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Multiple-Fault Detection and Isolation Based on Disturbance Attenuation Theory

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Aerospace Engineering

by

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ABSTRACT OF THE DISSERTATION

Multiple-Fault Detection and Isolation
Based on Disturbance Attenuation Theory

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Professor Jason L. Speyer, Chair

In this dissertation, a linear estimator for fault detection and isolation called the Game Theoretic Multiple-Fault Detection Filter is derived for both continuous and discrete systems. The detection filter uses a disturbance attenuation formulation to bound the transmission of disturbances to the output, approximately blocking all but one fault from each of a set of projected residuals. However, different from previous approximate methods for single-fault detection filters, the multiple-fault detection filter utilizes a secondary optimization problem to generate a solution for the estimator gain that achieves more advanced detection filter goals. Specifically, the current work examines an optimization that increases sensitivity of each projected residual to its target fault. For the continuous case, it is proven that the new detection filter approximates previous detection filters obtained from geometric and spectral theories and extends them to finite time-varying systems. Further, the detection filter is demonstrated via numerical examples.

The dissertation of Emmanuell Murray is approved.

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2012

This dissertation is dedicated to my parents,

Allie and Elijah Murray, Jr.

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LIST OF ACRONYMS

BJDF	Beard-Jones Detection Filter
DAP	disturbance attenuation problem
DGTFDF	Discrete Game Theoretic Fault Detection Filter
DGTMFDF	Discrete Game Theoretic Multiple-Fault Detection Filter
FDI	fault detection and isolation
GTFDF	Game Theoretic Fault Detection Filter
GTMFDF	Game Theoretic Multiple-Fault Detection Filter
OSFDF	Optimal Stochastic Fault Detection Filter
RDDF	Restricted-Diagonal Detection Filter
UIO	Unknown Input Observer

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In 2003, I moved across the country to California, much to the dismay of my parents who had never been more than an hour away before. What started out as a desire to obtain a master's degree and find a job, thanks to my advisor Prof. Speyer, gave way to an increasing interest in taking the much longer road to a PhD. I won't lie, there were times when I wondered if this interest had constituted a terrible lack of judgement. However, I had many supporters that saw me through to the end to whom I owe my sincerest thanks.

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PUBLICATIONS AND PRESENTATIONS

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Chapter 1

Introduction

Control of distributed vehicle systems is a technology that will enhance much of our currently overburdened transportation infrastructure by revolutionizing air traffic control and automated highways. Further, such systems enable autonomous aircraft to fly in formation for drag reduction and arrays of satellites to form large aperture observatories. Unfortunately, many vehicles are subject to possible degradation and failure of relative sensors and control actuators. There are many examples every day in which human controllers are able to overcome vehicle failures, such as blown car tires or jet engine failures. However, autonomous vehicle systems, especially those grouped in tight formations or clusters, suffer from dramatically increased sensitivity to such failures. Not only must the vehicle detect that a failure has occurred, but it must also do so quickly before the failure affects other parts of the distributed system. For the safety of the distributed system, vehicles must be able to maintain accurate relative control at all times, even in the presence of sensor and actuator failures. Therefore, complex distributed vehicle systems require the development of extremely reliable automated control technology. An enabling element of this technology is a distributed health management system that detects, isolates, and determines the time history of faults and then reconfigures the affected vehicles for continuous performance.

The proposed approach to the first step of a distributed health management system is the fault detection and isolation (FDI) system. FDI has received a great deal of attention over the last few decades. The earlier history of the field has been codified by several surveys [1, 2, 3, 4, 5, 6, 7], as well as the books [8, 9, 10]. FDI survey papers of the current century were written by Venkatasubramanian *et. al.* [11], Hwang *et. al.* [12] and Meskin and Khorasani [13]. Recently, two books have been published on model-based FDI methods by Isermann [14] and Ding [15]. The literature has produced several definitions of the terms fault detection and fault isolation. To clarify, in this work, *fault detection* refers to announcing the occurrence of a fault, while *fault isolation* refers to identifying the source of said fault.

In general, an FDI system utilizes redundant measures of the same quantity, then generates and analyzes an error residual composed by differencing pairs of redundant quantities. There are two types of redundancy: hardware redundancy and analytical redundancy. Hardware redundancy utilizes algebraic combinations of two or more redundant hardware signals (e.g. - two sensors that measure the same quantity) to detect faults. The signals are usually from identical sources so that dissimilar noise values do not affect the ability to detect changes in the mean residual. However, to enable fault isolation, multiple redundant pairs must be utilized since each is sensitive to more than one fault source. Analytical redundancy, on the other hand, uses mathematical models either to compare dissimilar hardware signals (e.g. - comparing the desired control signal with a sensor output measuring thrust, combining two sensor measurements to compare to a third) or to make predictions on the sensor outputs based on the assumed system dynamics and control signals. These mathematical predictions are then compared with sensor data, even sensors with different noise characteristics. If there is a variation between the measurement prediction and the sensor data (i.e. - the error residual has a nonzero mean), a fault is declared. To enable fault isolation, multiple error residuals are generated where each is made sensitive to only one fault. The overall FDI system may contain both hardware and analytical redundancy methods. The current approach utilizes analytical redundancy in what is know as a *fault detection filter* (a term that, in this work, is used interchangeably with *detection filter* and *FDI filter*).

1.1 Fault Detection Filters

A fault detection filter is a special class of linear observer. The observer produces an estimate of the system's output, which is then compared to the actual output to generate an error residual. Detection filters may be classified into two types: the *single-fault* detection filter whose residual is made sensitive to a single fault or subset of faults, and the *multiple-fault* detection filter whose residual is forced to take on specific directional properties depending on the fault source. By analyzing the residual direction, multiple, simultaneously occurring faults may be detected and isolated via a (single) multiple-fault detection filter. Both types have their own advantages with respect to each other. In general, asymptotically stable single-fault detection filters may be obtained for more general specifications of the dynamic system. On the other hand, a collection of single-fault detection filters are required to detect multiple faults. Thus, multiple-fault detection filters are more computationally efficient, requiring fewer states overall to detect and isolate the same number of faults as a bank of single-fault filters. This is true even when reduced-order detection filters may be obtained. For both the single-fault and multiple-fault detection filters, the raw residual is projected onto a specific residual subspace known *a priori* to be sensitive to only one fault. The multiple-fault filter uses multiple residual projectors to aid in the detection of different faults. In this section, a brief examination of previous fault detection filters is given for continuous and discrete systems.

1.1.1 Geometric and Spectral Theory-Based Methods

The detection filter using analytical redundancy for fault detection and isolation was first introduced by Beard [16] and Jones [17], now called the Beard-Jones Detection Filter (BJDF). The idea of the BJDF is to place the reachable subspace of each fault into non-overlapping invariant subspaces called detection spaces. Then, when a fault occurs, the resulting nonzero residual can be projected onto different residual subspaces associated with each fault so that the fault source can be identified. Each detection space includes both the reachable subspace of the associated fault and the directions associated with the transmission zeros (or invariant zeros) of the fault's transfer function. Constraints on the locations of the invariant zeros are imposed to guarantee the desired properties of the detection filter. Geometric and spectral analyses were given in [18] and [19], respectively. In [20], Chen re-examined the spectral analysis and showed that the structures of the detection filters generated from the spectral and geometric theories are equivalent. In [21], Chen generalized the BJDF to detect faults of arbitrary dimension. Design algorithms using right-eigenvectors were developed based on spectral theory in [19, 22, 23]. However, while the algorithm of [23] is simpler than White's algorithm in [19], an asymptotically stable detection filter can be obtained only for a more restricted class of faults. A closed-form version of White's algorithm was obtained in [20].

In [18], Massoumnia derived a slight generalization of the BJDF, called the Restricted-Diagonal Detection Filter (RDDF), using geometric theory. Instead of placing a particular fault into an invariant subspace like the BJDF, the RDDF places all but one fault into the unobservable subspace of a projected residual. When every fault is detected, the RDDF is equivalent to the BJDF. However, certain faults may not need to be detected, simply blocked from the residual output. The constraints on these faults are relaxed in the RDDF, making it potentially more robust than the BJDF [24]. The RDDF is *restricted* because it is a full-order observer and *diagonal* because the transfer function from the faults to the projected residuals is diagonal [18]. The RDDF was reformulated using spectral theory in [25]. Simple design algorithms using left-eigenstructure assignment were developed based on geometric theory in [24] and spectral theory in [25].

The Unknown Input Observer (UIO), originally used for fault-tolerant estimation, was applied to the detection filter problem for a single fault in [26]. The UIO simplifies the detection filter problem by requiring that all but one fault, the fault to be detected, be placed in an unobservable detection space that can be annihilated via model reduction. It was shown that constraints on the fault directions and on the location of invariant zeros are relaxed compared to the previous detection filters. A bank of UIOs can be used to detect multiple faults, trading relaxed constraints for possibly increased computational complexity. Banks of detection filter UIOs were applied to multi-vehicle actuator fault detection in [27] and to fault detection of Markovian jump linear systems in [28].

1.1.2 Robust Approximate Methods

The main drawback of the spectral and geometric methods is the rigidity of their structure and sensitivity to noise. In order to increase the flexibility of the detection filter problem, it has been approximated by relaxing the requirement of strict blocking of undesirable faults and noises. Recently, much attention has been devoted to approximating UIOs for fault detection using methods based on $\mathcal{H}_\infty$ estimation, which seeks to minimize the $\mathcal{H}_\infty$ norm of the disturbances' transfer functions. Enhanced sensitivity to the detected faults was examined in [29, 30, 31] and extended to time-varying systems [32] and detection filters for multiple faults [33, 34], though the latter is limited to certain special classes of faults. $\mathcal{H}_\infty$ methods were recently applied to fault detection for linear time-varying discrete systems in [35, 36]. The references cited therein discuss the results for time-invariant systems, with particular attention to linear matrix inequality methods.

Another robust approximation method is based on disturbance attenuation. This method generally applies to more complex systems and disturbances than the $\mathcal{H}_\infty$-based methods. In [37], a finite time-varying approximation of the UIO called the Game Theoretic Fault Detection Filter (GTFDF) was derived by applying a disturbance attenuation problem (DAP) for fault-tolerant estimation [38] to fault detection. The DAP was optimized with respect to the estimate and the disturbance inputs, including nuisance faults (or faults that must be blocked from the projected residual), noises, and initial condition error, via a differential game. The resulting detection filter gain is a function of the estimation error covariance, whose optimal value is the solution to a Riccati differential equation similar to those used in $\mathcal{H}_\infty$ control. As the disturbance attenuation bound goes to zero, methods from singular optimal control are used to derive the asymptotic detection filter. The geometric structure of the UIO is largely recovered in the limit. However, the invariant zero directions are not automatically included in the detection spaces created by the optimization, though they can be included artificially by modifying the fault directions [39]. The GTFDF was applied to the decentralized fault detection problem in [40]. A similar Riccati-based detection filter with enhanced sensitivity to the detected fault was derived in [41, 42].

The GTFDF was rederived for finite time-invariant, discrete systems in [43], resulting in the Discrete Game Theoretic Fault Detection Filter (DGTFDF). Similar to the GTFDF, the detection filter problem was modeled as a DAP and optimized with respect to the estimate and disturbances via a differential game. Further, the resulting filter gain is constrained by a discrete Riccati differential equation. However, unlike the continuous case, the curvature of the Riccati equation with respect to the fault directions is not lost as the disturbance attenuation bound goes to zero. Thus, the asymptotic detection filter is evaluated without requiring singular optimal control. Instead, it is shown that the Riccati equation develops *constant directions*, which behave similarly to an invariant subspace in the continuous case, along the observable subspace of the nuisance fault.

A stochastic robust approximation method called the Optimal Stochastic Fault Detection Filter (OSFDF) was introduced by in [39, 41, 44] and extended to the multiple-fault case in [41, 45]. The OSFDF uses a stochastic description of the estimation error covariance to construct a cost function. Specifically, the optimization chooses the filter gain to minimize the "transmissions", defined by the individual covariances, of the disturbance inputs and maximize transmission of the target fault (or fault to be detected) to the projected output error. In the multiple-fault case, multiple error covariances are constructed so that the detected faults can be isolated by their respective projected residuals. An example showed that the multiple-fault OSFDF automatically included the faults' invariant zero directions in their proper detection spaces. Thus, in the limit as the weight on the nuisance fault transmission goes to infinity, the structure of the RDDF is recovered. However, this approach assumes that all fault magnitudes are white noise processes and very little could be stated about the optimality of solutions in the general or limiting multiple-fault cases.

1.2 Contribution of the Dissertation

The focus of the dissertation is to develop an approximation of the multiple-fault detection filter that is robust to nuisance faults and to sensor noise. Because a disturbance attenuation-based formulation lends itself immediately to use in fault detection, it is the basis for the novel detection filter, called the Game Theoretic Multiple-Fault Detection Filter (GTMFDF). The proposed fault detection filter is derived for both continuous and discrete systems.

The GTMFDF has a solution similar to that of the single-fault GTFDF of the previous work. Both detection filters are obtained from disturbance attenuation and have Riccati-based solutions. Thus, both detection filters have reduced sensitivity to noise, parameter variations, and modeling errors compared to detection filters based on spectral and geometric theories. However, whereas the GTFDF gain is associated with a single DAP and constrained by a single Riccati equation, the GTMFDF uses multiple Riccati inequalities, one for each DAP, as the constraints on a secondary optimization to find the actual filter gain. The Riccati inequalities are very similar to the GTFDF Riccati equations, except for a new term that weights the difference between the filter gain and its optimal single-fault solution for each DAP. This added term is the key to blending multiple Riccati constraints to find a single filter gain via a secondary optimization. The added term also allows us to obtain alternative solutions to the single-fault problem, which may be used to account for system limitations or to tradeoff target fault sensitivity for nuisance fault insensitivity. The same occurs in the discrete case as well. In the limit as the disturbance attenuation bound goes to zero, both detection filters may satisfy identical constraints on the filter gain and Riccati variable. However, the GTMFDF satisfies these constraints over multiple DAPs.

To the best of knowledge of the author, the only comparable fault detection filter to date (for continuous-time systems) is the multiple-fault OSFDF by Chen and Speyer [41, 45]. However, the GTMFDF improves upon this previous work. While the cost function of the multiple-fault OSFDF is effective for penalizing nuisance fault and noise transmissions to the residual while promoting target fault transmission, the physical meaning of the sum of covariances is unclear. The disturbance attenuation problem, on the other hand, is very easily understood. Further, whereas target fault sensitivity was achieved through the constraint equations of the OSFDF, the GTMFDF achieves the same using the filter gain optimization itself. Also, the stochastic derivation of the multiple-fault OSFDF assumes that the fault magnitudes are white noise processes. The GTMFDF requires no such assumption.

1.2.1 Overview of the Dissertation

Chapter 2 revisits fault modeling and fault detection filters based on spectral and geometric theories for both continuous-time and discrete-time systems. Specifically for the continuous-time case, the RDDF problem is reviewed and used to rederive the state space structure. For discrete-time systems, a discrete detection filter analogous to the RDDF is presented.

Chapter 3 presents the GTMFDF. The RDDF is approximated by a set of DAPs, one associated with each projected residual, coupled by a single filter gain. The detection filter problem is to find constraints on the filter gain such that the disturbance attenuation bound is met under worst-case disturbance and fault conditions. The problem's solution poses a set of matrix Riccati differential inequalities on the estimation error covariances for each DAP and the filter gain. These Riccati inequalities become constraints on a secondary problem to optimize the covariances and filter gain with respect to a new, arbitrary cost function. Optimal solutions are generated either numerically or using calculus of variations. The latter implies a two-point boundary value problem that must be solved to obtain the filter gain. The chapter concludes with some results on the existence of solutions in the steady state case and a comparison to the previous Riccati-based approximate detection filter methods.

Chapter 4 examines the continuous detection filter in the limit as the disturbance attenuation bound goes to zero and the cost becomes singular with respect to the complementary fault. In this case, the detection filter problem generates Riccati inequality constraints (similar to those of the general case) along with a set of equality constraints that force the geometric structure of the detection filter to restrict faults to strict detection spaces. Thus, for most fault scenarios, the behavior of the asymptotic detection filter mirrors a finite time-varying version of the RDDF. However, unlike the RDDF, the asymptotic detection filter may not exist in general when subject to identical assumptions. A special case of the solution, for which sufficient conditions for existence are derived, has constraints identical to those of the single-fault problem for each DAP, with the difference that the filter gain must satisfy the constraints over multiple DAPs. Finally, by using the geometric structure, a method of obtaining a reduced-order detection filter is discussed.

Chapter 5 presents a parallel analysis of the GTMFDF for discrete-time systems, called the Discrete Game Theoretic Multiple-Fault Detection Filter (DGTMFDF). The detection filter problems and solutions for the continuous and discrete cases are extremely similar when the disturbance attenuation bound is nonzero. Similar Riccati inequality constraints are derived using calculus of variations and these inequalities become constraints on a secondary filter gain optimization. It is shown that the discrete detection filter generalizes the solution of the DGTFDF and extends it to the finite time-invariant multiple-fault case.

Finally, the GTMFDF is demonstrated via numerical examples in Chapter 6. In the second example, particular attention is paid to the eigenstructure of the detection filter. The example shows that the eigenstructure of the GTMFDF approximates that of the BJDF and RDDF. When each of the system's invariant zero directions can be included in the detection space of an individual fault, the detection filter eigenvalues are not tied to the invariant zeros. This is an improvement over the single-fault approximate detection filters, for which (nonpositive) eigenvalues are located at the system's invariant zeros or their mirror images.

Chapter 2

Fault Detection Filter Background

In this chapter, the fault detection filters used as a basis for the current research are discussed. In Section 2.1, fault modeling and the multiple-fault detection filter for continuous systems are reviewed. The GTMFDF in Chapter 3 approximates the geometric structure of the multiple-fault detection filter. In the limit as the disturbance attenuation bound goes to zero, the asymptotic solution to the GTMFDF in Chapter 4 emulates this same geometric structure. In Section 2.2, fault modeling and detection filtering for discrete systems are reviewed. Likewise, the DGTMFDF of Chapter 5 approximates the geometric structure of the discrete multiple-fault detection filter.

2.1 Continuous Multiple-Fault Detection Filtering

In this section, the continuous-time detection filters relevant to the current research are discussed in detail. First, the system and fault modeling techniques are discussed for the continuous-time case in Section 2.1.1. It is shown that actuator, sensor, and plant faults may all be treated as additive inputs to the system dynamics. Then, an overview of the RDDF is given in Section 2.1.2 to examine the invariant subspace structure of the detection filter.

2.1.1 Plant and Fault Modeling

In this section, models of the plant, sensor faults, and actuator faults are given [16, 19, 37, 41]. Consider a linear time-varying system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{2.1a}$$

$$y(t) = C(t)x(t) + D(t)u(t), \tag{2.1b}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^l$ is the control input, and $y(t) \in \mathbb{R}^m$ is the measurement.

The i^{th} actuator fault is modeled as an additive input to the state dynamics (2.1a) [16, 19] and to the measurement (2.1b) (when $D(t)$ is nonzero):

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + F_{a,i}(t)\mu_{a,i}(t)$$
$$y(t) = C(t)x(t) + D(t)u(t) + E_{a,i}(t)\mu_{a,i}(t)$$

where $F_{a,i}(t)$ and $E_{a,i}(t)$ are the *a priori* known actuator fault directions in the dynamics and measurement, respectively, and $\mu_{a,i}(t)$ is the actuator fault magnitude, which is an unknown, arbitrary function of time. For example, a stuck off i^{th} actuator has a fault direction of $F_{a,i}(t) = B_i(t)$, where $B_i(t)$ is the i^{th} column of $B(t)$, and a fault magnitude of $\mu_{a,i}(t) = -\bar{u}_i(t)$. Note that $\mu_{a,i}(t)$ is nonzero only when a fault is present.

The i^{th} sensor fault is modeled as an additive input to the measurement (2.1b) [16, 19]:

$$y(t) = C(t)x(t) + D(t)u(t) + F_{s,i}(t)\mu_{s,i}(t) \tag{2.2}$$

where $F_{s,i}(t)$ is the *a priori* known sensor fault direction, and $\mu_{s,i}(t)$ is the unknown, arbitrary sensor fault magnitude. Sensor faults are assumed to affect one sensor at a time. Thus, the fault direction $F_{s,i}(t)$ is a column vector of zeros except for a one in the i^{th} row. For example, a constant sensor bias in the i^{th} sensor has a fault magnitude of $\mu_{s,i}(t) = c_s$, where c_s is a constant.

To design an FDI filter, faults must be modeled as inputs to the system dynamics. Therefore, an input that drives the dynamics similarly to $\mu_{s,i}(t)$ in (2.2) is required. One method transforms the state so that it incorporates the sensor faults [37]. Define $f_{s,i}(t)$ and $e_{a,i}(t)$ to satisfy

$$F_{s,i}(t) = C(t)f_{s,i}(t)$$
$$E_{a,i}(t) = C(t)e_{a,i}(t).$$

In general, $f_{s,i}(t)$ and $e_{a,i}(t)$ are not unique. One solution satisfying the above conditions is

$$f_{s,i}(t) = C(t)^T \left(C(t)C(t)^T\right)^{-1} F_{s,i}(t)$$
$$e_{a,i}(t) = C(t)^T \left(C(t)C(t)^T\right)^{-1} E_{a,i}(t)$$

Then, a new state $\bar{x}(t)$ may be obtained where

$$\bar{x}(t) = x(t) + f_{s,i}(t)\mu_{s,i}(t) + e_{a,i}(t)\mu_{a,i}(t).$$

Therefore, (2.2) is rewritten

$$y(t) = C(t)\bar{x}(t)$$

and the dynamic equation of $\bar{x}(t)$ is

$$\dot{\bar{x}}(t) = A(t)\bar{x}(t) + B(t)u(t) + \begin{bmatrix} \dot{f}_{s,i}(t) - A(t)f_{s,i}(t) & f_{s,i}(t) \end{bmatrix} \begin{bmatrix} \mu_{s,i}(t) \\ \dot{\mu}_{s,i}(t) \end{bmatrix}$$
$$+ \begin{bmatrix} \dot{e}_{a,i}(t) - A(t)e_{a,i}(t) + F_{a,i}(t) & e_{a,i}(t) \end{bmatrix} \begin{bmatrix} \mu_{a,i}(t) \\ \dot{\mu}_{a,i}(t) \end{bmatrix}. \quad (2.3)$$

In (2.3), the faults magnitudes and system matrices are assumed differentiable. Thus, the sensor fault is modeled as a two-component additive term to (2.3), where $\dot{f}_{s,i}(t) - A(t)f_{s,i}(t)$ is the sensor fault magnitude direction and $f_{s,i}(t)$ is the sensor fault rate direction. When the measurement contains a direct feedthrough term, the actuator fault is modeled as a similar two-component term. Therefore, any fault may be treated as an additive term to the state dynamics, and for the remainder of the dissertation, it is assumed that all faults are modeled as such.

Remark 2.1: Plant faults can be modeled similarly by choosing fault directions along the changes in the system matrices ΔA, ΔB, ΔC, and ΔD. Then, these faults are represented as additive inputs to the dynamics using the same procedures as discussed above.

◇

2.1.2 Restricted-Diagonal Detection Filter (RDDF) Background

In this section, the geometric structure of the RDDF is derived as in [18, 24, 41, 25]. Consider a linear time-invariant system with q faults and m measurements

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i=1}^{q} F_i \mu_i(t) \tag{2.4a}$$

$$y(t) = Cx(t) + Du(t), \tag{2.4b}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^l$ is the control input, $y(t) \in \mathbb{R}^m$ is the measurement, $\mu_i(t) \in \mathbb{R}$ is an unknown, arbitrary scalar fault magnitude, and $F_i \in \mathbb{R}^n$ represents the *a priori* known fault direction of $\mu_i(t)$. It is assumed that the measurements are linearly independent, and so $C \in \mathbb{R}^{m\times n}$ is full row rank (with $m \leq n$). The detection filter is a linear observer of the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left(y(t) - C\hat{x}(t) - Du(t)\right) \tag{2.5a}$$

$$r(t) = y(t) - C\hat{x}(t) - Du(t), \tag{2.5b}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the state estimate, $L \in \mathbb{R}^{n\times m}$ is the filter gain, and $r(t) \in \mathbb{R}^m$ is the residual. Using (2.4) and (2.5), the state estimation error $e(t) \triangleq x(t) - \hat{x}(t)$ is subject to the dynamic system

$$\dot{e}(t) = (A - LC)e(t) + \sum_{i=1}^{q} F_i\mu_i(t) \tag{2.6a}$$

$$r(t) = Ce(t). \tag{2.6b}$$

In general, a detection filter is required to detect one or more *target faults*. However, some faults, called *nuisance faults* [24, 25], are disturbances that may not need to be detected explicitly, simply blocked from the residuals. Without loss of generality, assume that the first $s \leq q$ faults are target faults and the remaining $q - s$ faults are nuisance faults. The nuisance faults can be grouped into a single vector $\hat{\mu}(t) = [\ \mu_{s+1}(t)\ \ \ldots\ \ \mu_q(t)\ ]^T$. Several residuals can be generated, each sensitive to only one target fault, by multiplying $r(t)$ by a residual projector $\hat{H}_i$. Therefore, the dynamic model of the estimation error in (2.6) is rewritten as

$$\dot{e}(t) = (A - LC)e(t) + \sum_{i=1}^{s} F_i\mu_i(t) + \hat{F}\hat{\mu}(t) \tag{2.7a}$$

$$r_i(t) = \hat{H}_iCe(t). \tag{2.7b}$$

where $\hat{F} = [\ F_{s+1}\ \ \ldots\ \ F_q\ ]$ is the nuisance fault direction and $r_i(t)$ is the projected residual associated with fault $\mu_i(t)$.

Remark 2.2: For the remainder of the dissertation, the subscript i will be used to denote the target fault index where $i \in \{1, \ldots, s\}$. ♢

The RDDF problem is to choose L so that the detection filter satisfies the following objectives [18, 24, 41, 25]:

- When the target fault μ_i occurs, the residual $r(t)$ lies in a fixed subspace that is linearly independent from the subspace associated with the remaining faults.
- The projected residual $r_i(t)$ is nonzero if the target fault $\mu_i(t)$ occurs.
- The eigenvalues of the detection filter can be chosen arbitrarily.
- The steady-state residual response to a constant bias fault is nonzero.

These objectives lead to a description of the state space geometry that clarifies the structure of the RDDF problem. The remainder of this section is focused on deriving this geometry.

The invariant subspace structure of the RDDF is formulated around the occurrence of the complementary fault

$$\hat{\mu}_i(t) = \begin{bmatrix} \mu_1(t) & \ldots & \mu_{i-1}(t) & \mu_{i+1}(t) & \ldots & \mu_q(t) \end{bmatrix}$$

with direction

$$\hat{F}_i = \begin{bmatrix} F_1 & \ldots & F_{i-1} & F_{i+1} & \ldots & F_q \end{bmatrix}.$$

When it occurs, the estimation error lies in

$$\hat{\mathcal{W}}_i = \mathcal{W}_1 + \ldots + \mathcal{W}_{i-1} + \mathcal{W}_{i+1} + \ldots + \mathcal{W}_q \tag{2.8}$$

and the residual lies along $C\hat{\mathcal{W}}_i$ where for $j = 1, \ldots, q$

$$\mathcal{W}_j = \operatorname{Im} \begin{bmatrix} F_j & (A - LC)F_j & \ldots & (A - LC)^{n-1}F_j \end{bmatrix}.$$

Let δ_i be the smallest non-negative integer such that $C(A-LC)^{\delta_i}F_i \neq 0$. Therefore, for $k = 0, \ldots, \delta_i - 1$, $C(A-LC)^k F_i = 0$, which implies $CA^k F_i = 0$ and $C(A-LC)^{\delta_i}F_i = CA^{\delta_i}F_i$. So, when fault $\hat{\mu}_i$ occurs, the estimation error will lie in (2.8) where

$$\mathcal{W}_i = \text{Im}\left[\begin{array}{cccccccc} F_i & AF_i & \ldots & A^{\delta_i}F_i & (A-LC)A^{\delta_i}F_i & \ldots & (A-LC)^{n-\delta_i-1}A^{\delta_i}F_i \end{array}\right].$$

If the filter gain L is chosen such that $\delta_i + 1$ of the eigenvectors of $A - LC$ span

$$\mathcal{W}_i^* = \text{Im}\left[\begin{array}{cccc} F_i & AF_i & \ldots & A^{\delta_i}F_i \end{array}\right], \tag{2.9}$$

then $\mathcal{W}_i^*$ is the smallest reachable, observable subspace associated with F_i [18, 19] that satisfies

$$(A-LC)\mathcal{W}_i^* \subseteq \mathcal{W}_i^* \tag{2.10a}$$

$$\text{Im } F_i \subseteq \mathcal{W}_i^*. \tag{2.10b}$$

By (2.10), each $\mathcal{W}_i^*$ is invariant under $A - LC$. Hence, it is known as the *minimal* (C, A)-*invariant subspace of* F_i. Further, when the target fault μ_i occurs the residual lies along $C\mathcal{W}_i^*$.

For the RDDF, instead of choosing L so that a subset of the eigenvectors of $A - LC$ span $\mathcal{W}_i^*$, let a larger subset of the eigenvectors of $A-LC$ span the minimal (C, A)-invariant subspace $\hat{\mathcal{W}}_i^*$ of the multi-dimensional complementary fault $\hat{F}_i$. [1] Note that the algorithm is designed so that $\hat{\mathcal{W}}_i^*$ satisfies (2.10) for $\hat{F}_i$. Then, when the complementary fault $\hat{\mu}_i$ occurs, the residual lies along $C\hat{\mathcal{W}}_i^*$. To satisfy the first objective, $C\mathcal{W}_i^*$ and $C\hat{\mathcal{W}}_i^*$ must be linearly independent, i.e. the target faults must be (C, A) *output separable* from their complementary faults. Note that output separability implies that $m \geq q$.

[1] An equation to calculate $\hat{\mathcal{W}}_i^*$ and the algorithm from which it is derived are discussed in detail in Section 3.1.1 and Chapter 4, respectively.

To satisfy the second objective, the RDDF must generate a residual vector that is insensitive to the nuisance faults and can be used to uniquely identify any of the target faults when one occurs [18]. First, it is necessary to assume that the target fault directions are monic, i.e. $\mu_i(t) \neq 0$ implies $F_i \mu_i(t) \neq 0$ [37]. Otherwise, it would be impossible to observe the fault when it occurs. Then, to isolate a target fault, the detection filter is designed so that the associated complementary fault is placed in the unobservable subspace of the residual. For a given target fault, the associated residual projector $\hat{H}_i$ is defined as [41]

$$\hat{H}_i = I - C\hat{\mathcal{W}}_i^* \left[\left(C\hat{\mathcal{W}}_i^* \right)^T C\hat{\mathcal{W}}_i^* \right]^{-1} \left(C\hat{\mathcal{W}}_i^* \right)^T . \tag{2.11}$$

Thus, Ker $\hat{H}_i = C\hat{\mathcal{W}}_i^*$ and so the associated complementary fault is unobservable. Further, when the faults are output separable, $\hat{H}_i C\mathcal{W}_i^* \neq 0$ and so the target fault remains observable, satisfying the second objective. Since $\hat{H}_i$ is an orthogonal projector, it also satisfies $\hat{H}_i = \hat{H}_i^T = \hat{H}_i^2$.

While some of the eigenvalues can be chosen arbitrarily by letting the eigenvectors span $\mathcal{W}_1^*, \ldots, \mathcal{W}_s^*, \hat{\mathcal{W}}^*$, it is proven in [18] that the eigenvectors must also span the invariant zero directions associated with (C, A, F_i) and $(C, A, \hat{F})$. The following defines an invariant zero and its associated direction for a target fault direction F_i, though the definition is similar for the nuisance fault direction $\hat{F}$. The invariant zero directions $\nu_{i,j}$, $j = 1, \ldots, p_i$, satisfy

$$\begin{bmatrix} A - z_{i,j} I & F_i \\ C & 0 \end{bmatrix} \begin{bmatrix} \nu_{i,j} \\ \eta_{i,j} \end{bmatrix} = 0, \tag{2.12}$$

where $z_{i,j}$ is the invariant zero and $\eta_{i,j}$ is a scalar coefficient. For scaling purposes, it can be assumed that $\|\eta_{i,j}\| = 1$. If the detection filter eigenvectors associated with F_i do not span Im $\nu_{i,j}$, one of the eigenvalues of $A - LC$ will be located at $z_{i,j}$ [18].

Define the invariant zero subspace $\mathcal{V}_i$ such that $\nu \in \mathcal{V}_i, \xi \in \mathcal{W}_1^* \oplus \cdots \oplus \mathcal{W}_s^* \oplus \hat{\mathcal{W}}^* \Rightarrow \nu \perp \xi$. Similarly, invariant zeros $\hat{z}_j$ and directions $\hat{\nu}_j$ are calculated for $(C, A, \hat{F})$ where $\hat{\eta}_j$ is a vector coefficient. Note that the invariant zero subspace associated with the nuisance faults satisfies $\mathcal{V}_{s+1} + \ldots + \mathcal{V}_q \subseteq \hat{\mathcal{V}}$ since there may exist extra invariant zeros that cannot be associated with an individual nuisance fault direction. To satisfy the third objective, choose the filter gain such that n_i of the eigenvectors of $A - LC$ span

$$\mathcal{T}_i = \mathcal{W}_i^* \oplus \mathcal{V}_i \tag{2.13}$$

and $\hat{\mathcal{T}} = \hat{\mathcal{W}}^* \oplus \hat{\mathcal{V}}$. Thus, $\mathcal{T}_i$, called the *minimal* (C, A)*-unobservability subspace of* F_i [18] or the *detection space of* F_i [17], contains all and only the directions that satisfy (2.10) where by (2.13) and the second row of (2.12) $C\mathcal{T}_i = C\mathcal{W}_i^*$ [18].

Basic linear systems theory dictates that, if there exists an invariant zero at the origin, the steady state residual due to a constant input from the associated fault is zero [46]. Therefore, in order to satisfy the fourth objective, there can be no invariant zeros at the origin that are associated with (C, A, F_i).

Invariant zeros of $(C, A, [F_1 \ \ldots \ F_s \ \ \hat{F}])$ that are not associated with (C, A, F_i), or $(C, A, \hat{F})$ become eigenvalues of the detection filter. However, since these extra invariant zeros cannot be associated with a single fault, the resulting eigenvalues cannot be moved without altering the state space (one method of increasing the size of the state space to change the fault associations of the extra invariant zeros is discussed in [19]). Define the subspace of extra invariant zeros $\mathcal{V}_{\text{ext}}$ such that $\nu \in \mathcal{V}_{\text{ext}}, \xi \in \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_s \oplus \hat{\mathcal{T}} \Rightarrow \nu \perp \xi$. If $(C, A, [F_1 \ \ldots \ F_s \ \ \hat{F}])$ contains no more invariant zeros than (C, A, F_i) and $(C, A, \hat{F})$, then every eigenvalue of the detection filter can be specified arbitrarily and the faults are said to be *mutually detectable*.

With the detection spaces $\mathcal{T}_1, \ldots, \mathcal{T}_s, \hat{\mathcal{T}}$ and the subspace of extra invariant zeros $\mathcal{V}_{\text{ext}}$ defined, let the remainder of the state space be denoted as the complementary subspace $\mathcal{C}$. Define $\mathcal{C}$ so that it is orthogonal to $\mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_s \oplus \hat{\mathcal{T}} \oplus \mathcal{V}_{\text{ext}}$. Thus, the state space composition is determined as

$$\mathcal{T}_1 \oplus \ldots \oplus \mathcal{T}_s \oplus \hat{\mathcal{T}} \oplus \mathcal{V}_{\text{ext}} \oplus \mathcal{C} = \mathbb{R}^n. \tag{2.14}$$

Several system assumptions have been derived along with this state space description. To summarize, in order to generate an arbitrarily stable RDDF, the following must be satisfied:

Assumption 2.1: The system is detectable. ♣

Assumption 2.2: The target fault directions are monic and observable. ♣

Assumption 2.3: Each target fault is output separable from its complementary fault. ♣

Assumption 2.4: No target fault is associated with an invariant zero at the origin. ♣

Assumption 2.5: The faults are mutually detectable or else the extra invariant zeros are sufficiently stable so as not to destabilize the detection filter. ♣

Remark 2.3: Unobservable directions of (C, A) are eigenvectors of A that are also in the nullspace of C. Thus, they are solutions to (2.12) where $\eta_{i,j} = 0$. Since the directions are independent of the faults, the detection spaces overlap on the unobservable subspace of (C, A). To prevent this overlap, the system should be (C, A) observable. However, the detection filter can be obtained even when this condition is not satisfied. ♢

Remark 2.4: Mutual detectability is constraint necessary for finding an arbitrarily stable detection filter only in for the single-fault problem [26]. In this case, a detection space is formed only for the nuisance fault and not for the target fault. Since only one detection space is formed, the question of an invariant zero being associated with multiple detection spaces is moot. Thus, single-fault detection filters like the UIO may be applied to more systems than the multiple-fault filters. ♢

2.2 Discrete Multiple-Fault Detection Filtering

2.2.1 Plant and Fault Modeling

In this section, models of the plant, sensor faults, and actuator faults are given [43]. Consider a discrete linear time-varying system:

$$x(k+1) = \Phi(k+1|k)x(k) + B(k)u(k) \tag{2.15a}$$

$$y(k) = C(k)x(k) + D(k)u(k), \tag{2.15b}$$

where k is the time step, $x(k) \in \mathbb{R}^n$ is the state, $\Phi(k+1|k) \in \mathbb{R}^{n\times n}$ is the transition matrix from the k^{th} to the $(k+1)^{\text{th}}$ time step, $u(k) \in \mathbb{R}^l$ is the control input, $y(k) \in \mathbb{R}^m$ is the sensor measurement.

