$. Then $F^\cdot A^\cdot$ is called* filtered acyclic *if $gr^m A^\cdot$ is an acyclic complex for all $m \in \mathbb{Z}$.*

**Definition 6.1.15.** *Let $\mathcal{A}$ be an abelian category and let $F^\cdot A^\cdot$ be a filtered complex in $\mathcal{A}$. Then $F^\cdot A^\cdot$ is called* strong filtered acyclic *if $F^mA^\cdot$ is acyclic for all $m \in \mathbb{Z}$.*

**Lemma 6.1.16.** *Let $\mathcal{A}$ be an abelian category and suppose $F^\cdot A^\cdot$ is a strong filtered acyclic complex. Then $F^\cdot A^\cdot$ is filtered acyclic.*

*Proof.* Let $m \in \mathbb{Z}$ and consider the exact sequence of complexes in $\mathcal{A}$:

$$0 \to F^{m+1}A^\cdot \to F^mA^\cdot \to \mathrm{gr}^mA^\cdot \to 0$$

Because $F^{\cdot}A^{\cdot}$ is strong filtered acyclic, we have $H^n(F^mA^{\cdot}) = 0$ for all $n,m \in \mathbb{Z}$. Thus, in the long exact sequence of cohomology groups, $H^n(\mathrm{gr}^mA^{\cdot})$ is surrounded by terms that are zero. This forces $H^n(\mathrm{gr}^mA^{\cdot}) = 0$, and we have $F^{\cdot}A^{\cdot}$ is filtered acyclic.

$\square$

**Definition 6.1.17.** *Let $\mathcal{A}$ be an abelian category and let $F^{\cdot}I$ be a filtered object in $\mathcal{A}$. Then $F^{\cdot}I$ is called* filtered injective *if $gr^mI$ is an injective object of $\mathcal{A}$ for all $m \in \mathbb{Z}$.*

**Definition 6.1.18.** *Let $\mathcal{A}$ be an abelian category and let $F^{\cdot}I$ be a filtered object in $\mathcal{A}$. Then $F^{\cdot}I$ is called* strong filtered injective *if $F^mI$ is an injective object in $\mathcal{A}$ for all $m \in \mathbb{Z}$.*

Note the definitions below may conflict with [6], where there they would insist that $F^{\cdot}\varepsilon^{\cdot}$ is strict. Because strictness is not used in sections 6.4 and 6.5, I don't require it here.

**Definition 6.1.19.** *Let $\mathcal{A}$ be an abelian category and let $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ be a filtered chain map of filtered complexes in $\mathcal{A}$. Then $F^{\cdot}\varepsilon^{\cdot}$ is called a* filtered resolution *if for all $m \in \mathbb{Z}$, $gr^m\varepsilon^{\cdot} : gr^mA^{\cdot} \to gr^mB^{\cdot}$ is a resolution in the sense of Definition 3.2.15. That is, a filtered resolution is an injective filtered quasi-isomorphism. If $F^{\cdot}B^{\cdot}$ is filtered injective in each degree, has a terminating filtration in each degree, and $B^{\cdot}$ is bounded from below, then $F^{\cdot}\varepsilon^{\cdot}$ is called a* filtered injective resolution.

**Definition 6.1.20.** *Let $\mathcal{A}$ be an abelian category and let $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ be a filtered chain map of filtered complexes in $\mathcal{A}$. Then $F^{\cdot}\varepsilon^{\cdot}$ is called a* strong filtered resolution *if for all $m \in \mathbb{Z}$, $F^m\varepsilon^{\cdot} : F^mA^{\cdot} \to F^mB^{\cdot}$ is a resolution in the sense of Definition 3.2.15. That is, $F^{\cdot}\varepsilon^{\cdot}$ is an injective strong quasi-isomorphism. If $F^{\cdot}B^{\cdot}$ is strong filtered injective in each degree, has a terminating filtration in each degree, and $B^{\cdot}$ is bounded from below, then $F^{\cdot}\varepsilon^{\cdot}$ is called a* strong filtered injective resolution.

**Definition 6.1.21.** *Let $\mathcal{A}$ be an abelian category and suppose $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ are filtered chain complexes in $\mathcal{A}$. Let $F^{\cdot}g_1, F^{\cdot}g_2 : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ be two filtered chain maps in $\mathcal{A}$. A* filtered homotopy *between $F^{\cdot}g_1^{\cdot}$ and $F^{\cdot}g_2^{\cdot}$ is a family of filtered morphisms, $F^{\cdot}h^n : F^{\cdot}A^n \to F^{\cdot}B^{n-1}$ for each $n \in \mathbb{Z}$, such that $F^{\cdot}g_1^n - F^{\cdot}g_2^n = F^{\cdot}d_B^{n-1} \circ F^{\cdot}h^n + F^{\cdot}h^{n+1} \circ F^{\cdot}d_A^n$. If such a $F^{\cdot}h^{\cdot}$ exists, $F^{\cdot}g_1^{\cdot}$ and $F^{\cdot}g_2^{\cdot}$ are called* filtered homotopic.

**Lemma 6.1.22.** *Let $\mathcal{A}$ be an abelian category, and suppose $F^{\cdot}A^{\cdot}$, $F^{\cdot}B^{\cdot}$, and $F^{\cdot}C^{\cdot}$ are filtered complexes in $\mathcal{A}$. Let $F^{\cdot}g_1^{\cdot}, F^{\cdot}g_1^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ be two filtered chain maps that are filtered homotopic by a filtered homotopy $F^{\cdot}h^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B[-1]^{\cdot}$. Let $F^{\cdot}f^{\cdot} : F^{\cdot}B^{\cdot} \to F^{\cdot}C^{\cdot}$ be a filtered chain map. Then $F^{\cdot}f^{\cdot} \circ F^{\cdot}g_1^{\cdot}$ and $F^{\cdot}f^{\cdot} \circ F^{\cdot}g_2^{\cdot}$ are filtered homotopic by filtered homotopy $F^{\cdot}f^{\cdot} \circ F^{\cdot}h^{\cdot}$.*

*Proof.* By Lemma 3.1.15, $f^{\cdot} \circ g_1^{\cdot}$ and $f^{\cdot} \circ g_2^{\cdot}$ are homotopic by homotopy $f^{\cdot} \circ h^{\cdot}$. All that remains is to show that $F^{\cdot}f^{n-1} \circ F^{\cdot}h^{n}$ is a filtered morphism for all $n \in \mathbb{Z}$, and this is true, since it is a composition of filtered morphisms. $\square$

**Lemma 6.1.23.** *Let $\mathcal{A}$ be an abelian category, and suppose $F^{\cdot}A^{\cdot}$, $F^{\cdot}B^{\cdot}$, and $F^{\cdot}C^{\cdot}$ are filtered complexes in $\mathcal{A}$. Let $F^{\cdot}g_1^{\cdot}, F^{\cdot}g_1^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ be two filtered chain maps that are filtered homotopic by a filtered homotopy $F^{\cdot}h^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B[-1]^{\cdot}$. Let $F^{\cdot}f^{\cdot} : C^{\cdot} \to F^{\cdot}A^{\cdot}$ be a filtered chain map. Then $F^{\cdot}g_1^{\cdot} \circ F^{\cdot}f^{\cdot}$ and $F^{\cdot}g_2^{\cdot} \circ F^{\cdot}f^{\cdot}$ are filtered homotopic by filtered homotopy $F^{\cdot}h^{\cdot} \circ F^{\cdot}f^{\cdot}$.*

*Proof.* By Lemma 3.1.16, $g_1^{\cdot} \circ f^{\cdot}$ and $g_2^{\cdot} \circ f^{\cdot}$ are homotopic by homotopy $h^{\cdot} \circ f^{\cdot}$. All that remains is to show that $F^{\cdot}h^{n} \circ F^{\cdot}f^{n}$ is a filtered morphism for all $n \in \mathbb{Z}$, and this is true, since it is a composition of filtered morphisms. $\square$

**Definition 6.1.24.** *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $T : \mathcal{A} \to \mathcal{B}$ be a covariant left exact functor. Let $F^{\cdot}A$ be a filtered object of $\mathcal{A}$. Then we can induce a filtration on $T(A)$, where we define for all $m \in \mathbb{Z}$:*

$$F^{m}T(A) = T(F^{m}A)$$

*Note we need $T$ to be left exact in order to maintain the inclusions:*

$$T(F^{m}A) \hookrightarrow T(A)$$
$$T(F^{m+1}A) \hookrightarrow T(F^{m}A)$$

**Definition 6.1.25.** *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Let $F^{\cdot}A^{\cdot}$ be a filtered complex in $\mathcal{A}$. Then we may induce a filtration on $T(A^{\cdot})$ as in Definition 6.1.24 where we view $F^{\cdot}A^{\cdot}$ as a filtered object of the category $Comp(\mathcal{A})$ and $T$ as the induced functor $Comp(\mathcal{A}) \to Comp(\mathcal{B})$. With this we have:*

$$F^m T(A^n) = T(F^m A^n)$$

*for all $m, n \in \mathbb{Z}$.*

**Lemma 6.1.26.** *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $T : \mathcal{A} \to \mathcal{B}$ be an additive left exact functor. Let $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ be complexes in $\mathcal{A}$ and suppose $F^{\cdot}g_1^{\cdot}, F^{\cdot}g_2^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ are filtered chain maps that are filtered homotopic by filtered homotopy $F^{\cdot}h^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B[-1]^{\cdot}$. Then $F^{\cdot}T(g_1^{\cdot})$ and $F^{\cdot}T(g_2^{\cdot})$ are filtered homotopic by filtered homotopy $F^{\cdot}T(h^{\cdot})$.*

*Proof.* Because $T$ is additive, $T$ preserves homotopies. And because $F^{\cdot}h^n$ is a filtered morphism for all $n \in \mathbb{Z}$, so is $F^{\cdot}T(h^n)$. Thus $F^{\cdot}T(g_1^{\cdot})$ and $F^{\cdot}T(g_2^{\cdot})$ are filtered homotopic by filtered homotopy $F^{\cdot}T(h^{\cdot})$. □

The definition below is adapted from [6], near the beginning of Section 12.21 [1].

**Definition 6.1.27.** *Let $\mathcal{A}$ be an abelian category with sums. Let $F^{\cdot}K^{\cdot}$ be a filtered complex in $\mathcal{A}$. The objects associated with the spectral sequence for $F^{\cdot}K^{\cdot}$ are defined as follows. Let $r \geq 0$, $a, b \in \mathbb{Z}$:*

$$Z_r^{a,b} = \frac{F^a K^{a+b} \cap d_K^{-1}(F^{a+r}K^{a+b+1}) + F^{a+1}K^{a+b}}{F^{a+1}K^{a+b}}$$

$$B_r^{a,b} = \frac{F^a K^{a+b} \cap d_K(F^{a-r+1}K^{a+b-1}) + F^{a+1}K^{a+b}}{F^{a+1}K^{a+b}}$$

$$E_r^{a,b} = Z_r^{a,b} / B_r^{a,b}$$

$$d_r^{a,b} : E_r^{a,b} \to E_r^{a+r,b-r+1} \qquad z + F^{a+1}K^{a+b} \mapsto d_K(z) + F^{a+r+1}K^{a+b+1}$$

*The make things easier to work with, I define:*

$$\tilde{Z}_r^{a,b} = F^a K^{a+b} \cap d_K^{-1}(F^{a+r}K^{a+b+1}) + F^{a+1}K^{a+b}$$

[1] https://stacks.math.columbia.edu/tag/012K

$$\tilde{B}_r^{a,b} = F^a K^{a+b} \cap d_K(F^{a-r+1} K^{a+b-1}) + F^{a+1} K^{a+b}$$

*We have* $E_r^{a,b} \cong \tilde{Z}_r^{a,b} / \tilde{B}_r^{a,b}$. *Define:*

$$Z_\infty^{a,p} = \cap_r Z_r^{a,b}$$
$$B_\infty^{a,b} = \cup_r B_r^{a,b}$$

*When* $Z_\infty^{a,b}$ *and* $B_\infty^{a,b}$ *exist, we have:*

$$E_\infty^{a,b} = \frac{\cap_r \tilde{Z}_r^{a,b}}{\cup_r \tilde{B}_r^{a,b}}$$

*When there is more than one filtered complex present, I will use the notation,* $Z_r^{a,b}(F^\cdot K^\cdot)$, $B_r^{a,b}(F^\cdot K^\cdot)$, $E_r^{a,b}(F^\cdot K^\cdot)$, $\tilde{Z}_r^{a,b}(F^\cdot K^\cdot)$, *and* $\tilde{B}_r^{a,b}(F^\cdot K^\cdot)$, *to refer to the objects defined above.*

The below is from Lemma 12.21.4 of [6] [2].

**Lemma 6.1.28.** *Let* $\mathcal{A}$ *be an abelian category with sums, and let* $F^\cdot f^\cdot : F^\cdot K^\cdot \to F^\cdot L^\cdot$ *be a filtered chain map of filtered chain complexes in* $\mathcal{A}$. *Then* $F^\cdot f^\cdot$ *induces a family of morphisms between the spectral sequences of* $F^\cdot K^\cdot$ *and* $F^\cdot L^\cdot$, *where for* $r \geq 1$, $a, b \in \mathbb{Z}$, *one denotes:*

$$E_r^{a,b}(F^\cdot f^\cdot) : E_r^{a,b}(F^\cdot K^\cdot) \to E_r^{a,b}(F^\cdot L^\cdot)$$

*Proof.* Let $r \geq 1$ and $a, b \in \mathbb{Z}$. It must be shown that $f^\cdot(\tilde{Z}_r^{a,b}(F^\cdot K^\cdot)) \subseteq \tilde{Z}_r^{a,b}(F^\cdot L^\cdot)$ and $f^\cdot(\tilde{B}_r^{a,b}(F^\cdot K^\cdot)) \subseteq \tilde{B}_r^{a,b}(F^\cdot L^\cdot)$. Both of these statements follow from the fact that $F^\cdot f^\cdot$ is a filtered chain map. □

I expect the result below to hold in a general abelian category, but for the sake of simplicity I restrict to the category of sheaves of abelian groups. This way intersection and containment have a more obvious meaning.

[2] https://stacks.math.columbia.edu/tag/012O

**Lemma 6.1.29.** *Let $\mathcal{A}$ be the category of sheaves of abelian groups on a topological space $X$. Let $F^{\cdot}K^{\cdot}$ and $F^{\cdot}L^{\cdot}$ be filtered complexes in $\mathcal{A}$, and let $F^{\cdot}g_1^{\cdot}, F^{\cdot}g_2^{\cdot} : F^{\cdot}K^{\cdot} \to F^{\cdot}L^{\cdot}$ be two filtered chain maps in $\mathcal{A}$ that are filtered homotopic by filtered homotopy $F^{\cdot}h^{\cdot} : F^{\cdot}K^{\cdot} \to F^{\cdot}L[-1]^{\cdot}$. Then for all $r \geq 1$, $a, b \in \mathbb{Z}$, we have $E_r^{a,b}(F^{\cdot}g_1^{\cdot}) = E_r^{a,b}(F^{\cdot}g_2^{\cdot})$. That is, $F^{\cdot}g_1^{\cdot}$ and $F^{\cdot}g_2^{\cdot}$ induce the same morphism, $E_r^{a,b}(F^{\cdot}K^{\cdot}) \to E_r^{a,b}(F^{\cdot}L^{\cdot})$.*

*Proof.* Define $F^{\cdot}g^{\cdot} = F^{\cdot}g_1^{\cdot} - F^{\cdot}g_2^{\cdot}$, and let $r \geq 1$, $a, b \in \mathbb{Z}$. It suffices to show $E_r^{a,b}(F^{\cdot}g^{\cdot})$ is the zero map. That is, I must show:

$$F^{\cdot}g^{\cdot}(\tilde{Z}_r^{a,b}(F^{\cdot}K^{\cdot})) \subseteq \tilde{B}_r^{a,b}(F^{\cdot}L^{\cdot})$$

We have the relation $F^{\cdot}g^{\cdot} = F^{\cdot}d_L^{\cdot} \circ F^{\cdot}h^{\cdot} + F^{\cdot}h^{\cdot} \circ F^{\cdot}d_K^{\cdot}$. For the right summand above, we can show:

$$h^{a+b+1}(d_K^{a+b}(d_K^{-1}(F^{a+r}K^{a+b+1}))) \subseteq h^{a+b+1}(F^{a+r}K^{a+b+1}) \subseteq F^{a+r}L^{a+b} \subseteq F^{a+1}L^{a+b}$$

with the last inequality holding as long as $r \geq 1$. We also have:

$$d_L^{a+b-1}(h^{a+b}(F^aK^{a+b})) \subseteq d_L^{a+b-1}(F^aL^{a+b-1}) \subseteq F^aL^{a+b} \cap d_L(F^{a-r+1}L^{a+b-1})$$

with the last containment holding for $r \geq 1$. Now we can combine this all together. In the below I drop the degree superscript notation on the morphisms.

$$\begin{aligned}
g(\tilde{Z}_r^{a,b}(F^{\cdot}K^{\cdot})) &= g(F^aK^{a+b} \cap d_K^{-1}(F^{a+r}K^{a+b+1}) + F^{a+1}K^{a+b}) \\
&= g(F^aK^{a+b} \cap d_K^{-1}(F^{a+r}K^{a+b+1})) + g(F^{a+1}K^{a+b}) \\
&\subseteq g(F^aK^{a+b} \cap d_K^{-1}(F^{a+r}K^{a+b+1})) + F^{a+1}L^{a+b} \\
&\subseteq d_L(h(F^aK^{a+b})) + h(d_K(d_K^{-1}(F^{a+r}K^{a+b+1}))) + F^{a+1}L^{a+b} \\
&\subseteq d_L(F^aL^{a+b-1}) + h(F^{a+r}K^{a+b+1}) + F^{a+1}L^{a+b} \\
&\subseteq d_L(F^aL^{a+b-1}) + F^{a+r}L^{a+b} + F^{a+1}L^{a+b}
\end{aligned}$$

$$\subseteq F^a L^{a+b} \cap d_L(F^{a-r+1} L^{a+b-1}) + F^{a+1} L^{a+b} \qquad \text{Since } r \geq 1.$$
$$= \tilde{B}_r^{a,b}(F^\cdot L^\cdot)$$

Since $E_r^{a,b} = \tilde{Z}_r^{a,b} / \tilde{B}_r^{a,b}$, this shows $E_r^{a,b}(F^\cdot g^\cdot)$ is the zero map. Hence, $E_r^{a,b}(F^\cdot g_1^\cdot) = E_r^{a,b}(F^\cdot g_2^\cdot)$.

□

## 6.2 The Cone

**Definition 6.2.1.** *Let $\mathcal{A}$ be an abelian category with direct sums. Let $A^\cdot$ and $B^\cdot$ be complexes in $\mathcal{A}$. Suppose $f^\cdot : A^\cdot \to B^\cdot$ be a chain map. The cone of $f^\cdot$ is defined to be the following complex:*

$$Cone(f^\cdot)^n = A^{n+1} \oplus B^n$$

*with differential:*

$$d^n_{Cone(f^\cdot)} = \begin{bmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{bmatrix}$$

**Lemma 6.2.2.** *Let $\mathcal{A}$ be an abelian category with direct sums, $A^\cdot$ and $B^\cdot$ complexes in $\mathcal{A}$, and let $f^\cdot : A^\cdot \to B^\cdot$ be a chain map. Then $Cone(f^\cdot)^\cdot$ is a complex.*

*Proof.* Let $n \in \mathbb{Z}$. We have:

$$\begin{aligned} d^{n+1}_{\mathrm{Cone}(f^\cdot)} \circ d^n_{\mathrm{Cone}(f^\cdot)} &= \begin{bmatrix} -d_A^{n+2} & 0 \\ f^{n+2} & d_B^{n+1} \end{bmatrix} \begin{bmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{bmatrix} \\ &= \begin{bmatrix} d_A^{n+2} d_A^{n+1} & 0 \\ -f^{n+2} d_A^{n+1} + d_B^{n+1} f^{n+1} & d_B^{n+1} d_B^n \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

□

**Lemma 6.2.3.** *Let $\mathcal{A}$ be an abelian category with direct sums, and suppose $f^{\cdot}: A^{\cdot} \to B^{\cdot}$ is a chain map of complexes in $\mathcal{A}$. Then the inclusion $\iota_B^{\cdot}: B^{\cdot} \hookrightarrow Cone(f^{\cdot})^{\cdot}$ into the second component. Then $\iota_B^{\cdot}$ is a chain map.*

*Proof.* Let $n \in \mathbb{Z}$. We can represent $\iota_B^n$ in matrix notation as $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We have:

$$\begin{aligned} d^n_{\text{Cone}(f^{\cdot})} \circ \iota_B^n &= \begin{bmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ d_B^n \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} d_B^n \\ &= \iota_B^{n+1} \circ d_B^n \end{aligned}$$

□

**Lemma 6.2.4.** *Let $\mathcal{A}$ be an abelian category with direct sums, and suppose $f^{\cdot}: A^{\cdot} \to B^{\cdot}$ is a chain map of complexes in $\mathcal{A}$. Then the projection $\pi_A^{\cdot}: Cone(f^{\cdot})^{\cdot} \to A[1]^{\cdot}$ is a chain map.*

*Proof.* Let $n \in \mathbb{Z}$. We can represent $\pi_A^n$ using the matrix $\begin{bmatrix} 1 & 0 \end{bmatrix}$. We have:

$$\begin{aligned} \pi_A^{n+1} \circ d^n_{\text{Cone}(f^{\cdot})} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{bmatrix} \\ &= \begin{bmatrix} -d_A^{n+1} & 0 \end{bmatrix} \\ &= -d_A^{n+1} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ &= d^n_{A[1]} \circ \pi_A^n \end{aligned}$$

□

**Corollary 6.2.5.** *Let $\mathcal{A}$ be an abelian category with direct sums. Let $f^{\cdot} : A^{\cdot} \to B^{\cdot}$ be a chain map of complexes in $\mathcal{A}$. There is an exact sequence of complexes:*

$$0 \to B^{\cdot} \xrightarrow{\iota_B^{\cdot}} Cone(f^{\cdot})^{\cdot} \xrightarrow{\pi_A^{\cdot}} A[1]^{\cdot} \to 0$$

*Proof.* The exactness of the sequence in the middle is clear from the definition of $\iota_B^{\cdot}$ and $\pi_A^{\cdot}$. □

The below can be proven more elegantly, but I only need to use it in the category of abelian groups.

**Lemma 6.2.6.** *Let $\mathcal{A}$ be the category of abelian groups. Suppose $f^{\cdot} : A^{\cdot} \to B^{\cdot}$ is a quasi-isomorphism of complexes in $\mathcal{A}$. Then $Cone(f^{\cdot})^{\cdot}$ is acyclic.*

*Proof.* Let $(a^{n+1}, b^n) \in \ker(d^n_{\mathrm{Cone}(f^{\cdot})})$ with $a^{n+1} \in A^{n+1}$ and $b^n \in B^n$. I will show $(a^{n+1}, b^n) \in \mathrm{im}(d^{n-1}_{\mathrm{Cone}(f^{\cdot})})$. The condition $(a^{n+1}, b^n) \in \ker(d^n_{\mathrm{Cone}(f^{\cdot})})$ gives the equations:

$$\begin{aligned} -d_A^{n+1}(a^{n+1}) &= 0 \\ f^{n+1}(a^{n+1}) + d_B^n(b^n) &= 0 \end{aligned}$$

Since $a^{n+1} \in \ker(d_A^{n+1})$, $[a^{n+1}]$ is a class in $H^{n+1}(A^{\cdot})$. We have:

$$[f^{n+1}(a^{n+1})] = [-d_B^n(b^n)] = 0 \in H^{n+1}(B^{\cdot})$$

Since $f^{\cdot}$ is a quasi-isomorphism, $f^n$ induces an injective map on $H^{n+1}$, so the above implies $[a^{n+1}] = 0$ in $H^{n+1}(A^{\cdot})$. That is, there is a $\tilde{a}^n \in A^n$ such that $d_A^n(\tilde{a}^n) = a^{n+1}$. I claim $f^n(\tilde{a}^n) + b_n \in \ker(d_B^n)$. We have:

$$\begin{aligned} d_B^n(f^n(\tilde{a}^n) + b_n) &= f^{n+1}(d_A^n(\tilde{a}^n)) + d_B^n(b_n) \\ &= f^{n+1}(a^{n+1}) + d_B^n(b_n) = 0 \end{aligned}$$

Thus $[f^n(\tilde{a}^n)+b_n]$ is a class in $H^n(B^\cdot)$. Because $f^\cdot$ is a quasi-isomorphism, there is a $[\dot{a}^n] \in H^n(A^\cdot)$ such that $f^n([\dot{a}^n]) = [f^n(\tilde{a}^n)+b_n]$. That is, $\dot{a}^n \in \ker(d_A^n)$, and there is a $b^{n-1} \in B^{n-1}$ such that

$$f^n(\dot{a}^n) = f^n(\tilde{a}^n) + b^n - d_B^{n-1}(b^{n-1})$$

We now define $a^n = \dot{a}^n - \tilde{a}^n$. We have:

$$\begin{aligned}
-(d_A^n)(a^n) &= -d_A^n(\dot{a}^n) + d_A^n(\tilde{a}^n) = 0 + a^{n+1} = a^{n+1} \\
f^n(a^n) + d_B^{n-1}(b^{n-1}) &= f^n(\dot{a}^n) - f^n(\tilde{a}^n) + d_B^{n-1}(b^{n-1}) \\
&= (f^n(\tilde{a}^n) + b^n - d_B^{n-1}(\dot{b}^{n-1})) - f^n(\tilde{a}^n) + d_B^{n-1}(b^{n-1}) \\
&= b^n
\end{aligned}$$

These equations imply $d^{n-1}_{\mathrm{Cone}(f^\cdot)}(a^n, b^{n-1}) = (a^{n+1}, b^n)$. So we have shown $\mathrm{Cone}(f^\cdot)^\cdot$ is acyclic.

□

**Corollary 6.2.7.** *Let $\mathcal{A}$ be the category of sheaves of abelian groups on a topological space $X$. Let $f^\cdot : A^\cdot \to B^\cdot$ be a quasi-isomorphism of complexes in $\mathcal{A}$. Then Cone$(f^\cdot)^\cdot$ is acyclic.*

*Proof.* Let $x \in X$. We have $f_x^\cdot : A_x^\cdot \to B_x^\cdot$ is a quasi-isomorphism in the category of abelian groups, so by Lemma 6.2.6, $\mathrm{Cone}(f_x^\cdot)^\cdot$ is acyclic. Then since $\mathrm{Cone}(f_x^\cdot)^\cdot = \mathrm{Cone}(f^\cdot)_x^\cdot$, we have that $\mathrm{Cone}(f^\cdot)^\cdot$ is acyclic at all of its stalks. Hence, $\mathrm{Cone}(f^\cdot)^\cdot$ is acyclic. □

## 6.3 Inducing Maps

**Lemma 6.3.1.** *Let $\mathcal{A}$ be the category of abelian groups, and suppose $f^\cdot : A^\cdot \to B^\cdot$ is an injective quasi-isomorphism. Suppose there is an $a^n \in A^n$ and $b^{n-1} \in B^{n-1}$ satisfying $f^n(a^n) = d_B^{n-1}(b^{n-1})$. It follows that there is an $a^{n-1} \in A^{n-1}$ and $b^{n-2} \in B^{n-2}$ such that:*

$$d_A^{n-1}(a^{n-1}) = a^n$$

$$f^{n-1}(a^{n-1}) = b^{n-1} + d_B^{n-2}(b^{n-2})$$

*Proof.* We have that $a^n \in \ker(d_A^n)$:

$$\begin{aligned} f^{n+1}(d_A^n(a^n)) &= d_B^n(f^n(a^n)) \\ &= d_B^n(d_B^{n-1}(b^{n-1})) = 0 \end{aligned}$$

Since $f^{n+1}$ is an injective morphism, this implies $d_A^n(a^n) = 0$. At this point we have $(a^n, -b^{n-1}) \in \ker(d^{n-1}_{\mathrm{Cone}(f^\cdot)})$. Since $\mathrm{Cone}(f^\cdot)^\cdot$ is acyclic by Lemma 6.2.6, we can choose a pair $(\dot{a}^{n-1}, \dot{b}^{n-2}) \in \mathrm{Cone}(f^\cdot)^{n-2}$ such that $d^{n-2}_{\mathrm{Cone}(f^\cdot)}(\dot{a}^{n-1}, \dot{b}^{n-2}) = (a^n, -b^{n-1})$. That is, we have the equations:

$$\begin{aligned} -d_A^{n-1}(\dot{a}^{n-1}) &= a^n \\ f^{n-1}(\dot{a}^{n-1}) + d_B^{n-2}(\dot{b}^{n-2}) &= -b^{n-1} \end{aligned}$$

Set $a^{n-1} = -\dot{a}^{n-1}$ and $b^{n-2} = \dot{b}^{n-2}$. We have:

$$\begin{aligned} d_A^{n-1}(a^{n-1}) &= -d_A^{n-1}(\dot{a}^{n-1}) \\ &= a^n \\ f^{n-1}(a^{n-1}) &= -f^{n-1}(\dot{a}^{n-1}) \\ &= b^{n-1} + d_B^{n-2}(\dot{b}^{n-2}) \\ &= b^{n-1} + d_B^{n-2}(b^{n-2}) \end{aligned}$$

Now the lemma is complete.

□

The lemma below is somewhat weaker than Lemma 3.2.13. It's proved in detail here because the method will be used again in proving Lemma 6.5.2.

