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# THE SPINE OF THE $T$ -GRAPH OF THE HILBERT SCHEME OF POINTS IN THE PLANE

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**Abstract.** The torus  $T$  of projective space also acts on the Hilbert scheme of subschemes of projective space. The  $T$ -graph of the Hilbert scheme has vertices the fixed points of this action, and edges connecting pairs of fixed points in the closure of a one-dimensional orbit. In general this graph depends on the underlying field. We construct a subgraph, which we call the spine, of the  $T$ -graph of  $\text{Hilb}^m(\mathbb{A}^2)$  that is independent of the choice of infinite field. For certain edges in the spine we also give a description of the tropical ideal, in the sense of tropical scheme theory, of a general ideal in the edge. This gives a more refined understanding of these edges, and of the tropical stratification of the Hilbert scheme.

**Keywords.** Hilbert scheme,  $T$ -graph, tropical ideal

**Mathematics Subject Classifications.** 14C05, 14T10, 14L30

## 1. Introduction

The torus  $T \cong (K^*)^n$  of  $\mathbb{P}^n$  acts on the Hilbert scheme  $\text{Hilb}_P(\mathbb{P}^n)$  of subschemes of  $\mathbb{P}^n$ . There are finitely many fixed points of this action, but infinitely many one-dimensional orbits. The  $T$ -graph of the Hilbert scheme has vertices the fixed points of the  $T$ -action. There is an edge between two vertices if there is a one-dimensional  $T$ -orbit containing a  $K$ -rational point whose closure contains these two vertices. The  $T$ -graph provides a combinatorial skeleton of the Hilbert scheme; for example, the proof that  $\text{Hilb}_P(\mathbb{P}^n)$  is connected given by Peeva and Stillman [PS05] proceeds by showing the Borel-fixed subgraph of this graph is connected (the original proof by Hartshorne [Har66] has some moves which, while combinatorial, leave this graph). The  $T$ -graph of the Hilbert scheme was first systematically studied by Altmann–Sturmfels [AS05], who gave

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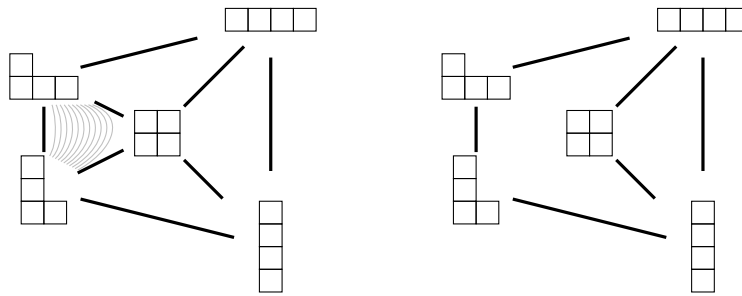


Figure 1.1: The  $T$ -graph and its spine when  $N = 4$ . The gray curves illustrate the fact that the edge from  $\langle x^3, xy, y^2 \rangle$  to  $\langle x^2, xy, y^3 \rangle$  corresponds to a one-dimensional family of  $T$ -orbits; taking a limit, the  $T$ -orbits degenerate into the union of two orbits, corresponding to the edges from  $\langle x^3, xy, y^2 \rangle$  to  $\langle x^2, y^2 \rangle$  and from  $\langle x^2, y^2 \rangle$  to  $\langle x^2, xy, y^3 \rangle$ .

an algorithm to compute it using Gröbner bases, and was studied combinatorially by Hering–Maclagan [HM12]. More generally,  $T$ -graphs arise in GKM theory [GKM97], where they are used to give a presentation of the equivariant cohomology ring of a variety with  $T$ -action.

The  $T$ -graph of the Hilbert scheme  $\text{Hilb}^4(\mathbb{A}^2)$  of 4 points in  $\mathbb{A}^2$  is shown on the left of Figure 1.1. Note that a single edge may correspond to multiple one-dimensional  $T$ -orbits, or even to a positive-dimensional family of them.

An additional complexity is given by the fact that the graph depends on the underlying field; the  $T$ -graph of  $\text{Hilb}^{10}(\mathbb{A}^2)$  differs for  $K = \mathbb{Q}$  and  $K = \mathbb{R}$ ; see [HM12, Example 2.11] and [Sil22, Theorem 5.11].

The first result of this paper is the construction of a subgraph of the  $T$ -graph of the Hilbert scheme  $\text{Hilb}^N(\mathbb{A}^2)$  that does not depend on the underlying field  $K$ , provided  $K$  is infinite.

A  $K$ -rational point of  $\text{Hilb}^N(\mathbb{A}^2)$  is a subscheme of  $\mathbb{A}^2$  of length  $N$ , given by an ideal  $I \subseteq S := K[x, y]$  with  $\dim_K S/I = N$ . Such an ideal  $I$  is a fixed point of the  $T$ -action if and only if it is a monomial ideal; these ideals are in bijection with Young diagrams with  $N$  boxes, with boxes corresponding to monomials not in  $I$ . A non-monomial ideal  $I$  lies on a one-dimensional orbit if and only if  $I$  is homogeneous with respect to a grading by  $\deg(x) = a$  and  $\deg(y) = b$ ; the subscheme of  $\mathbb{A}^2$  defined by  $I$  is stabilized by the subtorus  $\{(t^a, t^b) : t \in K^*\} \subseteq T$ . There are two  $T$ -fixed points in the closure of the orbit, so  $I$  corresponds to an edge of the  $T$ -graph, if and only if  $ab > 0$ .

In this latter case, denote by  $h : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  the Hilbert function of  $I$  with respect to this grading:  $h(d) := \dim_K (S/I)_d$ . Then  $I$  lies on the multigraded Hilbert scheme  $\text{Hilb}_S^h \subseteq \text{Hilb}^N(\mathbb{A}^2)$  parametrizing homogeneous ideals in  $S$  with Hilbert function  $h$  [HS04]. This multigraded Hilbert scheme is a  $T$ -invariant closed subscheme of  $\text{Hilb}^N(\mathbb{A}^2)$ , is smooth and irreducible [Eva04, Theorem 1] [MS10, Theorem 1.1], and has two distinguished  $T$ -fixed points: the “lex-most” and “lex-least” monomial ideals. See Section 2 and in particular Figure 2.1 for more details.

**Definition 1.1.** The *spine*  $G_N^*$  of the  $T$ -graph of  $\text{Hilb}^N(\mathbb{A}^2)$  is the graph with vertices the  $T$ -fixed points of  $\text{Hilb}^N(\mathbb{A}^2)$ , and an edge between two monomial ideals if they are the lex-most and lex-least ideals of  $\text{Hilb}_S^h$  with respect to some grading and Hilbert function.

Studying the spine was suggested in Remark 4.7 of [HM12]. Every one-dimensional  $T$ -orbit corresponding to an edge of the  $T$ -graph is in the closure of the set of  $T$ -orbits corresponding to edges in the spine. The spine for  $\text{Hilb}^4(\mathbb{A}^2)$  is shown on the right in Figure 1.1. Let  $G_N(K)$  denote the  $T$ -graph of  $\text{Hilb}^N(\mathbb{A}^2)$  over a field  $K$ . Our first theorem is the following.

**Theorem 1.2.** *For any infinite field  $K$ ,  $G_N^*$  is a subgraph of  $G_N(K)$ ; that is, if  $M^-$  and  $M^+$  are the lex-least and lex-most monomial ideals with respect to some grading and Hilbert function, then there exists an ideal  $I \subseteq K[x, y]$ , homogeneous with respect to this grading and Hilbert function, such that the closure of the  $T$ -orbit of  $I$  contains  $M^-$  and  $M^+$ .*

Our second result, Theorem 1.3 below, refines Theorem 1.2 for some edges by describing matroidal aspects of  $\text{Hilb}_S^h$  coming from tropical scheme theory. We now describe what we mean by this; for precise definitions, see Section 3.1.

The *tropicalization*  $\text{trop}(I)$  of an ideal  $I \subseteq K[x, y]$  is the ideal in the semiring of tropical polynomials obtained by tropicalizing every polynomial in the ideal. This is an example of a tropical ideal in the sense of tropical scheme theory [GG16, MR18, MR20, MR22]. When  $I$  is homogeneous, each degree- $d$  part of  $\text{trop}(I)$  determines a *matroid*  $\mathcal{M}(I_d)$  on the set  $\text{Mon}_d$  of degree- $d$  monomials.

