Title
Topological Quantum Gravity of the Ricci Flow

Author
Randall, Stephen

Publication Date
2021

Peer reviewed|Thesis/dissertation

Topological Quantum Gravity of the Ricci Flow

by

Stephen Randall

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Physics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Petr Hořava, Chair
Professor Ori Ganor
Professor John Lott

Summer 2021

Abstract

Topological Quantum Gravity of the Ricci Flow

by

Stephen Randall

Doctor of Philosophy in Physics

University of California, Berkeley

Professor Petr Hořava, Chair

In this thesis we construct a family of topological quantum gravity theories with the goal of finding a regime in which these theories' localization equations are Perelman's celebrated Ricci flow equations. The natural setting for these theories is a nonrelativistic $\mathcal{N} = 2$ superspace in which the spacetime is foliated by leaves of constant time. The basis for this construction is a "primitive" topological theory involving the spatial metric and its corresponding spatial diffeomorphism invariance, with $\mathcal{N} = 2$ BRST symmetry. The algebra of the BRST charges is chosen so that the localization equations derived from a BRST-invariant action are flow equations and the couplings in the action are chosen so that these flow equations take the exact form of Hamilton's Ricci flow equations. Gauging spatial diffeomorphisms and foliation-preserving time reparametrizations (initially in two separate steps, but later in a single sweeping step) leads us to a set of geometric constraints that produce three distinct classes of field content in the BRST multiplets involved. Notably, the superpotential of the gauged theory is precisely Perelman's $\mathcal{F}$-functional and the role of his dilaton is played by our nonrelativistic lapse function. Perelman's Ricci flow equations are then obtained as localization equations from the gauged theory in the same way as Hamilton's were obtained from the primitive theory (up to an interesting reframing), satisfying our goal.

Dedication

to Sarah, for everything

Acknowledgments

This thesis would not be possible without the tremendous support and guidance from a number of people.

First and foremost, to my wife Sarah: Across numerous changes in proximity and to somehow finding this golden ring on my finger, your love and warmth have been constant, much-needed sources of comfort and stability. I cannot tell you anything here you do not already know, but to record it for posterity — thank you for being who you are, and I hope that I can give you a fraction of what you have given me.

Second, to my advisor Petr: I could not have imagined a better teacher, colleague, and friend to have in your role. Your relentless work ethic was a standard I pushed myself to keep up with and your patience and guidance made our collaboration a wonderful experience. It was never of question of *if* you would go above and beyond what was required of you at any given time, but *how*. I feel so very lucky to count myself among your students.

Finally: to Alec and Erin, over Thanksgiving dinners and games of catch; to my family, over holidays and Zoom calls; to William, Cathy, and Luna, over (slowly-eaten) tacos and superspace discussions; to Julie, Zach, and Lauren, over game nights and hearty amounts of alcohol; to Mallory and Edgar, over Animondays and walks with 4-legged monsters; to everyone I have eaten, drunk, and celebrated with — thank you. A large part of my life has been wrapped up in the effort it has taken to produce this thesis, but a larger part — a more important part — is made up of you. You all truly mean the world to me.

Contents

Part I
Constructing the Theories

In this thesis, we bring together three distinct and previously rather unrelated subjects: The geometry of the Ricci-type flows on Riemannian manifolds, topological quantum field theory, and nonrelativistic Lifshitz-type quantum gravity.

The Ricci flow on Riemannian manifolds, governed by the Ricci flow equation

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}, \tag{0.1}$$

was introduced by Richard Hamilton in 1982 [1], as a potentially powerful tool for addressing some of the deep open questions in differential geometry and topology of low-dimensional manifolds. This program has been – and continues to be – very successful, leading to Grisha Perelman's celebrated proof [2–4] of the Poincaré conjecture, the proof of Thurston's geometrization conjecture for 3-manifolds, a new independent proof of the uniformization theorem for 2-manifolds [5], and more recently the proof of the generalized Smale conjecture [6–10]. One of the important stepping-stones was Perelman's addition of a "dilaton" field ϕ to the spatial metric, and his formulation of the combined flow equations of g_{ij} and ϕ as a gradient flow for the so-called $\mathcal{F}$-functional,

$$\mathcal{F}(g_{ij}, \phi) = 2\int d^D x\, e^{-\phi}\sqrt{g}\left\{R + g^{ij}\,\partial_i\phi\,\partial_j\phi\right\}. \tag{0.2}$$

In the process of proving the consequences of this flow, a truly impressive wealth of many geometric and topological results and insights has been accumulated in the past two decades, with many intriguing questions still remaining open and vigorous investigations being actively pursued. A comprehensive multi-volume introduction to the mathematics of Ricci flow can be found in [11–15].[1] Many excellent and mutually complementary mathematical reviews and surveys exist: [16–25]. Many of the foundational papers (including almost all of Hamilton's papers on the subject prior to 2002 and his influential 1995 survey [16]) are collected in [26].

[1] A comment about our list of references: Each of the three subjects that we connect in this thesis has a hugely extensive literature. Hence, our list of references is inevitably far from exhaustive; we focus on a relatively short list of papers and books that we find particularly relevant to our construction, plus a longer list of various illuminating reviews.

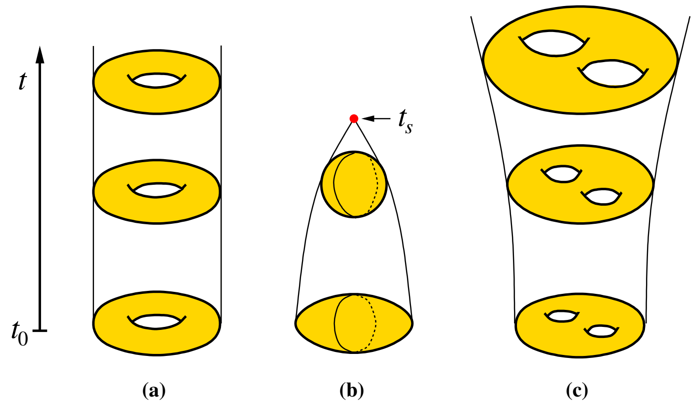

Figure 0.1: Simple illustrations of the typical behavior of the Ricci flow (0.1) in $3+1$ dimensions. **(a):** A Ricci-flat manifold stays constant with time. **(b):** A manifold with positive sectional curvatures, such as a slightly deformed sphere with bounded spatial inhomogeneities, will round itself out with time and uniformly collapse into a singularity at a finite instant t_s. **(c):** A hyperbolic manifold, with negative sectional curvatures, will expand forever.