**Lemma 6.3.2.** *Let $\mathcal{A}$ be an abelian category and suppose we have the solid diagram below:*

$$\begin{tikzcd} J^{\cdot} \arrow[r, dashed, "g^{\cdot}"] & I^{\cdot} \\ A^{\cdot} \arrow[u, hook, "\varepsilon^{\cdot}"] \arrow[ur, "f^{\cdot}"'] & \end{tikzcd}$$

*where we assume $I^{\cdot}$ is bounded below and injective in each degree, $f^{\cdot}$ is any chain map, and $\varepsilon^{\cdot}$ is an injective quasi-isomorphism. Then there exists a chain map $g^{\cdot}$ making the diagram commute.*

*Proof.* We construct $g^{\cdot}$ by increasing induction on the degree. Because $I^{\cdot}$ is bounded below, there is a $m \in \mathbb{Z}$ such that $I^n = 0$ for all $n \leq m$. Thus, we may define $g^n = 0$ for $n \leq m$. Now suppose there is a $n \in \mathbb{Z}$ such that for all $k \leq n$, $g^k$ is defined and the following relations hold:

$$f^k = g^k \circ \varepsilon^k$$
$$d_I^{k-1} \circ g^{k-1} = g^k \circ d_J^{k-1}$$

I must construct $g^{n+1}$ such that the following relations hold:

$$f^{n+1} = g^{n+1} \circ \varepsilon^{n+1}$$
$$d_I^n \circ g^n = g^{n+1} \circ d_J^n$$

That is, $g^{n+1}$ must be chosen so that both faces in the following diagram simultaneously commute:

$$\begin{tikzcd}
J^n \arrow[r, "g^n"] \arrow[d, "d_J^n"'] & I^n \arrow[d, "d_I^n"] \\
J^{n+1} \arrow[r, dashed, "g^{n+1}"] & I^{n+1} \\
A^{n+1} \arrow[u, "\varepsilon^{n+1}"] \arrow[ur, "f^{n+1}"'] &
\end{tikzcd}$$

We accomplish this with the following diagram:

$$\begin{tikzcd}
J^{n+1} \arrow[rrr, dashed, "g^{n+1}"] & & & I^{n+1} \\
& \frac{A^{n+1}\oplus J^n}{\ker(\varepsilon^{n+1}-d_J^n)} \arrow[ul, hook] \arrow[r, dashed, "\overline{1_A^{n+1}\oplus g^n}"] & \frac{A^{n+1}\oplus I^n}{\ker(f^{n+1}-d_I^n)} \arrow[ur, hook] & \\
A^{n+1}\oplus J^n \arrow[uu, "\varepsilon^{n+1}-d_J^n"] \arrow[ur, two heads] \arrow[rrr, "1_A^{n+1}\oplus g^n"] & & & A^{n+1}\oplus I^n \arrow[uu, "f^{n+1}-d_I^n"'] \arrow[ul, two heads]
\end{tikzcd}$$

I will first show that the quotient map $\overline{1_A^{n+1}\oplus g^n}$ is well defined by showing $(1_A^{n+1}\oplus g^n)(\ker(\varepsilon^{n+1}-d_J^n))\subseteq\ker(f^{n+1}-d_I^n)$. Because we are in the category of sheaves of abelian groups on a topological space, it suffices to verify the containment $(1_A^{n+1}\oplus g^n)(\ker(\varepsilon^{n+1}-d_J^n))\subseteq\ker(f^{n+1}-d_I^n)$ on all the stalks. Let $x\in X$. Suppose $(a^{n+1},j^n)\in\ker(\varepsilon_x^{n+1}-d_{J_x}^n)\subseteq A_x^{n+1}\oplus J_x^n$. That is, we have the relation:

$$\varepsilon_x^{n+1}(a^{n+1})=d_{J_x}^n(j^n)$$

We must show $(1_{A_x}^{n+1}\oplus g_x^n)(a^{n+1},j^n)\in\ker(f_x^{n+1}-d_{I_x}^n)$. That is, I must prove the relation:

$$f_x^{n+1}(a^{n+1})=d_{I_x}^n(g_x^n(j^n))$$

Because $\varepsilon_x^{\cdot}$ is an injective quasi-isomorphism, we may invoke Lemma 6.3.1 to find a $a^n\in A_x^n$ and $j^{n-1}\in J_x^{n-1}$ such that:

$$d^n_{A_x}(a^n) = a^{n+1}$$
$$\varepsilon^n_x(a^n) = j^n + d^{n-1}_{J_x}(j^{n-1})$$

Now we have:

$$\begin{aligned}
f^{n+1}_x(a^{n+1}) &= f^{n+1}_x(d^n_{A_x}(a^n)) \\
&= d^n_{I_x}(f^n_x(a^n)) \\
&= d^n_{I_x}(g^n_x(\varepsilon^n_x(a^n))) && \text{By induction} \\
&= d^n_{I_x}(g^n_x(j^n + d^{n-1}_{J_x}(j^{n-1}))) \\
&= d^n_{I_x}(g^n_x(j^n)) + d^n_{I_x}(g^n_x(d^{n-1}_{J_x}(j^{n-1}))) \\
&= d^n_{I_x}(g^n_x(j^n)) + d^n_{I_x}(d^{n-1}_{I_x}(g^{n-1}_x(j^{n-1}))) \\
&= d^n_{I_x}(g^n_x(j^n))
\end{aligned}$$

So the required relation is shown. This shows the containment $(1^{n+1}_{A_x} \oplus g^n_x)(\ker(\varepsilon^{n+1}_x - d^n_{J_x})) \subseteq \ker(f^{n+1}_x - d^n_{I_x})$. Since this containment holds for all $x \in X$, we have shown $(1^{n+1}_A \oplus g^n)(\ker(\varepsilon^{n+1} - d^n_J)) \subseteq \ker(f^{n+1} - d^n_I)$. So the quotient map $\overline{1^{n+1}_A \oplus g^n}$ in the diagram is well defined and makes the lower trapezoid commute. Now we define $g^{n+1}$ to be a morphism in the diagram making the upper trapezoid commute, which exists because the morphism $(A^{n+1} \oplus J^n)/\ker(\varepsilon^{n+1} - d^n_J) \hookrightarrow J^{n+1}$ is injective, and $I^{n+1}$ is an injective object. Because every face in the diagram commutes, the perimeter commutes, and the commutativity of the perimeter implies both the relations:

$$f^{n+1} = g^{n+1} \circ \varepsilon^{n+1}$$
$$d^{n+1}_I \circ g^n = g^{n+1} \circ d^n_J$$

We may now continue the induction to define $g^n$ for all $n \in \mathbb{Z}$ so that $g^\cdot$ is a chain map and $f^\cdot = g^\cdot \circ \varepsilon^\cdot$.

□

The lemmas below address the uniqueness of such a $g^\cdot$ in the lemma above. I will now work towards a proof of a weaker version of Lemma 3.2.14, since similar methods will be used in Section 6.5.

**Lemma 6.3.3.** *Let $\mathcal{A}$ be an abelian category. Let $K^\cdot$ be an acyclic complex in $\mathcal{A}$, let $I^\cdot$ be a bounded below complex in $\mathcal{A}$, injective in each degree, and let $g^\cdot : K^\cdot \to I^\cdot$ be any chain map. Then $g^\cdot$ is homotopic to zero.*

*Proof.* We will inductively define a family of morphisms $h^n : K^n \to I^{n-1}$ for $n \in \mathbb{Z}$, such that the following relation holds for all $n$:

$$g^n = d_I^{n-1} \circ h^n + h^{n+1} \circ d_K^n$$

Since $I^\cdot$ is bounded below, there is a $m \in \mathbb{Z}$ such that $I^n = 0$ for all $n \leq m$. Our only choice is to define $h^n = 0$ for $n \leq m+1$, and the required relation holds. Now suppose there is a $n$ such that for all $k \leq n$, $h^k$ is defined and the relation $g^{k-1} = d_I^{k-2} h^{k-1} + h^k d_K^{k-1}$ holds for all $k \leq n$. We must define $h^{n+1}$ such that $g^n = d_I^{n-1} h^n + h^{n+1} d_K^n$. Define $\phi^n : K^n \to I^n$ with:

$$\phi^n = g^n - d_I^{n-1} h^n$$

Consider the solid diagram below:

$$\begin{tikzcd}
 & K^{n+1} \arrow[ddr, dashed, "h^{n+1}"] & \\
 & \frac{K^n}{\ker(d_K^n)} \arrow[u, hook] \arrow[dr, dashed, "\overline{\phi}"'] & \\
K^n \arrow[uur, "d_K^n"] \arrow[ur, two heads] \arrow[rr, "\phi"'] & & I^n
\end{tikzcd}$$

I first must show that the quotient map $\overline{\phi}$ is well defined. That is, I must show $\phi(\ker(d_K^n)) = 0$. Since $K^\cdot$ is acyclic, it is sufficient to show the composition $\phi \circ d_K^{n-1}$ is zero. We have:

$$
\begin{aligned}
\phi^n d_K^{n-1} &= g^n d_K^{n-1} - d_I^{n-1} h^n d_K^{n-1} \\
&= g^n d_K^{n-1} - d_I^{n-1}(g^{n-1} - d_I^{n-2} h^{n-1}) && \text{By induction} \\
&= g^n d_K^{n-1} - d_I^{n-1} g^{n-1} + d_I^{n-1} d_I^{n-2} h^{n-1} \\
&= g^n d_K^{n-1} - d_I^{n-1} g^{n-1} \\
&= 0
\end{aligned}
$$

This shows $\phi^n(\mathrm{im}(d_K^{n-1})) = 0$, so we have $\phi^n(\ker(d_K^n)) = 0$. The quotient map $\overline{\phi}$ is then well defined and makes the lower triangle in the diagram commute. We then define $h^{n+1}$ to be a morphism that makes the top right triangle commute, which exists because the arrow $K^n/\ker(d_K^n) \hookrightarrow K^{n+1}$ is injective and $I^n$ is an injective object. Since all faces in the diagram commute, the perimeter commutes, which gives the relation:

$$h^{n+1} \circ d_K^n = \phi^n = g^n - d_I^{n-1} \circ h^n$$

This completes the inductive construction, and $h^n$ may be defined in this way for all $n \in \mathbb{Z}$, and by construction, $h^\cdot$ is a homotopy of $g^\cdot$ to zero.

□

**Lemma 6.3.4.** *Let $\mathcal{A}$ be the category of sheaves of abelian groups on a topological space $X$, let $A^\cdot$, $J^\cdot$, and $I^\cdot$ be chain complexes in $\mathcal{A}$, where $I^\cdot$ is bounded below and injective in each degree, and suppose we have the diagram below:*

$$
\begin{tikzcd}
J^\cdot \arrow[r, "g_1^\cdot, g_2^\cdot"] & I^\cdot \\
A^\cdot \arrow[u, "\varepsilon^\cdot"] \arrow[ur, "f^\cdot"'] &
\end{tikzcd}
$$

*where $\varepsilon^\cdot$ is a quasi-isomorphism, $f^\cdot$ is a chain map, and $g_1^\cdot, g_2^\cdot$ are two chain maps making the diagram commute. Then $g_1^\cdot$ and $g_2^\cdot$ are homotopic.*

*Proof.* For each $n \in \mathbb{Z}$, define $\gamma^n : \text{Cone}(\varepsilon^\cdot)^n \to J^n$ to be the projection onto the second component, where $\text{Cone}(\varepsilon^\cdot)^n = A^{n+1} \oplus J^n$. Note, $\gamma^\cdot$ is not necessarily a chain map. We can represent $\gamma^n$ with the matrix:

$$\gamma^n = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

For $n \in \mathbb{Z}$, define $\phi^n : \text{Cone}(\varepsilon^\cdot)^n \to I^n$ by the composition:

$$\phi^n = (g_1^n - g_2^n) \circ \gamma^n$$

I claim $\phi^\cdot$ is a chain map. Let $n \in \mathbb{Z}$:

$$\begin{aligned}
\phi^{n+1} \circ d^n_{\text{Cone}(\varepsilon^\cdot)} &= (g_1^{n+1} - g_2^{n+1}) \circ \gamma^{n+1} \circ d^n_{\text{Cone}(\varepsilon^\cdot)} \\
&= (g_1^n - g_2^n) \circ \begin{bmatrix} 0 & 1 \end{bmatrix} \circ \begin{bmatrix} -d_A^{n+1} & 0 \\ \varepsilon^{n+1} & d_J^n \end{bmatrix} \\
&= (g_1^n - g_2^n) \circ \begin{bmatrix} \varepsilon^{n+1} & d_J^n \end{bmatrix} \\
&= \begin{bmatrix} \left(g_1^{n+1}\varepsilon^{n+1} - g_2^{n+1}\varepsilon^{n+1}\right) & \left((g_1^{n+1} - g_2^{n+1})d_J^n\right) \end{bmatrix} \\
&= \begin{bmatrix} f^{n+1} - f^{n+1} & d_I^n(g_1^n - g_2^n) \end{bmatrix} \\
&= d_I^n \circ \begin{bmatrix} 0 & (g_1^n - g_2^n) \end{bmatrix} \\
&= d_I^n \circ (g_1^n - g_2^n) \circ \begin{bmatrix} 0 & 1 \end{bmatrix} \\
&= d_I^n \circ \phi^n
\end{aligned}$$

So $\phi^\cdot : \text{Cone}(\varepsilon^\cdot)^\cdot \to I^\cdot$ is a chain map. Because $\varepsilon^\cdot$ is a quasi-isomorphism, $\text{Cone}(\varepsilon^\cdot)^\cdot$ is acyclic by Lemma 6.2.6. By invoking Lemma 6.3.3, $\phi^\cdot$ is homotopic to zero by a homotopy $h^\cdot : \text{Cone}(\varepsilon^\cdot)^\cdot \to I[-1]^\cdot$. By Lemma 6.2.3, $\iota_J^\cdot : J^\cdot \hookrightarrow \text{Cone}(\varepsilon)^\cdot$ is a chain map. By Lemma 3.1.16, $\phi^\cdot \circ \iota_J^\cdot : J^\cdot \to I^\cdot$ is homotopic to zero by homotopy $h^\cdot \circ \iota_J^\cdot : J^\cdot \to I[-1]^\cdot$. But for all $n \in \mathbb{Z}$ we have:

$$\phi^n \circ \iota_J^n = (g_1^n - g_2^n) \circ \gamma^n \circ \iota_J^n$$

$$= (g_1^n - g_2^n) \circ \begin{bmatrix} 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= g_1^n - g_2^n$$

Since the above is homotopic to zero, we have shown $g_1^{\cdot}$ and $g_2^{\cdot}$ are homotopic. □

## 6.4 The Filtered Cone

**Definition 6.4.1.** *Let $\mathcal{A}$ be an abelian category with direct sums, and let $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ be filtered complexes in $\mathcal{A}$. Let $F^{\cdot}f^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ be a filtered chain map. Then $Cone(f^{\cdot})$ has an induced filtration, where for all $m, n \in \mathbb{Z}$:*

$$F^m Cone(F^{\cdot}f^{\cdot})^n = F^m A^{n+1} \oplus F^m B^n$$

*and differential given by:*

$$F^m d^n_{Cone(F^{\cdot}f^{\cdot})} = \begin{bmatrix} -F^m d_A^{n+1} & 0 \\ F^m f^n & F^m d_B^n \end{bmatrix}$$

*Since all the maps involved preserve filtration degree, this makes sense. We in fact have the following identification for all $m \in \mathbb{Z}$:*

$$F^m Cone(F^{\cdot}f^{\cdot})^{\cdot} = Cone(F^m f^{\cdot})^{\cdot}$$

**Lemma 6.4.2.** *Let $\mathcal{A}$ be an abelian category with direct sums, and suppose $F^{\cdot}f^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ is a filtered chain map of filtered complexes. Then the chain map $\iota_B^{\cdot} : B^{\cdot} \hookrightarrow Cone(f^{\cdot})^{\cdot}$ is a filtered chain map.*

*Proof.* For $m, n \in \mathbb{Z}$ we have:

$$F^m \iota^n_{F^{\cdot}B} : F^m B^n \to F^m \mathrm{Cone}(F^{\cdot}f^{\cdot})^n = F^m A^{n+1} \oplus F^m B^n$$

It is clear that $\iota_B^{\cdot}$ preserves filtration degree. □

**Lemma 6.4.3.** *Let $\mathcal{A}$ be an abelian category with direct sums and suppose $F^{\cdot}f^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ is a filtered chain map of filtered chain complexes in $\mathcal{A}$. Then the projection chain map $\pi_A^{\cdot}$ : $Cone(f^{\cdot})^{\cdot} \to A[1]^{\cdot}$ defined in Lemma 6.2.4 is a filtered chain map.*

*Proof.* For $m, n \in \mathbb{Z}$ we have:

$$F^m\pi_A^n : F^m\mathrm{Cone}(F^{\cdot}f^{\cdot})^n = F^mA^{n+1} \oplus F^mB^n \to F^mA^{n+1}$$

It is clear that that $F^{\cdot}\pi_A^{\cdot}$ preserves filtration degree. □

**Lemma 6.4.4.** *Let $\mathcal{A}$ be an abelian category with direct sums, let $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ be filtered complexes in $\mathcal{A}$, and suppose $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ is a strong filtered quasi-isomorphism as in Definition 6.1.8. Then $F^{\cdot}Cone(F^{\cdot}\varepsilon^{\cdot})^{\cdot}$ is strong filtered acyclic as in Definition 6.1.15.*

*Proof.* Let $m \in \mathbb{Z}$. Because $F^{\cdot}\varepsilon^{\cdot}$ is a strong filtered quasi-isomorphism, $F^m\varepsilon^{\cdot} : F^mA^{\cdot} \to F^mB^{\cdot}$ is a quasi-isomorphism. By Lemma 6.2.6, $\mathrm{Cone}(F^m\varepsilon^{\cdot})^{\cdot}$ is acyclic. This is identified with $F^m\mathrm{Cone}(F^{\cdot}\varepsilon^{\cdot})^{\cdot}$, so we have shown $F^m\mathrm{Cone}(F^{\cdot}\varepsilon^{\cdot})^{\cdot}$ is acyclic. Since this holds for all $m \in \mathbb{Z}$, $F^{\cdot}\mathrm{Cone}(F^{\cdot}\varepsilon^{\cdot})^{\cdot}$ is strong filtered acyclic. □

## 6.5 Inducing Filtered Maps

In this section I will proves lemmas corresponding to those of Section 6.3, but done with filtered complexes.

**Lemma 6.5.1.** *Let $\mathcal{A}$ be the category of abelian groups. Let $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ be filtered complexes in $\mathcal{A}$, and let $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ be an injective filtered quasi-isomorphism. Let $n, m \in \mathbb{Z}$ and suppose there are $a^n \in F^mA^n$, $b^{n-1} \in F^mB^{n-1}$, and $b^n \in F^{m+1}B^n$, such that:*

$$F^m\varepsilon^n(a^n) = F^md_B^{n-1}(b^{n-1}) + b^n$$

*I claim there exist $a^{n-1} \in F^mA^{n-1}$, $b^{n-2} \in F^mB^{n-2}$, $\dot{a}^n \in F^{m+1}A^n$, and $\dot{b}^{n-1} \in F^{m+1}B^{n-1}$, such that:*

$$a^n = F^md_A^{n-1}(a^{n-1}) + \dot{a}^n$$

$$F^m\varepsilon^{n-1}(a^{n-1}) = b^{n-1} + F^m d_B^{n-2}(b^{n-2}) + \dot{b}^{n-1}$$
$$F^{m+1}\varepsilon^n(\dot{a}^n) + F^{m+1}d_B^{n-1}(\dot{b}^{n-1}) = b^n$$

*Proof.* Because $F^{\cdot}\varepsilon^{\cdot}$ is a filtered quasi-isomorphism, $\mathrm{gr}^m\varepsilon^{\cdot} : \mathrm{gr}^m A^{\cdot} \to \mathrm{gr}^m B^{\cdot}$ is a quasi-isomorphism. Because $F^{\cdot}\varepsilon^{\cdot}$ is injective, so is $\mathrm{gr}^m\varepsilon^{\cdot}$ by Lemma 6.1.5. On the $m$th graded part, we have the relation:

$$\mathrm{gr}^m\varepsilon^n(\overline{a^n}) = \mathrm{gr}^m d_B^{n-1}(\overline{b^{n-1}})$$

Since $\mathrm{gr}^m\varepsilon^{\cdot}$ is an injective quasi-isomorphism, we can invoke Lemma 6.3.1 to find a $\overline{a^{n-1}} \in \mathrm{gr}^m A^{n-1}$ and $\overline{b^{n-2}} \in \mathrm{gr}^m B^{n-2}$ such that:

$$\mathrm{gr}^m d_A^{n-1}(\overline{a^{n-1}}) = \overline{a^n}$$
$$\mathrm{gr}^m\varepsilon^{n-1}(\overline{a^{n-1}}) = \overline{b^{n-1}} + \mathrm{gr}^m d_B^{n-2}(\overline{b^{n-2}})$$

That is, $a^{n-1} \in F^m A^{n-1}$, $b^{n-2} \in F^m B^{n-2}$, and there are $\dot{a}^n \in F^{m+1}A^n$, and $\dot{b}^{n-1} \in F^{m+1}B^{n-1}$, such that:

$$a^n = F^m d_A^{n-1}(a^{n-1}) + \dot{a}^n$$
$$F^m\varepsilon^{n-1}(a^{n-1}) = b^{n-1} + F^m d_B^{n-2}(b^{n-2}) + \dot{b}^{n-1}$$

From this we have:

$$F^m\varepsilon^n(a^n) = F^m d_B^{n-1}(b^{n-1}) + b^n$$
$$F^m\varepsilon^n(F^m d_A^{n-1}(a^{n-1}) + \dot{a}^n) = F^m d_B^{n-1}\left(F^m\varepsilon^{n-1}(a^{n-1}) - F^m d_B^{n-2}(b^{n-2}) - \dot{b}^{n-1}\right) + b^n$$
$$\begin{aligned} F^m\varepsilon^n(F^m d_A^{n-1}(a^{n-1})) + F^{m+1}\varepsilon^n(\dot{a}^n) &= F^m d_B^{n-1}(F^m\varepsilon^{n-1}(a^{n-1})) \\ &\quad - F^m d_B^{n-1}(F^m d_B^{n-2}(b^{n-2})) - F^{m+1}d_B^{n-1}(\dot{b}^{n-1}) + b^n \end{aligned}$$

$$F^m\varepsilon^n(F^m d_A^{n-1}(a^{n-1})) + F^{m+1}\varepsilon^n(\dot{a}^n) = F^m\varepsilon^n(F^m d_A^{n-1}(a^{n-1})) - F^{m+1}d_B^{n-1}(\dot{b}^{n-1}) + b^n$$
$$F^{m+1}\varepsilon^n(\dot{a}^n) = -F^{m+1}d_B^{n-1}(\dot{b}^{n-1}) + b^n$$

Thus, we also have the third relation:

$$F^{m+1}\varepsilon^n(\dot{a}^n) + F^{m+1}d_B^{n-1}(\dot{b}^{n-1}) = b^n$$

□

The Lemma below is related to Lemma 13.26.11 [3] of [6].

**Lemma 6.5.2.** *Let $\mathcal{A}$ be the category of sheaves of abelian groups on a topological space $X$. Let $F^\cdot A^\cdot$, $F^\cdot J^\cdot$, and $F^\cdot I^\cdot$ be filtered complexes in $\mathcal{A}$. Assume $F^\cdot I^\cdot$ is strong filtered injective, bounded below, and has a terminating filtration in each degree, as in Definition 6.1.4 (1). Assume $F^\cdot J^\cdot$ and $F^\cdot A^\cdot$ are exhaustively filtered, as in Definition 6.1.4 (4). Suppose we have the solid diagram below:*

$$\begin{tikzcd}
F^\cdot J^\cdot \arrow[r, dashed, "F^\cdot g^\cdot"] & F^\cdot I^\cdot \\
F^\cdot A^\cdot \arrow[u, hook, "F^\cdot \varepsilon^\cdot"] \arrow[ur, "F^\cdot f^\cdot"'] &
\end{tikzcd}$$

*where $F^\cdot\varepsilon^\cdot$ is an injective filtered quasi-isomorphism, and $F^\cdot f^\cdot$ is a filtered chain map. Then there exists a filtered chain map $F^\cdot g^\cdot$ making the diagram commute.*

*Proof.* The proof is an inductive construction of $F^m g^n$ for all $m, n \in \mathbb{Z}$. Because $I^\cdot$ is bounded below, there is a $N_0 \in \mathbb{Z}$ such that $I^n = 0$ for all $n \leq N_0$. Define $g^n = 0$ for $n \leq N_0$, and of course we have $F^m g^n = 0$ for all $m \in \mathbb{Z}$, $n \leq N_0$. Because $F^\cdot I^\cdot$ has a filtration that terminates in each degree, we have for all $n \in \mathbb{Z}$, there is a $m_n \in \mathbb{Z}$, such that $F^m I^n = 0$ for all $m \geq m_n$. Define $F^m g^n = 0$ for all $n \in \mathbb{Z}$, $m \geq m_n$. These definitions establish the basecase for what follows. Now suppose there is a $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ such that $F^m g^{n-1}$, $F^m g^{n-2}$, $F^{m+1} g^n$, and $F^{m+1} g^{n-1}$ are defined, satisfying the following relations:

[3] https://stacks.math.columbia.edu/tag/05TY

$$
\begin{aligned}
F^m f^{n-1} &= F^m g^{n-1} \circ F^m \varepsilon^{n-1} \\
F^m g^{n-1} \circ F^m d_J^{n-2} &= F^m d_I^{n-2} \circ F^m g^{n-2} \\
F^{m+1} f^n &= F^{m+1} g^n \circ F^{m+1} \varepsilon^n \\
F^{m+1} g^n \circ F^{m+1} d_J^{n-1} &= F^{m+1} d_I^{n-1} \circ F^{m+1} g^{n-1} \\
F^m g^{n-1}|_{F^{m+1}J^{n-1}} &= F^{m+1} g^{n-1}
\end{aligned}
$$

I must then choose $F^m g^n$ such that the following relations are satisfied:

$$
\begin{aligned}
F^m f^n &= F^m g^n \circ F^m \varepsilon^n \\
F^m g^n \circ F^m d_J^{n-1} &= F^m d_I^{n-1} \circ F^m g^{n-1} \\
(F^m g^n)|_{F^{m+1}J^n} &= F^{m+1} g^n
\end{aligned}
$$

That is, $F^m g^n$ must be chosen so that the following three diagrams simultaneously commute:

$$
\begin{tikzcd}
F^m J^n \arrow[r, dashed, "F^m g^n"] & F^m I^n \\
F^m A^n \arrow[u, hook, "F^m \varepsilon^n"] \arrow[ur, "F^m f^n"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
F^m J^n \arrow[r, dashed, "F^m g^n"] & F^m I^n \\
F^m J^{n-1} \arrow[u, "F^m d_J^{n-1}"] \arrow[r, "F^m g^{n-1}"] & F^m I^{n-1} \arrow[u, "F^m d_I^{n-1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F^m J^n \arrow[r, dashed, "F^m g^n"] & F^m I^n \\
F^{m+1} J^n \arrow[u, hook, "\mathrm{inc}_J^n"] \arrow[r, "F^{m+1} g^n"] & F^{m+1} I^n \arrow[u, hook, "\mathrm{inc}_I^n"']
\end{tikzcd}
$$

We accomplish this via the diagram below:

$$
\begin{tikzcd}
F^m J^n \arrow[rrr, dashed, "F^m g^n"] & & & F^m I^n \\
& C/\ker(\gamma) \arrow[ul, hook] \arrow[r, dashed, "\overline{\phi}"] & D/\ker(\psi) \arrow[ur, hook] & \\
C \arrow[uu, "\gamma"] \arrow[ur, two heads] \arrow[rrr, "\phi"] & & & D \arrow[uu, "\psi"'] \arrow[ul, two heads]
\end{tikzcd}
$$

where in the above we define:

$$C = F^m A^n \oplus F^m J^{n-1} \oplus F^{m+1} J^n$$
$$D = F^m A^n \oplus F^m I^{n-1} \oplus F^{m+1} I^n$$
$$\phi = F^m 1_A^n \oplus F^m g^{n-1} \oplus F^{m+1} g^n$$
$$\gamma = F^m \varepsilon^n - F^m d_J^{n-1} - \mathrm{inc}_J^n$$
$$\psi = F^m f^n - F^m d_I^{n-1} - \mathrm{inc}_I^n$$

In order to show the quotient map $\overline{\phi}$ is well defined, I must show $\phi(\ker(\gamma)) \subseteq \ker(\psi)$. It suffices to check this containment on the stalks. Let $x \in X$. Suppose $(a^n, j^{n-1}, j^n) \in \ker(\gamma_x)$. That is, $a^n \in F^m A_x^n$, $j^{n-1} \in F^m J_x^{n-1}$, and $j^n \in F^{m+1} J_x^n$, and we have the relation:

$$F^m \varepsilon_x^n(a^n) = F^m d_{J_x}^{n-1}(j^{n-1}) + j^n$$

I must show $\phi_x(a^n, j^{n-1}, j^n) \in \ker(\psi_x)$. That is, I must show the relation:

$$F^m f_x^n(a^n) = F^m d_{I_x}^{n-1}(F^m g_x^{n-1}(j^{n-1})) + F^{m+1} g_x^n(j^n)$$

Because of our given relation and the fact that $F^{\cdot}\varepsilon_x^{\cdot} : F^{\cdot}A_x^{\cdot} \to F^{\cdot}J_x^{\cdot}$ is an injective filtered quasi-isomorphism in the category of abelian groups, we can apply Lemma 6.5.1 to find a $a^{n-1} \in F^m A^{n-1}$, $j^{n-2} \in F^m J^{n-2}$, $\dot{a}^n \in F^{m+1} A^n$, and $\dot{j}^{n-1} \in F^{m+1} J^{n-1}$, such that the following relations hold:

$$a^n = F^m d_{A_x}^{n-1}(a^{n-1}) + \dot{a}^n$$
$$F^m \varepsilon_x^{n-1}(a^{n-1}) = j^{n-1} + F^m d_{J_x}^{n-2}(j^{n-2}) + \dot{j}^{n-1}$$
$$F^{m+1} \varepsilon_x^n(\dot{a}^n) + F^{m+1} d_{J_x}^{n-1}(\dot{j}^{n-1}) = j^n$$

We now have:

$$F^m f_x^n(a^n) = F^m f_x^n(F^m d_{A_x}^{n-1}(a^{n-1}) + \dot{a}^n)$$

$$
\begin{aligned}
&= F^m d_{I_x}^{n-1}(F^m f_x^{n-1}(a^{n-1})) + F^{m+1} f_x^n(\dot{a}^n) \\
&= F^m d_{I_x}^{n-1}(F^m g_x^{n-1}(F^m \varepsilon_x^{n-1}(a^{n-1}))) + F^{m+1} g_x^n(F^{m+1}\varepsilon^n(\dot{a}^n)) \\
&= F^m d_{I_x}^{n-1}(F^m g_x^{n-1}(j^{n-1} + F^m d_{J_x}^{n-2}(j^{n-2}) + \dot{j}^{n-1})) + F^{m+1} g_x^n(F^{m+1}\varepsilon^n(\dot{a}^n)) \\
&= F^m d_{I_x}^{n-1}(F^m g_x^{n-1}(j^{n-1})) + F^m d_{I_x}^{n-1}(F^m g_x^{n-1}(F^m d_{J_x}^{n-2}(j^{n-2}))) \\
&\qquad + F^{m+1} d_{I_x}^{n-1}(F^{m+1} g_x^{n-1}(\dot{j}^{n-1})) + F^{m+1} g_x^n(F^{m+1}\varepsilon^n(\dot{a}^n)) \\
&= F^m d_{I_x}^{n-1}(F^m g_x^{n-1}(j^{n-1})) + F^m d_{I_x}^{n-1}(F^m d_{I_x}^{n-2}(F^m g_x^{n-2}(j^{n-2}))) \\
&\qquad + F^{m+1} g_x^n(F^{m+1} d_{J_x}^{n-1}(\dot{j}^{n-1})) + F^{m+1} g_x^n(F^{m+1}\varepsilon^n(\dot{a}^n)) \\
&= F^m d_{I_x}^{n-1}(F^m g_x^{n-1}(j^{n-1})) + F^{m+1} g_x^n(F^{m+1} d_{J_x}^{n-1}(\dot{j}^{n-1})) + F^{m+1} g_x^n(F^{m+1}\varepsilon^n(\dot{a}^n)) \\
&= F^m d_{I_x}^{n-1}(F^m g_x^{n-1}(j^{n-1})) + F^{m+1} g_x^n(j^n)
\end{aligned}
$$

The required relation has been shown, so we have the containment, $\phi_x(\ker(\gamma_x)) \subseteq \ker(\psi_x)$. Since this holds for all $x \in X$, we have shown $\phi(\ker(\gamma)) \subseteq \ker(\psi)$. So the quotient map $\overline{\phi}$ is well defined and makes the lower trapezoid commute. Since the arrow $C/\ker(\gamma) \hookrightarrow F^m J^n$ is injective and $F^m I^n$ is an injective object, we can find a morphism $F^m g^n$ making the upper trapezoid commute. Since all faces in the diagram commute, the perimeter commutes, which implies the three required relations. We may now continue the induction and define $F^m g^n : F^m J^n \to F^m I^n$ for all $m,n \in \mathbb{Z}$. Since the filtration on $F^\cdot J^\cdot$ is exhaustive, this defines $g^\cdot : J^\cdot \to I^\cdot$. Since the filtration on $F^\cdot A^\cdot$ is also exhaustive, the relation $F^m f^\cdot = F^m g^\cdot \circ F^m \varepsilon^\cdot$ for all $m \in \mathbb{Z}$ implies $f^\cdot = g^\cdot \circ \varepsilon^\cdot$. We have now shown there is a filtered chain map $F^\cdot g^\cdot$ such that $F^\cdot f^\cdot = F^\cdot g^\cdot \circ F^\cdot \varepsilon^\cdot$.

