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POISSON BRACKET FORMULATION OF A HIGHER-ORDER, GEOMETRICALLY-EXACT BEAM

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Abstract. This paper investigates the Hamiltonian structure and Poisson bracket formulation of a higher-order, geometrically-exact Cosserat type beam with a deforming cross-section in terms of canonically conjugate variables.

1 Introduction

In our recent works, we investigated and refined the kinematics of Cosserat beams [1], and then we developed the variational and numerical formulation for a geometrically-exact beam with those improved kinematics [2]. This development incorporated fully-coupled Poisson's and warping effects, along with the classical deformation effects like bending, torsion, shear, and axial deformation for the case of *finite displacement and strain*; this allowed us to capture a three dimensional, multi-axial strain fields using single-manifold kinematics. Under the adapted kinematic model, a beam/rod is modeled by a framed spatial curve and a family of deformable cross-sections that can undergo both in-plane and out-of-plane deformation.

This paper deals with the *Poisson bracket* formulation associated with the beam kinematics discussed in [1]. The Poisson bracket formulation constitutes a part of the variational analysis of a mechanical system. In that regard, this paper is a theoretical extension to the variational formulation of the beam discussed in [2] that detailed the governing equation of motion, the associated weak form, and the numerical solution of the equation of motion. Therefore, we borrow the results abundantly from [2]. We define the Hamiltonian structure of the geometrically-exact beam with enhanced kinematics and deformable cross-section in terms of the canonical conjugate variables (as is indicated in Marsden and Hughes [3]). The paper by Simo et al. [4] discusses the Poisson bracket formulation for geometrically-exact beams with rigid cross-sections. The task of investigating the Hamiltonian structure of the generalized beam is substantially more challenging than the formulation described in [4], because the deformation map discussed in [1] is a function of higher-order derivatives of the mid-curve axial strain and the curvature tensor. Therefore, the canonical quantities associated with not only the quantities belonging to the tangent space of the beam but also their higher-order spatial derivatives have to be found, making this problem unique.

Section 2 briefly covers the description of beam configuration and its kinematics. Section 3 presents the governing equations of motion that are useful in the investigation of the Hamiltonian structure of the beam. Section 4 presents the Poisson bracket formulation. We discuss the cotangent space, phase space, and cotangent bundle associated with the beam configuration. We also define the Poisson bracket associated with the cotangent bundle or phase space of the system. Poisson brackets essentially help one study flows on phase space and the *generators* associated with such flows, and they facilitate the development of canonical transformations. Canonically transformed phase space coordinates preserve the Poisson geometry associated with the system or equivalently they preserve the Hamiltonian structure of the system. We obtain the Hamiltonian via a Legendre transformation of the Lagrangian. Finally, the Hamiltonian form of equilibrium equations are obtained. Section 5 concludes the paper.

2 Comprehensive kinematics and mathematical tools

We first present some preliminary definitions and notations. The dot product, ordinary vector product, and tensor product of two Euclidean vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are defined as $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \boldsymbol{v}_1^T \boldsymbol{v}_2$, $\boldsymbol{v}_1 \times \boldsymbol{v}_2$, and $\boldsymbol{v}_1 \otimes \boldsymbol{v}_2$ respectively. The n^{th} order partial derivative with respect to a scalar, e.g., ξ_1, is given by the operator $\partial^n_{\xi_1}$, with $\partial^1_{\xi_1} \equiv \partial_{\xi_1}$. The action of a tensor $\boldsymbol{A}$ onto the vector $\boldsymbol{v}$ is represented by $\boldsymbol{A}\boldsymbol{v} \equiv \boldsymbol{A}.\boldsymbol{v}$. The contraction between two tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is given by $\boldsymbol{A} : \boldsymbol{B} = A_{ij}B_{ij} = \text{trace}(\boldsymbol{B}^T.\boldsymbol{A})$. We note that the centered dot "$\cdot$" is meant for dot product between two vectors, whereas the action of a tensor onto the vector, the matrix multiplication or product of a scalar to a matrix (or a vector) is denoted by a lower dot ".".

2.1 Deformation map and configuration of the beam

Let an open set $\Omega_0 \subset \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^3$ with at least piecewise smooth boundaries $\mathfrak{S}_0$ and $\mathfrak{S}$ represent the undeformed and deformed configuration of the beam, respectively. The beam configuration is described by the mid-curve and a family of cross-sections. To lay the kinematic description of a beam, we assume the undeformed configuration Ω_0 to be *straight*.

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Let the fixed orthonormal reference basis be represented by $\{\boldsymbol{E}_i\}$ with origin at $(0, 0, 0)$. The regular curve $\boldsymbol{\varphi}_0 : [0, L] \longrightarrow \mathbb{R}^3$ represents the mid-curve associated with Ω_0. It is parameterized by the arc-length $\xi_1 \in [0, L]$. We assume that the undeformed configuration is made up of a continuously varying plane family of cross-sections $\mathscr{B}_0(\xi_1)$, such that $\boldsymbol{\varphi}_0 = \xi_1 \boldsymbol{E}_1$ is the locus of geometric centroid of the family of cross-sections $\mathscr{B}_0(\xi_1)$. The cross-section $\mathscr{B}_0(\xi_1)$ is spanned by the vectors $\boldsymbol{E}_2 - \boldsymbol{E}_3$ originating at $\boldsymbol{\varphi}_0(\xi_1)$ such that $(\xi_2, \xi_3) \in \mathscr{B}_0(\xi_1)$. Let $\Gamma_0(\xi_1)$ represent the peripheral boundary of $\mathscr{B}_0(\xi_1)$, such that $\mathfrak{S}_0 = \mathscr{B}_0(0) \cup \mathscr{B}_0(L) \cup_{\forall \xi_1} \Gamma_0(\xi_1)$. Any material point in the beam is defined by its material coordinate (ξ_1, ξ_2, ξ_3) with a position vector $\boldsymbol{R}_0 = \xi_i \boldsymbol{E}_i$. The final deformed state Ω defined by the mid-curve $\boldsymbol{\varphi}$ and a family of cross-section $\mathscr{B}(\xi_1) = \left\{ \left(W(\xi_1, \xi_2, \xi_3), \hat{\xi}_2, \hat{\xi}_3 \right) \in \mathbb{R}^3_{\xi_1} \right\}$. It incorporates a fully coupled Poisson's and warping effect. The deformation map for Ω is given by $\boldsymbol{\phi} : \boldsymbol{R}_0 \in \Omega_0 \longmapsto \boldsymbol{R} \in \Omega$, such that

$$\begin{aligned} \boldsymbol{\phi}(\boldsymbol{R}_0) &= \boldsymbol{R} = \boldsymbol{\varphi}(\xi_1) + \boldsymbol{r}; \\ \boldsymbol{r} &= \hat{\xi}_2 \boldsymbol{d}_2(\xi_1) + \hat{\xi}_3 \boldsymbol{d}_3(\xi_1) + W \boldsymbol{d}_1(\xi_1). \end{aligned} \tag{1}$$

Here, the vector $\boldsymbol{r}$ gives the position vector of a material point (ξ_2, ξ_3) in the deformed cross-section $\mathscr{B}(\xi_1)$ with respect to the point $\boldsymbol{\varphi}(\xi_1)$. Let $\Gamma(\xi_1)$ represent the boundary of cross-section $\mathscr{B}(\xi_1)$, such that $\mathfrak{S} = \mathscr{B}(0) \cup \mathscr{B}(L) \cup_{\forall \xi_1} \Gamma(\xi_1)$. The warping function W is defined as

$$W(\xi_1, \xi_2, \xi_3) = p(\xi_1)\Psi_1(\xi_2, \xi_3) + \partial_{\xi_1} \overline{\kappa}_2 . \Psi_2(\xi_2, \xi_3) + \partial_{\xi_1} \overline{\kappa}_3 . \Psi_3(\xi_2, \xi_3) = p(\xi_1)\Psi_1(\xi_2, \xi_3) + \partial_{\xi_1} \overline{\boldsymbol{\kappa}} \cdot \overline{\boldsymbol{\Psi}}_{23}. \tag{2}$$

In the equation above, $\overline{\boldsymbol{\Psi}}_{23} = \Psi_2(\xi_2, \xi_3)\boldsymbol{E}_2 + \Psi_3(\xi_2, \xi_3)\boldsymbol{E}_3$ and $\partial_{\xi_1} \overline{\boldsymbol{\kappa}} = \partial_{\xi_1} \overline{\kappa}_i . \boldsymbol{E}_i$ (defined in next section). The warping function W mentioned above is a modified version of the warping used in Simo and Vu-Quoc [5], where $p(\xi_1)$ gives the warping amplitude. The coefficients $\partial_{\xi_1} \overline{\kappa}_j$ $(j = 2, 3)$ incorporated bending induced non-uniform shear deformation. For the sake of computation, the cross-section dependent functions $\Psi_1(\xi_2, \xi_3)$, $\Psi_2(\xi_2, \xi_3)$ and $\Psi_3(\xi_2, \xi_3)$ are assumed to be known.

We define the planar cross-section $\mathscr{B}_3 = \left\{ (\hat{\xi}_2, \hat{\xi}_3) \in \mathbb{R}^2_{\xi_1} \right\}$ subjected to only in-plane Poisson's deformation. The coordinates $(\hat{\xi}_2, \hat{\xi}_3)$ are obtained by Poisson's transformation $P_{\xi_1} : \mathscr{B}_1(\xi_1) \longrightarrow \mathscr{B}_3(\xi_1)$, such that (refer to Eq. (34) of Chadha and Todd [1]),

$$\begin{aligned} &P_{\xi_1} : (\xi_2, \xi_3) \longmapsto (\hat{\xi}_2, \hat{\xi}_3); \\ &\hat{\xi}_i = \left(1 - \nu \left(\overline{\boldsymbol{\varepsilon}} \cdot \boldsymbol{E}_1 + \overline{\boldsymbol{\kappa}} \cdot (\xi_3 \boldsymbol{E}_2 - \xi_2 \boldsymbol{E}_3) + \partial_{\xi_1} p . \Psi_1 + \partial^2_{\xi_1} \overline{\boldsymbol{\kappa}} \cdot \overline{\boldsymbol{\Psi}}_{23} \right) \right) \xi_i \text{ for } i = 2, 3. \end{aligned} \tag{3}$$

In the equation above, ν represents Poisson's ratio and is assumed to be a constant (homogeneous material). The mid-curve axial strain is defined as $\boldsymbol{\varepsilon} = \partial_{\xi_1} \boldsymbol{\varphi} - \boldsymbol{d}_1 = \overline{\varepsilon}_i \boldsymbol{d}_i$. For more details on the kinematics, readers are referred to Section 2.1 of Chadha and Todd [2]. In next section we define the spatial curvatures $\boldsymbol{\kappa}$, and their material counterparts $\overline{\boldsymbol{\varepsilon}}$ and $\overline{\boldsymbol{\kappa}}$.