Topological quantum field theories (of the "cohomological" type relevant for this thesis) were introduced by Edward Witten in 1988: The first examples included topological Yang-Mills gauge theory [27] in $3+1$ dimensions, topological nonlinear sigma models [28] in $1+1$ dimensions which later became central in the construction of topological string theory, and the first version of topological gravity [29]. The central role in the construction is played by the BRST quantization and BRST cohomology.[2] An accessible introduction to the general concept of topological quantum field theories of this cohomological type is in [31]. Roughly, for any "interesting" differential equation, one can attempt to construct a topological quantum field theory of the cohomological type, whose path integral is expected to localize to the moduli space of the appropriate solutions of the equation. Ref. [31] provides if not an algorithm, then at least an itinerary how to do this. In this way, topological Yang-Mills theory is associated with the self-duality equation for the field strength of the Yang-Mills connection, and the instanton moduli space. Physical observables are related to Donaldson invariants of 4-manifolds. Similarly, the topological sigma model is associated with Gromov's pseudoholomorphic curve equation which describes worldsheet instantons in string theory. Observables lead to Gromov-Witten invariants.

[2] For the general overview of BRST symmetry, see for example [30].

In this thesis, our main goal is to to construct a topological quantum field theory associated with a generalized family of Ricci flow equations. The proper setting for this construction is in nonrelativistic quantum gravity, and its supersymmetric and topological generalizations. Nonrelativistic quantum gravity with anisotropic scaling (in the literature often referred to as Hořava-Lifshitz gravity; we will refer to it in this thesis as Lifshitz-type gravity) was introduced in [32–34]. It has been broadly studied as an example of quantum gravity with improved short-distance behavior, which can explain the numerical lattice results of the Causal Dynamical Triangulations approach to quantum gravity [35–37], and even be power-counting renormalizable in appropriate dimensions; as a tool for nonrelativistic holography, where it leads to a broader set of holographic duals of nonrelativistic systems than bulk relativistic gravity; and for cosmology [38].

The mathematical theory of the Ricci flow has been previously connected to

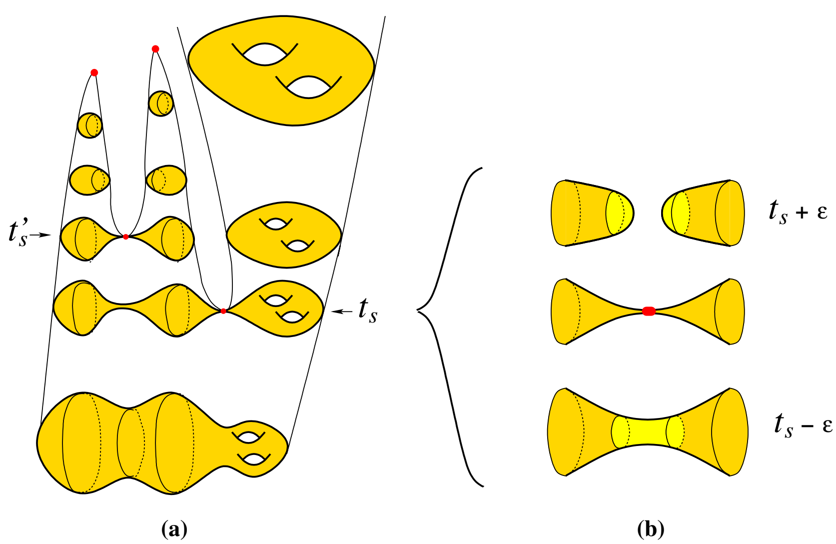

Figure 0.2: **(a):** Another illustration of the Ricci flow (0.1) in $3+1$ dimensions, now involving not only examples of the extinction singularity of the positively curved regions, but also two examples of a generic "neckpinch" singularity in finite time (here at t_s and t_s'). **(b):** The spatial topology change caused by the neckpinch singularity is handled by the geometrical technique of surgery on manifolds [39]. Zooming in on a small vicinity of the singularity, we find the spatial topology of $I \times S^2$ at time $t_s - \varepsilon$. Surgery replaces it with the union of two 3-balls $B_3 \cup B_3$ at $t_s + \varepsilon$, and restarts the Ricci flow.

physics in several ways. The relation to the renormalization group flow of nonlinear sigma models in two relativistic dimensions was already stressed and utilized by Perelman in [2]; for further developments of this connection, see [40]. Another useful connection has been made to numerical general relativity [41]. In this thesis, we find a new connection between Ricci flows and physics: We construct a topological quantum field theory of the cohomological type, whose path integral localizes to the solutions of a family of Ricci flow equations. This theory will inevitably take the form of a topological norelativistic quantum gravity. That such a topological theory of Lifshitz-type gravity associated with the evolution equations of the Ricci type should exist was first conjectured during the work on [32], see also the discussion in §1.3 of [42]. The purpose of this thesis is to fill this gap, and to present an explicit construction which links the mathematical theory of the Ricci flow to the physics of topological quantum field theory and quantum gravity.

This part of the thesis is organized as follows. We build our topological quantum gravity of the Ricci flow in stages, introducing a simplest version of nonrelativistic topological gravity first, and then bringing in additional steps and features needed to make contact with the Perelman theory of the Ricci flow.

In Section 1, we construct a "primitive" theory of topological nonrelativistic quantum gravity. The dynamical field in the primitive theory is the spatial metric $g_{ij}(t, x^k)$ on a $D+1$ dimensional spacetime, which carries a natural foliation structure by D-dimensional leaves Σ of constant time t. It is true that $D = 3$ appears to be the most immediately interesting case, both in physics and in mathematics, but our construction is more general than that, so we present it in D dimensions. The symmetries are all local topological deformations of g_{ij}. In addition to the topological BRST charge, we require the existence of an anti-BRST supercharge $\overline{Q}$, and construct the gauge-fixed primitive theory in an appropriately defined $\mathcal{N} = 2$ superspace. This theory is particularly interesting when the dynamical exponent z (which is a measure of the anisotropy between time and space) is equal to two: While it is well-known that Hamilton's Ricci flow equation (0.1) cannot be derived from a variational principle, we find that (0.1) *represents the locatization equation* in our primitive theory, for certain values of the coupling constants.

Much of modern theoretical physics is built around the concept of gauge symmetries. In the context of quantum gravity, it is natural to expect some form of spacetime diffeomorphism symmetry. The primitive theory constructed in Section 1 has no spacetime gauge symmetries: It is only invariant under time-independent spatial diffeomorphisms. In Section 2, we take the first step to remedy this, and we gauge spatial diffeomorphisms. This is again done in $\mathcal{N} = 2$ superspace, by introducing the shift vector $n^i(t, x^j)$ and its superpartners. It is in this theory with spatial diffeomorphisms promoted to a gauge symmetry where we find a natural setting for an important Ricci-flow technique known in the mathematical literature as the "DeTurck trick." It simply appears via possible choices of gauge fixing conditions.

In Section 3, we extend the gauge symmetry to include time reparametrizations, and thus promote the symmetries from spacetime-dependent spatial diffeomorphisms of Section 2 to the full gauge symmetry generally expected in Lifshitz-type quantum gravity: Foliation-preserving diffeomorphisms of spacetime. The gauging is accomplished by introducing the lapse function n and its $\mathcal{N} = 2$ superpartners, which can be either projectable (*i.e.*, dependent only on time), or nonprojectable, $n(t, x^i)$. We concentrate on the nonprojectable version of the theory, and reach two conclusions, which represent the central results of this thesis: (1) the role of Perelman's "dilaton" is played in our theory by the lapse function (more precisely, $\phi = -\log n$), and (2) Perelman's $\mathcal{F}$-functional arises simply as the $\mathcal{N} = 2$ superpotential in our topological gravity.