$\square$

The Lemma below is related to Lemma 13.26.10 [4] of [6].

**Lemma 6.5.3.** *Let $\mathcal{A}$ be the category of sheaves of abelian groups on a topological space $X$. Let $F^\cdot K^\cdot$ be a filtered complex in $\mathcal{A}$ that is strong filtered acyclic as in Definition 6.1.15 and exhaustively filtered as in Definition 6.1.4 (4). Let $F^\cdot I^\cdot$ be a filtered complex in $\mathcal{A}$ that is bounded below, strong filtered injective in each degree as in Definition 6.1.18, and whose filtration terminates in each degree as in Definition 6.1.4 (1). Let $F^\cdot g^\cdot : F^\cdot K^\cdot \to F^\cdot I^\cdot$ be any filtered chain map in $\mathcal{A}$. Then $F^\cdot g^\cdot$ is filtered homotopic to zero.*

[4] https://stacks.math.columbia.edu/tag/05TX

*Proof.* We will define a family of filtered morphisms $F^{\cdot}h^n : F^{\cdot}K^n \to F^{\cdot}I^{n-1}$ such that the following relation is satisfied for all $m,n \in \mathbb{Z}$:

$$F^m g^n = F^m d_I^{n-1} \circ F^m h^n + F^m h^{n+1} \circ F^m d_K^n$$

Since $I^{\cdot}$ is bounded below, there is a $N_0 \in \mathbb{Z}$ such that $I^n = 0$ for all $n \leq N_0$. Define $F^m h^n = 0$ for all $n \leq N_0 + 1$, $m \in \mathbb{Z}$. Because $F^{\cdot}I^{\cdot}$ has a terminating filtration in each degree, for every $n \in \mathbb{Z}$, there is a $m_n \in \mathbb{Z}$, such that $F^{m_n}I^n = 0$. Define $F^m h^n = 0$ for all $n \in \mathbb{Z}$, $m \geq m_{n-1}$. This establishes the basecase for the inductive construction. Now suppose there are $m,n \in \mathbb{Z}$, such that $F^m h^n$, $F^{m+1}h^n$, and $F^{m+1}h^{n+1}$ are defined, and satisfy the following relations:

$$\begin{aligned} F^m g^{n-1} &= F^m d_I^{n-2} \circ F^m h^{n-1} + F^m h^n \circ F^m d_K^{n-1} \\ F^{m+1} g^n &= F^{m+1} d_I^{n-1} \circ F^{m+1} h^n + F^{m+1} h^{n+1} \circ F^{m+1} d_K^n \\ F^m h^n|_{F^{m+1}K^n} &= F^{m+1} h^n \end{aligned}$$

I must construct $F^m h^{n+1}$ so that the following relations are satisfied:

$$\begin{aligned} F^m g^n &= F^m d_I^{n-1} \circ F^m h^n + F^m h^{n+1} \circ F^m d_K^n \\ F^m h^{n+1}|_{F^{m+1}K^{n+1}} &= F^{m+1} h^{n+1} \end{aligned}$$

That is, $F^m h^{n+1}$ must make the two following diagrams simultaneously commute:

$$\begin{tikzcd} F^m K^{n+1} \arrow[drr, dashed, "F^m h^{n+1}"] & & \\ F^m K^n \arrow[u, "F^m d_K^n"] \arrow[rr, "F^m g^n - F^m d_I^{n-1} \circ F^m h^n"'] & & F^m I^n \end{tikzcd} \qquad \begin{tikzcd} F^m K^{n+1} \arrow[r, dashed, "F^m h^{n+1}"] & F^m I^n \\ F^{m+1} K^{n+1} \arrow[u, hook, "\mathrm{inc}_K^{n+1}"] \arrow[r, "F^{m+1} h^{n+1}"'] & F^{m+1} I^n \arrow[u, hook, "\mathrm{inc}_I^n"'] \end{tikzcd}$$

We can accomplish this via the diagram below:

$$
\begin{tikzcd}
& F^m K^{n+1} \arrow[ddr, dashed, "F^m h^{n+1}"] & \\
& C/\ker(\gamma) \arrow[u, hook] \arrow[dr, dashed, "\bar{\phi}"'] & \\
C \arrow[uur, "\gamma"] \arrow[ur, two heads] \arrow[rr, "\phi"] & & F^m I^n
\end{tikzcd}
$$

where in the above we define:

$$
\begin{aligned}
C &= F^m K^n \oplus F^{m+1} K^{n+1} \\
\gamma &= F^m d_K^n - \mathrm{inc}_K^{n+1} \\
\phi &= (F^m g^n - F^m d_I^{n-1} \circ F^m h^n) - F^{m+1} h^{n+1}
\end{aligned}
$$

In order to show the quotient map $\bar{\phi}$ is well defined, I must show $\phi(\ker(\gamma)) = 0$. It suffices to check this condition on the stalks. Let $x \in X$, and suppose $(k^n, k^{n+1}) \in \ker(\gamma_x)$. That is, $k^n \in F^m K_x^n$, $k^{n+1} \in F^{m+1} K_x^{n+1}$, and we have the relation:

$$F^m d_{K_x}^n (k^n) = k^{n+1}$$

I must show $\phi_x(k^n, k^{n+1}) = 0$, which is the relation:

$$F^m g_x^n (k^n) = F^m d_{I_x}^{n-1} (h_x^n (k^n)) + F^{m+1} h_x^{n+1} (k^{n+1})$$

Let $\overline{k^n}$ denote the class of $k^n$ in $\mathrm{gr}^m K^n$. We have:

$$\mathrm{gr}^m d_{K_x}^n (\overline{k^n}) = \overline{k^{n+1}} = 0$$

Because $F^{\cdot} K_x^{\cdot}$ is strong filtered acyclic and Lemma 6.1.16, $F^{\cdot} K_x^{\cdot}$ is filtered acyclic, so $\mathrm{gr}^m K_x^{\cdot}$ is an acyclic complex. So we can find a $\overline{k^{n-1}} \in \mathrm{gr}^m K_x^{n-1}$ such that $\mathrm{gr}^m d_{K_x}^{n-1} (\overline{k^{n-1}}) = \overline{k^n}$. That is, there is a $k^{n-1} \in F^m K_x^{n-1}$ and a $\dot{k}^n \in F^{m+1} K_x^n$ such that:

$$k^n = F^m d_{K_x}^{n-1} (k^{n-1}) + \dot{k}^n$$

Note we also have the relation:

$$F^{m+1}d^n_{K_x}(\dot{k}^n) = F^m d^n_{K_x}(F^m d^{n-1}_{K_x}(k^{n-1}) + \dot{k}^n) = F^m d^n_{K_x}(k^n) = k^{n+1}$$

Now we have:

$$\begin{aligned}
F^m g^n_x(k^n) &= F^m g^n_x(F^m d^{n-1}_{K_x}(k^{n-1}) + \dot{k}^n) \\
&= F^m d^{n-1}_{I_x}(F^m g^{n-1}_x(k^{n-1})) + F^{m+1} g^n_x(\dot{k}^n) \\
&= F^m d^{n-1}_{I_x}\left[F^m d^{n-2}_{I_x}(F^m h^{n-1}_x(k^{n-1})) + F^m h^n_x(F^m d^{n-1}_{K_x}(k^{n-1}))\right] \\
&\quad + \left[F^{m+1} d^{n-1}_{I_x}(F^{m+1} h^n_x(\dot{k}^n)) + F^{m+1} h^{n+1}_x(F^{m+1} d^n_{K_x}(\dot{k}^n))\right] \\
&= F^m d^{n-1}_{I_x}\left[F^m h^n_x(F^m d^{n-1}_{K_x}(k^{n-1}))\right] \\
&\quad + \left[F^{m+1} d^{n-1}_{I_x}(F^{m+1} h^n_x(\dot{k}^n)) + F^{m+1} h^{n+1}_x(k^{n+1})\right] \\
&= F^m d^{n-1}_{I_x}(F^m h^n_x(F^m d^{n-1}_{K_x}(k^{n-1}) + \dot{k}^n)) + F^{m+1} h^{n+1}_x(k^{n+1}) \\
&= F^m d^{n-1}_{I_x}(F^m h^n_x(k^n)) + F^{m+1} h^{n+1}_x(k^{n+1})
\end{aligned}$$

The required relation has been shown, so we have $\phi_x(\ker(\gamma_x)) = 0$. Since this holds for all $x \in X$, we have $\phi(\ker(\gamma)) = 0$. Thus, the quotient map $\overline{\phi}$ is well defined and makes the lower triangle commute. We can then define $F^m h^{n+1}$ to be a morphism making the upper right triangle commute, which exists because the arrow $C/\ker(\gamma) \hookrightarrow F^m K^{n+1}$ is injective and $F^m I^n$ is an injective object. Since all faces in the diagram commute, the perimeter commutes, which implies the required relations. We may now continue the induction to define $F^m h^n$ for all $m,n \in \mathbb{Z}$, such that we have the following for all $m,n$:

$$\begin{aligned}
F^m g^n &= F^m d^{n-1}_I \circ F^m h^n + F^m h^{n+1} \circ F^m d^n_K \\
F^m h^n|_{F^{m+1}K^n} &= F^{m+1} h^n
\end{aligned}$$

Because $F^{\cdot}K^{\cdot}$ is exhaustively filtered, this defines all of $h^{\cdot} : K^{\cdot} \to I^{\cdot}$, and we have the relation $g^n = d^{n-1}_I \circ h^n + h^{n+1} \circ d^n_K$ for all $n \in \mathbb{Z}$. We have now shown $F^{\cdot}g^{\cdot}$ is filtered homotopic to zero by filtered homotopy $F^{\cdot}h^{\cdot}$.

$\square$

The below is related to Lemma 13.26.11 [5] of [6].

**Lemma 6.5.4.** *Let $\mathcal{A}$ be the category of sheaves of abelian groups on a topological space $X$. Let $F^{\cdot}A^{\cdot}$, $F^{\cdot}J^{\cdot}$, and $F^{\cdot}I^{\cdot}$ be filtered complexes in $\mathcal{A}$. We assume $F^{\cdot}I^{\cdot}$ is bounded below, strong filtered injective as in Definition 6.1.18, and has a filtration that terminates in each degree as in Definition 6.1.4 (1). Assume the filtrations on $F^{\cdot}A^{\cdot}$ and $F^{\cdot}J^{\cdot}$ are both exhaustive. Suppose we have the solid diagram below:*

$$\begin{tikzcd} F^{\cdot}J^{\cdot} \arrow[r, "F^{\cdot}g_1^{\cdot}, F^{\cdot}g_2^{\cdot}"] & F^{\cdot}I^{\cdot} \\ F^{\cdot}A^{\cdot} \arrow[u, hook, "F^{\cdot}\varepsilon^{\cdot}"] \arrow[ur, "F^{\cdot}f^{\cdot}"'] & \end{tikzcd}$$

*where $F^{\cdot}f^{\cdot}$ is a filtered chain map, $F^{\cdot}\varepsilon^{\cdot}$ is a strong filtered quasi-isomorphism, and $F^{\cdot}g_1^{\cdot}$ and $F^{\cdot}g_2^{\cdot}$ are two filtered chain maps that make the diagram commute. Then $F^{\cdot}g_1^{\cdot}$ and $F^{\cdot}g_2^{\cdot}$ are filtered homotopic.*

*Proof.* We can take the filtered cone, $F^{\cdot}\mathrm{Cone}(F^{\cdot}\varepsilon^{\cdot})^{\cdot}$. As was done in the proof of Lemma 6.3.4, we define the projection map, $\gamma : \mathrm{Cone}(\varepsilon^{\cdot}) \to J^{\cdot}$ to be the projection onto the second factor, which is not a chain map in general. In this case we have $\gamma^n$ is a filtered morphism for all $n \in \mathbb{Z}$, where we have:

$$F^m\gamma^n : F^m\mathrm{Cone}(F^{\cdot}\varepsilon)^n = F^mA^{n+1} \oplus F^mJ^n \to F^mJ^n$$

Define $F^{\cdot}\phi^{\cdot} : F^{\cdot}\mathrm{Cone}(F^{\cdot}\varepsilon^{\cdot})^{\cdot} \to F^{\cdot}J^{\cdot}$ by the composition:

$$F^{\cdot}\phi^{\cdot} = (F^{\cdot}g_1^{\cdot} - F^{\cdot}g_2^{\cdot}) \circ F^{\cdot}\gamma^{\cdot}$$

It was shown in the proof of Lemma 6.3.4 that $\phi^{\cdot}$ is a chain map, and here it is clear that $F^{\cdot}\phi^{\cdot}$ is a filtered chain map because it is a composition of filtered morphisms. Because $F^{\cdot}\varepsilon^{\cdot}$ is a strong filtered quasi-isomorphism, $F^{\cdot}\mathrm{Cone}(F^{\cdot}\varepsilon^{\cdot})^{\cdot}$ is strong filtered acyclic by Lemma 6.4.4. Because $F^{\cdot}A^{\cdot}$ and $F^{\cdot}J^{\cdot}$ are both exhaustively filtered, so is $F^{\cdot}\mathrm{Cone}(F^{\cdot}\varepsilon^{\cdot})^{\cdot}$. We can now

[5] https://stacks.math.columbia.edu/tag/05TY

invoke Lemma 6.5.3 to obtain that $F^{\cdot}\phi^{\cdot}$ is filtered homotopic to zero by filtered homotopy $F^{\cdot}h^{\cdot}$ : $F^{\cdot}\mathrm{Cone}(F^{\cdot}\varepsilon^{\cdot})^{\cdot} \to F^{\cdot}I[-1]^{\cdot}$. By Lemma 6.4.2, we have the injective filtered chain map, $F^{\cdot}\iota_J^{\cdot}$ : $F^{\cdot}J^{\cdot} \hookrightarrow F^{\cdot}\mathrm{Cone}(F^{\cdot}\varepsilon)^{\cdot}$ from Lemma 6.4.2. By Lemma 6.1.23, $F^{\cdot}\phi^{\cdot} \circ F^{\cdot}\iota_J^{\cdot}$ is filtered homotopic to zero. But as was the case in Lemma 6.3.4, we have $F^{\cdot}\phi^{\cdot} \circ F^{\cdot}\iota_J^{\cdot} = F^{\cdot}g_1^{\cdot} - F^{\cdot}g_2^{\cdot}$.

□

## 6.6 Bifiltrations

In this section I will define a bifiltered object and work out a few lemmas. Bifiltered objects arise when one takes the tensor product of two filtered objects over a field.

**Definition 6.6.1.** *Let $\mathcal{A}$ be an abelian category. A* bifiltered object *of $\mathcal{A}$ is an object $A$ of $\mathcal{A}$ and a collection of subjects $F^{p,q}A$ of $A$, such that for all $p,q,p',q' \in \mathbb{Z}$, $F^{p,q}A \cap F^{p',q'}A = F^{\max(p,p'),\max(q,q')}A$, where the intersection is taken inside of $A$. A morphism of bifiltered objects $F^{\cdot,\cdot}f : F^{\cdot,\cdot}A \to F^{\cdot,\cdot}B$ is a morphism $f : A \to B$ in $\mathcal{A}$ such that $f$ restricts to a morphism $F^{p,q}f$ : $F^{p,q}A \to F^{p,q}B$ for all $p,q \in \mathbb{Z}$. The category of bifiltered objects of $\mathcal{A}$ with bifiltered morphisms is denoted BiFil$(\mathcal{A})$.*

**Definition 6.6.2.** *Let $\mathcal{A}$ be an abelian category. A* bifiltered complex *of $\mathcal{A}$ is a bifiltered object in the category Comp$(\mathcal{A})$. That is, $A^{\cdot}$ is a complex in $\mathcal{A}$ and for every $p,q \in \mathbb{Z}$, $F^{p,q}A^{\cdot}$ is a subcomplex of $A^{\cdot}$. For all $p,q,p',q' \in \mathbb{Z}$, one has $F^{p,q}A^{\cdot} \cap F^{p',q'}A^{\cdot} = F^{\max(p,p'),\max(q,q')}A^{\cdot}$, where the intersection is taken inside of $A^{\cdot}$. A bifiltered chain map $F^{\cdot,\cdot}f^{\cdot} : F^{\cdot,\cdot}A^{\cdot} \to F^{\cdot,\cdot}B^{\cdot,\cdot}$ is a morphism in the category BiFil(Comp$(\mathcal{A})$). That is, $F^{\cdot,\cdot}f^{\cdot}$ is a chain map $f^{\cdot} : A^{\cdot} \to B^{\cdot}$ such that $f^{\cdot}$ restricts to a chain map $F^{p,q}f^{\cdot} : F^{p,q}A^{\cdot} \to F^{p,q}B^{\cdot}$ for all $p,q \in \mathbb{Z}$.*

**Definition 6.6.3.** *Let $\mathcal{A}$ be an abelian category with sums and let $F^{\cdot,\cdot}A$ be a bifiltered object of $\mathcal{A}$. Then one can define the* total filtration *of $A$ as follows:*

$$F^{m}A = \sum_{p+q=m} F^{p,q}A$$

*for all $m \in \mathbb{Z}$, where the sum is taken inside of the object $A$. In this way, any object of BiFil$(\mathcal{A})$ can be seen as an object of Fil$(\mathcal{A})$, as well as any morphism in BiFil$(\mathcal{A})$. The terms defined in Definition 6.1.2 will apply to $F^{\cdot,\cdot}A$ if they apply to the total filtration of $A$, $F^{\cdot}A$.*

**Definition 6.6.4.** *Let $\mathcal{A}$ be an abelian category and let $F^{\cdot,\cdot}A$ be a bifiltered object of $\mathcal{A}$. In addition to the total filtration, we define the following two filtrations of A. The* vertical filtration *of A is defined as follows, where for $m \in \mathbb{Z}$:*

$$F_v^m A = \bigcup_{p \in \mathbb{Z}} F^{p,m} A$$

*The* horizontal filtration *of A is defined by:*

$$F_h^m A = \bigcup_{q \in \mathbb{Z}} F^{m,q} A$$

**Definition 6.6.5.** *Let $\mathcal{A}$ be an abelian category with sums and let $F^{\cdot,\cdot}A$ be a bifiltered object of $\mathcal{A}$. Then one can define the total filtration of $A^{\cdot}$ as done in Definition 6.6.3 where one sees $A^{\cdot}$ as an object in $Comp(\mathcal{A})$. In each degree $n \in \mathbb{Z}$, and for all $m \in \mathbb{Z}$, we have:*

$$F^m A^n = \sum_{p+q=m} F^{p,q} A^n$$

**Lemma 6.6.6.** *Let $\mathcal{A}$ be the category of abelian groups, and suppose $F^{\cdot}A$ is a filtered object of $\mathcal{A}$. Suppose there are elements $x_i \in F^{m_i}A$ for $i = 1, \dots, k$, with $\sum_{i=1}^k x_i = 0$ in A. Then for each i, $x_i \in F^{M_i}A$, where $M_i = \min(\{m_j \mid j \neq i\})$.*

*Proof.* Choose an $i \in \{1, \dots, k\}$. We have:

$$x_i = -\sum_{j \neq i} x_j \in \sum_{j \neq i} F^{m_j} A = \bigcup_{j \neq i} F^{m_j} A = F^{M_i} A$$

$\square$

**Lemma 6.6.7.** *Let $\mathcal{A}$ be the category of abelian groups and suppose $F^{\cdot,\cdot}A$ is a bifiltered object of $\mathcal{A}$. Then for all $m \in \mathbb{Z}$, there is an exact sequence:*

$$0 \longrightarrow \bigoplus_{p+q=m+1} F^{p,q}A \xhookrightarrow{\iota} \bigoplus_{p+q=m} F^{p,q}A \xrightarrow{\pi} F^m A \longrightarrow 0$$

*where $\pi$ is the summation map, $\oplus_{p+q=m} F^{p,q}A \to \sum_{p+q=m} F^{p,q}A$, and $\iota$ is the sum of the maps:*

$$\iota^{p,q} : F^{p,q}A \to F^{p-1,q}A \oplus F^{p,q-1}A \qquad x \mapsto (x, -x)$$

*That is, for $(w^{p,q})_{p+q=m+1} \in \oplus_{p+q=m+1} F^{p,q}A$, we have:*

$$\iota(w)^{p,q} = w^{p+1,q} - w^{p,q+1}$$

*for all $p+q = m$.*

*Proof.* Abbreviate $S = \bigoplus_{p+q=m+1} F^{p,q}A$ and $T = \bigoplus_{p+q=m} F^{p,q}A$. We have that $\pi$ is surjective by definition. It is also straightforward to show that $\mathrm{im}(\iota) \subseteq \ker(\pi)$. Let $(w^{p,q})_{p+q=m+1} \in S$. Then:

$$\pi(\iota(w)) = \sum_{p+q=m} \iota(w)^{p,q} = \sum_{p+q=m} (w^{p+1,q} - w^{p,q+1}) = 0$$

To show right exactness, it remains to show $\ker(\pi) \subseteq \mathrm{im}(\iota)$. Let $(w^{p,q})_{p+q=m} \in \ker(\pi) \subseteq T$. There are only finitely many $p, q$ such that $w^{p,q} \neq 0$, since $w$ is an element of a direct sum. So we may assume that $(w^{p,q})_{p+q=m} = (w^{k+i,m-k-i})_{i=0,\ldots,n}$, for some $k \in \mathbb{Z}$ and $n \geq 0$. If all $w^{\cdot,\cdot}$ are zero we are done. Otherwise, because the sum is zero, there must be at least two non-zero $w$ values, hence $n \geq 1$. Suppose first that $n = 1$ and $(w^{p,q})_{p+q=m} = (w^{k,m-k}, w^{k+1,m-k-1})$. We have $w^{k,m-k} + w^{k+1,m-k-1} = 0$, So

$$w^{k,m-k} = -w^{k+1,m-k-1} \in F^{k,m-k}A \cap F^{k+1,m-k-1}A = F^{k+1,m-k}A$$

By setting $v^{k+1,m-k} = w^{k,m-k} \in F^{k+1,m-k}A$ in the $(k+1, m-k)$ component of $S$, we get:

$$\iota(v^{k+1,m-k}) = (w^{k,m-k}, -w^{k,m-k}) = (w^{k,m-k}, w^{k+1,m-k-1}) = (w^{p,q})_{p+q=m}$$

So $(w) \in \mathrm{im}(\iota)$ in the case $n = 1$. Now suppose $(w^{p,q})_{p+q=m} = (w^{k+i,m-k-i})_{i=0,\ldots,n}$ for a $n > 1$. I will reduce to the case $(w^{p,q})_{p+q=m} = (w^{k+i,m-k-i})_{i=1,\ldots,n}$. Consider $w^{k,m-k} \in F^{k,m-k}A$.

I claim we have $w^{k,m-k} \in F^{k+1,m-k}$. In the horizontal filtration, we have $w^{k+i,m-k-i} \in F_h^{k+i}(A)$, for $i = 0, \ldots, n$. Because $\sum_{i=0}^{n}(w^{k+i,m-k-i}) = 0$ in $A$, we may invoke Lemma 6.6.6 to get that $w^{k,m-k} \in F_h^{k+1}A$, since $\min\{k+i \mid i \neq 0\} = k+1$. Thus we have:

$$\begin{aligned}
w^{k,m-k} &\in F_h^{k+1}A \cap F^{k,m-k}(A) \\
&= (\bigcup_{q\in\mathbb{Z}} F^{k+1,q}A) \cap F^{k,m-k}(A) \\
&= \bigcup_{q\in\mathbb{Z}} (F^{k+1,q}A \cap F^{k,m-k}(A)) \\
&= \bigcup_{q\in\mathbb{Z}} F^{k+1,\max(q,m-k)}A \\
&= F^{k+1,m-k}A
\end{aligned}$$

Set $v^{k+1,m-k} = w^{k,m-k}$. We have

$$\iota(v^{k+1,m-k}) = (w^{k,m-k}, -w^{k,m-k}) \in F^{k,m-k}A \oplus F^{k+1,m-k-1}A$$

Let $(\tilde{w}^{\cdot,\cdot}) = (w^{\cdot,\cdot}) - \iota(v^{k+1,m-k})$, and note that $\tilde{w}^{k,m-k} = 0$. Therefore:

$$(\tilde{w}^{p,q})_{p+q=m} = (\tilde{w}^{(k+1)+i,m-(k+1)-i})_{i=0,\ldots,n-1}$$

Thus, by using $(\tilde{w}^{\cdot,\cdot})$ in place of $(w^{\cdot,\cdot})$, we reduce to the case where $(w^{\cdot,\cdot})$ has at most $n-1$ non-zero components. With this reduction we are done, and we have shown $\ker(\pi) \subseteq \operatorname{im}(\iota)$.

At this point we have shown that the sequence is right exact. To show left exactness we show $\iota$ is injective. Suppose $(w^{p,q})_{p+q=m+1} \in \ker(\iota)$. I must show $w^{p,q} = 0 \in F^{p,q}A$ for all $p+q = m+1$. For $p,q \in \mathbb{Z}$ with $p+q = m$, we have:

$$\iota(w)^{p,q} = w^{p+1,q} - w^{p,q+1} = 0 \in F^{p,q}A$$

since $\iota(w) = 0$, so we have the relation $w^{p+1,q} = w^{p,q+1}$ for all $p+q = m$. Thus by transitivity, $(w^{p,q})_{p+q=m+1}$ has the same entry for each component. But since only finitely many terms are nonzero, this forces $w^{p,q} = 0$ for all $p+q = m+1$. Thus, $(w^{p+q})_{p+q=m+1} = 0$, and we have shown $\iota$ is injective. Now left exactness is shown as well, so the sequence is exact.