2.2 Rotation and finite strain parameters

2.2.1 Finite rotation and curvature

Finite rotations are represented by an element of a proper orthogonal rotation group $SO(3)$. The $SO(3)$ manifold is a non-linear compact Lie group that has a linear skew-symmetric matrix as its Lie algebra, $so(3)$. The director triad $\{\boldsymbol{d}_i\}$ is related to the fixed reference triad $\{\boldsymbol{E}_i\}$ by means of an orthogonal tensor $\boldsymbol{Q} \in SO(3)$, such that $\boldsymbol{d}_i = \boldsymbol{Q}.\boldsymbol{E}_i$ and $\boldsymbol{Q} = \boldsymbol{d}_i \otimes \boldsymbol{E}_i$. Curvature defines the local change of the triad such that

$$\partial_{\xi_1} \boldsymbol{d}_i = \partial_{\xi_1} \boldsymbol{Q}.\boldsymbol{E}_i = \partial_{\xi_1} \boldsymbol{Q}.\boldsymbol{Q}^T.\boldsymbol{d}_i = \hat{\boldsymbol{\kappa}} \cdot \boldsymbol{d}_i. \tag{4}$$

Here, $\hat{\boldsymbol{\kappa}} = \partial_{\xi_1} \boldsymbol{Q}.\boldsymbol{Q}^T$ represents curvature tensor. It is an anti-symmetric matrix with the corresponding axial vector $\boldsymbol{\kappa} = \overline{\kappa}_i \boldsymbol{d}_i$, known as curvature vector. We define $T_{\boldsymbol{Q}}SO(3)$ as the tangent plane of non-linear $SO(3)$ manifold, such that $\partial_{\xi_1} \boldsymbol{Q} = \hat{\boldsymbol{\kappa}}.\boldsymbol{Q} \in T_{\boldsymbol{Q}}SO(3)$. We note that $so(3) = T_{\boldsymbol{I}_3}SO(3)$.

2.2.2 Material and spatial curvature

We define the quantity $\hat{\overline{\boldsymbol{\kappa}}} = \boldsymbol{Q}^T.\hat{\boldsymbol{\kappa}}.\boldsymbol{Q} = \boldsymbol{Q}^T.\partial_{\xi_1} \boldsymbol{Q} \in so(3)$ obtained by parallel transport of $\hat{\boldsymbol{\kappa}}.\boldsymbol{Q}$ from $T_{\boldsymbol{Q}}SO(3) \longrightarrow so(3)$. We call the quantities $\hat{\overline{\boldsymbol{\kappa}}}$ and $\overline{\boldsymbol{\kappa}}$ as *material representation*; and $\hat{\boldsymbol{\kappa}}$ and $\boldsymbol{\kappa}$ as *spatial representation* of the *curvature tensor* and the *curvature vector* respectively. Like with the curvature tensor, we may express a material form of other quantities like the deformation gradient tensor, angular velocity, etc. For instance, the material form of the axial strain vector and cross-section position vectors $\boldsymbol{r}$ is given by the following

$$\overline{\boldsymbol{\varepsilon}} = \overline{\varepsilon}_i \boldsymbol{E}_i = \boldsymbol{Q}^T.\boldsymbol{\varepsilon}; \tag{5a}$$

$$\overline{\boldsymbol{r}} = \boldsymbol{Q}^T.\boldsymbol{r} = \hat{\xi}_2 \boldsymbol{E}_2 + \hat{\xi}_3 \boldsymbol{E}_3 + W \boldsymbol{E}_1. \tag{5b}$$

From here on, we recognize any material quantity with a bar $\bar{(.)}$ over it. Consider a spatial and material vector $\boldsymbol{v} = \bar{v}_i \boldsymbol{d}_i$ and $\bar{\boldsymbol{v}} = \bar{v}_i \boldsymbol{E}_i$, respectively, such that $\boldsymbol{v} = \boldsymbol{Q}.\bar{\boldsymbol{v}}$. The derivative of these vectors are obtained as

$$\begin{gathered} \partial_{\xi_1}\boldsymbol{v} = \partial_{\xi_1}\bar{v}_i.\boldsymbol{d}_i + \bar{v}_i.\partial_{\xi_1}\boldsymbol{d}_i = \tilde{\partial}_{\xi_1}\boldsymbol{v} + \boldsymbol{\kappa}\times\boldsymbol{v}; \\ \partial_{\xi_1}\bar{\boldsymbol{v}} = \partial_{\xi_1}\bar{v}_i.\boldsymbol{E}_i = \boldsymbol{Q}^T.\tilde{\partial}_{\xi_1}\boldsymbol{v}. \end{gathered} \tag{6}$$

In the equation above, $\tilde{\partial}_{\xi_1}\boldsymbol{v}$ defines co-rotational derivative of spatial vector $\boldsymbol{v}$. Readers are referred to [6] for obtaining expressions for the higher-order and co-rotational derivatives of the curvature and rotation tensor.

2.3 Configuration and the state space

Adapting the kinematics discussed above, we find that there are three primary quantities required to define the configuration Ω: $\boldsymbol{\varphi} \in \mathbb{R}^3$, $\boldsymbol{Q} \in SO(3)$ and $p \in \mathbb{R}$. For the static case, the configuration space, tangent space, and tangent bundle of the beam Ω are defined as

$$\begin{aligned} &\mathbb{C} := \left\{\boldsymbol{\Phi} = (\boldsymbol{\varphi}, \boldsymbol{Q}, p) : [0, L] \longrightarrow \mathbb{R}^3 \times SO(3) \times \mathbb{R}\right\}; \\ &T_{\boldsymbol{\Phi}}\mathbb{C} := \left\{\tilde{\boldsymbol{\Phi}} = (\partial_{\xi_1}\boldsymbol{\varphi}, \partial_{\xi_1}\boldsymbol{Q}, \partial_{\xi_1}p) : [0, L] \longrightarrow \mathbb{R}^3 \times T_{\boldsymbol{Q}}SO(3) \times \mathbb{R}\right\}; \\ &T\mathbb{C} := \left\{(\boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}) | \boldsymbol{\Phi} \in \mathbb{C}, \tilde{\boldsymbol{\Phi}} \in T_{\boldsymbol{\Phi}}\mathbb{C}\right\}. \end{aligned} \tag{7}$$

3 Weak and Strong form of governing differential equation

In [2], we obtained both weak and strong form of the balance laws for higher-order, geometrically-exact beams. We briefly review the results here. For the current analysis, we ignore the external forces such that

$$G(\boldsymbol{\Phi}, \delta\boldsymbol{\Phi}) = \delta\mathrm{U}_{\text{strain}} + \delta\mathrm{W}_{\text{inertial}} = 0. \tag{8}$$

We reproduce the desirable (for this paper) expression of virtual strain energy $\delta\mathrm{U}_{\text{strain}}$, and the virtual inertial work $\delta\mathrm{W}_{\text{inertial}}$ as obtained in Eq. (23) and Eq, (25b) of the supplementary material in Appendix B of Chadha and Todd [2]

$$\begin{aligned} \delta\mathrm{U}_{\text{strain}} = &\int_0^L \partial_{\xi_1}\big(-\boldsymbol{\mathcal{N}}_{\boldsymbol{\varepsilon}} + \tilde{\partial}_{\xi_1}\boldsymbol{\mathcal{N}}_{\partial_{\xi_1}\boldsymbol{\varepsilon}}\big)\cdot\delta\boldsymbol{\varphi}\,\mathrm{d}\xi_1 + \int_0^L \big(\mathcal{N}_p - \partial_{\xi_1}\mathcal{N}_{\partial_{\xi_1}p} + \partial^2_{\xi_1}\mathcal{N}_{\partial^2_{\xi_1}p}\big).\delta p\,\mathrm{d}\xi_1 \\ &+ \int_0^L \Big(\partial_{\xi_1}\big(-\boldsymbol{\mathcal{N}}_{\boldsymbol{\kappa}} + \tilde{\partial}_{\xi_1}\boldsymbol{\mathcal{N}}_{\partial_{\xi_1}\boldsymbol{\kappa}} - \tilde{\partial}^2_{\xi_1}\boldsymbol{\mathcal{N}}_{\partial^2_{\xi_1}\boldsymbol{\kappa}} + \tilde{\partial}^3_{\xi_1}\boldsymbol{\mathcal{N}}_{\partial^3_{\xi_1}\boldsymbol{\kappa}}\big) - \partial_{\xi_1}\hat{\boldsymbol{\varphi}}.\big(\boldsymbol{\mathcal{N}}_{\boldsymbol{\varepsilon}} - \tilde{\partial}_{\xi_1}\boldsymbol{\mathcal{N}}_{\partial_{\xi_1}\boldsymbol{\varepsilon}}\big)\Big)\cdot\delta\boldsymbol{\alpha}\,\mathrm{d}\xi_1 + \delta\mathrm{U}^*_{\text{strain}}. \end{aligned} \tag{9a}$$

$$\begin{aligned} \delta\mathrm{W}_{\text{inertial}} = &\int_0^L \Big(\delta\boldsymbol{\varphi}\cdot\big(\boldsymbol{F}_{\boldsymbol{\varphi}} - \partial_{\xi_1}\boldsymbol{F}_{\boldsymbol{\varepsilon}}\big) + \delta p.\big(F_p - \partial_{\xi_1}F_{\partial_{\xi_1}p}\big) + \\ &\delta\boldsymbol{\alpha}\cdot\big(\boldsymbol{F}_{\boldsymbol{\alpha}} - \partial_{\xi_1}\hat{\boldsymbol{\varphi}}.\boldsymbol{F}_{\boldsymbol{\varepsilon}} - \partial_{\xi_1}(\boldsymbol{F}_{\boldsymbol{\kappa}} - \tilde{\partial}_{\xi_1}\boldsymbol{F}_{\partial_{\xi_1}\boldsymbol{\kappa}} + \tilde{\partial}^2_{\xi_1}\boldsymbol{F}_{\partial^2_{\xi_1}\boldsymbol{\kappa}})\big)\Big)\,\mathrm{d}\xi_1 + \delta\mathrm{W}^*_{\text{inertial}}. \end{aligned} \tag{9b}$$