The i^{th} actuator fault is modeled as an additive input to the state dynamics (2.15a) and to the measurement (2.15b) (when $D(k)$ is nonzero):

$$x(k+1) = \Phi(k+1|k)x(k) + B(k)u(k) + F_{a,i}(k)\mu_{a,i}(k)$$
$$y(k) = C(k)x(k) + D(k)u(k) + E_{a,i}(k)\mu_{a,i}(k)$$

where $F_{a,i}(k)$ and $E_{a,i}(k)$ are the *a priori* known actuator fault directions in the dynamics and measurement, respectively, and $\mu_{a,i}(k)$ is the actuator fault magnitude, which is an unknown, arbitrary function of time. Note that $\mu_{a,i}(k)$ is nonzero only when a fault is present.

The i^{th} sensor fault is modeled as an additive input to the measurement (2.15b):

$$y(k) = C(k)x(k) + D(k)u(k) + F_{s,i}(k)\mu_{s,i}(k) \tag{2.16}$$

where $F_{s,i}(k)$ is the *a priori* known sensor fault direction, and $\mu_{s,i}(k)$ is the unknown, arbitrary sensor fault magnitude. Sensor faults are assumed to affect one sensor at a time. Thus, the fault direction $F_{s,i}(k)$ is a column vector of zeros except for a one in the i^{th} row.

As in the continuous case, an input that drives the dynamics similarly to $\mu_{s,i}(k)$ in (2.16) is required. Define $f_{s,i}(k)$ and $e_{a,i}(k)$ to satisfy

$$F_{s,i}(k) = C(t)f_{s,i}(k)$$
$$E_{a,i}(k) = C(t)e_{a,i}(k).$$

In general, $f_{s,i}(k)$ and $e_{a,i}(k)$ are not unique. One solution satisfying the above conditions is

$$f_{s,i}(k) = C(k)^T \left(C(k)C(k)^T\right)^{-1} F_{s,i}(k)$$
$$e_{a,i}(k) = C(k)^T \left(C(k)C(k)^T\right)^{-1} E_{a,i}(k)$$

Then, a new state $\bar{x}(k)$ may be obtained where

$$\bar{x}(k) = x(k) + f_{s,i}(k)\mu_{s,i}(k) + e_{a,i}(k)\mu_{a,i}(k).$$

Therefore, (2.16) is rewritten

$$y(k) = C(k)\bar{x}(k)$$

and the dynamic equation of $\bar{x}(k)$ is

$$\bar{x}(k+1) = \Phi(k+1|k)\bar{x}(k) + B(k)u(k) + \begin{bmatrix} -\Phi(k+1|k)f_{s,i}(k) & f_{s,i}(k+1) \end{bmatrix} \begin{bmatrix} \mu_{s,i}(k) \\ \mu_{s,i}(k+1) \end{bmatrix}$$
$$+ \begin{bmatrix} F_{a,i}(k) - \Phi(k+1|k)e_{a,i}(k) & e_{a,i}(k+1) \end{bmatrix} \begin{bmatrix} \mu_{a,i}(k) \\ \mu_{a,i}(k+1) \end{bmatrix}. \tag{2.17}$$

Thus, the sensor fault is modeled as a two-component additive term to (2.17), where $-\Phi(k+1|k)f_{s,i}(k)$ is the direction associated with the current sensor fault magnitude and $f_{s,i}(k+1)$ is the direction associated with the future sensor fault magnitude. When the measurement contains a direct feedthrough term from a faulty actuator, the fault is modeled as a similar two-component additive term to the dynamics.

2.2.2 Discrete Multiple-Fault Detection Filter

In this section, the fault detection filter is derived for discrete systems [43]. Consider a discrete linear time-invariant system with q faults, s of which are target faults, and m measurements

$$x(k+1) = \Phi x(k) + Bu(k) + \sum_{i=1}^{q} F_i \mu_i(k) + \hat{F}\hat{\mu}(k) \tag{2.18a}$$

$$y(k) = Cx(k) + Du(k), \tag{2.18b}$$

where $x(k) \in \mathbb{R}^n$ is the state, $\Phi \in \mathbb{R}^{n \times n}$ is the state transition matrix, $u(k) \in \mathbb{R}^l$ is the control input, $y(k) \in \mathbb{R}^m$ is the measurement, $\mu_i(k) \in \mathbb{R}$ is the target fault magnitude associated with fault direction $F_i \in \mathbb{R}^n$, $\hat{\mu}(k) \in \mathbb{R}^{q-s}$ is the nuisance fault magnitude vector associated with fault direction $\hat{F} \in \mathbb{R}^{n \times (q-s)}$, and $C \in \mathbb{R}^{m \times n}$ is the full row rank measurement matrix. The detection filter is a Luenberger observer of the form

$$\bar{x}(k+1) = \Phi \hat{x}(k) + Bu(k) \tag{2.19a}$$

$$\hat{x}(k) = \bar{x}(k) + L\left(y(k) - C\bar{x}(k) - Du(k)\right) \tag{2.19b}$$

$$r(k) = y(k) - C\hat{x}(k) - Du(k), \tag{2.19c}$$

where $\bar{x}(k) \in \mathbb{R}^n$ is the *a priori* state estimate, $\hat{x}(k) \in \mathbb{R}^n$ is the *a posteriori* state estimate, $L \in \mathbb{R}^{n \times m}$ is the filter gain, and $r(k) \in \mathbb{R}^m$ is the residual. Using (2.18) and (2.19), the *a priori* and *a posteriori* state estimation errors $e(k) \triangleq x(k) - x(k)$ and $\hat{e}(k) \triangleq x(k) - \hat{x}(k)$, respectively, are subject to the dynamic system

$$e(k+1) = \Phi\hat{e}(k) + \sum_{i=1}^{q} F_i\mu_i(k) + \hat{F}\hat{\mu}(k) \tag{2.20a}$$

$$\hat{e}(k) = (I - LC)e(k) \tag{2.20b}$$

$$r_i(k) = \hat{H}_i C e(k). \tag{2.20c}$$

where $r_i(k)$ is the projected residual associated with the complementary fault $\mu_i(k)$.

To draw a clear parallel between the discrete and continuous detection filters, let the objectives be the same as for the RDDF in Section 2.1.2. The remainder of this section is focused on deriving the state space geometry with the goal of reaching a similar invariant subspace structure in the discrete case. The invariant subspace structure is formulated around the occurrence of the complementary fault

$$\hat{\mu}_i(k) = \begin{bmatrix} \mu_1(k) & \ldots & \mu_{i-1}(k) & \mu_{i+1}(k) & \ldots & \mu_q(k) \end{bmatrix}$$

with direction

$$\hat{F}_i = \begin{bmatrix} F_1 & \ldots & F_{i-1} & F_{i+1} & \ldots & F_q \end{bmatrix}.$$

When it occurs, the estimation error lies in $\hat{\mathcal{W}}_i$ as defined in (2.8) and the residual lies along $C\hat{\mathcal{W}}_i$.

Let δ_i be the smallest non-negative integer such that $C[\Phi(I-LC)]^{\delta_i}F_i \neq 0$. Therefore, for $k = 0, \ldots, \delta_i - 1$, $C[\Phi(I-LC)]^k F_i = 0$, which implies $C\Phi^k F_i = 0$ and $C[\Phi(I-LC)]^{\delta_i}F_i = C\Phi^{\delta_i}F_i$. So, when fault $\hat{\mu}_i$ occurs, the estimation error will lie in (2.8) where

$$\mathcal{W}_i = \text{Im}\begin{bmatrix} F_i & \Phi F_i & \ldots & \Phi^{\delta_i}F_i & \Phi(I-LC)\Phi^{\delta_i}F_i & \ldots & [\Phi(I-LC)]^{n-\delta_i-1}\Phi^{\delta_i}F_i \end{bmatrix}.$$

If the filter gain L is chosen such that $\delta_i + 1$ of the eigenvectors of $\Phi(I - LC)$ span

$$\mathcal{W}_i^* = \text{Im} \begin{bmatrix} F_i & \Phi F_i & \dots & \Phi^{\delta_i} F_i \end{bmatrix}, \tag{2.21}$$

then $\mathcal{W}_i^*$ is the smallest reachable, observable subspace associated with F_i that satisfies

$$\Phi(I - LC)\mathcal{W}_i^* \subseteq \mathcal{W}_i^* \tag{2.22a}$$

$$\text{Im } F_i \subseteq \mathcal{W}_i^*. \tag{2.22b}$$

By (2.22), $\mathcal{W}_i^*$ is invariant under $\Phi(I-LC)$, and so it is the minimal (C, Φ)-invariant subspace of F_i. Further, when the target fault μ_i occurs the residual lies along $C\mathcal{W}_i^*$.

Similar to the continuous case, instead of choosing L so that some of the eigenvectors of $\Phi(I - LC)$ span $\mathcal{W}_i^*$, let some of the eigenvectors of $\Phi(I - LC)$ span the minimal (C, Φ)-invariant subspace $\hat{\mathcal{W}}_i^*$ of the multi-dimensional complementary fault $\hat{F}_i$, which satisfies (2.10) for $\hat{F}_i$. Then, when the complementary fault $\hat{\mu}_i$ occurs, the residual lies along $C\hat{\mathcal{W}}_i^*$. To satisfy the first objective, $C\mathcal{W}_i^*$ and $C\hat{\mathcal{W}}_i^*$ must be linearly independent, which is identical to the output separability condition of the continuous case. Then, to satisfy the second objective, $\hat{\mathcal{W}}_i^*$ is placed in the unobservable subspace of the projected residual $r_i(k)$ using the residual projector $\hat{H}_i$ as defined in (2.11).

Finally, to examine the third and fourth objectives of the detection filter problem, the concept of invariant zeros is revisited. For discrete systems, the invariant zero directions $\nu_{i,j}$, $j = 1, \dots, p_i$, satisfy

$$\begin{bmatrix} \Phi(I - LC) - z_{i,j}I & F_i \\ C & 0 \end{bmatrix} \begin{bmatrix} \nu_{i,j} \\ \eta_{i,j} \end{bmatrix} = 0, \tag{2.23}$$

where $z_{i,j}$ is the invariant zero and $\eta_{i,j}$ is a scalar coefficient. For scaling purposes, it can be assumed that $\|\eta_{i,j}\| = 1$.

In order to satisfy the fourth objective, there can be no invariant zeros at the origin that are associated with (C, Φ, F_i). Otherwise, the transfer function from the target fault to the residual will be equal to zero [43]. Next, the third objective is explored. Note that the definition of an invariant zero for the nuisance fault direction $\hat{F}$ is similar to (2.23). To satisfy this objective, there are two requirements for the detection filter:

- The eigenvectors of $\Phi(I-LC)$ must span $\mathcal{W}_1^*, \ldots, \mathcal{W}_s^*, \hat{\mathcal{W}}^*$ as well as the invariant zero directions associated with (C, Φ, F_i) and $(C, \Phi, \hat{F})$

- There can be no invariant zeros associated with $(C, \Phi, [F_1 \ \ldots \ F_s \ \ \hat{F}])$ that are not associated with (C, Φ, F_i), or $(C, \Phi, \hat{F})$.

If either of the above requirements is not satisfied, then there will be an eigenvalue of the detection filter dynamics located at an invariant zero [43]. The proofs for both of these requirements are identical to their continuous counterparts. Note that the second requirement is clearly the mutual detectability constraint for discrete systems.

From the above analysis, the state space description for the discrete detection filter is identical to that of the continuous detection filter in (2.14). Further, the requirements discussed above are identical to Assumptions 2.1 - 2.5.

Chapter 3

The Game Theoretic Multiple-Fault Detection Filter

In this chapter, the Game Theoretic Multiple-Fault Detection Filter (GTMFDF) is derived. The GTMFDF extends the GTFDF to the multiple-fault case by modeling the detection filter problem as a set of DAPs to be optimized via a single differential game. However, since the globally optimal solution for the filter gain is difficult to obtain, sufficient conditions for satisfying the DAPs are derived instead. The resulting detection filter is similar to the multiple-fault OSFDF with a more general description of the fault magnitudes and simpler solution requirements. Thus, the flexibility, simplicity, and robustness of the GTFDF problem for single-fault detection is combined with the generality of the OSFDF, resulting in a multiple-fault detection filter with relatively few assumptions on the system and fault structure compared to the current literature.

This chapter is organized as follows. First, the RDDF structure is approximated by a set of DAPs Section 3.1. Then, the implied differential game problem is simplified into a feasibility problem to find the constraints on the filter gain such that the DAPs are satisfied given the worst case disturbances and faults. Sufficient conditions for satisfying the DAPs are obtained in Section 3.2. It is shown that these conditions require Riccati differential inequalities on the estimation error covariance to have nonnegative solutions. These inequalities become the constraints of a user-defined optimization function, which may be used to achieve desired secondary characteristics of the detection filter. Finally, the GTMFDF is compared directly to the previous Riccati-based robust detection filters (the GTFDF and the OSFDF) in Section 3.3.

3.1 Differential Game Problem Formulation

In this section, the detection filter problem for a given target fault input is formulated as a set of DAPs that can be optimized via a differential game problem. First, the RDDF problem is extended to the finite time-varying case and approximated by a set of DAPs in Section 3.1.1. Then, the DAPs are converted into a differential game feasibility problem and the required assumptions are discussed in Section 3.1.2. To simplify the derivation, this chapter considers only scalar target faults, though the results also apply to vector fault case.

3.1.1 Extension of the RDDF to the Finite Time-Varying and Approximate Cases

To approximate a multiple-fault detection filter for linear finite time-varying systems, a set of DAPs are formulated by requiring that the target faults be observable and relaxing the requirement on strict blocking implied by the first and second objectives of the RDDF problem. Instead, the transmissions of the complementary fault, sensor noise, and initial condition error are bounded above by a preset level. However, the target fault must remain observable to its projected residual. Further, the third RDDF objective is relaxed to requiring only that the detection filter dynamics be stable.

Define the time-varying dynamic system with s target faults, $q - s$ nuisance faults, m

measurements, and sensor noise $v(t)$ as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \sum_{i=1}^{s} F_i(t)\mu_i(t) + \hat{F}(t)\hat{\mu}(t) \tag{3.1a}$$

$$y(t) = C(t)x(t) + D(t)u(t) + v(t) \tag{3.1b}$$

is defined from initial time t_0 to final time $t_1 < \infty$. Recall from Section 2.1.1 that any fault in the plant, actuator, or sensor can be modeled as an additive input to (3.1a).

The time-varying extension of the minimal (C, A)-invariant subspace of F_i, denoted as $\mathcal{W}_i^*(t)$, is obtained by applying the RDDF objectives over the time interval from t_0 to t_1. Define

$$\mathcal{W}_i^*(t) = \text{Im} \begin{bmatrix} B_i^0(t) & B_i^1(t) & \ldots & B_i^{\beta_i}(t) \end{bmatrix} \tag{3.2}$$

where the columns of $\mathcal{W}_i^*(t)$ are constructed by [37]

$$\begin{aligned} B_i^0(t) &= F_i(t) \\ B_i^j(t) &= A(t)B_i^{j-1}(t) - \dot{B}_i^{j-1}(t). \end{aligned} \tag{3.3}$$

and β_i is the smallest nonnegative integer such that $C(t)B_i^{\beta_i}(t) \neq 0 \;\forall\, t \in [t_0, t_1]$. When $L(t)$ is chosen such that $\beta_i + 1$ of the eigenvectors of $A(t) - L(t)C(t)$ span $\mathcal{W}_i^*$, $\mathcal{W}_i^*$ is the minimal $(C(t), A(t))$-invariant subspace of $F_i(t)$ [18, 19, 37] and satisfies

$$(A(t) - L(t)C(t))\,\mathcal{W}_i^*(t) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_i^*(t) \subseteq \mathcal{W}_i^*(t) \tag{3.4a}$$

$$\text{Im } F_i(t) \subseteq \mathcal{W}_i^*(t). \tag{3.4b}$$

Further, Lemma 3.3 in Section 3.4.3 proves that when output separability is satisfied, $\hat{\mathcal{W}}_i^*(t)$ can be obtained as

$$\hat{\mathcal{W}}_i^*(t) = \mathcal{W}_1^*(t) \oplus \ldots \oplus \mathcal{W}_{i-1}^*(t) \oplus \mathcal{W}_{i+1}^*(t) \oplus \ldots \oplus \mathcal{W}_s^*(t) \oplus \hat{\mathcal{W}}^*(t). \tag{3.5}$$

For the i^{th} DAP, the transmissions of $\hat{\mu}_i$, v_i, and $e_i(t_0)$, henceforth collectively referred to as the *disturbance parameters*, are separated from the transmission of the target fault into their own state $x_i(t)$ where

$$\dot{x}_i(t) = A(t)x_i(t) + B(t)u(t) + \hat{F}_i(t)\hat{\mu}_i(t) \tag{3.6a}$$

$$y_i(t) = C(t)x_i(t) + D(t)u(t) + v_i(t). \tag{3.6b}$$

Since blocking the transmissions of the disturbance parameters is of primary interest, x_i is the most useful state vector. The conditions so that the target fault is not blocked along with the complementary fault are discussed later in this section. Note that each measurement $y_i(t)$ contains its own noise $v_i(t)$ in (3.6b), even though $v_1(t), \ldots, v_s(t)$ are physically identical. However, in the next section they will be considered as independent quantities in order to simplify the formulation of the GTMFDF problem. Further, in the absence of the target fault, $y_i(t)$ is identical to $y(t)$ in (3.1). Thus, let the detection filter be modeled as

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + L(t)\left(y_i(t) - C(t)\hat{x}(t) - D(t)u(t)\right) \tag{3.7a}$$

$$r_i(t) = y_i(t) - C(t)\hat{x}(t) - D(t)u(t). \tag{3.7b}$$

Using (3.6), the dynamics of the state estimation error $e_i(t) \triangleq x_i(t) - \hat{x}(t)$ and the estimation error residual $r_i(t)$ are

$$\dot{e}_i(t) = \left(A(t) - L(t)C(t)\right)e_i(t) + \hat{F}_i(t)\hat{\mu}_i(t) - L(t)v_i(t) \tag{3.8a}$$

$$r_i(t) = C(t)e_i(t) + v_i(t). \tag{3.8b}$$

Further, by multiplying $r_i(t)$ by the residual projector $\hat{H}_i(t)$, the projected residual is

$$\bar{r}_i(t) = \hat{H}_i(t)C(t)e_i(t) + \hat{H}_i(t)v_i(t). \tag{3.9}$$

where

$$\hat{H}_i(t) = I - C(t)\hat{\mathcal{W}}_i^*(t)\left[\left(C(t)\hat{\mathcal{W}}_i^*(t)\right)^T C(t)\hat{\mathcal{W}}_i^*(t)\right]^{-1}\left(C(t)\hat{\mathcal{W}}_i^*(t)\right)^T. \tag{3.10}$$

Since the projected residual contains a direct feedthrough term from the sensor noise, the projected output error $\hat{H}_i(t)C(t)e_i(t)$ is used instead of $\bar{r}_i(t)$ to represent the transmission of the disturbance parameters to the output. Thus, the i^{th} DAP is written as[1]

$$\frac{\int_{t_0}^{t_1} \|\hat{H}_i(t)C(t)e_i(t)\|_{Q_i}^2 \, \mathrm{d}t}{\int_{t_0}^{t_1}\left[\|\hat{\mu}_i(t)\|_{M_i^{-1}}^2 + \|v_i(t)\|_{\bar{V}^{-1}}^2\right] \mathrm{d}t + \|e_i(t_0)\|_{P_0^{-1}}^2} \leq \gamma, \tag{3.11}$$

subject to the dynamic system (3.8) for any $\hat{\mu}_i(t)$, $v_i(t)$, and $e_i(t_0)$ that satisfy $\int_{t_0}^{t_1} \|\hat{\mu}_i(t)\|^2 \mathrm{d}t < \infty$ and $\int_{t_0}^{t_1} \|v_i(t)\|^2 \mathrm{d}t < \infty$. The initial and final times are t_0 and t_1, respectively. $\gamma > 0$ is the arbitrary disturbance attenuation bound. $Q_i > 0$, $M_i > 0$, $\bar{V} > 0$, and $P_0 > 0$ are arbitrary symmetric design weighting matrices. However, $\bar{V}$ is typically chosen as the covariance of the measurement noise. Further, when the design weightings M_i, $\bar{V}$, and P_0 are chosen to be larger, the projected residual becomes less sensitive to the complementary fault, sensor noise, and initial condition error, respectively, which also can be achieved simultaneously by choosing Q_i to be larger.

Finally, in order to ensure that a stable detection filter exists that achieves the objectives of the approximate fault detection filter problem, assume the following:

Assumption 3.1: $(C(t), A(t))$ is uniformly observable over the interval $[t_0, t_1]$. ♣

Assumption 3.2: $F_i(t)$ is monic and $(C(t), A(t))$ output separable from $\hat{F}_i(t)$ $\forall\, i \in \{1, \ldots, s\}$ over the interval $[t_0, t_1]$. ♣

At first, Assumption 3.1 seems somewhat limiting. However, this assumption is necessary to formulate an asymptotically stable detection filter for time-varying systems that

[1]The notation of norms in this work is defined by $\|Y(t)\|_{Z(t)}^2 \triangleq Y^T(t)Z(t)Y(t)$ where $Y(t)$ is a vector and $Z(t)$ is a matrix of appropriate size. Note that $Z(t)$ is not required to have any particular sign-definiteness in general.

achieves the desired fault detection properties [45]. For infinite time-invariant systems, the unobservable subspace may be truncated from the detection filter state space at the beginning of the problem so that this assumption can be made without loss of generality. Assumption 3.2 guarantees that each target fault will remain observable in $(\hat{H}_i(t)C(t), A(t) - L(t)C(t))$ when the complementary fault is placed in the unobservable subspace of the projected residual. Since invariant zeros are not defined in the finite time-varying case, there is no assumption stated on their location or directional structure.

3.1.2 Problem Formulation

By multiplying both sides of (3.11) by the denominator of the left-hand side, subtracting the right-hand side, and setting the left-hand side equal to J_i, (3.11) is converted into the nonconvex cost function

$$J_i = \int_{t_0}^{t_1} \left[\left\| \hat{H}_i(t)C(t)e_i(t) \right\|_{Q_i}^2 - \|\hat{\mu}_i(t)\|_{\gamma M_i^{-1}}^2 - \|v_i(t)\|_{V^{-1}}^2 \right] \mathrm{d}t - \|e_i(t_0)\|_{\Pi_0}^2, \tag{3.12}$$

where $\Pi_0 \triangleq \gamma P_0^{-1}$ and $V \triangleq \gamma^{-1}\bar{V}$. The detection filter problem is modeled as a differential game optimization by summing (3.12) over i, minimizing the sum with respect to the filter gain, and maximizing the sum with respect to the disturbance parameters. Therefore, the differential game problem is

$$\min_{L(t)} \max_{v_1(t),\ldots,v_s(t)} \max_{\hat{\mu}_1(t),\ldots,\hat{\mu}_s(t)} \max_{e_1(t_0),\ldots,e_s(t_0)} \sum_{i=1}^{s} J_i(L(t), v_i(t), \hat{\mu}_i(t), e_i(t_0)) \tag{3.13}$$

subject to (3.8a). Recall from their definitions that $\hat{\mu}_1(t), \ldots, \hat{\mu}_s(t)$ are co-dependent since they share common elements (e.g. - $\hat{\mu}_1$ and $\hat{\mu}_2$ both contain $\mu_3, \ldots, \mu_q$).

Since the detection filter gain $L(t)$ does not appear in the game cost (3.12) and enters linearly into the constraint (3.8a), (3.13) is singular with respect to $L(t)$ [47]. This makes the process of finding a globally optimal solution for $L(t)$ that will generate the desired fault detection properties very complex. However, in order to satisfy the DAPs (3.11), it is only required that (3.12) be nonpositive for any value of $\hat{\mu}_i(t)$, $v_i(t)$, and $e_i(t_0)$ $\forall\ t \in [t_0, t_1]$. Thus, it is required only to solve a feasibility problem to find $L(t)$ such that (3.12) is nonpositive. To further simplify the problem statement, assume that all of the disturbance parameters are independent. This assumption only affects the problem statement, not the equations to eventually solve for the filter gain. Therefore, to determine a filter gain sufficient to satisfy (3.11), the following simplified problem is solved:

Problem 3.1: Find $L(t)$ such that

$$\max_{v_i(t)} \max_{\hat{\mu}_i(t)} \max_{e_i(t_0)} J_i \leq 0$$

subject to (3.8a) and (3.12) $\forall\ i \in \{1, \ldots, s\}$. ♡

3.2 Detection Filter Problem Solution

In this section, solutions for the GTMFDF gain $L(t)$ in Problem 3.1 are determined for the general case where $\gamma > 0$. First, the necessary and sufficient conditions for optimality of (3.12) are determined in Section 3.2.1. To this end, a set of Riccati differential inequalities are derived for the estimation error covariances, each of which must have a nonnegative solution. The Riccati inequalities, which are functions of the filter gain, become constraints for a secondary filter gain optimization problem. This new problem is solved in general and for an example cost function in Section 3.2.2. Next, some results for the existence of solutions in the infinite-time case are discussed in Section 3.2.3. Finally, it is shown that this derivation generalizes and clarifies the GTFDF [37] and OSFDF [45] in Section 3.3.

3.2.1 Conditions for Nonpositivity of the Game Cost

In this section, the necessary and sufficient conditions for optimality of the game cost are considered. By appending the estimation error dynamics to the cost function and completing the squares, obvious optimality conditions are determined with respect to the disturbances parameters. The sufficient conditions for optimality are derived as Riccati inequality constraints on the estimation error covariance associated with each DAP. A valid solution for the filter gain in Problem 3.1 is any for which the Riccati inequalities are nonpositive and the estimation error covariances are nonnegative-definite, implying that the DAPs are also satisfied. For compactness, the matrices' and variables' time dependence will no longer be shown.

First, the estimation error dynamics (3.8a) are appended to the cost function (3.12) using the LaGrange multiplier $e_i^T \Pi_i$, which yields

$$J_i = \int_{t_0}^{t_1} \left[\left\| \hat{H}_i C e_i \right\|_{Q_i}^2 - \|\hat{\mu}_i\|_{\gamma M_i^{-1}}^2 - \|v_i\|_{V^{-1}}^2 + e_i^T \Pi_i \left((A - LC) e_i + \hat{F}_i \hat{\mu}_i - L v_i - \dot{e}_i \right) \right] \mathrm{d}t - \|e_i(t_0)\|_{\Pi_0}^2 .$$

By integrating $\int_{t_0}^{t_1} e_i^T \Pi_i \dot{e}_i \, \mathrm{d}t$ by parts, substituting (3.8a), and collecting terms,

$$J_i = \int_{t_0}^{t_1} \left[\|e_i\|_{\dot{\Pi}_i + \Pi_i(A-LC) + (A-LC)^T \Pi_i + C^T \hat{H}_i Q_i \hat{H}_i C}^2 + e_i^T \Pi_i \left(\hat{F}_i \hat{\mu}_i - L v_i \right) + \left(\hat{F}_i \hat{\mu}_i - L v_i \right)^T \Pi_i e_i - \|\hat{\mu}_i\|_{\gamma M_i^{-1}}^2 - \|v_i\|_{V^{-1}}^2 \right] \mathrm{d}t - \|e_i(t_0)\|_{\Pi_0 - \Pi_i(t_0)}^2 - \|e_i(t_1)\|_{\Pi_i(t_1)}^2 .$$

Finally, by adding and subtracting $\int_{t_0}^{t_1} \|e_i\|_{\Pi_i \left(\frac{1}{\gamma} \hat{F}_i M_i \hat{F}_i^T + L V L^T \right) \Pi_i}^2 \mathrm{d}t$ and collecting terms again,

$$J_i = \int_{t_0}^{t_1} \left[\|e_i\|_{\Psi_i(\Pi_i, L, t)}^2 - \left\| \hat{\mu}_i - \frac{1}{\gamma} M_i \hat{F}_i^T \Pi_i e_i \right\|_{\gamma M_i^{-1}}^2 - \|v_i + V L^T \Pi_i e_i\|_{V^{-1}}^2 \right] \mathrm{d}t - \|e_i(t_0)\|_{\Pi_0 - \Pi_i(t_0)}^2 - \|e_i(t_1)\|_{\Pi_i(t_1)}^2 \tag{3.14}$$

where

$$\Psi_i(\Pi_i, L, t) = \dot{\Pi}_i + \Pi_i(A - LC) + (A - LC)^T \Pi_i + \Pi_i \left(\frac{1}{\gamma} \hat{F}_i M_i \hat{F}_i^T + L V L^T \right) \Pi_i + C^T \hat{H}_i Q_i \hat{H}_i C. \tag{3.15}$$

Clearly, if there exist L and Π_i, $\forall\ i \in \{1, \ldots, s\}$, such that

$$0 \geq \Psi_i(\Pi_i, L, t) \tag{3.16}$$

$$0 \leq \Pi_0 - \Pi_i(t_0) \tag{3.17}$$

$$0 \leq \Pi_i(t_1), \tag{3.18}$$

then the game cost is nonpositive for all possible (real) values of the disturbance parameters, implying that the faults are placed in approximate detection spaces so that they can be isolated by each projected residual. Further, the constraints above imply that $\Pi_i \geq 0$ over the entire time interval when $A - LC$ is asymptotically stable (see Proposition 3.4 in Section 3.4.3).

Therefore, Problem 3.1 requires a solution to the coupled Riccati inequalities (3.16) given (3.15) with boundary conditions (3.17) and (3.18). Since the degree of complementary fault blocking can be changed by adjusting γ, the structure of the GTMFDF is less constrained than the detection filters based on spectral [19, 24] and geometric [18, 26] theories.

3.2.2 Filter Gain Optimization

In this section, the filter gain L is optimized with respect to a new cost function. Since any solutions L and $\Pi_i \geq 0$ to (3.16) and (3.17) automatically implies that (3.12) is nonpositive, the specific cost function used at this stage is arbitrary. Therefore, let the optimal filter gain minimize the cost function $\bar{J}$, defined as

$$\min_L \bar{J} = \min_L \sum_{i=1}^{s} \int_{t_0}^{t_1} \text{tr}\ \Omega_i\ \ \text{d}t \tag{3.19}$$

where the integrand $\Omega_i \in \mathbb{R}^{n \times n}$ is a symmetric, differentiable function chosen by the user. For convenience, assume that Ω_i is a function of Π_i, L, and t only. At the end of the section, suggestions on choosing Ω_i such that (3.19) has a non-trivial solution are discussed and an example is presented.

To determine the first-order necessary conditions for optimality of (3.19), use the Lagrange multiplier Δ_i to append $\Psi_i(\Pi_i, L, t) = 0$ to $\bar{J}$ to obtain[2]

$$\bar{J} = \sum_{i=1}^{s} \int_{t_0}^{t_1} \mathrm{tr}\left\{\Delta_i \Psi_i(\Pi_i, L, t) + \Omega_i\left(\Pi_i, L, t\right)\right\} \, \mathrm{d}t.$$

Substituting (3.15) and integrating $\sum_{i=1}^{s} \int_{t_0}^{t_1} \mathrm{tr}(\Delta_i \dot{\Pi}_i) \, \mathrm{d}t$ by parts,

$$\begin{aligned}\bar{J} =& \sum_{i=1}^{s} \int_{t_0}^{t_1} \mathrm{tr}\left\{\Delta_i\left[\Pi_i(A - LC) + (A - LC)^T \Pi_i + \Pi_i\left(\frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T + LVL^T\right)\Pi_i + C^T \hat{H}_i Q_i \hat{H}_i C\right]\right. \\ & \left. -\dot{\Delta}_i \Pi_i + \Omega_i\left(\Pi_i, L, t\right)\right\} \mathrm{d}t + \mathrm{tr}\left\{\Delta_i(t_1)\Pi_i(t_1) - \Delta_i(t_0)\Pi_i(t_0)\right\}.\end{aligned}$$

Taking the first-order variation with respect to L and Π_i,

$$\begin{aligned}\delta\bar{J} =& \sum_{i=1}^{s} \int_{t_0}^{t_1} \left\{\left[-\dot{\Delta}_i + \left(\frac{\delta\left[\mathrm{tr}\ \Omega_i\left(\Pi_i, L, t\right)\right]}{\delta \Pi_i}\right)^T + \left(A + \frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T \Pi_i + L\left(VL^T\Pi_i - C\right)\right)\Delta_i\right.\right. \\ & \left. + \Delta_i\left(A + \frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T \Pi_i + L\left(VL^T \Pi_i - C\right)\right)^T\right] \delta\Pi_i \\ & \left. + \left[2\left(VL^T\Pi_i - C\right)\Delta_i \Pi_i + \left(\frac{\delta\left[\mathrm{tr}\ \Omega_i\left(\Pi_i, L, t\right)\right]}{\delta L}\right)^T\right]\delta L\right\} \mathrm{d}t + \Delta_i(t_1)\delta\Pi_i(t_1) - \Delta_i(t_0)\delta\Pi_i(t_0).\end{aligned}$$

Thus, the first-order necessary conditions for optimality of (3.19) are

$$0 = \sum_{i=1}^{s} \left[2\left(VL^{*T}\Pi_i^* - C\right)\Delta_i \Pi_i^* + \left(\frac{\delta\left[\mathrm{tr}\ \Omega_i\left(\Pi_i^*, L^*, t\right)\right]}{\delta L^*}\right)^T\right] \tag{3.20}$$

$$\begin{aligned}\dot{\Delta}_i =& \left(A + \frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T \Pi_i^* + L^*(VL^{*T}\Pi_i^* - C)\right)\Delta_i + \Delta_i\left(A + \frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T \Pi_i^* + L^*(VL^{*T}\Pi_i^* - C)\right)^T \\ & + \left(\frac{\delta\left[\mathrm{tr}\ \Omega_i\left(\Pi_i^*, L^*, t\right)\right]}{\delta \Pi_i^*}\right)^T\end{aligned} \tag{3.21}$$

$$0 = \Delta_i(t_1) \tag{3.22}$$

[2]If the inequality form of the dynamics constraint (3.16) is desired, simply add a nonnegative term G_i to Ψ_i such that $0 = \Psi_i + G_i$ and append to the cost function.

where L^* is the optimal strategy for the filter gain and Π_i^* is the Riccati variable using L^*. Therefore, since Ω_i is symmetric by assumption, Δ_i is the solution of a Lyapunov differential equation. The optimal filter gain is determined by solving a two-point boundary value problem which includes a set of Riccati equations (3.15) equal to zero and Lyapunov equations (3.21) coupled by (3.20) with boundary conditions (3.17) equal to zero and (3.22).

Certain choices for Ω_i may lead to trivial solutions for the detection filter problem. Every term in (3.21) is dependent on Δ_i except for $\frac{\delta\Omega_i}{\delta\Pi_i}$. If $\frac{\delta\Omega_i}{\delta\Pi_i} = 0$, then the solution to (3.21) is $\Delta_i = 0$ because of the terminal constraint (3.22). Also, (3.20) is satisfied trivially in this case, providing no information on how to choose L^*. Therefore, Ω_i must be chosen such that $\frac{\delta\Omega_i}{\delta\Pi_i} \neq 0$. Further, it is generally unnecessary and undesirable to choose $\frac{\delta\Omega_i}{\delta L} \neq 0$, as very simple solutions for L^* may be obtained when $\frac{\delta\Omega_i}{\delta L} = 0$. In general, most enhancements to the detection filter problem may be achieved by choosing $\frac{\delta\Omega_i}{\delta\Pi_i} \neq 0$ and $\frac{\delta\Omega_i}{\delta L} = 0$.

Finally, an example cost function is minimized with respect to L. It was proven in [37] that as $\gamma \to 0$, Π_i obtains a nullspace that contains $\hat{\mathcal{W}}_i$, signifying that the nuisance fault will be blocked from the projected residual. Thus, the optimization should attempt to minimize the transmission of each complementary fault by placing $\hat{F}_i$ approximately in the nullspace of Π_i. Further, the target fault direction F_i should remain in the range space of Π_i so that it is not blocked along with the complementary fault. Thus, choose Ω_i as

$$\text{tr}\ \Omega_i = \text{tr}\ \frac{1}{\gamma} K_i \hat{F}_i^T \Pi_i \hat{F}_i - \text{tr}\ N_i F_i^T \Pi_i F_i \tag{3.23}$$

where K_i and N_i are design weightings on the complementary fault and target fault transmissions, respectively. Thus, the optimization problem (3.19) attempts to choose L such that Π_i has the aforementioned desired structure. When K_i is large, the transmission from $\hat{F}_i$ to the projected residual $\bar{r}_i$ is smaller. When N_i is large, the transmission from F_i to the residual is larger. By differentiating Ω_i with respect to Π_i and substituting into (3.21), Δ_i

is subject to the matrix differential equation

$$\dot{\Delta}_i = \left(A + \frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T \Pi_i^* + L^*(V L^{*T} \Pi_i^* - C)\right) \Delta_i + \Delta_i \left(A + \frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T \Pi_i^* + L^*(V L^{*T} \Pi_i^* - C)\right)^T$$
$$+ \frac{1}{\gamma}\hat{F}_i K_i \hat{F}_i^T - F_i N_i F_i^T, \qquad (3.24)$$

with boundary condition (3.22). From (3.20), the optimal filter gain is

$$L^* = \left(\sum_{i=1}^{s} \Pi_i^* \Delta_i \Pi_i^*\right)^{-1} \left[\sum_{i=1}^{s} \Pi_i^* \Delta_i C^T V^{-1}\right]. \qquad (3.25)$$

Remark 3.1: During the derivation of the GTMFDF constraints and optimal filter gain, it has not been necessary to assume that the Riccati solutions are invertible. For the example filter gain optimization cost function above, the solution exists as long as the nullspaces of $\Pi_i^* \Delta_i \Pi_i^*$ do not overlap over all DAPs. However, there is an exception in the single-fault case, in which Π_i must be invertible to obtain a solution. ◇

3.2.3 Steady-State Detection Filter

In this section, the steady-state (infinite time-invariant) results for the GTMFDF are discussed. First, two theorems useful for generating numerical solutions to the filter optimization problem are derived. In the previous sections, it was assumed that the design parameters Q_i, M_i, $\bar{V}$, and Π_0 are chosen so that there exists a real, symmetric, nonnegative-definite solution to (3.15) over the time interval $[t_0, t_1]$. In steady-state, conditions on the existence of real, symmetric, nonnegative-definite solutions are linked to the stability of $A - LC$. Then, steady-state equations to obtain an analytical solution to the example detection filter problem in Section 3.2.2 are presented.