□

**Corollary 6.6.8.** *Let $\mathcal{A}$ be the category of sheaves of abelian groups on a topological space $X$, and let $F^{\cdot,\cdot}A$ be a bifiltered object of $X$. Let $m \in \mathbb{Z}$. Then there is an exact sequence:*

$$0 \longrightarrow \bigoplus_{p+q=m+1} F^{p,q}A \xhookrightarrow{\iota} \bigoplus_{p+q=m} F^{p,q}A \xrightarrow{\pi} F^m A \longrightarrow 0$$

*where $\iota$ and $\pi$ are as defined in Lemma 6.6.7.*

*Proof.* Let $x \in X$. We have the following sequence of abelian groups:

$$0 \longrightarrow \bigoplus_{p+q=m+1} F^{p,q}A_x \xrightarrow{\iota} \bigoplus_{p+q=m} F^{p,q}A_x \xrightarrow{\pi} F^m A_x \longrightarrow 0$$

By Lemma 6.6.7, this sequence is exact. Since the sequence

$$0 \longrightarrow \bigoplus_{p+q=m+1} F^{p,q}A \xrightarrow{\iota} \bigoplus_{p+q=m} F^{p,q}A \xrightarrow{\pi} F^m A \longrightarrow 0$$

is exact on all stalks, it is exact.

□

**Corollary 6.6.9.** *Let $\mathcal{A}$ be the category of sheaves of abelian groups on topological space $X$. Let $F^{\cdot,\cdot}A^{\cdot}$ be a bifiltered complex in $\mathcal{A}$. Then for all $m \in \mathbb{Z}$ we have the exact sequence of complexes in $\mathcal{A}$:*

$$0 \longrightarrow \bigoplus_{p+q=m+1} F^{p,q}A^{\cdot} \xrightarrow{\iota^{\cdot}} \bigoplus_{p+q=m} F^{p,q}A^{\cdot} \xrightarrow{\pi^{\cdot}} F^m A^{\cdot} \longrightarrow 0$$

*Proof.* For each $n \in \mathbb{Z}$ we have the following exact sequence from Corollary 6.6.8:

$$0 \longrightarrow \bigoplus_{p+q=m+1} F^{p,q}A^n \xrightarrow{\iota^n} \bigoplus_{p+q=m} F^{p,q}A^n \xrightarrow{\pi^n} F^m A^n \longrightarrow 0$$

Because $\pi^{\cdot}$ and $\iota^{\cdot}$ are both naturally chain maps, the result is shown.

□

**Definition 6.6.10.** *Let $\mathcal{A}$ be an abelian category and let $F^{\cdot,\cdot}A^{\cdot}$ and $F^{\cdot,\cdot}B^{\cdot}$ be bifiltered complexes of $\mathcal{A}$. A* strong bifiltered quasi-isomorphism *from $F^{\cdot,\cdot}A^{\cdot}$ to $F^{\cdot,\cdot}B^{\cdot}$ is a bifiltered chain map $F^{\cdot,\cdot}\varepsilon^{\cdot}$ : $F^{\cdot,\cdot}A^{\cdot} \to F^{\cdot,\cdot}B^{\cdot}$ such that $F^{p,q}\varepsilon^{\cdot} : F^{p,q}A^{\cdot} \to F^{p,q}B^{\cdot}$ is a quasi-isomorphism for all $p,q \in \mathbb{Z}$.*

**Lemma 6.6.11.** *Let $\mathcal{A}$ be the category of sheaves of abelian groups on a topological space $X$, let $F^{\cdot,\cdot}A^{\cdot}$ and $F^{\cdot,\cdot}B^{\cdot}$ be bifiltered complexes in $\mathcal{A}$, and let $F^{\cdot,\cdot}\varepsilon^{\cdot} : F^{\cdot,\cdot}A^{\cdot} \to F^{\cdot,\cdot}B^{\cdot}$ be a strong bifiltered quasi-isomorphism. Then $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}B^{\cdot}$ is a strong filtered quasi-isomorphism, where $F^{\cdot}$ denotes the total filtration.*

*Proof.* Let $m \in \mathbb{Z}$. By Corollary 6.6.8, the rows in the diagram below are exact:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{p+q=m+1} F^{p,q}A^{\cdot} \arrow[r, hook, "\iota_A^{\cdot}"] \arrow[d, "\bigoplus_{p+q=m+1} F^{p,q}\varepsilon^{\cdot}"] & \bigoplus_{p+q=m} F^{p,q}A^{\cdot} \arrow[r, two heads, "\pi_A^{\cdot}"] \arrow[d, "\bigoplus_{p+q=m} F^{p,q}\varepsilon^{\cdot}"] & F^m A^{\cdot} \arrow[r] \arrow[d, "F^m\varepsilon^{\cdot}"] & 0 \\
0 \arrow[r] & \bigoplus_{p+q=m+1} F^{p,q}B^{\cdot} \arrow[r, hook, "\iota_B^{\cdot}"] & \bigoplus_{p+q=m} F^{p,q}B^{\cdot} \arrow[r, two heads, "\pi_B^{\cdot}"] & F^m B^{\cdot} \arrow[r] & 0
\end{tikzcd}$$

and the squares are commutative. Since direct sums of quasi-isomorphisms are quasi-isomorphisms, the left and middle downward arrows are quasi-isomorphisms. Hence, so is the third. Since $F^m\varepsilon^{\cdot}$ is a quasi-isomorphism for all $m \in \mathbb{Z}$, we have $F^{\cdot}\varepsilon^{\cdot}$ is a strong filtered quasi-isomorphism.

□

## 6.7 Tensor Products

For the entirety of this chapter I will assume the tensor product preserves subobject relations. That is, if $A \hookrightarrow C$ and $B \hookrightarrow D$ are injections, then I assume we have the injection:

$$A \otimes B \hookrightarrow C \otimes D$$

In other words I make the assumption that $\otimes$ is exact. This will be true when we are working in the category of sheaves of $k$ vector spaces on a topological space $X$, for $k$ a field.

**Definition 6.7.1.** *Let $\mathcal{A}$ be an abelian category with exact tensor product and sums. Let $F^{\cdot}A$ and $F^{\cdot}B$ be two filtered objects of $\mathcal{A}$. Then one induces a filtration on $A \otimes B$ as follows:*

$$F^m(A \otimes B) = \sum_{p+q=m} F^pA \otimes F^qB$$

*The sum above takes place inside of $A \otimes B$. Note because we have assumed $\otimes$ is exact, we have $F^pA \otimes F^qB$ is a subobject of $A \otimes B$ for all $p, q \in \mathbb{Z}$.*

**Definition 6.7.2.** *Let $\mathcal{A}$ be an abelian category with exact tensor product and sums. Let $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ be two filtered complexes in $\mathcal{A}$. Then one induces a filtration on the total complex $Tot^{\cdot}(A^{\cdot} \otimes B^{\cdot})$ as follows:*

$$F^m Tot^{\cdot}(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot}) = \sum_{p+q=m} Tot^{\cdot}(F^pA^{\cdot} \otimes F^qB^{\cdot})$$

*In each degree, one has:*

$$F^m Tot^n(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot}) = \bigoplus_{p'+q'=n} \left( \sum_{p+q=m} F^pA^{p'} \otimes F^qB^{q'} \right)$$

**Lemma 6.7.3.** *Let $\mathcal{A}$ be an abelian category with exact tensor product and sums, and suppose $F^{\cdot}A$ and $F^{\cdot}B$ are filtered objects of $\mathcal{A}$ with terminating filtrations. Then the induced filtration on $F^{\cdot}(F^{\cdot}A \otimes F^{\cdot}B)$ terminates.*

*Proof.* There is a $m_1 \in \mathbb{Z}$ such that $F^mA = 0$ for all $m \geq m_1$, and there is a $m_2 \in \mathbb{Z}$ such that $F^mB = 0$ for all $m \geq m_2$. I claim the filtration on $F^{\cdot}(F^{\cdot}A \otimes F^{\cdot}B)$ terminates at $m_1 + m_2$. We have:

$$
\begin{aligned}
F^{m_1+m_2}(F^{\cdot}A \otimes F^{\cdot}B) &= \sum_{p+q=m_1+m_2} F^pA \otimes F^qB \\
&= \sum_{i\in\mathbb{Z}} F^{m_1+i}A \otimes F^{m_2-i}B \\
&= \sum_{i\geq 0}(F^{m_1+i}A \otimes F^{m_2-i}B) + \sum_{i<0}(F^{m_1+i}A \otimes F^{m_2-i}B) \\
&= \sum_{i\geq 0}(0 \otimes F^{m_2-i}B) + \sum_{i<0}(F^{m_1+i}A \otimes 0) = 0
\end{aligned}
$$

□

**Lemma 6.7.4.** *Let $\mathcal{A}$ be an abelian category with exact tensor products and sums, and suppose $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ are filtered complexes whose filtrations terminate in each degree, and $A^{\cdot}$ and $B^{\cdot}$ are either both bounded below or both bounded above. Then the induced filtration on $F^{\cdot}(Tot^{\cdot}(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot}))$ terminates in each degree.*

*Proof.* Let $n \in \mathbb{Z}$. Since we have assumed that $A^{\cdot}$ and $B^{\cdot}$ are either both bounded below or both bounded above, there are only finitely many $p, q \in \mathbb{Z}$ with $p+q=n$ that satisfy $A^p \neq 0$ and $B^q \neq 0$. Because $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ have terminating filtrations in each degree, for each of these finite values for $p$ and $q$, there are $m_p \in \mathbb{Z}$ and $m'_q \in \mathbb{Z}$ such that $F^{m_p}A^p = 0$ and $F^{m'_q}B^q = 0$. Because there are only finitely many $m_p$ and $m'_q$, we may set $M = \max(\{m_p, m'_q | p+q=n,\ A^p \neq 0,\ B^q \neq 0\})$. That is, we choose $M$ large enough so that $F^MA^p = 0$ and $F^MB^q = 0$ for all $p+q=n$ with $A^p \neq 0$, $B^q \neq 0$. We then have:

$$
\begin{aligned}
F^{2M}\mathrm{Tot}^n(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot}) &= \sum_{i+j=2M,\ p+q=n} F^iA^p \otimes F^jB^q \\
&= \sum_{i\in\mathbb{Z},\ p+q=n} F^{M+i}A^p \otimes F^{M-i}B^q \\
&= \sum_{i\geq 0,\ p+q=n}(F^{M+i}A^p \otimes F^{M-i}B^q) + \sum_{i<0,\ p+q=n}(F^{M+i}A^p \otimes F^{M-i}B^q) \\
&= \sum_{i\geq 0,\ p+q=n}(0 \otimes F^{M-i}B^q) + \sum_{i<0,\ p+q=n}(F^{M+i}A^p \otimes 0) = 0
\end{aligned}
$$

So the filtration on $F^{\cdot}\mathrm{Tot}^{\cdot}(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot})$ terminates in each degree. $\square$

**Lemma 6.7.5.** *Let $\mathcal{A}$ be an abelian category with exact tensor product and sums, and suppose $F^{\cdot}A$ and $F^{\cdot}B$ are filtered objects of $\mathcal{A}$ whose filtrations begin. Then $F^{\cdot}(A \otimes B)$ has a filtration that begins.*

*Proof.* There are $m_1, m_2 \in \mathbb{Z}$ such that $F^{m_1}A = A$ and $F^{m_2}B = B$. Thus:

$$A \otimes B = F^{m_1}A \otimes F^{m_2}B \subseteq \sum_{p+q=m_1+m_2} F^pA \otimes F^qB = F^{m_1+m_2}(A \otimes B)$$

Thus $F^{m_1+m_2}(A \otimes B) = A \otimes B$, so the filtration on $F^{\cdot}(A \otimes B)$ begins. $\square$

**Lemma 6.7.6.** *Let $\mathcal{A}$ be an abelian category with exact tensor product and sums, and let $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ be filtered complexes in $\mathcal{A}$ whose filtrations begin in each degree and are either both bounded above or both bounded below. Then the filtration on $F^{\cdot}Tot^{\cdot}(A^{\cdot} \otimes B^{\cdot})$ begins in each degree.*

*Proof.* Let $n \in \mathbb{Z}$. Because $A^{\cdot}$ and $B^{\cdot}$ are either both bounded above or both bounded below, there are only finitely many non-zero terms in the direct sum below:

$$\mathrm{Tot}^n(A^{\cdot} \otimes B^{\cdot}) = \bigoplus_{p+q=n} A^p \otimes B^q$$

For all $p \in \mathbb{Z}$ there is an $i_p \in \mathbb{Z}$ such that $F^{i_p}A^p = A^p$, and for all $q \in \mathbb{Z}$ there is a $j_q \in \mathbb{Z}$ such that $F^{j_q}B^q = B^q$. Set:

$$M = \min\{i_p, j_q \mid A^p \neq 0 \text{ and } B^q \neq 0\}$$

We then have:

$$\begin{aligned}
\mathrm{Tot}^n(A^{\cdot} \otimes B^{\cdot}) &= \bigoplus_{p+q=n} A^p \otimes B^q \\
&= \bigoplus_{p+q=n} F^{i_p}A^p \otimes F^{j_q}B^q \\
&\subseteq \bigoplus_{p+q=n} F^M A^p \otimes F^M B^q
\end{aligned}$$

$$\subseteq F^{2M}\mathrm{Tot}^n(A^{\cdot}\otimes B^{\cdot})$$

So the filtration on $F^{\cdot}\mathrm{Tot}^{\cdot}(A^{\cdot}\otimes B^{\cdot})$ begins in each degree. □

**Corollary 6.7.7.** *Let $\mathcal{A}$ be an abelian category with exact tensor product and sums. Let $F^{\cdot}A$ and $F^{\cdot}B$ be two finitely filtered objects of $\mathcal{A}$. Then $F^{\cdot}(A\otimes B)$ is finitely filtered.*

*Proof.* Follows from Lemmas 6.7.3 and 6.7.5. □

**Corollary 6.7.8.** *Let $\mathcal{A}$ be an abelian category with exact tensor product and sums. Let $F^{\cdot}A^{\cdot}$ be a finitely filtered complex in $\mathcal{A}$. Then for all $k\geq 1$, $F^{\cdot}Tot^{\cdot}((F^{\cdot}A^{\cdot})^{[k]})$ is finitely filtered.*

*Proof.* Inductive application of Corollary 6.7.7. □

**Corollary 6.7.9.** *Let $\mathcal{A}$ be an abelian category with exact tensor product and sums. Let $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ be two filtered complexes in $\mathcal{A}$ that are finitely filtered in each degree, and either both bounded above or both bounded below. Then $F^{\cdot}(A^{\cdot}\otimes B^{\cdot})$ is finitely filtered in each degree.*

*Proof.* Follows from Lemmas 6.7.4 and 6.7.6. □

**Lemma 6.7.10.** *For simplicity, let $\mathcal{A}$ be the category of sheaves of $k$ vector spaces on a topological space $X$. Let $F^{\cdot}A$ and $F^{\cdot}B$ be two filtered objects of $\mathcal{A}$ with exhaustive filtrations. Then the induced filtration on $F^{\cdot}(F^{\cdot}A\otimes F^{\cdot}B)$ is exhaustive.*

*Proof.* I must show $A\otimes B\subseteq\cup_{m\in\mathbb{Z}}F^m(F^{\cdot}A\otimes F^{\cdot}B)$. This containment may be checked on the stalks. Let $x\in X$. I claim $A_x\otimes B_x\subseteq\cup_{m\in\mathbb{Z}}F^m(F^{\cdot}A_x\otimes F^{\cdot}B_x)$. Let $z\in A_x\otimes B_x$. We can write $z=\sum_{i=1}^k a_i\otimes b_i$, for $a_i\in A_x$ and $b_i\in B_x$. Since the filtrations on $F^{\cdot}A_x$ and $F^{\cdot}B_x$ are exhaustive, there are $m_i\in\mathbb{Z}$ and $m_i'\in\mathbb{Z}$ such that $a_i\in F^{m_i}A_x$ and $b_i\in F^{m_i'}B_x$. Since there are only finitely many $m_i,m_i'$, we may set $M=\min_{i=1}^k(m_i)$ and $M'=\min_{i=1}^k(m_i')$. Since each $a_i\in F^MA$ and $b_i\in F^{M'}B$, we have $z\in F^MA_x\otimes F^{M'}B_x$. Thus, $z\in F^{M+M'}(F^{\cdot}A_x\otimes F^{\cdot}B_x)$. Since $z$ was an arbitrary element of $A_x\otimes B_x$, we have shown $A_x\otimes B_x\subseteq\cup_{m\in\mathbb{Z}}F^m(F^{\cdot}A_x\otimes F^{\cdot}B_x)$. Since this containment holds for all $x\in X$, we have shown $A\otimes B\subseteq\cup_{m\in\mathbb{Z}}F^m(F^{\cdot}A\otimes F^{\cdot}B)$. Thus $F^{\cdot}(F^{\cdot}A\otimes F^{\cdot}B)$ is exhaustively filtered.

□

**Lemma 6.7.11.** *For simplicity, let $\mathcal{A}$ be the category of sheaves of k vector spaces on a topological space $X$. Let $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ be filtered complexes in $\mathcal{A}$ that are exhaustively filtered as in Definition 6.1.4 (4). Then $F^{\cdot}Tot^{\cdot}(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot})$ is exhaustively filtered.*

*Proof.* Let $n \in \mathbb{Z}$. I must show $\mathrm{Tot}^n(A^{\cdot} \otimes B^{\cdot}) \subseteq \cup_{m\in\mathbb{Z}} F^m \mathrm{Tot}^n(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot})$. For each $p, q \in \mathbb{Z}$ with $p+q=n$, we have $F^{\cdot}A^p$ and $F^{\cdot}B^q$ are exhaustively filtered. By Lemma 6.7.10, we have $A^p \otimes B^q \subseteq \cup_{m\in\mathbb{Z}} F^m(F^{\cdot}A^p \otimes F^{\cdot}B^q) \subseteq \cup_{m\in\mathbb{Z}} F^m \mathrm{Tot}^n(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot})$. Since $Tot^n(A^{\cdot} \otimes B^{\cdot}) = \oplus_{p+q=n} A^p \otimes B^q$, this shows $\mathrm{Tot}^n(A^{\cdot} \otimes B^{\cdot}) \subseteq \cup_{m\in\mathbb{Z}} F^m \mathrm{Tot}^n(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot})$. Thus $F^{\cdot}\mathrm{Tot}^{\cdot}(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot})$ is exhaustively filtered in each degree. $\square$

**Corollary 6.7.12.** *Let $\mathcal{A}$ be the category of sheaves of k vector spaces on a topological space $X$, and suppose $F^{\cdot}A$ and $F^{\cdot}B$ are two inductively filtered objects of $\mathcal{A}$ as in Definition 6.1.2 (6). Then $F^{\cdot}(F^{\cdot}A \otimes F^{\cdot}B)$ is inductively filtered.*

*Proof.* By Lemma 6.7.3, $F^{\cdot}(F^{\cdot}A \otimes F^{\cdot}B)$ has a terminating filtration, and by Lemma 6.7.10, $F^{\cdot}(F^{\cdot}A \otimes F^{\cdot}B)$ has an exhaustive filtration. Thus, $F^{\cdot}(F^{\cdot}A \otimes F^{\cdot}B)$ is inductively filtered. $\square$

**Corollary 6.7.13.** *Let $\mathcal{A}$ be the category of sheaves of k vector spaces on a topological space $X$, and suppose $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ are inductively filtered in each degree, with $A^{\cdot}$ and $B^{\cdot}$ either both bounded below or both bounded above. Then $F^{\cdot}Tot^{\cdot}(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot})$ is inductively filtered.*

*Proof.* Because $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ both have filtrations that terminate in each degree, and $A^{\cdot}$ and $B^{\cdot}$ are either both bounded below or both bounded above, we may apply Lemma 6.7.4 to obtain that the filtration on $F^{\cdot}\mathrm{Tot}^{\cdot}(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot})$ terminates in each degree. Since the filtrations on $F^{\cdot}A^{\cdot}$ and $F^{\cdot}B^{\cdot}$ are both exhaustive, the filtration on $F^{\cdot}(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot})$ is exhaustive by Lemma 6.7.11. Thus, the filtration on $F^{\cdot}\mathrm{Tot}^{\cdot}(F^{\cdot}A^{\cdot} \otimes F^{\cdot}B^{\cdot})$ is inductive in each degree. $\square$

The two Lemmas below are elementary.

**Lemma 6.7.14.** *Let $k$ be a field and $V$ a $k$ vector space. Suppose $\{v_i\}_{i\in I}$ is a linearly independent subset of $V$. Let $J, K \subset I$. Then we have equality of subspaces of $V$:*

$$Span(v_i)_{i\in J} \cap Span(v_i)_{i\in K} = Span(v_i)_{i\in J\cap K}$$

*Proof.* The containment $\mathrm{Span}(v_i)_{i\in J\cap K} \subseteq \mathrm{Span}(v_i)_{i\in J} \cap \mathrm{Span}(v_i)_{i\in K}$ is clear. Suppose:

$$w \in \mathrm{Span}(v_i)_{i\in J} \cap \mathrm{Span}(v_i)_{i\in K}$$

Then there are $c_i \in k$ and $d_i \in k$ such that:

$$w = \sum_{i\in J} c_i v_i = \sum_{i\in K} d_i v_i$$

Thus we have:

$$0 = \sum_{i\in J\setminus K} c_i v_i + \sum_{i\in J\cap K} (c_i - d_i) v_i + \sum_{i\in J\setminus K} -d_i v_i$$

Since $\{v_i\}_{i\in I}$ is a linearly independent subset of $V$, this implies $c_i = 0$ for $i \in J\setminus K$, $c_i = d_i$ for $i \in J\cap K$, and $d_i = 0$ for $i \in J\setminus K$. Thus $w = \sum_{i\in J\cap K} c_i v_i \in \mathrm{Span}(v_i)_{i\in J\cap K}$ and this completes the proof.

$\square$

**Lemma 6.7.15.** *Let $k$ be a field, $V,W$ $k$ vector spaces, and let $A,C \subseteq V$ and $B,D \subseteq W$ be subspaces. Then we have the equality of subspaces of $V \otimes_k W$:*

$$(A \otimes_k B) \cap (C \otimes_k D) = (A\cap C) \otimes_k (B \cap D)$$

*Proof.* Let $\{v_i\}_{i\in L_1}$ be a basis for $A\cap C$. Let $\{v_i\}_{i\in J_1}$ be a basis for a complement to $A\cap C$ inside of $A$. Let $\{v_i\}_{i\in K_1}$ be a basis for a complement to $A\cap C$ inside of $C$. Then $\{v_i\}_{i\in L_1\cup J_1}$ is a basis for $A$ and $\{v_i\}_{i\in L_1\cup K_1}$ is a basis for $C$. Similarly let $\{w_i\}_{i\in L_2}$ be a basis for $B\cap D$. Let $\{w_i\}_{i\in J_2}$ be a basis for a complement to $B\cap D$ inside of $B$. Let $\{w_i\}_{i\in K_2}$ be a basis to a complement to $B\cap D$ inside of $D$. Then $\{w_i\}_{i\in L_2\cup J_2}$ is a basis for $B$, and $\{w_i\}_{i\in L_2\cup K_2}$ is a basis for $D$. We have the basis for $(A\cap C)\otimes(B\cap D)$:

$$\beta_{(A\cap C)\otimes(B\cap D)} = \{v_i \otimes w_j \mid i \in L_1, j \in L_2\}$$

The following is a basis for $A \otimes B$:

$$\beta_{A\otimes B} = \{v_i \otimes w_j \mid i \in L_1 \cup J_1, j \in L_2 \cup J_2\}$$

And the following is a basis for $C \otimes D$:

$$\beta_{C\otimes D} = \{v_i \otimes w_j \mid i \in L_1 \cup K_1, j \in L_2 \cup K_2\}$$

By Lemma 6.7.14, we can compute a basis for $(A \otimes B) \cap (C \otimes D)$ as:

$$\begin{aligned}\beta_{(A\otimes B)\cap(C\otimes D)} &= \beta_{A\otimes B} \cap \beta_{C\otimes D} \\ &= \{v_i \otimes w_j \mid i \in L_1, j \in L_2\} \\ &= \beta_{(A\cap C)\otimes(B\cap D)}\end{aligned}$$

Since this set is a basis for both $(A \cap C) \otimes (B \cap D)$ and $(A \otimes B) \cap (C \otimes D)$, it follows $(A \cap C) \otimes (B \cap D) = (A \otimes B) \cap (C \otimes D)$.

□

**Corollary 6.7.16.** *Let $k$ be a field, $X$ a topological space, and suppose $F, G$ are sheaves of $k$ vector spaces on $X$, with $A, C \subseteq F$ and $B, D \subseteq G$ subsheaves. Then we have the equality of subsheaves of $F \otimes_k G$:*

$$(A \otimes_k B) \cap (C \otimes_k D) = (A \cap C) \otimes_k (B \cap D)$$

*Proof.* Let $x \in X$. Then $A_x, C_x \subseteq F_x$ and $B_x, D_x \subseteq G_x$ are subvector spaces. By Lemma 6.7.15, we have:

$$(A_x \otimes_k B_x) \cap (C_x \otimes_k D_x) = (A_x \cap C_x) \otimes_k (B_x \cap D_x)$$

Since this holds for all $x \in X$, we have the required equality of sheaves. □

**Lemma 6.7.17.** *Let $k$ be a field, and let $\mathcal{A} = Sh_k(X)$, the category of sheaves of $k$ vector spaces on a topological space $X$. Let $F^{\cdot}A$ and $F^{\cdot}B^{\cdot}$ be filtered objects of $\mathcal{A}$. Let $C = A \otimes_k B$. Then $F^{\cdot,\cdot}C$ is a bifiltered object, where one defines:*

$$F^{p,q}C = F^pA \otimes_k F^qB$$

*Proof.* It is clear that $F^pA \otimes_k F^qB$ is a subobject of $A \otimes_k B$ for all $p,q \in \mathbb{Z}$. Let $p,q,p',q' \in \mathbb{Z}$. It must be shown that:

$$(F^pA \otimes_k F^qB) \cap (F^{p'}A \otimes_k F^{q'}B) = F^{\max(p,p')}A \otimes_k F^{\max(q,q')}B$$

By Corollary 6.7.16, we have:

$$\begin{aligned}(F^pA \otimes_k F^qB) \cap (F^{p'}A \otimes_k F^{q'}B) &= (F^pA \cap F^{p'}A) \otimes_k (F^qB \cap F^{q'}B) \\ &= F^{\max(p,p')}A \otimes_k F^{\max(q,q')}B\end{aligned}$$

Thus, $F^{\cdot,\cdot}C$ is a bifiltered object.

$\square$

**Lemma 6.7.18.** *Let $k$ be a field, $X$ a topological space, and let $\mathcal{A} = Sh_k(X)$. Let $F^{\cdot}A^{\cdot}$, $F^{\cdot}B^{\cdot}$, $F^{\cdot}J^{\cdot}$, and $F^{\cdot}K^{\cdot}$ be filtered complexes in $\mathcal{A}$, with $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}J^{\cdot}$ and $F^{\cdot}\gamma : F^{\cdot}B^{\cdot} \to F^{\cdot}K^{\cdot}$ two strong filtered quasi-isomorphisms. Then:*

$$F^{\cdot,\cdot}(F^{\cdot}\varepsilon^{\cdot} \otimes_k F^{\cdot}\gamma^{\cdot}) : F^{\cdot,\cdot}Tot^{\cdot}(F^{\cdot}A^{\cdot} \otimes_k F^{\cdot}B^{\cdot}) \to F^{\cdot,\cdot}Tot^{\cdot}(F^{\cdot}J^{\cdot} \otimes_k F^{\cdot}K^{\cdot})$$

*is a strong bifiltered quasi-isomorphism as in Definition 6.6.10.*

*Proof.* Let $p,q \in \mathbb{Z}$. Since $F^{\cdot}\varepsilon^{\cdot}$ and $F^{\cdot}\gamma$ are both strong quasi-isomorphisms, $F^p\varepsilon^{\cdot} : F^pA^{\cdot} \to F^pJ^{\cdot}$ and $F^q\gamma : F^qB^{\cdot} \to F^qK^{\cdot}$ are both quasi-isomorphisms. By Lemma 3.2.16, we have:

$$\mathrm{Tot}^{\cdot}(F^p\varepsilon^{\cdot} \otimes_k F^q\gamma^{\cdot}) : \mathrm{Tot}^{\cdot}(F^pA^{\cdot} \otimes_k F^qB^{\cdot}) \to \mathrm{Tot}^{\cdot}(F^pJ^{\cdot} \otimes_k F^qK^{\cdot})$$

is a quasi-isomorphism. Since the above chain map is precisely $F^{p,q}(F^{\cdot}\varepsilon^{\cdot} \otimes_k F^{\cdot}\gamma^{\cdot})$ and $p,q \in \mathbb{Z}$ were arbitrary, this shows that $F^{\cdot,\cdot}(F^{\cdot}\varepsilon^{\cdot} \otimes_k F^{\cdot}\gamma^{\cdot})$ is a strong bifiltered quasi-isomorphism.