From Theorem 1 of the Section 2.3.2 of the supplementary material in Appendix B of Chadha and Todd [2],

$$\delta\mathrm{U}^*_{\text{strain}} + \delta\mathrm{W}^*_{\text{inertial}} = 0. \tag{10}$$

Here, the terms $\boldsymbol{\mathcal{N}}_{(.)}$, and $\boldsymbol{F}_{(.)}$ represents the reduced internal and inertial force vectors, respectively. They are defined in Eq. (57) and (132) of [2].

The strong form essentially represents the local balance laws governing the deformation of the beam. Integration by part of the weak form yields,

$$G(\boldsymbol{\Phi}, \delta\boldsymbol{\Phi}) = \int_0^L \delta\boldsymbol{\Phi}^T[\boldsymbol{\mathscr{E}}_{\boldsymbol{\varphi}}; \boldsymbol{\mathscr{E}}_{\boldsymbol{\alpha}}; \mathscr{E}_p]\,\mathrm{d}\xi_1 = \int_0^L \delta\boldsymbol{\varphi}\cdot\boldsymbol{\mathscr{E}}_{\boldsymbol{\varphi}} + \delta\boldsymbol{\alpha}\cdot\boldsymbol{\mathscr{E}}_{\boldsymbol{\alpha}} + \delta p.\mathscr{E}_p\,\mathrm{d}\xi_1 = 0, \tag{11}$$

In accordance with the fundamental Lemma of variational calculus, the strong form can be obtained from Eq. (11) as:

$$\boldsymbol{\mathscr{E}}_{\boldsymbol{\varphi}} = \partial_{\xi_1}\boldsymbol{n} - \boldsymbol{F}_{\boldsymbol{\varphi}} = 0; \tag{12a}$$

$$\boldsymbol{\mathscr{E}}_{\boldsymbol{\alpha}} = \partial_{\xi_1}\boldsymbol{m} + \partial_{\xi_1}\hat{\boldsymbol{\varphi}}.\boldsymbol{n} - \boldsymbol{F}_{\boldsymbol{\alpha}} = 0; \tag{12b}$$

$$\mathscr{E}_p = \partial_{\xi_1}M_\Psi - \mathcal{N}_p - F_p = 0. \tag{12c}$$

Here we define the reduced cross-section force, moment vector, and the bi-moment as

$$\boldsymbol{n} = \Big(\big(\boldsymbol{\mathcal{N}}_{\boldsymbol{\varepsilon}} - \tilde{\partial}_{\xi_1}\boldsymbol{\mathcal{N}}_{\partial_{\xi_1}\boldsymbol{\varepsilon}}\big) + \big(\boldsymbol{F}_{\boldsymbol{\varepsilon}} - \boldsymbol{N}_{\boldsymbol{\varepsilon}}\big)\Big); \tag{13a}$$

$$\begin{aligned} \boldsymbol{m} =& \left(\boldsymbol{\mathcal{N}}_{\boldsymbol{\kappa}} - \tilde{\partial}_{\xi_1}\boldsymbol{\mathcal{N}}_{\partial_{\xi_1}\boldsymbol{\kappa}} + \tilde{\partial}^2_{\xi_1}\boldsymbol{\mathcal{N}}_{\partial^2_{\xi_1}\boldsymbol{\kappa}} - \tilde{\partial}^3_{\xi_1}\boldsymbol{\mathcal{N}}_{\partial^3_{\xi_1}\boldsymbol{\kappa}}\right) + \left(\boldsymbol{F}_{\boldsymbol{\kappa}} - \tilde{\partial}_{\xi_1}\boldsymbol{F}_{\partial_{\xi_1}\boldsymbol{\kappa}} + \tilde{\partial}^2_{\xi_1}\boldsymbol{F}_{\partial^2_{\xi_1}\boldsymbol{\kappa}}\right) \\ &- \left(\boldsymbol{N}_{\boldsymbol{\kappa}} - \tilde{\partial}_{\xi_1}\boldsymbol{N}_{\partial_{\xi_1}\boldsymbol{\kappa}} + \tilde{\partial}^2_{\xi_1}\boldsymbol{N}_{\partial^2_{\xi_1}\boldsymbol{\kappa}}\right); \end{aligned} \tag{13b}$$

$$M_\Psi = \left(\left(\mathcal{N}_{\partial_{\xi_1} p} - \partial_{\xi_1}\mathcal{N}_{\partial^2_{\xi_1} p}\right) + \left(F_{\partial_{\xi_1} p} - N_{\partial_{\xi_1} p}\right)\right). \tag{13c}$$

These can be further simplified (as shown in Section 2.3.3 of the supplementary material in Appendix B of Chadha and Todd [2]) into

$$\boldsymbol{n} = \int_{\mathscr{B}_0} \boldsymbol{P}_1 \, \mathrm{d}\mathscr{B}_0; \boldsymbol{m} = \int_{\mathscr{B}_0} \boldsymbol{r} \times \boldsymbol{P}_1 \, \mathrm{d}\mathscr{B}_0; M_\Psi = \int_{\mathscr{B}_0} \Psi_1 \boldsymbol{d}_1 \cdot \boldsymbol{P}_1 \, \mathrm{d}\mathscr{B}_0. \tag{14}$$

In the equation above $\boldsymbol{P}_1$ represents the longitudinal stress vector corresponding the first PK stress tensor. The terms $\boldsymbol{n}$, $\boldsymbol{m}$, and M_Ψ represent the reduced section force, couple, and bi-moment, respectively.

4 The Poisson bracket formulation

4.1 The cotangent space, phase space, and cotangent bundle

To define phase space associated with the configuration space of the beam, we need to describe the cotangent space $T^*_{\boldsymbol{\Phi}}\mathbb{C}$ (identified with the product space $(\mathbb{R}^3)^* \times T^*_{\boldsymbol{Q}}SO(3) \times \mathbb{R}^*$) dual to the tangent space $T_{\boldsymbol{\Phi}}\mathbb{C}$.

Consider $\boldsymbol{v}^* = v_i \boldsymbol{E}^*_i \in (\mathbb{R}^3)^*$ and $\boldsymbol{u} = u_i \boldsymbol{E}_i \in \mathbb{R}^3$. Here, $\boldsymbol{E}^*_i$ is the one-form (or covector) associated with the vector $\boldsymbol{E}_i$ such that $\boldsymbol{E}^*_i(\boldsymbol{E}_j) = \boldsymbol{E}_i \cdot \boldsymbol{E}_j = \delta_{ij}$. We define the duality $\langle .,. \rangle_{\mathbb{R}^3} : (\mathbb{R}^3)^* \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ by means of the dot product, such that

$$\langle \boldsymbol{v}^*, \boldsymbol{u} \rangle_{\mathbb{R}^3} = \boldsymbol{v}^*(\boldsymbol{u}) = \boldsymbol{v} \cdot \boldsymbol{u}. \tag{15}$$

Here, $\boldsymbol{v} = v_i \boldsymbol{E}_i$ is dual to $\boldsymbol{v}^*$. From here on, any quantity with $*$ as a superscript represents the covector. Essentially the duality defined above is an identity metric on the tangent space of $\mathbb{R}^3$. Therefore, we may identify $(\mathbb{R}^3)^* \equiv \mathbb{R}^3$ via the Euclidean dot product. Similarly, we realize that $\mathbb{R}^* \equiv \mathbb{R}$. However, to avoid confusion, we maintain our nomenclature of using $*$ as a superscript representing an element of dual space. Therefore, if $v^* \in \mathbb{R}^*$ (with $v^* = v$) and $u \in \mathbb{R}$, the duality $\langle .,. \rangle_{\mathbb{R}} : \mathbb{R}^* \times \mathbb{R} \longrightarrow \mathbb{R}$ is expressed by means of a product as

$$\langle v^*, u \rangle_{\mathbb{R}} = v^*(u) = vu. \tag{16}$$

We define $so(3)^* \equiv T^*_{\boldsymbol{I}_3}SO(3)$ as the cotangent space to $so(3)$ such that for $\hat{\boldsymbol{A}}^* = \hat{A}_{ij}\boldsymbol{E}^*_i \otimes \boldsymbol{E}^*_j \in so(3)^*$ and $\hat{\boldsymbol{B}} = \hat{B}_{ij}\boldsymbol{E}_i \otimes \boldsymbol{E}_j \in so(3)$, we define the duality $\langle .,. \rangle_{so(3)} : so(3)^* \times so(3) \longrightarrow \mathbb{R}$ as follows

$$\langle \hat{\boldsymbol{A}}^*, \hat{\boldsymbol{B}} \rangle_{so(3)} = \hat{\boldsymbol{A}}^*(\hat{\boldsymbol{B}}) = \frac{1}{2}\hat{\boldsymbol{A}} : \hat{\boldsymbol{B}} = \frac{1}{2}\hat{A}_{ij}\hat{B}_{ij} = \boldsymbol{A} \cdot \boldsymbol{B}. \tag{17}$$