This construction induces a *tropical stratification* of  $\text{Hilb}_S^h$ ; two ideals are in the same stratum if and only if their tropicalizations coincide. This can be thought of as a generalization of the matroid stratification of the Grassmannian [GGMS87]. Very little is known about the tropical stratification; see [Sil22, FGG24].

When  $n = 2$ , and the grading is the standard one  $\deg(x) = \deg(y) = 1$ , the Hilbert scheme  $\text{Hilb}_S^h$  is irreducible [Eva04, MS10], and hence has a unique open (largest) stratum. Our second main theorem, Theorem 1.3 below, describes this stratum; in other words, it describes the tropicalization of a general ideal  $I$  in  $\text{Hilb}_S^h$ .

**Theorem 1.3.** *Let  $S = K[x, y]$  be graded by  $\deg(x) = \deg(y) = 1$ . For any  $d \geq 0$ , the degree- $d$  matroid  $\mathcal{M}(I_d)$  of a general ideal  $I$  in  $\text{Hilb}_S^h$  is the uniform matroid  $U_{h(d), d+1}$ . Furthermore,  $I$  can be taken to be a  $K$ -rational point of  $\text{Hilb}_S^h$ , provided  $K$  is infinite.*

Theorem 1.3 fails in the nonstandard grading; see Section 3.5.

There are comparatively few explicit examples of tropical ideals; see [Zaj18, AR22]. One important aspect of Theorem 1.3 is thus that it provides a large class of new examples for which all matroids are understood.

Theorem 1.3 refines Theorem 1.2 as follows. For a fixed grading and Hilbert function, the ideals  $I \subseteq K[x, y]$  whose orbit contains  $M^-$  and  $M^+$  comprise an open set  $U_1 \subseteq \text{Hilb}_S^h$ , which is nonempty by Theorem 1.2. Meanwhile, the ideals  $I \subseteq K[x, y]$  such that the conclusion of Theorem 1.3 holds for all  $d \geq 0$  also comprise a nonempty open set  $U_2 \subseteq \text{Hilb}_S^h$ , and we have the containment  $U_2 \subseteq U_1$ ; see Remark 3.23.

The structure of this paper is as follows. In Section 2 we give more precise definitions of the main objects of study, and prove Theorem 1.2. Theorem 1.3 is proved in Section 3.

## 2. The spine of the $T$ -graph

In this section we recall previous work on the  $T$ -graph, and prove Theorem 1.2.

Let  $K$  be an infinite field. Recall that a  $K$ -rational point of  $\text{Hilb}^N(\mathbb{A}^2)$  is given by an ideal  $I \subseteq S := K[x, y]$  with  $\dim_K(S/I) = N$ . The  $T \cong (K^*)^2$  action on  $\mathbb{A}^2$  induces a  $T$ -action on  $\text{Hilb}^N(\mathbb{A}^2)$ . Such an ideal is a fixed point of the  $T \cong (K^*)^2$  action on  $\text{Hilb}^N(\mathbb{A}^2)$  if and only if it is a monomial ideal, and lies on a one-dimensional  $T$ -orbit if and only if it is homogeneous with respect to a  $\mathbb{Z}$ -grading by  $\deg(x) = a$  and  $\deg(y) = b$ . The closure of a one-dimensional  $T$ -orbit has either one or two  $T$ -fixed points; if there are two  $T$ -fixed points in the closure we have  $ab > 0$ .

**Notation 2.1.** We set  $S = K[x, y]$ . We grade  $S$  by  $\deg(x) = a$ ,  $\deg(y) = b$ , for positive integers  $a, b$ , and denote this as an  $(a, b)$ -grading. From now on we restrict to  $a, b > 0$  and  $\gcd(a, b) = 1$ ; this makes no material difference, and will simplify our notation.

Let  $I \in \text{Hilb}^N(\mathbb{A}^2)$  be an ideal that is homogeneous with respect to the grading by  $(a, b) \in \mathbb{Z}_{>0}^2$ . The Hilbert function  $h : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  of  $I$  is defined by  $h(d) = \dim_K(S/I)_d$ . Note that  $\sum_{d \geq 0} h(d) = N$ . The point  $I \in \text{Hilb}^N(\mathbb{A}^2)$  is contained in the closed subscheme  $\text{Hilb}_S^h$  parametrizing ideals in  $S$  that are homogeneous with respect to the  $(a, b)$ -grading and have Hilbert function  $h$ . This subscheme is a *multigraded Hilbert scheme* in the sense of [HS04]. Furthermore, for any grading  $(a, b) \in \mathbb{Z}_{>0}^2$  and for any Hilbert function  $h : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  with  $\sum_{d \geq 0} h(d) = N$ , if the scheme  $\text{Hilb}_S^h$  is nonempty, it is a smooth irreducible  $T$ -invariant subvariety of  $\text{Hilb}^N(\mathbb{A}^2)$  [Eva04, MS10]. The union of the subvarieties  $\text{Hilb}_S^h$ , as  $(a, b)$  and  $h$  vary, is precisely the set of ideals corresponding to vertices and edges of the  $T$ -graph, and any intersection of two different  $\text{Hilb}_S^h$  is either empty, or consists of a single point corresponding to a monomial ideal.

Each multigraded Hilbert scheme  $\text{Hilb}_S^h$  inherits a  $T$ -action from  $\text{Hilb}^N(\mathbb{A}^2)$ . We may define the  $T$ -graph  $G_h(K)$  of  $\text{Hilb}_S^h$  in exact analogy with that of  $\text{Hilb}^N(\mathbb{A}^2)$ : the vertices of  $G_h(K)$  are zero-dimensional  $T$ -orbits, which are monomial ideals whose Hilbert function with respect to  $(a, b)$  is  $h$ , and two vertices are connected by an edge if there is a one-dimensional  $T$ -orbit in  $\text{Hilb}_S^h$  containing a  $K$ -rational point whose closure contains those vertices. Note that  $G_h(K)$  is naturally a subgraph of  $G_N(K)$ . Moreover, we have the following decomposition:

**Proposition 2.2** ([HM12], Corollary 2.6). *The  $T$ -graph  $G_N(K)$  is the union of the subgraphs  $G_h(K)$  as  $(a, b)$  and  $h$  vary, and these subgraphs have disjoint edge sets.*

In light of Proposition 2.2, in order to determine  $G_N(K)$  it is sufficient to study the graded Hilbert schemes  $\text{Hilb}_S^h$  separately. Thus from now on, fix a grading  $(a, b) \in \mathbb{Z}_{>0}^2$  with  $\gcd(a, b) = 1$ , and a Hilbert function  $h : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  with  $\sum_{d \geq 0} h(d) < \infty$ .

Note that the 1-parameter subtorus  $T_{a,b} := \{(t^a, t^b) \mid t \in K^*\} \subseteq T$  acts trivially on  $\text{Hilb}_S^h$ , so we need only consider the action of the one-dimensional torus  $T/T_{a,b}$ . Since  $\text{Hilb}_S^h$  is smooth and projective, the Białynicki-Birula decomposition of  $\text{Hilb}_S^h$  with respect to  $T/T_{a,b}$  decomposes  $\text{Hilb}_S^h$  as a union of affine spaces, each consisting of points whose limit under the subtorus action is a given fixed point. We describe these affine spaces algebraically as follows. Set  $\prec$  to

be the lexicographic order with  $x \prec y$ . A  $T$ -fixed point corresponds to a monomial ideal  $M$ . The Białyński-Birula cell associated to  $M$  is

$$C_{\prec}(M) = \{I \in \text{Hilb}_S^h \mid \text{in}_{\prec}(I) = M\},$$

where  $\text{in}_{\prec}(I)$  is the initial ideal in the sense of Gröbner bases; see [CLO15]. An explicit parameterization of  $C_{\prec}(M)$  was given by Evain [Eva04]; we recall this in Section 3.2. For two monomial ideals  $M, M' \in \text{Hilb}_S^h$ , the *edge-scheme* between  $M$  and  $M'$  is the scheme-theoretic intersection

$$E(M, M') = C_{\prec}(M) \cap C_{\prec^{opp}}(M'),$$

where  $\prec^{opp}$  is the lexicographic order with  $x \succ y$ . This was first studied computationally by Altmann and Sturmfels [AS05]. There is an edge between  $M$  and  $M'$  in the  $T$ -graph if and only if one of  $E(M, M')$  and  $E(M', M)$  has a  $K$ -rational point.