$\square$

**Corollary 6.7.19.** *Let $k$ be a field, $X$ a topological space, and let $\mathcal{A} = Sh_k(X)$. Let $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}J^{\cdot}$ and $F^{\cdot}\gamma : F^{\cdot}B^{\cdot} \to F^{\cdot}K^{\cdot}$ be strong filtered quasi-isomorphisms of filtered complexes in $\mathcal{A}$. Then*

$$F^{\cdot}(F^{\cdot}\varepsilon^{\cdot} \otimes_k F^{\cdot}\gamma^{\cdot}) : F^{\cdot}Tot^{\cdot}(F^{\cdot}A^{\cdot} \otimes_k F^{\cdot}B^{\cdot}) \to F^{\cdot}Tot^{\cdot}(F^{\cdot}J^{\cdot} \otimes_k F^{\cdot}K^{\cdot})$$

*is a strong filtered quasi-isomorphism.*

*Proof.* We have $F^{\cdot}(F^{\cdot}\varepsilon^{\cdot} \otimes_k F^{\cdot}\gamma^{\cdot})$ is precisely the total filtration of the bifiltered chain map, $F^{\cdot,\cdot}(F^{\cdot}\varepsilon^{\cdot} \otimes_k F^{\cdot}\gamma^{\cdot})$, which was shown to be a strong bifiltered quasi-isomorphism in Lemma 6.7.18. Since $F^{\cdot,\cdot}(F^{\cdot}\varepsilon^{\cdot} \otimes_k F^{\cdot}\gamma^{\cdot})$ is a strong bifiltered quasi-isomorphism, we have $F^{\cdot}(F^{\cdot}\varepsilon^{\cdot} \otimes_k F^{\cdot}\gamma^{\cdot})$ is a strong filtered quasi-isomorphism by Lemma 6.6.11, and this is what we wanted to show. □

**Corollary 6.7.20.** *Let $k$ be a field, $X$ a topological space, and let $\mathcal{A} = Sh_k(X)$. Let $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}J^{\cdot}$ and $F^{\cdot}\gamma^{\cdot} : F^{\cdot}B^{\cdot} \to F^{\cdot}K^{\cdot}$ be strong filtered resolutions in the sense of Definition 6.1.20. Then:*

$$F^{\cdot}(F^{\cdot}\varepsilon^{\cdot} \otimes_k F^{\cdot}\gamma^{\cdot}) : F^{\cdot}Tot^{\cdot}(F^{\cdot}A^{\cdot} \otimes_k F^{\cdot}B^{\cdot}) \to F^{\cdot}Tot^{\cdot}(F^{\cdot}J^{\cdot} \otimes_k F^{\cdot}K^{\cdot})$$

*is a strong filtered resolution.*

*Proof.* Because $F^{\cdot}\varepsilon^{\cdot}$ and $F^{\cdot}\gamma^{\cdot}$ are both strong filtered quasi-isomorphisms, $F^{\cdot}(F^{\cdot}\varepsilon^{\cdot} \otimes_k F^{\cdot}\gamma^{\cdot})$ is a strong filtered quasi-isomorphism by Corollary 6.7.19. And since $\varepsilon^{\cdot}$ and $\gamma^{\cdot}$ are both injective in each degree, $\varepsilon^{\cdot} \otimes_k \gamma^{\cdot} : \mathrm{Tot}^{\cdot}(A^{\cdot} \otimes_k B^{\cdot}) \to \mathrm{Tot}^{\cdot}(J^{\cdot} \otimes_k K^{\cdot})$ is injective in each degree by Lemma 3.2.16 (1). Thus, $F^{\cdot}(F^{\cdot}\varepsilon^{\cdot} \otimes_k F^{\cdot}\gamma^{\cdot})$ is a strong filtered resolution. □

**Corollary 6.7.21.** *Let $k$ be a field, $X$ a topological space, and let $\mathcal{A} = Sh_k(X)$. Let $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}J^{\cdot}$ be a strong filtered resolution and let $k \geq 1$. Then:*

$$F^{\cdot}Tot^{\cdot}((F^{\cdot}\varepsilon^{\cdot})^{[k]}) : F^{\cdot}Tot^{\cdot}((F^{\cdot}A^{\cdot})^{[k]}) \to F^{\cdot}Tot^{\cdot}((F^{\cdot}J^{\cdot})^{[k]})$$

*is a strong filtered resolution.*

*Proof.* Inductive application of Corollary 6.7.20. □

## 6.8 Filtered Injective Resolutions

This section will be concerned with the existence of filtered injective resolutions in abelian categories that have enough injectives. Because the lemmas of Section 6.5 are able to make claims with filtrations that aren't necessarily finite in each degree, it may be useful to develop this section with the same generality if possible. For now I just cite lemmas from [6], which stay within the context of complexes that are finitely filtered in each degree.

The below is from Definition 12.16.3 [6]. of [6].

**Definition 6.8.1.** *Let $\mathcal{A}$ be an abelian category. A morphism $f : A \to B$ of filtered objects in $\mathcal{A}$ is said to be strict if $f(F^iA) = f(A) \cap F^iB$ for all $i \in \mathbb{Z}$.*

The following is Lemma 13.26.2 [7] from [6]. Recall $\mathrm{Fil}^f(\mathcal{A})$ is the category of finitely filtered objects of $\mathcal{A}$.

**Lemma 6.8.2.** *Let $\mathcal{A}$ be an abelian category. An object $I$ of $\mathrm{Fil}^f(\mathcal{A})$ is filtered injective (as in Definition 6.1.17) if and only if there exist $a \leq b$, injective objects $I_n$, $a \leq n \leq b$ of $\mathcal{A}$ and an isomorphism $I \cong \oplus_{a \leq n \leq b} I_n$, such that $F^pI = \oplus_{n \geq p} I_n$.*

We have the following corollary.

**Corollary 6.8.3.** *Let $\mathcal{A}$ be an abelian category and let $F^{\cdot}I$ be a finitely filtered object of $\mathcal{A}$ that is filtered injective. Then $F^{\cdot}I$ is strong filtered injective, and $I$ is injective.*

*Proof.* Let $a, b$ and $I_n$ be as in Lemma 6.8.2, and let $p \in \mathbb{Z}$. We have $F^pI = \oplus_{p \geq n} I_n$ is a finite direct sum of injective objects, and hence, is injective. In the case $p = a$ we have $I = F^aI$ is an injective object. □

**Corollary 6.8.4.** *Let $\mathcal{A}$ be an abelian category, let $F^{\cdot}A^{\cdot}$ and $F^{\cdot}I^{\cdot}$ be filtered complexes in $\mathcal{A}$ that are finitely filtered in each degree and suppose $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}I^{\cdot}$ is a filtered injective resolution as in Definition 6.1.19. Then $F^{\cdot}\varepsilon^{\cdot}$ is a strong filtered resolution as in Definition 6.1.20 and $\varepsilon^{\cdot}$ is an injective resolution as in Definition 3.2.15.*

*Proof.* Because $F^{\cdot}\varepsilon^{\cdot}$ is a filtered quasi-isomorphism of complexes that are finitely filtered in each degree, we have by Corollary 6.1.13, that $F^{\cdot}\varepsilon^{\cdot}$ is a strong filtered quasi-isomorphism and $\varepsilon^{\cdot}$ is a quasi-isomorphism. Because $F^{\cdot}I^{\cdot}$ is filtered injective in each degree and finitely filtered in each degree, we can apply Corollary 6.8.3 in each degree to get that $F^{\cdot}I^{\cdot}$ is strong filtered

[6] https://stacks.math.columbia.edu/tag/0123
[7] https://stacks.math.columbia.edu/tag/05TP

injective in each degree and $I^{\cdot}$ is injective in each degree. We have by hypothesis that $I^{\cdot}$ is bounded below, and that $\varepsilon^{\cdot}$ is injective in each degree. At this point we can conclude by the definition that $F^{\cdot}\varepsilon^{\cdot}$ is a strong filtered injective resolution and $\varepsilon^{\cdot}$ is an injective resolution. $\square$

The below is Lemma 13.26.5 [8] from [6].

**Lemma 6.8.5.** *Let $\mathcal{A}$ be an abelian category with enough injectives. For any object $A$ of $Fil^f(\mathcal{A})$ there exists a strict monomorphism $A \to I$ where $I$ is a filtered injective object.*

The below is Lemma 13.26.6 [9] from [6].

**Lemma 6.8.6.** *Let $\mathcal{A}$ be an abelian category with enough injectives. For any object $A$ of $Fil^f(\mathcal{A})$ there exists a filtered quasi-isomorphism $A[0] \to I^{\cdot}$ where $I^{\cdot}$ is a complex of filtered injective objects with $I^n = 0$ for $n < 0$.*

The below is Lemma 13.26.9 [10] from [6].

**Lemma 6.8.7.** *Let $\mathcal{A}$ be an abelian category with enough injectives. For every $K^{\cdot} \in K^+(Fil^f(\mathcal{A}))$ there exists a filtered quasi-isomorphism $K^{\cdot} \to I^{\cdot}$ with $I^{\cdot}$ bounded below, each $I^n$ a filtered injective object, and each $K^n \to I^n$ a strict monomorphism.*

It should be noted that in [6], being finitely filtered in each degree is part of the definition of being filtered injective, and so in the above $F^{\cdot}I^{\cdot}$ is finitely filtered in each degree as well.

## 6.9 Hom and Resolutions

**Definition 6.9.1.** *Let $\mathcal{A}$ be an abelian category, and let $F^{\cdot}A$ and $F^{\cdot}B$ be filtered objects of $\mathcal{A}$. We can induce a filtration on $Hom(A,B)$ as follows, where for all $m \in \mathbb{Z}$, we define:*

$$F^m Hom(F^{\cdot}A, F^{\cdot}B) = \{f \in Hom(A,B) \mid f(F^iA) \subseteq F^{i+m}B,\ \forall i \in \mathbb{Z}\}$$

*That is, for each $i \in \mathbb{Z}$, $f$ restricts to a morphism $F^if : F^iA \to F^{i+m}B$. Note that a filtered morphism $F^{\cdot}f : F^{\cdot}A \to F^{\cdot}B$ is an element of $F^0Hom(F^{\cdot}A, F^{\cdot}B)$.*

[8] https://stacks.math.columbia.edu/tag/05TS
[9] https://stacks.math.columbia.edu/tag/05TT
[10] https://stacks.math.columbia.edu/tag/05TW

**Lemma 6.9.2.** *In the case that $A$ is not a filtered object of $\mathcal{A}$ we can give $A$ the trivial filtration of $F^0 = A$ and $F^1 A = 0$. In this case we have:*

$$F^m Hom(A, F^{\cdot} B) = Hom(A, F^m B)$$

*Proof.* Let $f \in F^m \mathrm{Hom}(A, F^{\cdot} B)$. Then we have $f(F^0 A) \in f(F^m B)$, and since $F^0 A = A$, this shows $f \in \mathrm{Hom}(A, F^m B)$. Now suppose $f \in \mathrm{Hom}(A, F^m B)$. Let $i \in \mathbb{Z}$. For $i > 0$, we have $F^i A = 0$, and we trivially have $f(F^i A) \subseteq F^{m+i}(B)$. For $i \leq 0$ we have:

$$f(F^i A) = f(A) \subseteq F^m B \subseteq F^{m+i} B$$

So we have $f \in \mathrm{Hom}^m(A, F^{\cdot} B)$. At this point we have shown the equality of sets, $F^m \mathrm{Hom}(A, F^{\cdot} B) = \mathrm{Hom}(A, F^m B)$.

$\square$

**Definition 6.9.3.** *Let $\mathcal{A}$ be an abelian category, and let $F^{\cdot} A^{\cdot}$ and $F^{\cdot} B^{\cdot}$ be filtered complexes in $\mathcal{A}$. Then one may induce a filtration on $Hom^{\cdot}(A^{\cdot}, B^{\cdot})$ as follows, where for $m, n \in \mathbb{Z}$:*

$$F^m Hom^n(F^{\cdot} A^{\cdot}, F^{\cdot} B^{\cdot}) = \prod_{i \in \mathbb{Z}} F^m Hom(F^{\cdot} A^i, F^{\cdot} B^{i+n})$$

*where $F^m Hom(A^i, B^{i+n})$ is defined in Definition 6.9.1. That is, if $f^{\cdot} \in F^m Hom^n(A^{\cdot}, B^{\cdot})$, we have $f^i : A^i \to B^{n+i}$ for all $i \in \mathbb{Z}$, and for all $j \in \mathbb{Z}$:*

$$f^i(F^j A^i) \subseteq F^{j+m} B^{i+n}$$

**Corollary 6.9.4.** *Let $\mathcal{A}$ be an abelian category, let $A^{\cdot}$ be a complex in $\mathcal{A}$, and let $F^{\cdot} B^{\cdot}$ be a filtered complex in $\mathcal{A}$. Give $A^{\cdot}$ the trivial filtration with $F^0 A^{\cdot} = A^{\cdot}$ and $F^1 A^{\cdot} = 0^{\cdot}$. Then for all $m \in \mathbb{Z}$ we have:*

$$F^m Hom^{\cdot}(A^{\cdot}, F^{\cdot} B^{\cdot}) = Hom^{\cdot}(A^{\cdot}, F^m B^{\cdot})$$

*Proof.* For all $m, n \in \mathbb{Z}$ we have:

$$F^m \mathrm{Hom}^n(A^{\cdot}, F^{\cdot} B^{\cdot}) = \prod_{i \in \mathbb{Z}} F^m \mathrm{Hom}(A^i, F^{\cdot} B^{i+n})$$

$$= \prod_{i\in\mathbb{Z}} \mathrm{Hom}(A^i, F^m B^{i+n}) \qquad \text{By Lemma 6.9.2}$$

$$= \mathrm{Hom}^n(A^\cdot, F^m B^\cdot)$$

$\square$

**Lemma 6.9.5.** *Let $G$ be a finite group, $\Lambda$ a commutative ring, and let $A$, $B$, and $C$ be $\Lambda G$ modules. Let $G$ act diagonally on $Hom_\Lambda(B,C)$ and $A \otimes_\Lambda B$. Then the adjoint isomorphism $\Phi$ from Lemma 3.1.2 is a filtered isomorphism. That is, $F^\cdot\Phi$ and $F^\cdot\Phi$ are filtered morphisms.*

*Proof.* Let $m \in \mathbb{Z}$ and $f \in F^m\mathrm{Hom}_{\Lambda G}(F^\cdot(A \otimes B), F^\cdot C)$. I must show:

$$\Phi(f) \in F^m\mathrm{Hom}_{\Lambda G}(A, \mathrm{Hom}_\Lambda(B,C))$$

Let $a \in F^iA$ and $b \in F^jB$. Note $a \otimes b \in F^{i+j}(A \otimes B)$. We have:

$$\Phi(f)(a)(b) = f(a \otimes b) \in F^{i+j+m}C$$

Since $\Phi(f)(a) \in \mathrm{Hom}_\Lambda(B,C)$ and $b \in F^jB$ was arbitrary, this shows

$$\Phi(f)(a) \in F^{m+i}\mathrm{Hom}_\Lambda(B,C)$$

Since $a \in F^iA$ was arbitrary, we have shown:

$$\Phi(f)(F^iA) \subseteq F^{m+i}\mathrm{Hom}_\Lambda(B,C)$$

Thus, $\Phi(f) \in \mathrm{Hom}^m(A, \mathrm{Hom}_\Lambda(B,C))$. Since $m \in \mathbb{Z}$ was arbitrary, this shows $F^\cdot\Phi$ is a filtered morphism.

The argument for showing $\Phi^{-1}$ is a filtered morphism is similar. Let

$$g \in F^m\mathrm{Hom}_{\Lambda G}(A, \mathrm{Hom}_\Lambda(B,C)), \qquad a \in F^iA, \qquad b \in F^jB$$

Then we have: $\Phi^{-1}(g)(a \otimes b) = g(a)(b)$. We have $g(a) \in F^{m+i}\mathrm{Hom}_\Lambda(B,C)$, so $g(a)(b) \in F^{m+i+j}C$. This shows $\Phi^{-1}(g)(F^{i+j}(A \otimes B)) \subseteq F^{m+i+j}C$. Thus,

$$\Phi^{-1}(g) \in F^m\mathrm{Hom}_{\Lambda G}(A \otimes B, C)$$

and this shows $\Phi^{-1}$ is a filtered morphism as well. □

**Corollary 6.9.6.** *Let $G$ be a finite group and $\Lambda$ a commutative ring. Let $A^\cdot$, $B^\cdot$, and $C^\cdot$ be complexes of $\Lambda G$ modules. Then the adjoint isomorphism $\Phi^\cdot$ from Lemma 3.1.5 if a filtered isomorphism of filtered complexes.*

*Proof.* Follows from Lemma 6.9.5. □

**Corollary 6.9.7.** *Let $G$ be a finite group, and $\Lambda$ a commutative ring. Let $F^\cdot A^\cdot$, $F^\cdot B^\cdot$, and $F^\cdot C^\cdot$ be filtered complexes of $\Lambda G$ modules. Suppose we have the following filtered chain maps in ${}_{\Lambda G}Mod$:*

$$F^\cdot f^\cdot, F^\cdot g^\cdot : F^\cdot A^\cdot \to F^\cdot Hom^\cdot_\Lambda(F^\cdot B^\cdot, F^\cdot C^\cdot)$$

*that are filtered homotopic by filtered homotopy:*

$$F^\cdot h^\cdot : F^\cdot A^\cdot \to F^\cdot Hom_\Lambda(F^\cdot B^\cdot, F^\cdot C^\cdot)[-1]^\cdot$$

*By Corollary 6.9.6, we have the filtered chain maps in ${}_{\Lambda G}Mod$:*

$$F^\cdot \Phi^{-1}(f^\cdot), F^\cdot \Phi^{-1}(g^\cdot) : F^\cdot Tot^\cdot(F^\cdot A^\cdot \otimes F^\cdot B^\cdot) \to F^\cdot C^\cdot$$

*I claim these filtered chain maps are filtered homotopic by the filtered homotopy:*

$$F^\cdot \Phi^{-1}(F^\cdot h^\cdot) : F^\cdot Tot^\cdot(F^\cdot A^\cdot \otimes F^\cdot B^\cdot) \to F^\cdot C[-1]^\cdot$$

*Proof.* We already have that $f^\cdot$ and $g^\cdot$ are homotopic by homotopy $\Phi^{-1}(h^\cdot)$ by Lemma 3.1.6. So it suffices to show that $\Phi^{-1}(h^\cdot)$ is a filtered morphism. Since $h^\cdot$ is a filtered morphism, we have:

$$F^\cdot h^\cdot \in F^0\mathrm{Hom}^{-1}_{\Lambda G}(F^\cdot A^\cdot, F^\cdot \mathrm{Hom}^\cdot_\Lambda(F^\cdot B^\cdot, F^\cdot C^\cdot))$$

Since $F^\cdot \Phi\cdot$ is a filtered chain map, we have:

$$F^{\cdot}\Phi^{-1}(h^{\cdot}) \in F^0\mathrm{Hom}^{-1}_{\Lambda G}(F^{\cdot}\mathrm{Tot}^{\cdot}(F^{\cdot}A^{\cdot} \otimes_{\Lambda} F^{\cdot}B^{\cdot}), F^{\cdot}C)$$

Thus $\Phi^{-1}(h^{\cdot})$ is a filtered morphism, and we have shown $F^{\cdot}\Phi^{-1}(f^{\cdot})$ and $F^{\cdot}\Phi^{-1}(g^{\cdot})$ are filtered homotopic.

□

For the remainder of this section the filtration on the left argument of Hom is assumed to be trivial.

**Lemma 6.9.8.** *Let $\mathcal{A}$ be an abelian category, let $A$ be an object of $\mathcal{A}$, and let $F^{\cdot}B$ be a filtered object of $\mathcal{A}$ with a terminating filtration. Then the induced filtration on $F^{\cdot}Hom(A, F^{\cdot}B)$ terminates.*

*Proof.* Let $k \in \mathbb{Z}$ such that $F^mB = 0$ for all $m \geq k$. Then for all $m \geq k$:

$$F^m\mathrm{Hom}(A, F^{\cdot}B) = \mathrm{Hom}(A, F^mB) = \mathrm{Hom}(A, 0) = 0$$

□

**Lemma 6.9.9.** *Let $\mathcal{A}$ be an abelian category and let $A^{\cdot}$ be a bounded above complex in $\mathcal{A}$, and let $F^{\cdot}B^{\cdot}$ be a filtered complex in $\mathcal{A}$ whose filtration terminates in each degree, and $B^{\cdot}$ is bounded below. Then $F^{\cdot}Hom^{\cdot}(A^{\cdot}, F^{\cdot}B^{\cdot})$ has a filtration that terminates in each degree.*

*Proof.* Let $n \in \mathbb{Z}$. By Lemma 3.2.18, there are only finitely non-zero terms in the product below:

$$\mathrm{Hom}^n(A^{\cdot}, B^{\cdot}) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}(A^i, B^{i+n})$$

So for each $n$, there is a $k_n$ and $l_n$ such that:

$$\mathrm{Hom}^n(A^{\cdot}, B^{\cdot}) = \prod_{i=0,\ldots,l_n} \mathrm{Hom}(A^{k_n+i}, B^{k_n+i+n})$$

Because $F^{\cdot}B^{\cdot}$ has a filtration that terminates in each degree, for each $i$ there is a $m_{k_n+i+n} \in \mathbb{Z}$ such that $F^jB^{k_n+i+n} = 0$ for all $j \geq m_{k_n+i+n}$. Set:

$$M_n = \max_{i=0,\ldots,l_n} m_{k_n+i+n}$$

This ensures $F^j B^{k_n+i+n} = 0$ for all $j \geq M_n$ and $i = 0, \ldots, l_n$. For all $j \geq M_n$, we have:

$$\begin{aligned} F^j \mathrm{Hom}^n(A^{\cdot}, F^{\cdot}B^{\cdot}) &= \prod_{i=0,\ldots,l_n} \mathrm{Hom}(A^{k_n+i}, F^j B^{k_n+i+n}) \\ &= \prod_{i=0,\ldots,l_n} \mathrm{Hom}(A^{k_n+i}, 0) \\ &= 0 \end{aligned}$$

Thus, the filtration of $F^{\cdot}\mathrm{Hom}^n(A^{\cdot}, F^{\cdot}B^{\cdot})$ terminates at $M_n$. Since $n \in \mathbb{Z}$ was arbitrary, we have shown the filtration of $F^{\cdot}\mathrm{Hom}^{\cdot}(A^{\cdot}, F^{\cdot}B^{\cdot})$ terminates in each degree.

□

**Lemma 6.9.10.** *Let $\mathcal{A}$ be an abelian category, $V$ a projective object of $\mathcal{A}$, and $F^{\cdot}B$ a filtered object of $\mathcal{A}$. For all $m \in \mathbb{Z}$, we have a natural isomorphism:*

$$gr^m Hom(V, F^{\cdot}B) \cong Hom(V, gr^m B)$$

*Proof.* We have the exact sequence:

$$0 \to F^{m+1}B \xrightarrow{\iota} F^m B \xrightarrow{\pi} \mathrm{gr}^m B \to 0$$

Because $V$ is projective, we obtain the exact sequence:

$$0 \to \mathrm{Hom}(V, F^{m+1}B) \xrightarrow{\iota_*} \mathrm{Hom}(V, F^m B) \xrightarrow{\pi_*} \mathrm{Hom}(V, \mathrm{gr}^m B) \to 0$$

Hence:

$$\mathrm{Hom}(V, \mathrm{gr}^m B) \cong \mathrm{Hom}(V, F^m B)/\mathrm{Hom}(V, F^{m+1}B) = \mathrm{gr}^m \mathrm{Hom}(V, F^{\cdot}B)$$

□

**Lemma 6.9.11.** *Let $\mathcal{A}$ be an abelian category with products, let $A^{\cdot}$ be a complex in $\mathcal{A}$, and let $F^{\cdot}B^{\cdot}$ be a filtered complex in $\mathcal{A}$. Then for all $m \in \mathbb{Z}, n \in \mathbb{Z}$ we have the identity:*

$$gr^m Hom^n(A^\cdot, F^\cdot B^\cdot) \cong \prod_{i\in\mathbb{Z}} gr^m Hom(A^i, F^\cdot B^{n+i})$$

*Proof.* We have:

$$\begin{aligned}
\mathrm{gr}^m \mathrm{Hom}^n(A^\cdot, F^\cdot B^\cdot) &= \mathrm{gr}^m \prod_{i\in\mathbb{Z}} \mathrm{Hom}(A^i, F^m B^{n+i}) \\
&= \left(\prod_{i\in\mathbb{Z}} \mathrm{Hom}(A^i, F^m B^{n+i})\right) / \left(\prod_{i\in\mathbb{Z}} \mathrm{Hom}(A^i, F^{m+1} B^{n+i})\right) \\
&= \prod_{i\in\mathbb{Z}} \left(\mathrm{Hom}(A^i, F^m B^{n+i}) / \mathrm{Hom}(A^i, F^{m+1} B^{n+i})\right) \\
&= \prod_{i\in\mathbb{Z}} \mathrm{gr}^m \mathrm{Hom}(A^i, F^\cdot B^{n+i})
\end{aligned}$$

□

**Lemma 6.9.12.** *Let $\mathcal{A}$ be an abelian category, let $V.$ be a complex in in $\mathcal{A}$ that is projective in each degree, and let $F^\cdot B^\cdot$ be a filtered complex in $\mathcal{A}$. Then for all $m \in \mathbb{Z}$ we have:*

$$gr^m Hom^\cdot(V., F^\cdot B^\cdot) \cong Hom^\cdot(V., gr^m B^\cdot)$$

*Proof.* Let $n \in \mathbb{Z}$. Starting with Lemma 6.9.11, we have:

$$\begin{aligned}
\mathrm{gr}^m \mathrm{Hom}^n(V., F^\cdot B^\cdot) &= \prod_{i\in\mathbb{Z}} \mathrm{gr}^m \mathrm{Hom}(V_i, F^\cdot B^{n-i}) \\
&\cong \prod_{i\in\mathbb{Z}} \mathrm{Hom}(V_i, \mathrm{gr}^m B^{n-i}) && \text{By Lemma 6.9.10} \\
&= \mathrm{Hom}^n(V., \mathrm{gr}^m B^\cdot)
\end{aligned}$$

□

**Definition 6.9.13.** *Let $X$ be a topological space, $\Lambda$ be a commutative ring, and $G$ a finite group. Suppose $F^\cdot A$ is a filtered object in $Sh_\Lambda(X)$, and let $M$ be a finitely generated $\Lambda G$ module. Recall* *the object* $\mathbf{Hom}_\Lambda(M,A)$ *in* $Sh_{\Lambda G}(X)$ *from Definition 3.2.1. Then we may induce a filtration on this object as follows, where for* $m \in \mathbb{Z}$*:*

$$F^m\mathbf{Hom}_\Lambda(M,F^{\cdot}A) = \mathbf{Hom}_\Lambda(M,F^mA)$$

*One could also give a definition for when* $M$ *is a filtered object as in Definition 6.9.1.*

**Definition 6.9.14.** *Let* $X$ *be a topological space,* $\Lambda$ *a commutative ring, and* $G$ *a finite group. Let* $M.$ *be a complex of* $\Lambda G$ *modules and* $F^{\cdot}A^{\cdot}$ *a filtered complex in* $Sh_\Lambda(X)$*. We have the complex* $\mathbf{Hom}^{\cdot}_\Lambda(M.,A^{\cdot})$ *in* $Sh_{\Lambda G}(X)$ *from Definition 3.2.7. We may induce a filtration on this complex as follows, where for* $m \in \mathbb{Z}$*:*

$$F^m\mathbf{Hom}^{\cdot}_\Lambda(M.,F^{\cdot}A^{\cdot}) = \mathbf{Hom}^{\cdot}_\Lambda(M.,F^mA^{\cdot})$$

*One could also give a definition for when* $M.$ *is a filtered complex, as was done in Definition 6.9.3.*

**Lemma 6.9.15.** *Let* $X$*,* $\Lambda$*, and* $G$ *be as in Definition 6.9.13, let* $M$ *be a finitely generated projective* $\Lambda G$ *module, and let* $F^{\cdot}A$ *be a filtered object in* $Sh_\Lambda(X)$ *whose filtration terminates. Then* $F^{\cdot}\mathbf{Hom}_\Lambda(M,F^{\cdot}A)$ *has a terminating filtration.*

*Proof.* Proof is the same as that of Lemma 6.9.8. □

**Lemma 6.9.16.** *Let* $X$*,* $\Lambda$*, and* $G$ *be as in Definition 6.9.14, let* $M.$ *be a (lowered index) bounded below complex of finitely generated* $\Lambda G$ *modules. Let* $F^{\cdot}A^{\cdot}$ *be a filtered complex in* $Sh_\Lambda(X)$ *that has a terminating filtration in each degree and* $A^{\cdot}$ *is bounded below. Then* $F^{\cdot}\mathbf{Hom}^{\cdot}_\Lambda(M.,F^{\cdot}A^{\cdot})$ *has a terminating filtration.*

*Proof.* Proof is the same as Lemma 6.9.9, but cite Corollary 3.2.20 instead of Lemma 3.2.18 to get that there are only finitely many non zero terms in the product below for each $n \in \mathbb{Z}$:

$$\mathbf{Hom}^n_\Lambda(M.,A^{\cdot}) = \prod_{i\in\mathbb{Z}}\mathbf{Hom}_\Lambda(M_i,A^{n-i})$$

□

**Lemma 6.9.17.** *Let $X$, $\Lambda$, and $G$ be as in Definition 6.9.14. Let $V$ be a finitely generated projective $\Lambda G$ module, and let $F^{\cdot}A$ be a filtered object of $Sh_{\Lambda}(X)$. Then for all $m \in \mathbb{Z}$, we have a natural isomorphism:*

$$gr^m \mathbf{Hom}_{\Lambda}(V, F^{\cdot}A) \cong \mathbf{Hom}_{\Lambda}(V, gr^m A)$$

*Proof.* The steps are identical to the proof of Lemma 6.9.10, but with $\mathbf{Hom}_{\Lambda}$ used in place of Hom. □

**Lemma 6.9.18.** *Let $X$, $\Lambda$, and $G$ be as in Definition 6.9.14. Let $M.$ be a complex of finitely $\Lambda G$ modules, and let $F^{\cdot}A^{\cdot}$ be a filtered complex in $Sh_{\Lambda}(X)$. Then for all $m, n \in \mathbb{Z}$ we have:*

$$gr^m \mathbf{Hom}^n_{\Lambda}(M., F^{\cdot}A^{\cdot}) = \prod_{i \in \mathbb{Z}} gr^m \mathbf{Hom}^n_{\Lambda}(M_i, F^{\cdot}A^{n-i})$$

*Proof.* Steps are the same as the ones used in the proof of Lemma 6.9.11. □

**Lemma 6.9.19.** *Let $X$, $\Lambda$, and $G$ be as in Definition 6.9.14. Let $V.$ be a complex of finitely generated projective $\Lambda G$ modules, and let $F^{\cdot}A^{\cdot}$ be a filtered complex in $Sh_{\Lambda}(X)$. Then for all $m \in \mathbb{Z}$, we have the natural isomorphism of complexes in $Sh_{\Lambda G}(X)$:*

$$gr^m \mathbf{Hom}^{\cdot}_{\Lambda}(V., F^{\cdot}A^{\cdot}) \cong \mathbf{Hom}^{\cdot}_{\Lambda}(V., gr^m A^{\cdot})$$

*Proof.* The same steps used in the proof of Lemma 6.9.12 work here as well. □

**Lemma 6.9.20.** *Let $X$ be a topological space, $k$ a field, and $G$ a finite group. Let $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}I^{\cdot}$ be a strong filtered injective resolution in $Sh_k(X)$, as in Definition 6.1.20. That is, $F^{\cdot}A^{\cdot}$ and $F^{\cdot}I^{\cdot}$ are filtered complexes in $Sh_k(X)$, $F^{\cdot}I^{\cdot}$ is bounded below, has a terminating filtration in each degree, is strong filtered injective in each degree, and $F^{\cdot}\varepsilon^{\cdot}$ is an injective strong filtered quasi-isomorphism. Let $V.$ be a projective resolution of $k$ in ${}_{kG}Mod$, finitely generated in each degree, with surjective quasi-isomorphism $\pi. : V. \to k[0].$. We have the filtered complex $F^{\cdot}\mathbf{Hom}^{\cdot}_k(V., F^{\cdot}I^{\cdot})$. Define the filtered chain map:*

$$F^{\cdot}\nu : F^{\cdot}A^{\cdot} = F^{\cdot}\mathbf{Hom}^{\cdot}_k(k[0]., F^{\cdot}A^{\cdot}) \to F^{\cdot}\mathbf{Hom}^{\cdot}_k(V., F^{\cdot}I^{\cdot})$$

*to be the map induced by precomposition with $\pi.$ and postcomposition with $F^{\cdot}\varepsilon^{\cdot}$. Then $F^{\cdot}\nu^{\cdot}$ is a strong filtered injective resolution.*

*Proof.* Let $m \in \mathbb{Z}$. We have that $F^m\varepsilon^{\cdot} : F^mA^{\cdot} \to F^mI^{\cdot}$ is an injective resolution in $\mathrm{Sh}_k(X)$. So by Lemma 3.2.21, $F^m\nu^{\cdot} : F^mA^{\cdot} = \mathbf{Hom}^{\cdot}_k(k[0]_{\cdot}, F^mA^{\cdot}) \to \mathbf{Hom}^{\cdot}_k(V_{\cdot}, F^mI^{\cdot})$ is an injective resolution in $\mathrm{Sh}_{kG}(X)$. This implies $F^m\varepsilon^{\cdot}$ is a quasi-isomorphism for all $m \in \mathbb{Z}$, so $F^{\cdot}\varepsilon^{\cdot}$ is a strong filtered quasi-isomorphism. We have $\nu$ is injective because it is induced by precomposition with a surjection and postcomposition with an injection, so we now have $F^{\cdot}\varepsilon^{\cdot}$ is a strong filtered resolution.