In the steady-state case, the filter gain optimization problem becomes

$$\lim_{t_1 \to \infty} \min_L \bar{J} = \min_L \left(\sum_{i=1}^{s} \text{tr } \Omega_i\right)$$

subject to

$$0 \geq \Pi_i(A-LC)+(A-LC)^T\Pi_i+\Pi_i\left(\frac{1}{\gamma}\hat{F}_iM_i\hat{F}_i^T+LVL^T\right)\Pi_i+C^T\hat{H}_iQ_i\hat{H}_iC,\ \forall\, i \in \{1,\ldots,s\}. \tag{3.26}$$

For a numerical optimization, since the detection filter dynamics $A-LC$ are chosen by the user, assume that the dynamics are asymptotically stable. Then, the following theorem and corollary prove that there exists a real, symmetric, nonnegative solution to (3.26) $\forall\, i \in \{1,\ldots,s\}$ when the associated Hamiltonian has no eigenvalues on the imaginary axis. They also show that Ker Π_i is equivalent to either the unobservable subspace of $(\hat{H}_iC, A-LC)$ (when (3.26) equals zero), denoted as $\mathcal{M}_{uo}(\hat{H}_iC, A-LC)$, or a similar and possibly smaller unobservable subspace (when (3.26) is nonpositive). When (3.26) satisfies a strict inequality, Corollary 3.2 implies that Ker Π_i is equal to zero.

Theorem 3.1: Assume that (3.26) satisfies equality and the associated Hamiltonian has no eigenvalues on the imaginary axis. If $A-LC$ is asymptotically stable, then there exists a real, symmetric, and nonnegative solution for Π_i such that $A-LC+\left(\frac{1}{\gamma}\hat{F}_iM_i\hat{F}_i^T+LVL^T\right)\Pi_i \leq 0$. Further, Ker Π_i is equal to $\mathcal{M}_{uo}(\hat{H}_iC, A-LC)$. ▮

Proof: See Section 3.4.1. **QED**

Corollary 3.2: Define $G=\bar{G}\bar{G}^T \geq 0$ where (3.26) equals $-G$ and $\bar{G}$ has full column rank. Also, assume the associated Hamiltonian has no eigenvalues on the imaginary axis. If $A-LC$ is asymptotically stable, then there exists a real, symmetric, and nonnegative solution for Π_i such that $A-LC+\left(\frac{1}{\gamma}\hat{F}_iM_i\hat{F}_i^T+LVL^T\right)\Pi_i \leq 0$. Further, Ker Π_i is equal to $\mathcal{M}_{uo}\left(\begin{bmatrix}\bar{G}^T\\ \hat{H}_iC\end{bmatrix}, A-LC\right)$. ▮

Proof: See Section 3.4.2 **QED**

For a typical Riccati equation obtained for optimal control problems, where the quadratic term is nonpositive-definite, the Hamiltonian has no imaginary eigenvalues when the system satisfies certain stabilizability and detectability conditions found in [48]. However, because the quadratic term in (3.26) is nonnegative-definite (resembling Riccati equations used for $\mathcal{H}_\infty$ control [49]), the Hamiltonian eigenvalues must be verified directly to have nonzero real parts to satisfy the sufficient conditions for the existence of a real, symmetric, stabilizing solution.

For the example in Section 3.2.2, the steady-state detection filter problem is to find

$$\min_L \sum_{i=1}^{s} \operatorname{tr}\left[\left(\frac{1}{\gamma}\hat{F}_i K_i \hat{F}_i^T - F_i N_i F_i^T\right)\Pi_i\right] \tag{3.27}$$

subject to

$$0 = \Pi_i(A-LC)+(A-LC)^T\Pi_i+\Pi_i\left(\frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T + LVL^T\right)\Pi_i+C^T\hat{H}_i Q_i \hat{H}_i C,\ \forall\, i\in\{1,\ldots,s\}. \tag{3.28}$$

The numerical examples in Chapter 6 obtain solutions to the above problem. Alternatively, an analytical solution may be derived as (3.20)

$$L^* = \left(\sum_{i=1}^{s}\Pi_i^*\Delta_i\Pi_i^*\right)^{-1}\left[\sum_{i=1}^{s}\Pi_i^*\Delta_i C^T V^{-1}\right]$$

subject to (3.28) and

$$\begin{aligned}0 = &\left(A+\frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T\Pi_i^*+L^*(VL^{*T}\Pi_i^*-C)\right)\Delta_i+\Delta_i\left(A+\frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T\Pi_i^*+L^*(VL^{*T}\Pi_i^*-C)\right)^T\\ &+\frac{1}{\gamma}\hat{F}_i K_i \hat{F}_i^T - F_i N_i F_i^T.\end{aligned}$$

Remark 3.2: The theorems above assume that the Hamiltonians have no eigenvalues on the imaginary axis. Mathematically, however, this condition is not always required in order to find a real or symmetric solution to a steady-state Riccati equation. However, it is a necessary condition for the existence of a stabilizing solution such that $A - LC + \left(\hat{F}_i M_i \hat{F}_i^T + LVL^T\right)\Pi_i < 0$ [48]. Further, when the associated Hamiltonian has eigenvalues on the imaginary axis, Π_i itself becomes complex, implying that the solution has a finite escape time. Thus, it is meaningless to generalize the results above for imaginary eigenvalues. ♢

3.3 Comparison to Previous Detection Filters

In this section, the GTMFDF is compared to the previous (Riccati-based) robust approximate detection filter methods. First, it is shown that the GTMFDF problem formulation generalizes the solution of the single-fault problem examined in [37]. Further, though the constraint equations of the current problem are similar to those of the mulitple-fault OSFDF [45], the GTMFDF problem is shown to be clearer and more general.

In [37], it was proven that the single-fault DAP (assumes $s = 1$) is satisfied when the Riccati variable Γ is propagated by the differential equation

$$0 = \dot{\Gamma} + \Gamma A + A^T\Gamma + \frac{1}{\gamma}\Gamma\hat{F}M\hat{F}^T\Gamma + C^T\left(\hat{H}Q\hat{H} - V^{-1}\right)C \tag{3.29}$$

where $\Gamma(t_0) = \Gamma_0$ and the solution for the filter gain L is

$$L = \Gamma^{-1}C^TV^{-1}. \tag{3.30}$$

To compare to the current work, add and subtract $C^TV^{-1}C$ from (3.15) to obtain

$$\begin{aligned}\Psi_i = {} & \dot{\Pi}_i + \Pi_i A + A^T\Pi_i + \frac{1}{\gamma}\Pi_i\hat{F}_i M_i\hat{F}_i^T\Pi_i + C^T\left(\hat{H}_i Q_i\hat{H}_i - V^{-1}\right)C \\ & + \left(\Pi_i L - C^TV^{-1}\right)V\left(\Pi_i L - C^TV^{-1}\right)^T\end{aligned} \tag{3.31}$$

Clearly, (3.31) is simply (3.29) with an added quadratic term. When $L = \Pi_i^{-1}C^TV^{-1}$, this extra term is zero and (3.31) becomes identical to (3.29). Thus, (3.29) and (3.30) are a special case of the solution to (3.16) given (3.31).

The advantage of (3.29) and (3.30) is that Γ can be computed independently of L, thereby simplifying the calculation of the filter gain. However, the GTMFDF problem generalizes the solution to the single-fault problem in two ways. First, by adding an additional term to the Riccati constraint as in (3.31), the filter gain can be chosen achieve secondary objectives. For example, let

$$L = \Pi_i^{-1} C^T V^{-1} + L_i$$

where $L_i \in \mathbb{R}^{m \times n}$ is an arbitrary matrix. Then, (3.31) becomes

$$\Psi_i = \dot{\Pi}_i + \Pi_i A + A^T \Pi_i + \Pi_i \left[\frac{1}{\gamma} \hat{F}_i M_i \hat{F}_i^T + L_i V L_i^T \right] \Pi_i + C^T \left(\hat{H}_i Q_i \hat{H}_i - V^{-1} \right) C,$$

and it is possible to choose or optimize L_i to improve transmission of the target fault (though at the expense of nuisance fault blocking and/or dynamic stability). Such optimization of the filter gain was the subject of Section 3.2.2. Second, since the Riccati constraint of the GTMFDF problem is an inequality constraint, one may determine a range of possible (suboptimal and optimal) solutions to the single-fault problem. This is important, for example, in cases where the optimal solution is physically unattainable (e.g. - due to mechanical limitations).

Next, to compare the GTMFDF derivation to the multiple-fault OSFDF derivation in [45], the Riccati differential inequality (3.16) is rewritten in terms of $P_i = \Pi_i^{-1}$. Assuming that (3.15) equals zero, pre- and post-multiplying by P_i, and substituting $P_i = \Pi_i^{-1}$,

$$\dot{P}_i = (A - LC) P_i + P_i (A - LC)^T + \frac{1}{\gamma} \hat{F}_i M_i \hat{F}_i^T + L V L^T + P_i C^T \hat{H}_i Q_i \hat{H}_i C P_i, \qquad (3.32)$$

where $P_i(t_0) = \gamma P_0$. In [45], the filter gain of the multiple-fault OSFDF is optimized for a certain cost function subject to the Lyapunov differential equation[3]

$$\dot{W}_i = (A - LC)\,W_i + W_i\,(A - LC)^T + \frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T + LVL^T - F_i N_i F_i^T, \tag{3.33}$$

where N_i is a design weighting on the transmission of the target fault to the residual.

The GTMFDF and multiple-fault OSFDF are similar in that, when an analytical solution is obtained, they both require the solution to a two-point boundary value problem coupled by an equation for the filter gain. Further, the two constraint equations (3.32) and (3.33) are similar except that (3.32) compares complementary fault to projected residual size while (3.33) compares complementary fault to target fault size. However, the constraint for the OSFDF is a Lyapunov equation rather than a Riccati equation, and so solutions exist more often than for the GTMFDF constraint. This becomes a bigger factor for numerical algorithms in which the constraints must have solutions to proceed with the optimization.

Though numerical solutions are more easily obtained for the multiple-fault OSFDF, the GTMFDF improves upon its design in several ways. First, the optimization problem of the GTMFDF is derived in a more coherent way than that of the OSFDF. The cost function for the OSFDF is a sum of covariances whose physical meaning is unclear. The disturbance attenuation problem, on the other hand, is very easily understood and lends itself immediately to the FDI problem. Second, the OSFDF requires the optimal solution to a specific cost function. On the other hand, the GTMFDF only requires a solution to a feasibility problem for the constraints (3.32) and $P_i \geq 0$. If desired, a secondary cost function can be used to achieve other objectives, such as enhancing sensitivity to the target faults, thereby increasing the flexibility of the detection filter problem. Finally, the multiple-fault OSFDF derivation assumes that the disturbances are modeled as white noise processes. The GTMFDF requires no such assumption, proving that constraints like (3.32) and (3.33) are applicable to more general systems.

[3] In [45], the optimization is actually subject to two differential equations, one Riccati and one Lyapunov, for each target fault. For comparison, the two have been summed into a single equation with a single variable for each target fault in (3.33).

Remark 3.3: It is possible to introduce target fault sensitivity to the DAPs by adding $\int_{t_0}^{t_1} \|\mu_i\|^2_{N_i^{-1}}\, \mathrm{d}t$ to the numerator of (3.11) to obtain

$$\frac{\int_{t_0}^{t_1}\left[\|\hat{H}_i C\bar{e}_i\|^2_{Q_i} + \|\mu_i\|^2_{N_i^{-1}}\right]\mathrm{d}t}{\int_{t_0}^{t_1}\left[\|\hat{\mu}_i\|^2_{M_i^{-1}} + \|v_i\|^2_{V^{-1}}\right]\mathrm{d}t + \|\bar{e}_i(t_0)\|^2_{P_0^{-1}}} \leq \gamma,$$

subject to

$$\dot{\bar{e}}_i = (A - LC)\,\bar{e}_i + F_i\mu_i + \hat{F}_i\hat{\mu}_i - Lv_i$$

where $\mu_1, \hat{\mu}_1, \mu_2, \hat{\mu}_2, \ldots, \mu_s, \hat{\mu}_s$ are assumed independent. The game cost (3.12) becomes

$$J_i = \int_{t_0}^{t_1}\left[\left\|\hat{H}_i C\bar{e}_i\right\|^2_{Q_i} + \|\mu_i\|^2_{N_i^{-1}} - \|\hat{\mu}_i\|^2_{\gamma M_i^{-1}} - \|v_i\|^2_{V^{-1}}\right]\mathrm{d}t - \|\bar{e}_i(t_0)\|^2_{\Pi_0} \tag{3.34}$$

and the resulting differential game problem additionally requires minimization with respect to μ_i. In the single-fault case, (3.34) resembles the cost function of the $\mathcal{H}_\infty$ controller synthesis problem where μ_i is the control [50]. However, since the user does not have control over $\mu_i(t)$, $\mathcal{H}_\infty$ results cannot be guaranteed and it is unclear how target fault detection is affected. When $Q_i = 0$ (similar to the problems in [42, 45]), a constraint identical to (3.33) can be obtained without the assumption of white noise disturbance processes. Thus, using the methods of the current work, it is possible to generalize the multiple-fault OSFDF problem to arbitrary fault types. ♢

3.4 Proofs of Theorems, Lemmas, and Propositions

3.4.1 Proof of Theorem 3.1

Rewrite (3.26) as

$$0 = \Pi_i(A - LC) + (A - LC)^T\Pi_i + \Pi_i\left(\frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T + LVL^T\right)\Pi_i + C^T\hat{H}_i Q_i \hat{H}_i C. \tag{3.35}$$

The Hamiltonian H_i of (3.35) is defined as [48]

$$H_i = \begin{bmatrix} A - LC & R_i \\ -\bar{Q}_i & -(A - LC)^T \end{bmatrix} \tag{3.36}$$

where

$$R_i = \frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T + LVL^T$$
$$\bar{Q}_i = C^T \hat{H}_i Q_i \hat{H}_i C.$$

R_i is a nonnegative symmetric matrix and by assumption H_i has no imaginary eigenvalues and $A - LC$ is asymptotically stable. Therefore, by using Theorems 13.5 and 13.6 of [48], there exists a real, symmetric, stabilizing solution Π_i. Further, by rewriting (3.35) as

$$\Pi_i(A - LC) + (A - LC)^T \Pi_i = -\Pi_i R_i \Pi_i - \bar{Q}_i \leq 0,$$

it is clear that since $A - LC$ is asymptotically stable, $\Pi_i \geq 0$.

Finally, it is proven that Ker $\Pi_i \neq 0$ if and only if $\mathcal{M}_{uo}(\hat{H}_i C, A - LC) \neq 0$.

($\Rightarrow$) Assume that Ker $\Pi_i \neq 0$. Then, there exists $x \in$ Ker Π_i where $x \neq 0$. Pre-multiply (3.35) by x^T and post-multiply by x to get

$$0 = x^T \bar{Q}_i x$$

which implies that

$$\hat{H}_i C x = 0. \tag{3.37}$$

Now post-multiply (3.35) by x to get

$$\Pi_i(A - LC)x = 0.$$

Thus, Ker Π_i is an $(A-LC)$-invariant subspace, and so there exists a λ such that $(A-LC)x = \lambda x$. By combining this with (3.37), $x \in \mathcal{M}_{uo}(\hat{H}_i C, A-LC)$.

($\Leftarrow$) Assume that $x \in \mathcal{M}_{uo}(\hat{H}_i C, A-LC)$. This implies that $\hat{H}_i C x = 0$ and $(A-LC)x = \lambda x$ where $\lambda < 0$ by assumption. By post-multiplying (3.35) by x,

$$0 = (A - LC + R_i \Pi_i + \lambda I)^T \, \Pi_i x.$$

Thus, $x \in$ Ker Π_i and/or $-\lambda$ is an eigenvalue of $(A - LC + R_i\Pi_i)^T$ with eigenvector $\Pi_i x$. However, $A - LC + R_i \Pi_i \leq 0$, and so $\Pi_i x = 0$.

3.4.2 Proof of Corollary 3.2

Rewrite (3.26) as

$$0 = \Pi_i(A-LC) + (A-LC)^T\Pi_i + \Pi_i \left(\frac{1}{\gamma} \hat{F}_i M_i \hat{F}_i^T + LVL^T \right) \Pi_i + C^T \hat{H}_i Q_i \hat{H}_i C + \bar{G}\bar{G}^T. \quad (3.38)$$

From Theorem 3.1, a nonnegative solution for Π_i clearly exists for the general problem if H_i has no eigenvalues on the imaginary axis. Therefore, it is required to show only that Ker $\Pi_i \neq 0$ if and only if $\mathcal{M}_{uo}\left(\begin{bmatrix} \bar{G}^T \\ \hat{H}_i C \end{bmatrix}, A - LC \right) \neq 0$.

($\Rightarrow$) Assume that Ker $\Pi_i \neq 0$. Then, there exists $x \in$ Ker Π_i where $x \neq 0$. Pre-multiply (3.38)by x^T and post-multiply by x to get

$$0 = x^T \left(\bar{Q}_i + \bar{G}\bar{G}^T \right) x$$

which implies that

$$0 = \hat{H}_i C x \quad (3.39a)$$

$$0 = \bar{G}^T x. \quad (3.39b)$$

Now post-multiply (3.38) by x to get

$$\Pi_i(A - LC)x = 0.$$

Thus, Ker Π_i is an $(A-LC)$-invariant subspace, and so there exists a λ such that $(A-LC)x = \lambda x$. By combining this with (3.39), x is an unobservable mode of $\left(\begin{bmatrix} \bar{G}^T \\ \hat{H}_i C \end{bmatrix}, A - LC\right)$.

($\Leftarrow$) Assume that x is an unobservable mode $\left(\begin{bmatrix} \bar{G}^T \\ \hat{H}_i C \end{bmatrix}, A - LC\right)$. This implies that $\hat{H}_i Cx = 0$, $\bar{G}^T x = 0$, and $(A-LC)x = \lambda x$ where $\lambda < 0$ by assumption. Then, the remainder of the proof is identical to that of Theorem 3.1.

3.4.3 Lemmas and Propositions

Lemma 3.3: If $F_1, \ldots, F_s, \hat{F}$ are (C, A)-output separable, then

$$\hat{\mathcal{W}}_i^* = \mathcal{W}_1^* \oplus \ldots \oplus \mathcal{W}_{i-1}^* \oplus \mathcal{W}_{i+1}^* \oplus \ldots \oplus \mathcal{W}_s^* \oplus \hat{\mathcal{W}}^*.$$

▌

Proof: Let

$$\bar{\mathcal{W}}_i^* \triangleq \mathcal{W}_1^* \oplus \ldots \oplus \mathcal{W}_{i-1}^* \oplus \mathcal{W}_{i+1}^* \oplus \ldots \oplus \mathcal{W}_s^* \oplus \hat{\mathcal{W}}^*. \tag{3.40}$$

The proof is divided into two parts. First, it is proven that

$$\hat{\mathcal{W}}_i^* \supseteq \bar{\mathcal{W}}_i^* \tag{3.41}$$

where $\hat{\mathcal{W}}_i^*$ is the minimal (C, A)-invariant subspace associated with $\hat{F}_i$ as calculated via Algorithm 4.1 and (4.18) in Chapter 4. Then, $\bar{\mathcal{W}}_i^*$ is proven to be a (C, A)-invariant subspace.

Contrary to (3.41), assume $\exists\ x \in \mathcal{W}_j^*, x \notin \hat{\mathcal{W}}_i^*, i \neq j$. From (3.2) and (3.3), $x \in \mathcal{W}_j^*$ implies

$$x = \sum_{k=0}^{\beta_j} \alpha_k B_j^k \tag{3.42}$$

where $CB_j^0 = CB_j^1 = \ldots = CB_j^{\beta_j - 1} = 0$ and $\alpha_0, \ldots, \alpha_{\beta_j}$ are scalar coefficients. Let $B_j^k = \hat{B}_i^k \zeta_{i,j}\ \forall\ k \leq \beta_j$ for some vector coefficient $\zeta_{i,j}$. Then, from Algorithm 4.1 and (3.42),

$$\text{Im} \left[\begin{array}{ccc} \hat{B}_i^0 & \ldots & \hat{B}_i^{\hat{\beta}_i} \end{array} \right] \subseteq \hat{\mathcal{W}}_i^* \Rightarrow \sum_{k=0}^{\beta_j} \alpha_k \hat{B}_i^k \zeta_{i,j} \in \hat{\mathcal{W}}_i^* \Rightarrow \sum_{k=0}^{\beta_j} \alpha_k B_j^k \in \hat{\mathcal{W}}_i^* \Rightarrow x \in \hat{\mathcal{W}}_i^*. \tag{3.43}$$

However, (3.43) contradicts the assumption that $x \notin \hat{\mathcal{W}}_i^*$. The same can be shown for the multi-dimensional nuisance fault $\hat{F}$. Therefore, $\hat{\mathcal{W}}_i^*$ must satisfy (3.41).

Now, $\bar{\mathcal{W}}_i^*$ is shown to be a (C, A)-invariant subspace. Since F_j and $\hat{F}$ satisfy (3.4) $\forall\ j \in \{1, \ldots, s\}$, Im $F_j \subseteq \bar{\mathcal{W}}_i^*$ and Im $\hat{F} \subseteq \bar{\mathcal{W}}_i^*$. Also, since the faults directions are assumed linearly independent, the ranks of F_j and $\hat{F}$ are equivalent to those of $C\mathcal{W}_j^*$ and $C\hat{\mathcal{W}}^*$, respectively. Further, since $F_1, \ldots, F_s, \hat{F}$ are assumed (C, A)-output separable, $C\mathcal{W}_1^*, \ldots, C\mathcal{W}_s^*, C\hat{\mathcal{W}}^*$ are linearly independent. Using these along with (3.40),

$$\begin{aligned}
\text{rank } C\bar{\mathcal{W}}_i^* &= \text{rank } C \left(\mathcal{W}_1^* \oplus \ldots \oplus \mathcal{W}_{i-1}^* \oplus \mathcal{W}_{i+1}^* \oplus \ldots \oplus \mathcal{W}_s^* \oplus \hat{\mathcal{W}}^* \right) \\
&= \text{rank} \left(C\mathcal{W}_1^* \oplus \ldots \oplus C\mathcal{W}_{i-1}^* \oplus C\mathcal{W}_{i+1}^* \oplus \ldots \oplus C\mathcal{W}_s^* \oplus C\hat{\mathcal{W}}^* \right) \\
&= \text{rank } C\mathcal{W}_1^* + \ldots + \text{rank } C\mathcal{W}_{i-1}^* + \text{rank } C\mathcal{W}_{i+1}^* + \ldots + \text{rank } C\mathcal{W}_s^* \\
&\quad + \text{rank } C\hat{\mathcal{W}}^* \\
&= \text{rank } F_1 + \ldots + \text{rank } F_{i-1} + \text{rank } F_{i+1} + \ldots + \text{rank } F_s + \text{rank } \hat{F} \\
&= \text{rank} \left[\begin{array}{cccccc} F_1 & \ldots & F_{i-1} & F_{i+1} & \ldots & F_s & \hat{F} \end{array} \right] \\
&= \text{rank } \hat{F}_i.
\end{aligned}$$

Therefore, $\bar{\mathcal{W}}_i^*$ is a (C, A)-invariant subspace. Using this along with (3.41), the minimal (C, A)-invariant subspace associated with $\hat{F}_i$ satisfies

$$\hat{\mathcal{W}}_i^* = \bar{\mathcal{W}}_i^*.$$

QED

Proposition 3.4: Assume that $A - LC$ is asymptotically stable and that Π_i satisfies the constraints (3.16) given (3.15). Then, $\Pi_i \geq 0$ over the time interval $[t_0, t_1]$. ∎

Proof: Substituting (3.15) into (3.16), the inequality may be rewritten as

$$\dot{\Pi}_i + \Pi_i(A - LC) + (A - LC)^T\Pi_i \leq -\left[\Pi_i\left(\frac{1}{\gamma}\hat{F}_i M_i \hat{F}_i^T + LVL^T\right)\Pi_i + C^T\hat{H}_i Q_i \hat{H}_i C\right] \leq 0$$

Since $A - LC$ is asymptotically stable by assumption, the Lyapunov stability criterion for finite time-varying systems implies that $\Pi \geq 0$. **QED**

Chapter 4

The Asymptotic Game Theoretic Multiple-Fault Detection Filter

The GTMFDF problem was motivated as a finite time-varying approximation of detection filters based on the spectral and geometric theories (like the RDDF). Thus, to truly generalize these detection filters, the GTMFDF must recover their detection space structure in the limit as the disturbance attenuation bound goes to zero. In this chapter, the problem of obtaining the asymptotic solution to the GTMFDF problem is examined. It is shown that the sufficient conditions for disturbance attenuation subject the filter gain to a set of Riccati inequalities (similar to those of the general case of the GTMFDF) along with a new set of equality conditions. The equality conditions enforce the desired detection filter structure in which faults are completely blocked from the projected residuals. A special case of the solution is obtained by using the similar constraints to those derived for the single-fault problem. Sufficient conditions for obtaining this filter gain solution are subject to restrictions on the geometric properties of the Riccati solutions. Each pair of inequality and equality constraints specifies a portion of the filter gain solution. However, with nuisance faults and/or a complementary subspace, the constraints do not completely define the filter gain. Thus, a reduced-order solution is obtained in which the nuisance faults and complementary subspace are projected out of the detection filter's state space.

This chapter is arranged as follows. The GTMFDF problem is examined in the limit as the disturbance attenuation bound goes to zero and rewritten using singular optimal control methods in Section 4.1. Next, the detection filter gain constraints are derived in Section 4.2. It is shown that the geometric structure of the asymptotic solution extends that of the RDDF to the finite time-varying case. Sufficient conditions for the existence of a special case of the filter gain are also derived. Finally, the geometric structure is used to generate a reduced-order version of the special case filter gain in Section 4.3.

4.1 The Asymptotic Differential Game Problem

The goal of the asymptotic detection filter is to obtain an invariant subspace structure in which the nuisance fault is blocked from the projected residual. To that end, the presence of the initial condition error and measurement noise are not important, and so their covariances are allowed to go to zero along with γ. Then, from their previous definitions, $V \triangleq \lim_{\gamma\to 0}\gamma^{-1}\bar{V}$ and $\Pi_0 \triangleq \lim_{\gamma\to 0}\gamma P_0^{-1}$. Therefore, the i^{th} DAP (3.11) becomes

$$\frac{\int_{t_0}^{t_1}\|\hat{H}_i C e_i\|_{Q_i}^2 \mathrm{d}t}{\int_{t_0}^{t_1}\|\hat{\mu}_i\|_{M_i^{-1}}^2 \mathrm{d}t} = 0, \tag{4.1}$$

implying that $\hat{\mu}_i$ would be blocked completely from the projected residual if there exists a solution for the asymptotic detection filter. Further, the i^{th} game cost (3.12) becomes

$$J_i' = \lim_{\gamma\to 0} J_i = \int_{t_0}^{t_1}\left[\|\hat{H}_i C e_i\|_{Q_i}^2 - \|v_i\|_{V^{-1}}^2\right]\mathrm{d}t - \|e_i(t_0)\|_{\Pi_0}^2 \tag{4.2}$$

subject to (3.8a). Examination of (4.2) reveals that in the limit the cost is singular not only with respect to L, but also with respect to the complementary fault $\hat{\mu}_i$. In its singular form, (4.2) cannot be used to generate a detection filter gain that will block $\hat{\mu}_i$ from the i^{th} projected residual. However, the Goh transformation may be used to convert (4.2) into a non-singular cost function with a new control variable that is related to $\hat{\mu}_i$.

In this section, a new version of Problem 3.1 is obtained as $\gamma \to 0$ for the singular cost function (4.2). In order to relate the complementary fault directions $\hat{F}_i$ to the new detection filter, a new state and control obtained via the Goh transformation are used to the convert the function into a non-singular form in Section 4.1.1. Then, the asymptotic detection filter problem is formulated in Section 4.1.2 as a feasibility problem to determine a filter gain L such that the game cost is nonpositive for the worst-case disturbance parameter values.

4.1.1 Conversion to a Non-Singular Problem

In this section, the Goh transformation is applied to the singular cost function (4.2) to eliminate the singularity with respect to the complementary fault $\hat{\mu}_i$. Define the new fault input and state estimation error

$$\hat{\phi}_i^1 \triangleq \int_{t_0}^{t} \hat{\mu}_i(\tau)\,\mathrm{d}\tau \tag{4.3}$$

$$\epsilon_i^1 \triangleq e_i - \hat{F}_i \hat{\phi}_i^1. \tag{4.4}$$

Differentiating (4.4) yields

$$\dot{\epsilon}_i^1 = A\epsilon_i^1 + \hat{B}_i^1 \hat{\phi}_i^1 - Lr_i \tag{4.5}$$

where

$$\hat{B}_i^1 = \left[\; B_1^1 \;\dots\; B_{i-1}^1 \;\; B_{i+1}^1 \;\dots\; B_q^1 \;\right]$$

and from (3.3)

$$B_j^1 = AF_j - \dot{F}_j.$$

Thus, (4.5) is the new estimation error dynamics constraint for the i^{th} cost function and (4.3) is the new fault input. Substituting (3.8b), (4.4), and $\hat{H}_i C \hat{F}_i = 0$ into the singular game cost (4.2) yields

$$J_i' = \int_{t_0}^{t_1} \left[\left\| \hat{H}_i C \epsilon_i^1 \right\|_{Q_i}^2 - \left\| r_i - C \left(\epsilon_i^1 + \hat{F}_i \hat{\phi}_i^1 \right) \right\|_{V^{-1}}^2 \right] \mathrm{d}t - \left\| \epsilon_i^1(t_0^+) + \hat{F}_i(t_0^+) \hat{\phi}_i^1(t_0^+) \right\|_{\Pi_0}^2 . \quad (4.6)$$

In (4.6), t_0^+ replaces t_0 because there is an impulsive jump in $\hat{\phi}_i^1$ at $t = t_0$ in order to reach the singular surface [37]. If Im $\hat{F}_i \bigcap$ Ker $C = 0$, then (4.6) is non-singular with respect to $\hat{\phi}_i^1$ and can be optimized with respect to $\hat{\phi}_i^1$ and $\epsilon_i^1(t_0^+)$. On the other hand, if Im $\hat{F}_i \bigcap$ Ker $C \neq 0$, then the problem remains singular with respect to the fault input.

If the problem remains singular after the first application of the Goh transformation, then it must be applied again, though only to the remaining singular component. Choose the transformation $T_{i,1}$ such that

$$\begin{aligned} \hat{F}_i T_{i,1}^T &= \begin{bmatrix} \hat{F}_{i,1} & \hat{F}_{i,2} \end{bmatrix} \\ \hat{B}_i^1 T_{i,1}^T &= \begin{bmatrix} \hat{B}_{i,1}^1 & \hat{B}_{i,2}^1 \end{bmatrix} \\ &= \begin{bmatrix} A\hat{F}_{i,1} - \dot{\hat{F}}_{i,1} & A\hat{F}_{i,2} - \dot{\hat{F}}_{i,2} \end{bmatrix} \\ T_{i,1} \hat{\phi}_i^1 &= \begin{bmatrix} \hat{\phi}_{i,1}^1 \\ \hat{\phi}_{i,2}^1 \end{bmatrix} \end{aligned}$$

where $\hat{F}_{i,2} \triangleq$ Im $\hat{F}_i \bigcap$ Ker C and Im $[\ \hat{F}_{i,1} \quad \hat{F}_{i,2}\ ] =$ Im $\hat{F}_i$. The Goh transformation applied to the singular component, $\hat{\phi}_{i,2}^1$, generates the new fault input and state estimation error

$$\begin{aligned} \bar{\phi}_{i,2}^2 &\triangleq \int_{t_0}^{t} \hat{\phi}_{i,2}^1(\tau) \, \mathrm{d}\tau \\ \epsilon_i^2 &\triangleq \epsilon_i^1 - \hat{B}_{i,2}^1 \bar{\phi}_{i,2}^2 . \end{aligned} \quad (4.7)$$

Differentiating (4.7) yields

$$\dot{\epsilon}_i^2 = A\epsilon_i^2 + \hat{B}_i^2 \hat{\phi}_i^2 - Lr_i \quad (4.8)$$

where

$$\hat{B}_i^2\hat{\phi}_i^2 = \left[\begin{array}{cc} \hat{B}_{i,1}^1 & A\hat{B}_{i,2}^1 - \dot{\hat{B}}_{i,2}^1 \end{array}\right] \left[\begin{array}{c} \hat{\phi}_{i,1}^1 \\ \bar{\phi}_{i,2}^2 \end{array}\right].$$

Further, by substituting (4.7) into (4.6) and using $C\hat{F}_i\hat{\phi}_i^1 = C\hat{F}_{i,1}\hat{\phi}_{i,1}^1$,

$$J_i' = \int_{t_0}^{t_1} \left[\left\| \hat{H}_i C \epsilon_i^2 \right\|_{Q_i}^2 - \left\| r_i - C\left(\epsilon_i^2 + \tilde{B}_i^1 \hat{\phi}_i^2\right) \right\|_{V^{-1}}^2 \right] \mathrm{d}t - \left\| \epsilon_i^2(t_0^+) + \bar{B}_i^1(t_0^+)\bar{\phi}_i^2(t_0^+) \right\|_{\Pi_0}^2 \quad (4.9)$$

where

$$\tilde{B}_i^1 = \left[\begin{array}{cc} \hat{F}_{i,1} & \hat{B}_{i,2}^1 \end{array}\right]$$

$$\bar{B}_i^1\bar{\phi}_i^2 = \left[\begin{array}{cc} \hat{F}_i & \hat{B}_{i,2}^1 \end{array}\right] \left[\begin{array}{c} \hat{\phi}_i^1 \\ \bar{\phi}_{i,2}^2 \end{array}\right].$$

If Im $\tilde{B}_i^1 \bigcap$ Ker $C = 0$, then (4.9) is non-singular with respect to $\hat{\phi}_i^2$ and can be optimized with respect to r_i, $\hat{\phi}_i^2$, $\bar{\phi}_i^2(t_0^+)$, and $\epsilon_i^2(t_0^+)$. On the other hand, if Im $\tilde{B}_i^1 \bigcap$ Ker $C \neq 0$, then the problem remains singular with respect to the fault input.