The vertices of  $G_N(K)$  are purely combinatorial: colength- $N$  monomial ideals in  $S = K[x, y]$  correspond to partitions of  $N$ , and requiring that the Hilbert function with respect to  $(a, b)$  is  $h$  is a combinatorial condition on partitions of  $N = \sum_{d \geq 0} h(d)$ . The edges, however, depend on the field  $K$ , as the following examples show.

**Example 2.3.** There is an edge between the two monomial ideals  $M = \langle x^5, y^2 \rangle$  and  $M' = \langle x^2, y^5 \rangle$  when viewed as ideals in  $\mathbb{R}[x, y]$ , but not when viewed as ideals in  $\mathbb{Q}[x, y]$ , so the  $T$ -graph of  $\text{Hilb}^{10}(\mathbb{A}^2)$  differs over  $\mathbb{R}$  and  $\mathbb{Q}$ . This is the case because every ideal in  $E(M, M')$  has the form  $I = \langle y^2 + \alpha xy + \beta x^2, x^5 \rangle$ , with  $\alpha^4 - 3\alpha^2\beta + \beta^2 = 0$ , by [HM12, Example 2.11]. This is the union of two one-dimensional  $T$ -orbits, which have  $\mathbb{R}$ -rational points, but no  $\mathbb{Q}$ -rational points. Note that the edge scheme  $E(M, M')$  is a subscheme of  $\text{Hilb}_S^h$ , where the grading is  $(1, 1)$ , and  $h = (1, 2, 2, 2, 2, 1, 0, 0, \dots)$ .

This example is generalized in [Sil22, Theorem 5.11], which shows that if  $m > k > 0$ , and we define  $M = \langle x^m, y^k \rangle$  and  $M' = \langle x^k, y^m \rangle$  in  $\text{Hilb}^{mk}(\mathbb{A}^2)$ , then the edge-scheme  $E(M, M')$  is one dimensional and reducible over  $\mathbb{C}$ , with the number of irreducible components equal to the number of binary necklaces with  $k$  black and  $m - k$  white beads. The proof actually shows that these edges have  $K$ -rational points whenever there exists  $c \in K$  such that  $x^m + cy^m$  has a degree- $k$  factor with coefficients in  $K$ .

By contrast, the definition of the spine of the  $T$ -graph, given in Definition 1.1, is purely combinatorial.

**Definition 2.4.** Let  $M, M'$  be monomial ideals in  $\text{Hilb}_S^h$ . We define  $M \preceq M'$  if for each degree  $d$  there is a degree-preserving bijection  $f$  from the monomials in  $M$  to the monomials in  $M'$  with  $m \succeq f(m)$  for all monomials  $m \in M$ , where  $\prec$  is the lexicographic order with  $x \prec y$ . This defines a partial ordering on the monomial ideals in  $\text{Hilb}_S^h$ .

*Remark 2.5.* The partial order of Definition 2.4 may be regarded as a graded version of the dominance order for partitions. Recall that the monomial ideals  $M, M'$  correspond to partitions, or alternatively, to Young diagrams. Under this correspondence,  $M \preceq M'$  if and only if  $M'$  can be obtained from  $M$  by moving boxes of the Young diagram up and to the left, along lines of slope  $-a/b$ . See Figure 2.1. Compare this to the usual dominance order for partitions, which is identical after removing the slope restriction.

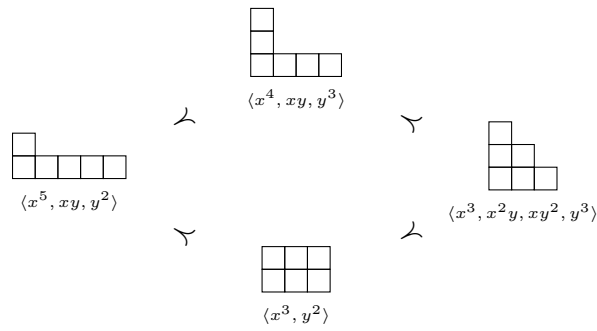


Figure 2.1: The poset of monomial ideals in  $\text{Hilb}_S^h$ , where  $h = (1, 1, 2, 1, 1, 0, 0, \dots)$  with respect to the  $(1, 2)$ -grading. The leftmost ideal is the lex-least, and the rightmost ideal is the lex-most. Note that  $M_1 \prec M_2$  if and only if  $M_2$  can be obtained from  $M_1$  by moving boxes of the Young diagram up-and-left along lines of slope  $-1/2$ .

A necessary, but not sufficient, condition for  $E(M, M')$  to be nonempty is that  $M \preceq M'$  with respect to this partial order; this is a straightforward special case of [HM12, Thm. 1.3]. In particular, if  $M \neq M'$ , then at most one of  $E(M, M')$  and  $E(M', M)$  is nonempty. We may therefore regard  $G_N(K)$  as a directed graph, with an edge from  $M$  to  $M'$  if  $E(M, M')$  is nonempty; the necessary condition above implies that the resulting directed graph is acyclic.

It was first noted by Evain [Eva04, Theorem 19] that this poset has a unique maximal element, which we denote by  $M^+$ , and a unique minimal element, which we denote by  $M^-$ ; see also [MS10, Proposition 3.12]. We call  $M^+$  the lex-most ideal with Hilbert function  $h$ , and  $M^-$  the lex-least such ideal.

As defined in Definition 1.1, the spine  $G_N^*$  of the  $T$ -graph of  $\text{Hilb}^N(\mathbb{A}^2)$  is the graph with vertices monomial ideals  $M$  in  $S$  with  $\dim_K(S/M) = N$ , and an edge joining two ideals  $M, M'$  if  $M = M^-$ , and  $M' = M^+$  for some grading  $(a, b)$  and Hilbert function. Figure 2.2 shows  $G_6(\mathbb{C})$  and  $G_6^*$ .

**Theorem 2.6.** *Let  $K$  be an infinite field. If there is an edge connecting two monomial ideals  $M, M'$  in  $G_N^*$ , then there is an edge in the  $T$ -graph  $G_N(K)$  connecting  $M, M'$ .*

*Proof.* Suppose  $M$  and  $M'$  are connected by an edge in  $G_N^*$ . By definition, there exists a grading  $(a, b)$  with respect to which the Hilbert functions of  $M$  and  $M'$  agree, and after possibly renaming  $M$  and  $M'$ , we have  $M = M^+$  and  $M' = M^-$ .

By [Eva04, Thm. 1], [MS10, Thm. 1.1],  $\text{Hilb}_S^h$  is smooth, projective, and irreducible. Since  $M^+$  is a source, and  $M^-$  is a sink, of the  $T/T_{a,b}$  action, the Białyński–Birula cells  $C_{\rightarrow\text{opp}}(M^+)$  and  $C_{\leftarrow}(M^-)$  are Zariski open and isomorphic to affine spaces ([BB73][Eva04, Thm. 11]). Note that this follows from [BB73] only when the field  $K$  is algebraically closed, but this assumption is unnecessary — for a discussion see [Bro05, §3]. Also, while [Eva04] assumes  $K$  is algebraically closed, this is never used in the proofs. Thus  $C_{\rightarrow\text{opp}}(M^+) \cap C_{\leftarrow}(M^-)$  is isomorphic to an open subset of an affine space over  $K$ ; since  $K$  is infinite, it follows that  $C_{\rightarrow\text{opp}}(M^+) \cap C_{\leftarrow}(M^-)$  contains a  $K$ -rational point.  $\square$