Let $m \in \mathbb{Z}$ again. Because $F^m\varepsilon^{\cdot} : F^mA \to F^m\mathbf{Hom}^{\cdot}_k(V_{\cdot}, F^{\cdot}I^{\cdot})$ is an injective resolution, we have $F^m\mathbf{Hom}^{\cdot}_k(V_{\cdot}, F^{\cdot}I^{\cdot})$ is injective in each degree. Hence, $F^{\cdot}\mathbf{Hom}^{\cdot}_k(V_{\cdot}, F^{\cdot}I^{\cdot})$ is strong filtered injective in each degree. Because $V_{\cdot}$ is (lowered index) bounded below, $I^{\cdot}$ is bounded below, and $F^{\cdot}I^{\cdot}$ has a filtration that terminates in each degree, we have by Lemma 6.9.16 that $F^{\cdot}\mathbf{Hom}^{\cdot}_k(V_{\cdot}, I^{\cdot})$ has a filtration that terminates in each degree. We also have by Lemma 3.2.20 that $\mathbf{Hom}^{\cdot}_k(V_{\cdot}, I^{\cdot})$ is bounded below. At this point we have shown all the conditions required for $F^{\cdot}\nu^{\cdot} : F^{\cdot}A^{\cdot} \to \mathbf{Hom}^{\cdot}_k(V_{\cdot}, F^{\cdot}I^{\cdot})$ to be a strong filtered injective resolution in the category $\mathrm{Sh}_{kG}(X)$. $\square$

**Lemma 6.9.21.** *Let $X$ be a topological space, $k$ a field, and $G$ a finite group. Let $F^{\cdot}\varepsilon^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}I^{\cdot}$ be a filtered injective resolution in $Sh_k(X)$, as in Definition 6.1.19. That is, $F^{\cdot}A^{\cdot}$ and $F^{\cdot}I^{\cdot}$ are filtered complexes in $Sh_k(X)$, $I^{\cdot}$ is bounded below, $F^{\cdot}I^{\cdot}$ is filtered injective in each degree, and $F^{\cdot}I^{\cdot}$ has a filtration that terminates in each degree. We also have $F^{\cdot}\varepsilon^{\cdot}$ is an injective filtered quasi-isomorphism. Let $V_{\cdot}$ be a $G$ projective resolution of $k$ in ${}_{kG}Mod$, with $V_{\cdot}$ finitely generated in each degree, and augmentation $\pi_{\cdot} : V_{\cdot} \to k[0]_{\cdot}$. We have the $kG$ chain map $F^{\cdot}\nu^{\cdot}$ induced by precomposition with $\pi_{\cdot}$ and postcomposition with $F^{\cdot}\varepsilon^{\cdot}$.*

$$F^{\cdot}\nu^{\cdot} : F^{\cdot}A = F^{\cdot}\mathbf{Hom}^{\cdot}_k(k[0]_{\cdot}, F^{\cdot}A^{\cdot}) \to F^{\cdot}\mathbf{Hom}^{\cdot}_k(V_{\cdot}, F^{\cdot}I^{\cdot})$$

*Then $F^{\cdot}\nu^{\cdot}$ is a filtered injective resolution in the category $Sh_{kG}(X)$.*

*Proof.* Let $m \in \mathbb{Z}$. By invoking the identification of Lemma 6.9.19, we have:

$$\mathrm{gr}^m\nu^{\cdot} : \mathrm{gr}^mA^{\cdot} = \mathbf{Hom}^{\cdot}_k(k[0]_{\cdot}, \mathrm{gr}^mA^{\cdot}) \to \mathbf{Hom}^{\cdot}_k(V_{\cdot}, \mathrm{gr}^mI^{\cdot})$$

is induced by precomposition with $\pi_{\cdot}$ and postcomposition with $\mathrm{gr}^m\varepsilon^{\cdot}$. Since $\mathrm{gr}^m\varepsilon^{\cdot} : \mathrm{gr}^mA^{\cdot} \to \mathrm{gr}^mI^{\cdot}$ is an injective quasi-isomorphism, and $\mathrm{gr}^mI^{\cdot}$ is a bounded below complex that is injective in each degree, we have by Lemma 3.2.21 that $\mathrm{gr}^m\nu^{\cdot}$ is an injective resolution in the category $\mathrm{Sh}_{kG}(X)$. Thus, $\mathrm{gr}^m\nu^{\cdot}$ is a quasi-isomorphism for all $m \in \mathbb{Z}$. Because $\nu^{\cdot}$ is induced by precomposition with a surjection and postcomposition with an injection, $\nu^{\cdot}$ is injective. Thus $F^{\cdot}\nu^{\cdot}$ is a filtered resolution.

Let $m \in \mathbb{Z}$ be arbitrary again. Because $\mathrm{gr}^m\nu^{\cdot} : \mathrm{gr}^m A^{\cdot} \to \mathrm{gr}^m \mathbf{Hom}^{\cdot}_k(V_{\cdot}, I^{\cdot})$ is an injective resolution, we have $\mathrm{gr}^m\mathbf{Hom}^{\cdot}_k(V_{\cdot}, I^{\cdot})$ is injective in each degree. Because $V_{\cdot}$ is (lowered index) bounded below and $I^{\cdot}$ is bounded below, $\mathbf{Hom}^{\cdot}_k(V_{\cdot}, I^{\cdot})$ is bounded below by Corollary 3.2.20. Because $V_{\cdot}$ is (lowered index) bounded below and $F^{\cdot}I^{\cdot}$ is bounded below and has a filtration that terminates in each degree, we have by Lemma 6.9.16 that the filtration on $F^{\cdot}\mathbf{Hom}^{\cdot}_k(V_{\cdot}, I^{\cdot})$ terminates in each degree. We have now shown enough to conclude that $F^{\cdot}\nu^{\cdot}$ is a filtered injective resolution in the category $\mathrm{Sh}_{kG}(X)$.

$\square$

# Chapter 7

# Steenrod Operations on Spectral Sequences

In this chapter I will show that the Steenrod operations constructed on the algebraic De Rham cohomology groups from Chapter 5 can be constructed in a way so that they also act on the first and infinite pages of the Hodge to De Rham spectral sequence. The construction is general, and can apply to other spectral sequences as well.

For the remainder of the chapter, we will fix the following. Let $X$ be a topological space. Let $F^{\cdot}A^{\cdot}$ be a filtered complex of graded commutative $\mathbb{F}_p$ algebras on $X$ that is bounded below and finitely filtered in each degree. Here we insist that the product map:

$$F^{\cdot}m^{\cdot} : F^{\cdot}\mathrm{Tot}^{\cdot}(F^{\cdot}A^{\cdot} \otimes_{\mathbb{F}_p} F^{\cdot}A^{\cdot}) \to F^{\cdot}A^{\cdot}$$

is a filtered chain map of complexes in $\mathrm{Sh}_{\mathbb{F}_p}(X)$. By Lemma 6.8.7 there is a strict injective filtered quasi-isomorphism $F^{\cdot}\iota^{\cdot} : F^{\cdot}A^{\cdot} \to F^{\cdot}I^{\cdot}$ where $F^{\cdot}I^{\cdot}$ is a filtered complex in $\mathrm{Sh}_{\mathbb{F}_p}(X)$, with $F^{\cdot}I^{\cdot}$ bounded below, finitely filtered in each degree, and filtered injective in each degree. By applying Corollary 6.8.4, we get that $F^{\cdot}\iota^{\cdot}$ is a strong filtered injective resolution and $\iota^{\cdot}$ is an injective resolution. Let $T$ denote the global section functor, $\mathrm{Sh}_{\mathbb{F}_p}(X) \to \mathrm{Vect}(\mathbb{F}_p)$. Because $T$ is left exact, we have the filtered chain complex in $\mathrm{Vect}(\mathbb{F}_p)$, $F^{\cdot}K^{\cdot} = F^{\cdot}T(I^{\cdot})$ as in Definition 6.1.25. The cohomology groups of $K^{\cdot}$ compute the sheaf hypercohomology of $X$ with coefficients in $A^{\cdot}$,

$$H^{\cdot}(K^{\cdot}) = \mathbf{H}^{\cdot}(X, A^{\cdot})$$

as was the case in Chapter 5. But now we also have the filtration on $F^{\cdot}K^{\cdot}$ which induces a spectral sequence $E_{\cdot}^{\cdot,\cdot}(F^{\cdot}K^{\cdot})$ that converges to $H^{\cdot}(K^{\cdot})$. In this chapter we will go through the same construction in Chapter 5 but in a way that is compatible with the filtrations involved, and we will end up with operations that act on $E_1^{\cdot,\cdot}(F^{\cdot}K^{\cdot})$ and $E_{\infty}^{\cdot,\cdot}(F^{\cdot}K^{\cdot})$, in a way that must be compatible with the operations constructed on $H^{\cdot}(K^{\cdot})$ from Chapter 5.

## 7.1 Spectral Sequence Classes

For this section let $F^{\cdot}K^{\cdot}$ denote any filtered chain complex of $\mathbb{F}_p$ vector spaces. In [5], May defines a collection of maps, $D_i : H^q(K^{\cdot}) \to H^{pq-i}(K^{\cdot})$ for each $q \in \mathbb{Z}$, where a cohomology class $[x] \in H^q(K^{\cdot})$ is mapped to $\theta^{\cdot}([e_i \otimes_{\pi} x^{[p]}])$, where $e_i$ is the generator of the free rank one $\mathbb{F}_p\pi$ module $W_i$. For this definition to make sense we must have that $[e_i \otimes_{\pi} x^{[p]}]$ is a well defined cohomology class in $\mathrm{Tot}^{\cdot}(W_{\cdot} \otimes_{\pi} (K^{\cdot})^{[p]})$, which is something May verifies. In order to define the Steenrod operations on the spectral sequence for $F^{\cdot}K^{\cdot}$, it would be convenient if we had the following analogous result:

**Question 7.1.1.** *Let $i, a, b \geq 0$, $r > 0$, and suppose $[x] \in E_r^{a,b}(F^{\cdot}K^{\cdot})$. Is $[e_i \otimes_{\pi} x^{[p]}]$ is a well defined element of $E_r^{ap,bp-i}(F^{\cdot}Tot^{\cdot}(W_{\cdot} \otimes_{\pi} (K^{\cdot})^{[p]}))$?*

If the above was true, then I could define a map like $D_i$ on each page of the spectral sequence by the following composition, where the first map takes $[x]$ to $[e_i \otimes_{\pi} x^{[p]}]$, and $F^{\cdot}\theta^{\cdot}$ is a filtered chain map to be defined in the next section.

$$\begin{array}{ccc} E_r^{a,b}(F^{\cdot}K^{\cdot}) & \longrightarrow & E_r^{ap,bp-i}(F^{\cdot}\mathrm{Tot}^{\cdot}(W_{\cdot} \otimes (K^{\cdot})^{[p]})) \\ & & \Big\downarrow {\scriptstyle E_r^{ap,bp-i}(F^{\cdot}\theta^{\cdot})} \\ & & E_r^{ap,bp-i}(F^{\cdot}K^{\cdot}) \end{array}$$

Then since the Steenrod operations are defined in terms of the $D_i$ maps with a reindexing and sign, this would define Steenrod operations on all pages of the spectral sequence. However it appears that the above question is not true in general. We do at least have the following two lemmas, whose proofs are a bit detailed and will be included later in this section.

**Lemma 7.1.2.** *Let $i,a,b \geq 0$, $r > 0$, and suppose $x \in Z_r^{a,b}(F^\cdot K^\cdot)$. Then $e_i \otimes x^{[p]}$ is an element of $Z_r^{ap,bp-i}(F^\cdot Tot^\cdot(W_\cdot \otimes_\pi (K^\cdot)^{[p]}))$.*

Although we have the above, it appears that $[e_i \otimes x^{[p]}]$ is not always a well defined element of $E_r^{ap,bp-i}(F^\cdot \mathrm{Tot}^\cdot(W_\cdot \otimes_\pi (K^\cdot)^{[p]}))$. Instead, we have the following:

**Lemma 7.1.3.** *Let $i,a,b \geq 0$, $r > 0$, and suppose $[x] = [y]$ in $E_r^{a,b}(F^\cdot K^\cdot)$. Then $e_i \otimes (x^{[p]} - y^{[p]})$ is an element of $B_{r+(r-1)(p-1)}^{ap,bp-i}(F^\cdot Tot^\cdot(W_\cdot \otimes_\pi (K^\cdot)^{[p]}))$.*

To ease notation, let $L^\cdot = \mathrm{Tot}^\cdot(W_\cdot \otimes_\pi (K^\cdot)^{[p]})$.

**Corollary 7.1.4.** *If $[x] \in E_r^{a,b}$, then $[w_i \otimes x^{[p]}]$ is a well defined element of $Z_r^{ap,bp-i}(F^\cdot L^\cdot)/B_{r+(r-1)(p-1)}^{ap,bp-i}(F^\cdot L^\cdot)$. If we have $B_r^{ap,bp-i}(F^\cdot L^\cdot) = B_{r+(r-1)(p-1)}^{ap,bp-i}(F^\cdot L^\cdot)$, then $[w_i \otimes x^{[p]}]$ is a well defined element of $E_r^{ap,bp-i}(F^\cdot L^\cdot)$.*

*Proof.* Follows from Lemma 7.1.3. □

**Corollary 7.1.5.** *Let $a,b,i \geq 0$ and let $[x] \in E_1^{a,b}(F^\cdot K^\cdot)$. Then $[e_i \otimes x^{[p]}]$ is a well defined element of $E_1^{ap,bp-i}(F^\cdot L^\cdot)$.*

*Proof.* This is Corollary 7.1.4 when $r = 1$, where we have $r + (r-1)(p-1) = 1$. □

**Corollary 7.1.6.** *Let $a,b,i \geq 0$, and let $[x] \in E_\infty^{a,b}(F^\cdot K^\cdot)$. Then $[e_i \otimes x^{[p]}]$ is a well defined element of $E_\infty^{ap,bp-i}(F^\cdot L^\cdot)$.*

*Proof.* Recall from Definition 6.1.27:

$$E_\infty^{a,b} = \frac{\cap_r \tilde{Z}_r^{a,b}}{\cup_r \tilde{B}_r^{a,b}}$$

Since $x \in \cap_r \tilde{Z}_r^{a,b}(F^\cdot K^\cdot)$, we can apply Lemma 7.1.2 for each $r \geq 0$ to obtain $e_i \otimes x^{[p]} \in \cap_r \tilde{Z}_r^{ap,bp-i}(F^\cdot L^\cdot)$. Now suppose there are $x, y \in \cap_r \tilde{Z}_r^{a,b}(F^\cdot K^\cdot)$ with $x - y \in \cup_r \tilde{B}_r^{a,b}(F^\cdot K^\cdot)$. I must show $e_i \otimes (x^{[p]} - y^{[p]}) \in \cup_r \tilde{B}_r^{a,b}(F^\cdot L^\cdot)$. There is a $r' \geq 0$ such that $x - y \in \tilde{B}_{r'}^{a,b}(F^\cdot K^\cdot)$. Since $x, y \in \tilde{Z}_{r'}^{a,b}(F^\cdot K^\cdot)$ as well, we have by Lemma 7.1.3 that $e_i \otimes (x^{[p]} - y^{[p]}) \in \tilde{B}_{r'+(r'-1)(p-1)}^{ap,bp-i}(F^\cdot L^\cdot) \subseteq \cup_r \tilde{B}_r^{ap,bp-i}(F^\cdot L^\cdot)$. With this, we are done.

□

I will now move towards proving Lemma 7.1.2 and Lemma 7.1.3. In what follows, let $I^{\cdot}$ be a complex of $\mathbb{F}_p$ vector spaces in which $I^0$ is free of rank two generated by $e_0$ and $e_1$, and $I^{-1}$ is free of rank one generated by $e$. We define $d(e) = e_1 - e_0$. In a sense, $I$ represents a line segment $e$ connecting the two vertices $e_0$ and $e_1$. Unfortunately $e_i$ is also used to denote the generators of the $\mathbb{F}_p\pi$ complex $W.$. For the remainder of this section I will use $w_i$ to denote the generators of $W.$. The following is Lemma 1.1 from [5], on page 156.

**Lemma 7.1.7.** *Let $\Lambda$ denotes a commutative ring which we take to be $\mathbb{F}_p$. Let $V.$ be a positive $\Lambda\pi$-free complex.*

1. *There exists a $\Lambda\pi$-morphism $h : I \otimes_\Lambda V \to V \otimes_\Lambda I^{[p]}$ such that $h(e_i \otimes v) = v \otimes e_i^{[p]}$ for $i = 0, 1$ for all $v \in V_j$ and $j \geq 0$.*
2. *If $f, g : K \to L$ are $\Lambda$-homotopic morphisms of $\Lambda$-complexes, then $1 \otimes f^{[p]}, 1 \otimes g^{[p]} : V \otimes_\Lambda K^{[p]} \to V \otimes_\Lambda L^{[p]}$ are $\Lambda\pi$-homotopic morphisms of $\Lambda\pi$ complexes.*
3. *If $\Lambda$ is a field and $K$ is a $\Lambda$-complex, then $K$ is $\Lambda$-homotopy equivalent to $H(K)$ and $V \otimes_\Lambda K^{[p]}$ is $\Lambda\pi$-homotopy equivalent to $V \otimes_\Lambda H(K)^{[p]}$.*
4. *Let $v \in V$ satisfy $d(v \otimes_\pi 1) = 0$ in $V \otimes_\pi \Lambda$; let $K$ be a $\Lambda$-complex and let $x, y \in K^q$ be homologous cycles. Then $v \otimes x^{[p]}$ and $v \otimes y^{[p]}$ are homologous cycles of $Tot^{\cdot}(V. \otimes_\pi (K^{\cdot})^{[p]})$.*

I will need to generalize statement 4 in the above lemma. May proves (4) by using (1), where he defines a morphism of $\Lambda$-complexes $f^{\cdot} : I^{\cdot} \to K[q]^{\cdot}$, where $f(e_1) = x$, $f(e_0) = y$, and $f(e) = (-1)^q z$ where $z \in K^{q-1}$ satisfies $d(z) = x - y$. Then one has that the element $\zeta = (1 \otimes f^{[p]})(h(e \otimes v))$ satisfies $d(\zeta) = \pm(v \otimes_\pi x^{[p]} - v \otimes_\pi y^{[p]})$ (omitting the sign here), which shows $v \otimes_\pi x^{[p]}$ and $v \otimes_\pi y^{[p]}$ are cohomologous. I will now repeat this argument in the context of the spectral sequence of a filtered complex. Recall from Definition 6.1.27:

$$\tilde{Z}_r^{a,b} = F^a K^{a+b} \cap d^{-1}(F^{a+r} K^{a+b+1}) + F^{a+1} K^{a+b}$$
$$\tilde{B}_r^{a,b} = F^a K^{a+b} \cap d(F^{a-r+1} K^{a+b-1}) + F^{a+1} K^{a+b}$$
$$E_r^{a,b} \cong \tilde{Z}_r^{a,b} / \tilde{B}_r^{a,b}$$

**Lemma 7.1.8.** *Suppose $F^{\cdot}X^{\cdot}$ and $F^{\cdot}Y^{\cdot}$ are filtered complexes and denote $T^{\cdot} = Tot^{\cdot}(X^{\cdot} \otimes Y^{\cdot})$. Let $x \in \tilde{Z}_r^{a,b}(F^{\cdot}X^{\cdot})$ and $y \in \tilde{Z}_r^{c,d}(F^{\cdot}Y^{\cdot})$. Then $x \otimes y \in \tilde{Z}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$.*

*Proof.* We have $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1 \in F^aX^{a+b} \cap d^{-1}(F^{a+r}X^{a+b+1})$, $x_2 \in F^{a+1}X^{a+b}$, $y_1 \in F^cY^{c+d} \cap d^{-1}(F^{c+r}Y^{c+d+1})$, and $y_2 \in F^{c+1}(Y^{c+d})$. We can show $x_1 \otimes y_2$, $x_2 \otimes y_1$, $x_2 \otimes y_2$ are in $\tilde{Z}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$ simply because of their filtration degrees. We have:

$$x_1 \otimes y_2 \in F^aX^{a+b} \otimes F^{c+1}Y^{c+d} \subseteq F^{(a+c)+1}T^{(a+c)+(b+d)} \subseteq \tilde{Z}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$$
$$x_2 \otimes y_1 \in F^{a+1}X^{a+b} \otimes F^{c}Y^{c+d} \subseteq F^{(a+c)+1}T^{(a+c)+(b+d)} \subseteq \tilde{Z}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$$
$$x_2 \otimes y_2 \in F^{a+1}X^{a+b} \otimes F^{c+1}Y^{c+d} \subseteq F^{(a+c)+2}T^{(a+c)+(b+d)} \subseteq \tilde{Z}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$$

Now I show $x_1 \otimes y_1 \in F^{a+c}T^{(a+c)+(b+d)} \cap d^{-1}(F^{(a+c)+r}T^{(a+c)+(b+d)+1})$. We have:

$$d(x_1) \otimes y_1 \in F^{a+r}X^{a+b+1} \otimes F^cY^{c+d} \subseteq F^{(a+c)+r}(T^{(a+c)+(b+d)+1})$$
$$x_1 \otimes d(y_1) \in F^{a}X^{a+b} \otimes F^{c+r}Y^{c+d+1} \subseteq F^{(a+c)+r}(T^{(a+c)+(b+d)+1})$$

Since $d(x_1 \otimes y_1) = d(x_1) \otimes y_1 + (-1)^{a+b} x_1 \otimes d(y_1)$, this shows

$$x_1 \otimes y_1 \in d^{-1}(F^{(a+c)+r}T^{(a+c)+(b+d)+1})$$

Thus:

$$x_1 \otimes y_1 \in F^{a+c}T^{(a+c)+(b+d)} \cap d^{-1}(F^{(a+c)+r}T^{(a+c)+(b+d)+1}) \subseteq \tilde{Z}_r^{a+c,b+d}$$

Hence, $x \otimes y \in \tilde{Z}_r^{a+c,b+d}$.

$\square$

**Lemma 7.1.9.** *Suppose $F^{\cdot}X^{\cdot}$ and $F^{\cdot}Y^{\cdot}$ are filtered complexes with $x \in \tilde{B}_r^{a,b}(F^{\cdot}X^{\cdot})$ and $y \in \tilde{Z}_r^{c,d}(F^{\cdot}Y^{\cdot})$. Then $x \otimes y \in \tilde{B}_r^{a+c,b+d}(F^{\cdot}Tot^{\cdot}(X^{\cdot} \otimes Y^{\cdot}))$.*

*Proof.* We have $x = x_1 + x_2$ and $y = y_1 + y_2$ where $x_1 \in F^a X^{a+b} \cap d(F^{a-r+1}X^{a+b-1})$, $x_2 \in F^{a+1}X^{a+b}$, $y_1 \in F^c Y^{c+d} \cap d^{-1}(F^{c+r}Y^{c+d+1})$, and $y_2 \in F^{c+1}Y^{c+d}$. Like before, we have $x_1 \otimes y_2$, $x_2 \otimes y_1$, and $x_2 \otimes y_2$ are in $\tilde{B}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$ because of their filtration degrees:

$$x_1 \otimes y_2 \in F^a X^{a+b} \otimes F^{c+1}Y^{c+d} \subseteq F^{(a+c)+1}T^{(a+c)+(b+d)} \subseteq \tilde{B}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$$
$$x_2 \otimes y_1 \in F^{a+1} X^{a+b} \otimes F^{c}Y^{c+d} \subseteq F^{(a+c)+1}T^{(a+c)+(b+d)} \subseteq \tilde{B}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$$
$$x_2 \otimes y_2 \in F^{a+1} X^{a+b} \otimes F^{c+1}Y^{c+d} \subseteq F^{(a+c)+2}T^{(a+c)+(b+d)} \subseteq \tilde{B}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$$

Now I will show $x_1 \otimes y_1 \in \tilde{Z}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$. There is an $x' \in F^{a-r+1}X^{a+b-1}$ such that $d(x') = x_1$. Consider the element:

$$x' \otimes y_1 \in F^{a-r+1}X^{a+b-1} \otimes F^c Y^{c+d} \subseteq F^{(a+c)-r+1}T^{(a+c)+(b+d)-1}$$

Thus, $d(x' \otimes y_1) \in d(F^{(a+c)-r+1}T^{(a+c)+(b+d)-1})$. We also have:

$$x' \otimes d(y_1) \in F^{a-r+1}X^{a+b-1} \otimes F^{c+r}(Y^{c+d+1}) \subseteq F^{(a+c)+1}T^{(a+c)+(b+d)} \subseteq \tilde{B}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$$

Since $d(x' \otimes y_1) = x_1 \otimes y_1 + (-1)^{a+b-1}x' \otimes d(y_1)$, this implies

$$d(x' \otimes y_1) \in F^{a+c}T^{(a+c)+(b+d)} \cap d(F^{(a+c)-r+1}T^{(a+c)+(b+d)}) \subseteq \tilde{B}_r^{a+c,b+d}$$

Thus:

$$x_1 \otimes y_1 = d(x') \otimes y_1 = d(x' \otimes y_1) - (-1)^{a+b-1}x' \otimes d(y_1) \in \tilde{B}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$$

It now follows $x \otimes y \in \tilde{B}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$.

□

**Lemma 7.1.10.** *Suppose $F^{\cdot}X^{\cdot}$ and $F^{\cdot}Y^{\cdot}$ are filtered complexes with $x \in \tilde{Z}_r^{a,b}(F^{\cdot}X^{\cdot})$ and $y \in \tilde{B}_r^{c,d}(F^{\cdot}Y^{\cdot})$. Then $x \otimes y \in \tilde{B}_r^{a+c,b+d}(F^{\cdot}Tot^{\cdot}(X^{\cdot} \otimes Y^{\cdot}))$.*

*Proof.* Symmetric to Lemma 7.1.9. ☐

**Corollary 7.1.11.** *Suppose* $[x_1] = [x_2]$ *in* $E_r^{a,b}(F^{\cdot}X^{\cdot})$ *and* $y \in \tilde{Z}_r^{c,d}(F^{\cdot}Y^{\cdot})$. *Then* $[x_1 \otimes y] = [x_2 \otimes y]$ *in* $E_r^{a+c,b+d}(F^{\cdot}Tot^{\cdot}(X^{\cdot} \otimes Y^{\cdot}))$.