If the problem remains singular after the second application of the Goh transformation, then the process above can be repeated as necessary to convert it into a non-singular problem. Assume that converting the i^{th} problem from singular to non-singular requires $\hat{\beta}_i + 1$ iterations. The cost function at the end of the $\hat{\beta}_i^{\text{th}}$ iteration is

$$J_i' = \int_{t_0}^{t_1} \left[\left\| \hat{H}_i C \epsilon_i^{\hat{\beta}_i} \right\|_{Q_i}^2 - \left\| r_i - C\left(\epsilon_i^{\hat{\beta}_i} + \tilde{B}_i^{\hat{\beta}_i - 1} \hat{\phi}_i^{\hat{\beta}_i}\right) \right\|_{V^{-1}}^2 \right] \mathrm{d}t - \left\| \epsilon_i^{\hat{\beta}_i}(t_0^+) + \bar{B}_i^{\hat{\beta}_i - 1}(t_0^+)\bar{\phi}_i^{\hat{\beta}_i}(t_0^+) \right\|_{\Pi_0}^2 \quad (4.10)$$

where Im $\tilde{B}_i^{\hat{\beta}_i - 1} \bigcap$ Ker $C \neq 0$. For the $(\hat{\beta}_i + 1)^{\text{th}}$ and final iteration, choose the transformation $T_{i,\hat{\beta}_i}$ such that

$$\tilde{B}_i^{\hat{\beta}_i} T_{i,\hat{\beta}_i}^T = \left[\begin{array}{cc} \tilde{B}_{i,1}^{\hat{\beta}_i - 1} & \tilde{B}_{i,2}^{\beta_i - 1} \end{array}\right]$$

$$\hat{B}_i^{\hat{\beta}_i} T_{i,\hat{\beta}_i}^T = \left[\begin{array}{cc} \hat{B}_{i,1}^{\beta_i} & \hat{B}_{i,2}^{\hat{\beta}_i - 1} \end{array}\right] = \left[\begin{array}{cc} A\tilde{B}_{i,1}^{\hat{\beta}_i - 1} - \dot{\tilde{B}}_{i,1}^{\hat{\beta}_i - 1} & A\tilde{B}_{i,2}^{\hat{\beta}_i - 1} - \dot{\tilde{B}}_{i,2}^{\hat{\beta}_i - 1} \end{array}\right]$$

$$T_{i,\hat{\beta}_i}\hat{\phi}_i^{\hat{\beta}_i} = \begin{bmatrix} \hat{\phi}_{i,1}^{\hat{\beta}_i} \\ \hat{\phi}_{i,2}^{\hat{\beta}_i} \end{bmatrix}$$

where $\tilde{B}_{i,2}^{\hat{\beta}_i-1} \triangleq \text{Im } \tilde{B}_i^{\hat{\beta}_i-1} \bigcap \text{Ker } C$ and $\text{Im } [\ \tilde{B}_{i,1}^{\hat{\beta}_i-1} \quad \tilde{B}_{i,2}^{\hat{\beta}_i-1}\ ] = \text{Im } \tilde{B}_i^{\hat{\beta}_i-1}$. Define

$$\bar{\phi}_{i,2}^{\hat{\beta}_i+1} \triangleq \int_{t_0}^{t} \hat{\phi}_{i,2}^{\hat{\beta}_i}(\tau)\,\mathrm{d}\tau \tag{4.11}$$

$$\epsilon_i^{\hat{\beta}_i+1} \triangleq \epsilon_i^{\hat{\beta}_i} - \hat{B}_{i,2}^{\hat{\beta}_i}\bar{\phi}_{i,2}^{\hat{\beta}_i+1}. \tag{4.12}$$

Differentiating (4.12) yields

$$\dot{\epsilon}_i^{\hat{\beta}_i+1} = A\epsilon_i^{\hat{\beta}_i+1} + \hat{B}_i^{\hat{\beta}_i+1}\hat{\phi}_i^{\hat{\beta}_i+1} - Lr_i \tag{4.13}$$

where

$$\hat{B}_i^{\hat{\beta}_i+1}\hat{\phi}_i^{\hat{\beta}_i+1} = \begin{bmatrix} \hat{B}_{i,1}^{\hat{\beta}_i} & A\hat{B}_{i,2}^{\hat{\beta}_i} - \dot{\hat{B}}_{i,2}^{\hat{\beta}_i} \end{bmatrix} \begin{bmatrix} \hat{\phi}_{i,1}^{\hat{\beta}_i} \\ \bar{\phi}_{i,2}^{\hat{\beta}_i+1} \end{bmatrix}.$$

Further, by substituting (4.12) into (4.10) and using $C\tilde{B}_i^{\hat{\beta}_i-1}\hat{\phi}_i^{\hat{\beta}_i} = C\tilde{B}_{i,1}^{\hat{\beta}_i-1}\hat{\phi}_{i,1}^{\hat{\beta}_i}$,

$$\begin{aligned} J_i' = & \int_{t_0}^{t_1} \left[\left\| \hat{H}_i C \epsilon_i^{\hat{\beta}_i+1} \right\|_{Q_i}^2 - \left\| r_i - C\left(\epsilon_i^{\hat{\beta}_i+1} + \tilde{B}_i^{\hat{\beta}_i}\hat{\phi}_i^{\hat{\beta}_i+1}\right) \right\|_{V^{-1}}^2 \right] \mathrm{d}t \\ & - \left\| \epsilon_i^{\hat{\beta}_i+1}(t_0) + \bar{B}_i^{\hat{\beta}_i}(t_0^+)\bar{\phi}_i^{\hat{\beta}_i+1}(t_0^+) \right\|_{\Pi_0}^2 \end{aligned} \tag{4.14}$$

where

$$\tilde{B}_i^{\hat{\beta}_i} = \begin{bmatrix} \tilde{B}_{i,1}^{\hat{\beta}_i-1} & \hat{B}_{i,2}^{\hat{\beta}_i} \end{bmatrix}$$

$$\bar{B}_i^{\hat{\beta}_i}\bar{\phi}_i^{\hat{\beta}_i+1} = \begin{bmatrix} \bar{B}_i^{\hat{\beta}_i-1} & \hat{B}_{i,2}^{\hat{\beta}_i} \end{bmatrix} \begin{bmatrix} \bar{\phi}_i^{\hat{\beta}_i} \\ \bar{\phi}_{i,2}^{\hat{\beta}_i+1} \end{bmatrix}$$

and $\text{Im } \tilde{B}_i^{\hat{\beta}_i} \bigcap \text{Ker } C = 0$. Since (4.14) is non-singular with respect to $\hat{\phi}_i^{\hat{\beta}_i+1}$, it can be optimized with respect to r_i, $\hat{\phi}_i^{\hat{\beta}_i+1}$, $\bar{\phi}_i^{\hat{\beta}_i+1}(t_0^+)$, and $\epsilon_i^{\hat{\beta}_i+1}(t_0)$.

4.1.2 Problem Formulation

In this section, the asymptotic detection filter problem is formulated using the asymptotic cost function (4.14) subject to (4.13). Reaching the $(\hat{\beta}_i + 1)^{\text{th}}$ iteration from Section 4.1.1 describes the following iterative algorithm for the detection filter's system matrices:

Algorithm 4.1:

Step 1: Define the initial conditions $\hat{B}_i^0 = \tilde{B}_i^0 = \bar{B}_i^0 = \hat{F}_i$ and set $k = 0$.

Step 2: Calculate $\hat{B}_i^1 = A\hat{F}_i - \dot{\hat{F}}_i$.

Step 3: If Im $\tilde{B}_i^k \bigcap$ Ker $C = 0$, stop here and set $\hat{\beta}_i = k$.

Step 4: Set $k = k + 1$. Determine T_k such that $\tilde{B}_i^{k-1} T_k = \left[\begin{array}{cc} \tilde{B}_{i,1}^{k-1} & \tilde{B}_{i,2}^{k-1} \end{array} \right]$ where $\tilde{B}_{i,2}^{k-1} \triangleq$ Im $\tilde{B}_i^{k-1} \bigcap$ Ker C and Im $\left[\begin{array}{cc} \tilde{B}_{i,1}^{k-1} & \tilde{B}_{i,2}^{k-1} \end{array} \right] =$ Im $\tilde{B}_i^{k-1}$.

Step 5: Obtain $\hat{B}_{i,1}^k$ and $\hat{B}_{i,2}^k$ from $\hat{B}_i^k T_k = \left[\begin{array}{cc} \hat{B}_{i,1}^k & \hat{B}_{i,2}^k \end{array} \right] = \left[\begin{array}{cc} A\tilde{B}_{i,1}^{k-1} - \dot{\tilde{B}}_{i,1}^{k-1} & A\tilde{B}_{i,2}^{k-1} - \dot{\tilde{B}}_{i,2}^{k-1} \end{array} \right]$.

Step 6: Construct

$$\tilde{B}_i^k = \left[\begin{array}{cc} \tilde{B}_{i,1}^{k-1} & \hat{B}_{i,2}^k \end{array} \right]$$
$$\bar{B}_i^k = \left[\begin{array}{cc} \bar{B}_i^{k-1} & \hat{B}_{i,2}^k \end{array} \right].$$

Step 7: Calculate

$$\hat{B}_i^{k+1} = \left[\begin{array}{cc} \hat{B}_{i,1}^k & A\hat{B}_{i,2}^k - \dot{\hat{B}}_{i,2}^k \end{array} \right] = A\tilde{B}_i^k - \dot{\tilde{B}}_i^k.$$

Step 8: Go to Step 3.

♡

To simplify the notation used in the rest of the section, the superscripts are removed from the system matrices obtained at the end of Algorithm 4.1. Therefore, (4.14) is rewritten as

$$J_i' = \int_{t_0}^{t_1} \left[\left\| \hat{H}_i C \epsilon_i \right\|_{Q_i}^2 - \left\| r_i - C\left(\epsilon_i + \tilde{B}_i \hat{\phi}_i\right) \right\|_{V^{-1}}^2 \right] \mathrm{d}t - \left\| \epsilon_i(t_0) + \bar{B}_i(t_0^+)\bar{\phi}_i(t_0^+) \right\|_{\Pi_0}^2 \quad (4.15)$$

subject to the notationally simplified form of (4.13)

$$\dot{\epsilon}_i = A\epsilon_i + \hat{B}_i \hat{\phi}_i - L r_i. \quad (4.16)$$

The new differential game problem is therefore

$$\min_{L} \max_{r_1,\ldots,r_s} \max_{\hat{\phi}_1,\ldots,\hat{\phi}_s} \max_{\bar{\phi}_1(t_0^+),\ldots,\bar{\phi}_s(t_0^+)} \max_{\epsilon_1(t_0^+),\ldots,\epsilon_s(t_0^+)} \sum_{i=1}^{s} J_i'(L, r_i, \hat{\phi}_i, \bar{\phi}_i(t_0^+), \epsilon_i(t_0)) \quad (4.17)$$

subject to (4.16). Recall from the general case that $\hat{\mu}_1, \ldots, \hat{\mu}_s$ are not independent since they all share elements. Thus, $\hat{\phi}_1, \ldots, \hat{\phi}_s$ and $\bar{\phi}_1(t_0^+), \ldots, \bar{\phi}_s(t_0^+)$ are similarly interconnected.

Like the general case, the detection filter gain L does not appear in the modified game cost (4.15) and enters linearly into the constraint (4.16). Therefore, (4.17) remains singular with respect to the filter gain. However, in order to satisfy the DAPs (4.1), it is only required that (4.2) be nonpositive for the maximizing values of r_i, $\hat{\phi}_i$, $\bar{\phi}_i(t_0^+)$, and $\epsilon_i(t_0)$ $\forall t \in [t_0, t_1]$. To further simplify the problem statement and obtain sufficient conditions for fault isolation as in the general case, assume that all of the disturbance parameters are independent. Therefore, a solution to the following simplified problem is required:

Problem 4.1: Find $L(t)$ such that

$$\max_{r_i} \max_{\hat{\phi}_i} \max_{\bar{\phi}_i(t_0^+)} \max_{\epsilon_i(t_0)} J_i' \leq 0$$

subject to (4.16) and (4.2) $\forall\, i = 1, \ldots, s$. ♡

Remark 4.1: The solution for $\bar{B}_i$ derived in Algorithm 4.1 is identical to the minimal (C, A)-invariant subspace associated with the multi-dimensional complementary fault direction $\hat{F}_i$ [21]. Therefore, in the remainder of this chapter, let

$$\hat{\mathcal{W}}_i^* \triangleq \text{Im } \bar{B}_i \tag{4.18}$$

where

$$C\hat{\mathcal{W}}_i^* = \text{Im } C\tilde{B}_i. \tag{4.19}$$

Though each target fault is assumed to be scalar, Algorithm 4.1 can be used to calculate the minimal (C, A)-invariant subspaces of multi-dimensional target faults as well by replacing $\hat{F}_i$ with F_i in Step 1.

$\diamondsuit$

4.2 Asymptotic Detection Filter Problem Solution

In this section, the asymptotic solution to the GTMFDF is derived. The conditions for the optimality of the asymptotic problem's cost function are discussed in Section 4.2.1. It is shown that the sufficient conditions for optimality may be written as a set of Riccati inequality constraints similar to the general case, along with a set of equality constraints that force the Riccati solutions to obtain the desired nullspace. The invariant subspace structure generated by the Riccati solutions is derived in Section 4.2.2. This structure is an essential component in the detection filter design, giving it the required fault isolation properties. A special case of the detection filter solution is obtained in Section 4.2.3, in which the optimal filter gain is constrained by Riccati differential equations that are identical to those derived for the single-fault GTFDF [37]. However, the asymptotic filter gain may not exist in general, as it requires a solution to a set of coupled, possibly contradictory matrix constraints. Conditions for the existence of the asymptotic filter gain are discussed in Section 4.2.4. It is shown that the existence of the special case filter gain is subject to a condition on the state space description.

4.2.1 Conditions for Optimality of the Game Cost

In this section, the necessary and sufficient conditions for optimality of the game cost are derived. By appending the estimation error dynamics to the cost function and using calculus of variations, necessary conditions for optimality are determined with respect to the disturbances parameters. Then, by examining the second-order variation of each cost function [51], sufficient conditions for optimality are derived as algebraic Riccati inequality constraints on the estimation error covariance associated with each DAP, along with a set of equality constraints that enforce the desired detection space structure. A valid solution for the filter gain in Problem 4.1 is any for which the Riccati inequalities are nonpositive, the estimation error covariances are nonnegative-definite, and the equality constraints are satisfied, implying that the DAPs are also satisfied.

Necessary Conditions for Optimality

First, the modified estimation error dynamics (4.16) are appended to the cost function (4.15) using the LaGrange multiplier $2\lambda_i^T$, which yields

$$\begin{aligned} J_i' = \int_{t_0}^{t_1} \left[\left\| \hat{H}_i C \epsilon_i \right\|_{Q_i}^2 - \left\| r_i - C\left(\epsilon_i + \tilde{B}_i \hat{\phi}_i\right) \right\|_{V^{-1}}^2 + 2\lambda_i^T \left(A\epsilon_i + \hat{B}_i \hat{\phi}_i - Lr_i - \dot{\epsilon}_i \right) \right] \mathrm{d}t \\ - \left\| \epsilon_i(t_0) + \bar{B}_i(t_0^+)\bar{\phi}_i(t_0^+) \right\|_{\Pi_0}^2 . \end{aligned}$$

By integrating $\int_{t_0}^{t_1} 2\lambda_i^T \dot{\epsilon}_i \, \mathrm{d}t$ by parts, substituting (4.16), and collecting terms,

$$\begin{aligned} J_i' = \int_{t_0}^{t_1} \Bigg[\left\| \hat{H}_i C \epsilon_i \right\|_{Q_i}^2 + 2\lambda_i^T \left(A\epsilon_i + \hat{B}_i \hat{\phi}_i - Lr_i \right) + 2\left(A\epsilon_i + \hat{B}_i \hat{\phi}_i - Lr_i \right)^T \lambda_i \\ - \left\| r_i - C\left(\epsilon_i + \tilde{B}_i \hat{\phi}_i\right) \right\|_{V^{-1}}^2 \Bigg] \mathrm{d}t - \left\| \epsilon_i(t_0) + \bar{B}_i(t_0^+)\bar{\phi}_i(t_0^+) \right\|_{\Pi_0}^2 - 2\lambda_i^T(\tau)\epsilon_i(\tau)\Big|_{\tau=t_0}^{t_1} . \end{aligned}$$

By taking the first-order variation with respect to r_i, $\hat{\phi}_i$, $\bar{\phi}_i(t_0^+)$, and $\epsilon_i(t_0)$,

$$
\begin{aligned}
\delta J_i' = 2\int_{t_0}^{t_1} \Big\{ & \delta\epsilon_i^T \Big(C^T \hat{H}_i Q_i \hat{H}_i C \epsilon_i + C^T V^{-1} \left[r_i - C \left(\epsilon_i + \tilde{B}_i \hat{\phi}_i \right) \right] + A^T \lambda_i + \dot{\lambda}_i \Big) \\
& + \delta\hat{\phi}_i^T \left(\hat{B}_i^T \lambda_i + \tilde{B}_i^T C^T V^{-1} \left[r_i - C \left(\epsilon_i + \tilde{B}_i \hat{\phi}_i \right) \right] \right) \\
& - \delta r_i^T \left(L^T \lambda_i + V^{-1} \left[r_i - C \left(\epsilon_i + \tilde{B}_i \hat{\phi}_i \right) \right] \right) \Big\} \, \mathrm{d}t \\
& + 2\delta\epsilon_i^T(t_0) \left\{ \lambda_i(t_0) - \Pi_0 \left[\epsilon_i(t_0) + \bar{B}_i(t_0^+) \bar{\phi}_i(t_0^+) \right] \right\} - 2\delta\epsilon_i^T(t_1)\lambda_i(t_1) \\
& - 2\delta\bar{\phi}_i^T(t_0^+) \left\{ \bar{B}_i^T(t_0^+) \Pi_0 \left[\epsilon_i(t_0) + \bar{B}_i(t_0^+) \bar{\phi}_i(t_0^+) \right] \right\}
\end{aligned}
$$

The necessary conditions to maximize (4.15) with respect to r_i, $\hat{\phi}_i$, $\bar{\phi}_i(t_0^+)$, and $\epsilon_i(t_0)$ are

$$0 = L^T \lambda_i + V^{-1} \left[r_i^* - C \left(\epsilon_i^* + \tilde{B}_i \hat{\phi}_i^* \right) \right] \tag{4.20}$$

$$0 = \hat{B}_i^T \lambda_i + \tilde{B}_i^T C^T V^{-1} \left[r_i^* - C \left(\epsilon_i^* + \tilde{B}_i \hat{\phi}_i^* \right) \right] \tag{4.21}$$

$$0 = \bar{B}_i^T(t_0^+) \Pi_0 \left[\epsilon_i^*(t_0) + \bar{B}_i(t_0^+) \bar{\phi}_i^*(t_0^+) \right] \tag{4.22}$$

$$0 = \dot{\lambda}_i + A^T \lambda_i + C^T \hat{H}_i Q_i \hat{H}_i C \epsilon_i^* + C^T V^{-1} \left[r_i^* - C \left(\epsilon_i^* + \tilde{B}_i \hat{\phi}_i^* \right) \right] \tag{4.23}$$

$$0 = \lambda_i(t_0) - \Pi_0 \left[\epsilon_i^*(t_0) + \bar{B}_i(t_0^+) \bar{\phi}_i^*(t_0^+) \right] \tag{4.24}$$

$$0 = \lambda_i(t_1) \tag{4.25}$$

$\forall i \in \{1, \ldots, s\}$, where the * denotes the optimal strategy for the given variable. Note that ϵ_i^* is the state using $\hat{\phi}_i^*$ and $\bar{\phi}_i^*(t_0^+)$. Upon further examination of (4.22), the optimal strategy for $\bar{\phi}_i(t_0^+)$ is

$$\bar{\phi}_i^*(t_0^+) = - \left[\bar{B}_i^T(t_0^+) \Pi_0 \bar{B}_i(t_0^+) \right]^{-1} \bar{B}_i^T(t_0^+) \Pi_0 \epsilon_i^*(t_0). \tag{4.26}$$

Also, by substituting (4.20) into (4.21),

$$0 = \left(\hat{B}_i - LC\tilde{B}_i \right)^T S_i \epsilon_i^* \tag{4.27}$$

This constraint will be useful later in the analysis.

The first-order necessary conditions for optimality are simplified using an assumed form of the LaGrange multiplier, where

$$\lambda_i = S_i \epsilon_i^*. \tag{4.28}$$

The differential equation for $S_i \in \mathbb{R}^{n\times n}$ must be determined so that (4.28) is consistent with the optimal solution. Substituting (4.28) into (4.21), the optimal strategy for $\hat{\phi}_i$ is

$$\hat{\phi}_i^* = \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i\right)^{-1} \left[\hat{B}_i^T S_i \epsilon_i^* + \tilde{B}_i^T C^T V^{-1} \left(r_i^* - C\epsilon_i^*\right)\right] \tag{4.29}$$

Then, by substituting (4.28) and (4.29) into (4.20),

$$0 = \left[L^T - V^{-1} C \tilde{B}_i \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i\right)^{-1} \hat{B}_i^T\right] S_i \epsilon_i^* + \bar{H}_i^T V^{-1} \bar{H}_i \left(r_i^* - C\epsilon_i^*\right). \tag{4.30}$$

where

$$\bar{H}_i = I - C\tilde{B}_i(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i)^{-1} \tilde{B}_i^T C^T V^{-1}. \tag{4.31}$$

Note that (4.31) describes a projector $\bar{H}_i$ where Ker $\bar{H}_i = C\tilde{B}_i$ (the same nullspace as $\hat{H}_i$) and with the properties $\bar{H}_i^2 = \bar{H}_i$ and $V^{-1}\bar{H}_i = \bar{H}_i^T V^{-1} \bar{H}_i$ [37]. Next, by substituting (4.28) and (4.29) into (4.23) and using the dynamic constraint (4.16),

$$0 = \tilde{\Upsilon}_i \epsilon_i^* - \left[S_i L - S_i \hat{B}_i \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i\right)^{-1} \tilde{B}_i^T C^T V^{-1} - C^T \bar{H}_i^T V^{-1} \bar{H}_i\right] r_i^* \tag{4.32}$$

where

$$\begin{aligned} \tilde{\Upsilon}_i = \dot{S}_i &+ S_i \left(A - \hat{B}_i \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i\right)^{-1} \tilde{B}_i^T C^T V^{-1} C\right) + S_i \hat{B}_i \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i\right)^{-1} \hat{B}_i^T S_i \\ &+ \left(A - \hat{B}_i \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i\right)^{-1} \tilde{B}_i^T C^T V^{-1} C\right)^T S_i + C^T \left(\hat{H}_i Q_i \hat{H}_i - \bar{H}_i^T V^{-1} \bar{H}_i\right) C. \end{aligned} \tag{4.33}$$

The Riccati term (4.33) is identical to that of the asymptotic single-fault problem [37]. The necessary conditions (4.30) and (4.32) may be combined and simplified using $\bar{H}_i C \tilde{B}_i = 0$ as

$$0 = \Upsilon_i \epsilon_i^* - S_i \left(LC\tilde{B}_i - \hat{B}_i \right) \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i \right)^{-1} \tilde{B}_i^T C^T V^{-1} r_i^*. \tag{4.34}$$

where

$$\Upsilon_i(S_i, L, t) = \tilde{\Upsilon}_i + S_i \left(L - L_i^* \right) V \left(L - L_i^* \right)^T S_i \tag{4.35}$$

and

$$S_i L_i^* = S_i \hat{B}_i \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i \right)^{-1} \tilde{B}_i^T C^T V^{-1} + C^T \bar{H}_i^T V^{-1} \bar{H}_i. \tag{4.36}$$

The filter gain constraint (4.36) is identical to that of the asymptotic single-fault problem [37]. Finally, the optimal boundary conditions are obtained by substituting (4.28) into (4.24) and (4.25) and using (4.26), resulting in

$$S_i(t_0^+)\epsilon_i^*(t_0) = \left(\Pi_0 - \Pi_0 \bar{B}_i(t_0^+) \left[\bar{B}_i^T(t_0^+) \Pi_0 \bar{B}_i(t_0^+) \right]^{-1} \bar{B}_i^T(t_0^+) \Pi_0 \right) \epsilon_i^*(t_0) \tag{4.37}$$

$$S_i(t_1)\epsilon_i^*(t_1) = 0, \tag{4.38}$$

where the maximizing value of the initial state estimation error $\epsilon_i^*(t_0)$ is chosen such that $\epsilon_i^*(t_1) \in \text{Ker } S_i(t_1)$ in accordance with (4.38). The boundary constraints (4.37) and (4.38) are also identical to the boundary constraints of the single-fault problem [37]. Thus, like the general case, the Riccati term for each DAP of the asymptotic GTMFDF (4.35) is the Riccati of the single-fault problem with an added term to account for the difference between the optimal multiple-fault filter gain and the optimal single-fault filter gain.

Next, it is confirmed that the optimal cost is in fact equal to zero. As a preliminary step, the cost function (4.15) is converted into one that uses the necessary conditions more efficiently. The dynamic constraint (4.16) is appended to (4.15) using $\epsilon_i^T S_i$, which yields

$$J_i' = \int_{t_0}^{t_1} \left[\left\| \hat{H}_i C \epsilon_i \right\|_{Q_i}^2 - \left\| r_i - C \left(\epsilon_i + \tilde{B}_i \hat{\phi}_i \right) \right\|_{V^{-1}}^2 + \epsilon_i^T S_i \left(A\epsilon_i + \hat{B}_i \hat{\phi}_i - L r_i - \dot{\epsilon}_i \right) \right] \text{d}t$$
$$- \left\| \epsilon_i(t_0) + \bar{B}_i(t_0^+) \bar{\phi}_i(t_0^+) \right\|_{\Pi_0}^2 .$$

By integrating $\int_{t_0}^{t_1} \epsilon_i^T S_i \dot{\epsilon}_i \, \mathrm{d}t$ by parts, substituting (4.16), and collecting terms,

$$J_i' = \int_{t_0}^{t_1} \Big[\|\epsilon_i\|^2_{\dot{S}_i + S_i A + A^T S_i + C^T \hat{H}_i Q_i \hat{H}_i C} + \epsilon_i^T S_i \Big(\hat{B}_i \hat{\phi}_i - L r_i \Big) + \Big(\hat{B}_i \hat{\phi}_i - L r_i \Big)^T S_i \epsilon_i$$
$$- \Big\| r_i - C \left(\epsilon_i + \tilde{B}_i \hat{\phi}_i \right) \Big\|^2_{V^{-1}} \Big] \, \mathrm{d}t - \big\| \epsilon_i(t_0) + \bar{B}_i(t_0^+) \bar{\phi}_i(t_0^+) \big\|^2_{\Pi_0} - \|\epsilon_i(\tau)\|^2_{S_i(\tau)} \Big|_{\tau=t_0}^{t_1}.$$

Then, by adding and subtracting $\int_{t_0}^{t_1} \|\hat{B}_i^T S_i \epsilon_i + \tilde{B}_i^T C^T V^{-1}(r_i - C\epsilon_i)\|^2_{(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i)^{-1}} \, \mathrm{d}t$, expanding $\int_{t_0}^{t_1} \|r_i - C(\epsilon_i + \tilde{B}_i \hat{\phi}_i)\|^2_{V^{-1}} \, \mathrm{d}t$, and collecting terms

$$J_i' = \int_{t_0}^{t_1} \Big[\|\epsilon_i\|^2_{\dot{S}_i + S_i A + A^T S_i + C^T \left(\hat{H}_i Q_i \hat{H}_i - V^{-1} \right) C} + \Big\| \hat{B}_i^T S_i \epsilon_i + \tilde{B}_i^T C^T V^{-1} (r_i - C\epsilon_i) \Big\|^2_{\left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i \right)^{-1}}$$
$$- \|r_i\|^2_{V^{-1}} - \Big\| \hat{\phi}_i - \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i \right)^{-1} \left[\hat{B}_i^T S_i \epsilon_i + \tilde{B}_i^T C^T V^{-1} (r_i - C\epsilon_i) \right] \Big\|^2_{\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i}$$
$$- \epsilon_i^T (S_i L - C V^{-1}) \, r_i - r_i^T (S_i L - C V^{-1})^T \epsilon_i \Big] \mathrm{d}t - \big\| \epsilon_i(t_0) + \bar{B}_i(t_0^+) \bar{\phi}_i(t_0^+) \big\|^2_{\Pi_0} - \|\epsilon_i(\tau)\|^2_{S_i(\tau)} \Big|_{\tau=t_0}^{t_1}.$$

By expanding $\int_{t_0}^{t_1} \|\hat{B}_i^T S_i \epsilon_i + \tilde{B}_i^T C^T V^{-1}(r_i - C\epsilon_i)\|^2_{(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i)^{-1}} \, \mathrm{d}t$, adding and subtracting $\int_{t_0}^{t_1} \| (L - L_i^*)^T S_i \epsilon_i\|^2_V \, \mathrm{d}t$, and collecting terms

$$J_i' = \int_{t_0}^{t_1} \Big[\|\epsilon_i\|^2_{\Upsilon_i(S_i, L, t)} - \big\| r_i + V (L - L_i^*)^T S_i \epsilon_i \big\|^2_{V^{-1}} + \|r_i\|^2_{V^{-1} - \bar{H}_i^T V^{-1} \bar{H}_i}$$
$$- \Big\| \hat{\phi}_i - \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i \right)^{-1} \left[\hat{B}_i^T S_i \epsilon_i + \tilde{B}_i^T C^T V^{-1} (r_i - C\epsilon_i) \right] \Big\|^2_{\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i} \Big] \, \mathrm{d}t$$
$$- \big\| \epsilon_i(t_0) + \bar{B}_i(t_0^+) \bar{\phi}_i(t_0^+) \big\|^2_{\Pi_0} - \|\epsilon_i(\tau)\|^2_{S_i(\tau)} \Big|_{\tau=t_0}^{t_1} \tag{4.39}$$

Finally, we show that the optimal cost is equal to zero. To prove this, substitute (4.29)-(4.31) into (4.39) to obtain

$$J_i' = \int_{t_0}^{t_1} \Big[\|\epsilon_i\|^2_{\Upsilon_i(S_i, L, t)} \Big] \, \mathrm{d}t - \big\| \epsilon_i(t_0) + \bar{B}_i(t_0^+) \bar{\phi}_i(t_0^+) \big\|^2_{\Pi_0} - \|\epsilon_i(\tau)\|^2_{S_i(\tau)} \Big|_{\tau=t_0}^{t_1}$$

Then, substitute (4.27), (4.34), and the boundary conditions (4.26), (4.37), and (4.38) to obtain $J_i' = 0$. Therefore, the optimal value of the cost satisfies Problem 4.1.

Sufficient Conditions for Optimality

In order to obtain sufficient conditions to satisfy the DAP (4.1), sufficient conditions for optimality are generated from the second-order variation of the cost (4.39). The second-order variation of (4.39) may be written as

$$\delta^2 J_i' = \int_{t_0}^{t_1} \begin{bmatrix} \delta\epsilon_i^T & \delta r_i^T & \delta\hat{\phi}_i^T \end{bmatrix} \hat{\Upsilon}_i \begin{bmatrix} \delta\epsilon_i \\ \delta r_i \\ \delta\hat{\phi}_i \end{bmatrix} \mathrm{d}t - \begin{bmatrix} \delta\epsilon_i^T(t_0) & \delta\bar{\phi}_i^T(t_0^+) & \delta\epsilon_i^T(t_1) \end{bmatrix} \bar{\Upsilon}_i \begin{bmatrix} \delta\epsilon_i(t_0) \\ \delta\bar{\phi}_i(t_0^+) \\ \delta\epsilon_i(t_1) \end{bmatrix},$$

where, so that r_i^*, $\hat{\phi}_i^*$, $\bar{\phi}_i^*(t_0^+)$, and $\epsilon_i^*(t_0)$ maximize the cost (i.e. $\delta^2 J_i' \leq 0$),

$$\hat{\Upsilon}_i = \begin{bmatrix} \dot{S}_i + S_i A + A^T S_i + C^T\left(\hat{H}_i Q_i \hat{H}_i - V^{-1}\right) C & C^T V^{-1} - S_i L & S_i \hat{B}_i - C^T V^{-1} C \tilde{B}_i \\ V^{-1} C - L^T S_i & -V^{-1} & V^{-1} C \tilde{B}_i \\ \hat{B}_i^T S_i - \tilde{B}_i^T C^T V^{-1} C & \tilde{B}_i^T C^T V^{-1} & -\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i \end{bmatrix} \leq 0 \tag{4.40}$$

$$\bar{\Upsilon}_i = \begin{bmatrix} \Pi_0 - S_i(t_0^+) & \Pi_0 \bar{B}_i(t_0^+) & 0 \\ \bar{B}_i^T(t_0^+)\Pi_0 & \bar{B}_i^T(t_0^+)\Pi_0 \bar{B}_i(t_0^+) & 0 \\ 0 & 0 & S_i(t_1) \end{bmatrix} \geq 0. \tag{4.41}$$

By applying the Schur complement formula for semi-definite matrices

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \leq 0,\ V_{22} < 0 \Leftrightarrow V_{11} - V_{12} V_{22}^{-1} V_{21} \leq 0,\ V_{22} < 0 \tag{4.42}$$

to (4.40) and using (4.33) and (4.36), we obtain

$$\begin{bmatrix} \tilde{\Upsilon}_i & S_i\left(L_i^* - L\right) \\ \left(L_i^* - L\right)^T S_i & -\bar{H}_i^T V^{-1} \bar{H}_i \end{bmatrix} \leq 0. \tag{4.43}$$

Thus, (4.43) is the matrix inequality form of the sufficient condition for optimality.

Since the bottom-right term of (4.43) is not full rank, the Schur complement formula (4.42) cannot be used directly to simplify the inequality. However, it can still be rewritten in a form similar to the Riccati inequality constraint (3.31) by using Theorem 4.1 below.

Theorem 4.1: The sufficient condition for optimality (4.43) for the i^{th} asymptotic DAP is equivalent to

$$0 = S_i \left(\hat{B}_i - LC\tilde{B}_i \right) \tag{4.44}$$

$$0 \geq \Upsilon_i(S_i, L, t) \tag{4.45}$$

where Υ_i is as defined in (4.35). ∎

Proof: See Section 4.4.1. **QED**

Further, by applying (4.42) twice consecutively to the sufficient conditions for optimality (4.41) at the boundaries,

$$S_i(t_0^+) \leq \Pi_0 - \Pi_0 \bar{B}_i(t_0^+) \left[\bar{B}_i^T(t_0^+) \Pi_0 \bar{B}_i(t_0^+) \right]^{-1} \bar{B}_i^T(t_0^+) \Pi_0 \tag{4.46}$$

$$S_i(t_1) \geq 0. \tag{4.47}$$

Therefore, the sufficient conditions for optimality, and thus the solution constraints for Problem 4.1, have two components: an equality condition (4.44) and a matrix Riccati inequality condition (4.45) with boundary conditions (4.46) and (4.47). The inequality condition is similar to that of the general case where $\gamma > 0$. Any solution satisfying these conditions $\forall i \in \{1, \ldots, s\}$ places each complementary fault into its associated detection space so that it may be blocked from the associated projected residual.

4.2.2 Asymptotic Detection Filter Structure

In this section, the invariant subspace structure of the asymptotic GTMFDF is discussed. The main result is that Ker S_i forms an invariant subspace that contains the minimal (C, A)-invariant subspace associated with the complementary fault $\hat{\mu}_i$. Therefore, in the absence of sensor noise and the target fault μ_i, if the state estimation error starts in Ker S_i it will remain there even when $\hat{\mu}_i$ occurs. Further, Ker S_i is unobservable to the associated projected residual of the detection filter.

First, given the sufficient conditions for optimality (4.44)-(4.47), Theorem 4.2 proves that the minimal (C, A)-invariant subspace $\hat{\mathcal{W}}_i^*$ associated with $\hat{F}_i$ is contained in Ker S_i. Next, Theorem 4.3 proves that when the complementary fault occurs, in the absence of the i^{th} target fault and sensor noise, the i^{th} state estimation error will remain in Ker S_i. Since the theorem also shows CKer $S_i = \text{Im } C\tilde{B}_i$, Ker S_i is unobservable to $(\hat{H}_i C, A - L^* C)$.

Theorem 4.2: Assume that $S_i \geq 0$ for the entire interval $[t_0, t_1]$. Then, given the sufficient conditions for optimality (4.44)-(4.47), $S_i \bar{B}_i = 0$. ▮

Proof: See Section 4.4.2. **QED**

Theorem 4.3: When the complementary fault $\hat{\mu}_i$ occurs, the state error remains in Ker S_i and the residual remains in CKer $S_i = \text{Im } C\tilde{B}_i$ in the absence of sensor noise. ▮

Proof: See Section 4.4.3. **QED**

Together, Theorems 4.2 and 4.3 imply that Ker S_i forms a (C, A)-invariant subspace that contains the complementary fault direction and is unobservable to $(\hat{H}_i C, A - L^* C)$. Therefore, the asymptotic GTMFDF blocks $\hat{\mu}_i$ from the projected residual $\bar{r}_i$, extending the behavior of fault detection filters derived via spectral and geometric theories to the finite time-varying case.

4.2.3 Analytical Solution

Next, this section focuses on a specific analytical solution to Problem 4.1, for which we are able to derive sufficient conditions for existence, subject to (4.2), (4.35), and the constraints (4.44)-(4.47). Assume that there exists $L = L^*$ such that

$$S_i L^* = S_i \hat{B}_i \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i \right)^{-1} \tilde{B}_i^T C^T V^{-1} + C^T \bar{H}_i^T V^{-1} \bar{H}_i \tag{4.48}$$

$$0 = \Upsilon_i \left(S_i, L^*, t \right) \tag{4.49}$$

subject to the boundary conditions

$$S_i(t_0^+) = \Pi_0 - \Pi_0 \bar{B}_i(t_0^+) \left[\bar{B}_i^T(t_0^+) \Pi_0 \bar{B}_i(t_0^+) \right]^{-1} \bar{B}_i^T(t_0^+) \Pi_0 \tag{4.50}$$

$$S_i(t_1) \geq 0 \tag{4.51}$$

$\forall i \in \{1, \ldots, s\}$ where $\Pi_0 > 0$. Note that L^* is identical to the optimal solution for the single-fault GTFDF (4.36) for each individual DAP. However, it must now satisfy the constraint for multiple DAPs simultaneously. Substituting (4.48) into (4.44), we find that L^* satisfies the equality constraint of the sufficient conditions for optimality. Further, by substituting (4.48) into (4.35), the Riccati constraint (4.49) becomes

$$\begin{aligned} -\dot{S}_i = {} & S_i \left(A - L^* C \right) + \left(A - L^* C \right)^T S_i + S_i \hat{B}_i \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i \right)^{-1} \hat{B}_i^T S_i \\ & + C^T \left(\hat{H}_i Q_i \hat{H}_i + \bar{H}_i^T V^{-1} \bar{H}_i \right) C, \end{aligned} \tag{4.52}$$

which is the Riccati constraint for the GTFDF.

Therefore, if L^* exists, it is a solution to Problem 4.1. Note, however, that L^* is not unique, nor is it guaranteed to exist in general. Sufficient conditions for the existence of the filter gain are discussed in the next section.

4.2.4 Filter Gain Existence Conditions

In this section, the existence conditions for the finite time-varying detection filter of Section 4.2.3 are discussed. First, an additional assumption for finite time-varying systems is presented to simplify the asymptotic detection filter problem. Then, the central theorems of the section are derived. It is shown that the simplest condition requires that there be no complementary subspace. Finally, the implied problem constraints are discussed.

In order to obtain a simple solution to the time-varying detection filter problem, one additional assumption (on top of Assumptions 3.1-3.2) is required:

Assumption 4.1: The dimensions of both Ker S_i and $\hat{\mathcal{W}}_i^*$ are fixed over $t = (t_0, t_1]\ \forall\ i \in \{1, \ldots, s\}$. ♣

In this case, by using (4.50) where $\Pi_0 > 0$, which implies

$$\text{Ker } S_i(t_0^+) = \text{Im } \bar{B}_i(t_0^+) = \hat{\mathcal{W}}_i^*(t_0^+),$$

it may be assumed that

$$\text{Ker } S_i = \hat{\mathcal{W}}_i^* \tag{4.53}$$

for the entire time interval. Assumption 4.1 is used throughout the remainder of the chapter.