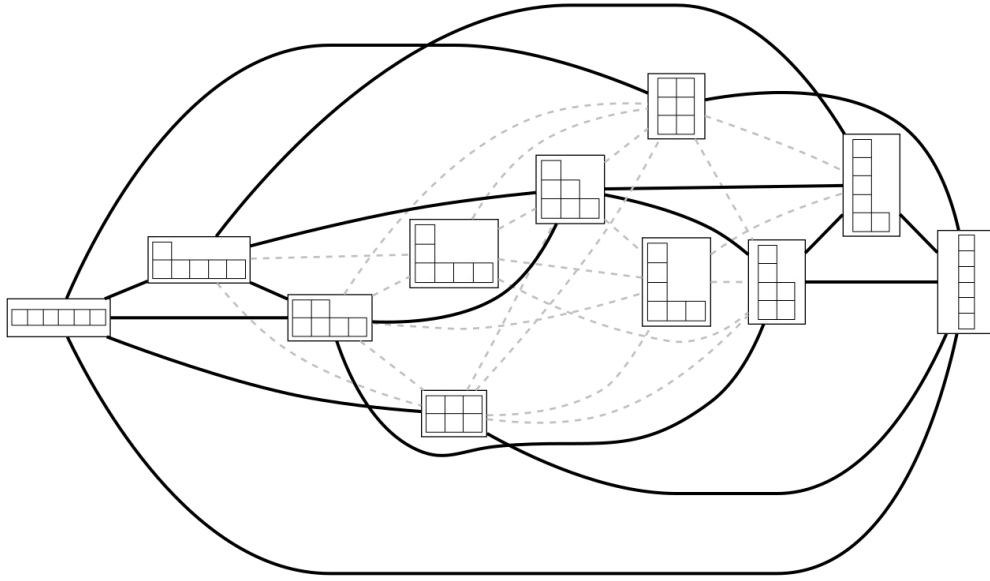


Figure 2.2: The  $T$ -graph  $G_6(\mathbb{C})$ , where the edges of  $G_6^*$  are solid.

### 3. The tropical ideal of an edge of the spine

In this section we prove Theorem 1.3.

#### 3.1. Tropicalizations of ideals

We first recall the concept of tropicalization of ideals, and the tropical stratification of the Hilbert scheme.

Let  $\mathbb{B} = (\{0, \infty\}, \oplus, \odot)$  be the Boolean semiring, with the operations of tropical addition (minimum) and tropical multiplication (addition). The tropicalization of  $f = \sum c_{ij}x^i y^j \in K[x, y]$  is  $\text{trop}(f) = \oplus_{c_{ij} \neq 0} x^i y^j \in \mathbb{B}[x, y]$ . The tropicalization of an ideal  $I \subseteq K[x, y]$  is

$$\text{trop}(I) = \langle \text{trop}(f) : f \in I \rangle \subseteq \mathbb{B}[x, y].$$

This is the trivial valuation case of tropicalizing ideals in the sense of tropical scheme theory [GG16, MR18, MR20, MR22].

Note that a polynomial in the semiring  $\mathbb{B}[x, y]$  can be identified with its support. When  $I \subseteq S$  is graded, the polynomials in  $\text{trop}(I)$  of degree  $d$  of minimal support are the *circuits* of a matroid  $\mathcal{M}(I_d)$  on the ground set  $\text{Mon}_d$  of degree- $d$  monomials. We call this the degree- $d$  matroid of  $I$ . See, for example, [Ox11] for more on matroids.

We will primarily focus on the *basis* characterization of matroids. When  $I \subseteq S$  is homogeneous with Hilbert function  $h$ , a collection  $E$  of  $h(d)$  monomials of degree  $d$  is a basis for  $\mathcal{M}(I_d)$  if there is no polynomial in  $I$  with support in  $E$ . The matroid  $\mathcal{M}(I_d)$  is *uniform* if every collection of  $h(d)$  monomials of degree  $d$  is a basis. In this case we write  $\mathcal{M}(I_d) = U_{h(d), \text{mon}_d}$ , where  $\text{mon}_d = |\text{Mon}_d|$ .



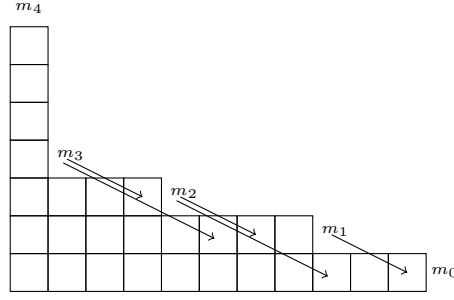


Figure 3.1: The ideal  $M = \langle x^{11}, x^8y, x^4y^2, xy^3, y^7 \rangle$  has  $T^+(M) = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2)\}$  with respect to the grading  $(1, 2)$ .

The assignment  $I \mapsto \text{trop}(I)$  defines a stratification of  $\text{Hilb}_S^h$ , called the *matroid stratification* or *tropical stratification*. A stratum of this stratification consists of all ideals with a fixed tropicalization. If  $\sum_d h(d) < \infty$  (as will always be true in this paper), then there are finitely many strata, and they are Zariski-locally closed. In general, there may be countably many strata; see [Sil22].

### 3.2. Evain's parameterization of the Białynicki-Birula cells

In this section we recall Evain's parameterization of the Białynicki-Birula cells. This relies on the combinatorial decomposition of the tangent space to the Hilbert scheme at a monomial ideal given by *significant arrows*.

**Notation 3.1.** Let  $\prec$  denote the lexicographic order on monomials in  $S = K[x, y]$  with  $x \prec y$ . We set  $r$  to be the Laurent monomial  $x^b/y^a$ . When  $(a, b) = (1, 1)$ , we have  $r = x/y$ .

**Definition 3.2.** Let  $M \subseteq S$  be a finite-colenlength monomial ideal. Write the minimal generators for  $M$  as  $m_0 \prec m_1 \prec \cdots \prec m_e$ , so  $m_0$  is a power of  $x$ , and  $m_e$  is a power of  $y$ . For  $1 \leq i \leq e$  set  $w_i = \text{lcm}(m_i, m_{i-1})$ .

The set  $T^+(M)$  is

$$T^+(M) = \{(i, \ell) : 1 \leq i \leq e, \ell \in \mathbb{Z}_{\geq 1}, m_i r^\ell \in S \setminus M, w_i r^\ell \in M\}.$$

Elements of  $T^+(M)$  are often drawn as arrows from  $m_i$  to  $m_i r^\ell$ , and are called *positive significant arrows*. This is illustrated in Figure 3.1.

The set  $T^-(M)$  of *negative significant arrows* has arrows pointing in the other direction:

$$T^-(M) = \{(i, \ell) : 0 \leq i \leq e - 1, \ell \in \mathbb{Z}_{\leq -1}, m_i r^\ell \in S \setminus M, w_{i+1} r^\ell \in M\}.$$

*Remark 3.3.* The use of arrows as a combinatorial basis for the tangent space of the Hilbert scheme of points at a monomial ideal was introduced by Haiman in [Hai98]. In Haiman's formulation there is an equivalence class of arrows; we follow the convention introduced in [Eva04] to choose a particular representative of this class that starts at a minimal generator of the ideal, and use the notation from [MS10].

The lex-most and lex-least ideals  $M^+$  and  $M^-$  defined in Section 2 can be characterized as the unique ideals with  $T^+(M^+) = T^-(M^-) = \emptyset$ . This was first shown in [Eva04], and generalized in [MS10].

We next recall the construction of the universal ideal over  $C_{\prec}(M)$ .

**Definition 3.4.** For a monomial  $m \in M$ , we define

$$j^+(m) = \max\{i \mid m_i \text{ divides } m\}, \text{ and}$$

$$j^-(m) = \min\{i \mid m_i \text{ divides } m\}.$$

Note  $j^+$  is denoted by  $j$  in [HM12]. We form the polynomial ring  $K[\{c_i^\ell \mid (i, \ell) \in T^+(M)\}]$  with variables indexed by  $T^+(M)$ , and recursively define polynomials

$$f_0, \dots, f_e \in K[\{c_i^\ell \mid (i, \ell) \in T^+(M)\}][x, y]$$

by  $f_0 = m_0$  and

$$f_i = \frac{m_i}{m_{i-1}} f_{i-1} + \sum_{(i, \ell) \in T^+(M)} c_i^\ell \frac{m_i r^\ell}{m_{j^+(w_i r^\ell)}} f_{j^+(w_i r^\ell)}.$$

Note that the initial (leading) term of  $f_i$  with respect to  $\prec$  is  $m_i$ .

**Theorem 3.5** ([Eva04], Theorem 11). *The set  $\{f_0, \dots, f_e\}$  is a Gröbner basis for the universal ideal  $I$  over  $C_{\prec}(M)$ . The induced map  $\mathbb{A}^{|T^+(M)|} \rightarrow H_S^h$  is injective with image  $C_{\prec}(M)$ .*

We will work directly with the coefficients of  $f_i$ . To do so, we will use a combinatorial non-recursive description of these coefficients given in [HM12], which we now describe.