Our construction leads to a multi-parameter family of topological quantum gravities, whose localization equations represent a multi-parameter generalization of Perelman's Ricci flow equations for the fields g_{ij} and ϕ, parametrized by the values of general couplings in our topological gravity Lagrangian with $z = 2$ dynamical scaling. We list some open questions and challenges in Section 4.

1 The primitive theory

In Lifshitz-type gravity, one can describe the dynamics of spacetime geometry using the fields of the ADM formalism, first developed in the Hamiltonian description of general relativity [43]. These ADM variables consist of the spatial metric g_{ij}, the shift vector n^i, and the lapse function n,[3] and they were originally viewed as a decomposition of the full relativistic spacetime metric. In Lifshitz-type gravity, these fields define *two distinct* geometric length elements $d\tau$ and $d\sigma$ on spacetime,

$$d\tau = n\, dt, \qquad d\sigma^2 = g_{ij}(dx^i + n^i dt)(dx^j + n^j dt). \tag{1.1}$$

A priori, these two elements are unrelated, and the distances they define are measured in two different units: The spatial length scale L and the time scale T. In such a

[3]The lapse and shift variables are usually denoted in the literature by the capital letters N and N^i. In this thesis, we reserve N and N^i to denote the superfields whose lowest components are the lapse and shift n and n^i.

theory with two separate scales, the scaling properties of fields and their derivatives undergo the appropriate refinement in comparison to their relativistic counterparts. In the traditional way of assigning classical dimensions to the building blocks of Lifshitz gravity, one assigns the coordinate element dx^i the dimension of length, $[dx^i] = L$, and the time element the dimension of time, $[dt] = T$. Since the physical distances $d\sigma$ and $d\tau$ also have those same dimensions, $[d\sigma] = L$ and $[d\tau] = T$, we see from (1.1) that g_{ij} and n are both dimensionless, and that the dimension of the shift vector is $[n^i] = L/T$.

In general relativity, the speed of light c is a dimensionful constant of nature which relates space and time distances to each other in a canonical way, and $d\tau$ and $d\sigma$ combine to form the unique spacetime metric, which transforms covariantly under the symmetries of general relativity. We can naturally set $c = 1$ for convenience, which canonically relates $L = T$, and the spacetime metric is then

$$ds^2_{\mathrm{GR}} = -d\tau^2 + d\sigma^2 = -(n^2 - n_i n^i)dt^2 + 2n_i dx^i dt + g_{ij}dx^i dx^j. \tag{1.2}$$

The theory now has only one scale in which dimensions of fields and their derivatives are measured.

In contrast, in nonrelativistic Lifshitz gravity no such canonical constant of nature c is present, and the two length elements (1.1) and (1.2) cannot be canonically combined into a unique spacetime element. A relation between the two scales L and T is typically generated by the renormalization-group fixed point appropriate for the system in a particular regime. Typically, the short-distance physics is dominated by one fixed point, characterized by the relation $T \sim L^z$, where z is the dynamical critical exponent characterizing the short-distance anisotropy between time and space; usually, we have $z > 1$. The long-distance physics is typically governed by another fixed point, usually with $z = 1$, resulting from the natural renormalization-group flow of the theory. Whether the theory is short-distance complete or at least power-counting renormalizable is governed by the value of z at the short-distance fixed point, and the spatial dimension D.

1.1 Preliminaries: the structure of spacetime

The main purpose of this thesis will be to construct an appropriately supersymmetric version of nonrelativistic Lifshitz-type gravity on spacetime manifold $\mathcal{M}$ of dimension $D + 1$, equipped with the further structure of a codimension-one foliation $\mathcal{M}_{\mathcal{F}}$ by spatial slices Σ of dimension D, which can be thought of as slices of constant time. There is a natural projection π from $\mathcal{M}$ to the time dimension $\mathbf{R}$, by simply forgetting the location along the leaf Σ. We will use coordinates $(t, x^i; i \in 1, \ldots, D)$ on $\mathcal{M}$, naturally adapted to the foliation $\mathcal{M}_{\mathcal{F}}$ so that $\pi : (t, x^i) \mapsto t$.

Specific solutions of Ricci-type flow equations often develop interesting singularities, with some simple examples illustrated in Fig. 0.1 and Fig. 0.2. Therefore, they may only be defined – without surgery or some other prescription for continuing through the singularities – on some open time interval $\mathcal{I} = (t_0, t_1) \subset \mathbf{R}$, where one or both of t_0 and t_1 might be finite. Alternatively, one studies the initial-value problem, on $[t_0, t_1)$ with t_0 being the initial time and t_1 chosen such that no singularity is encountered for $t < t_1$ in this interval. In this thesis, we focus on constructing the description of our quantum gravity theories on such a smooth patch, and leave the fascinating question of the singularities (such as the exctinction singularities of Fig. 0.1, or the topology-changing "neckpinch" singularities of Fig. 0.2) and their physical interpretation for future study. Our time manifold $\mathcal{M}_0$ should therefore be interpreted as either $\mathbf{R}$ when time extends for all eternity, or an open interval $\mathcal{I} \subset \mathbf{R}$, or the intial-value problem interval $[t_0, t_1)$, as appropriate.

For simplicity, in this thesis we focus on the case of compact Σ. We fully expect our theory to describe the noncompact case as well (when Σ is a complete Riemannian manifold), but the precise formulation would require a careful discussion of the suitable behavior near the appropriately defined spacetime infinity (in the sense of [44]), which goes beyond the scope of the work reported here.

In this section, we begin with a simpler task, and construct a more primitive topological gravity theory for the special case when the spacetime manifold is canonically a direct product, $\mathcal{M} = \Sigma \times \mathcal{M}_0$, with the time manifold $\mathcal{M}_0 \subset \mathbf{R}$ as explained above. This construction is simpler because the only dynamical field is the spatial metric $g_{ij}(t, x^k)$ and its superpartners implied by the topological symmetry. The time dimension is assumed to carry a constant nondynamical metric, and there is no secondary spacetime gauge invariance besides the topological symmetry. We will refer to this theory as the "primitive" theory.

1.2 Fields and symmetries

The only dynamical field of the primitive theory will be the spacetime-dependent spatial metric $g_{ij}(t, x^k)$. (We will use Penrose's "abstract index" notation throughout, for all our fields.) The gauge symmetry will be the topological gauge symmetry, given by all local deformations of the metric,

$$\delta g_{ij}(t,x^k)=\xi_{ij}(t,x^k). \tag{1.3}$$

We anticipate that due to this very large gauge symmetry, our theory will have no propagating local degrees of freedom (such as gravitons), but it may still have a nontrivial global structure.