Theorem 4.4 below gives a sufficient condition for the existence of the asymptotic filter gain constrained by (4.48).

Theorem 4.4: Assume that S_i, $\forall\ i \in \{1, \ldots, s\}$, is a solution to (4.52) with boundary conditions (4.50) and (4.51). If

$$\text{Im } S_i \bigcap \text{Im } S_j = 0,\ i \neq j,\ \forall i, j \in \{1, \ldots, s\}, \tag{4.54}$$

then $\exists\ L^*$ subject to (4.48) $\forall\ i \in \{1, \ldots, s\}$. ∎

Proof: See Section 4.4.4. **QED**

Therefore, to ensure that the asymptotic filter gain solution exists, we must determine if and when the problem may be converted into one such that (4.54) is satisfied. This requires a preliminary step as the geometric structure of the detection filter must be discussed further.

Similar to state space geometry of the infinite time-invariant case (2.14), that of the finite time-varying case is described by

$$\mathbb{R}^n = \mathcal{W}_1^* \oplus \ldots \oplus \mathcal{W}_s^* \oplus \hat{\mathcal{W}}^* \oplus \mathcal{C}, \tag{4.55}$$

where the complementary subspace $\mathcal{C}$ is defined such that

$$\zeta \in \mathcal{C},\ \xi \in \mathcal{W}_i^* \oplus \hat{\mathcal{W}}_i^* \Rightarrow \zeta \perp \xi. \tag{4.56}$$

Recall that invariant zeros are not defined for finite time-varying systems, and so they are not included explicitly in (4.55). However, $\mathcal{C}$ may contain directions that satisfy the invariant zero equation (2.12) in the finite time-varying case during a subinterval of $[t_0, t_1]$.

Denote the minimal (C, A)-invariant subspace of a new complementary fault direction $\tilde{F}_i$ as $\tilde{\mathcal{W}}_i^*$ and that of a new nuisance fault direction $\tilde{F}$ as $\tilde{\mathcal{W}}^*$. Further, define $\tilde{S}_i \geq 0$ as a solution to (4.52) given the boundary conditions (4.50) and (4.51) for this new set of faults. Let the faults be chosen such that $\tilde{S}_i$ satisfies (4.54) $\forall\ i, j \in \{1, \ldots, s\}$. Theorem 4.5 reveals an important implication of (4.54) to the state space geometry.

Theorem 4.5: $\tilde{S}_i$ and $\tilde{S}_j$, $i \neq j$, satisfy (4.54) $\forall\ i, j \in \{1, \ldots, s\}$ if and only if there is no complementary subspace. ∎

Proof: See Section 4.4.5. **QED**

By applying Theorem 4.5 to (4.55), the minimal (C, A)-invariant subspaces (equivalent to the detection spaces for time-varying systems) of the target and nuisance faults span the entire state space, such that

$$\mathbb{R}^n = \mathcal{W}_1^* \oplus \ldots \oplus \mathcal{W}_s^* \oplus \tilde{\mathcal{W}}^*. \tag{4.57}$$

However, in general, the state space may contain a complementary subspace. Then, it is necessary to augment the set of nuisance fault directions so that (4.54) holds. To this end, choose $\tilde{F}$ such that $\tilde{\mathcal{W}}^*$ includes both the detection space of $\hat{F}$ as well as $\mathcal{C}$, i.e.

$$\tilde{\mathcal{W}}^* = \hat{\mathcal{W}}^* \oplus \mathcal{C}. \tag{4.58}$$

In this case, (4.55) and (4.57) are clearly equivalent.

In order to prevent target faults from being blocked along with the modified nuisance fault, $\tilde{F}$ must be (C, A)-output separable from the target faults over the time interval $[t_0, t_1]$. Further, $\hat{\mathcal{W}}^*$ must be (C, A)-invariant satisfying (3.4) over the time interval. Otherwise, when $\tilde{\mathcal{W}}^*$ is truncated from the detection filter state space, the invariant subspace structure will be destroyed, leading to incorrect fault isolation.

Therefore, when the nuisance fault subspace and complementary subspace[1] combine to form a single (C, A)-invariant subspace that is linearly independent from the target fault subspaces, the asymptotic detection filter gain L^* and Riccati solutions $\tilde{S}_i$ satisfy (4.48) $\forall\ i \in \{1, \ldots, s\}$. Then, since $\tilde{S}_i \geq 0$ over the time interval, these constraint equations may be summed to generate a single constraint on the asymptotic filter gain

$$\left(\sum_{i=1}^{s} \tilde{S}_i\right) L^* = \sum_{i=1}^{s} \left[\tilde{S}_i \hat{B}_i \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i\right)^{-1} \tilde{B}_i^T C^T V^{-1} + C^T \bar{H}_i^T V^{-1} \bar{H}_i\right]. \tag{4.59}$$

However, it is possible that (4.59) specifies L^* in only a subspace of the state space since, from Theorem 4.2, $\tilde{\mathcal{W}}^* \subseteq \text{Ker} \sum_{i=1}^{s} \tilde{S}_i$. Thus, the full-order detection filter cannot be implemented in general without a numerical optimization or pole placement techniques to

[1]Both subspaces may be obtained by examining the range spaces and nullspaces of the Riccati solutions S_i as discussed in Section 4.3.

define the undefined eigenstructure. In Section 4.3, a reduced-order state space and detection filter are obtained for which a simple solution for the filter gain exist.

Remark 4.2: Up to now, the GTMFDF has been discussed in the finite time-varying case. However, the RDDF was derived only for infinite time-invariant systems. Thus, in order to compare the GTMFDF directly to the RDDF, it is important to discuss the steady-state results for the GTMFDF in the limit as $\gamma \to 0$. Specifically of interest is whether the invariant zero directions of the system are included in the appropriate detection spaces generated by the Riccati solutions and if the associated invariant zeros become eigenvalues of the detection filter. Chung and Speyer proved in [37] that the directions associated with unstable invariant zeros were included in the detection space of the GTFDF generated by the Riccati solution. As it turns out, this is not the case for directions associated with stable invariant zeros (as is shown in the example in Section 6.2). Further, it was shown in [41] for the OSFDF that the invariant zeros or their mirror images over the imaginary axis become eigenvalues of the single-fault detection filter when γ is small. Thus, the existence of stable invariant zeros becomes a limiting factor in obtaining solutions for the asymptotic Riccati-based detection filters. However, as discussed in [44], the invariant zero directions may be artificially augmented onto the associated faults' directions, eliminating the invariant zeros.

On the other hand, it is also shown in Section 6.2 that neither invariant zeros nor their mirror images become eigenvalues of the multiple-fault detection filter. Examining the derivation of the asymptotic GTMFDF, the eigenstructure of the multiple-fault filter is a combination of the eigenstructures of single-fault filters. It is believed that this combination moves the eigenvalues away from the invariant zeros. However, the exact mechanism that produces this result has yet to be determined. $\diamondsuit$

4.3 The Reduced-Order Asymptotic Detection Filter

So that the detection filter blocks each complementary fault from its associated projected residual while allowing the target fault to remain observable, the detection filter gain L^* must be chosen such that it satisfies (4.59). Further, for simplicity, the solution should be determined analytically. However, when projected residuals contain a common unobservable subspace, as is the case when there are nuisance faults, (4.48) cannot fully specify L^*. In [20], the unconstrained dimensions of the filter gain were specified via an arbitrary matrix chosen using pole-placement techniques. However, these dimensions have no effect on the detection filter problem other than to ensure dynamic stability. Alternatively, a reduced-order detection filter may be obtained by truncating the common unobservable subspace from the state space, similar to the single-fault case [37]. Since the detection filter is an observer, removing the unobservable subspace minimally affects the behavior of detection filter with respect to target faults (assuming that the system is modeled approximately correctly). Further, Theorem 4.3 shows that Ker S_i is contained in the unobservable subspace of the associated projected residual. Therefore, the common nullspace of the Riccati solutions S_i is unobservable to all projected residuals and may be truncated from the state space so that (4.59) fully specifies a reduced-order filter gain.

In this section, the reduced-order asymptotic detection filter is obtained as follows. First, the reduced-order problem is formulated in Section 4.3.1 by truncating the minimal (C, A)-invariant subspace associated with the nuisance fault from the state space. Then, the reduced-order detection filter gain is obtained in Section 4.3.2.

4.3.1 Problem Formulation

In this section, the reduced-order asymptotic detection filter problem is derived. Since $\tilde{S}_i \geq 0$ has a nontrivial nullspace, a nonsingular transformation T_0 may be chosen such that

$$\tilde{S}_i = T_0^T \begin{bmatrix} \bar{S}_i & 0 \\ 0 & 0 \end{bmatrix} T_0 \tag{4.60}$$

where $\bar{S}_i \geq 0$. Let T_0 be constructed as

$$T_0 = \begin{bmatrix} T_{0,1} & T_{0,2} \end{bmatrix}^{-1} \tag{4.61}$$

where $T_{0,2}$ is a set of linearly independent vectors such that

$$\text{Im } T_{0,2} = \text{Ker} \begin{bmatrix} \tilde{S}_1 \\ \vdots \\ \tilde{S}_s \end{bmatrix} = \bigcap_{i=1}^{s} \text{Ker } \tilde{S}_i \tag{4.62}$$

and $T_{0,1}$ is composed of $n - \dim(T_{0,2})$ vectors that are linearly independent from $T_{0,2}$ so that T_0 is full rank. By applying (3.5) to (4.53) and using $\tilde{S}_i \geq 0$, (4.62) may be rewritten as

$$\text{Im } T_{0,2} = \text{Ker} \sum_{i=1}^{s} \tilde{S}_i = \hat{\mathcal{W}}^*. \tag{4.63}$$

Further, by using (4.62),

$$\begin{aligned} T_0^{-T} \tilde{S}_i T_0^{-1} &= \begin{bmatrix} T_{0,1}^T \\ T_{0,2}^T \end{bmatrix} \tilde{S}_i^{1/2} \tilde{S}_i^{1/2} \begin{bmatrix} T_{0,1} & T_{0,2} \end{bmatrix} = \begin{bmatrix} T_{0,1}^T \tilde{S}_i^{1/2} \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{S}_i^{1/2} T_{0,1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} T_{0,1}^T \tilde{S}_i T_{0,1} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus, T_0 satisfies (4.60).

Pre-multiplying (4.48) by T_0^{-T} and using $T_0^{-1}T_0 = I$,

$$\begin{bmatrix} \bar{S}_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L_1^* \\ L_2^* \end{bmatrix} = \begin{bmatrix} \bar{S}_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{B}_{i,1} \\ \hat{B}_{i,2} \end{bmatrix} \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i\right)^{-1} \tilde{B}_i^T C^T V^{-1} + \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} \bar{H}_i^T V^{-1} \bar{H}_i \tag{4.64}$$

where

$$T_0 L^* = \begin{bmatrix} L_1^* \\ L_2^* \end{bmatrix}, \quad T_0 \hat{B}_i = \begin{bmatrix} \hat{B}_{i,1} \\ \hat{B}_{i,2} \end{bmatrix}, \quad C T_0^{-1} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

From (4.64), two equations emerge:

$$\bar{S}_i L_1^* = \bar{S}_i \hat{B}_{i,1} \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i\right)^{-1} \tilde{B}_i^T C^T V^{-1} + C_1^T \bar{H}_i^T V^{-1} \bar{H}_i \tag{4.65}$$

$$0 = \bar{H}_i^T V^{-1} \bar{H}_i C_2. \tag{4.66}$$

Eqn. (4.65) is the constraint equation for the reduced-order detection filter gain L_1^*, while (4.66) shows how the transformation isolates the common unobservable subspace Im $T_{0,2}$. Therefore, the reduced-order asymptotic detection filter problem is:

Problem 4.2: Find L_1^* that satisfies (4.65) $\forall\, i \in \{1, \ldots, s\}$. ♡

In the next section, a solution to Problem 4.2 is derived.

4.3.2 Reduced-Order Detection Filter Solution

In this section, the reduced-order asymptotic detection filter is obtained. By summing (4.65) over $i = 1, \ldots, s$,

$$\sum_{i=1}^{s} \bar{S}_i L_1^* = \sum_{i=1}^{s} \bar{S}_i \hat{B}_{i,1} \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i\right)^{-1} \tilde{B}_i^T C^T V^{-1} + C_1^T \bar{H}_i^T V^{-1} \bar{H}_i. \tag{4.67}$$

Note that (4.67) is simply the reduced order form of (4.59). By pre-multiplying (4.67) by $\left(\sum_{i=1}^{s} \bar{S}_i\right)^{-1}$ (which exists according to Proposition 4.6 in Section 4.4.6), the reduced-order detection filter gain solution is obtained as

$$L_1^* = \left(\sum_{i=1}^{s} \bar{S}_i\right)^{-1} \left(\sum_{i=1}^{s} \bar{S}_i \hat{B}_{i,1} \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i\right)^{-1} \tilde{B}_i^T C^T V^{-1} + C_1^T \bar{H}_i^T V^{-1} \bar{H}_i\right). \tag{4.68}$$

The detection filter itself is obtained using a similar reduced-order structure. First, a Riccati differential equation is derived for $\bar{S}_i$. By pre-multiplying (4.52) by T_0^{-T}, post-multiplying by T_0^{-1}, and using (4.66),

$$\begin{aligned} -\dot{\bar{S}}_i &= \bar{S}_i \left(A_{1,1} - L_1^* C_1\right) + \left(A_{1,1} - L_1^* C_1\right)^T \bar{S}_i + \bar{S}_i \hat{B}_{i,1} \left(\tilde{B}_i^T C^T V^{-1} C \tilde{B}_i\right)^{-1} \hat{B}_{i,1}^T \bar{S}_i \\ &\quad + C_1^T \left(\hat{H}_i Q_i \hat{H}_i + \bar{H}_i^T V^{-1} \bar{H}_i\right) C_1 && (4.69) \\ 0 &= \bar{S}_i \left(A_{1,2} - L_1^* C_2\right) && (4.70) \end{aligned}$$

where

$$T_0 A T_0^{-1} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}.$$

Eqn. (4.69) is the Riccati differential equation for $\bar{S}_i$, while (4.70) is a constraint so that the nuisance fault remains unobservable to all projected residuals. The boundary conditions for (4.69) are obtained by applying the transformation to (4.50) and (4.51), resulting in

$$\bar{S}_i(t_0^+) = T_{0,1}^T \tilde{S}_i(t_0^+) T_{0,1} \tag{4.71}$$

$$\bar{S}_i(t_1) \geq 0. \tag{4.72}$$

Next, the reduced-order detection filter equations are derived. Pre-multiplying (3.7a) by $\tilde{S}_i$

and summing over $i = 1, \ldots, s$,

$$\sum_{i=1}^{s} \tilde{S}_i \dot{\hat{x}} = \sum_{i=1}^{s} \tilde{S}_i \left[(A - LC)\, \hat{x} + Bu - Ly \right]. \tag{4.73}$$

Then, by pre-multiplying by T_0^{-T}, (4.73) reduces to

$$\sum_{i=1}^{s} \bar{S}_i \dot{\hat{x}}_1 = \sum_{i=1}^{s} \bar{S}_i \left[(A_{1,1} - L_1 C_1)\, \hat{x}_1 + (A_{1,2} - L_1 C_2)\, \hat{x}_2 + B_1 u - L_1 y \right] \tag{4.74}$$

where

$$T_0 \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}, \; T_0 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Finally, by substituting $L_1 = L_1^*$ into (4.74), using (4.68) and (4.70), and pre-multiplying by $\left(\sum_{i=1}^{s} \bar{S}_i \right)^{-1}$,

$$\dot{\hat{x}}_1 = A_{1,1} \hat{x}_1 + B_1 u - L_1^* \left(y - C_1 \hat{x}_1 \right). \tag{4.75}$$

Thus, (4.75) is the reduced-order asymptotic detection filter subject to (4.69) with boundary conditions (4.71) and (4.72) where, by using Theorem 4.4, (4.68) is a solution to Problem 4.2.

4.4 Proofs of Theorems and Propositions

4.4.1 Proof of Theorem 4.1

($\Rightarrow$): First, recall that Ker $\hat{H}_i =$ Ker $\bar{H}_i = C\tilde{B}_i$. Then,

$$0 = \begin{bmatrix} 0 & \tilde{B}_i^T C^T \end{bmatrix} \begin{bmatrix} \tilde{\Upsilon}_i & S_i \left(L_i^* - L \right) \\ \left(L_i^* - L \right)^T S_i & -\bar{H}_i^T V^{-1} \bar{H}_i \end{bmatrix} \begin{bmatrix} 0 \\ C\tilde{B}_i \end{bmatrix},$$

which implies

$$0 = \begin{bmatrix} \tilde{\Upsilon}_i & S_i \left(L_i^* - L \right) \\ \left(L_i^* - L \right)^T S_i & -\bar{H}_i^T V^{-1} \bar{H}_i \end{bmatrix} \begin{bmatrix} 0 \\ C\tilde{B}_i \end{bmatrix}$$

since the matrix in (4.43) is nonpositive. Therefore, by using (4.48) we obtain

$$0 = S_i \left(L_i^* - L\right) C\tilde{B}_i = S_i \left(\hat{B}_i - LC\tilde{B}_i\right). \tag{4.76}$$

Next, (4.43) also implies

$$\begin{aligned} 0 &\geq \begin{bmatrix} I & S_i \left(L_i^* - L\right) V \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{\Upsilon}_i & S_i \left(L_i^* - L\right) \\ \left(L_i^* - L\right)^T S_i & -\bar{H}_i^T V^{-1} \bar{H}_i \end{bmatrix} \begin{bmatrix} I & 0 \\ V \left(L_i^* - L\right)^T S_i & I \end{bmatrix} \\ &\geq \begin{bmatrix} \Upsilon_i & S_i \left(L_i^* - L\right) \left(I - \bar{H}_i\right) \\ \left(L_i^* - L\right)^T S_i & -\bar{H}_i^T V^{-1} \bar{H}_i \end{bmatrix} \begin{bmatrix} I & 0 \\ V \left(L_i^* - L\right)^T S_i & I \end{bmatrix}. \end{aligned}$$

By substituting (4.31) and (4.76),

$$\begin{aligned} 0 &\geq \begin{bmatrix} \Upsilon_i & 0 \\ \left(L_i^* - L\right)^T S_i & -\bar{H}_i^T V^{-1} \bar{H}_i \end{bmatrix} \begin{bmatrix} I & 0 \\ V \left(L_i^* - L\right)^T S_i & I \end{bmatrix} \\ &\geq \begin{bmatrix} \Upsilon_i & 0 \\ \left(I - \bar{H}_i\right)^T \left(L_i^* - L\right)^T S_i & -\bar{H}_i^T V^{-1} \bar{H}_i \end{bmatrix} \\ &\geq \begin{bmatrix} \Upsilon_i & 0 \\ 0 & -\bar{H}_i^T V^{-1} \bar{H}_i \end{bmatrix}. \end{aligned}$$

Therefore,

$$0 \geq \Upsilon_i.$$

($\Leftarrow$): Sufficiency is proven by performing the steps above in reverse.

4.4.2 Proof of Theorem 4.2

First, by pre-multiplying (4.50) by $\bar{B}_i^T(t_0^+)$ and post-multiplying by $\bar{B}_i(t_0^+)$,

$$0 \geq \bar{B}_i^T(t_0^+) S_i(t_0^+) \bar{B}_i(t_0^+).$$

Since $S_i \geq 0$ by assumption, this implies

$$S_i(t_0^+)\bar{B}_i(t_0^+) = 0. \tag{4.77}$$

Next, the sufficient conditions (4.44) and (4.45) are rewritten in a more useful form for this analysis. By substituting $\hat{B}_i = A\tilde{B}_i - \dot{\tilde{B}}_i$ from Algorithm 4.1 into (4.44),

$$S_i(A - LC)\tilde{B}_i = S_i\dot{\tilde{B}}_i. \tag{4.78}$$

Then, by substituting (4.35) and (4.44) into (4.45), the Riccati inequality constraint is simplified to

$$0 \geq \dot{S}_i + S_i\left(A - LC\right) + \left(A - LC\right)^T S_i + S_i LVL^T S_i + C^T\hat{H}_i Q_i \hat{H}_i C. \tag{4.79}$$

Further, it can be shown by using Algorithm 4.1 that

$$\text{Im } \bar{B}_i = \text{Im}\left[\begin{array}{ccccc} \tilde{B}_i & \tilde{B}_{i,2}^{\hat{\beta}_i - 1} & \ldots & \tilde{B}_{i,2}^1 & \tilde{B}_{i,2}^0 \end{array}\right] = \text{Im}\left[\begin{array}{ccccc} \hat{B}_{i,2}^{\hat{\beta}_i} & \hat{B}_{i,2}^{\hat{\beta}_i - 1} & \ldots & \hat{B}_{i,2}^1 & \hat{F}_i \end{array}\right]$$

where $\text{Im } C\bar{B}_i = \text{Im } C\tilde{B}_i$. By using (4.18), proving the theorem is equivalent to proving $\text{Im } \bar{B}_i \subseteq \text{Ker } S_i$.

Pre-multiplying (4.79) by $\tilde{B}_i^T$ and post-multiplying by $\tilde{B}_i$, substituting (4.78), and using $\hat{H}_i C\tilde{B}_i = 0$ from (3.10),

$$0 \geq \frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{B}_i^T S_i \tilde{B}_i\right) + \tilde{B}_i^T S_i LVL^T S_i \tilde{B}_i. \tag{4.80}$$

Using (4.77), let there exist a vector ν such that $L^T(\tau^+)S_i(\tau^+)\tilde{B}_i(\tau^+)\nu \neq 0$ where $\tau > t_0$ and $S_i(t)\tilde{B}_i(t)\nu = 0 \; \forall t \leq \tau$. In this case, (4.80) and $V > 0$ imply

$$0 > \frac{\mathrm{d}}{\mathrm{d}t}\left[\nu^T \tilde{B}_i^T(\tau^+)S_i(\tau^+)\tilde{B}_i(\tau^+)\nu\right],$$

which implies that $\nu^T \tilde{B}_i^T S_i \tilde{B}_i \nu < 0$ at some time $t > \tau$. This contradicts the assumption that $S_i \geq 0$. Therefore, $\tilde{B}_i \in \text{Ker } S_i$ over the time interval $(t_0, t_1]$. Further, note that since $\tilde{B}_i = \begin{bmatrix} \tilde{B}_{i,1}^{\hat{\beta}_i - 1} & \hat{B}_{i,2}^{\hat{\beta}_i} \end{bmatrix}$, $S_i \hat{B}_{i,2}^{\hat{\beta}_i} = 0$.

Next, by pre-multiplying (4.79) by $\left(\tilde{B}_{i,2}^{\hat{\beta}_i-1}\right)^T$ and post-multiplying by $\tilde{B}_{i,2}^{\hat{\beta}_i-1}$, substituting $\hat{B}_{i,2}^{\hat{\beta}_i} = A\tilde{B}_{i,2}^{\hat{\beta}_i-1} - \dot{\tilde{B}}_{i,2}^{\hat{\beta}_i-1}$ from Algorithm 4.1, and using $S_i \hat{B}_{i,2}^{\hat{\beta}_i} = 0$ and $\tilde{B}_{i,2}^{\hat{\beta}_i-1} \in \text{Ker } C$

$$0 \geq \frac{\mathrm{d}}{\mathrm{d}t} \left[\left(\tilde{B}_{i,2}^{\hat{\beta}_i-1}\right)^T S_i \tilde{B}_{i,2}^{\hat{\beta}_i-1} \right] + \left(\tilde{B}_{i,2}^{\hat{\beta}_i-1}\right)^T S_i L V L^T S_i \tilde{B}_{i,2}^{\hat{\beta}_i-1}. \tag{4.81}$$

Using (4.77), let there exist a vector ν such that $L^T(\tau^+) S_i(\tau^+) \tilde{B}_{i,2}^{\hat{\beta}_i-1}(\tau^+)\nu \neq 0$ where $\tau > t_0$ and $S_i(t)\tilde{B}_{i,2}^{\hat{\beta}_i-1}(t)\nu = 0 \ \forall t \leq \tau$. In this case, (4.81) and $V > 0$ imply

$$0 > \frac{\mathrm{d}}{\mathrm{d}t} \left[\nu^T \left(\tilde{B}_{i,2}^{\hat{\beta}_i-1}(\tau^+)\right)^T S_i(\tau^+) \tilde{B}_{i,2}^{\hat{\beta}_i-1}(\tau^+)\nu \right],$$

which implies that $\nu^T \left(\tilde{B}_{i,2}^{\hat{\beta}_i-1}\right)^T S_i \tilde{B}_{i,2}^{\hat{\beta}_i-1}\nu < 0$ at some time $t > \tau$. This contradicts the assumption that $S_i \geq 0$. Therefore, by combining $S_i \tilde{B}_{i,2}^{\hat{\beta}_i-1}\nu = 0$ with $S_i \tilde{B}_{i,1}^{\hat{\beta}_i-1} = 0$, we find that $\tilde{B}_i^{\hat{\beta}_i-1} \in \text{Ker } S_i$ over the time interval $(t_0, t_1]$. Further, note that since $\tilde{B}_i^{\hat{\beta}_i-1} = \begin{bmatrix} \tilde{B}_{i,1}^{\hat{\beta}_i-1} & \tilde{B}_{i,2}^{\hat{\beta}_i-1} \end{bmatrix} = \begin{bmatrix} \tilde{B}_{i,1}^{\hat{\beta}_i-2} & \hat{B}_{i,2}^{\hat{\beta}_i-1} \end{bmatrix}$, $S_i \hat{B}_{i,2}^{\hat{\beta}_i-1} = 0$.

Next, by pre-multiplying (4.79) by $\left(\tilde{B}_{i,2}^{\hat{\beta}_i-2}\right)^T$ and post-multiplying by $\tilde{B}_{i,2}^{\hat{\beta}_i-2}$, substituting $\hat{B}_{i,2}^{\hat{\beta}_i-1} = A\tilde{B}_{i,2}^{\hat{\beta}_i-2} - \dot{\tilde{B}}_{i,2}^{\hat{\beta}_i-2}$ from Algorithm 4.1, and using $S_i \hat{B}_{i,2}^{\hat{\beta}_i-1} = 0$ and $\tilde{B}_{i,2}^{\hat{\beta}_i-2} \in \text{Ker } C$

$$0 \geq \frac{\mathrm{d}}{\mathrm{d}t} \left[\left(\tilde{B}_{i,2}^{\hat{\beta}_i-2}\right)^T S_i \tilde{B}_{i,2}^{\hat{\beta}_i-2} \right] + \left(\tilde{B}_{i,2}^{\hat{\beta}_i-2}\right)^T S_i L V L^T S_i \tilde{B}_{i,2}^{\hat{\beta}_i-2}. \tag{4.82}$$

Using (4.77), let there exist a vector ν such that $L^T(\tau^+) S_i(\tau^+) \tilde{B}_{i,2}^{\hat{\beta}_i-2}(\tau^+)\nu \neq 0$ where $\tau > t_0$ and $S_i(t)\tilde{B}_{i,2}^{\hat{\beta}_i-2}(t)\nu = 0 \ \forall t \leq \tau$. In this case, (4.82) and $V > 0$ imply

$$0 > \frac{\mathrm{d}}{\mathrm{d}t} \left[\nu^T \left(\tilde{B}_{i,2}^{\hat{\beta}_i-2}(\tau^+)\right)^T S_i(\tau^+) \tilde{B}_{i,2}^{\hat{\beta}_i-2}(\tau^+)\nu \right],$$

which implies that $\nu^T \left(\tilde{B}_{i,2}^{\hat{\beta}_i-2}\right)^T S_i \tilde{B}_{i,2}^{\hat{\beta}_i-2}\nu < 0$ at some time $t > \tau$. This contradicts the assumption that $S_i \geq 0$. Therefore, by combining $S_i \tilde{B}_{i,2}^{\hat{\beta}_i - 2} \nu = 0$ with $S_i \tilde{B}_{i,2}^{\hat{\beta}_i - 2} = 0$, we find that $\tilde{B}_i^{\hat{\beta}_i - 2} \in \text{Ker } S_i$ over the time interval $(t_0, t_1]$. By repeating this pattern for the remaining components of $\bar{B}_i$, the claim is proven.

Remark 4.3: Given that the Riccati solution S_i obtains a nullspace that includes $\bar{B}_i$, the asymptotic detection filter's Riccati inequality constraint (4.79) is actually identical to the constraint in the general case (3.15) when $\Pi_i \hat{F}_i = 0$. ♢

4.4.3 Proof of Theorem 4.3

First, it is shown that $\hat{H}_i C \text{Ker } S_i = 0$. Let $x \in \text{Ker } S_i$ over the time interval $(t_0, t_1]$. Pre-multiplying (4.79) by x^T, post-multiplying by x, and using $S_i x = 0$,

$$0 \geq x^T \dot{S}_i x + x^T C^T \hat{H}_i Q_i \hat{H}_i C x. \tag{4.83}$$

Let $\hat{H}_i(\tau^+) C(\tau^+) x(\tau^+) \neq 0$ where $\tau > t_0$ and $\hat{H}_i(t) C(t) x(t) \neq 0 \ \forall t \leq \tau$. In this case, (4.83) and $Q_i > 0$ imply

$$0 > x^T(\tau^+) \dot{S}_i(\tau^+) x(\tau^+).$$

However, this implies that $S_i x \neq 0$ at some time $t > \tau$, which contradicts the assumption $S_i x = 0$. Therefore, $\hat{H}_i C x = 0$, which implies $\hat{H}_i C \text{Ker } S_i = 0$.

Next, it is shown that $C \text{Ker } S_i = \text{Im } C\tilde{B}_i$. Let $x \in \text{Ker } S_i$ over the time interval $(t_0, t_1]$. Then, using $\hat{H}_i C x = 0$,

$$\hat{H}_i C x = \left\{ I - C\tilde{B}_i \left[\left(C\tilde{B}_i \right)^T C\tilde{B}_i \right]^{-1} \left(C\tilde{B}_i \right)^T \right\} C x$$
$$\Rightarrow C x = C\tilde{B}_i \left[\left(C\tilde{B}_i \right)^T C\tilde{B}_i \right]^{-1} \left(C\tilde{B}_i \right)^T C x.$$

Since $\left[\left(C\tilde{B}_i \right)^T C\tilde{B}_i \right]^{-1} \left(C\tilde{B}_i \right)^T C x$ is a column vector, Cx is a linear combination of the column vectors of $C\tilde{B}_i$, which implies $C \text{Ker } S_i \subseteq \text{Im } C\tilde{B}_i$. However, since $\text{Im } \tilde{B}_i \subseteq \text{Ker } S_i$

from Theorem 4.2, this implies $C\text{Ker } S_i = \text{Im } C\tilde{B}_i$.

Finally, the claims of the theorem are proven. By taking the time derivative of $e_i^T S_i e_i$, substituting (3.8a) in the absence of sensor noise, and using (4.79) and Theorem 4.2,

$$0 \geq \frac{\mathrm{d}}{\mathrm{d}t}\left(e_i^T S_i e_i\right) + e_i^T \left(S_i L V L^T S_i + C^T \hat{H}_i Q_i \hat{H}_i C\right) e_i. \tag{4.84}$$

Assume that $S_i(t_0^+)e_i(t_0) = 0$. If $e_i^T \left(S_i L V L^T S_i + C^T \hat{H}_i Q_i \hat{H}_i C\right) e_i \neq 0$ at time $t = \tau^+$ where $\tau > t_0$, then, since $V > 0$ and $Q_i > 0$, $e_i^T \left(S_i L V L^T S_i + C^T \hat{H}_i Q_i \hat{H}_i C\right) e_i > 0$. This implies

$$0 > \frac{\mathrm{d}}{\mathrm{d}t}\left(e_i^T(\tau^+) S_i(\tau^+) e_i(\tau^+)\right).$$

However, this implies that $e_i^T S_i e_i < 0$ at some time $t > \tau$, which contradicts the assumption $S_i \geq 0$. Then, $L^T S_i e_i = 0$ and $\hat{H}_i C e_i = 0$, which when combined with (4.84) and $S_i \geq 0$ imply $S_i e_i = 0$. Hence, (4.84) implies that if the state estimation error starts in Ker S_i it will remain there. Further, the residual r_i in (3.8b) will remain in Im $C\tilde{B}_i$ because $C\text{Ker } S_i = \text{Im } C\tilde{B}_i$.

4.4.4 Proof of Theorem 4.4

By substituting (4.68) into (4.65), a sufficient condition for (4.68) to be a solution to Problem 4.2 is

$$\bar{S}_j \left(\sum_{i=1}^{s} \bar{S}_i\right)^{-1} \bar{S}_k = \begin{cases} \bar{S}_j, & j = k \\ 0, & j \neq k \end{cases} \tag{4.85}$$

$\forall\ j, k \in \{1, \ldots, s\}$. Assume that S_i satisfies (4.54) $\forall\ i \in \{1, \ldots, s\}$. First, it must be shown that $\bar{S}_i$ also satisfies (4.54) $\forall\ i \in \{1, \ldots, s\}$. To the contrary, assume $\exists\ v \neq 0 \ni$: $v \in$ Im $\bar{S}_j$ and $v \in$ Im $\bar{S}_k$. Then, $\exists\ \gamma_j, \gamma_k \neq 0 \ni$:

$$v = \bar{S}_j \gamma_j = \bar{S}_k \gamma_k.$$

This implies

$$\begin{bmatrix} \bar{S}_j\gamma_j & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{S}_k\gamma_k & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \bar{S}_j & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \gamma_j \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{S}_k & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \gamma_k \\ 0 \end{bmatrix} \Rightarrow S_j\bar{\gamma}_j = S_k\bar{\gamma}_k.$$

However, this contradicts (4.54), and so $\bar{S}_i$ satisfies (4.54).

Then, for $i = 1$, there exists an invertible transformation Γ and an invertible matrix $\hat{Z}_1$ such that

$$\text{Ker } \bar{S}_1 = \Gamma \begin{bmatrix} 0 \\ \hat{Z}_1 \end{bmatrix} \Rightarrow \bar{S}_1\Gamma \begin{bmatrix} 0 \\ \hat{Z}_1 \end{bmatrix} = 0 \Rightarrow \Gamma^T\bar{S}_1\Gamma \begin{bmatrix} 0 \\ \hat{Z}_1 \end{bmatrix} = 0$$

where $\hat{Z}_1 = \text{diag}(Z_2, \ldots, Z_s)$ and Z_i, $i = 1, \ldots, s$, is an invertible matrix with dimension equal to dim(Im $\bar{S}_i$). Since $\hat{Z}_1$ is invertible and $\Gamma^T\bar{S}_1\Gamma$ is symmetric,

$$\Gamma^T\bar{S}_1\Gamma = \begin{bmatrix} \check{S}_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This can be shown similarly for $\bar{S}_2, \ldots, \bar{S}_s$ such that

$$\begin{bmatrix} \bar{S}_1 & \ldots & \bar{S}_s \end{bmatrix} = \Gamma \begin{bmatrix} \check{S}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \check{S}_s \end{bmatrix}.$$

Therefore,

$$\Gamma^T\bar{S}_j\Gamma \left(\sum_{i=1}^{s} \Gamma^T\bar{S}_i\Gamma\right)^{-1} \Gamma^T\bar{S}_k\Gamma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \check{S}_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \check{S}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \check{S}_s \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \check{S}_k & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{cases} \Gamma^T \bar{S}_j \Gamma, & j = k \\ 0, & j \neq k. \end{cases} \tag{4.86}$$

By pre-multiplying (4.86) by Γ^{-T} and post-multiplying by Γ^{-1}, (4.85) is recovered.

Remark 4.4: This proof is similar to the proofs of Lemmas 4.1 and A.1 in [45]. $\diamondsuit$

4.4.5 Proof of Theorem 4.5

By applying (3.5) and substituting $\tilde{\mathcal{W}}^*$ for $\hat{\mathcal{W}}^*$, (4.53) implies

$$\text{Ker } \tilde{S}_i = \mathcal{W}_1^* \oplus \ldots \oplus \mathcal{W}_{i-1}^* \oplus \mathcal{W}_{i+1}^* \oplus \ldots \oplus \mathcal{W}_s^* \oplus \tilde{\mathcal{W}}^* = \tilde{\mathcal{W}}_i^*. \tag{4.87}$$

($\Rightarrow$) Substituting $\tilde{S}_i$ for S_i in Proposition 4.7,

$$\mathcal{C} \subset \text{Im } \tilde{S}_i, \forall\, i \in \{1, \ldots, s\}.$$

However, by assumption $\tilde{S}_i$ and $\tilde{S}_j$, $i \neq j$, satisfy (4.54), and so $\mathcal{C} = 0$.