**Definition 3.6** ([HM12, Definition 4.10]). A path from a generator  $m_i \in M$  is a sequence of positive significant arrows  $P = ((i_1, \ell_1), (i_2, \ell_2), \dots, (i_d, \ell_d))$ , such that:

- (a)  $i_1 \leq i$ , and
- (b) if  $d \geq 2$ , then  $((i_2, \ell_2), \dots, (i_d, \ell_d))$  is a path from  $m_{j^+(w_{i_1} r^{\ell_1})}$ .

The length of  $P$  is  $l(P) = \ell_1 + \dots + \ell_d$ . We also associate to  $P$  the monomial  $c_P = c_{i_1}^{\ell_1} \dots c_{i_d}^{\ell_d}$  in  $K[C_{\prec}(M)] = K[c_i^\ell : (i, \ell) \in T^+(M)]$ . If the sequence is empty,  $P$  is the empty path, which has length 0, and  $c_P = 1$ .

**Example 3.7.** In Figure 3.1, the paths from  $m_3$  are as follows:

Length	Paths
1	<b>((3,1))</b> , ((2,1)), ((1,1))
2	<b>((3,2))</b> , ((3,1),(2,1)), ((3,1),(1,1)), ((2,2)), ((2,1),(1,1))
3	<b>((3,2),(1,1))</b> , ((3,1),(2,2)), ((3,1),(2,1),(1,1))

The boldfaced paths are the direct paths, defined in Definition 3.10.

**Theorem 3.8** ([HM12], Lemma 4.12). *We have the following alternate characterization of  $f_i$ :*

$$f_i = \sum_{\substack{P \text{ a path} \\ \text{from } m_i}} c_P m_i r^{l(P)}.$$

Note that the term in this sum corresponding to the empty path is  $m_i$ .

**Example 3.9.** Continuing Example 3.7, we have  $f_0 = m_0 = x^{11}$ ,  $f_1 = x^8 y + c_1^1 x^{10}$ ,  $f_2 = x^4 y^2 + (c_1^1 + c_2^1) x^6 y + (c_2^1 c_1^1 + c_2^2) x^8$ , and  $f_3 = xy^3 + (c_1^1 + c_2^1 + \mathbf{c}_3^1) x^3 y^2 + (c_2^1 c_1^1 + c_2^2 + c_3^1 c_1^1 + c_3^1 c_2^1 + \mathbf{c}_3^2) x^5 y + (c_3^1 c_2^1 c_1^1 + c_3^1 c_2^2 + \mathbf{c}_3^3 c_1^1) x^7$ . The boldfaced monomials  $c_P$  correspond to direct paths, defined in Definition 3.10.

We will focus on one monomial  $c_P$  in each term of  $f_i$ , as follows.

**Definition 3.10.** Fix  $k \geq 1$ . For all  $j \leq k$ , let  $\ell_j$  be the longest length of a significant arrow  $(j, \ell') \in T^+(M^-)$ . We construct a sequence  $(z_1, z_2, \dots)$  of variables  $c_i^{\ell'}$  as follows. Set  $m = m_k$ , and  $l = 1$ . If  $\ell_k > 0$ , set  $z_1 = c_k^{\ell_k}$ ,  $i = j^+(w_k r^{\ell_k})$ , and  $l = 2$ . Otherwise set  $i = k - 1$ . We now iterate. Given  $m, l, i$ , if  $\ell_i > 0$ , set  $z_l = c_i^{\ell_i}$ ,  $i = j^+(w_i r^{\ell_i})$ , and  $l = l + 1$ . Otherwise set  $i = i - 1$ . This procedure stops when  $i \leq 0$ .

A path  $P$  is called a *direct path* from  $m_k$  if it is of one of the two forms  $(z_1, z_2, \dots, z_s)$ , with  $s > 0$ , or  $(z_1, z_2, \dots, z_s, c_{i'}^{\ell'})$ , with  $s \geq 0$ , where the index  $i'$  agrees with the index of  $z_{s+1}$ , and  $\ell' < \ell_{i'}$ .

*Remark 3.11.* Note the following properties:

1. There is at most one direct path from  $m_k$  of a given length  $\ell$ . This is because a choice of path is determined by  $s$  and  $\ell'$ , and the corresponding path has length  $\ell = \sum_{i=1}^s \ell_i + \ell' < \sum_{i=1}^{s+1} \ell_i$ . We refer to this path, when it exists, as  $p_{k,\ell}$ .
2. For a fixed  $k$ , and a fixed positive significant arrow  $c_i^{\ell'}$ , there is at most one  $\ell$  such that  $c_i^{\ell'}$  is the *last* step in a direct path  $p_{k,\ell}$ , in the sense that for any other positive significant arrow  $c_{i'}^{\ell''}$  in  $p_{k,\ell}$ , we have  $i' > i$ .
3. If  $P$  is a direct path from  $m_k$  of length  $\ell > \ell_k$ , then the path  $P'$  obtained by deleting the first step of  $P$  is a direct path of length  $\ell - \ell_k$  from  $m_{j^+(w_k r^{\ell_k})}$ .

When  $S$  has the standard grading, we next show that direct paths of all possible lengths exist from certain monomials  $m_k$ . This uses the following properties of the lex-most and lex-least ideals.

*Remark 3.12.* In the standard grading  $\deg(x) = \deg(y) = 1$ , the lex-most ideal  $M^+$  is the *lexicographic ideal*, also known as the *lexsegment ideal*, with respect to the order  $x \succ y$ ; see [BH93, Chapter 4]. This is the monomial ideal whose degree  $d$  part is the span of the  $(d + 1) - h(d)$  largest monomials in lexicographic order. The lex-least ideal  $M^-$  is the lexicographic ideal for the opposite order of the variables  $x \prec y$ . A monomial ideal is lex-least with respect to the standard grading if and only if the rows of its Young diagram are *strictly* decreasing in length, and similarly is lex-most if and only if the columns of its Young diagram are strictly decreasing in

length. This means that for  $M^-$  we have  $m_k = x^i y^k$  for some  $i$ , so  $m_k r^\ell \in S$  for  $0 \leq \ell \leq k$ . We also have by symmetry that if  $M^-$  and  $M^+$  are the lex-least and lex-most monomial ideals with a given Hilbert function  $h$  respectively, then the Young diagrams of  $M^-$  and  $M^+$  are transposes of each other.

Another standard-graded fact about  $M^-$  that we need, which is not true for nonstandard gradings, is that  $w_k/m_{k-1} = y$  for all  $k \geq 1$ .

**Proposition 3.13.** *Fix the standard  $(1, 1)$ -grading for  $S$ . Fix  $k \geq 0$  with  $m_k r \in S \setminus M^-$ . If for some  $0 < \ell \leq k$  we have that  $m_k r^\ell \in S \setminus M^-$ , then there is a direct path of length  $\ell$  from  $m_k$ .*

*Proof.* The proof is by induction on  $k$ . When  $k = 0$ , there is no such  $\ell$ , so the claim holds. Now assume that the claim is true for all  $k' < k$ . Let  $\ell'$  be maximal such that  $(k, \ell') \in T^+(M^-)$ . If  $\ell' \geq \ell$ , then we claim that  $(k, \ell) \in T^+(M^-)$ . This follows from the fact that  $w_k r^{\ell'} \in M^-$ , so since  $w_k r^{\ell'} \preceq w_k r^\ell$ , we have  $w_k r^\ell \in M^-$ . In this case  $c_k^\ell$  is the required direct path. Otherwise,  $(k, \ell) \notin T^+(M^-)$ , so  $w_k r^\ell \in S \setminus M^-$ . Let  $k' = j^+(w_k r^{\ell'}) = k - \ell'$ . We have  $w_k r^{\ell'} = x^i m_{k'}$  for some  $i \geq 0$ . Thus  $m_{k'} r^{\ell - \ell'} x^i = w_k r^\ell$ , so since  $w_k r^\ell \in S \setminus M^-$ , the same is true for  $m_{k'} r^{\ell - \ell'}$ , and  $\ell - \ell' \leq k'$ . Since  $\ell' < \ell \leq k$ , we have  $k' > 1$  and  $\ell' + 1 \leq k$ . This means that  $w_k r^{\ell'+1} \in S$ , and  $m_{k'} r \in S \setminus M^-$ , as otherwise we would have  $m_{k'} x^i r = w_k r^{\ell'+1} \in M^-$ , so  $(k, \ell' + 1)$  would be in  $T^+(M^-)$ . By induction there is a direct path  $c_P$  from  $m_{k'}$  of length  $\ell - \ell'$ , so  $c_k^{\ell'} c_P$  is a direct path of length  $\ell$  from  $m_k$ .  $\square$

### 3.3. The structure of the Macaulay matrix

For the rest of this section, we fix the standard  $(1, 1)$  grading on  $S = K[x, y]$ , and a Hilbert function  $h : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ . Let  $I \subseteq K[c_i^\ell : (i, \ell) \in T^+(M^-)][x, y]$  be the ideal of the universal family over  $C_{\prec}(M^-)$  as in Theorem 3.5. Note that there are  $\text{mon}_d = d + 1$  monomials of degree  $d$  in  $S$ .