The only action that is invariant under the topological gauge symetry (1.3) would be a sum of topological invariants built from the spatial metric, and therefore does not yet define a meaningful path integral. The path-integral representation of this theory comes entirely from the "gauge-fixing" of (1.3) using the BRST method: One replaces the local gauge symmetry with a global symmetry, generated by a supercharge Q which squares to zero,

$$Q^2=0. \tag{1.4}$$

This BRST supercharge Q maps g_{ij} to a ghost field $\psi_{ij}(t,x^k)$ which is the section of the same bundle as the gauge transformation parameter ξ_{ij}, but carries the opposite (*i.e.*, fermionic) statistics. Thus, our first BRST multiplet is

$$Qg_{ij}=\psi_{ij},\qquad Q\psi_{ij}=0. \tag{1.5}$$

The next step is to choose a gauge fixing condition: a local functional $\mathscr{F}^J$ of g_{ij} and its derivatives, designed such that the path integral of the theory will localize to the space of solutions to $\mathscr{F}^J=0$. For judiciously chosen $\mathscr{F}^J$, the space of such solutions is finite-dimensional, and typically of great geometric interest. In the process of choosing $\mathscr{F}^J$, one chooses the bundle on spacetime, and $\mathscr{F}^J$ will be a section of this bundle. To implement the gauge fixing and to make sense of the path integral, one then introduces a trivial BRST multiplet consisting of a fermion "antighost" χ_J and the bosonic auxiliary field B_J,

$$Q\chi_J=B_J,\qquad QB_J=0. \tag{1.6}$$

We assign a "ghost number" gh: the ghost and antighost are assigned $\mathsf{gh}(\psi_{ij})=1$ and $\mathsf{gh}(\chi_J)=-1$, while $\mathsf{gh}(g_{ij})=\mathsf{gh}(B_J)=0$. Consequently, the supercharge Q has $\mathsf{gh}(Q)=1$. Classically, one may start with the requirement that gh be conserved; quantum mechanically, however, there are often anomalies in this global symmetry, which play an important role in determining the dimensions of the moduli spaces to which the path integral is localized, and what insertions of various observables may be needed to make any correlation function non-vanishing.

With these fields, one then constructs an action

$$S = \int dt\, d^D x\, \{Q, \Psi\}, \tag{1.7}$$

where the "gauge-fixing fermion" Ψ is $\sim \chi_J \mathscr{F}^J$. We require that S preserve the ghost number symmetry, therefore the gauge-fixing fermion must have $\mathsf{gh}(\Psi) = -1$. States and physical operators in this theory are defined as the cohomology classes of the BRST charge Q on the spaces of all states and operators built from the available fields.

1.3 Extended BRST superalgebra

In general, the antighost field does not have to be (and typically indeed is not) the section of the same bundle as the ghost field. In our topological gravity, we wish to choose as our gauge fixing condition a functional whose vanishing will imply the Ricci-type flow of the metric g_{ij},

$$\mathscr{F}^J \sim \frac{\partial g_{ij}}{\partial t} + 2R_{ij} + \dots. \tag{1.8}$$

Hence, in this case, we have $J \equiv (ij)$, and the ghost and antighost fields *are* sections of the same bundle.

Since the ghost and antighost fields are sections of the same bundle, it is possible to demand that our theory has an additional symmetry, which exchanges the ghosts with antighosts. Some early examples of topological field theories with this additional ghost-antighost symmetry include the harmonic topological sigma models and the topological rigid string [45, 46] (see also [47, 48]). The partition function in such theories typically evaluates the appropriately defined Euler number of the moduli space of solutions of the localization equation [49]. Topological field theories with the ghost-antighost symmetry later became known as "balanced theories" [50]. We will indeed take advantage of this possibility, and simply *postulate* that our theory has a second real supercharge $\overline{Q}$, which also squares to zero,

$$Q^2 = 0, \qquad \overline{Q}^2 = 0. \tag{1.9}$$

We will refer to $\overline{Q}$ as the "anti-BRST charge." It carries $\mathsf{gh}(\overline{Q}) = -1$. Note that as a consequence of the symmetry between the ghosts and antighosts, the global symmetry associated with the ghost number gh will be non-anomalous.

In order to complete our superalgebra of supercharges Q and $\overline{Q}$, we need to decide what their anticommutator should be. One option would be to simply set it to zero. Indeed, the anti-BRST charge and the extended BRST algebra was first discovered in the context of gauge-fixing relativistic Yang-Mills gauge theories [51, 52], where $\overline{Q}$ was found to anticommute with Q. For our purposes it will be crucial to choose another, more interesting possibility consistent with (1.9), whereby the supercharges anticommute up to a time translation generator,

$$\{Q, \overline{Q}\} = \partial_t. \tag{1.10}$$

This algebra is a natural deformation of the extended BRST anti-BRST algebra found originally in the relativistic setting of Yang-Mills theories [51, 52]. In the relativistic case, there simply is no suitable candidate, consistent with Lorentz invariance, for a bosonic symmetry generator that could appear on the right-hand side of (1.10). In the nonrelativistic theory, the time translation generator can naturally appear, and our topological gravity will take advantage of this possibility.[4]

Requiring the existence of the second supercharge $\overline{Q}$ and the extended superalgebra is beneficial for two reasons: It not only allows us to make it easier to implement the ghost-antighost symmetry but, more importantly, it will also guarantee that the flow equations on which the path integral localizes are *gradient* flow equations.

1.4 $\mathcal{N} = 2$ superspace extension of time

In order to proceed in the most efficient way, it is very natural to organize all component fields into superfields. Thus, we extend the spacetime manifold into a supermanifold $\mathscr{M}$ of dimension $D + 1|2$, with coordinates $(t, x^i, \theta, \overline{\theta})$, where θ and $\overline{\theta}$ are two *real* anticommuting coordinates.[5] On this supermanifold, we combine the spatial metric, and its ghost, antighost and bosonic auxilary field into the spatial metric superfield

$$G_{ij} = g_{ij} + \theta\psi_{ij} + \overline{\theta}\chi_{ij} + \theta\overline{\theta}B_{ij}. \tag{1.11}$$

[4] The supersymmetric structure bears formal similarity to the supersymmetric treatment of stochastic quantization. The analogy between Lifshitz-type gravities satisfying the detailed balance condition, and stochastic quantization of a gravity theory in one lower dimension, was pointed out in [32, 33]; see also [53] and [54].

[5] We stress that in this thesis, we utilize the oft-used physics convention, in which the bar on top of θ etc. is simply an additional index, and never a (complex) conjugation operation. Thus, θ and $\overline{\theta}$ denote two *real* Grassmannian variables, Q and $\overline{Q}$ are two independent real supercharges, and so on.