($\Leftarrow$) Assume that there is no complementary subspace, i.e. $\mathcal{C} = 0$. Define $\mathcal{V} = \text{Im } \tilde{S}_i \bigcap \text{Im } \tilde{S}_j, i \neq j$. Then, choose the vector coefficients γ_i and γ_j such that

$$\nu = \tilde{S}_i \gamma_i = \tilde{S}_j \gamma_j \tag{4.88}$$

where $\nu \in \mathcal{V}$. Combining $\mathcal{C} = 0$ with (4.55), any vector may be written as a linear combination of vectors in $\mathcal{W}_1^*, \ldots, \mathcal{W}_s^*$, and $\tilde{\mathcal{W}}^*$. Thus, γ_i and γ_j in (4.88) may be rewritten using vectors that span these subspaces. However, by using (4.87), $\tilde{S}_i \tilde{\mathcal{W}}_i^* = 0$ and $\tilde{S}_i \mathcal{W}_i^*$ is full rank. Therefore, without loss of generality, let

$$\nu = \tilde{S}_i B_i^k = \tilde{S}_j B_j^k \tag{4.89}$$

where B_i^k and B_j^k are vectors in $\mathcal{W}_i^*$ and $\mathcal{W}_j^*$, respectively (see (3.2)). Then, pre-multiply (4.89) by $(B_i^k)^T$ and use $\mathcal{W}_i^* \subseteq \text{Ker } \tilde{S}_j$ to obtain

$$\left(B_i^k\right)^T \tilde{S}_i B_i^k = 0. \tag{4.90}$$

Since $\tilde{S}_i \geq 0$ is a square, symmetric matrix, (4.90) implies

$$\tilde{S}_i B_i^k = 0 \Rightarrow \nu = 0$$

Therefore, $\text{Im } \tilde{S}_i \bigcap \text{Im } \tilde{S}_j = 0$.

4.4.6 Propositions

Proposition 4.6: $\sum_{i=1}^s \bar{S}_i$ is full rank. ∎

Proof: Define $v \ni: T_0^{-1} v \in \text{Ker} \sum_{i=1}^s S_i$ where by using (4.61) and letting $v = [v_1^T \;\; v_2^T]^T$

$$T_0^{-1} v = T_{0,1} v_1 + T_{0,2} v_2.$$

By summing (4.60) over $i = 1, \ldots, s$,

$$T_0^{-T} \left(\sum_{i=1}^s S_i \right) T_0^{-1} = \begin{bmatrix} \sum_{i=1}^s \bar{S}_i & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.91}$$

Post-multiply (4.91) by v to obtain

$$T_0^T \left(\sum_{i=1}^s S_i \right) T_0 v = \begin{bmatrix} \sum_{i=1}^s \bar{S}_i & 0 \\ 0 & 0 \end{bmatrix} v = 0. \tag{4.92}$$

Applying (4.62) to (4.92),

$$T_0^{-T}\left(\sum_{i=1}^{s} S_i\right)(T_{0,1}v_1 + T_{0,2}v_2) = \sum_{i=1}^{s} \bar{S}_i v_1 = 0. \tag{4.93}$$

Recall that, from (4.62), the columns of $T_{0,2}$ form a basis for Ker $\sum_{i=1}^{s} S_i$ and T_0 is full rank. Therefore,

$$T_0^{-T}\left(\sum_{i=1}^{s} S_i\right) T_{0,1}v_1 = \sum_{i=1}^{s} \bar{S}_i v_1 = 0.$$

Since $T_{0,1}$ is full column rank and its columns are linearly independent from those of $T_{0,2}$, $v_1 \neq 0$ implies that $\left(\sum_{i=1}^{s} S_i\right) T_{0,1}v_1 \neq 0$. Therefore, $\sum_{i=1}^{s} \bar{S}_i v_1 = 0$ implies $v_1 = 0$, and so $\sum_{i=1}^{s} \bar{S}_i$ is full rank. **QED**

Proposition 4.7: $\bigcap_{i=1}^{s} \text{Im } S_i = \mathcal{C}$. ▮

Proof: First, it must be shown that $\bigcap_{i=1}^{s} \text{Im } S_i \subseteq \mathcal{C}$, or equivalently, $\nu^T\mathcal{C} = 0 \Rightarrow \nu \in \bigcup_{i=1}^{s} \text{Ker } S_i$. From (4.56), $\nu^T\mathcal{C} = 0$ implies $\nu \in \mathcal{W}_j^* \oplus \hat{\mathcal{W}}_j^*, j \in \{1, \ldots, s\}$. By using Theorem 4.2 with (3.5) and (4.53), this implies $\nu \in \bigcup_{i=1}^{s} \text{Ker } S_i$.

Next, it must be shown that $\bigcap_{i=1}^{s} \text{Im } S_i \supseteq \mathcal{C}$, or equivalently, $\nu \in \text{Ker } S_i \Rightarrow \nu^T\mathcal{C} = 0, \forall\ i \in \{1, \ldots, s\}$. By using (4.53), $\nu \in \text{Ker } S_i \Rightarrow \nu \in \hat{\mathcal{W}}_i^*$. Then, (4.56) implies $\nu^T\mathcal{C} = 0$. **QED**

Chapter 5

The Discrete Game Theoretic Multiple-Fault Detection Filter

In this chapter, the Discrete Game Theoretic Multiple-Fault Detection Filter (DGTMFDF) is derived. The DGTMFDF extends the DGTFDF to the multiple-fault case by modeling the detection filter problem as a set of algebraic DAPs to be optimized via a single differential game. However, like the continuous case, the globally optimal solution for the filter gain is difficult to obtain and so sufficient conditions for satisfying the DAPs are derived instead. Then, these conditions are treated as constraints on a secondary optimization to determine the filter gain. Given the similarities between the continuous and discrete problems, this chapter is presented as a parallel to Chapter 3. However, because time-varying results for singular optimal control have been difficult to obtain, the DGTMFDF is currently limited to the time-invariant case.

This chapter is organized as follows. First, the structure of the discrete detection filter of Section 2.2.2 is approximated by a set of DAPs in Section 5.1. Then, the implied differential game problem is simplified into a feasibility problem to find a filter gain that satisfies the DAPs given the worst case disturbances and faults. Sufficient conditions for satisfying the DAPs are obtained in Section 5.2 as a set of Riccati differential inequalities with nonnegative solutions. These inequalities become constraints of a user-defined optimization function, which may be used to achieve desired secondary characteristics of the detection filter. Finally, the DGTMFDF is compared directly to the DGTFDF in Section 5.3.

5.1 Differential Game Problem Formulation

In this section, the detection filter problem for a given target fault input is formulated as a set of DAPs that can be optimized via a differential game problem. First, the discrete detection filter problem is approximated by a set of DAPs in Section 5.1.1. Then, the DAPs are converted into a differential game feasibility problem and the required assumptions are discussed in Section 5.1.2. To simplify the derivation, this chapter considers only scalar target faults, though the results apply to vector fault case as well.

5.1.1 Approximation of the Discrete Detection Filter

To approximate a multiple-fault detection filter for linear time-invariant systems, a set of DAPs are formulated by requiring that the target faults be observable and relaxing the requirement on strict blocking implied by the first and second objectives of the detection filter problem. Instead, the transmissions of the disturbance parameters are bounded above by a preset level. However, the target fault must remain observable to its projected residual. Further, the third detection filter objective is relaxed to requiring only that the detection filter dynamics be stable.

Define the time-invariant dynamic system with s target faults, $q-s$ nuisance faults, m measurements, and sensor noise $v(k)$ as

$$x(k+1) = \Phi x(k) + Bu(k) + \sum_{i=1}^{s} F_i \mu_i(k) + \hat{F}\hat{\mu}(k) \tag{5.1a}$$

$$y(k) = Cx(k) + Du(k) + v(k) \tag{5.1b}$$

is defined from initial time step k_0 to final time step $k_1 < \infty$. Let the time step size be represented by Δt. Recall from Section 2.2.1 that any fault in the plant, actuator, or sensor can be modeled as an additive input to (5.1a).

For the i^{th} DAP, the transmissions of the disturbance parameters are separated from the transmission of the target fault into their own state $x_i(k)$ where

$$x_i(k+1) = \Phi x(k) + Bu(k) + \hat{F}_i \hat{\mu}_i(k) \tag{5.2a}$$

$$y_i(k) = C x_i(k) + Du(k) + v_i(k). \tag{5.2b}$$

Since blocking the transmissions of the disturbance parameters is of primary interest, x_i is the most useful state vector. The conditions so that the target fault is not blocked along with the complementary fault are discussed later in this section. Note that each measurement $y_i(k)$ contains its own noise $v_i(k)$ in (5.2b), even though $v_1(k), \ldots, v_s(k)$ are physically identical. However, in the next section they will be considered as independent quantities in order to simplify the formulation of the DGTMFDF problem. Further, in the absence of the target fault, $y_i(k)$ is identical to $y(k)$ in (5.1). Thus, let the detection filter be modeled as

$$\bar{x}(k+1) = \Phi \hat{x}(k) + Bu(k) \tag{5.3a}$$

$$\hat{x}(k) = \bar{x}(k) + L\left(y_i(k) - C\bar{x}(k) - Du(k)\right) \tag{5.3b}$$

$$r_i(k) = y_i(k) - C\bar{x}(k) - Du(k). \tag{5.3c}$$

Using (5.2), the *a priori* and *a posteriori* state estimation errors $e_i(k) \triangleq x_i(k) - x(k)$ and $\hat{e}_i(k) \triangleq x_i(k) - \hat{x}(k)$, respectively, and the estimation error residual $r_i(k)$ satisfy

$$e(k+1) = \Phi \hat{e}_i(k) + \hat{F}_i \hat{\mu}_i(k) \tag{5.4a}$$

$$\hat{e}_i(k) = (I - LC)\, e_i(k) - L v_i(k) \tag{5.4b}$$

$$r_i(k) = C e_i(k) + v_i(k). \tag{5.4c}$$

Combining (5.4a) and (5.4b), $e_i(k)$ is propagated by

$$e_i(k+1) = \Phi\left[(I - LC)\, e_i(k) - Lv_i(k)\right] + \hat{F}_i \hat{\mu}_i(k). \tag{5.5}$$

It is assumed that Φ and $I - LC$ in (5.5) are both invertible. This assumption is in general true for Φ given sufficiently small values of Δt. Later, L will simply be chosen such that $I - LC$ is also invertible. Further, by multiplying $r_i(k)$ by the residual projector $\hat{H}_i$, the projected residual is

$$\bar{r}_i(k) = \hat{H}_i C e_i(k) + \hat{H}_i v_i(k). \tag{5.6}$$

where $\hat{H}_i$ is the residual projector as defined in (2.11). Further, Lemma 5.1 in Section 5.4 proves that when output separability is satisfied, $\hat{\mathcal{W}}_i^*$ can be obtained as

$$\hat{\mathcal{W}}_i^* = \mathcal{W}_1^* \oplus \ldots \oplus \mathcal{W}_{i-1}^* \oplus \mathcal{W}_{i+1}^* \oplus \ldots \oplus \mathcal{W}_s^* \oplus \hat{\mathcal{W}}^*(k). \tag{5.7}$$

Since the projected residual contains a direct feedthrough term from the sensor noise, the projected output error $\hat{H}_i C e_i(k)$ is used instead of $\bar{r}_i(k)$ to represent the transmission of the disturbance parameters to the output. Thus, the i^{th} DAP is written as [43]

$$\frac{\sum_{k=k_0}^{k_1-1} \|\hat{H}_i C e_i(k)\|_{Q_i}^2}{\|\hat{e}_i(k_0)\|_{P_0^{-1}}^2 + \sum_{k=k_0}^{k_1-1}\left[\|\hat{\mu}_i(k)\|_{M_i^{-1}}^2 + \|v_i(k)\|_{\bar{V}^{-1}}^2\right]} \le \gamma, \tag{5.8}$$

subject to the dynamic system (5.4) for any $\hat{\mu}_i(k)$, $v_i(k)$, and $\hat{e}_i(k_0)$ that satisfy

$$\sum_{k=k_0}^{k_1-1} \|\hat{\mu}_i(k)\|^2 < \infty$$

and

$$\sum_{k=k_0}^{k_1-1} \|v_i(k)\|^2 < \infty.$$

$\gamma > 0$ is the arbitrary disturbance attenuation bound. $Q_i \ge 0$, $M_i > 0$, $\bar{V} > 0$, and $P_0 > 0$

are arbitrary symmetric design weighting matrices. However, $\bar{V}$ is typically chosen as the covariance of the measurement noise. Further, when the design weightings M_i, $\bar{V}$, and P_0 are chosen to be larger, the projected residual becomes less sensitive to the complementary fault, sensor noise, and initial condition error, respectively, which also can be achieved simultaneously by choosing Q_i to be larger.

Finally, in order to ensure that a stable detection filter exists that achieves the objectives of the approximate fault detection filter problem, assume the following:

Assumption 5.1: (C, Φ) is detectable. ♣

Assumption 5.2: F_i is monic and (C, Φ)-observable $\forall\ i \in \{1, \ldots, s\}$. ♣

Assumption 5.3: F_i is (C, Φ) output separable from $\hat{F}_i$ $\forall\ i \in \{1, \ldots, s\}$. ♣

Assumption 5.4: (C, Φ, F_i) has no invariant zeros at the origin. ♣

Assumption 5.5: $F_1, \ldots, F_q$ are mutually detectable or else the extra invariant zeros are stable. ♣

Assumptions 5.1 and 5.5 are necessary to formulate an asymptotically stable detection filter for time-invariant systems that achieves the desired fault detection properties. The unstable, unobservable subspace may be truncated from the detection filter state space at the beginning of the problem so that this assumption can be made without loss of generality. Assumptions 5.2 and 5.3 guarantee that each target fault will remain observable in $(\hat{H}_i C, \Phi(I - LC))$ when the complementary fault is placed in the unobservable subspace of the projected residual. Finally, Assumption 5.4 ensures that the induced steady state residuals will be non-zero.

5.1.2 Problem Formulation

By multiplying both sides of (5.8) by the denominator, subtracting the right-hand side, and setting the left-hand side equal to J_i, (5.8) is converted into the nonconvex cost function

$$J_i = -\|\hat{e}_i(k_0)\|^2_{\Pi_0} + \sum_{k=k_0}^{k_1-1} \left[\left\| \hat{H}_i C e_i(k) \right\|^2_{Q_i} - \|\hat{\mu}_i(k)\|^2_{\gamma M_i^{-1}} - \|v_i(k)\|^2_{V^{-1}} \right], \tag{5.9}$$

where $\Pi_0 \triangleq \gamma P_0^{-1}$ and $V \triangleq \gamma^{-1}\bar{V}$. The detection filter problem is modeled as a differential game optimization by summing (5.9) over i, minimizing the sum with respect to the filter gain, and maximizing the sum with respect to the disturbance parameters. Therefore, the differential game problem is

$$\min_{L} \max_{v_1(k),\ldots,v_s(k)} \max_{\hat{\mu}_1(k),\ldots,\hat{\mu}_s(k)} \max_{\hat{e}_1(k_0),\ldots,\hat{e}_s(k_0)} \sum_{i=1}^{s} J_i(L, v_i(k), \hat{\mu}_i(k), \hat{e}_i(k_0)) \tag{5.10}$$

subject to (5.5). Recall from their definitions that $\hat{\mu}_1(k), \ldots, \hat{\mu}_s(k)$ are not independent since they share some common elements.

Since the detection filter gain L does not appear in the game cost (5.9) and enters linearly into the constraint (5.5), (5.10) is singular with respect to L [47]. This makes the process of finding a globally optimal solution for L that will generate the desired fault detection properties very complex. However, in order to satisfy the DAPs (5.8), it is only required that (5.9) be nonpositive for any value of $\hat{\mu}_i(k)$, $v_i(k)$, and $\hat{e}_i(k_0)$ $\forall\, k \in \{k_0, \ldots, k_1\}$. Thus, only a solution to a feasibility problem in L such that (5.9) is nonpositive is required. To further simplify the problem statement, assume that all of the disturbance parameters are independent. This assumption only affects the problem statement, not the equations to eventually solve for the filter gain. Therefore, to determine a filter gain sufficient to satisfy (5.8), the following simplified problem is solved:

Problem 5.1: Find L such that

$$\max_{v_i(k)} \max_{\hat{\mu}_i(k)} \max_{\hat{e}_i(k_0)} J_i \leq 0$$

subject to (5.5) and (5.9) for all $i = 1, \ldots, s$. ♡

5.2 Detection Filter Problem Solution

In this section, solutions for the DGTMFDF gain L in Problem 5.1 are determined for the general case where $\gamma > 0$. First, the conditions under which (5.9) is nonpositive are determined in Section 5.2.1. To this end, a set of discrete Riccati differential inequalities are derived for the estimation error covariances, each of which must have a nonnegative solution. The Riccati inequalities, which are functions of the filter gain, become constraints for a secondary filter gain optimization problem. This new problem is solved in general and for an example cost function in Section 5.2.2.

5.2.1 Conditions for Optimality of the Game Cost

In this section, the necessary and sufficient conditions for optimality of the game cost are derived. By appending the estimation error dynamics to the cost function and using calculus of variations, necessary conditions for optimality are determined with respect to the disturbances parameters. Then, by examining the second-order variation of each cost function [51], sufficient conditions for optimality are derived as discrete algebraic Riccati inequality (DARI) constraints on the estimation error covariance associated with each DAP. A valid solution for the filter gain in Problem 5.1 is any for which the Riccati inequalities are nonpositive and the estimation error covariances are nonnegative-definite, implying that the DAPs are also satisfied. For compactness, the matrices' and variables' time dependence will no longer be shown. Further, a tilde above a variable will refer to the $(k+1)^{\text{th}}$ time step of the variable. For example,

$$e_i = e_i(k)$$

$$\tilde{e}_i = e_i(k+1).$$

Necessary Conditions for Optimality

First, the estimation error dynamics (5.5) are appended to the cost function (5.9) using the LaGrange multiplier $2\tilde{\lambda}_i^T$, which yields

$$J_i = \sum_{k=k_0}^{k_1-1}\left[\left\|\hat{H}_i C e_i\right\|_{Q_i}^2 - \|\hat{\mu}_i\|_{\gamma M_i^{-1}}^2 - \|v_i\|_{V^{-1}}^2 + 2\tilde{\lambda}_i^T\left(\Phi\left[(I-LC)\,e_i - Lv_i\right] + \hat{F}_i\hat{\mu}_i - \tilde{e}_i\right)\right] - \|\hat{e}_i(k_0)\|_{\Pi_0}^2.$$

The factor of 2 is included with the LaGrange multiplier to simplify the optimality conditions later. By changing the upper and lower indices of the term $\sum_{k=k_0}^{k_1-1} 2\tilde{\lambda}_i^T\tilde{e}_i$,

$$J_i = \sum_{k=k_0}^{k_1-1}\left[\left\|\hat{H}_i C e_i\right\|_{Q_i}^2 - \|\hat{\mu}_i\|_{\gamma M_i^{-1}}^2 - \|v_i\|_{V^{-1}}^2 + 2\tilde{\lambda}_i^T\left(\Phi\left[(I-LC)\,e_i - Lv_i\right] + \hat{F}_i\hat{\mu}_i\right) - 2\lambda_i^T e_i\right]$$
$$- \|\hat{e}_i(k_0)\|_{\Pi_0}^2 + 2\lambda_i^T(k_0)e_i(k_0) - 2\lambda_i^T(k_1)e_i(k_1).$$

Next, assume that $e_i(k_0) = \hat{e}_i(k_0)$.[1] By taking the first-order variation with respect to the disturbance parameters $\hat{e}_i(k_0)$, $\hat{\mu}_i$, and v_i,

$$\delta J_i = 2\sum_{k=k_0}^{k_1-1}\left\{\left[e_i^T C^T \hat{H}_i Q_i \hat{H}_i C - \lambda_i^T + \tilde{\lambda}_i^T\Phi(I-LC)\right]\delta e_i - \left[\gamma\hat{\mu}_i^T M_i^{-1} - \tilde{\lambda}_i^T\hat{F}_i\right]\delta\hat{\mu}_i\right.$$
$$\left. - \left[v_i^T V^{-1} + \tilde{\lambda}_i^T\Phi L\right]\delta v_i\right\} + 2\left[\lambda_i^T(k_0) - \hat{e}_i^T(k_0)\Pi_0\right]\delta\hat{e}_i(k_0) - 2\lambda_i^T(k_1)\delta e_i(k_1).$$

[1] The *a posteriori* estimation error values are usually generated by updating the *a priori* error with the measurement. However, at $k = k_0$, the detection filter is started only after the first measurement is received. Therefore, the initial *a priori* error is arbitrary and may be set equal to the initial *a posteriori* error.

Therefore, the first-order necessary conditions for optimality are

$$\hat{\mu}_i^* = \frac{1}{\gamma} M_i \hat{F}_i^T \lambda_i \tag{5.11}$$

$$v_i^* = -V L^T \Phi^T \lambda_i \tag{5.12}$$

$$\tilde{\lambda}_i = \Phi^{-T}(I - LC)^{-T} \left(\lambda_i - C^T \hat{H}_i Q_i \hat{H}_i C e_i^*\right) \tag{5.13}$$

$$\lambda_i(k_0) = \Pi_0 \hat{e}_i^*(k_0) \tag{5.14}$$

$$\lambda_i(k_1) = 0, \tag{5.15}$$

$\forall\ i \in \{1, \ldots, s\}$, where the asterisks denote the optimal strategies for the given variables.

The necessary conditions for optimality are rewritten using an assumed form of the LaGrange multiplier, where

$$\lambda_i = \Pi_i e_i^*. \tag{5.16}$$

Then, propagation and update equations for Π_i must be determined so that (5.16) is consistent with the optimal solution. To simplify the ensuing analysis, define

$$\Gamma_i = \Pi_i - C^T \hat{H}_i Q_i \hat{H}_i C \tag{5.17}$$

$$\hat{\Phi} = \Phi(I - LC)$$

$$\hat{\Gamma}_i = \hat{\Phi}^{-T} \Gamma_i \hat{\Phi}^{-1}$$

$$G_i = \frac{1}{\gamma} \hat{F}_i M_i \hat{F}_i^T + \Phi L V L^T \Phi^T.$$

Substituting (5.16) into (5.13),

$$\tilde{\Pi}_i \tilde{e}_i^* = \Phi^{-T}(I - LC)^{-T} \Gamma_i e_i^*.$$

Applying the backward discrete dynamics

$$e_i = \hat{\Phi}^{-1} \left(\tilde{e}_i - \hat{F}_i \mu_i + \Phi L v_i\right), \tag{5.18}$$

we obtain

$$\tilde{\Pi}_i \tilde{e}_i^* = \hat{\Gamma}_i \left(\tilde{e}_i^* - \hat{F}_i \mu_i^* + \Phi L v_i^* \right).$$

Finally, by substituting the optimal strategies for $\hat{\mu}_i^*$ and v_i^* from (5.11) and (5.12), respectively, and subtracting the right-hand side,

$$0 = \Psi_i \tilde{e}_i^* \tag{5.19}$$

where we choose

$$\Psi_i(\Pi_i, L, k) = \tilde{\Pi}_i - \hat{\Gamma}_i \left(I - G_i \tilde{\Pi}_i \right). \tag{5.20}$$

It will be shown that the primary Riccati constraint for the DGTMFDF problem is directly related to (5.20).

Next, it is confirmed that the optimal value of the cost is in fact equal to zero. As a preliminary step, the cost function (5.9) is converted into one that uses the necessary conditions more efficiently. By adding the identically zero term

$$0 = \|e_i(k_0)\|^2_{\Pi_i(k_0)} - \|e_i(k_1)\|^2_{\Pi_i(k_1)} + \sum_{k=k_0}^{k_1-1} \left[\|\tilde{e}_i\|^2_{\tilde{\Pi}_i} - \|e_i\|^2_{\Pi_i} \right]$$

to (5.9) assuming $\hat{e}_i(k_0) = e_i(k_0)$,

$$J_i = -\|\hat{e}_i(k_0)\|^2_{\Pi_0 - \Pi_i(k_0)} - \|e_i(k_1)\|^2_{\Pi_i(k_1)} + \sum_{k=k_0}^{k_1-1} \left[\|\tilde{e}_i\|^2_{\tilde{\Pi}_i} - \|e_i\|^2_{\Gamma_i} - \|\hat{\mu}_i\|^2_{\gamma M_i^{-1}} - \|v_i\|^2_{V^{-1}} \right].$$

Then, by applying (5.18),

$$\begin{aligned} J_i = &- \|\hat{e}_i(k_0)\|^2_{\Pi_0 - \Pi_i(k_0)} - \|e_i(k_1)\|^2_{\Pi_i(k_1)} \\ &+ \sum_{k=k_0}^{k_1-1} \left[\|\tilde{e}_i\|^2_{\tilde{\Pi}_i} - \left\| \tilde{e}_i - \hat{F}_i \hat{\mu}_i + \Phi L v_i \right\|^2_{\hat{\Gamma}_i} - \|\hat{\mu}_i^*\|^2_{\gamma M_i^{-1}} - \|v_i\|^2_{V^{-1}} \right]. \end{aligned} \tag{5.21}$$

Now, the necessary constraints may be used to confirm the optimal value of the cost. By substituting (5.11) and (5.12) into (5.21), the cost function is defined entirely in terms of the optimal estimation error e_i^*. Collecting terms,

$$J_i^* = \|\hat{e}_i^*(k_0)\|^2_{\Pi_i(k_0)-\Pi_0} - \|e_i^*(k_1)\|^2_{\Pi_i(k_1)} + \sum_{k=k_0}^{k_1-1} \|\tilde{e}_i^*\|^2_{\Psi_i^T\left(I-G_i\tilde{\Pi}_i\right)}. \tag{5.22}$$

Assuming (5.24), (5.25), and (5.30), it is easily verified that the remaining terms in (5.22) are nonpositive. Further, by substituting (5.14)-(5.16) and (5.19) into (5.22), the maximum value of the cost is zero.

Sufficient Conditions for Optimality

In order to obtain sufficient conditions to satisfy the DAP (5.8), sufficient conditions for optimality are generated from the second-order variation of the cost (5.21). So that $\hat{\mu}_i^*$, v_i^*, and $\hat{e}_i^*(k_0)$ maximize the cost, the second derivative with respect to the associated variables must be nonpositive. The second variation of (5.21) is

$$\delta^2 J_i = \delta\hat{e}_i^T(k_0)\left[\Pi_i(k_0)-\Pi_0\right]\delta\hat{e}_i(k_0) - \delta e_i^T(k_1)\Pi_i(k_1)\delta e_i(k_1) + \sum_{k=k_0}^{k_1-1}\begin{bmatrix} \delta\tilde{e}_i^T & \delta\hat{\mu}_i^T & \delta v_i^T \end{bmatrix}\hat{\Psi}_i\begin{bmatrix} \delta\tilde{e}_i \\ \delta\hat{\mu}_i \\ \delta v_i \end{bmatrix}.$$

where

$$\hat{\Psi}_i = \begin{bmatrix} \tilde{\Pi}_i - \hat{\Gamma}_i & \hat{\Gamma}_i\hat{F}_i & -\hat{\Gamma}_i\Phi L \\ \hat{F}_i^T\hat{\Gamma}_i & -\left(\gamma M_i^{-1} + \hat{F}_i^T\hat{\Gamma}_i\hat{F}_i\right) & \hat{F}_i\hat{\Gamma}_i\Phi L \\ -L^T\Phi^T\hat{\Gamma}_i & L^T\Phi^T\hat{\Gamma}_i\hat{F}_i & -\left(V^{-1} + L^T\Phi^T\hat{\Gamma}_i\Phi L\right) \end{bmatrix}$$

Note that the factor of 2 in the second-order variation has been removed for brevity. Clearly, if there exist L and Π_i, $\forall\, i \in \{1,\ldots,s\}$, such that

$$0 \geq \hat{\Psi}_i \tag{5.23}$$

$$0 \leq \Pi_0 - \Pi_i(k_0) \tag{5.24}$$

$$0 \leq \Pi_i(k_1), \tag{5.25}$$

then the second-order variation is nonpositive. Since the optimal value of the cost is zero, the game cost (5.9) is nonpositive when the above inequality constraints are satisfied, implying that the faults are placed in approximate detection spaces so that they can be isolated by each projected residual.

The coefficient matrix $\hat{\Psi}_i$ is not very useful by itself. However, by using the Schur complement formula, (5.23) implies

$$0 < \begin{bmatrix} \gamma M_i^{-1} + \hat{F}_i^T \hat{\Gamma}_i \hat{F}_i & -\hat{F}_i \hat{\Gamma}_i \Phi L \\ -L^T \Phi^T \hat{\Gamma}_i \hat{F}_i & V^{-1} + L^T \Phi^T \hat{\Gamma}_i \Phi L \end{bmatrix} \tag{5.26}$$

$$0 \geq \tilde{\Pi}_i - \hat{\Gamma}_i + \hat{\Gamma}_i \begin{bmatrix} \hat{F}_i & -\Phi L \end{bmatrix} \begin{bmatrix} \gamma M_i^{-1} + \hat{F}_i^T \hat{\Gamma}_i \hat{F}_i & -\hat{F}_i \hat{\Gamma}_i \Phi L \\ -L^T \Phi^T \hat{\Gamma}_i \hat{F}_i & V^{-1} + L^T \Phi^T \hat{\Gamma}_i \Phi L \end{bmatrix}^{-1} \begin{bmatrix} \hat{F}_i^T \\ -L^T \Phi^T \end{bmatrix} \hat{\Gamma}_i. \tag{5.27}$$

Eqns. (5.26) and (5.27) become constraints on L and Π_i that are equivalent to (5.23). The estimation error covariance propagation function is (5.27) and the update equation is (5.17). Further, when combined with (5.17), (5.27) is a DARI. Also, (5.27) implies that $\Pi_i \geq 0$ and $\Gamma_i \geq 0$ over the entire time interval (see Proposition 5.2 in Section 5.4).

To further simplify the propagation equation (5.27), apply the matrix inversion lemma

$$\left(V_{11} + V_{12} V_{22}^{-1} V_{21}\right)^{-1} = V_{11}^{-1} - V_{11}^{-1} V_{12} \left(V_{22} + V_{21} V_{11}^{-1} V_{12}\right)^{-1} V_{21} V_{11}^{-1} \tag{5.28}$$

where

$$V_{11} = I$$
$$V_{12} = \hat{\Gamma}_i \begin{bmatrix} \hat{F}_i & -\Phi L \end{bmatrix}$$

$$V_{21} = \left[\begin{array}{cc} \hat{F}_i & -\Phi L \end{array} \right]^T$$

$$V_{22} = \left[\begin{array}{cc} \frac{1}{\gamma} M_i & 0 \\ 0 & V \end{array} \right]$$

to obtain

$$0 \geq \tilde{\Pi}_i - \left(I + \hat{\Gamma}_i G_i \right)^{-1} \hat{\Gamma}_i. \tag{5.29}$$

Note that $I + \hat{\Gamma}_i G_i$ is invertible since $\hat{\Gamma}_i G_i \geq 0$. By casting (5.27) in the form on the right-hand side of (5.28), it is apparent that (5.26) is automatically satisfied since it involves a sum of positive-definite and nonnegative-definite matrices. Next, by pre-multiplying (5.29) by $I + \hat{\Gamma}_i G_i$ and using (5.20) from the first-order necessary conditions for optimality,

$$0 \geq \left(I + \hat{\Gamma}_i G_i \right) \tilde{\Pi}_i - \hat{\Gamma}_i = \Psi_i \tag{5.30}$$

Thus, three equivalent forms of the DARI constraint for the DGTMFDF problem have been generated in this section: a matrix inequality (5.23), a block matrix differential inequality (5.27), and a simple matrix inequality related to the first-order necessary conditions (5.30). Eqn. (5.30) will be the most useful for the filter gain optimization in the next section.

Therefore, Problem 5.1 requires a solution to the DARIs (5.30) with measurement update (5.17), coupled by the filter gain L, with boundary conditions (5.24) and (5.25). Since the degree of complementary fault blocking can be changed by adjusting γ, the structure of the DGTMFDF is less constrained than the detection filters based on spectral and geometric theories. Further, the DGTMFDF extends the DGTFDF to the multiple-fault case.

Remark 5.1: An analysis of the steady-state case of the DGTMFDF was attempted by examining the discrete Riccati equation implied by the necessary conditions for optimality where (5.20) is set equal to zero. Defining $\bar{Q}_i = -C^T \hat{H}_i Q_i \hat{H}_i C \leq 0$, substituting into

(5.17) and (5.20), and setting (5.20) equal to zero,

$$\begin{aligned} 0 &= \Pi_i - \left(I + \hat{\Gamma}_i G_i\right)^{-1} \hat{\Gamma}_i \\ &= \Gamma_i - \bar{Q}_i - \hat{\Phi}^{-T} \left(I + \Gamma_i \hat{\Phi}^{-1} G_i \hat{\Phi}^{-T}\right)^{-1} \Gamma_i \hat{\Phi}^{-1}. \end{aligned}$$

Note that in steady-state, $\tilde{\Pi}_i = \Pi_i$. The current literature discusses solutions to discrete Riccati equations of this form only when $G_i \geq 0$ and $\bar{Q}_i \geq 0$ [48]. However, the DGTMFDF constraints do not meet the latter condition. Thus, there are currently no existence conditions for steady-state solutions to the current problem. $\diamondsuit$

5.2.2 Filter Gain Optimization

In this section, the filter gain L is optimized with respect to a new cost function. Since any solutions L and $\Pi_i \geq 0$ to (5.27) and (5.24) automatically imply that (5.9) is nonpositive, the specific cost function used at this stage is arbitrary. Therefore, let the optimal filter gain minimize the cost function $\bar{J}$, defined as

$$\min_L \bar{J} = \min_L \sum_{i=1}^{s} \sum_{k=k_0}^{k_1-1} \text{tr } \Omega_i \tag{5.31}$$

where $\Omega_i \in \mathbb{R}^{n \times n}$ is a symmetric function chosen by the user. For convenience, assume that Ω_i is a function of Π_i, L, and k only. At the end of the section, suggestions on choosing Ω_i such that (5.31) has a non-trivial solution are discussed and an example is presented.

To determine the first-order necessary conditions for optimality of (5.31), use the Lagrange multiplier $\tilde{\Delta}_i$ to append the dynamic constraint

$$0 = \Psi_i = \tilde{\Pi}_i - \left(I + \hat{\Gamma}_i G_i\right)^{-1} \hat{\Gamma}_i \tag{5.32}$$

to $\bar{J}$ to obtain[2]

$$\bar{J} = \sum_{i=1}^{s}\sum_{k=k_0}^{k_1-1} \text{tr}\left\{\Omega_i\left(\Pi_i, L, k\right) + \tilde{\Delta}_i\left[\tilde{\Pi}_i - \left(I + \hat{\Gamma}_i G_i\right)^{-1}\hat{\Gamma}_i\right]\right\}.$$

Changing the upper and lower indices of $\sum_{k=k_0}^{k_1-1} \text{tr}\ \tilde{\Delta}_i\tilde{\Pi}_i$,

$$\bar{J} = \sum_{i=1}^{s} \text{tr}\left\{\Delta_i(k_1)\Pi_i(k_1) - \Delta_i(k_0)\Pi_i(k_0) + \sum_{k=k_0}^{k_1-1}\left[\Omega_i(\Pi_i, L, t) + \Delta_i\Pi_i - \tilde{\Delta}_i\left(I + \hat{\Gamma}_i G_i\right)^{-1}\hat{\Gamma}_i\right]\right\}.$$

Taking the first-order variation with respect to L and Π_i,

$$\begin{aligned}\delta\bar{J} = &\sum_{i=1}^{s} \Delta_i(k_1)\Pi_i(k_1) - \Delta_i(k_0)\Pi_i(k_0) + \sum_{k=k_0}^{k_1-1}\left\{\left(\frac{\delta\left[\text{tr}\ \Omega_i\left(\Pi_i, L, k\right)\right]}{\delta L}\right)^T \delta L\right.\\ &+ 2\left[\left(\left[V L^T \Phi^T + C\hat{\Phi}^{-1}G_i\right]\left(I + \hat{\Gamma}_i G_i\right)^{-1}\hat{\Gamma}_i - C\hat{\Phi}^{-1}\right)\tilde{\Delta}_i\left(I + \hat{\Gamma}_i G_i\right)^{-1}\hat{\Gamma}_i\Phi\right]\delta L\\ &+ \left.\left[\Delta_i - \hat{\Phi}^{-1}\left(I - G_i\left(I + \hat{\Gamma}_i G_i\right)^{-1}\hat{\Gamma}_i\right)\tilde{\Delta}_i\left(I + \hat{\Gamma}_i G_i\right)^{-1}\hat{\Phi}^{-T} + \left(\frac{\delta\left[\text{tr}\ \Omega_i\left(\Pi_i, L, k\right)\right]}{\delta\Pi_i}\right)^T\right]\delta\Pi_i\right\}.\end{aligned}$$

Thus, by using two variants of (5.28)

$$\left(I + V_{12}V_{21}\right)^{-1} = I - V_{12}\left(I + V_{21}V_{12}\right)^{-1}V_{21} \tag{5.33a}$$

$$\left(I + V_{12}V_{21}\right)^{-1}V_{12} = V_{12}\left(I + V_{21}V_{12}\right)^{-1} \tag{5.33b}$$

and substituting (5.32), the first-order necessary conditions for optimality of (5.31) are

$$0 = \sum_{i=1}^{s}\left[2\left(V L^{*T}\Phi^T\tilde{\Pi}_i^* - C\hat{\Phi}^{*-1}\left(I + \hat{\Gamma}_i^* G_i^*\right)^{-T}\right)\tilde{\Delta}_i\tilde{\Pi}_i^*\Phi + \left(\frac{\delta\left[\text{tr}\ \Omega_i\left(\Pi_i^*, L^*, k\right)\right]}{\delta L^*}\right)^T\right] \tag{5.34}$$

$$0 = \Delta_i - \hat{\Phi}^{*-1}\left(I + \hat{\Gamma}_i^* G_i^*\right)^{-T}\tilde{\Delta}_i\left(I + \hat{\Gamma}_i^* G_i^*\right)^{-1}\hat{\Phi}^{*-T} + \left(\frac{\delta\left[\text{tr}\ \Omega_i\left(\Pi_i^*, L^*, k\right)\right]}{\delta\Pi_i^*}\right)^T \tag{5.35}$$

$$0 = \Delta_i(k_1) \tag{5.36}$$

[2]If the inequality form of the dynamic constraint (5.30) is desired, simply add a nonnegative term to Ψ_i such that the sum is equal to zero and append to the cost function.

where L^* is the optimal strategy for the filter gain and Π_i^* is the Riccati variable using L^*. Note that G_i, $\hat{\Gamma}_i$, and $\hat{\Phi}$ are all functions of L, and so they also use L^* in the necessary conditions above. Therefore, since Ω_i is symmetric by assumption, Δ_i is the solution of a discrete Lyapunov differential equation (5.35) [48]. The optimal filter gain is then determined by solving a two-point boundary value problem which includes a set of discrete Riccati equations (5.32) and Lyapunov equations (5.35) coupled by (5.34) with initial condition (5.24) set equal to zero and final condition (5.36).