Here, $\boldsymbol{A} = A_{ij}\boldsymbol{E}_i \otimes \boldsymbol{E}_j \in T_{\boldsymbol{Q}}SO(3)$ is the tangent vector dual to $\boldsymbol{A}^*$. Since $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{B}}$ are skew-symmetric, let $\boldsymbol{A}$ and $\boldsymbol{B}$ represent the associated axial vectors. Let $\boldsymbol{A}_{\boldsymbol{Q}} = \boldsymbol{Q}.\hat{\boldsymbol{A}} \in T_{\boldsymbol{Q}}SO(3)$ and $\boldsymbol{B}_{\boldsymbol{Q}} = \boldsymbol{Q}.\hat{\boldsymbol{B}} \in T_{\boldsymbol{Q}}SO(3)$ be obtained by left translating the quantities $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{B}}$. We note that the quantities $\boldsymbol{A}_{\boldsymbol{Q}}$ and $\boldsymbol{B}_{\boldsymbol{Q}}$ are not skew-symmetric. For the cotangent vector $\boldsymbol{A}^*_{\boldsymbol{Q}} \in T^*_{\boldsymbol{Q}}SO(3)$, dual to the tangent vector $\boldsymbol{A}_{\boldsymbol{Q}}$, we define the duality $\langle .,. \rangle_{T_{\boldsymbol{Q}}SO(3)} : T^*_{\boldsymbol{Q}}SO(3) \times T_{\boldsymbol{Q}}SO(3) \longrightarrow \mathbb{R}$ as

$$\langle \boldsymbol{A}^*_{\boldsymbol{Q}}, \boldsymbol{B}_{\boldsymbol{Q}} \rangle_{T_{\boldsymbol{Q}}SO(3)} = \boldsymbol{A}^*_{\boldsymbol{Q}}(\boldsymbol{B}_{\boldsymbol{Q}}) = \frac{1}{2}\boldsymbol{A}_{\boldsymbol{Q}} : \boldsymbol{B}_{\boldsymbol{Q}}. \tag{18}$$

We also observe the left-invariant nature of the metric (or duality) discussed in Eq. (17) and (18) such that

$$\langle \boldsymbol{A}^*_{\boldsymbol{Q}}, \boldsymbol{B}_{\boldsymbol{Q}} \rangle_{T_{\boldsymbol{Q}}SO(3)} = \langle \hat{\boldsymbol{A}}^*, \hat{\boldsymbol{B}} \rangle_{so(3)}. \tag{19}$$

Similarly, the duality associated with $T^*_{\boldsymbol{\Phi}}\mathbb{C}$ and $T_{\boldsymbol{\Phi}}\mathbb{C}$ is given by

$$\langle .,. \rangle_{T_{\boldsymbol{\Phi}}\mathbb{C}} = \langle .,. \rangle_{\mathbb{R}^3} + \langle .,. \rangle_{T_{\boldsymbol{Q}}SO(3)} + \langle .,. \rangle_{\mathbb{R}}. \tag{20}$$

We note that the dualities discussed above are commutative in the sense that

$$\langle \boldsymbol{A}^*_{\boldsymbol{Q}}, \boldsymbol{B}_{\boldsymbol{Q}} \rangle_{T_{\boldsymbol{Q}}SO(3)} = \langle \boldsymbol{B}^*_{\boldsymbol{Q}}, \boldsymbol{A}_{\boldsymbol{Q}} \rangle_{T_{\boldsymbol{Q}}SO(3)} \text{ and } \langle \boldsymbol{v}^*, \boldsymbol{u} \rangle_{\mathbb{R}^3} = \langle \boldsymbol{u}^*, \boldsymbol{v} \rangle_{\mathbb{R}^3}. \tag{21}$$

This brings us to the definition of the cotangent bundle $T^*\mathbb{C}$ dual to $T\mathbb{C}$ associated with the configuration $\mathbb{C}$. For $\tilde{\boldsymbol{\Phi}}^* \in T^*_{\boldsymbol{\Phi}}\mathbb{C}$ and $\boldsymbol{\Phi} \in \mathbb{C}$, we have

$$T^*\mathbb{C} := \left\{(\mathbf{\Phi}, \tilde{\mathbf{\Phi}}^*) | \mathbf{\Phi} \in \mathbb{C}, \tilde{\mathbf{\Phi}}^* \in T^*_{\mathbf{\Phi}}\mathbb{C}\right\}. \tag{22}$$

Like the tangent bundle $T\mathbb{C}$, the cotangent bundle $T^*\mathbb{C}$ is not a product space. Secondly, the $T\mathbb{C}$ gives the state space and $T^*\mathbb{C}$ gives the phase space. For simplicity, we assume displacement prescribed boundary conditions and no external force for the analysis in the forthcoming sections.

4.2 The Lagrangian and Hamiltonian

Usually, the Lagrangian is defined as $\mathbb{L} : T\mathbb{C} \longrightarrow \mathbb{R}$. The Hamiltonian $\mathbb{H} : T^*\mathbb{C} \longrightarrow \mathbb{R}$ is obtained by means of Legendre transformation of Lagrangian via the change of variables $(\mathbf{\Phi}, \tilde{\mathbf{\Phi}}) \mapsto (\mathbf{\Phi}, \tilde{\mathbf{\Phi}}^*)$. However, the kinematics of the beam at hand not only depends on the configuration space but also on the spatial (with respect to ξ_1) derivatives of $(\boldsymbol{\varphi}, \boldsymbol{Q}, p)$. Therefore, we take a more general approach to obtain the Hamiltonian from the Lagrangian. We start with defining the Lagrangian in terms of *passive* and *active* coordinates. The coordinates that takes part in Legendre transformation are called as *active* coordinates (refer to chapter 6 of Lanczos [7]).

Definition 1: Let the set $\mathfrak{q}$ and $\mathfrak{a}$ define the field of passive and active variables respectively with $\overline{\mathfrak{q}}$ and $\overline{\mathfrak{a}}$ being their respective material forms. These sets are given by

$$\begin{aligned}
\mathfrak{q} &= \{\boldsymbol{\varphi}, \boldsymbol{Q}, p, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}, \boldsymbol{Q}.\partial_{\xi_1}\overline{\boldsymbol{\varepsilon}}, \boldsymbol{Q}.\partial_{\xi_1}\overline{\boldsymbol{\kappa}}, \boldsymbol{Q}.\partial^2_{\xi_1}\overline{\boldsymbol{\kappa}}, \boldsymbol{Q}.\partial^3_{\xi_1}\overline{\boldsymbol{\kappa}}, \partial_{\xi_1}p, \partial^2_{\xi_1}p\};\\
\overline{\mathfrak{q}} &= \{\boldsymbol{Q}^T.\boldsymbol{\varphi}, \boldsymbol{I}_3, p, \overline{\boldsymbol{\varepsilon}}, \overline{\boldsymbol{\kappa}}, \partial_{\xi_1}\overline{\boldsymbol{\varepsilon}}, \partial_{\xi_1}\overline{\boldsymbol{\kappa}}, \partial^2_{\xi_1}\overline{\boldsymbol{\kappa}}, \partial^3_{\xi_1}\overline{\boldsymbol{\kappa}}, \partial_{\xi_1}p, \partial^2_{\xi_1}p\};\\
\mathfrak{a} &= \{\partial_t\boldsymbol{\varphi}, \boldsymbol{\omega}, \partial_t p, \tilde{\partial}_t\boldsymbol{\varepsilon}, \tilde{\partial}_t\boldsymbol{\kappa}, \boldsymbol{Q}.\partial_t(\partial_{\xi_1}\overline{\boldsymbol{\kappa}}), \boldsymbol{Q}.\partial_t(\partial^2_{\xi_1}\overline{\boldsymbol{\kappa}}), \partial_t(\partial_{\xi_1}p)\};\\
\overline{\mathfrak{a}} &= \{\boldsymbol{Q}^T.\partial_t\boldsymbol{\varphi}, \overline{\boldsymbol{\omega}}, \partial_t p, \partial_t\overline{\boldsymbol{\varepsilon}}, \partial_t\overline{\boldsymbol{\kappa}}, \partial_t(\partial_{\xi_1}\overline{\boldsymbol{\kappa}}), \partial_t(\partial^2_{\xi_1}\overline{\boldsymbol{\kappa}}), \partial_t(\partial_{\xi_1}p)\},
\end{aligned} \tag{23}$$

where t represents time. We note that $(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3) \in \mathbb{C}$ and $(\mathfrak{a}_1, \mathfrak{a}_{2Q}, \mathfrak{a}_3) \in T_{\mathbf{\Phi}}\mathbb{C}$, where $\mathfrak{a}_{2Q} = \hat{\mathfrak{a}}_2.\boldsymbol{Q}$. Finally, we define $\hat{\overline{\mathfrak{a}}}_2 = \boldsymbol{Q}^T.\hat{\mathfrak{a}}_2.\boldsymbol{Q}$.