*Proof.* We have $x_1, x_2 \in \tilde{Z}_r^{a,b}(F^{\cdot}X^{\cdot})$ and $y \in \tilde{Z}_r^{c,d}(F^{\cdot}Y^{\cdot})$. By Lemma 7.1.8, $x_1 \otimes y$ and $x_2 \otimes y$ are in $\tilde{Z}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$. To show $[x_1 \otimes y] = [x_2 \otimes y]$ in $E_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$, note $x_1 - x_2 \in \tilde{B}_r^{a,b}(F^{\cdot}X^{\cdot})$. By Lemma 7.1.9 we have $(x_1 - x_2) \otimes y \in \tilde{B}_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$. Thus $[x_1 \otimes y] = [x_2 \otimes y]$ in $E_r^{a+c,b+d}(F^{\cdot}T^{\cdot})$. ☐

**Corollary 7.1.12.** *Suppose* $[y_1] = [y_2]$ *in* $E_r^{c,d}(F^{\cdot}Y^{\cdot})$ *and* $x \in \tilde{Z}_r^{a,b}(F^{\cdot}X^{\cdot})$. *Then* $[x \otimes y_1] = [x \otimes y_2]$ *in* $E_r^{a+c,b+d}(F^{\cdot}Tot^{\cdot}(X^{\cdot} \otimes Y^{\cdot}))$.

*Proof.* Symmetric to Corollary 7.1.11. ☐

**Corollary 7.1.13.** *Suppose* $[x_1] = [x_2]$ *in* $E_r^{a,b}(F^{\cdot}X^{\cdot})$ *and* $[y_1] = [y_2] \in E_r^{c,d}(F^{\cdot}Y^{\cdot})$. *Then* $[x_1 \otimes y_1] = [x_2 \otimes y_2] = [x_1 \otimes y_2] = [x_2 \otimes y_1]$ *in* $E_r^{a+c,b+d}(F^{\cdot}Tot^{\cdot}(X^{\cdot} \otimes Y^{\cdot}))$.

*Proof.* By Corollary 7.1.11, $[x_1 \otimes y_1] = [x_2 \otimes y_1]$ and $[x_1 \otimes y_2] = [x_2 \otimes y_2]$. By Corollary 7.1.12, $[x_1 \otimes y_1] = [x_1 \otimes y_2]$ and $[x_2 \otimes y_1] = [x_2 \otimes y_2]$. Thus, they are all equal. ☐

**Corollary 7.1.14.** *Suppose* $[x_1] = [x_2]$ *in* $E_r^{a,b}(F^{\cdot}X^{\cdot})$, *and* $m \geq 1$. *Then* $[x_1^{[m]}] = [x_2^{[m]}]$ *in* $E_r^{am,bm}(F^{\cdot}Tot^{\cdot}((X^{\cdot})^{[m]}))$.

*Proof.* Apply Corollary 7.1.13 $m$ times. ☐

I will now prove Lemmas 7.1.2 and 7.1.3.

**Proof of Lemma 7.1.2**:

*Proof.* Let $a, b, i \geq 0$, $r > 0$, and let $x \in \tilde{Z}_r^{a,b}(F^{\cdot}K^{\cdot})$. I must show $w_i \otimes_\pi x^{[p]} \in \tilde{Z}_r^{ap,bp-i}(F^{\cdot}L^{\cdot})$, where $L^{\cdot} = \mathrm{Tot}^{\cdot}(W_{\cdot} \otimes_\pi (K^{\cdot})^{[p]})$. We have $x = x_1 + x_2$ where $x_1 \in F^a K^{a+b} \cap d^{-1}(F^{a+r}K^{a+b+1})$ and $x_2 \in F^{a+1}K^{a+b}$. Define $\varepsilon = x^{[p]} - x_1^{[p]}$ so that $x^{[p]} = x_1^{[p]} + \varepsilon$. I will show $w_i \otimes x_1^{[p]} \in F^{ap}L^{ap+bp-i} \cap d^{-1}(F^{ap+r}L^{ap+bp-i+1})$ and $w_i \otimes \varepsilon \in F^{ap+1}L^{ap+bp-i}$. We have that $\varepsilon$ consists of sums of tensors where at least one term is $x_2$ and the remaining terms are $x_1$. Thus, the filtration degree of $\varepsilon$ is at least $(a+1)\cdot 1 + a \cdot (p-1) = ap+1$. Thus $\varepsilon \in F^{ap+1}\mathrm{Tot}^{ap+bp}((K^{\cdot})^{[p]})$, and we have:

$$w_i \otimes_\pi \varepsilon \in W_i \otimes_\pi (F^{ap+1}\mathrm{Tot}^{ap+bp}((K^{\cdot})^{[p]})) \subseteq F^{ap+1}L^{ap+bp-i} \subseteq \tilde{Z}_r^{ap,bp-i}(F^{\cdot}L^{\cdot})$$

Now I will show $w_i \otimes_\pi x_1^{[p]} \in F^{ap}L^{ap+bp-i} \cap d^{-1}(F^{ap+r}L^{ap+bp-i+1})$. Note that:

$$w_i \otimes_\pi x_1^{[p]} \in W_i \otimes_\pi (F^a K^{a+b})^{[p]} \subseteq W_i \otimes_\pi F^{ap}(\mathrm{Tot}^{ap+bp}((K^{\cdot})^{[p]})) \subseteq F^{ap}L^{ap+bp-i}$$

Because $x_1^{[p]}$ is a homogeneous tensor product, it is $\pi$ invariant. Let $\sigma$ denote the generator $(1\ 2\ 3\ \cdots\ p)$ of $\pi$. The differential on $W_{\cdot}$ is defined by: $d(w_{2i+1}) = (\sigma - 1)w_{2i}$ and $d(w_{2i}) = (1+\sigma+\cdots+\sigma^{p-1})w_{2i-1}$. In both cases we have $d(w_i) \otimes_\pi x_1^{[p]} = 0$ because $x_1^{[p]}$ is fixed by $\sigma$. Note, the fact that $\sigma$ has even sign for $p > 2$ is important, due to the signs incurred when $\sigma$ transposes the $x_1$ terms. In the case $p = 2$, we are working in $\mathbb{F}_2$ vector spaces and signs don't matter. Thus:

$$\begin{aligned} d(w_i \otimes_\pi x_1^{[p]}) &= d(w_i) \otimes_\pi x_1^{[p]} + (-1)^i w_i \otimes_\pi d(x_1^{[p]}) \\ &= (-1)^i w_i \otimes_\pi d(x_1^{[p]}) \end{aligned}$$

Note the above only holds when the tensor product is over $\pi$. We have $d(x_1^{[p]})$ is a sum of tensors that consist of 1 $d(x_1)$ term and $p-1$ $x_1$ terms. Hence, it has filtration degree $1 \cdot (a+r) + (p-1) \cdot a = ap + r$. Thus, $d(x_r^{[p]}) \in F^{ap+r}(\mathrm{Tot}^{ap+bp+1}((K^{\cdot})^{[p]}))$. Therefore:

$$w_i \otimes_\pi d(x_1^{[p]}) \in W_i \otimes_\pi (F^{ap+r}(\mathrm{Tot}^{ap+bp+1}((K^{\cdot})^{[p]}))) \subseteq F^{ap+r}L^{ap+bp-i+1}$$

This shows $w_i \otimes_\pi x_1^{[p]} \in d^{-1}(F^{ap+r}L^{ap+bp-i+1})$. Now we have:

$$w_i \otimes_\pi x_1^{[p]} \in F^{ap}L^{ap+bp-i} \cap d^{-1}(F^{ap+r}L^{ap+bp-i+1}) \subseteq \tilde{Z}_r^{ap,bp-i}(F^{\cdot}L^{\cdot})$$

So we have shown:

$$w_i \otimes_\pi x^{[p]} = w_i \otimes_\pi x_1^{[p]} + w_i \otimes_\pi \varepsilon \in \tilde{Z}_r^{ap,bp-i}(F^\cdot L^\cdot)$$

$\square$

**Proof of Lemma 7.1.3**:

*Proof.* Let $a, b, i \geq 0$, $r > 0$, and let $x, y \in \tilde{Z}_r^{a,b}(F^\cdot K^\cdot)$ with $x - y \in \tilde{B}_r^{a,b}(F^\cdot K^\cdot)$. I must show $w_i \otimes_\pi (x^{[p]} - y^{[p]}) \in \tilde{B}_{r+(r-1)(p-1)}^{ap,bp-i}(F^\cdot L^\cdot)$. We have $x = x_1 + x_2$ and $y = y_1 + y_2$ where $x_1, y_1 \in F^a K^{a+b} \cap d^{-1}(F^{a+r} K^{a+b+1})$ and $x_2, y_2 \in F^{a+1} K^{a+b}$. Let $z = x - y$. Because $z \in \tilde{B}_r^{a,b}(F^\cdot K^\cdot)$, we have $z = z_1 + z_2$ where $z_1 \in F^a K^{a+b} \cap d(F^{a-r+1} K^{a+b-1})$ and $z_2 \in F^{a+1} K^{a+b}$. We have the relation $z_1 = (x_1 + \varepsilon) - y_1$, where we set $\varepsilon = x_2 - y_2 - z_2 \in F^{a+1} K^{a+b}$. Let $\gamma \in F^{a-r+1} K^{a+b-1}$ such that $d(\gamma) = z_1$. First note that:

$$w_i \otimes (x^{[p]} - y^{[p]}) = w_i \otimes (x_1^{[p]} - y_1^{[p]}) + w_i \otimes \Big( ((x_1 + x_2)^{[p]} - x_1^{[p]}) - ((y_1 + y_2)^{[p]} - y_1^{[p]}) \Big)$$

We have $(x_1 + x_2)^{[p]} - x_1^{[p]}$ consists of sums of tensors in which at least one term is $x_2$ and the remaining terms are $x_1$. Hence this element has filtration degree at least $1 \cdot (a+1) + (p-1) \cdot a = ap + 1$. Hence, $(x_1 + x_2)^{[p]} - x_1^{[p]} \in F^{ap+1}(\mathrm{Tot}^{ap+bp}((K^\cdot)^{[p]}))$, and similarly for $(y_1 + y_2)^{[p]} - y_1^{[p]}$. This shows:

$$\begin{aligned} w_i \otimes \Big( ((x_1 + x_2)^{[p]} - x_1^{[p]}) - ((y_1 + y_2)^{[p]} - y_1^{[p]}) \Big) &\in W_i \otimes F^{ap+1}(\mathrm{Tot}^{ap+bp}((K^\cdot)^{[p]})) \\ &\subseteq F^{ap+1} L^{ap+bp-i} \\ &\subseteq \tilde{B}_{r+(r-1)(p-1)}^{ap,bp-i}(F^\cdot L^\cdot) \end{aligned}$$

So we are now reduced to showing $w_i \otimes (x_1^{[p]} - y_1^{[p]}) \in \tilde{B}_{r+(r-1)(p-1)}^{ap,bp-i}$. Recall $d(\gamma) = z_1 = (x_1 + \varepsilon) - y_1$. We have $W.$ is a positive (lowered index) complex of free $\mathbb{F}_p\pi$ modules. Invoking part 1 of Lemma 7.1.7, there is a chain map $h : I^\cdot \otimes W. \to W. \otimes (I^\cdot)^{[p]}$ such that $h(e_j \otimes w_i) = w_i \otimes e_j^{[p]}$ for $j = 0, 1$ and $i \geq 0$. Recall $e_0, e_1 \in I^0$ and $e \in I^{-1}$ satisfies $d(e) = e_1 - e_0$. Because $h$ is a chain map, the element $\nu = h(e \otimes w_i)$ satisfies:

$$
\begin{aligned}
d(\nu) &= d(h(e \otimes w_i)) \\
&= h(d(e \otimes w_i)) \\
&= h(d(e) \otimes w_i + (-1)e \otimes d(w_i)) \\
&= h(d(e) \otimes w_i) && \sigma \text{ acts trivially on } e. \\
&= h((e_1 - e_0) \otimes w_i) \\
&= w_i \otimes (e_1^{[p]} - e_0^{[p]})
\end{aligned}
$$

Define the map, $f : I^{\cdot} \to K[a+b]^{\cdot}$ via $f(e_1) = x_1 + \varepsilon$, $f(e_0) = y_1$, and $f(e) = (-1)^{a+b}\gamma$. We have $d_{K[a+b]}(f(e)) = d_{K[a+b]}((-1)^{a+b}\gamma) = d_K(\gamma) = z_1 = (x_1 + \varepsilon) - (y_1) = f(e_1) - f(e_0) = f(d_I(e))$. So $f$ is a chain map. We have the chain map:

$$1 \otimes f^{[p]} : \mathrm{Tot}^{\cdot}(W_{\cdot} \otimes (I^{\cdot})^{[p]}) \to \mathrm{Tot}^{\cdot}(W_{\cdot} \otimes (K[a+b]^{\cdot})^{[p]}) = \mathrm{Tot}(W_{\cdot} \otimes (K^{\cdot})^{[p]})[p(a+b)]^{\cdot}$$

Define the element:

$$\zeta = (-1)^{p(a+b)}(1 \otimes f^{[p]})(\nu)$$

We have:

$$
\begin{aligned}
d_{W_{\cdot} \otimes (K^{\cdot})^{[p]}}(\zeta) &= (-1)^{p(a+b)} d_{W_{\cdot} \otimes (K[a+b]^{\cdot})^{[p]}} \left((-1)^{p(a+b)}(1 \otimes f^{[p]})(\nu)\right) \\
&= (1 \otimes f^{[p]})(d_{W_{\cdot} \otimes (K^{\cdot})^{[p]}}(\nu)) \\
&= (1 \otimes f^{[p]})(w_i \otimes (e_1^{[p]} - e_0^{[p]})) \\
&= w_i \otimes (f(e_1)^{[p]} - f(e_0)^{[p]}) \\
&= w_i \otimes ((x_1 + \varepsilon)^{[p]} - y_1^{[p]})
\end{aligned}
$$

I now calculate the filtration degree and complex degree of $\zeta$. We have $e \otimes w_i \in \mathrm{Tot}^{-i-1}(I^{\cdot} \otimes W_{\cdot})$. Thus $\nu = h(e \otimes w_i) \in \mathrm{Tot}^{-i-1}(W_{\cdot} \otimes (I^{\cdot})^{[p]})$. Then we have $(1 \otimes f^{[p]})$ is a degree $p(a+b)$ chain map. So $\zeta = (-1)^{p(a+b)}(1 \otimes f^{[p]})(\nu) \in \mathrm{Tot}^{ap+bp-i-1}(W_{\cdot} \otimes (K^{\cdot})^{[p]})$. For calculating the filtration degree, we have:

$$\nu = (w_{i+1} \otimes_{\pi} t_0) + (w_i \otimes_{\pi} t_1) + \cdots + (w_{i-p+1} \otimes_{\pi} t_p) = \sum_{j=0}^{p} w_{i-j+1} \otimes_{\pi} t_j$$

where $t_j \in \mathrm{Tot}^{-j}(I^{\cdot})$ consists of sums of tensors with exactly $j$ $e$ terms and the rest $e_0$ or $e_1$. Thus:

$$\zeta = \sum_{j=0}^{p} w_{i-j+1} \otimes s_j$$

where $s_j$ is a sum of tensors with exactly $j$ $(-1)^{a+b}\gamma$ terms and the remaining $p-j$ terms are $(x_1+\varepsilon)$ or $y_1$. Thus, $s_j$ has filtration degree $(a-r+1)\cdot j + a\cdot(p-j) = ap + j(1-r)$, and complex degree $(a+b-1)\cdot j + (a+b)\cdot(p-j) = ap+bp-j$. Thus:

$$w_{i-j+1} \otimes s^j \in W_{i-j+1} \otimes (F^{ap+j(1-r)}(\mathrm{Tot}^{ap+bp-j}((K^{\cdot})^{[p]}))) \subseteq F^{ap+j(1-r)}L^{ap+bp-i-1}$$

Since the filtration on $L$ is decreasing, and $1-r \leq 0$, we have each $w_{i-j+1} \otimes s_j \in F^{ap+p(1-r)}L^{ap+bp-i-1}$ for $j = 0, \ldots, p$. So we have:

$$\zeta \in F^{ap+p(1-r)}L^{ap+bp-i-1}$$

We can rewrite the filtration degree above as $ap - (r + (p-1)(r-1)) + 1$, so we now have:

$$d(\zeta) = w_i \otimes ((x_1+\varepsilon)^{[p]} - y_1^{[p]}) \in d(F^{ap-(r+(p-1)(r-1))+1}L^{ap+bp-1})$$

We also have:

$$w_i \otimes_{\pi} ((x_1+\varepsilon)^{[p]} - y_1^{[p]}) \in W_i \otimes_{\pi} (F^a K^{a+b})^{[p]} \subseteq F^{ap}L^{ap+bp-i}$$

Thus:

$$w_i \otimes_{\pi} ((x_1+\varepsilon)^{[p]} - y_1^{[p]}) \in F^{ap}L^{ap+bp-i} \cap d(F^{ap-(r+(r-1)(p-1))+1}L^{ap+bp-i-1})$$

$$\subseteq \tilde{B}^{ap,bp-i}_{r+(r-1)(p-1)}(F^{\cdot}L^{\cdot})$$

We can now finish the proof by observing:

$$w_i \otimes_\pi (x_1^{[p]} - y_1^{[p]}) = w_i \otimes_\pi ((x_1+\varepsilon)^{[p]} - y_1^{[p]}) - w_i \otimes_\pi ((x_1+\varepsilon)^{[p]} - x_1^{[p]})$$

We have $(x_1+\varepsilon)^{[p]} - x_1^{[p]}$ consists of sums of tensors in which at least one term is $\varepsilon$ and the remaining terms are $x_1$. So this element has filtration degree at least $1\cdot(a+1)+(p-1)\cdot a = ap+1$, and hence:

$$\begin{aligned} w_i \otimes_\pi ((x_1+\varepsilon)^{[p]} - x_1^{[p]}) &\in W_i \otimes_\pi F^{ap+1}(\mathrm{Tot}^{ap+bp}((K^{\cdot})^{[p]})) \\ &\subseteq F^{ap+1}L^{ap+bp-i} \\ &\subseteq \tilde{B}^{ap,bp-i}_{r+(r-1)(p-1)}(F^{\cdot}L^{\cdot}) \end{aligned}$$

This shows $w_i \otimes_\pi (x_1^{[p]} - y_1^{[p]}) \in \tilde{B}^{ap,bp-i}_{r+(r-1)(p-1)}(F^{\cdot}L^{\cdot})$, and by the reduction at the beginning, we now have $w_i \otimes_\pi (x^{[p]} - y^{[p]}) \in \tilde{B}^{ap,bp-i}_{r+(r-1)(p-1)}$.

□

Now that these lemmas have been proven, the corollaries at the beginning of this section are established.

## 7.2 The Product on $F^{\cdot}K^{\cdot}$

In this section I will show that the product defined in Section 5.1 can be constructed in a way so that there is an induced cup product on each page of the spectral sequence $E^{\cdot,\cdot}_{\cdot}(F^{\cdot}K^{\cdot})$. The construction in this section is actually a special case of the construction from Section 5.1.

**Definition 7.2.1.** *We will define a filtered product on $F^{\cdot}K^{\cdot}$. Let $F^{\cdot}m^{\cdot} : F^{\cdot}Tot^{\cdot}(F^{\cdot}A^{\cdot} \otimes F^{\cdot}A^{\cdot})$ denote the filtered graded product on $F^{\cdot}A^{\cdot}$. We have the solid diagram below of filtered complexes in $Sh_{\mathbb{F}_p}(X)$:*

$$\begin{tikzcd} F^{\cdot}Tot^{\cdot}(F^{\cdot}I^{\cdot}\otimes F^{\cdot}I^{\cdot}) \arrow[r, dashed, "F^{\cdot}\tilde{m}^{\cdot}"] & F^{\cdot}I^{\cdot} \\ F^{\cdot}Tot^{\cdot}(F^{\cdot}A^{\cdot}\otimes F^{\cdot}A^{\cdot}) \arrow[u, hook, "F^{\cdot}Tot^{\cdot}(\iota^{\cdot}\otimes\iota^{\cdot})"] \arrow[r, "F^{\cdot}m^{\cdot}"] & F^{\cdot}A^{\cdot} \arrow[u, hook, "F^{\cdot}\iota^{\cdot}"'] \end{tikzcd}$$

*We have by construction that $F^{\cdot}I^{\cdot}$ is bounded below, has a terminating filtration in each degree, and is strong filtered injective in each degree. Because $F^{\cdot}A^{\cdot}$ and $F^{\cdot}I^{\cdot}$ are finitely filtered in each degree and both bounded below, we have by Corollary 6.7.9 that $F^{\cdot}Tot^{\cdot}(A^{\cdot}\otimes A^{\cdot})$ and $F^{\cdot}Tot^{\cdot}(I^{\cdot}\otimes I^{\cdot})$ are both finitely filtered in each degree. Because $F^{\cdot}\iota^{\cdot}$ is a strong filtered resolution, we have by Corollary 6.7.20 that $F^{\cdot}Tot^{\cdot}(\iota^{\cdot}\otimes\iota^{\cdot})$ is a strong filtered resolution. In particular, this implies $F^{\cdot}Tot^{\cdot}(\iota^{\cdot}\otimes\iota^{\cdot})$ is an injective filtered quasi-morphism. We can now apply Lemma 6.5.2 to obtain a filtered chain map $F^{\cdot}\tilde{m}^{\cdot}$ making the diagram commute. By Lemma 6.5.4, $F^{\cdot}\tilde{m}^{\cdot}$ is unique up to filtered homotopy. By applying the global section functor $T$, we obtain the filtered chain map in Vect($\mathbb{F}_p$):*

$$F^{\cdot}T(\tilde{m}^{\cdot}):F^{\cdot}T(Tot^{\cdot}(F^{\cdot}I^{\cdot}\otimes F^{\cdot}I^{\cdot}))\to F^{\cdot}T(I^{\cdot})$$

*We have the natural filtered chain map in Vect($\mathbb{F}_p$):*

$$F^{\cdot}\gamma:F^{\cdot}Tot^{\cdot}(F^{\cdot}T(I^{\cdot})\otimes F^{\cdot}T(I^{\cdot}))\to F^{\cdot}T(Tot^{\cdot}(F^{\cdot}I^{\cdot}\otimes F^{\cdot}I^{\cdot}))$$

*induced by the map, $T(C)\otimes T(D)\to T(C\otimes D)$, when $C$ and $D$ are sheaves. We now define $F^{\cdot}M^{\cdot}$ by the composition:*

$$F^{\cdot}M^{\cdot}:F^{\cdot}Tot^{\cdot}(F^{\cdot}T(I^{\cdot})\otimes F^{\cdot}T(I^{\cdot}))\xrightarrow{F^{\cdot}\gamma}F^{\cdot}T(Tot^{\cdot}(F^{\cdot}I^{\cdot}\otimes F^{\cdot}I^{\cdot}))\xrightarrow{F^{\cdot}T(\tilde{m}^{\cdot})}F^{\cdot}T(I^{\cdot})$$

*By Lemma 6.1.26, the uniqueness of $F^{\cdot}\tilde{m}^{\cdot}$ up to filtered homotopy implies the uniqueness of $F^{\cdot}T(\tilde{m}^{\cdot})$ up to filtered homotopy. Then because $F^{\cdot}\gamma$ is a filtered chain map, we have by Lemma 6.1.23, that $F^{\cdot}M^{\cdot}=F^{\cdot}T(\tilde{m}^{\cdot})\circ F^{\cdot}\gamma$ is unique up to filtered homotopy. This implies the product on cohomology and on the pages of the spectral sequence are unique. That is, this construction gives well defined maps for all $n\in\mathbb{Z}$:*

$$H^n(M^\cdot) : H^n(Tot^\cdot(K^\cdot \otimes K^\cdot)) \to H^n(K^\cdot)$$

*And by Lemmas 6.1.28 and 6.1.29, we have well defined morphisms for all* $a, b \in \mathbb{Z}$ *and* $r \geq 1$*:*

$$E_r^{a,b}(F^\cdot M^\cdot) : E_r^{a,b}(F^\cdot Tot^\cdot(F^\cdot K^\cdot \otimes F^\cdot K^\cdot)) \to E_r^{a,b}(F^\cdot K^\cdot)$$

Because the construction of $F^\cdot M^\cdot$ is a special case of the construction of $M^\cdot$ from Section 5.1, we have the cup product from Definition 5.1.2:

$$\cup^{n,m} : H^n(K^\cdot) \otimes H^m(K^\cdot) \to H^{n+m}K^\cdot$$

We also have a cup product defined on each page of the spectral sequence:

**Definition 7.2.2.** *For* $a, b, c, d \in \mathbb{Z}$ *and* $r \geq 1$ *we have the spectral sequence cup product:*

$$\cup_r^{a,b,c,d} : E_r^{a,b}(F^\cdot K^\cdot) \otimes E_r^{c,d}(F^\cdot K^\cdot) \to E_r^{a+c,b+d}(F^\cdot K^\cdot)$$

*defined by the composition:*

$$\begin{array}{ccc}
\cup_r^{a,b,c,d} : E_r^{a,b}(F^\cdot K^\cdot) \otimes E_r^{c,d}(F^\cdot K^\cdot) & \xrightarrow{\psi'} & E_r^{a+c,b+d}(F^\cdot Tot^\cdot(F^\cdot K^\cdot \otimes F^\cdot K^\cdot)) \\
 & & \Big\downarrow {\scriptstyle E_r^{a+c,b+d}(F^\cdot M^\cdot)} \\
 & & E_r^{a+c,b+d}(F^\cdot K^\cdot)
\end{array}$$

*where* $\psi'$ *is the map* $[x] \otimes [y] \mapsto [x \otimes y]$*, for* $x \in \tilde{Z}_r^{a,b}(F^\cdot K^\cdot)$ *and* $y \in \tilde{Z}_r^{c,d}(F^\cdot K^\cdot)$*. The map* $\psi'$ *was shown to be well defined in Corollary 7.1.13.*

**Lemma 7.2.3.** *The cup product* $\cup^{\cdot,\cdot}$ *from Definition 5.1.2 and the cup product* $\cup_\cdot^{\cdot,\cdot,\cdot,\cdot}$ *of Definition 7.2.2 agree. That is, for all* $a, b, c, d \in \mathbb{Z}$ *and* $r \geq 1$*, when* $x \in Z^{a+b}(K^\cdot) \cap \tilde{Z}_r^{a,b}(F^\cdot K^\cdot)$ *and* $y \in Z^{c+d}(K^\cdot) \cap \tilde{Z}_r^{c,d}(F^\cdot K^\cdot)$*, we have* $\cup_r^{a,b,c,d}([x] \otimes [y])$ *and* $\cup^{a+b,c+d}([x] \otimes [y])$ *are cosets that are both represented by the common element,* $M^{a+b+c+d}(x \otimes y) \in Z^{a+b+c+d}(K^\cdot)$*.*

*Proof.* Let $[x]$ and $[y]$ denote the cosets of $x$, and $y$ in $H^{a+b}(K^\cdot)$ and $H^{c+d}(K^\cdot)$ respectively, and let $[x \otimes y]$ denote the coset in $H^{a+b+c+d}(\mathrm{Tot}^\cdot(K^\cdot \otimes K^\cdot))$. Then for the cup product from Definition 5.1.2, we have:

$$\cup^{a+b,c+d}([x] \otimes [y]) = H^{a+b+c+d}(M^\cdot)([x \otimes y]) = [M^{a+b+c+d}(x \otimes y)] \in H^{a+b+c+d}(K^\cdot)$$

Now denote $[x] \in E_r^{a,b}(F^\cdot K^\cdot)$ and $[y] \in E_r^{c,d}(F^\cdot K^\cdot)$. Let $[x \otimes y]$ denote the coset in $E_r^{a+c,b+d}(F^\cdot \mathrm{Tot}^\cdot(F^\cdot K^\cdot \otimes F^\cdot K^\cdot))$. Then the cup product from Definition 7.2.2 evaluates as:

$$\cup_r^{a,b,c,d}([x] \otimes [y]) = E_r^{a+c,b+d}(F^\cdot M^\cdot)([x \otimes y]) = [M^{a+b+c+d}(x \otimes y)] \in E_r^{a+c,b+d}(F^\cdot K^\cdot)$$

Thus, both cosets are represented by $M^{a+b+c+d}(x \otimes y)$.