For any  $d \geq 0$ , and for any basis of  $I_d$ , we may write the coefficients of the basis as the columns of a matrix  $R$  with entries in  $K[C_{\prec}(M^-)]$ ; such a matrix is called a degree- $d$  Macaulay matrix for  $I$ , and has size  $(d + 1) \times (d + 1 - h(d))$ . For any collection  $E$  of  $h(d)$  monomials of degree  $d$ , there is a polynomial in  $I$  with support in  $E$  if and only if the minor indexed by rows corresponding to monomials not in  $E$  is zero. The matroid  $\mathcal{M}(I_d)$  is thus exactly characterized by which maximal minors of  $R$  vanish.

We begin by choosing a basis for  $I_d$  via the combinatorial set-up given in Section 3.2. For each of the  $d + 1 - h(d)$  monomials  $m \in M_d^-$ , the polynomial  $g_m := \frac{m}{m_{j^-(m)}} f_{j^-(m)} \in I_d$  has initial term  $m$  with respect to  $x \prec y$ .

As the polynomials  $\{g_m\}_{m \in M_d^-}$  have distinct initial terms, they are all linearly independent. Since  $\dim(I_d) = d + 1 - h(d) = |M_d^-|$ , we conclude that  $\{g_m\}_{m \in M_d^-}$  is a basis for  $I_d$ .

Let  $R$  be the matrix with columns the coefficient vectors of the polynomials  $g_m$ . This is a degree- $d$  Macaulay matrix for  $I$ . We index the rows by  $\text{Mon}_d$  in increasing order with respect to  $x \prec y$ , and index the columns by the monomials in  $M_d^-$  in increasing order with respect to  $x \prec y$ , so  $R_{m', m}$  is the coefficient of  $m'$  in  $g_m$ .

We now make a series of observations about the matrix  $R$ .

$$m^* \rightarrow \begin{pmatrix} * & \times & \cdots & \times & \times \\ * & * & \cdots & \times & \times \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & \times \\ 1 & * & \cdots & * & * \\ 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Figure 3.2: The structure of the Macaulay matrix  $R$ . In the base-change  $\bar{R}$  to the ring  $K[C_{\prec}(M^-)]/\langle Y \rangle$ , the  $\times$  entries are zero.

**Property 3.14.** The matrix  $R$  is upper triangular in the following sense. If  $m' \succ m$ , then  $R_{m',m} = 0$  as  $g_m$  has initial term  $m$ . Since  $M^-$  is the lexicographic ideal with respect to  $x \prec y$  when  $(a, b) = (1, 1)$ , the monomials  $m \in M_d^-$  are  $\prec$ -consecutive; this means that the entries  $R_{m,m}$  comprise a diagonal of  $R$ . We conclude that all entries below this diagonal are zero. The entries along this diagonal are all 1, corresponding to the fact that  $m_k$  has coefficient 1 in  $f_k$ . See Figure 3.2.

**Property 3.15.** Theorem 3.8 gives a combinatorial description of the entry  $R_{m',m}$ . Namely, let  $\ell$  be such that  $m' = mr^\ell$ . Then

$$R_{mr^\ell, m} = \sum_{\substack{P \text{ a path from} \\ m_{j^-(m)} \text{ of length } \ell}} c_P.$$

**Property 3.16.** Let  $m^*$  be the smallest monomial in  $M_d^-$  with respect to  $x \prec y$ . It will be convenient to consider the entries of  $R$  in a *quotient* of  $K[C_{\prec}(M^-)]$  where some variables have been set to zero. Let  $Y = \{c_i^\ell : (i, \ell) \in T^+(M^-), i > j^-(m^*)\}$ . Let  $\bar{R}$  be the base-change of the matrix  $R$  to  $K[C_{\prec}(M^-)]/\langle Y \rangle$ . That is,  $\bar{R}$  is a Macaulay matrix for the universal ideal over the *coordinate subspace* defined by  $\langle Y \rangle$  in  $C_{\prec}(M^-) \cong \mathbb{A}^{|T^+(M^-)|}$ .

The reason for using this quotient is as follows. Suppose  $m \in M_d^-$ , so  $m \succeq m^*$ . Then  $j^-(m) \geq j^-(m^*)$ . By Definition 3.6, a nonempty path  $P$  from  $m_{j^-(m)}$  either (1) contains an element of  $Y$ , in which case  $c_P = 0 \in K[C_{\prec}(M^-)]/\langle Y \rangle$ , or (2) is a path from  $m_{j^-(m^*)}$ . It follows that  $\bar{R}$  has entries

$$\bar{R}_{mr^\ell, m} = \sum_{\substack{P \text{ a path from} \\ m_{j^-(m^*)} \text{ of length } \ell}} c_P.$$

In particular,  $\bar{R}$  is *lower* triangular in the following sense. Note that for  $\ell_0 = j^-(m^*)$  we have that  $m_{j^-(m^*)}r^{\ell_0}$  is a power of  $x$ . Then for  $\ell' > \ell_0$ , we have  $\bar{R}_{mr^{\ell'}, m} = 0$ . This implies the

vanishing of all entries of  $\bar{R}$  that lie above the main diagonal; see Figure 3.2. Furthermore, it follows from Proposition 3.13 and Property 3.15 that all entries on the main diagonal are nonzero.

**Property 3.17.** Define a grading on  $K[C_{\prec}(M^{-})]$  by  $\deg(c_i^\ell) = \ell$ . Then  $R_{mr^\ell, m}$  is homogeneous of degree  $\ell$ . In particular, the degree is constant along diagonals of  $R$ , and satisfies  $\deg(R_{mr, m'}) = \deg(R_{m, m'^{r-1}}) = \deg(R_{m, m'}) + 1$ . This also implies that for every square submatrix  $R'$  of  $R$ , the minor  $\det(R')$  is a homogeneous polynomial. Additionally, for any maximal square submatrix  $R'$  the degrees of the diagonal entries of  $R'$  are a nonincreasing sequence (read starting at the top left as usual); this follows from the fact that  $R'$  is obtained by deleting only rows (and no columns) from  $R$ . The same holds for  $\bar{R}$ .

**Example 3.18.** Consider the monomial ideal  $M^{-} = \langle x^6, x^4y, x^2y^2, xy^3, y^4 \rangle$ . We have  $T^{-}(M^{-}) = \emptyset$ . The chosen degree-4 Macaulay matrix for  $C_{\prec}(M^{-})$  is

$$R = \begin{matrix} & x^2y^2 & xy^3 & y^4 \\ \begin{matrix} x^4 \\ x^3y \\ x^2y^2 \\ xy^3 \\ y^4 \end{matrix} & \begin{pmatrix} c_1^1c_2^1 + c_2^2 & c_1^1c_3^2 & c_1^1c_4^3 \\ c_1^1 + c_2^1 & c_1^1c_2^1 + c_2^2 + c_3^2 & c_1^1c_3^2 + c_4^3 \\ 1 & c_1^1 + c_2^1 & c_1^1c_2^1 + c_2^2 + c_3^2 \\ 0 & 1 & c_1^1 + c_2^1 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

The base-change to  $K[C_{\prec}(M^{-})]/\langle Y \rangle$  is

$$\bar{R} = \begin{matrix} & x^2y^2 & xy^3 & y^4 \\ \begin{matrix} x^4 \\ x^3y \\ x^2y^2 \\ xy^3 \\ y^4 \end{matrix} & \begin{pmatrix} c_1^1c_2^1 + c_2^2 & 0 & 0 \\ c_1^1 + c_2^1 & c_1^1c_2^1 + c_2^2 & 0 \\ 1 & c_1^1 + c_2^1 & c_1^1c_2^1 + c_2^2 \\ 0 & 1 & c_1^1 + c_2^1 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

Observe how the various properties above apply to these matrices:

- Both are upper triangular in the sense of Property 3.14: there are zeros below the diagonal  $m = m'$ .
- The matrix  $\bar{R}$  is lower triangular in the sense of Property 3.16.
- The homogeneity of Property 3.17 is satisfied with  $\deg(c_1^1) = \deg(c_2^1) = 1$ ,  $\deg(c_2^2) = \deg(c_3^2) = 2$ , and  $\deg(c_4^3) = 3$ .