Definition 2: The Lagrangian $\mathbb{L} : (\mathfrak{q}; \mathfrak{a}) \mapsto \mathbb{R}$ associated with the beam is defined as

$$\mathbb{L} = \mathrm{T}(\mathfrak{a}) - \mathrm{U}_{\text{strain}}(\mathfrak{q}). \tag{24}$$

Here, T and $\mathrm{U}_{\text{strain}}$ gives the kinetic energy and strain energy stored in the system, respectively. T can be obtained using the result (1) as

$$\mathrm{T} = \int_{\Omega_0} \rho_0 \partial_t \boldsymbol{R}.\partial_t \boldsymbol{R}\, \mathrm{d}\Omega_0 = \frac{1}{2}\int_0^L \sum_{i=1}^{8} \mathfrak{a}_i.(\mathbb{I}_i \mathfrak{a}_i)\, \mathrm{d}\xi_1 = \frac{1}{2}\int_0^L \sum_{i=1}^{8} \overline{\mathfrak{a}}_i.(\overline{\mathbb{I}}_i \overline{\mathfrak{a}}_i)\, \mathrm{d}\xi_1. \tag{25}$$

Here,

$$\begin{aligned}
\mathbb{I}_1 &= \int_{\mathscr{B}_0} \rho_0\, \mathrm{d}\mathscr{B}_0; & \mathbb{I}_5 &= \int_{\mathscr{B}_0} \rho_0 (\boldsymbol{L}^{\lambda_1}_{\partial_{\xi_1}\boldsymbol{\kappa}})^T.\boldsymbol{L}^{\lambda_1}_{\partial_{\xi_1}\boldsymbol{\kappa}}\, \mathrm{d}\mathscr{B}_0;\\
\mathbb{I}_2 &= \int_{\mathscr{B}_0} \rho_0 \hat{\boldsymbol{r}}^T.\hat{\boldsymbol{r}}\, \mathrm{d}\mathscr{B}_0; & \mathbb{I}_6 &= \int_{\mathscr{B}_0} \rho_0 (\boldsymbol{L}^{\lambda_1}_{\partial^2_{\xi_1}\boldsymbol{\kappa}})^T.\boldsymbol{L}^{\lambda_1}_{\partial^2_{\xi_1}\boldsymbol{\kappa}}\, \mathrm{d}\mathscr{B}_0;\\
\mathbb{I}_3 &= \int_{\mathscr{B}_0} \rho_0 \boldsymbol{L}^{\lambda_1}_{\partial_{\xi_1}p}.\boldsymbol{L}^{\lambda_1}_{\partial_{\xi_1}p}\, \mathrm{d}\mathscr{B}_0; & \mathbb{I}_7 &= \int_{\mathscr{B}_0} \rho_0 (\boldsymbol{L}^{\lambda_1}_{\partial^3_{\xi_1}\boldsymbol{\kappa}})^T.\boldsymbol{L}^{\lambda_1}_{\partial^3_{\xi_1}\boldsymbol{\kappa}}\, \mathrm{d}\mathscr{B}_0;\\
\mathbb{I}_4 &= \int_{\mathscr{B}_0} \rho_0 (\boldsymbol{L}^{\lambda_1}_{\partial_{\xi_1}\boldsymbol{\varepsilon}})^T.\boldsymbol{L}^{\lambda_1}_{\partial_{\xi_1}\boldsymbol{\varepsilon}}\, \mathrm{d}\mathscr{B}_0. & \mathbb{I}_8 &= \int_{\mathscr{B}_0} \rho_0 \boldsymbol{L}^{\lambda_1}_{\partial^2_{\xi_1}p}.\boldsymbol{L}^{\lambda_1}_{\partial^2_{\xi_1}p}\, \mathrm{d}\mathscr{B}_0.
\end{aligned} \tag{26}$$

Refer to Section 1.1 of the supplementary material in Appendix B of Chadha and Todd [2] for the definition of $\boldsymbol{L}^{\lambda_1}_{(.)}$. The material form is then defined as $\overline{\mathbb{I}}_i = \boldsymbol{Q}^T.\mathbb{I}_i.\boldsymbol{Q}$. For $i \in \{1, 3, 8\}$, we have $\overline{\mathbb{I}}_i = \mathbb{I}_i$. We observe that $\overline{\mathbb{I}}_2$ is dependent on $(p, \overline{\varepsilon}_1, \overline{\boldsymbol{\kappa}}, \partial_{\xi_1}\overline{\boldsymbol{\kappa}}, \partial^2_{\xi_1}\overline{\boldsymbol{\kappa}})$.

The strong form of equations defined in 12 can also be obtained by making the action stationary, provided $\delta\boldsymbol{u}(t_1) = \delta\boldsymbol{u}(t_2) = 0$, such that (refer section 6 of [8]),

$$\delta \int_{t_1}^{t_2} \mathbb{L}\, \mathrm{d}t = 0. \tag{27}$$

To obtain canonical coordinates using the Legendre transformation, we assume each $\mathfrak{a}_i$ as independent quantities and we note the following result that can be easily proved using the chain rule.

Proposition 1: For a function of form $g(\mathfrak{a}_i) = \overline{g}(\boldsymbol{Q}.\overline{\mathfrak{a}}_i)$ for $i \in \{1, 2, 4, 5, 6, 7\}$ and a function of form $f(\mathfrak{a}_{2Q}) = \overline{f}(\hat{\overline{\mathfrak{a}}}_2)$, the following are true

$$\frac{\partial g}{\partial \mathfrak{a}_i} = \boldsymbol{Q}.\frac{\partial \overline{g}}{\partial \overline{\mathfrak{a}}_i}; \frac{\partial f}{\partial \mathfrak{a}_{2Q}} = \boldsymbol{Q}.\frac{\partial \overline{f}}{\partial \hat{\overline{\mathfrak{a}}}_2}. \tag{28}$$

Definition 3: Define the set $\mathfrak{p}$ of canonical momentum coordinates corresponding to the active variable set $\mathfrak{a}$ obtained by Legendre transformation $\mathbb{F}$ as

$$\mathbb{F}\mathbb{L}(\mathfrak{a}_i) = \mathfrak{p}_i = \partial_{\xi_1}\frac{\partial \mathbb{L}}{\partial \mathfrak{a}_i}. \tag{29}$$

Using the symmetry of $\bar{\mathbb{I}}_i$ and the result in proposition 1, we obtain $\mathfrak{p}_i = \mathbb{I}_i\mathfrak{a}_i$ (Einstein summation is suppressed). Let $\overline{\mathfrak{p}}$ represent the material form of canonical momentum coordinates. We note that the kinetic energy depends on the first component of $\partial_t\bar{\boldsymbol{\varepsilon}}$ and on the second and third components of $\partial_t\overline{\boldsymbol{\kappa}}, \partial_t(\partial_{\xi_1}\overline{\boldsymbol{\kappa}})$ and, $\partial_t(\partial^2_{\xi_1}\overline{\boldsymbol{\kappa}})$. We assume that the canonical momentum coordinate for all the zero active coordinates (for example, $\partial_t\overline{\kappa}_1 = 0$) is zero, for example $\overline{\mathfrak{p}}_4 = (\bar{\mathbb{I}}_{4_{11}}\partial_t\bar{\varepsilon}_1, 0, 0)^T$ and $\mathfrak{p}_4 = \boldsymbol{Q}.(\bar{\mathbb{I}}_{4_{11}}\partial_t\bar{\varepsilon}_1, 0, 0)^T$. As such, the non-zero active coordinates can then be uniquely defined as a function of their corresponding canonical coordinate and vice-versa. This is equivalent to the fact that if the active coordinate consists of non-zero terms only, then the determinant of the Hessian of the Lagrangian with respect to the active coordinate is non-zero. Using the result (28), we get the following

$$\begin{gathered}\frac{\partial\overline{f}}{\partial\overline{\mathfrak{a}}_2} = (\bar{\mathbb{I}}_2\overline{\boldsymbol{\omega}}) \implies \frac{\partial\overline{f}}{\partial\hat{\overline{\mathfrak{a}}}_2} = \widehat{(\bar{\mathbb{I}}_2\overline{\boldsymbol{\omega}})};\\ \mathfrak{p}_{2\boldsymbol{Q}} = \frac{\partial\mathbb{L}}{\partial\mathfrak{a}_{2\boldsymbol{Q}}} = \boldsymbol{Q}.\widehat{(\bar{\mathbb{I}}_2\overline{\boldsymbol{\omega}})}.\end{gathered} \tag{30}$$

Definition 4: The Hamiltonian $\mathbb{H} : (\mathfrak{q}, \mathfrak{p}^*) \mapsto \mathbb{R}$ is defined in terms of canonical coordinates as

$$\mathbb{H} = \int_0^L \sum_{i=1}^{8} \mathfrak{p}_i \cdot \mathfrak{a}_i \, \mathrm{d}\xi_1 - \mathbb{L} = \int_0^L H \, \mathrm{d}\xi_1 = \mathrm{T}(\mathfrak{p}) + \mathrm{U}_{\text{strain}}(\mathfrak{q}) = \text{total energy}. \tag{31}$$

Here, $H(\mathfrak{q}, \mathfrak{p})$ is energy per unit arc length or energy density.

Definition 5: Define the inverse Legendre transformation $\mathbb{F}^{-1}$ as

$$\mathbb{F}^{-1}\mathbb{H}(\mathfrak{p}_i) = \mathfrak{a}_i = \partial_{\xi_1}\frac{\partial\mathbb{H}}{\partial\mathfrak{p}_i} = \frac{\partial H}{\partial\mathfrak{p}_i}. \tag{32}$$

4.3 Canonical bracket

Poisson brackets are defined on phase space. The definition of Poisson's bracket consists of a mix of *partial derivatives of the functional* of form $f(\mathfrak{q};\mathfrak{p})$ (example of such a function is the Hamiltonian) with respect to parameters defining configuration space $(\boldsymbol{\varphi}, \boldsymbol{Q}, p) \equiv (\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3)$ and parameters defining cotangent space $(\mathfrak{p}_1, \mathfrak{p}_{2\boldsymbol{Q}}, \mathfrak{p}_3)$. Therefore, in order to state Poisson bracket on $T^*\mathbb{C}$, we first define *partial functional derivatives* of such functional (we consider Hamiltonian as the functional of interest). Refer to appendix A of Engel et al. [9] for detailed discussion on functional derivatives.

Definition 6: The varied passive and canonical variables are defined as $\mathfrak{q}_{i\varepsilon} = \mathfrak{q}_i + \epsilon\delta\mathfrak{q}_i$ and $\mathfrak{p}_{i\varepsilon} = \mathfrak{p}_i + \epsilon\delta\mathfrak{p}_i$. We have $\mathfrak{q}_\epsilon = \{\mathfrak{q}_{i\varepsilon}\}$ and $\mathfrak{p}_\epsilon = \{\mathfrak{p}_{i\varepsilon}\}$ such that $\mathfrak{p}_{2\boldsymbol{Q}\epsilon} = \boldsymbol{Q}.\hat{\overline{\mathfrak{p}}}_{2\epsilon} = \boldsymbol{Q}.\widehat{(\boldsymbol{Q}^T.\mathfrak{p}_{2\epsilon})}$.