$\square$

**Lemma 7.2.4.** *The filtered product $F^\cdot M^\cdot$ on $F^\cdot K^\cdot$ makes $F^\cdot K^\cdot$ a filtered homotopy associative differential graded $\mathbb{F}_p$ algebra. The induced cup product $\cup_r^{\cdot,\cdot,\cdot,\cdot}$ on $E_r^{\cdot,\cdot}(F^\cdot K^\cdot)$ is associative.*

*Proof.* Like in the proof of Lemma 5.1.3, we consider the diagram below of filtered complexes in $\mathrm{Sh}_{\mathbb{F}_p}(X)$. Recall ${}^{[l]}$ denotes the $l$-fold tensor product over $\mathbb{F}_p$.

$$\begin{tikzcd}
F^\cdot \mathrm{Tot}^\cdot((F^\cdot I^\cdot)^{[3]}) \arrow[r, shift left, "F^\cdot(1\otimes \tilde{m}^\cdot)"] \arrow[r, shift right, "F^\cdot(\tilde{m}^\cdot\otimes 1)"'] & F^\cdot \mathrm{Tot}^\cdot((F^\cdot I^\cdot)^{[2]}) \arrow[r, "F^\cdot \tilde{m}^\cdot"] & F^\cdot I^\cdot \\
F^\cdot \mathrm{Tot}^\cdot((F^\cdot A^\cdot)^{[3]}) \arrow[u, hook, "F^\cdot(\iota^\cdot)^{[3]}"] \arrow[r, shift left, "F^\cdot(1\otimes m^\cdot)"] \arrow[r, shift right, "F^\cdot(m^\cdot\otimes 1)"'] & F^\cdot \mathrm{Tot}^\cdot((F^\cdot A^\cdot)^{[2]}) \arrow[u, hook, "F^\cdot(\iota^\cdot)^{[2]}"] \arrow[r, "F^\cdot m^\cdot"] & F^\cdot A^\cdot \arrow[u, hook, "F^\cdot \iota^\cdot"]
\end{tikzcd}$$

Because the product on $F^\cdot A^\cdot$ is associative, the two compositions along the bottom row, $m^\cdot \circ (1 \otimes m^\cdot)$ and $m^\cdot \circ (m^\cdot \otimes 1)$, are equal. Thus, the two compositions along the top row, $\tilde{m}^\cdot \circ (1 \otimes \tilde{m}^\cdot)$ and $\tilde{m}^\cdot \circ (\tilde{m}^\cdot \otimes 1)$ are two filtered chain maps making the perimeter of the diagram commute. We have by Corollary 6.7.21 that $F^\cdot(\iota^\cdot)^{[3]}$ is a strong filtered resolution. We have by Corollary 6.7.8 that $F^\cdot \mathrm{Tot}^\cdot((F^\cdot A^\cdot)^{[3]})$ and $F^\cdot \mathrm{Tot}^\cdot((F^\cdot I^\cdot)^{[3]})$ are both finitely filtered in each degree. Thus, $F^\cdot(\iota^\cdot)^{[3]}$ is a filtered quasi-isomorphism, and $F^\cdot \mathrm{Tot}^\cdot((F^\cdot A^\cdot)^{[3]})$ and $F^\cdot \mathrm{Tot}^\cdot((F^\cdot I^\cdot)^{[3]})$ are both exhaustively filtered. $F^\cdot I^\cdot$ is still strong injective in each degree and has a filtration that terminates in each degree. Now we invoke Lemma 6.5.4 to obtain a filtered homotopy:

$$F^{\cdot}h^{\cdot}: F^{\cdot}\mathrm{Tot}^{\cdot}((F^{\cdot}I^{\cdot})^{[3]}) \to F^{\cdot}I[-1]^{\cdot}$$

between $F^{\cdot}\tilde{m}^{\cdot} \circ F^{\cdot}(\tilde{m}^{\cdot} \otimes 1)$ and $F^{\cdot}\tilde{m}^{\cdot} \circ F^{\cdot}(1 \otimes \tilde{m}^{\cdot})$. Then by Lemma 6.1.26, we have $F^{\cdot}T(h^{\cdot})$ is a filtered homotopy between the filtered chain maps:

$$\begin{matrix} F^{\cdot}T(\tilde{m}^{\cdot}) \circ F^{\cdot}T(\tilde{m}^{\cdot} \otimes 1), \\ F^{\cdot}T(\tilde{m}^{\cdot}) \circ F^{\cdot}T(1 \otimes \tilde{m}^{\cdot}) \end{matrix} : F^{\cdot}T(\mathrm{Tot}^{\cdot}(F^{\cdot}I^{\cdot})^{[3]}) \to F^{\cdot}T(I^{\cdot})$$

Let $F^{\cdot}\gamma_3$ be the natural filtered chain map:

$$F^{\cdot}\gamma_3 : F^{\cdot}\mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[3]}) \to F^{\cdot}T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[3]}))$$

By Lemma 6.1.23, $F^{\cdot}T(h^{\cdot}) \circ F^{\cdot}\gamma_3$ is a filtered homotopy between the filtered chain maps:

$$\begin{matrix} F^{\cdot}T(\tilde{m}^{\cdot}) \circ F^{\cdot}T(\tilde{m}^{\cdot} \otimes 1) \circ F^{\cdot}\gamma_3, \\ F^{\cdot}T(\tilde{m}^{\cdot}) \circ F^{\cdot}T(1 \otimes \tilde{m}^{\cdot}) \circ F^{\cdot}\gamma_3 \end{matrix} : F^{\cdot}\mathrm{Tot}^{\cdot}(F^{\cdot}T(I^{\cdot})^{[3]}) \to F^{\cdot}T(I^{\cdot})$$

And these filtered homotopic chain maps are equal to $F^{\cdot}M^{\cdot} \circ F^{\cdot}(M^{\cdot} \otimes 1)$ and $F^{\cdot}M^{\cdot} \circ F^{\cdot}(1 \otimes M^{\cdot})$ respectively. We have now shown the filtered product on $F^{\cdot}K^{\cdot}$ is filtered homotopy associative. By Lemma 6.1.29, the following induced morphisms are equal for all $a, b \in \mathbb{Z}$ and $r \geq 1$:

$$\begin{matrix} E_r^{a,b}(F^{\cdot}M^{\cdot} \circ F^{\cdot}(M^{\cdot} \otimes 1)), \\ E_r^{a,b}(F^{\cdot}M^{\cdot} \circ F^{\cdot}(1 \otimes M^{\cdot})) \end{matrix} : E_r^{a,b}(F^{\cdot}\mathrm{Tot}^{\cdot}(K^{\cdot})^{[3]}) \to E_r^{a,b}(F^{\cdot}K^{\cdot})$$

This implies the following are equal, for all $a, b, c, d, e, f \in \mathbb{Z}$:

$$\cup_r^{a+c,b+d,e,f} \circ (\cup_r^{a,b,c,d} \otimes 1_{E_r^{e,f}(F^{\cdot}K^{\cdot})}) = \cup_r^{a,b,c+e,d+f} \circ (1_{E_r^{a,b}(F^{\cdot}K^{\cdot})} \otimes \cup_r^{c,d,e,f})$$

$$E_r^{a,b}(F^{\cdot}K^{\cdot}) \otimes E_r^{c,d}(F^{\cdot}K^{\cdot}) \otimes E_r^{e,f}(F^{\cdot}K^{\cdot}) \to E_r^{a+c+e,b+d+f}(F^{\cdot}K^{\cdot})$$

That is, the cup product on $E_r^{\cdot,\cdot}(F^{\cdot}K^{\cdot})$ is associative for all $r \geq 1$.

□

## 7.3 Construction of $F^{\cdot}\theta^{\cdot}$

In this section we will construct the filtered chain map $F^{\cdot}\theta^{\cdot} : F^{\cdot}\mathrm{Tot}^{\cdot}(W_{\cdot} \otimes (F^{\cdot}K^{\cdot})^{[p]}) \to F^{\cdot}K^{\cdot}$. Consider the solid diagram below:

$$
\begin{tikzcd}
F^{\cdot}\mathrm{Tot}^{\cdot}((F^{\cdot}I^{\cdot})^{[p]}) \arrow[r, dashed, "F^{\cdot}\beta^{\cdot}"] & F^{\cdot}\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, F^{\cdot}I^{\cdot}) \\
F^{\cdot}\mathrm{Tot}^{\cdot}((F^{\cdot}A^{\cdot})^{[p]}) \arrow[u, hook, "F^{\cdot}(\iota^{\cdot})^{[p]}"] \arrow[r, "F^{\cdot}m^{\cdot}_p"] & F^{\cdot}A^{\cdot} \arrow[u, hook, "F^{\cdot}\nu^{\cdot}"']
\end{tikzcd}
$$

Where in the above $F^{\cdot}m^{\cdot}_p$ is the $p$-iterated product on $F^{\cdot}A^{\cdot}$, which is a filtered chain map. In the above we let $\pi$, the cyclic group of order $p$, act on tensors by permuting with graded signs, and the action on $F^{\cdot}A^{\cdot}$ is trivial. The action on $\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, F^{\cdot}I^{\cdot})$ is induced by the action on $W_{\cdot}$. We have that both $F^{\cdot}(\iota^{\cdot})^{[p]}$ and $F^{\cdot}\nu^{\cdot}$ are filtered chain maps in $\mathrm{Sh}_{\mathbb{F}_p\pi}(X)$. Because $F^{\cdot}\iota^{\cdot}$ is a strong injective resolution in $\mathrm{Sh}_{\mathbb{F}_p}(X)$, we have by Lemma 6.9.20 the strong filtered injective resolution $F^{\cdot}\nu^{\cdot}$ in $\mathrm{Sh}_{\mathbb{F}_p\pi}(X)$. Note that $F^{\cdot}\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, F^{\cdot}I^{\cdot})$ is strong filtered injective in each degree in $\mathrm{Sh}_{\mathbb{F}_p\pi}(X)$, is bounded below, and has a terminating filtration in each degree. By Corollary 6.7.8, $F^{\cdot}\mathrm{Tot}^{\cdot}((F^{\cdot}A^{\cdot})^{[p]})$ and $F^{\cdot}\mathrm{Tot}^{\cdot}((F^{\cdot}I^{\cdot})^{[p]})$ are both finitely filtered in each degree. Hence their filtrations are exhaustive. By Corollary 6.7.21, $F^{\cdot}(\iota^{\cdot})^{[p]}$ is a strong filtered resolution, and hence, $F^{\cdot}(\iota^{\cdot})^{[p]}$ is an injective filtered quasi-isomorphism. We now have the conditions to invoke Lemma 6.5.2 to find a filtered chain map $F^{\cdot}\beta^{\cdot}$ in $\mathrm{Sh}_{\mathbb{F}_p\pi}(X)$ that makes the square commute. By Lemma 6.5.4, $F^{\cdot}\beta^{\cdot}$ is unique up to filtered homotopy. Because the global section functor $T$ is left exact, by Definition 6.1.24 we have the filtered chain map in ${}_{\mathbb{F}_p\pi}\mathrm{Mod}$:

$$F^{\cdot}T(\beta^{\cdot}) : F^{\cdot}T(\mathrm{Tot}^{\cdot}((F^{\cdot}I^{\cdot})^{[p]})) \to F^{\cdot}T(\mathbf{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, F^{\cdot}I^{\cdot})) = F^{\cdot}\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W_{\cdot}, F^{\cdot}T(I^{\cdot}))$$

By Lemma 6.1.26, $F^{\cdot}T(\beta^{\cdot})$ is unique up to filtered homotopy. There is a natural filtered chain map:

$$F^{\cdot}\gamma^{\cdot}_p : F^{\cdot}\mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}) \to F^{\cdot}T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]}))$$

Define $F^{\cdot}\hat{\theta}^{\cdot}$ by the composition:

$$F^{\cdot}\hat{\theta}^{\cdot} : F^{\cdot}\mathrm{Tot}^{\cdot}(T(F^{\cdot}I^{\cdot})^{[p]}) \xrightarrow{F^{\cdot}\gamma_p} F^{\cdot}T(\mathrm{Tot}^{\cdot}((I^{\cdot})^{[p]})) \xrightarrow{F^{\cdot}T(\beta^{\cdot})} F^{\cdot}\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W.,F^{\cdot}T(I^{\cdot}))$$

By Lemma 6.1.23, the filtered homotopy uniqueness of $F^{\cdot}T(\beta^{\cdot})$ implies the filtered homotopy uniqueness of $F^{\cdot}\hat{\theta}^{\cdot}$. We have the adjoint isomorphism of Lemma 3.1.5, which was shown to be a filtered isomorphism in Corollary 6.9.6.

$$F^{\cdot}\Phi^{\cdot} : F^{\cdot}\mathrm{Hom}^{\cdot}_{\mathbb{F}_p\pi}(\mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}),\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W.,T(I^{\cdot}))) \to F^{\cdot}\mathrm{Hom}^{\cdot}_{\mathbb{F}_p\pi}((\mathrm{Tot}^{\cdot}T(I^{\cdot})^{[p]})\otimes_{\mathbb{F}_p}W.,T(I^{\cdot}))$$

We have:

$$F^{\cdot}\hat{\theta}^{\cdot} \in F^0Z^0(\mathrm{Hom}^{\cdot}_{\mathbb{F}_p\pi}(\mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}),\mathrm{Hom}^{\cdot}_{\mathbb{F}_p}(W.,T(I^{\cdot}))))$$

In the above $F^{\cdot}\hat{\theta}^{\cdot}$ has filtration degree zero because it is a filtered morphism, as mentioned in Definition 6.9.1, and $F^{\cdot}\hat{\theta}^{\cdot}$ is in $Z^0$ because it is a chain map. Thus:

$$F^0\Phi^0(F^{\cdot}\hat{\theta}^{\cdot}) \in F^0Z^0(\mathrm{Hom}^{\cdot}_{\mathbb{F}_p\pi}(\mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]})\otimes W.,T(I^{\cdot})))$$

because $F^{\cdot}\Phi\cdot$ is a filtered chain map. Thus, $\Phi^0(F^{\cdot}\hat{\theta}^{\cdot})$ is a filtered chain map. By Corollary 6.9.7, the filtered homotopy uniqueness of $F^{\cdot}\hat{\theta}^{\cdot}$ implies $F^{\cdot}\Phi^0(\hat{\theta}^{\cdot})$ is unique up to filtered homotopy. We have the filtered isomorphism that swaps tensors with sign based upon grading:

$$F^{\cdot}U^{\cdot} : \mathrm{Tot}^{\cdot}(W.\otimes_{\mathbb{F}_p}T(I^{\cdot})^{[p]}) \to \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}\otimes_{\mathbb{F}_p}W.)$$

And now we define $F^{\cdot}\theta^{\cdot}$ by the composition:

$$F^{\cdot}\theta^{\cdot} : \mathrm{Tot}^{\cdot}(W.\otimes_{\mathbb{F}_p}T(I^{\cdot})^{[p]}) \xrightarrow{F^{\cdot}U^{\cdot}} \mathrm{Tot}^{\cdot}(T(I^{\cdot})^{[p]}\otimes_{\mathbb{F}_p}W.) \xrightarrow{F^{\cdot}\Phi^0(\hat{\theta}^{\cdot})} T(I^{\cdot})$$

By Lemma 6.1.23, the filtered homotopy uniqueness of $F^{\cdot}\hat{\theta}^{\cdot}$ implies $F^{\cdot}\theta^{\cdot}$ is unique up to filtered homotopy. The $F^{\cdot}\theta^{\cdot}$ constructed here is actually a special case of the construction of $\theta^{\cdot}$ from Section 5.2, but now we have that it preserves the filtration. Because $F^{\cdot}\theta^{\cdot}$ is a filtered chain map, there are induced morphisms for all $a,b \in \mathbb{Z}$, $r \geq 1$:

$$E_r^{a,b}(F^{\cdot}\theta^{\cdot}) : E_r^{a,b}(F^{\cdot}\mathrm{Tot}^{\cdot}(W_{\cdot} \otimes (F^{\cdot}K^{\cdot})^{[p]})) \to E_r^{a,b}(F^{\cdot}K^{\cdot})$$

Because $F^{\cdot}\theta^{\cdot}$ is unique up to filtered homotopy, these morphisms are uniquely defined by Lemma 6.1.29.

## 7.4 Construction of Operations

It should be noted that we defined $F^{\cdot}\theta^{\cdot}$ to be a filtered $\mathbb{F}_p\pi$ chain map:

$$F^{\cdot}\theta^{\cdot} : \mathrm{Tot}^{\cdot}(W_{\cdot} \otimes_{\mathbb{F}_p} T(I^{\cdot})^{[p]}) \to F^{\cdot}T(I^{\cdot})$$

but since the action of $\pi$ on $F^{\cdot}T(I^{\cdot})$ is trivial, we have by Lemma 3.1.3 that $F^{\cdot}\theta^{\cdot}$ also specifies a canonical morphism:

$$F^{\cdot}\theta^{\cdot} : \mathrm{Tot}^{\cdot}(W_{\cdot} \otimes_{\pi} T(I^{\cdot})^{[p]}) \to F^{\cdot}T(I^{\cdot})$$

**Definition 7.4.1.** *We can now define the maps $D_i$ on the first and infinite pages of the spectral sequence for $F^{\cdot}K^{\cdot}$. Let $r = 1$ or $\infty$ and let $a,b \in \mathbb{Z}$. Let $[x] \in E_r^{a,b}(F^{\cdot}K^{\cdot})$. Then by Lemma 7.1.1, $[e_i \otimes_{\pi} x^{[p]}]$ is a well defined element of $E_r^{ap,bp-i}(F^{\cdot}Tot^{\cdot}(W_{\cdot} \otimes_{\pi} (K^{\cdot})^{[p]}))$, and we can apply the morphism $F^{\cdot}\theta^{\cdot}$ induces on the spectral sequence:*

$$D_i([x]) = E_r^{ap,bp-i}(F^{\cdot}Tot^{\cdot}(\theta^{\cdot,\cdot}))([e_i \otimes_{\pi} x^{[p]}]) \in E_r^{ap,bp-i}(F^{\cdot}K^{\cdot})$$

*Now using the definition of $P^{\cdot}$ and $\beta P^{\cdot}$ from Corollary 2.0.8, we have induced Steenrod operations on the spectral sequence $E_{\cdot}^{\cdot,\cdot}(F^{\cdot}K^{\cdot})$. Let $a,b \in \mathbb{Z}$ and $r = 1$ or $\infty$. Let $[x] \in E_r^{a,b}(F^{\cdot}K^{\cdot})$. For $p = 2$ we have:*

$$P^s([x]) = D_{a+b-s}([x]) \in E_r^{2a,2b-(a+b-s)}(F^{\cdot}K^{\cdot}) = E_r^{2a,b-a+s}(F^{\cdot}K^{\cdot})$$

*For $p > 2$ we have:*

$$P^s([x]) = (-1)^s \nu(-(a+b)) D_{((a+b)-2s)(p-1)}([x])$$

$$\begin{aligned}&\in E_r^{ap,bp-(a+b-2s)(p-1)}(F^{\cdot}K^{\cdot})\\&=E_r^{ap,b+(2s-a)(p-1)}(F^{\cdot}K^{\cdot})\end{aligned}$$

$$\begin{aligned}\beta P^s([x])&=(-1)^s\nu(-(a+b))D_{((a+b)-2s)(p-1)-1}([x])\\&\in E_r^{ap,bp-[(a+b-2s)(p-1)-1]}(F^{\cdot}K^{\cdot})\\&=E_r^{ap,b+(2s-a)(p-1)+1}(F^{\cdot}K^{\cdot})\end{aligned}$$

*where* $\nu(-q)=(-1)^j(m!)^{\varepsilon}$, *with* $q=2j-\varepsilon$, *and* $\varepsilon=0$ *or 1.*

**Lemma 7.4.2.** *For all r in which the Steenrod operations are defined on* $E_r^{\cdot,\cdot}(F^{\cdot}K^{\cdot})$*, the operations agree with those previously defined on* $H^{\cdot}(K^{\cdot})$*.*

*Proof.* The proof of this lemma is analogous to the proof of Lemma 7.2.3. Let $a,b\in\mathbb{Z}$ and suppose $r\geq 1$. Suppose $x\in Z^{a+b}(K^{\cdot})\cap\tilde{Z}_r^{a,b}(F^{\cdot}K^{\cdot})$. Then when one regards $[x]\in H^{a+b}(K^{\cdot})$, we have the $D_i$ map from Definition 2.0.7:

$$D_i([x])=[\theta^{ap+bp-i}(e_i\otimes_{\pi}x^{[p]})]\in H^{ap+bi-i}(K^{\cdot})$$

And when one regards $[x]\in E_r^{a,b}(F^{\cdot}K^{\cdot})$ and uses the $D_i$ map from Definition 7.4.1, we have:

$$D_i([x])=[F^{ap}\theta^{ap+bp-i}(e_i\otimes_{\pi}x^{[p]})]\in E_r^{ap,bp-i}(F^{\cdot}K^{\cdot})$$

Since both cosets are represented by the same element, the forms of $D_i$ agree for all $i$. Since the Steenrod operations $P^s$ and $\beta P^s$ are defined in the same way in terms of $D_i$, this implies the operations also agree in this sense. □

**Theorem 7.4.3.** *Let X be a topological space and k a field of characteristic p. Suppose* $F^{\cdot}A^{\cdot}$ *is a bounded below filtered complex of sheaves of graded commutative* $\mathbb{F}_p$ *algebras on X, where* $A^{\cdot}$ *is finitely filtered in each degree and the product on* $A^{\cdot}$ *preserves the filtration. That is,* $F^{m_1}A^{\cdot}\cdot F^{m_2}A^{\cdot}\subseteq F^{m_1+m_2}A^{\cdot}$ *for all* $m_1,m_2\in\mathbb{Z}$*. Then the canonically defined Steenrod operations on the hypercohomology groups,* $\mathbf{H}^n(X,A^{\cdot})$*, from Theorem 5.3.5 also act in a canonical and compatible way on the* $E_1$ *and* $E_\infty$ *pages of the spectral sequence:*

$$E_r^{a,b}(F^\cdot K^\cdot) \Longrightarrow H^{a+b}(K^\cdot) = \mathbf{H}^{a+b}(X, A^\cdot)$$

*where $F^\cdot A^\cdot \hookrightarrow F^\cdot I^\cdot$ is a filtered injective resolution in $Sh_{\mathbb{F}_p}(X)$, $F^\cdot K^\cdot = F^\cdot T(I^\cdot)$, and $E_r^{a,b}(F^\cdot K^\cdot)$ is as defined in Definition 6.1.27.*

*Proof.* The construction of $F^\cdot\theta$ in this chapter is actually a more specific construction of the $\theta^\cdot$ from Chapter 5. So the Steenrod operations from Theorem 5.3.5 can actually be induced by this $F^\cdot\theta^\cdot$, which in turn induces canonical operations on $E_1^{\cdot,\cdot}(F^\cdot K^\cdot)$ and $E_\infty^{\cdot,\cdot}(F^\cdot K^\cdot)$ as in Definition 7.4. By Lemma 7.4.2, the operations on $E_1^{\cdot,\cdot}(F^\cdot K^\cdot)$ and $E_\infty^{\cdot,\cdot}(F^\cdot K^\cdot)$ are compatible with the operations $F^\cdot\theta^\cdot$ induces on $H^\cdot(K^\cdot) = \mathbf{H}^\cdot(X, A^\cdot)$. $\square$

## 7.5 Applications

In this section I will apply Theorem 7.4.3 to a few different bounded below complexes of sheaves of differential graded commutative $\mathbb{F}_p$ algebras that are finitely filtered in each degree.

### 7.5.1 The Stupid Filtration

Let $X$ be a topological space and let $A^\cdot$ be complex of sheaves of differential graded commutative $\mathbb{F}_p$ algebras on $X$, with $A^\cdot$ concentrated in non-negative degree. We may give $A^\cdot$ the stupid filtration, defined as follows, for all $n, m \in \mathbb{Z}$:

$$F^i A^n = \begin{cases} A^n & \text{when } n \geq i \\ 0 & \text{otherwise} \end{cases}$$

We have that the filtered complex $F^\cdot A^\cdot$ is finitely filtered in each degree. Because $A^\cdot$ is concentrated in non-negative degree and the product is graded, we have that the multiplication map

$$m^\cdot : \mathrm{Tot}^\cdot(A^\cdot \otimes A^\cdot) \to A^\cdot$$

is filtered, since $m^{i+j}(A^i \otimes A^j) \subseteq A^{i+j}$ for all $i, j \geq 0$. Since $F^\cdot A^\cdot$ is finitely filtered in each degree and $F^\cdot m^\cdot$ is a filtered chain map, we can apply Theorem 7.4.3 to obtain Steenrod operations on $\mathbf{H}^{\cdot}(X,A^{\cdot})$ and $E_1^{\cdot,\cdot}(F^{\cdot}K^{\cdot})$, $E_{\infty}(F^{\cdot}K^{\cdot})$, where $F^{\cdot}A^{\cdot} \hookrightarrow F^{\cdot}I^{\cdot}$ is a filtered injective resolution in $\mathrm{Sh}_{\mathbb{F}_p}(X)$, and $F^{\cdot}K^{\cdot} = F^{\cdot}T(I^{\cdot})$.

As a side note, in choosing a filtered injective resolution $F^{\cdot}\iota^{\cdot} : F^{\cdot}A^{\cdot} \hookrightarrow F^{\cdot}I^{\cdot}$, we can in fact choose a Cartan Eilenberg resolution, $A^{\cdot} \hookrightarrow J^{\cdot,\cdot}$, with embedding $\varepsilon^{\cdot} : A^{\cdot} \to J^{\cdot,0}$, and then define $I^{\cdot} = \mathrm{Tot}^{\cdot}(J^{\cdot,\cdot})$. Let $\iota^{\cdot} : A^{\cdot} \hookrightarrow I^{\cdot}$ be induced by $\varepsilon^{\cdot}$. We give $I^{\cdot}$ the first filtration, where one has for all $i,n \in \mathbb{Z}$:

$$F^iI^n = \bigoplus_{a+b=n,\ a\geq i} J^{a,b}$$

With $F^{\cdot}I^{\cdot}$ filtered as such and $F^{\cdot}A^{\cdot}$ given the stupid filtration, the chain map $\iota^{\cdot}$ is a filtered morphism. In fact, $F^{\cdot}\iota^{\cdot}$ is an injective filtered quasi-isomorphism by the construction of the Cartan Eilenberg resolution. Because both $F^{\cdot}A^{\cdot}$ and $F^{\cdot}I^{\cdot}$ are finitely filtered, $F^{\cdot}\iota^{\cdot}$ is a strong filtered injective resolution, and an injective resolution in the non-filtered sense.

Setting $K^{\cdot} = T(I^{\cdot})$ with $T$ the global section functor, we have $H^{\cdot}(K^{\cdot})$ computes the hypercohomology groups of $X$ with coefficients in $A^{\cdot}$:

$$H^n(K^{\cdot}) = \mathbf{H}^n(X,A^{\cdot})$$

And the filtration on $K^{\cdot}$ induces a spectral sequence that converges to these hypercohomology groups:

$$E_r^{a,b}(F^{\cdot}K^{\cdot}) \Longrightarrow \mathbf{H}^{a+b}(X,A^{\cdot})$$

In fact, with the stupid filtration on $A^{\cdot}$, we have:

$$E_1^{a,b}(F^{\cdot}K^{\cdot}) = H^b(X,A^a)$$

The above is evident when one chooses $F^{\cdot}I^{\cdot} = F^{\cdot}\mathrm{Tot}^{\cdot}(J^{\cdot,\cdot})$ to be the total complex of a Cartan Eilenberg resolution of $A^{\cdot}$, given the first filtration.

### 7.5.2 The Hodge to De Rham Spectral Sequence

This is a special case of Section 7.5.1. Let $X$ be a smooth projective variety over a field $k$ of characteristic $p$. We let $A^{\cdot} = \Omega^{\cdot}_{X/k}$, be the De Rham complex of $X$, where $\Omega^{\cdot}_{X/k}$ is concentrated in non-negative degree and the wedge product makes $\Omega^{\cdot}_{X/k}$ a sheaf of differential graded commutative $\mathbb{F}_p$ algebras on $X$. We may follow through the steps of Section 7.5.1 to obtain a filtered injective resolution $F^{\cdot}\iota^{\cdot} : F^{\cdot}A^{\cdot} \hookrightarrow F^{\cdot}I^{\cdot}$. Setting $F^{\cdot}K^{\cdot} = F^{\cdot}T(I^{\cdot})$, we have that $H^{\cdot}(K^{\cdot})$ computes the algebraic De Rham cohomology of $X$ over $k$.

$$H^n(K^{\cdot}) = \mathbf{H}^n(X, \Omega^{\cdot}_{X/k}) = H^n_{\mathrm{DR}}(X/k)$$

Meanwhile, the spectral sequence:

$$E_r^{a,b}(F^{\cdot}K^{\cdot}) \Longrightarrow H^{a+b}_{\mathrm{DR}}(X/k)$$

is in fact the Hodge to De Rham spectral sequence, where we have:

$$E_1^{a,b} = H^b(X, \Omega^a_{X/k})$$

Theorem 7.4.3 gives Steenrod operations that act in a compatible way on $H^{\cdot}_{\mathrm{DR}}(X/k)$, $E_1^{\cdot,\cdot}$, and $E_\infty^{\cdot,\cdot}$.

### 7.5.3 A Spectral Sequence of Katz and Oda

Let $k$ be a field of characteristic $p$ and let $\pi : X \to S$ be a smooth $k$ morphism of smooth varieties over $k$, with $S$ affine. We may filter the De Rham complex $\Omega^{\cdot}_{X/k}$ as done by Katz and Oda in [4], page 202:

$$F^i\Omega^n_{X/k} = \mathrm{im}(\Omega^{n-i}_{X/k} \otimes_{O_X} \pi^*(\Omega^i_{S/k}) \to \Omega^n_{X/k})$$

where the map in the above image calculation is the wedge product on $\Omega^{\cdot}_{X/k}$. Katz and Oda verify that this filtration is finite in each degree and compatible with the wedge product on page 202 of [4]. Thus, we can apply Theorem 7.4.3 to obtain Steenrod operations acting on $H^{\cdot}_{\mathrm{DR}}(X/k)$ and in a compatible way on $E_1^{\cdot,\cdot}(F^{\cdot}K^{\cdot})$ and $E_\infty^{\cdot,\cdot}(F^{\cdot}K^{\cdot})$, where $F^{\cdot}\Omega^{\cdot}_{X/k} \hookrightarrow F^{\cdot}I^{\cdot}$ is a filtered injective resolution in $\mathrm{Sh}_{\mathbb{F}_p}(X)$ and $F^{\cdot}K^{\cdot} = F^{\cdot}T(I^{\cdot})$. On page 210, Katz and Oda compute the $E_1$ page of this spectral sequence:

$$E_1^{a,b} \cong \Gamma_S(\Omega^a_{S/k} \otimes_{O_S} \mathcal{H}^b_{\mathrm{DR}}(X/S))$$

Thus the Steenrod operations on $H^{\cdot}_{\mathrm{DR}}(X/k)$ also induce operations as described in Definition 7.4.1 on these objects as well.

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