### 3.4. Proof of Theorem 1.3

We now prove:

**Theorem 3.19.** Fix the standard  $(1, 1)$ -grading on  $S$ , and a Hilbert function  $h$ , and fix  $d \geq 0$  such that  $h(d) < d + 1$ . Then every  $(d + 1 - h(d)) \times (d + 1 - h(d))$  minor of  $R$  is a nonzero polynomial in  $K[C_{\prec}(M^{-})]$ .

*Proof.* For convenience, in this proof let  $n_0 = d + 1 - h(d) > 0$ . As in Property 3.16, we define  $m^*$  to be the smallest monomial (with respect to  $x \prec y$ ) in  $M_d^-$ . Again as in Property 3.16, we work with the Macaulay matrix  $\bar{R}$  over  $K[C_{\prec}(M^-)]/\langle Y \rangle$ ; if a minor is nonzero in this ring, it is also nonzero in  $K[C_{\prec}(M^-)]$ .

Fix an  $n_0 \times n_0$  submatrix  $R'$  of  $\bar{R}$ . By Property 3.16, the  $(i, j)$ th entry  $R'_{i,j}$  of  $R'$  is of the form

$$\sum_{\substack{P \text{ a path from} \\ m_{j-(m^*)} \text{ of length } \ell_{i,j}}} c_P$$

for some  $\{\ell_{i,j}\}_{1 \leq i,j \leq n_0}$ . Note that the sum is zero if  $\ell_{i,j} < 0$ . We have  $\ell_{j,j} \geq 0$  for all  $1 \leq j \leq n_0$  by Property 3.14.

The chosen minor is then

$$\det(R') = \sum_{\sigma \in S_{n_0}} (-1)^{\text{sgn}(\sigma)} \prod_{j=1}^{n_0} \sum_{\substack{P \text{ a path from} \\ m_{j-(m^*)} \text{ of length } \ell_{j,\sigma(j)}}} c_P. \quad (3.1)$$

By Proposition 3.13, the path  $p_{j-(m^*),\ell_{j,j}}$  exists for all  $1 \leq j \leq n_0$ . Hence we may define:

$$Q = \prod_{j=1}^{n_0} c_{p_{j-(m^*),\ell_{j,j}}}.$$

Then  $Q$  is a monomial in  $K[C_{\prec}(M^-)]/\langle Y \rangle$ , and  $Q$  appears as a term of the right side of (3.1) when  $\sigma = \text{id}$ . We will show that in fact,  $Q$  appears with coefficient 1 in  $\det(R')$ , with the only contribution coming from that term.

**Claim 3.20.** *Suppose  $C = \prod_{j=1}^s c_{p_{k,\ell_j}}$ , with  $\ell_1 \geq \ell_2 \geq \dots \geq \ell_s \geq 0$ , and we also have  $C = \prod_{i=1}^s c_{P_i}$ , where each  $P_i$  is a path from  $m_k$ , and  $l(P_1) \geq l(P_2) \geq \dots \geq l(P_s) \geq 0$ . Then we have the following inequality with respect to the lexicographic order on  $\mathbb{Z}^s$ :*

$$(\ell_1, \dots, \ell_s) \succeq (l(P_1), \dots, l(P_s)).$$

*Proof of Claim 3.20.* If  $C = 1$  then all paths have length zero, and the claim follows. We now assume that  $C \neq 1$ . The proof is by induction on  $s$ . The base case is  $s = 1$ , in which case we must have  $P_1 = p_{k,\ell_1}$ , so the inequality is an equality. Suppose now that  $s > 1$ , and the result is true for smaller values of  $s$ . Recall from Definition 3.10 that every variable  $c_i^\ell$  dividing  $C$  has  $i$  occurring in some  $z_n$  as defined there. Let  $c_i^{\ell'}$  be the variable with  $i$  minimal dividing  $c_{P_1}$ . Since  $c_{P_1}$  divides  $C$ , we have  $c_i^{\ell'}$  dividing  $c_{p_{k,\ell_j}}$  for some  $1 \leq j \leq s$ . We claim that the length of the part of  $p_{k,\ell_j}$  before the step  $c_i^{\ell'}$  is at least as long as the part of the path  $P_1$  before  $c_i^{\ell'}$ , so  $\ell_j \geq l(P_1)$ , with equality only if  $p_{k,\ell_j} = P_1$ . To see this, note that the part of  $P_1$  before  $c_i^{\ell'}$  contains only variables  $c_{i'}^\ell$ , where  $i' < i$  is the index of some  $z_n$ , while the part of  $p_{k,\ell_j}$  before  $c_i^{\ell'}$  contains every  $z_n$  with  $i' < i$ . Since the length  $\ell$  of  $c_{i'}^\ell$  is at most the length of the associated  $z_n$ , we have  $\ell_j \geq l(P_1)$ , and so  $\ell_1 \geq l(P_1)$ , with equality only if  $j = 1$  and  $p_{k,\ell_1} = P_1$ . When the inequality is strict we have the strict inequality  $(\ell_1, \dots, \ell_s) \succ (l(P_1), \dots, l(P_s))$ , while otherwise the induction hypothesis applied to  $C/c_{p_{k,\ell_1}}$  yields the desired inequality.  $\square$



**Claim 3.21.** For  $\sigma \in S_{n_0}$ , let  $\Pi(\sigma)$  denote the integer partition  $\ell_{1,\sigma(1)} + \ell_{2,\sigma(2)} + \dots + \ell_{n_0,\sigma(n_0)}$ . We treat  $\Pi(\sigma)$  as a nonincreasing list of integers, whose sum is the degree of  $\det(R')$  with respect to the grading in Property 3.17. Then for all  $\sigma \in S_{n_0}$ , we have  $\Pi(\sigma) \succeq \Pi(\text{id})$  with respect to the lexicographic order on  $\mathbb{Z}_{\geq 0}^{n_0}$ , with equality only if  $\sigma = \text{id}$ .

*Proof of Claim 3.21.* Suppose  $\sigma \neq \text{id}$  and  $\Pi(\sigma) \preceq \Pi(\text{id})$ . Let  $i \in \{1, \dots, n_0\}$  be minimal such that  $\sigma(i) > i$ . Then  $\ell_{1,1}, \dots, \ell_{i-1,i-1}$  are parts of both  $\Pi(\sigma)$  and  $\Pi(\text{id})$ . By Property 3.17,  $\ell_{j,j}$  is nonincreasing as  $j$  increases, so  $\ell_{1,1}, \dots, \ell_{i-1,i-1}$  are the  $i - 1$  largest parts of  $\Pi(\text{id})$ . Since  $\Pi(\sigma) \preceq \Pi(\text{id})$ , we must have that  $\ell_{1,1}, \dots, \ell_{i-1,i-1}$  are the  $i - 1$  largest parts of  $\Pi(\sigma)$ . The next largest part of  $\Pi(\text{id})$  is  $\ell_{i,i}$ , but we know that  $\ell_{i,\sigma(i)} > \ell_{i,i}$ , since  $\sigma(i) > i$  and, by Property 3.17,  $\ell_{i,j}$  strictly increases as  $j$  increases. This contradicts  $\Pi(\sigma) \preceq \Pi(\text{id})$ .  $\square$

Claims 3.20 and 3.21 together show that in the sum (3.1), the monomial  $Q$  appears only in the term  $\sigma = \text{id}$ , which is the product

$$\prod_{j=1}^{n_0} R'_{j,j} = \prod_{j=1}^{n_0} \sum_{\substack{P \text{ a path from} \\ m_{j-(m^*)} \text{ of length } \ell_{j,j}}} c_P \tag{3.2}$$

Finally, we argue that the coefficient of  $Q$  in (3.2) is 1. Order the variables  $c_i^\ell$  so that  $c_i^\ell \succ c_j^{\ell'}$  if  $i > j$  or  $i = j$  and  $\ell > \ell'$ . Then  $c_{p_{j-(m^*),\ell_{j,j}}}$  is the largest monomial  $c_P$  in the resulting lexicographic order, when  $P$  varies over all paths from  $m_{j-(m^*)}$  of length  $\ell_{j,j}$ . The initial term of  $R'_{j,j}$  is thus  $c_{p_{j-(m^*),\ell_{j,j}}}$  with coefficient 1. The initial term of the product (3.2) is the product of the initial terms, namely  $Q$ . Thus  $Q$  appears in  $\det(R')$  with coefficient 1, so we conclude that  $\det(R')$  is a nonzero element of  $K[C_{\prec}(M^-)]$ , for any field  $K$ .  $\square$

Theorem 3.19 is the key to proving Theorem 1.3.