Definition 7: For a pure displacement-specified boundary, the following are the partial functional derivative $\frac{\delta H}{\delta\mathfrak{p}_i}$ of Hamiltonian (density) $H(\mathfrak{q};\mathfrak{p})$ with respect to parameters defining cotangent space $(\mathfrak{p}_1, \mathfrak{p}_{2\boldsymbol{Q}}, \mathfrak{p}_3)$:

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathbb{H}(\mathfrak{q};(\mathfrak{p}_{1\epsilon},\mathfrak{p}_i))\bigg|_{\epsilon=0} = \int_0^L \left\langle \frac{\delta H}{\delta\mathfrak{p}_1}^*, \delta\mathfrak{p}_1 \right\rangle_{\mathbb{R}^3} \mathrm{d}\xi_1 = \int_0^L \frac{\delta H}{\delta\mathfrak{p}_1}\cdot\delta\mathfrak{p}_1 \, \mathrm{d}\xi_1 \tag{33a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathbb{H}(\mathfrak{q};(\mathfrak{p}_{2\boldsymbol{Q}\epsilon},\mathfrak{p}_i))\bigg|_{\epsilon=0} = \int_0^L \left\langle \frac{\delta H}{\delta\mathfrak{p}_{2\boldsymbol{Q}}}^*, \delta\mathfrak{p}_{2\boldsymbol{Q}} \right\rangle_{T_{\boldsymbol{Q}}SO(3)} \mathrm{d}\xi_1 = \frac{1}{2}\int_0^L \frac{\delta H}{\delta\mathfrak{p}_{2\boldsymbol{Q}}} : \delta\mathfrak{p}_{2\boldsymbol{Q}} \, \mathrm{d}\xi_1 \tag{33b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathbb{H}(\mathfrak{q};(\mathfrak{p}_{3\epsilon},\mathfrak{p}_i))\bigg|_{\epsilon=0} = \int_0^L \left\langle \frac{\delta H}{\delta\mathfrak{p}_3}^*, \delta\mathfrak{p}_3 \right\rangle_{\mathbb{R}} \mathrm{d}\xi_1 = \int_0^L \frac{\delta H}{\delta\mathfrak{p}_3}.\delta\mathfrak{p}_3 \, \mathrm{d}\xi_1 \tag{33c}$$

Like the result in Eq. (28), we have

$$\frac{\delta H}{\delta\mathfrak{p}_{2\boldsymbol{Q}}} = \boldsymbol{Q}.\frac{\partial H}{\partial\hat{\overline{\mathfrak{p}}}_2} \tag{34}$$

Remark 1: This result holds because $\mathfrak{p}_i$ for $i \neq 2$ does not have any functional dependence on $\mathfrak{p}_2$. However, the elements of $\mathfrak{q}$ and $\mathfrak{p}$ do have dependence on the spatial and time derivatives of $(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3)$. To define partial functional derivatives of H with respect to $(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3)$, we treat the pairs $(\mathfrak{q}_1, \mathfrak{p}_1)$, $(\mathfrak{q}_2, \mathfrak{p}_{2\boldsymbol{Q}})$ and $(\mathfrak{q}_3, \mathfrak{p}_3)$ as independent quantities. This is crucial as it allows us to operate on cotangent bundle. As a result, even though, for example, $\partial_t\boldsymbol{\varphi}$ is functionally dependent on $\boldsymbol{\varphi}$, the corresponding canonical quantity $\mathfrak{p}_1$ is considered to be independent of $\boldsymbol{\varphi}$. On the other hand, the direct dependence of $\mathfrak{p}_2$ on $\mathfrak{q}_3$ is consider while evaluating $\frac{\delta H}{\delta p}$. We also note that since $(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3)$ defines the configuration space, we do not consider $\mathfrak{p}_i$ for $i > 3$ to be functionally independent on the configuration space. As was pointed in section 3 of Simo et al. [4], defining the functional derivative of $H(\mathfrak{q}; \mathfrak{p})$ with respect to parameters on configuration space requires some caution. This is because the cotangent bundle is not a simple product space. Accordingly definition 5 can be written as

$$\mathbb{F}^{-1}\mathbb{H}(\mathfrak{p}_i) = \mathfrak{a}_i = \frac{\delta H}{\delta \mathfrak{p}_i}. \tag{35}$$

Definition 8: For a change $\mathfrak{q}_i \mapsto \mathfrak{q}_{i\epsilon} = \mathfrak{q}_i + \epsilon\delta\mathfrak{q}_i$ (with $i \in 1, 2, 3$), let $\mathfrak{p}_{(\mathfrak{q}_{i\epsilon})} = \{\mathfrak{p}_{j(\mathfrak{q}_{i\epsilon})}\}$ and $\mathfrak{q}_{j(\mathfrak{q}_{i\epsilon})}$ (for $j \neq 1, 2, 3$) define the associated canonical and passive quantities, respectively.

Definition 9: For a pure displacement-specified boundary, the partial functional derivatives $\frac{\delta H}{\delta \mathfrak{q}_i}$ of Hamiltonian density $H(\mathfrak{q}; \mathfrak{p})$ with respect to parameters defining cotangent space $(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3)$ are given as:

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathbb{H}((\mathfrak{q}_{1\epsilon}, \mathfrak{q}_{i(\mathfrak{q}_{1\epsilon})}); \mathfrak{p}_{(\mathfrak{q}_{1\epsilon})})\Big|_{\epsilon=0} = \int_0^L \left\langle \frac{\delta H}{\delta \mathfrak{q}_1}^*, \delta\mathfrak{q}_1 \right\rangle_{\mathbb{R}^3} \mathrm{d}\xi_1 = \int_0^L \frac{\delta H}{\delta \boldsymbol{\varphi}} \cdot \delta\boldsymbol{\varphi}\, \mathrm{d}\xi_1; \tag{36a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathbb{H}((\mathfrak{q}_{2\epsilon}, \mathfrak{q}_{i(\mathfrak{q}_{2\epsilon})}); \mathfrak{p}_{(\mathfrak{q}_{2\epsilon})})\Big|_{\epsilon=0} = \int_0^L \left\langle \frac{\delta H}{\delta \mathfrak{q}_2}^*, \delta\mathfrak{q}_2 \right\rangle_{T_Q SO(3)} \mathrm{d}\xi_1 = \frac{1}{2}\int_0^L \frac{\delta H}{\delta \boldsymbol{Q}} : \delta\boldsymbol{Q}\, \mathrm{d}\xi_1; \tag{36b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathbb{H}((\mathfrak{q}_{3\epsilon}, \mathfrak{q}_{i(\mathfrak{q}_{3\epsilon})}); \mathfrak{p}_{(\mathfrak{q}_{2\epsilon})})\Big|_{\epsilon=0} = \int_0^L \left\langle \frac{\delta H}{\delta \mathfrak{q}_3}^*, \delta\mathfrak{q}_3 \right\rangle_{\mathbb{R}} \mathrm{d}\xi_1 = \int_0^L \frac{\delta H}{\delta p}.\delta p\, \mathrm{d}\xi_1. \tag{36c}$$

Proposition 2: The following holds:

$$\frac{\mathrm{d}\mathfrak{p}_1}{\mathrm{dt}} = \boldsymbol{F}_{\boldsymbol{\varphi}}; \tag{37a}$$

$$\frac{\mathrm{d}\mathfrak{p}_2}{\mathrm{dt}} = \boldsymbol{F}_{\boldsymbol{\alpha}}; \tag{37b}$$

$$\frac{\mathrm{d}\mathfrak{p}_3}{\mathrm{dt}} = \frac{\mathrm{d}^2 p}{\mathrm{dt}^2}.\int_{\mathscr{B}_0} \rho_0 \Psi_1^2\, \mathrm{d}\mathscr{B}_0 = \mathbb{I}_3\frac{\mathrm{d}^2 p}{\mathrm{dt}^2} = F_p. \tag{37c}$$

Proof: Proof of proposition 2 follows from a straightforward calculation and application of chain rule. We leave proving (37a) and (37c) to the readers. Realizing $\tilde{\partial}_t \mathbb{I}_2 = \mathbf{0}_3$ and $\tilde{\partial}_t \boldsymbol{\omega} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{dt}} - \boldsymbol{\omega} \times \boldsymbol{\omega} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{dt}}$, we can prove the result (37b) as

$$\frac{\mathrm{d}\mathfrak{p}_2}{\mathrm{dt}} = \frac{\mathrm{d}\mathbb{I}_2.\boldsymbol{\omega}}{\mathrm{dt}} = \tilde{\partial}_t(\mathbb{I}_2.\boldsymbol{\omega}) + \boldsymbol{\omega} \times \mathbb{I}_2.\boldsymbol{\omega} = \mathbb{I}_2.\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{dt}} + \boldsymbol{\omega} \times \mathbb{I}_2.\boldsymbol{\omega} = \boldsymbol{F}_{\boldsymbol{\alpha}}. \tag{38}$$

This completes the proof. □

Proposition 3: With the definition of Hamiltonian and its partial functional derivatives in equation set (31) and (36), respectively, the following holds

$$\frac{\delta H}{\delta \mathfrak{q}_1} = -(\boldsymbol{\mathscr{E}}_{\boldsymbol{\varphi}} + \boldsymbol{F}_{\boldsymbol{\varphi}}); \tag{39a}$$

$$\frac{\delta H}{\delta \mathfrak{q}_{2\boldsymbol{Q}}} = -\boldsymbol{Q}.(\hat{\bar{\boldsymbol{\mathscr{E}}}}_{\boldsymbol{\alpha}} + \hat{\bar{\boldsymbol{F}}}_{\boldsymbol{\alpha}}), \text{ where } \hat{\bar{\boldsymbol{\mathscr{E}}}}_{\boldsymbol{\alpha}} = \widehat{\boldsymbol{Q}^T.\boldsymbol{\mathscr{E}}_{\boldsymbol{\alpha}}} \text{ and } \hat{\bar{\boldsymbol{F}}}_{\boldsymbol{\alpha}} = \widehat{\boldsymbol{Q}^T.\boldsymbol{F}_{\boldsymbol{\alpha}}}; \tag{39b}$$

$$\frac{\delta H}{\delta \mathfrak{q}_3} = -(\mathscr{E}_p + F_p); \tag{39c}$$

$$\frac{\delta H}{\delta \mathfrak{q}_2} = -(\boldsymbol{\mathscr{E}}_{\boldsymbol{\alpha}} + \boldsymbol{F}_{\boldsymbol{\alpha}}). \tag{39d}$$