Some choices for Ω_i may lead to trivial solutions for the detection filter problem. To explore this concept, rewrite the Lyapunov equation (5.35) as

$$\Lambda_i = \Delta_i + \left(\frac{\delta \left[\text{tr } \Omega_i \left(\Pi_i^*, L^*, k \right) \right]}{\delta \Pi_i^*} \right)^T \tag{5.37}$$

$$\tilde{\Delta}_i = \left(I + \hat{\Gamma}_i^* G_i^* \right) \hat{\Phi}^{*T} \Lambda_i \hat{\Phi}^* \left(I + \hat{\Gamma}_i^* G_i^* \right)^T \tag{5.38}$$

where Δ_i is the *a priori* (propagation) solution and Λ_i is the *a posteriori* (update) solution. The update stage (5.37) is based entirely on $\frac{\delta \Omega_i}{\delta \Pi_i}$. If $\frac{\delta \Omega_i}{\delta \Pi_i} = 0$, then the update is trivial and the solution for Δ_i is based on the propagation alone. However, because of the terminal constraint (5.36), the solution to (5.38) is $\Delta_i = 0$. Also, (5.34) is satisfied trivially in this case, providing no information on how to choose L^*. Therefore, Ω_i must be chosen such that $\frac{\delta \Omega_i}{\delta \Pi_i} \neq 0$. Further, it is generally unnecessary and undesirable to choose $\frac{\delta \Omega_i}{\delta L} \neq 0$, as very simple solutions for L^* may be obtained when $\frac{\delta \Omega_i}{\delta L} = 0$. In general, most enhancements to the detection filter problem may be achieved by choosing $\frac{\delta \Omega_i}{\delta \Pi_i} \neq 0$.

Finally, an example cost function is minimized with respect to L. It was proven in [43] that as $\gamma \to 0$, Π_i obtains a nullspace that contains $\hat{\mathcal{W}}_i$, signifying that the nuisance fault will be blocked from the projected residual. Thus, the optimization should attempt to minimize the transmission of each complementary fault by placing $\hat{F}_i$ approximately in the nullspace of Π_i. Further, the target fault direction F_i should remain in the range space of Π_i so that it is not blocked along with the complementary fault. Thus, choose Ω_i as

$$\text{tr }\Omega_i = \text{tr }\frac{1}{\gamma}K_i\hat{F}_i^T\Pi_i\hat{F}_i - \text{tr }N_iF_i^T\Pi_iF_i \tag{5.39}$$

where K_i and N_i are design weightings on the complementary fault and target fault transmissions, respectively. Thus, the optimization problem (5.31) attempts to choose L such that Π_i has the aforementioned desired structure. When K_i is large, the transmission from $\hat{F}_i$ to the projected residual $\bar{r}_i$ is smaller. When N_i is large, the transmission from F_i to the residual is larger. By differentiating Ω_i with respect to Π_i and substituting into (5.37), Δ_i is subject to the update equation

$$\Lambda_i = \Delta_i + \frac{1}{\gamma}\hat{F}_iK_i\hat{F}_i^T - F_iN_iF_i^T, \tag{5.40}$$

with propagation equation (5.38) and boundary condition (5.36). By substituting (5.32) and (5.38) into (5.34) and using the variants of the matrix inversion lemma (5.33),

$$\begin{aligned}
0 &= \sum_{i=1}^{s}\left[VL^{*T}\Phi^T\tilde{\Pi}_i^* - C\hat{\Phi}^{*-1}\left(I+\hat{\Gamma}_i^*G_i^*\right)^{-T}\right]\tilde{\Delta}_i\tilde{\Pi}_i^*\Phi \\
&= \sum_{i=1}^{s}\left[VL^{*T}\Phi^T\hat{\Gamma}_i\hat{\Phi}^* - C\right]\hat{\Phi}^{*-1}\left(I+\hat{\Gamma}_i^*G_i^*\right)^{-T}\tilde{\Delta}_i\tilde{\Pi}_i^*\Phi \\
&= \sum_{i=1}^{s}\left[VL^{*T}\left(I-L^*C\right)^{-T}\Gamma_i^* - C\right]\Lambda_i\Gamma_i^* \\
&= \sum_{i=1}^{s}\left[V\left(I-CL^*\right)^{-T}L^{*T}\Gamma_i^* - C\right]\Lambda_i\Gamma_i^* \\
&= \sum_{i=1}^{s}\left[L^{*T}\Gamma_i^* - \left(I-CL^*\right)^T V^{-1}C\right]\Lambda_i\Gamma_i^* \\
&= \sum_{i=1}^{s}\left[L^{*T}\left(\Gamma_i^* + C^TV^{-1}C\right) - V^{-1}C\right]\Lambda_i\Gamma_i^*
\end{aligned}$$

Thus, the optimal filter gain is

$$L^* = \left[\sum_{i=1}^{s} \Gamma_i^* \Lambda_i \left(\Gamma_i^* + C^T V^{-1} C \right) \right]^{-1} \left[\sum_{i=1}^{s} \Gamma_i^* \Lambda_i C^T V^{-1} \right]. \tag{5.41}$$

Remark 5.2: In order to numerically solve for the detection filter gain and error covariance, it is desirable to use linear matrix inequalities. However, (5.23) is nonlinear in Π_i and L. Further, it cannot be expanded into an LMI since $\tilde{\Pi}_i$ and Γ_i are both positive or nonnegative-definite quantities. Thus, a gradient descent algorithm is the best option to obtain a numerical solution. ♢

Remark 5.3: During the derivation of the DGTMFDF constraints and optimal filter gain, it has not been necessary to assume that the Riccati solutions are invertible. For the example filter gain optimization cost function above, the solution exists as long as the nullspaces of $\Gamma_i^* \Lambda_i \left(\Gamma_i^* + C^T V^{-1} C \right)$ do not overlap over all DAPs. However, the single-fault case is the exception in which Γ_i must be invertible to obtain the optimal solution.

♢

5.3 Comparison to the Single-Fault Detection Filter

In this section, the DGTMFDF is compared to the only previous discrete Riccati-based robust fault detection method, the (single-fault) DGTFDF [43]. It is shown that the DGTMFDF problem formulation generalizes that of the single-fault detection filter.

In [43], it was proven that the single-fault DAP (assumes $s = 1$) is satisfied when the Riccati variable Θ is propagated by the update and propagation equations

$$\Sigma = \Theta - C^T \left(\hat{H} Q \hat{H} - V^{-1} \right) C \tag{5.42a}$$

$$\tilde{\Theta} = \left[\Phi \Lambda^{-1} \Phi^T + \frac{1}{\gamma} \hat{F} M \hat{F}^T \right]^{-1} \tag{5.42b}$$

where $\Theta(t_0) = \Theta_0$ and the solution for the filter gain L is

$$L = \Sigma^{-1} C^T V^{-1}. \tag{5.43}$$

Through some algebraic manipulation, (5.32) may be rewritten using a new *a posteriori* Riccati solution Λ_i as

$$\Lambda_i = \Gamma_i + C^T V^{-1} C = \Pi_i - C^T \left(\hat{H}_i Q_i \hat{H}_i - V^{-1} \right) C \tag{5.44a}$$

$$\tilde{\Pi}_i = \left(\Phi \left[\Lambda_i^{-1} + \left(LV - \Lambda_i^{-1} C^T \right) \left(V - C \Lambda_i^{-1} C^T \right)^{-1} \left(V L^T - C \Lambda_i^{-1} \right) \right] \Phi^T + \frac{1}{\gamma} \hat{F}_i M_i \hat{F}_i^T \right)^{-1} \tag{5.44b}$$

Clearly, (5.44) is simply (5.42) with an added quadratic term. When $L = \Lambda_i^{-1} C^T V^{-1}$, this extra term is zero and (5.44b) becomes identical to (5.42b). Thus, (5.42) and (5.43) are a special case of the solution to (5.30).

The advantage of (5.42) and (5.43) is that Θ can be computed independently of L, thereby simplifying the calculation of the filter gain. However, the DGTMFDF problem generalizes the solution to the single-fault problem in two ways. First, by adding an additional term to the Riccati constraint as in (5.44), the filter gain can be chosen achieve secondary objectives. For example, let

$$L = \Lambda_i^{-1} C^T V^{-1} + L_i$$

where $L_i \in \mathbb{R}^{m \times n}$ is an arbitrary matrix. Then, (5.44b) becomes

$$\tilde{\Pi}_i = \left(\Phi \left[\Lambda_i^{-1} + L_i V \left(V - C \Lambda_i^{-1} C^T \right)^{-1} V L_i^T \right] \Phi^T + \frac{1}{\gamma} \hat{F}_i M_i \hat{F}_i^T \right)^{-1},$$

and it may be possible to choose or optimize L_i to improve transmission of the target fault (though possibly at the expense of nuisance fault blocking and/or dynamic stability). Such optimization of the filter gain was the subject of Section 5.2.2. Second, since the Riccati constraint of the DGTMFDF problem is an inequality constraint, one may determine a range of possible (sub-optimal and optimal) solutions to the single-fault problem. This is important, for example, in cases where the optimal solution is physically unattainable (e.g. - due to mechanical limitations).

5.4 Lemmas and Propositions

Lemma 5.1: If $F_1, \ldots, F_s, \hat{F}$ are (C, Φ)-output separable, then

$$\hat{\mathcal{W}}_i^* = \mathcal{W}_1^* \oplus \ldots \oplus \mathcal{W}_{i-1}^* \oplus \mathcal{W}_{i+1}^* \oplus \ldots \oplus \mathcal{W}_s^* \oplus \hat{\mathcal{W}}^*.$$

∎

Proof: The proof is identical to that of Lemma 3.3, replacing the continuous time-varying state dynamics matrix A with the discrete time-invariant state transition matrix Φ.

QED

Proposition 5.2: Assuming that Γ_i and Π_i satisfy the constraints (5.23) and (5.25),

$$\Gamma_i \geq 0,\ \Pi \geq 0\ \forall\ k \in \{k_0, \ldots, k_1\}.$$

∎

Proof: To the contrary, let there exist a vector ν_1 such that

$$\Pi_i \nu_1 = \lambda_1 \nu_1,\ \lambda_1 < 0.$$

Pre-multiplying (5.17) by ν_1^T and post-multiplying by ν_1 yields

$$\begin{aligned}
\nu_1^T \Gamma_i \nu_1 &= \nu_1^T \left(\Pi_i - C^T \hat{H}_i Q_i \hat{H}_i C \right) \nu_1 \\
&= \nu_1^T \left(\lambda_1 I - C^T \hat{H}_i Q_i \hat{H}_i C \right) \nu_1 \leq 0
\end{aligned}$$

Therefore, if Π_i has a negative eigenvalue, then Γ_i, and by extension $\hat{\Gamma}_i$, has one as well.

Next, define the vector ν_2 such that

$$\hat{\Gamma}_i \nu_2 = \lambda_2 \nu_2, \;\; \lambda_2 < 0.$$

Pre-multiplying (5.27) by ν_2^T and post-multiplying by ν_2 yields

$$\begin{aligned}
\nu_2^T \tilde{\Pi}_i \nu_2 &\leq \nu_2^T \left(\hat{\Gamma}_i - \hat{\Gamma}_i \begin{bmatrix} \hat{F}_i & -\Phi L \end{bmatrix} S_i^{-1} \begin{bmatrix} \hat{F}_i^T \\ -L^T \Phi^T \end{bmatrix} \hat{\Gamma}_i \right) \nu_2 \\
&\leq \nu_2^T \left(\lambda_2 I - \lambda_2^2 \begin{bmatrix} \hat{F}_i & -\Phi L \end{bmatrix} S_i^{-1} \begin{bmatrix} \hat{F}_i^T \\ -L^T \Phi^T \end{bmatrix} \right) \nu_2 \leq 0
\end{aligned}$$

where from (5.26)

$$S_i = \begin{bmatrix} \gamma M_i^{-1} + \hat{F}_i^T \hat{\Gamma}_i \hat{F}_i & -\hat{F}_i \hat{\Gamma}_i \Phi L \\ -L^T \Phi^T \hat{\Gamma}_i \hat{F}_i & V^{-1} + L^T \Phi^T \hat{\Gamma}_i \Phi L \end{bmatrix} > 0.$$

Therefore, if Γ_i has a negative eigenvalue, then $\tilde{\Pi}_i$ has one as well.

The two statements above imply that there will always exist a negative eigenvalue associated with Π_i and Γ_i. However, the terminal boundary constraint (5.25) requires that Π_i reach a nonnegative value, and so a contradiction is reached. Therefore, there can be no negative eigenvalues associated with Π_i and Γ_i. **QED**

Chapter 6

Numerical Examples

6.1 Example 1

In this section, a linear time-invariant numerical example for the F16XL aircraft [24, 41, 45] is used to demonstrate the performance of the GTMFDF from Chapter 3. The system has four states (longitudinal velocity x_u, normal velocity x_w, pitch rate x_q, and pitch angle x_θ), one control input (elevon deflection angle u_δ), four measurements (longitudinal velocity y_u, normal velocity y_w, pitch rate y_q and pitch angle y_θ), one disturbance input (wind gust μ_{wg}), and sensor noise v. The system matrices are

$$A = \begin{bmatrix} -0.0674 & 0.0430 & -0.8886 & -0.5587 \\ 0.0205 & -1.4666 & 16.5800 & -0.0299 \\ 0.1377 & -1.6788 & -0.6819 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_\delta = \begin{bmatrix} -0.1672 \\ -1.5179 \\ -9.7842 \\ 0 \end{bmatrix}, \quad B_{wg} = \begin{bmatrix} 0.0430 \\ -1.4666 \\ -1.6788 \\ 0 \end{bmatrix},$$

and $C = I$. Three faults, including faults in the pitch angle sensor y_θ, elevon deflector u_δ, and wind gust u_{wg}, are considered. The fault directions used for the detection filter design are

$$F_\theta = \begin{bmatrix} 0 & 0.5587 \\ 0 & 0.0299 \\ 0 & 0 \\ 1.0000 & 0 \end{bmatrix}, \quad F_\delta = B_\delta, \quad F_{wg} = B_{wg}.$$

Note that the pitch angle sensor fault enters the measurement equation (3.1b) as an additive input $F_{s,\theta}\mu_\theta$ with fault direction $F_{s,\theta} = [0\ 0\ 0\ 1]^T$. However, by using the method described in Section 2.1.1, the sensor fault is converted into an additive input to the dynamics $F_\theta[\,\dot{\mu}_\theta\ \mu_\theta\,]^T$ with fault direction $F_\theta = [\ f_\theta\ \ -Af_\theta\ ]$ where $Cf_\theta = F_{s,\theta}$. It can be verified that the three faults are output separable, mutually detectable, and have no invariant zeros at the origin. Further, assume that the covariance of the measurement noise is $\bar{V} = 10^{-6}I$.

The GTMFDF is demonstrated for this problem using three different methods. In Section 6.1.1, the detection filter problem is solved using three single-fault detection filters, like a bank of UIOs. For simplicity, the solution is made equivalent to the GTFDF solution (3.29) and (3.30). In Section 6.1.2, the detection filter is derived as a multiple-fault detection filter that detects all modeled faults, like the BJDF. In Section 6.1.3, the detection filter is derived as a multiple-fault detection filter that detects only two of the modeled faults, blocking the third from the residual. Finally, in Section 6.1.4, target fault sensitivity is enhanced for the UIO-type and RDDF-type detection filters. For all of the implementations below, assume the disturbance attenuation bound is $\gamma = 10^{-6}$.

Remark 6.1: The example is simulated in MATLAB. Typically, the optimal multiple-fault filter gain may be obtained using the "fminunc" function, which computes a numerical solution to an unconstrained minimization problem (using the continuous algebraic Riccati equation function "care" to solve for the required Riccati solutions at each iteration). This function also requires an initial guess for the filter gain, which may be calculated via the suggestion in [45]. However, the "care" function returns an error when the eigenvalues of the problem's Hamiltonian stray too close to the imaginary axis. Thus, to facilitate convergence to an asymptotically stable numerical solution, two additions to this basic algorithm are introduced using the constrained functional minimization algorithm "fmincon". These include:

- A constraint so that the detection filter eigenvalues remain asymptotically stable
- A constraint so that the Hamiltonian eigenvalues do not fall on the imaginary axis

From Theorem 3.1, a nonnegative-definite stabilizing solution for the Riccati equation exists when the above constraints hold. ♢

6.1.1 Unknown Input Observer

In this section, three detection filters, one to detect each fault, are derived in the form of a bank of UIOs for which $s = 1$. Let $F_1 = F_\theta$, $F_2 = F_\delta$, and $F_3 = F_{wg}$. The weights on the complementary faults directions $\hat{F}_i$ $\forall i \in \{1, 2, 3\}$ for each Riccati constraint (3.29) are $M_i = 0.01 \cdot I$ where I is the appropriately dimensioned identity matrix. Note that $\hat{F}_1 = [\, F_2 \;\; F_3 \,]$, $\hat{F}_2 = [\, F_1 \;\; F_3 \,]$, and $\hat{F}_3 = [\, F_1 \;\; F_2 \,]$. The weights on the projected output error for each Riccati constraint are $Q_i = 0.0001 \cdot I$. The weight on the measurement noise is $V = \frac{1}{\gamma}\bar{V} = I$.

The steady-state solutions to (3.29) are obtained for $i = 1, 2, 3$, respectively. Then, three single-fault detection filters are obtained via (3.30). Recall that the aforementioned equations are a special case of the solution to (3.28). Each detection filter has only one projected residual $\hat{H}_i r_i$ to detect fault F_i, where $\hat{H}_i$ is defined *a priori* by (3.10). The detection filter is demonstrated by integrating the dynamic system (3.1) and detection filter (3.7) simultaneously with a unit bias fault starting at $t = 2s$ in y_θ, u_δ, or u_{wg}. Note that it is assumed that the nominal control input $u_\delta = 0$.

In Fig. 6.1, the frequency responses of the three detection filters are graphed with respect to each fault direction. The blue and green lines represent the rate and magnitude of the pitch angle sensor fault, respectively, which should be detected by the first detection filter.

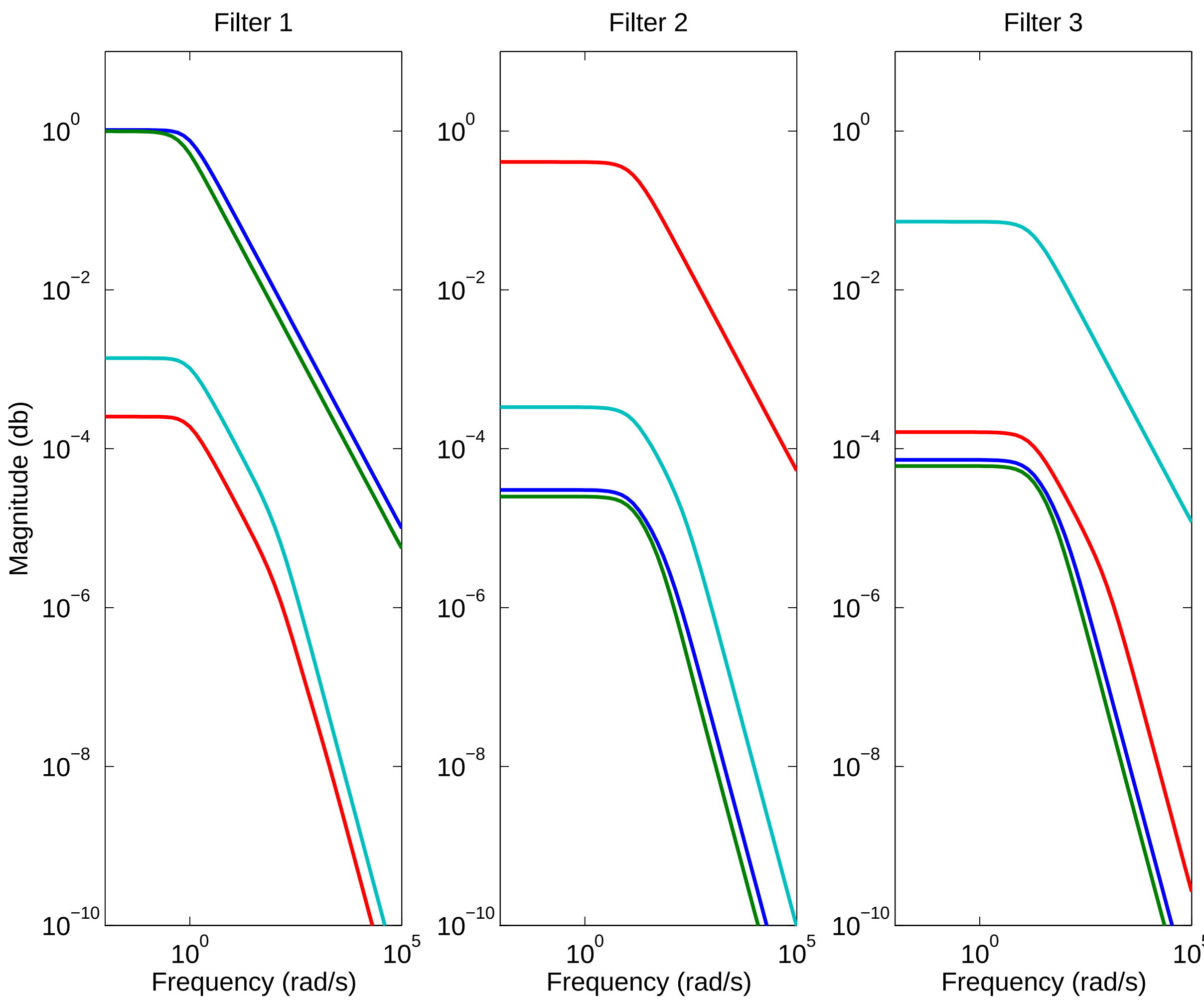

Figure 6.1: Projected residual frequency responses for UIO-type filter

The red line represents the elevon deflector fault, which should be detected by the second filter. The cyan line represents the wind gust fault, which should be detected by the third filter. The figure shows that the projected residual of each detection filter is sensitive only to its associated target fault. This notion is supported by Fig. 6.2, which graphs the time histories of the projected residuals to the unit bias faults. The figure shows that the target fault in each case is clearly detectable, even in the presence of measurement noise.

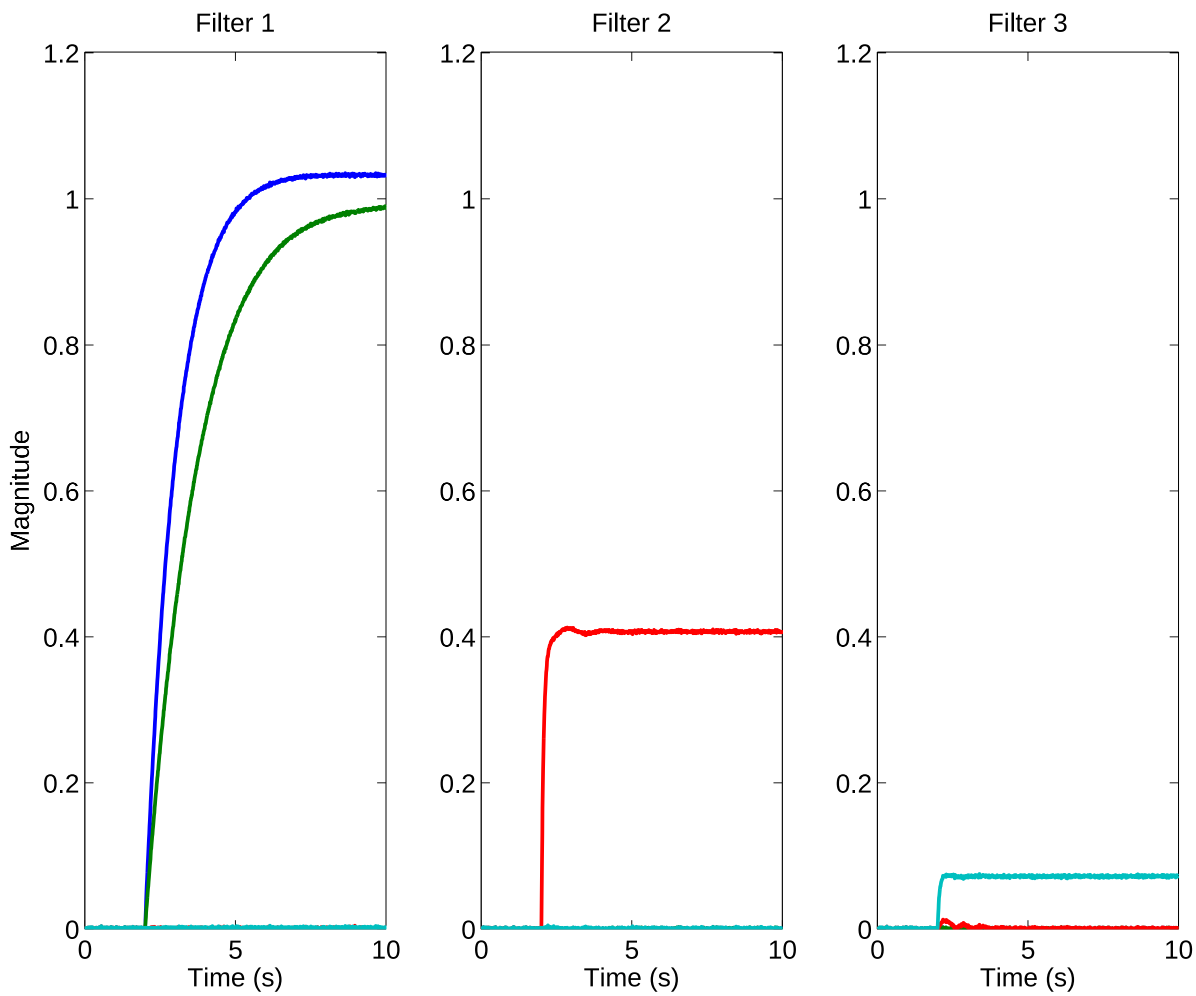

Figure 6.2: Projected residual time histories for UIO-type filter

6.1.2 Beard-Jones Detection Filter

In this section, a single detection filter to detect all three faults is derived in the form of a BJDF for which $s = 3$. The Riccati constraint weights are kept the same as for the UIO-type filters above. The filter gain is obtained by numerically solving the optimization problem (3.27) subject to the steady-state Riccati constraint (3.28). The weights on the complementary fault transmissions for the cost function are $K_1 = 0.01 \cdot I$, $K_2 = 0.01 \cdot I$, and $K_3 = 0.001 \cdot I$, respectively. The weights on the target fault transmissions are $N_1 = I$, $N_2 = 1.5 \cdot I$, and $N_3 = 0.1 \cdot I$, respectively.

In Fig. 6.3, the frequency response of the detection filter is graphed with respect to each projected residual and each fault direction. The figure shows that each projected residual is sensitive only to its associated target fault. This notion is supported by Fig. 6.4, which graphs the time histories of the projected residuals to unit step inputs in each fault. The figure shows that the target fault in each case is clearly detectable.

6.1.3 Restricted Diagonal Detection Filter

In this section, a single detection filter to detect two of the three faults is derived in the form of a RDDF for which $s = 2$. It is desired to detect the pitch angle sensor and elevon deflector faults in the presence of a wind gust disturbance and sensor noise. Thus, $F_1 = F_\theta$ and $F_2 = F_\delta$ are the target fault directions and $\hat{F} = F_{wg}$ is the nuisance fault direction. Two residual projectors are obtained using (3.10) to isolate the two target faults. The Riccati constraint and cost function weights remain the same as for the BJDF-type filter except that for this case $Q_1 = 0.0002 \cdot I$.

In Fig. 6.5, the frequency response of the detection filter is graphed with respect to each projected residual and each fault direction. The figure shows that each projected residual is sensitive only to its associated target fault. This notion is supported by Fig. 6.6, which graphs the time histories of the projected residuals to unit step inputs in each fault. The

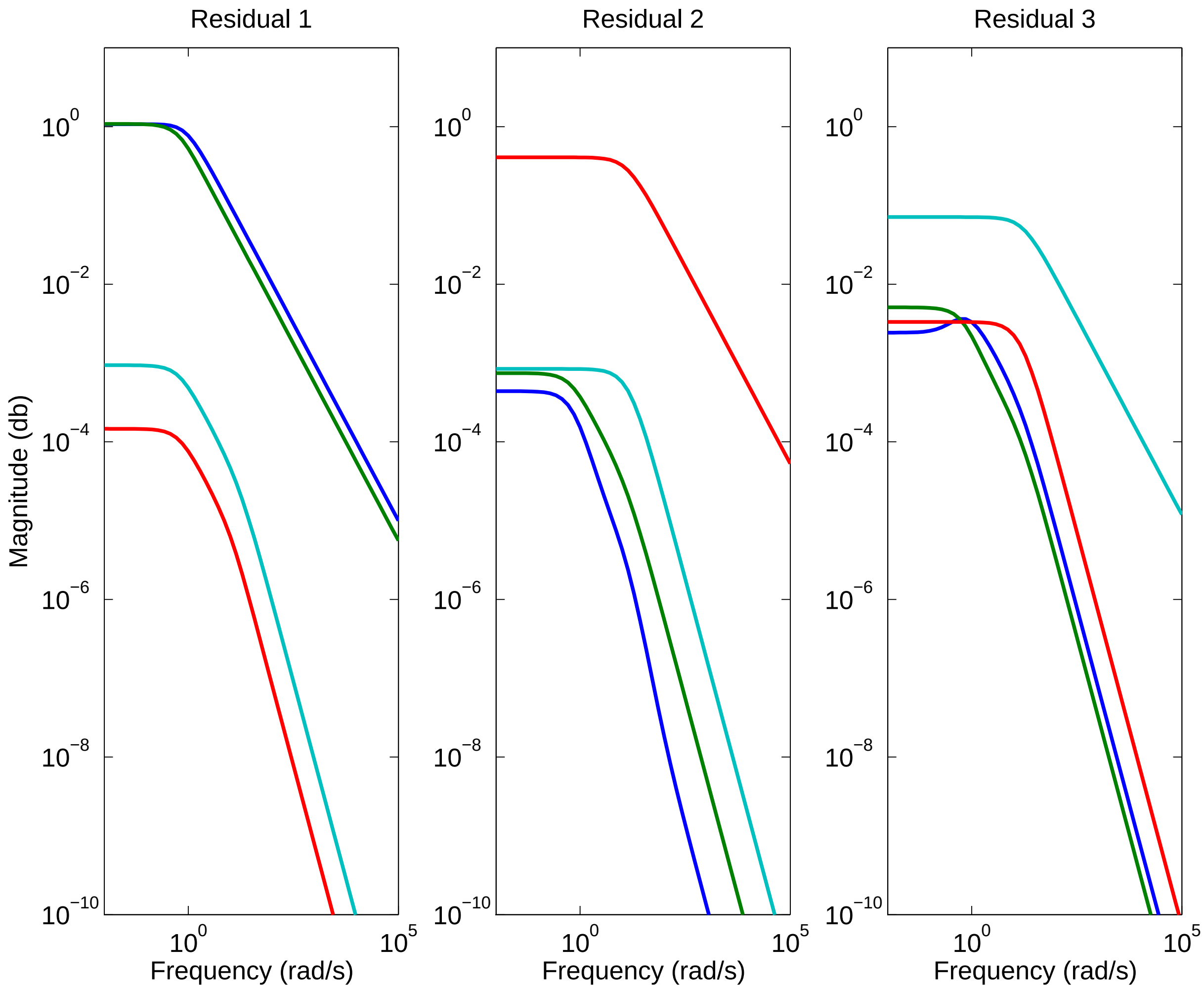

Figure 6.3: Projected residual frequency responses for BJDF-type filter

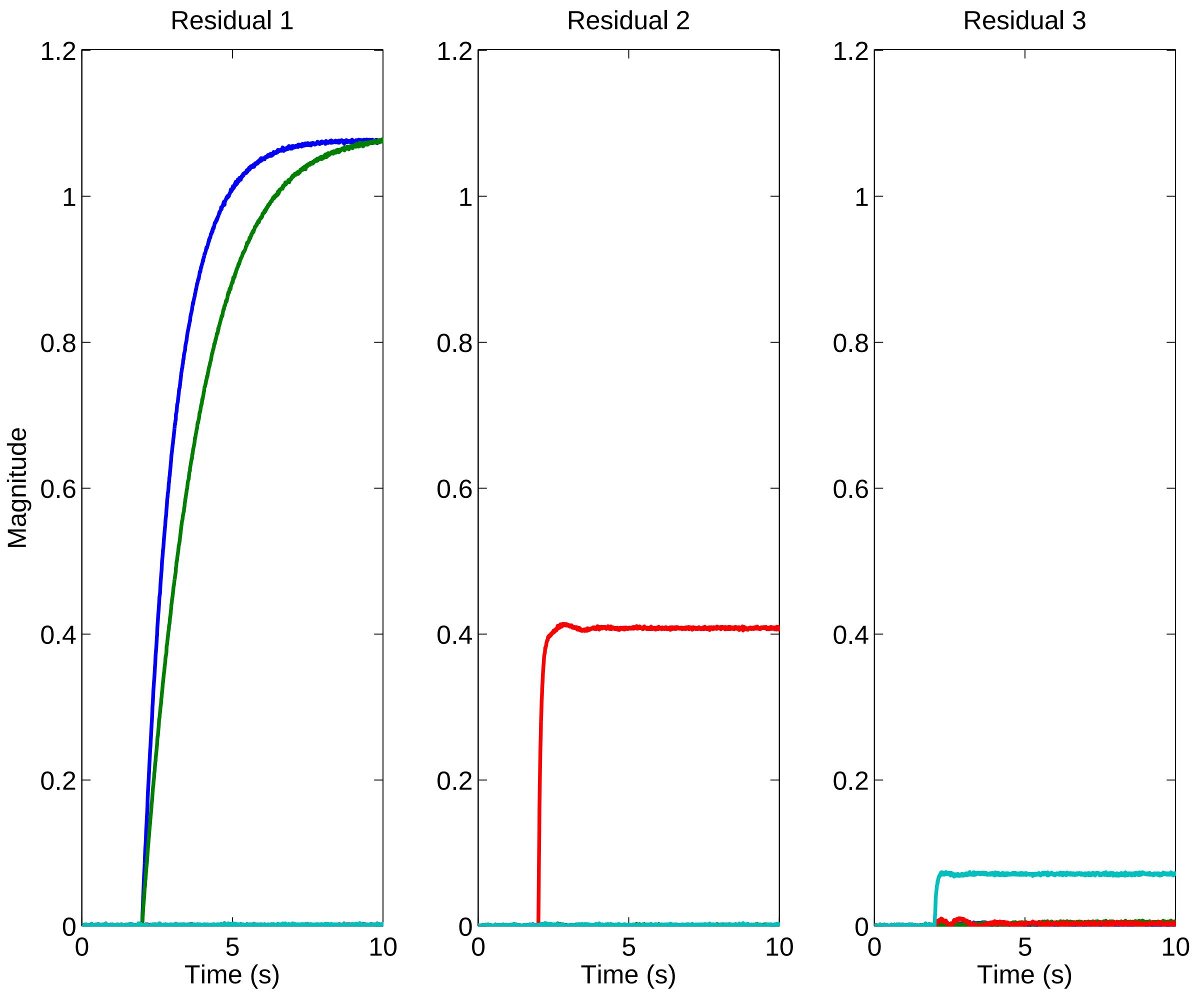

Figure 6.4: Projected residual time histories for BJDF-type filter

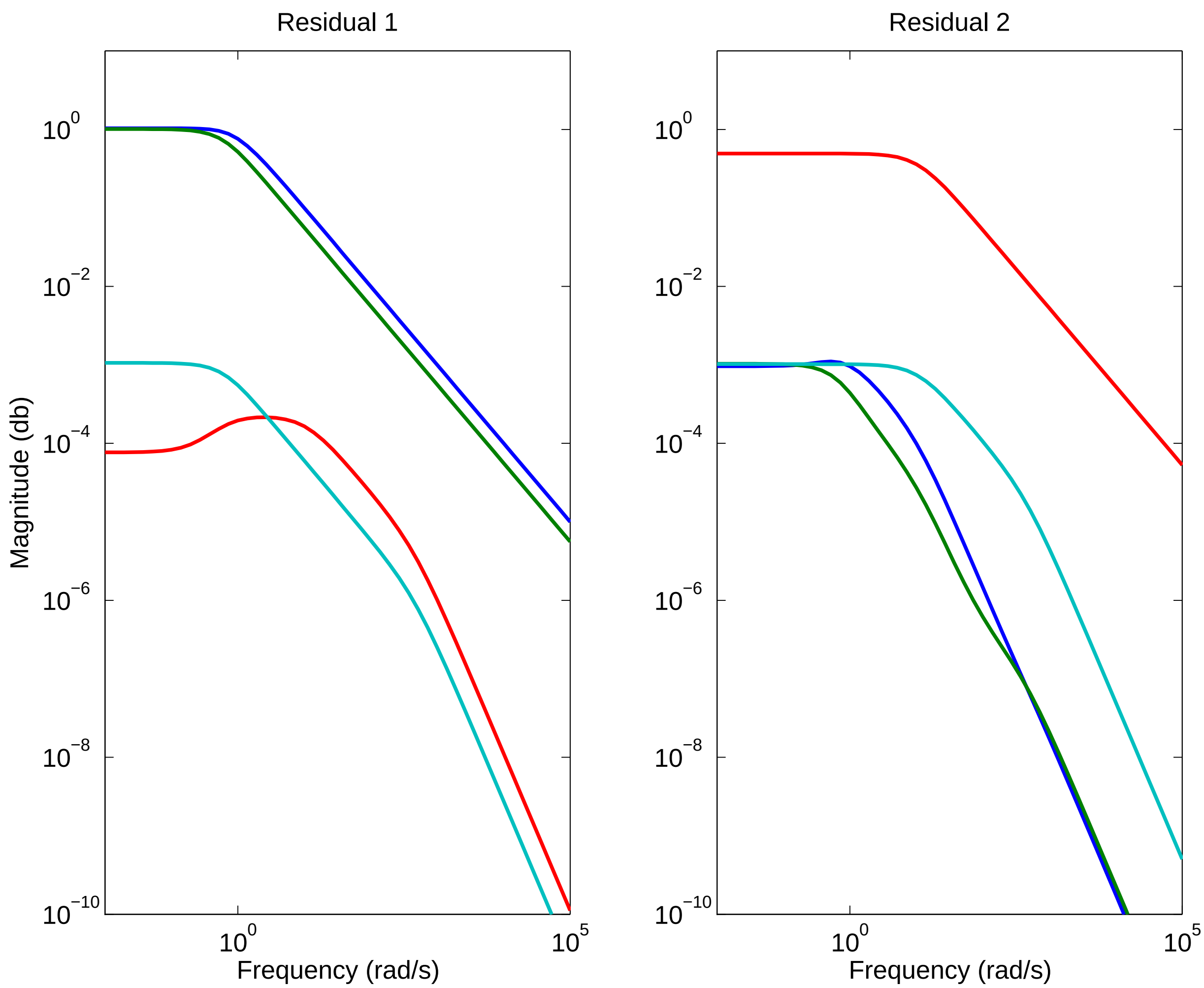

Figure 6.5: Projected residual frequency responses for RDDF-type filter

figure shows that the target fault in each case is clearly detectable. Therefore, the GTMFDF sufficiently isolates the target faults and blocks the nuisance fault.