Our construction of topological gravity theory involves supersymmetry with two supercharges Q and $\overline{Q}$. It will be convenient to formulate the theory directly in the language of superfields and other geometric objects and operations on $\mathscr{M}$, instead of using the cumbersome component field formulation. The superspace $\mathscr{M}$ inherits a natural foliation $\mathscr{M}_{\mathcal{F}}$, again by leaves of the bosonic space Σ, and therefore is a codimension-$(1|2)$ foliation. Thus, our bosonic time dimension $\mathcal{M}_0 \subset \mathbf{R}$ is promoted to a supermanifold $\mathscr{M}_0$ of dimension $(1|2)$, with coordinates $(t, \theta, \overline{\theta})$, which we will naturally refer to as "supersymmetric time", or "supertime" for short. The projection from $\mathscr{M}$ to the supertime $\mathscr{M}_0$ is given in coordinates by $\pi : (t, \theta, \overline{\theta}, x^i) \mapsto (t, \theta, \overline{\theta})$. We will sometimes refer to the coordinates $(t, \theta, \overline{\theta})$ on supertime collectively as τ^M, with the coordinate index $M \in \{t, \theta, \overline{\theta}\}$.

Note that the dimensions θ and $\overline{\theta}$ are two real Grassmannian coordinates, and they supersymmetrize only the time dimension; the bosonic spatial coordinates x^i parametrize the leaves of the foliation, and can often be viewed as spectators from the perspective of the supersymmetrized time. In what follows, we will use interchangeably ∂_t and $\dot{}$ to denote the time derivative $\partial/\partial t$.

The theories we will be interested in will exhibit $\mathcal{N} = 2$ supersymmetry,[6] with supercharges realized on $\mathscr{M}$ as differential operators

$$Q = \frac{\partial}{\partial \theta}, \qquad \overline{Q} = \frac{\partial}{\partial \overline{\theta}} + \theta \partial_t, \tag{1.12}$$

and satisfying the superalgebra

$$\{Q, \overline{Q}\} = \partial_t, \qquad Q^2 = \overline{Q}^2 = 0. \tag{1.13}$$

In this $\mathcal{N} = 2$ superalgebra, we intend to identify Q to be our BRST charge. Thus, physical states and operators in our topological gravity theory will be determined from the cohomology of Q. However, for the time being we suspend this underlying BRST interpretation, and simply construct our theory as a supersymmetric theory with the rigid $\mathcal{N} = 2$ superalgebra (1.13).

The superderivatives that anticommute appropriately with the supercharges are:

$$\mathrm{D} = \frac{\partial}{\partial \theta} - \overline{\theta} \partial_t, \qquad \overline{\mathrm{D}} = \frac{\partial}{\partial \overline{\theta}}; \tag{1.14}$$

[6]In our conventions, $\mathcal{N}$ counts the number of individual real supercharges.

they satisfy

$$\{\mathrm{D},\overline{\mathrm{D}}\}=-\partial_t,\qquad \mathrm{D}^2=\overline{\mathrm{D}}^2=0, \tag{1.15}$$

and

$$\{\mathrm{D},Q\}=\{\mathrm{D},\overline{Q}\}=\{\overline{\mathrm{D}},Q\}=\{\overline{\mathrm{D}},\overline{Q}\}=0, \tag{1.16}$$

and of course D and $\overline{\mathrm{D}}$ both commute with the spatial derivative $\partial_i\equiv\partial/\partial x^i$.

1.5 The action

Our primitive theory is a topological theory of the component fields contained in the spatial metric superfield G_{ij}. It will have no gauge symmetries, and it will respect the $\mathcal{N}=2$ supersymmetry algebra described above. In addition, we will require that the theory be invariant under time-independent spatial diffeomorphisms of the spatial slices Σ. Since this symmetry does not depend on time, it is better not to interpret it as a gauge symmetry, despite its dependence on the location along Σ. In the present setting, Diff(Σ) essentially represents an infinite-dimensional global symmetry.

Under these symmetry assumptions, we now write the superspace action as a sum of two terms,

$$S=\frac{1}{\kappa^2}\left(S_K-S_{\mathcal{W}}\right). \tag{1.17}$$

The kinetic term S_K is a sum of all the invariants that contain at least one supertime derivative, while the potential term $S_{\mathcal{W}}$ contains all the invariants with only spatial derivatives but no supertime derivatives. In the component form, this decomposition will translate into S_K containing at least one time derivative, and $S_{\mathcal{W}}$ including all the terms without time derivatives. Extending the customary physics terminology to this case, we will refer to $S_{\mathcal{W}}$ as the "superpotential." Both S_K and $S_{\mathcal{W}}$ are integrals of a local Lagrangian density over all of superspace,[7]

$$S_K=\int dt\, d^Dx\, d^2\theta\, \mathcal{L}_K,\qquad S_{\mathcal{W}}=\int dt\, d^Dx\, d^2\theta\, \mathcal{L}_{\mathcal{W}}. \tag{1.18}$$

We will require that they preserve the ghost number symmetry, $\mathsf{gh}(S_K)=\mathsf{gh}(S_{\mathcal{W}})=0$. Note that for future convenience, we have factored out one overall coupling constant, κ^2, in front of the entire action.

The terms that can appear in S_K and $S_{\mathcal{W}}$ can be usefully organized by their increasing classical scaling dimensions. Until or unless we commit to a particular

[7] As usual, we define the measure $d^2\theta$ and the Berezin integral over the anticommuning coordinates by linearity together with $\int d^2\theta\,\theta\overline{\theta}=1$ and $\int d^2\theta\,\theta=\int d^2\theta\,\overline{\theta}=\int d^2\theta\,1=0$.

value of the dynamical scaling exponent z, time and space scaling is unrelated (as we reviewed briefly at the beginning of Section 1), and we assign classical scaling dimensions to the ingredients appearing in the action as follows: $[\partial_i] = L^{-1}, [\partial_t] = T^{-1}, [G_{ij}] = 0$. The superalgebra implies that $[\mathrm{D}] + [\overline{\mathrm{D}}] = T^{-1}$.[8] The terms in S_K of the lowest scaling dimension (*i.e.*, with the lowest number of derivatives) will be of dimension $T^{-1}L^0$. The first obvious candidate would be $\int \sqrt{G} G^{ij} \dot{G}_{ij}$, but that term is a total derivative, $\int \partial_t (2\sqrt{G})$, and hence gives no local dynamics. A nontrivial leading-order kinetic term of this dimension can indeed be constructed; it contains two superderivatives,

$$S_K = \int dt\, d^2\theta\, d^D x\, \sqrt{G} \left\{ \left(\lambda_\perp G^{ik} G^{j\ell} - \lambda G^{ij} G^{k\ell} \right) \overline{\mathrm{D}} G_{ij}\, \mathrm{D} G_{k\ell} + \dots \right\} \tag{1.19}$$

(The "..." stand as a reminder that there may be terms of higher scaling dimension that one may wish to include.) This kinetic term depends on two coupling constants λ and $\lambda_\perp$, which we take to be of scaling dimension zero: $L^0 T^0$. This in turn implies that the scaling dimension of κ^2 is $[\kappa^2] = L^D T^{-1}$. Clearly, $\lambda_\perp$ is redundant, and one usually sets $\lambda_\perp = 1$. We will do so from now on, but we wish to point out that the implicit assumption leading to this step is that $\lambda_\perp$ is positive, while in some circumstances these types of theories can also be studied in the regime where $\lambda_\perp \leq 0$.