Proof: Recall *remark 1*, which stated the need to consider $(\mathfrak{q}_i, \mathfrak{p}_i)$ as independent quantities while considering partial functional derivative of H with respect to $\mathfrak{q}_1, \mathfrak{q}_{2Q}, \mathfrak{q}_3$. With that in mind, for the curve $\epsilon : \boldsymbol{\Phi} \mapsto \boldsymbol{\Phi} + \epsilon\delta\boldsymbol{\Phi}$ (keeping the respective canonical coordinates fixed), the variation of Hamiltonian in the direction of $\delta\boldsymbol{\Phi}$ is given as

$$\delta\mathbb{H}|_{(\{\mathfrak{p}_1, \mathfrak{p}_{2Q}, \mathfrak{p}_3\}=\text{fixed})} = \int_0^L \left\langle \frac{\delta H}{\delta \mathfrak{q}_1}^*, \delta\mathfrak{q}_1 \right\rangle_{\mathbb{R}^3} + \left\langle \frac{\delta H}{\delta \mathfrak{q}_2}^*, \delta\mathfrak{q}_2 \right\rangle_{T_Q SO(3)} + \left\langle \frac{\delta H}{\delta \mathfrak{q}_3}^*, \delta\mathfrak{q}_3 \right\rangle_{\mathbb{R}} \mathrm{d}\xi_1 \tag{40}$$

Since strain energy does not have any dependence on the canonical quantities $\mathfrak{p}$, we have

$$\delta \mathrm{U}_{\text{strain}}|_{(\{\mathfrak{p}_1, \mathfrak{p}_{2Q}, \mathfrak{p}_3\}=\text{fixed})} = \delta \mathrm{U}_{\text{strain}}. \tag{41}$$

Substituting for the expression of $\boldsymbol{R}$ defined in 1 into Eq. (25) and carrying out integration by parts yields:

$$\delta \mathrm{T} = \delta W_{\text{inertial}}. \tag{42}$$

However, the terms $-\int_0^L \delta\boldsymbol{\varphi}\cdot\boldsymbol{F}_{\boldsymbol{\varphi}} + \delta\boldsymbol{\alpha}\cdot\boldsymbol{F}_{\boldsymbol{\alpha}} + \delta p.F_p \,\mathrm{d}\xi_1$ in the expression of $\delta W_{\text{inertial}}$ are obtained by considering the terms $\partial_t\boldsymbol{\varphi} = \mathfrak{a}_1 = \mathbb{I}_1^{-1}\mathfrak{p}_1$, $\boldsymbol{\omega} = \mathfrak{a}_2 = \mathbb{I}_2^{-1}\mathfrak{p}_2$ and $\partial_t p = \mathfrak{a}_3 = \mathbb{I}_3^{-1}\mathfrak{p}_3$ to be functionally dependent on the configuration space. Therefore, we can obtain $\delta\mathrm{T}|_{(\{\mathfrak{p}_1,\mathfrak{p}_{2\boldsymbol{Q}},\mathfrak{p}_3\}=\text{fixed})}$ by ignoring these terms, yielding

$$\begin{aligned}\delta\mathrm{T}|_{(\{\mathfrak{p}_1,\mathfrak{p}_{2\boldsymbol{Q}},\mathfrak{p}_3\}=\text{fixed})} = \int_0^L \delta\boldsymbol{\varphi}\cdot\Big(-\partial_{\xi_1}\boldsymbol{F}_{\boldsymbol{\varepsilon}}\Big) + \delta p.\big(-\partial_{\xi_1}F_{\partial_{\xi_1}p}\big) + \\ \delta\boldsymbol{\alpha}\cdot\Big(-\partial_{\xi_1}\hat{\boldsymbol{\varphi}}\cdot\boldsymbol{F}_{\boldsymbol{\varepsilon}} - \partial_{\xi_1}\big(\boldsymbol{F}_{\boldsymbol{\kappa}} - \tilde{\partial}_{\xi_1}\boldsymbol{F}_{\partial_{\xi_1}\boldsymbol{\kappa}} + \tilde{\partial}^2_{\xi_1}\boldsymbol{F}_{\partial^2_{\xi_1}\boldsymbol{\kappa}}\big)\Big)\Big)\,\mathrm{d}\xi_1 + \delta\mathrm{W}^*_{\text{inertial}}.\end{aligned} \tag{43}$$

From the definition of the Hamiltonian in Eq. (31), we have

$$\delta\mathbb{H}|_{(\{\mathfrak{p}_1,\mathfrak{p}_{2\boldsymbol{Q}},\mathfrak{p}_3\}=\text{fixed})} = \delta\mathrm{T}|_{(\{\mathfrak{p}_1,\mathfrak{p}_{2\boldsymbol{Q}},\mathfrak{p}_3\}=\text{fixed})} + \delta\mathrm{U}_{\text{strain}}|_{(\{\mathfrak{p}_1,\mathfrak{p}_{2\boldsymbol{Q}},\mathfrak{p}_3\}=\text{fixed})}. \tag{44}$$

We use the expression of $\delta\mathrm{U}_{\text{strain}}$ Eq. (9a) and the results in Eq. set (12) and (13) to obtain

$$\delta\mathbb{H}|_{(\{\mathfrak{p}_1,\mathfrak{p}_{2\boldsymbol{Q}},\mathfrak{p}_3\}=\text{fixed})} = -\int_0^L (\boldsymbol{\mathscr{E}}_{\boldsymbol{\varphi}} + \boldsymbol{F}_{\boldsymbol{\varphi}})\cdot\delta\boldsymbol{\alpha} + (\boldsymbol{\mathscr{E}}_{\boldsymbol{\alpha}} + \boldsymbol{F}_{\boldsymbol{\alpha}})\cdot\delta\boldsymbol{\alpha} + (\mathscr{E}_p + F_p).\delta p\,\mathrm{d}\xi_1. \tag{45}$$

Eq. (40) and (44) proves the results (39a) and (39c). To prove (39b), we consider

$$(\boldsymbol{\mathscr{E}}_{\boldsymbol{\alpha}} + \boldsymbol{F}_{\boldsymbol{\alpha}})\cdot\delta\boldsymbol{\alpha} = (\overline{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\alpha}} + \overline{\boldsymbol{F}}_{\boldsymbol{\alpha}})\cdot\delta\overline{\boldsymbol{\alpha}} = (\hat{\overline{\boldsymbol{\mathscr{E}}}}_{\boldsymbol{\alpha}} + \hat{\overline{\boldsymbol{F}}}_{\boldsymbol{\alpha}}).\delta\hat{\overline{\boldsymbol{\alpha}}} = \Big\langle \boldsymbol{Q}.(\hat{\overline{\boldsymbol{\mathscr{E}}}}_{\boldsymbol{\alpha}} + \hat{\overline{\boldsymbol{F}}}_{\boldsymbol{\alpha}}), \delta\boldsymbol{Q}\Big\rangle_{T_{\boldsymbol{Q}}SO(3)}. \tag{46}$$

This proves the result (39b). Using chain rule, like Eq. (34), we have

$$\frac{\delta H}{\delta\mathfrak{p}_{2\boldsymbol{Q}}} = \boldsymbol{Q}.\frac{\delta H}{\delta\hat{\overline{\mathfrak{p}}}_2};\ \text{and}\ \frac{\delta H}{\delta\mathfrak{p}_2} = \boldsymbol{Q}.\frac{\delta H}{\delta\overline{\mathfrak{p}}_2}. \tag{47}$$

Since $\hat{\overline{\mathfrak{p}}}_2 \in so(3)$, we realize that $\frac{\delta H}{\delta\hat{\overline{\mathfrak{p}}}_2} = \widehat{\left(\frac{\delta H}{\delta\overline{\mathfrak{p}}_2}\right)}$. Using the result obtained above and in (39b), we have

$$\frac{\delta H}{\delta\hat{\overline{\mathfrak{p}}}_2} = (\hat{\overline{\boldsymbol{\mathscr{E}}}}_{\boldsymbol{\alpha}} + \hat{\overline{\boldsymbol{F}}}_{\boldsymbol{\alpha}}) \implies \frac{\delta H}{\delta\overline{\mathfrak{p}}_2} = -(\overline{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\alpha}} + \overline{\boldsymbol{F}}_{\boldsymbol{\alpha}}) \implies \frac{\delta H}{\delta\mathfrak{p}_2} = -(\boldsymbol{\mathscr{E}}_{\boldsymbol{\alpha}} + \boldsymbol{F}_{\boldsymbol{\alpha}}). \tag{48}$$

This completes the proof. □

Note that a more direct approach towards obtaining partial functional derivatives of the Hamiltonian with respect to the configuration space is by considering a general function $\mathbb{H} = \int_0^L H(\mathfrak{q},\mathfrak{p})\,\mathrm{d}\xi_1$ and obtaining $\delta\mathbb{H}|_{(\{\mathfrak{p}_1,\mathfrak{p}_{2\boldsymbol{Q}},\mathfrak{p}_3\}=\text{fixed})}$ by carrying integration by parts of all functionally dependent quantities (keeping $(\mathfrak{p}_1,\mathfrak{p}_{2\boldsymbol{Q}},\mathfrak{p}_3)$ fixed) to obtain result of form (40). Such proof would require defining strain energy in an integral form using, for example, a free-energy function characterizing hyperelastic response. Readers are recommended to refer section 5 of Simo et al. [4] that deploys this approach for a beam with rigid cross-section.