6.1.4 Enhancement of Target Fault Sensitivity

In this section, the ability of the GTMFDF to enhance sensitivity to the target fault is demonstrated for single-fault and multiple-fault problems.

First, the set of UIO-type filters in Section 6.1.1 is derived with new Riccati weights. The new weights on the complementary fault directions for each Riccati constraint are $M_1 = 0.08 \cdot I$, $M_2 = 0.01 \cdot I$, and $M_3 = 0.1 \cdot I$. The weights on the projected output error for each

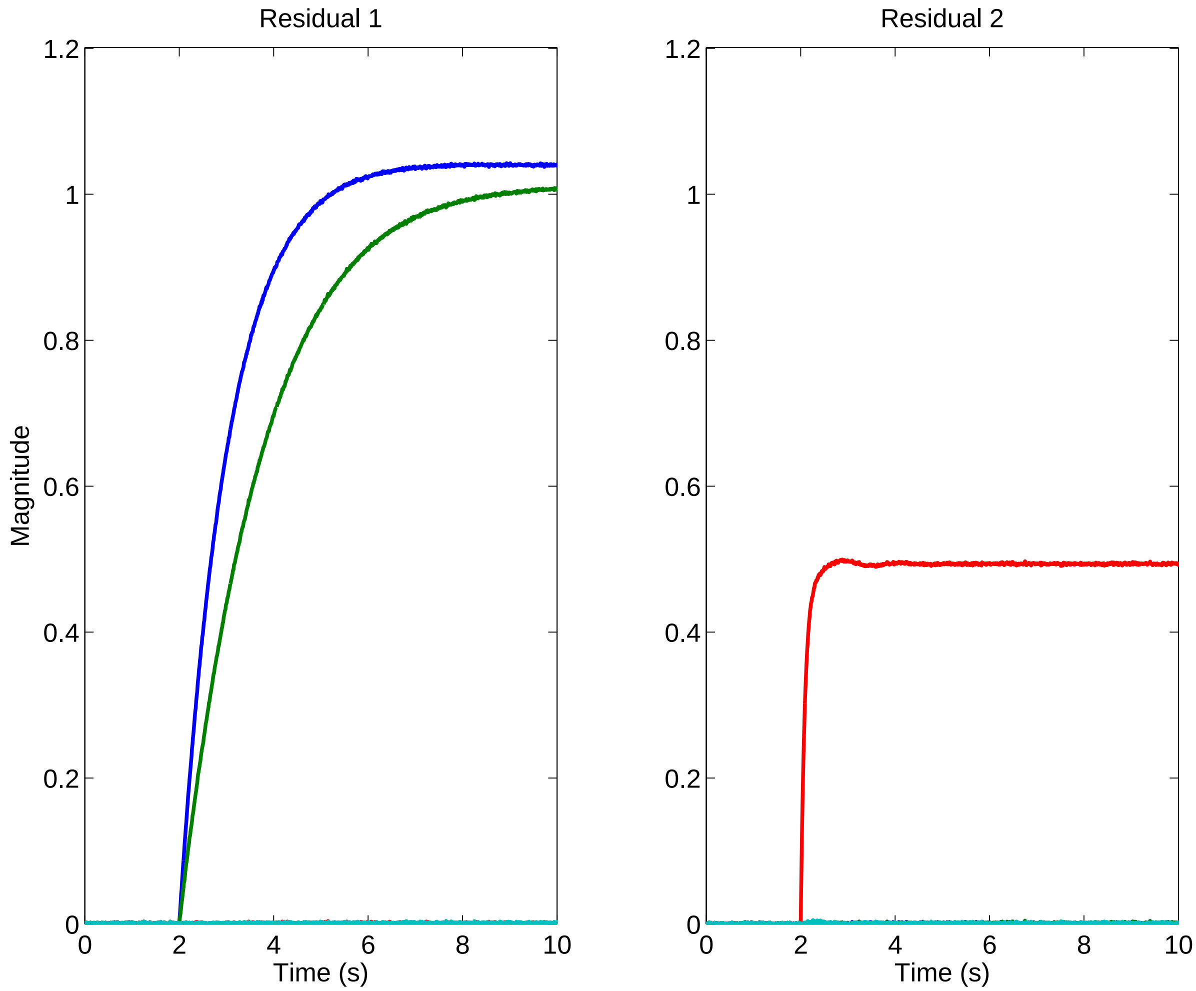

Figure 6.6: Projected residual time histories for RDDF-type filter

Riccati constraint are $Q_1 = 0.0002 \cdot I$, $Q_2 = 0.001 \cdot I$, and $Q_3 = 0.0001 \cdot I$.

In order to enhance target fault sensitivity, the optimization problem (3.27) is applied to the single-fault detection filter problem. The weights on the complementary fault transmissions for the cost function are $K_1 = 0.01 \cdot I$, $K_2 = 0.01 \cdot I$, and $K_3 = 0.001 \cdot I$. The weights on the target fault transmissions are $N_1 = 200 \cdot I$, $N_2 = 50 \cdot I$, and $N_3 = 10000 \cdot I$.

In Fig. 6.7, the projected residual frequency responses to their associated target faults for the three new detection filters are compared to those of the detection filters in Section 6.1.1. The color schemes are the same as before. The solid lines represent the responses of the new detection filters while the dashed lines represent the old. The figure shows that new weights have increased the frequency responses of the detection filters. This notion is supported by Fig. 6.8, which graphs a comparison of the time histories of the projected residuals to unit bias faults for the old and new filters. The figure shows that the target fault responses are much larger than before. In each case, the target fault is clearly detectable compared to both the old and the new nuisance fault responses.

Next, the RDDF-type filter in Section 6.1.3 is derived with new Riccati and cost function weights. The new weights on the complementary fault directions for each Riccati constraint are $M_1 = 0.01 \cdot I$ and $M_2 = 0.1 \cdot I$. The weights on the projected output error for each Riccati constraint are $Q_1 = 0.0002 \cdot I$ and $Q_2 = 0.001 \cdot I$. The weights on the complementary fault transmissions for the cost function are $K_1 = I$ and $K_2 = I$. The weights on the target fault transmissions are $N_1 = 12 \cdot I$ and $N_2 = 100 \cdot I$.

In Fig. 6.9, the two projected residual frequency responses to their associated target faults for the new detection filter are compared to those of the detection filter in Section 6.1.3. As before, the solid lines represent the responses of the new detection filter while the dashed lines represent the old. The figure shows that new weights have increased the frequency responses of the detection filter. This notion is supported by Fig. 6.10, which graphs a comparison of the time histories of the projected residuals to unit bias faults for the old and new filters. The figure shows that the target fault responses are much larger than before.

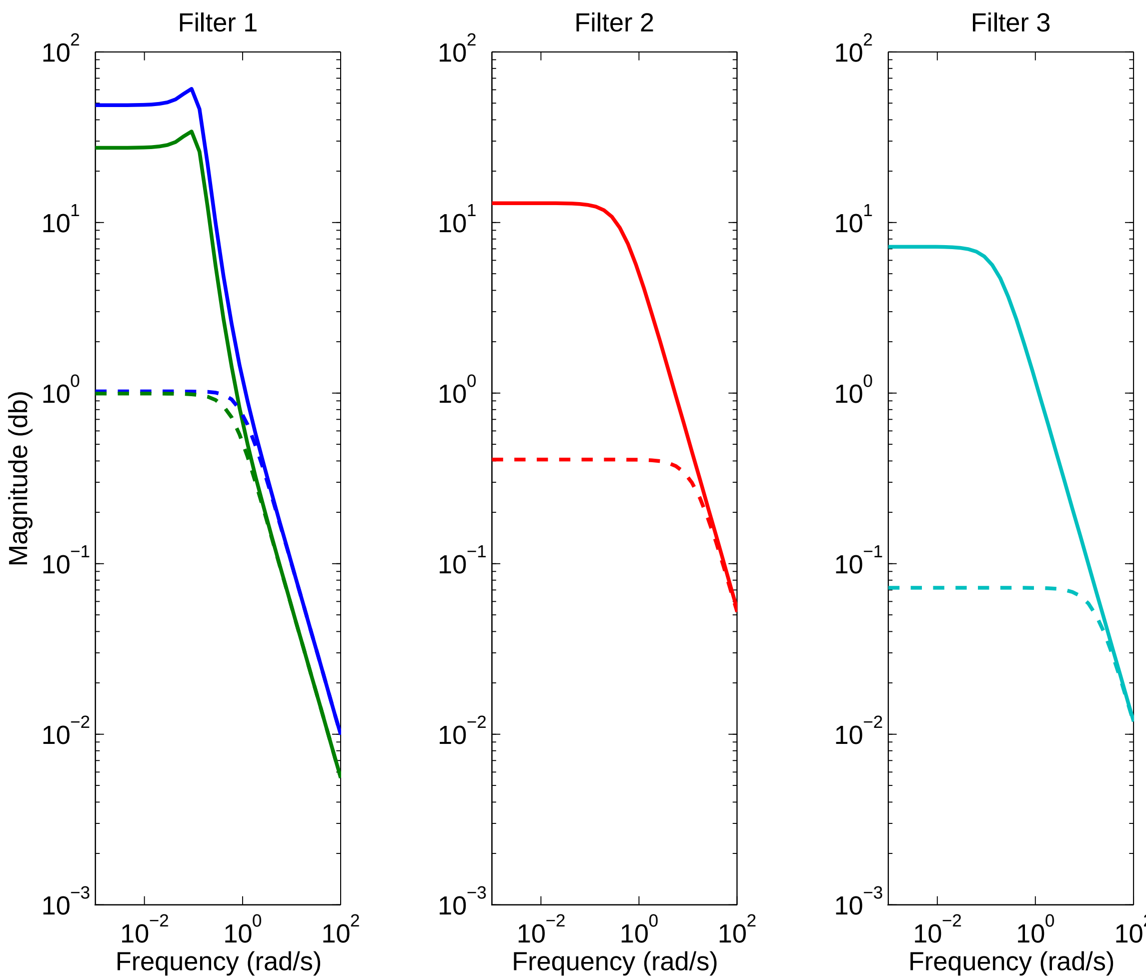

Figure 6.7: Comparison of target fault projected residual frequency responses for UIO-type filter

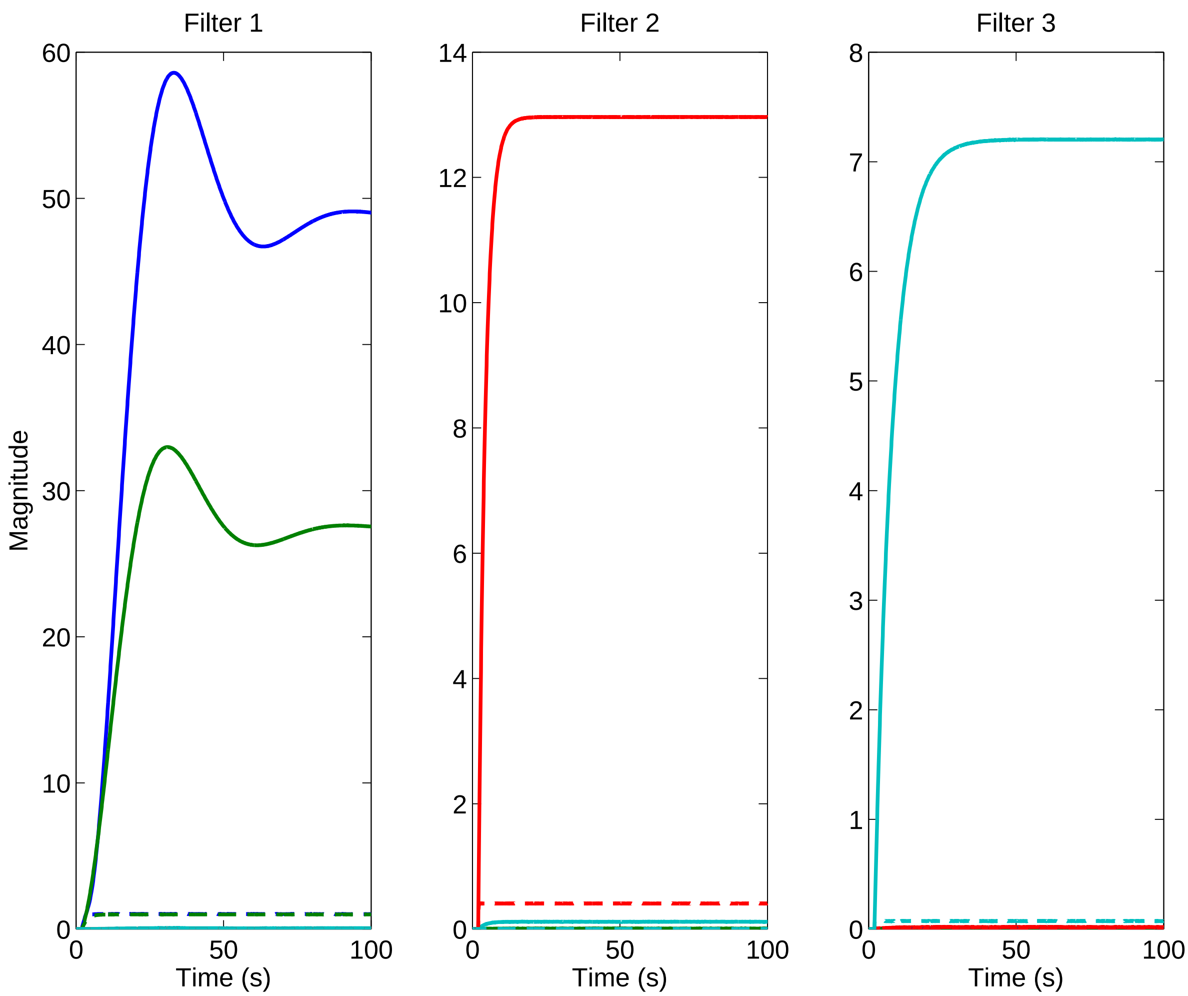

Figure 6.8: Comparison of projected residual time histories for UIO-type filter

In each case, the target fault is clearly detectable compared to both the old and the new nuisance fault responses.

Remark 6.2: In the previous cases of the GTMFDF, solutions were obtained using the MATLAB function "fmincon" using the constraints mentioned at the beginning of this section. However, attempts to increase target fault sensitivity were often met with algorithmic failure due to the constraints being violated and "care" returning an error. To facilitate convergence to a solution in the enhanced sensitivity case, when "care" returned an error, the cost function was artificially forced to take on an extremely large value. In that way, the solution would be driven away from values that cause algorithmic failure. The weights in this section were chosen as acceptable after careful monitoring of solution convergence during the algorithm. $\diamondsuit$

6.2 Example 2

In this section, a linear time-invariant numerical example taken from [19, 41, 45] is used to demonstrate that the GTMFDF has behaviors similar to the RDDF and BJDF with respect to invariant zeros. The system matrices are

$$A = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 2 & 5 \end{bmatrix}, \quad B = 0, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Further, assume that the covariance of the measurement noise is $\bar{V} = 10^{-6} I$ and let the disturbance attenuation bound be $\gamma = 10^{-6}$.

The GTMFDF is demonstrated for four different sets of faults. In Section 6.2.1, the detection filter is derived when one of the faults is associated with an stable invariant zero. In Section 6.2.2, the detection filter is derived when one of the faults is associated with an

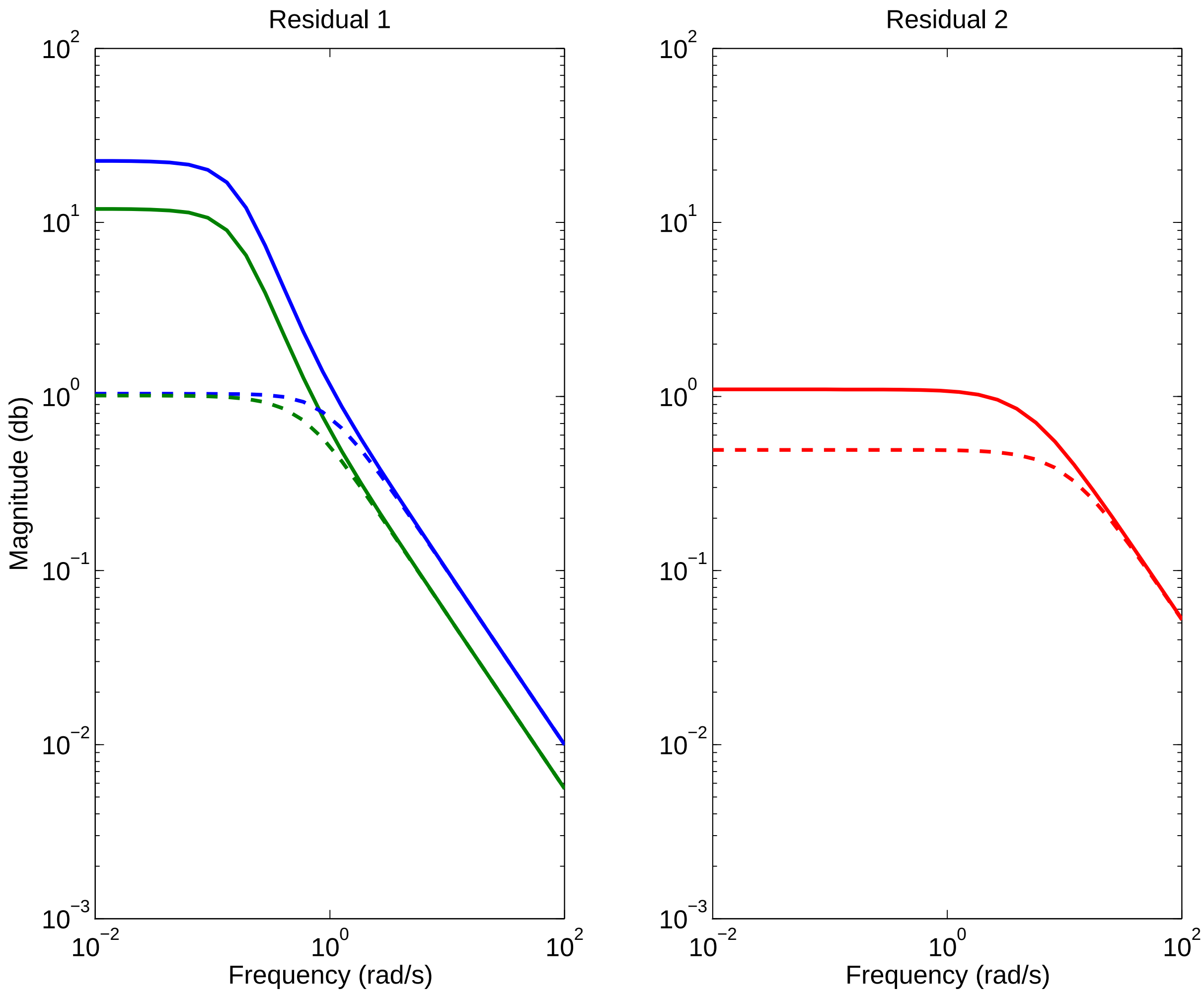

Figure 6.9: Comparison of target fault projected residual frequency responses for RDDF-type filter

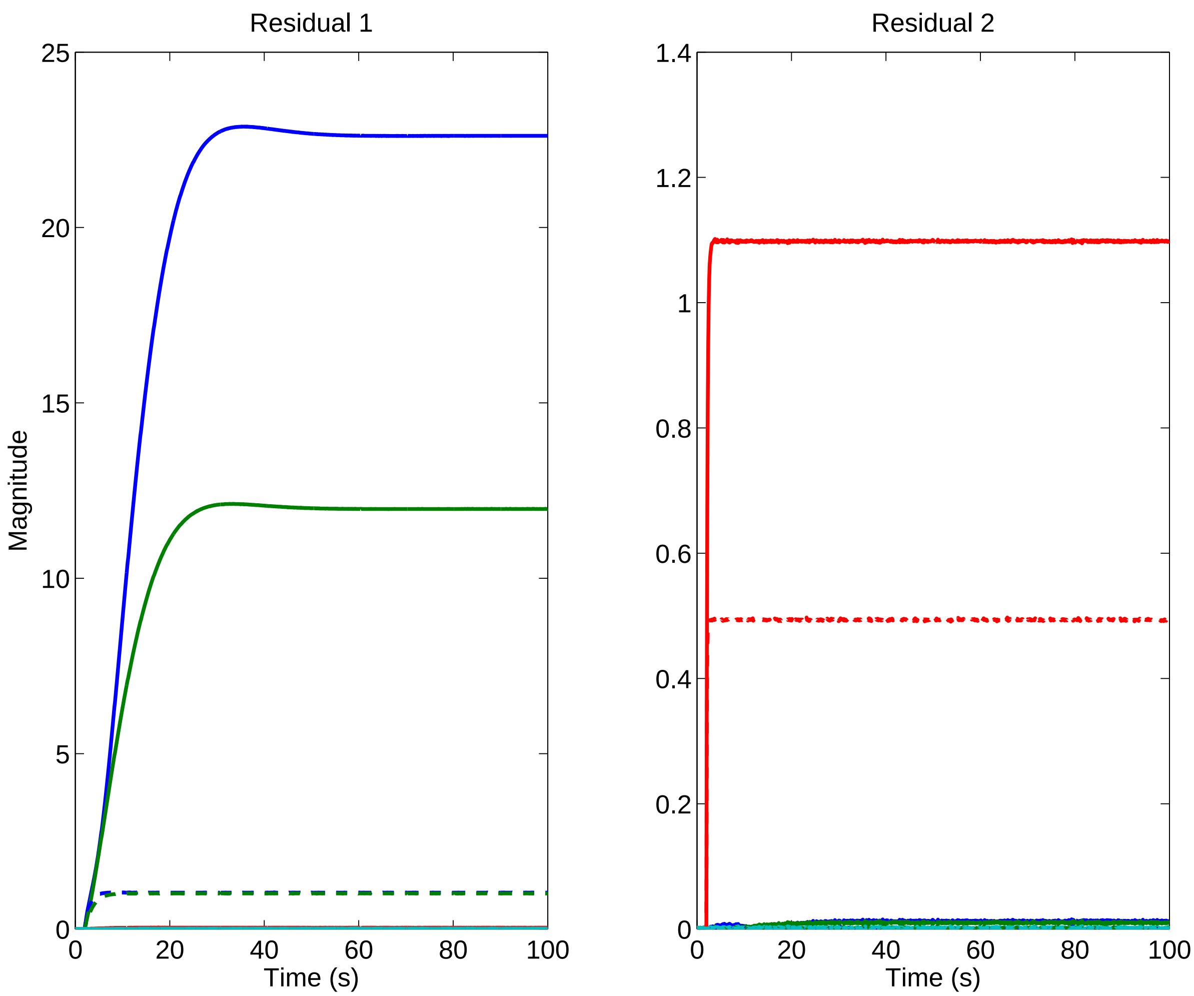

Figure 6.10: Comparison of projected residual time histories for RDDF-type filter

unstable invariant zero. In Section 6.2.3, the detection filter is derived when the faults are not mutually detectable and have a stable invariant zero. In Section 6.2.4, the detection filter is derived when the faults are not mutually detectable and have an unstable invariant zero. For each case below, the first fault direction is defined as

$$F_1 = \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix}$$

and the second fault is modified to produce the intended invariant zero.

6.2.1 Stable Invariant Zero

In this section, the second fault direction is defined as

$$F_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

It can be shown that the two faults are output separable and mutually detectable. F_2 is associated with an invariant zero at -3 with direction $\nu = [1\ 0\ 0]^T$. By using (2.13), $\mathcal{T}_1 = \hat{\mathcal{T}}_2 = \text{Im}\ F_1$ and $\mathcal{T}_2 = \hat{\mathcal{T}}_1 = \text{Im}[\,F_2\ \nu\,]$. Since $\mathcal{T}_1 \oplus \mathcal{T}_2 = \mathbb{R}^{3\times 3}$, there is no complementary subspace.

The detection filter to detect both faults is derived in the form of a BJDF for which $s = 2$. The weights on the complementary faults directions $\hat{F}_i\ \forall i \in \{1, 2\}$ for the Riccati constraint (3.28) are $M_i = 0.05$. Note that M_i is a scalar since $\hat{F}_i$ is a vector. The weights on the projected output error for each target fault are $Q_1 = 0.001 \cdot I$ and $Q_2 = 0.002 \cdot I$. The weight on the measurement noise is $V = \frac{1}{\gamma}\bar{V} = I$. The filter gain is obtained similarly to the BJDF-type filter in Section 6.1.2. The weights on the complementary fault transmission for the cost function of (3.27) are $K_i = 0.1 \ \forall i \in \{1, 2\}$. The weights on the target fault transmissions are $N_1 = 5$ and $N_2 = 10$, respectively.

The eigenvectors of the detection filter dynamics are very close to $\mathcal{T}_1$ and $\mathcal{T}_2$. Further, the eigenvalues are -0.6779, -5.1079, and -6.8872. Since the invariant zero direction is approximately included in $\mathcal{T}_2$, none of the eigenvalues are close to the invariant zero at -3.

Remark 6.3: That the GTMFDF includes the directions associated stable invariant zeros in its detection spaces represents a major improvement over the single-fault GTFDF. It can be shown that the GTFDF does not include stable invariant zeros in its nuisance fault detection space. This is confirmed by designing the single-fault detection filter to detect F_1 with Riccati weights identical to those above, producing a detection filter with eigenvalues at -3.0003, -5.4041, and -223.63. Note that there is an eigenvalue very close to the invariant zero at -3. However, because the GTMFDF automatically includes these directions in its detection spaces, none of the eigenvalues are close to the invariant zero. Thus, the stability of the closed-loop system is unaffected by this property of the open-loop system. ♢

6.2.2 Unstable Invariant Zero

In this section, the second fault direction is defined as

$$F_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

It can be shown that the two faults are output separable and mutually detectable. F_2 is associated with an invariant zero at $+3$ with direction $\nu = [1 \ 0 \ 0]^T$. By using (2.13), $\mathcal{T}_1 = \hat{\mathcal{T}}_2 = \text{Im}\ F_1$ and $\mathcal{T}_2 = \hat{\mathcal{T}}_1 = \text{Im}[\,F_2 \ \nu\,]$. Since $\mathcal{T}_1 \oplus \mathcal{T}_2 = \mathbb{R}^{3\times 3}$, there is no complementary subspace.

The detection filter is derived as before with the same Riccati constraint and cost function weights. Once again, the eigenvectors of the detection filter dynamics are very close to $\mathcal{T}_1$ and $\mathcal{T}_2$. The eigenvalues are identical to those in the stable invariant zero case. Since the invariant zero direction is approximately included in $\mathcal{T}_2$, none of the eigenvalues are close to the invariant zero at $+3$ or its mirror image over the imaginary axis -3.

Remark 6.4: It can be shown that, even though the GTFDF includes unstable invariant zeros in its nuisance fault detection space, a subset of the eigenvalues of the filter are very close to the mirror images of these invariant zeros over the imaginary axis [41]. This is confirmed by designing the single-fault detection filter to detect F_1 with Riccati weights identical to those above, producing a detection filter with eigenvalues at -3.0003, -5.4041, and -223.63. Note that there is an eigenvalue very close to the mirror image of the invariant zero at $+3$. The GTMFDF automatically includes these invariant zero directions in its detection spaces as well. However, none of the eigenvalues are close to these invariant zeros or their mirror images. Thus, the stability of the closed-loop system is unaffected by this property of the open-loop system. ♢

6.2.3 Nonmutually Detectable Faults with Stable Invariant Zero

In this section, the second fault direction is defined as

$$F_2 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

It can be shown that the two faults are output separable. However, F_1 and F_2 are not mutually detectable, and an invariant zero at -1.5 is associated with $(C, A, [\, F_1 \;\, F_2 \,])$. By using (2.13), $\mathcal{T}_1 = \hat{\mathcal{T}}_2 = \text{Im } F_1$ and $\mathcal{T}_2 = \hat{\mathcal{T}}_1 = \text{Im } F_2$. Since $\mathcal{T}_1 \oplus \mathcal{T}_2 \subset \mathbb{R}^{3\times 3}$, there is a complementary subspace.

The detection filter is derived as before with the same Riccati constraint and cost function weights. Once again, the eigenvectors of the detection filter dynamics are very close to $\mathcal{T}_1$ and $\mathcal{T}_2$. However, in this case the eigenvalues are -1.5001, -4.4615, and -4.6708. Since the invariant zero direction is not approximately included in $\mathcal{T}_1$ or $\mathcal{T}_2$, one of the eigenvalues is close to the invariant zero at -1.5.

Remark 6.5: This behavior with respect to nonmutually detectable invariant zeros is identical to the behaviors of the BJDF [19] and RDDF [18]. However, the system may be modified as in [19] so that the faults become mutually detectable. In that case, a detection filter may still be obtained with eigenvalues independent of the invariant zeros. $\diamondsuit$

Remark 6.6: Recall that mutual detectability is not a constraint for the single-fault problem since only one detection space is formed for the nuisance fault. Thus, invariant zeros associated with both the target and nuisance faults do not become eigenvalues of the detection filter. This is confirmed by designing the single-fault detection filter to detect F_1 with Riccati weights identical to those above, producing a detection filter with eigenvalues at $-3.5305 + 1.7895j$, $-3.5305 - 1.7895j$, and -316.27. Note that none of the eigenvalues are near the nonmutually detectable invariant zero at -1.5. $\diamondsuit$

6.2.4 Nonmutually Detectable Faults with Unstable Invariant Zero

In this section, the second fault direction is defined as

$$F_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It can be shown that the two faults are output separable. However, F_1 and F_2 are not mutually detectable, and an invariant zero at $+2$ is associated with $(C, A, [\, F_1 \;\; F_2 \,])$. By using (2.13), $\mathcal{T}_1 = \hat{\mathcal{T}}_2 = \text{Im } F_1$ and $\mathcal{T}_2 = \hat{\mathcal{T}}_1 = \text{Im } F_2$. Since $\mathcal{T}_1 \oplus \mathcal{T}_2 \subset \mathbb{R}^{3\times 3}$, there is a complementary subspace.

The detection filter is derived as before with the same cost function weights. The Riccati constraint weights are changed to $M_i = 0.01$ and $Q_i = 0.0001 \cdot I$. Once again, the eigenvectors of the detection filter dynamics are very close to $\mathcal{T}_1$ and $\mathcal{T}_2$. Further, in this case the eigenvalues are -0.2540, -3062, and -6795. In its attempt to obtain a stable detection filter, the optimization produces very large detection filter eigenvalues. Also, while the disturbance attenuation bounds are met for both projected residuals, the projected residuals are also both insensitive to their associated target fault. Thus, no stable detection filter that isolates the can be obtained when there exists an unstable nonmutually detectable invariant zero.

Remark 6.7: This behavior with respect to unstable nonmutually detectable invariant zeros is identical to the behaviors of the BJDF [19] and RDDF [18]. For those filters, a solution exists, but has an eigenvalue at the unstable invariant zero. However, the system may be modified as in [19] so that the faults become mutually detectable. In that case, a detection filter may still be obtained with stable eigenvalues. ♢

Remark 6.8: Recall that mutual detectability is not a constraint for the single-fault problem since only one detection space is formed for the nuisance fault. Thus, invariant zeros associated with both the target and nuisance faults do not become eigenvalues of the detection filter. This is confirmed by designing the single-fault detection filter to detect F_1 with Riccati weights identical to those above, producing a detection filter with eigenvalues at -1.1925, -4.1878, and -100.14. Note that none of the eigenvalues are near the nonmutually detectable invariant zero at $+2$. Further, the projected residual of the single-fault detection filter is sensitive to the target fault. ♢

Chapter 7

Conclusions

7.1 Summary of Contribution

The purpose of this dissertation is to develop advanced methodologies for the detection of multiple, possibly simultaneously occurring faults using a single fault detection filter. Previous Riccati-based detection filters were obtained from the application of game theory to the disturbance attenuation problem. The approach of the current work is to extend these detection filters to the multiple-fault case.

In this dissertation, the Game Theoretic Multiple-Fault Detection Filter was derived for both continuous-time and discrete-time systems. The detection filters for the two system descriptions displayed very similar modifications and advancements compared to their single-fault counterparts. Specifically, it was shown that the detection filter problem has a range of possible solutions for both the single-fault and multiple-fault cases. Thus, the detection filter gain may be optimized to achieve secondary objectives, such as increased sensitivity to the target fault. The continuous detection filter's ability to detect multiple faults, enhance target fault sensitivity for the single-fault problem, and reach arbitrarily stable steady-state solutions that do not depend on the systems invariant zeros was demonstrated via examples.

For continuous systems, the multiple-fault detection filter was also derived in the limit as the disturbance attenuation bound goes to zero. It was shown that, when the detection filter exists, the sufficient conditions for optimality force the desired structure of the detection filter so that the time-varying complementary faults are completely blocked from their associated projected residuals. Thus, the Riccati-based approximate detection filter approximates the detection filters derived from the spectral and geometric theories. A specific case of the solution was examined in which the asymptotic full-order multiple-fault detection filter satisfies the same constraints as the asymptotic full-order single-fault detection filter for each complementary fault. Necessary and sufficient conditions for the existence of this special case were derived. Further, an algorithm for generating a reduced-order detection filter was obtained.

7.2 Future Work

In this section, several avenues to extend the analysis of the game-theoretic multiple-fault detection filter are reviewed.

Alternative Solutions to the Asymptotic GTMFDF

In the current work, the existence conditions and model reduction algorithm for the asymptotic solution of the GTMFDF have been evaluated for a specific case. Namely, the Riccati equations and filter gain of the evaluated solution mirror that of the single-fault detection filter, with the difference that the filter gain must satisfy one such set of constraints for each complementary fault. Given the current analysis, other solutions to the asymptotic detection filter problem may exist. Thus, a secondary problem to optimize the filter gain may have merit for the asymptotic filter.

Analysis of the Asymptotic DGTMFDF

To determine if the DGTMFDF approximates discrete detection filters derived from the spectral and geometric theories, an analysis of the asymptotic detection filter is required.

For the single-fault filter, it was shown in the previous work that the Riccati equation of the discrete asymptotic filter develops a set of *invariant directions* in the limit. These directions work like an analog to the invariant subspaces of the continuous-time case: they become constant with respect to the Riccati solution after a certain number of time steps. It was also shown that the minimal (C, Φ)-invariant subspace associated with the nuisance faults is contained in the set of invariant directions after one time step. A similar analysis is required for the DGTMFDF.

In the single-fault case, it was also shown that the Riccati equations retain curvature with respect to the nuisance fault directions, even as the disturbance attenuation bound goes to zero. This curvature was the basis of the asymptotic detection filter analysis, meaning that it was unnecessary to examine a singular optimal control problem as for the continuous case. However, the remaining curvature was assumed to be full rank, an assumption that was never completely substantiated. If the curvature is indeed not full rank, the Riccati equation becomes singular with respect to a subset of the nuisance faults and an analysis in the vein of singular optimal control may be required.

Extension of the DGTMFDF to Time-Varying Systems

Currently, the problem of extending the discrete game-theoretic fault detection filters to the time-varying case has not been examined. Extension of the DGTMFDF to time-varying systems is trivial when the disturbance attenuation bound is non-zero. However, an issue of particular importance to asymptotic filter how we define the minimal $(C(k), \Phi(k+1|k))$-invariant subspace and the invariant directions for the more complex system. With answers to these questions, a complete analysis of the time-varying version DGTMFDF should be accessible.

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