The superpotential terms can be similarly organized by the number of increasing spatial derivatives. Focusing on the terms with up to two derivatives, we find two terms respecting all our global symmetries: the Ricci scalar of G_{ij} and the cosmological constant term,

$$S_{\mathcal{W}} = \int dt\, d^2\theta\, d^D x\, \sqrt{G} \left\{ \dots + \alpha_R R^{(G)} + \alpha_\Lambda \right\}. \tag{1.20}$$

We will always refer to the various couplings in the superpotential as α, with an appropriate subscript indicating the term each particular coupling is associated with. Thus, here α_R is the coupling associated with the spatial Einstein-Hilbert term in superspace, and α_Λ is the superspace cosmological constant. We organized the terms

[8]Sometimes, in various dynamical regimes, it is convenient to choose a specific value of the dynamical exponent z, which relates the scaling of time and space, so that T scales as $T \sim L^z$. In that case it is then conventional to assign the classical "scaling dimension" Δ to any object $\mathcal{O}$ if $\mathcal{O}$ scales as $T^{-\Delta}$, *i.e.*, to measure the scaling dimension in the units of energy. Also, since in this thesis we are focusing on the basic set-up of the path integral representation of the theory, and do not calculate any quantum corrections to classical scaling dimensions, all our scaling dimensions will be classical. We will follow these conventions throughout.

in the order of their increasing scaling dimension from the right to the left, with the "..." on the left standing for all terms with more than two derivatives. We of course assume the perspective and logic of effective quantum field theory here, implying that all terms consistent with the underlying symmetries are in principle present. In some cases, only a finite number of terms up to a certain "critical" dimension is sufficient to make the theory perturbatively renormalizable, or perhaps even short-distance complete, without the need for higher-derivative terms. The analysis of possible short-distance completeness of the topological gravity theories presented in this thesis is a fascinating open question for future research.[9]

Now we are ready to see the relation between our primitive supersymmetric theory and the Ricci flow equations. We perform the $d^2\theta$ integral in S_K and $S_{\mathcal{W}}$ to obtain the action in component form. In components, the action (1.17) with S_K given in (1.19) and with a general superpotential term $S_{\mathcal{W}}$ given by (1.18) takes the following form,

$$S = \frac{1}{\kappa^2}\int dt\, d^D x\, \left\{\sqrt{g}(g^{ik}g^{j\ell} - \lambda g^{ij}g^{k\ell})B_{ij}\,(\dot{g}_{k\ell} - B_{k\ell}) - B_{ij}\frac{\delta\mathcal{F}}{\delta g_{ij}} + \text{fermions}\right\}. \tag{1.21}$$

Here we defined $\mathcal{F}$ to be the (bosonic) spacetime integral of the lowest component $\mathcal{L}_{\mathcal{F}}$ of the $\mathcal{L}_{\mathcal{W}}$ superfield in the $\theta, \overline{\theta}$ expansion:

$$\mathcal{L}_{\mathcal{W}} = \mathcal{L}_{\mathcal{F}} + \text{higher orders in } \theta, \overline{\theta}, \tag{1.22}$$

$$\mathcal{F} = \int dt\, d^D x\, \mathcal{L}_{\mathcal{F}}. \tag{1.23}$$

The auxiliary field B_{ij} can be integrated out, and the bosonic part of the action then becomes

$$S_{\text{bose}} = \frac{1}{4\kappa^2}\int dt\, d^D x\,\sqrt{g}(g^{ik}g^{j\ell} - \lambda g^{ij}g^{k\ell})\left[\dot{g}_{ij} - \frac{1}{\sqrt{g}}(g_{im}g_{jn} - \widetilde{\lambda} g_{ij}g_{mn})\frac{\delta\mathcal{F}}{\delta g_{mn}}\right] \tag{1.24}$$

$$\times\left[\dot{g}_{k\ell} - \frac{1}{\sqrt{g}}(g_{kr}g_{\ell s} - \widetilde{\lambda} g_{k\ell}g_{rs})\frac{\delta\mathcal{F}}{\delta g_{rs}}\right], \tag{1.25}$$

[9] The naive scaling properties of free-field fixed points suggest that in $3+1$ dimensions, power-counting renormalizability requires $z = 3$, implying in turn that all terms up to three derivatives would need to be included in the superpotential. This would include the gravitational Chern-Simons 3-form built out of the Levi-Cività connection of G_{ij}, and would lead to a generalization of the Ricci flow involving the Cotton tensor [33, 55, 56]. In $2+1$ dimensions, only terms up to two derivatives are sufficent for power-counting renormalizability.

where $\widetilde{\lambda}$ is given, as usual in Lifshitz gravity [32], by

$$\widetilde{\lambda} = \frac{\lambda}{D\lambda - 1}. \tag{1.26}$$

Clearly, for values of $\lambda \leq 1/D$, this action is bounded from below by zero, and this bound is saturated when the metric satisfies the appropriate flow equation, of first order in time derivatives.

The fermionic component contributions to (1.21) are straightforward to determine, but they look a little cumbersome and we suppress them for the ease of the presentation, as is often done in supergravity theories. Perhaps the most important thing to remember about the fermions is that they also have a non-degenerate kinetic term,

$$\chi_{ij}(g^{ik}g^{j\ell} - \lambda g^{ij}g^{k\ell})\dot{\psi}_{k\ell} + \ldots, \tag{1.27}$$

and therefore our entire theory can be treated in a perturbative expansion using standard Feynman diagram techniques.

The quantum theory of our primitive topological gravity is formally defined via the path integral as a sum over all appropriate histories,

$$\mathcal{Z} = \int \mathscr{D}\mu[G_{ij}] \, \exp\left\{ -\frac{1}{\hbar\kappa^2}(S_K - S_{\mathcal{W}}) \right\}. \tag{1.28}$$

Here $\mathscr{D}\mu[G_{ij}]$ is the $\mathcal{N} = 2$ supersymmetric measure on the space of the metric fields.

A few comments about some salient features of this path integral seem in order. Many of them will be relevant also to the more sophisticated cousins of the primitive theory, which we will develop below.

- In order to become well-defined even by the physics standard of rigor, this path integral requires that appropriate boundary conditions be specified at the boundaries of spacetime, even in the absence of singularities. What is the correct question to ask must be guided by physics principles: We must first decide what is the appropriate set of probability amplitudes and observables that are meaningful in this context of time-dependent quantum gravity and cosmology. We might be interested in choosing the initial surface Σ and calculating the Hartle-Hawking-type wavefunction of the Universe. Or perhaps one might wish to evaluate the transition amplitudes between physical states at an initial and finite time. Besides calculating the partition function $\mathcal{Z}$ or transition amplitudes with such boundary conditions, one can define correlation functions of BRST-invariant local operators, or of observables associated with extended submanifolds in spacetime. This question of observables is beyond the scope of the present thesis, but represents an intriguing opportunity to find a new window into quantum gravity and quantum cosmology far from equilibrium, at least in the topological setting.