Corollary 1: Proposition 2 and 3 along with the strong form of equilibrium equation stated in equation set 12 yields

$$\frac{\mathrm{d}\mathfrak{p}_i}{\mathrm{d}t} = -\frac{\delta H}{\delta\mathfrak{q}_i}\ \text{for}\ i = 1,2,3. \tag{49}$$

The equation set (49) along with the inverse Legendre transformation (35) gives the *Hamiltonian equation of motion*. Note that there are 7 equations constituting the strong form (3 for linear momentum conservation, 3 for angular momentum conservation, and 1 for the balance of bi-moment and bi-shear), whereas there are 14 equations constituting the Hamiltonian form. This brings us to the definition of the Poisson bracket. □

Definition 10: Consider $(\boldsymbol{\Phi},\tilde{\boldsymbol{\Phi}}) \in T^*\mathbb{C}$ such that $\boldsymbol{\Phi} = \{\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3\} \in \mathbb{C}$ and $\tilde{\boldsymbol{\Phi}}^* = \{\mathfrak{p}_1^*,\mathfrak{p}_{2\boldsymbol{Q}}^*,\mathfrak{p}_3^*\} \in T^*_{\boldsymbol{\Phi}}\mathbb{C}$. For the functions of form $F,G : T^*\mathbb{C} \longrightarrow \mathbb{R}$ or $F,G \in \mathfrak{f}(T^*\mathbb{C})$, such that $F(\boldsymbol{\Phi},\tilde{\boldsymbol{\Phi}}) = \int_0^L f(\boldsymbol{\Phi},\tilde{\boldsymbol{\Phi}})\,\mathrm{d}\xi_1$ and $G(\boldsymbol{\Phi},\tilde{\boldsymbol{\Phi}}) = \int_0^L g(\boldsymbol{\Phi},\tilde{\boldsymbol{\Phi}})\,\mathrm{d}\xi_1$, the Poisson bracket $\{.,.\} : \mathfrak{f}(T^*\mathbb{C})\times\mathfrak{f}(T^*\mathbb{C}) \longrightarrow \mathbb{R}$ is defined as

$$
\begin{aligned}
\{F,G\} &= \int_0^L \left\langle \frac{\delta f}{\delta \mathbf{\Phi}}^*, \frac{\delta g}{\delta \tilde{\mathbf{\Phi}}} \right\rangle_{T_{\mathbf{\Phi}} C} - \left\langle \frac{\delta g}{\delta \mathbf{\Phi}}^*, \frac{\delta f}{\delta \tilde{\mathbf{\Phi}}} \right\rangle_{T_{\mathbf{\Phi}} C} \mathrm{d}\xi_1 \\
\{F,G\} &= \int_0^L \left(\left\langle \frac{\delta f}{\delta \mathfrak{q}_1}^*, \frac{\delta g}{\delta \mathfrak{p}_1} \right\rangle_{\mathbb{R}^3} - \left\langle \frac{\delta f}{\delta \mathfrak{q}_1}^*, \frac{\delta g}{\delta \mathfrak{p}_1} \right\rangle_{\mathbb{R}^3} \right) + \left(\left\langle \frac{\delta f}{\delta \mathfrak{q}_2}^*, \frac{\delta g}{\delta \mathfrak{p}_{2Q}} \right\rangle_{T_Q SO(3)} - \left\langle \frac{\delta g}{\delta \mathfrak{q}_2}^*, \frac{\delta f}{\delta \mathfrak{p}_{2Q}} \right\rangle_{T_Q SO(3)} \right) \\
&\quad + \left(\left\langle \frac{\delta f}{\delta \mathfrak{p}_3}^*, \frac{\delta g}{\delta \mathfrak{q}_3} \right\rangle_{\mathbb{R}} - \left\langle \frac{\delta f}{\delta \mathfrak{p}_3}^*, \frac{\delta g}{\delta \mathfrak{q}_3} \right\rangle_{\mathbb{R}} \right) \mathrm{d}\xi_1 \\
\{F,G\} &= \int_0^L \frac{\delta f}{\delta \boldsymbol{\varphi}} . \frac{\delta g}{\delta \mathfrak{p}_1} - \frac{\delta g}{\delta \boldsymbol{\varphi}} . \frac{\delta f}{\delta \mathfrak{p}_1} \mathrm{d}\xi_1 + \frac{1}{2} \int_0^L \frac{\delta f}{\delta \boldsymbol{Q}} : \frac{\delta g}{\delta \mathfrak{p}_{2Q}} - \frac{\delta g}{\delta \boldsymbol{Q}} : \frac{\delta f}{\delta \mathfrak{p}_{2Q}} \mathrm{d}\xi_1 + \int_0^L \frac{\delta f}{\delta p} . \frac{\delta g}{\delta \mathfrak{p}_3} - \frac{\delta g}{\delta p} . \frac{\delta f}{\delta \mathfrak{p}_3} \mathrm{d}\xi_1
\end{aligned}
\tag{50}
$$

Theorem 1: The following are equivalent

1. The strong form of equilibrium equations ($\boldsymbol{\mathscr{E}}_{\boldsymbol{\varphi}} = \mathbf{0}_1, \boldsymbol{\mathscr{E}}_{\boldsymbol{\alpha}} = \mathbf{0}_1, \mathscr{E}_p = 0$);
2. Hamilton's principle of stationary action defined by Eq. (27);
3. The Hamiltonian equation of motion given by equation set (35) and (49);
4. Hamiltonian equation in their Poisson bracket formulation given by $\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, \mathbb{H}\}$ for all $F = \int_0^L f \, \mathrm{d}\xi_1 \in \mathfrak{f}(T^*\mathbb{C})$.

Proof: We used the strong form (statement 1) to establish the Hamiltonian equation (statement 3) in corollary 1. We can obtain the strong form using equation (27) by substituting for the expression of virtual kinetic energy and virtual strain energy in (42) and(9a) respectively. We leave the proof to readers (refer, section 6 of Chadha and Todd [8]). We prove statement 4. By the chain rule, we have

$$
\frac{\mathrm{d}F}{\mathrm{d}t} = \int_0^L \left(\frac{\delta f}{\delta \mathfrak{q}_1} \cdot \mathfrak{a}_1 + \frac{1}{2} \frac{\delta f}{\delta \mathfrak{q}_2} : (\boldsymbol{Q} \hat{\mathfrak{a}}_{2Q}) + \frac{\delta f}{\delta \mathfrak{q}_3} . \mathfrak{a}_3 \right) + \left(\frac{\delta f}{\delta \mathfrak{p}_1} \cdot \frac{\mathrm{d}\mathfrak{p}_1}{\mathrm{d}t} + \frac{\delta f}{\delta \mathfrak{p}_2} . \frac{\mathrm{d}\mathfrak{p}_2}{\mathrm{d}t} + \frac{\delta f}{\delta \mathfrak{p}_3} . \frac{\mathrm{d}\mathfrak{p}_3}{\mathrm{d}t} \right) \mathrm{d}\xi_1 \tag{51}
$$

Using Hamiltonian equations (35) and (49), the equation above simplifies to $\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, \mathbb{H}\}$, completing the proof. □

Remark 2: The Poisson bracket defined in (50) satisfies the properties of anti-commutativity, bilinearity, Leibniz's rule, and the Jacobi identity (refer to the chapter on *canonical transformation* in Goldstein et al. [10]). Using the anti-commutative property, we arrive at an energy conservation law as $\frac{\mathrm{d}\mathbb{H}}{\mathrm{d}t} = \{\mathbb{H}, \mathbb{H}\} = 0 \implies \frac{\mathrm{d}\mathbb{H}}{\mathrm{d}t} = 0$. This is true because the energy density H (or the total energy $\mathbb{H}$ and the Lagrangian $\mathbb{L}$) does not have explicit time dependence, thereby implying time-invariant symmetry. Thus, the equations derived in the last section are for *scleronomic* system. However, if we consider time-dependent external forces (for example, non-conservative forces like follower loads) and damping, it would imply the presence of unaccounted sources of energy, such that $\frac{\partial \mathbb{H}}{\partial t} \neq 0$. Therefore, the general Poisson bracket form of equilibrium equation is $\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, \mathbb{H}\} + \frac{\partial F}{\partial t}$. Lastly, we note that for infinitesimal motion considered on phase space and using the Hamiltonian form of equations, we have

$$
\begin{aligned}
\mathbf{\Phi}(t) &= \mathbf{\Phi}(t=0) + t \frac{\mathrm{d}\mathbf{\Phi}}{\mathrm{d}t}\bigg|_{t=0} = \mathbf{\Phi}(t=0) + t \frac{\delta H}{\delta \tilde{\mathbf{\Phi}}}\bigg|_{t=0} \\
\tilde{\mathbf{\Phi}}(t) &= \tilde{\mathbf{\Phi}}(t=0) + t \frac{\mathrm{d}\tilde{\mathbf{\Phi}}}{\mathrm{d}t}\bigg|_{t=0} = \tilde{\mathbf{\Phi}}(t=0) - t \frac{\delta H}{\delta \mathbf{\Phi}}\bigg|_{t=0}.
\end{aligned}
\tag{52}
$$

Thus, $\left(\frac{\delta H}{\delta \tilde{\mathbf{\Phi}}}, -\frac{\delta H}{\delta \mathbf{\Phi}} \right)$ can be thought as two components of the tangent vector to the curve representing the time evolution of the system in phase space at $t = 0$. Therefore, we can consider time evolution as a canonical transformation on coordinates $(\mathbf{\Phi}(t=0), \tilde{\mathbf{\Phi}}(t=0)) \longrightarrow (\mathbf{\Phi}(t), \tilde{\mathbf{\Phi}}(t))$ generated by the Hamiltonian.

5 Summary and conclusion

This paper considers the Hamiltonian structure of geometrically-exact beams with enhanced kinematics. The phase space and the associated duality (or metric) are defined. The Hamiltonian is obtained from the Lagrangian via change of coordinates from state space to phase space carried out using a Legendre transformation. The Hamiltonian form of the equations is obtained, the Poisson bracket formulation is described, and the equivalence between various forms of balance laws are stated.

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