*Proof of Theorem 1.3.* For an ideal  $I \in \text{Hilb}_S^h$ , we have  $\mathcal{M}(I_d) = U_{h(d),d+1}$  for all  $d \geq 0$  if and only all maximal minors of all Macaulay matrices for  $I$  are nonzero in degrees where  $h(d) > 0$ . By Theorem 3.19, these minors are *nonzero* polynomials in  $K[C_{\prec}(M)]$ , so  $\mathcal{M}(I_d) = U_{h(d),d+1}$  for all  $d \geq 0$  if and only if  $I$  is in the complement of the vanishing sets of these finitely many polynomials. The set of such  $I$  forms a nonempty open subset of  $C_{\prec}(M^-)$ , and hence of  $\text{Hilb}_S^h$ . This implies the main claim of Theorem 1.3; the second claim in Theorem 1.3 follows from the standard fact that if  $K$  is an infinite field, then any nonempty open subset of  $\mathbb{A}_K^n$  contains a  $K$ -point.  $\square$

*Remark 3.22.* The proofs of Theorems 1.2 and 1.3 rely on  $K$  being infinite, but we do not know of a counterexample to either one with  $K$  finite.

*Remark 3.23.* If  $I \in \text{Hilb}_S^h$  satisfies the conclusion of Theorem 1.3, then necessarily  $\text{in}_{\prec}(I) = M^-$  and  $\text{in}_{\prec^{opp}}(I) = M^+$ ; that is,  $I \in E(M^-, M^+)$ . Indeed, for  $I \in C_{\prec}(M^-)$ ,  $\text{in}_{\prec}(I)_d$  is the span of the monomials corresponding to leading ones in the reduced column-echelon form of  $R$ . Thus the condition  $\text{in}_{\prec^{opp}}(I) = M^+$  is equivalent to the nonvanishing of a *single* maximal minor of  $R$ . This is a strictly weaker condition than the nonvanishing of *all* maximal minors, as guaranteed by Theorem 1.3.

*Remark 3.24.* Theorem 1.3 determines  $\mathcal{M}(I_d)$  when  $I$  is a general element of  $E(M^-, M^+)$ . It is natural to ask if the theorem can be generalized to determine the matroid of a general element of an arbitrary edge-scheme  $E(M, M')$ , at least when  $E(M, M')$  is irreducible. In small examples, even when  $E(M, M')$  is irreducible,  $\mathcal{M}(I_d)$  is often non-uniform. For example, this occurs in  $N = 6$ , in the edge in Figure 2.2 connecting  $(4, 1, 1)$  and  $(3, 1, 1, 1)$ , where  $\mathcal{M}(I_2)$  has  $xy$  as a loop.

### 3.5. Discussion of other gradings

In this section we show that Theorem 1.3 does not hold for all edges in the spine, so the standard-graded hypothesis is necessary.

We first note that the degree- $d$  matroid can have *loops* and *coloops* in degrees where the entire matroid is not trivial.

**Example 3.25.** Let  $(a, b) = (2, 3)$ . Then the two monomial ideals  $M^- = (x^7, xy, y^4)$  and  $M^+ = (x^6, xy, y^5)$  share a Hilbert function  $h$ . (Here  $N = 10$ .) The ideal  $M^-$  has the unique positive significant arrow  $(2, 2)$ , and the universal ideal  $I$  over  $C_{\prec}(M^-)$  is thus  $\langle x^7, xy, y^4 + c_2^2 x^6 \rangle$ .

The degree-12 Macaulay matrix is

$$\begin{array}{c} x^6 \\ x^3 y^2 \\ y^4 \end{array} \begin{array}{cc} x^3 y^2 & y^4 \\ \left( \begin{array}{cc} 0 & c_2^2 \\ 1 & 0 \\ 0 & 1 \end{array} \right) \end{array}.$$

The matroid  $\mathcal{M}(I_{12})$  on ground set  $\{x^6, x^3 y^2, y^4\}$  has circuits  $\{\{x^3 y^2\}, \{x^6, y^4\}\}$ . In particular,  $\mathcal{M}(I_{12})$  is not a uniform matroid, due to the existence of the loop  $x^3 y^2$ . This loop is forced to exist since  $h(5) = 0$ , so  $xy \in I$ , and thus  $x^3 y^2 \in I$ , for any ideal  $I$  with Hilbert function  $h$ .

Furthermore, the degree-8 Macaulay matrix is

$$\begin{array}{c} x^4 \\ xy^2 \end{array} \begin{array}{c} xy^2 \\ \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \end{array},$$

so the matroid  $\mathcal{M}(I_8)$  on ground set  $\{x^4, xy^2\}$  has the unique circuit  $\{xy^2\}$ . Again,  $\mathcal{M}(I_8)$  is not a uniform matroid. In addition to the loop  $xy^2$ , there is also the coloop  $x^4$ , which is forced to exist by the structure of  $h$ . To see this, note that since  $xy \in I$  as noted above, we have  $xy^2 \in I$ . As  $h(8) = 1$ , we must have  $x^4 \notin I$  for any ideal  $I$  with Hilbert function  $h$ , so the matroid  $\mathcal{M}(I_8)$  has a coloop.

We now see, however, that loops and coloops do not entirely account for the failure of Theorem 1.3.

**Example 3.26.** Let  $(a, b) = (2, 3)$ . Let  $h$  be the Hilbert function of the monomial ideal  $M^- = \langle x^{10}, x^7 y, x^2 y^3, xy^5, y^6 \rangle$ . Then  $M^+ = \langle x^9, x^5 y, x^4 y^3, xy^5, y^7 \rangle$ . (Here  $N = 29$ .) The ideal  $M^-$  has the positive significant arrows

$$T^+(M^-) = \{(2, 1), (4, 2), (4, 3)\}.$$

Thus the universal ideal  $I$  over  $C_{\prec}(M^{-})$  is

$$\langle x^{10}, x^7y, x^2y^3 + c_2^1x^5y, xy^5 + c_2^1x^4y^3, y^6 + c_2^1x^3y^4 + c_4^2x^6y^2 + c_4^3x^9 \rangle.$$

The degree-18 Macaulay matrix is

$$\begin{matrix} & x^3y^4 & y^6 \\ x^9 & \begin{pmatrix} 0 & c_4^3 \\ c_2^1 & c_4^2 \\ 1 & c_2^1 \\ 0 & 1 \end{pmatrix} \\ x^6y^2 & \\ x^3y^4 & \\ y^6 & \end{matrix}.$$

The degree-18 matroid of  $I$  thus has rank 2 on the ground set  $\{x^9, x^6y^2, x^3y^4, y^6\}$ , with circuits

$$\{\{x^3y^4, x^6y^2\}, \{x^9, x^3y^4, y^6\}, \{x^9, x^6y^2, y^6\}\}.$$

This is not the uniform matroid, and does not have any loops or coloops. This is the smallest example we know in which a matroid appears that is not the direct sum of a uniform matroid with a collection of loops and coloops.

*Remark 3.27.* In Example 3.26,  $\text{trop}(I)$  is “maximally general”, in the following sense. Let  $J \subseteq \mathbb{B}[x, y]$  be any tropical ideal with Hilbert function  $h$ , in the sense of [MR18]. Then for all  $d \geq 0$ , the matroid  $\mathcal{M}(J_d)$  is a weak image of  $\text{trop}(I)_d$ .

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