- Standard arguments of topological quantum field theory apply [27, 31], at least formally: The overall coupling κ^2 plays the role of $\hbar$. The semiclassical approximation at small κ is "exact" at one loop, and the path integral localizes to the space of solutions of the localization equation, which in our case is a Ricci-type flow equation for g_{ij}. A similar argument implies that the physical observables (such as the partition function) are independent of the small changes in the coupling constants; here "small" means roughly those changes which do not lead to degeneracies in the action.

- Note that our theory is formally defined in "imaginary time". One might also be interested instead in the "real-time" path integral, which would have the integrand $\exp(iS)$ instead of the $\exp(-S)$ appearing in (1.28). This possibility is already interesting for the primitive theory, but will become even more relevant for the more sophisticated versions of topological quantum gravity constructed below, which have some form of spacetime diffeomorphism invariance. We will further comment on this possibility of continuing to real time in Section 3.4.3.

- Already this simplest "primitive" theory depends on several coupling constants: λ, α_R and α_Λ (and perhaps more, if we choose to add higher-derivative terms), and the classical localization equations thus represent a multi-parameter generalization of the standard Ricci flow. It will be important to subject this "landscape" of topological gravity theories to a closer study, to see what limits on the values of the coupling constants naturally emerge from requiring that the formal path integral satisfy various physical consistency conditions. In particular, not for all values of the couplings will the solutions of the localization equations be as well-behaved as those of Hamilton's Ricci flow, putting bounds on the range of the couplings. Of course, this broader family of generalized flow equations has been much less studied in the mathematical literature, and much less is known exactly.

1.6 Localization and Hamilton's Ricci flow

To see that Hamilton's original Ricci flow indeed appears in the landscape of our theory, let us take a closer look at the localization equation, obtained from (1.21):

$$\frac{\partial g_{ij}}{\partial t} = \frac{1}{\sqrt{g}}(g_{ik}g_{j\ell} - \widetilde{\lambda} g_{ij}g_{k\ell})\frac{\delta \mathcal{F}}{\delta g_{k\ell}}, \tag{1.29}$$

With the specific form of the superpotential given in (1.20), this becomes

$$\frac{\partial g_{ij}}{\partial t} = -\alpha_R R_{ij} + \frac{\alpha_R}{2}\left[1 - \widetilde{\lambda}(D-2)\right] g_{ij}R + \frac{\alpha_\Lambda}{2} g_{ij}. \tag{1.30}$$

We observe that setting

$$\alpha_R = 2, \qquad \alpha_\Lambda = 0, \qquad \lambda = \frac{1}{2} \tag{1.31}$$

in the action of the primitive theory reduces the localization equation (1.30) to

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}. \tag{1.32}$$

Thus, for the values of the couplings given in (1.31), the original Ricci flow equation (0.1) of Hamilton's appears as the localization equation in our theory of topological quantum gravity, despite the fact that it is not a gradient flow equation.

Interestingly, the value $\lambda = 1/2$ that leads to Hamilton's Ricci flow is *not* in the range of λ in which the action is positive definite. The proper treatment of the path integral would require a rather subtle analytic continuation. If we wanted to make sense of this continuation, we would be facing a situation very analogous to relativistic Euclidean quantum gravity [57], in which the Euclidean action is also not bounded from below, due to the contributions from the spacetime scale factor of the metric. In the nonrelativistic context relevant here, the culprit is the scale factor of the spatial metric g_{ij}.

2 The gauge theory: Gauging spatial diffeomorphisms

In the next step, we wish to incorporate some of the gauge symmetries expected of quantum gravity into our topological theory. We begin by gauging the time-independent symmetries of spatial diffeomorphisms Diff(Σ) exhibited by the primitive theory.

The basic field of the primitive theory was the spatial metric $g_{ij}(t, x^k)$. Under an infinitesimal time-dependent spatial diffeomorphism $\xi^i(t, x^j)$, the metric tensor would transform as

$$\delta g_{ij} = \xi^k \partial_k g_{ij} + g_{kj} \partial_i \xi^k + g_{ik} \partial_j \xi^k. \tag{2.1}$$

Note that in this relation, the time coordinate plays the role of a spectator: not only the time-independent but also the time-dependent spatial diffeomophisms act via (2.1) leaf-by-leaf, at each fixed t, as ordinary spatial diffeomorphisms.

In the primitive theory, we supersymmetrized the spatial metric by promoting $g_{ij}(t, x^k)$ to an unconstrained $\mathcal{N} = 2$ superfield $G_{ij}(t, \theta, \overline{\theta}, x^k)$, whose component expansion we recall here,

$$G_{ij} = g_{ij} + \theta \psi_{ij} + \overline{\theta} \chi_{ij} + \theta \overline{\theta} B_{ij}. \tag{2.2}$$

In order to gauge the spatial diffeomorphisms consistently with the $\mathcal{N} = 2$ supersymmetry, we follow the strategy familiar from supersymmetric Yang-Mills theories in superspace: We promote the diffeomorphism generator ξ^i into a superfield,

$$\Xi^i = \xi^i + \theta \zeta^i + \overline{\theta} \eta^i + \theta \overline{\theta} \alpha^i. \tag{2.3}$$

(Later on, we will impose various chirality constraints on Ξ^i, but for now we will treat it as unconstrained.) Under the spatial superdiffeomorphisms, the metric superfield G_{ij} transforms in a straightforward generalization of (2.1), as

$$\delta G_{ij} = \Xi^k \partial_k G_{ij} + G_{kj} \partial_i \Xi^k + G_{ik} \partial_j \Xi^k. \tag{2.4}$$

Note that in this transformation rule, both t and $\theta, \overline{\theta}$ again play the role of spectators, and (2.4) acts at each fixed value of the specator supercoordinates (defining an individual leaf of the foliation) as a spatial superdiffeomorphism of the metric superfield along the leaf.

2.1 ABCs of supersymmetrizations of the Diff(Σ) symmetry

We begin with the primitive theory, and G_{ij} as the only superfield. In order to promote the transformations of the spatial Diff(Σ) symmetries into a gauge symmetry, one must introduce the appropriate gauge fields which allow us to covariantize the time derivatives of the spatial metric.

Let us first recall how this works in bosonic gravity. The role of such gauge fields is played by the famous "shift vector" n^i (in the terminology of the ADM formalism), which transforms as

$$\delta n^i = \dot{\xi}^i + \xi^k \partial_k n^i - n^k \partial_k \xi^i. \tag{2.5}$$

The interpretation of the three terms in (2.5) is very clear: The first term, viewed for each fixed value of the spatial index i is exactly the transformation of an Abelian gauge field under a time-dependent gauge transformation with parameter ξ^i, with one such Abelian symmetry for each spatial dimension. And the second plus third term are a nonlinear correction to this leading gauge transformation, which ensure that ξ^i are not independent Abelian symmetries but represent the nonlinear transformations of spatial diffeomorphisms. These two terms make sure that n^i transform correctly as components of a spatial one-vector under time-independent spatial diffeomorphisms.

Using the shift vector n^i, one can now covariantize $\dot{g}_{ij}$ to

$$\nabla_t g_{ij} \equiv \dot{g}_{ij} - n^k \partial_k g_{ij} - g_{kj} \partial_i n^k - g_{ik} \partial_j n^k, \tag{2.6}$$

and show that this covariantization transforms correctly, as a spatial two-tensor, under time-dependent spatial diffeomorphisms $\xi^i(t, x^k)$:

$$\delta(\nabla_t g_{ij}) = \xi^k \partial_k (\nabla_t g_{ij}) + (\nabla_t g_{kj}) \partial_i \xi^k + (\nabla_t g_{ik}) \partial_j \xi^k. \tag{2.7}$$

The supersymmetrization of the covariant time derivative is straightforward: We promote the shift vector n^i into a superfield

$$N^i = n^i + \theta \psi^i + \overline{\theta} \chi^i + \theta \overline{\theta} B^i, \tag{2.8}$$

and postulate that N^i transform under superdiffeomorphisms Ξ^i as

$$\delta N^i = \dot{\Xi}^i + \Xi^k \partial_k N^i - N^k \partial_k \Xi^i. \tag{2.9}$$

For now, we treat N^i as an unconstrained superfield, but will see below that it might be consistent with various chirality constraints. We extend the definition of the covarantized time derivative ∇_t to the superfield G_{ij},

$$\nabla_t G_{ij} \equiv \dot{G}_{ij} - N^k \partial_k G_{ij} - G_{kj} \partial_i N^k - G_{ik} \partial_j N^k, \tag{2.10}$$

and observe that the superfield $\nabla_t G_{ij}$ transforms under Ξ^i as a spatial two-tensor,

$$\delta(\nabla_t G_{ij}) = \xi^k \partial_k (\nabla_t G_{ij}) + (\nabla_t G_{kj}) \partial_i \xi^k + (\nabla_t G_{ik}) \partial_j \xi^k. \tag{2.11}$$

Indeed, this is a simple consequence of (2.5) together with the fact that $\theta, \overline{\theta}$ play the role of spectators in our construction of the covariant time derivative.

Having covariantized the time derivative of G_{ij}, we must now covariantize the superderivatives $\mathrm{D}G_{ij}$ and $\overline{\mathrm{D}}G_{ij}$. We first introduce gauge superfields S^i and $\overline{S}^i$, of the opposite statistics to N^i, and such that they transform under the gauge supertransformations Ξ^i as

$$\delta S^i = \mathrm{D}\Xi^i + \Xi^k \partial_k S^i - S^k \partial_k \Xi^i, \tag{2.12}$$

$$\delta \overline{S}^i = \overline{\mathrm{D}}\Xi^i + \Xi^k \partial_k \overline{S}^i - \overline{S}^k \partial_k \Xi^i. \tag{2.13}$$

With such superconnections, we now define the covariantized superderivatives of G_{ij},

$$\mathcal{D}G_{ij} \equiv \mathrm{D}G_{ij} - S^k \partial_k G_{ij} - G_{kj}\partial_i S^k - G_{ik}\partial_j S^k, \tag{2.14}$$

$$\overline{\mathcal{D}}G_{ij} \equiv \overline{\mathrm{D}}G_{ij} - \overline{S}^k \partial_k G_{ij} - G_{kj}\partial_i \overline{S}^k - G_{ik}\partial_j \overline{S}^k, \tag{2.15}$$

and see that they transform correctly under $\Xi^i(t, \theta, \overline{\theta}, x^k)$, as spatial two-tensors.

2.1.1 Type C: The chiral theory

Before studying in more detail this general case, we first observe that one can consistently restrict Ξ^i to be chiral superfields,

$$\overline{\mathrm{D}}\Xi^i = 0. \tag{2.16}$$

This will define what we will refer to as "Type C theory" (here "C" naturally stands for "chiral"). In Type C theory, the ordinary superderivative $\overline{\mathrm{D}}$ is already covariant, and no $\overline{S}^i$ superconnection is needed. Only S^i must be introduced, to covariantize D into $\mathcal{D}$. Still, having both N^i and S^i without any relation between them would lead to too many gauge field components (for example, both N^i and S^i contain a bosonic component that transforms as the bosonic shift vector n^i). In order to find a suitable constraint that relates them, note first that $-\overline{\mathrm{D}}S^i$ transforms as N^i. This leads us to expect that in Type C theory,

$$N^i = -\overline{\mathrm{D}}S^i. \tag{2.17}$$

In fact, this constraint has a very clear geometric origin, closely reminiscent of similar constraints in supersymmetric Yang-Mills gauge theories: It simply states that the action of the anticommutator of the covariantized superderivatives $\{\mathcal{D}, \overline{\mathrm{D}}\}$ on G_{ij} (or, indeed, on any symmetric 2-tensor T_{ij}) in Type C theory reproduces the action of

$-\nabla_t$ on G_{ij} (or T_{ij}). Here we must be careful of the order of terms when evaluating $\mathcal{D}$ of an odd two-tensor; the correct formula that works regardless of the statistics of T_{ij} is

$$\mathcal{D}T_{ij} \equiv \mathrm{D}T_{ij} - \partial_i S^k T_{kj} - \partial_j S^k T_{ik} - S^k \partial_k T_{ij}. \tag{2.18}$$

Note that since N^i in Type C theory is $\overline{\mathrm{D}}$ of something, it is automatically chiral:

$$\overline{\mathrm{D}} N^i = 0. \tag{2.19}$$

This is pleasing, since such a chiral N^i (for each fixed i) contains one real bosonic component n^i and one real fermionic component, which matches the number of independent component gauge transformation contained in a chiral Ξ^i. We can thus plan on eliminating the fermionic component of N^i by going to the analog of Wess-Zumino gauge [58].

We would similarly expect that S^i should have only two independent components in the $\theta, \overline{\theta}$ expansion. However, we clearly cannot impose the antichirality condition and simply set $\mathrm{D}S^i$ to be zero: This would be inconsistent with the fact that under a chiral Ξ^i, the transformation δS^i is *not* antichiral (even though it would be so at the linearized level). So, either S^i is unconstrained, and therefore contains four independent components two of which would have to be gauge invariant (which would be unpleasant), or there is another constraint that can be consistently imposed on S^i. The correct constraint turns out to be nonlinear,

$$\mathrm{D}S^i = S^k \partial_k S^i, \tag{2.20}$$

and it represents a covariant version of the antichirality condition. We will return to its precise geometric interpretation in Section 2.2.

2.1.2 Type A: The antichiral theory

Instead of postulating that the spacetime superdiffeomorphisms Ξ^i are chiral as in Type C theory, we could start with the antichirality condition,

$$\mathrm{D}\Xi^i = 0. \tag{2.21}$$

The entire construction will go through in the same way as in Type C theory, with all chiralities and antichiralities reversed at all the relevant steps. We will refer to this construction as "Type A theory" (with "A" standing for "antichiral"). Since the theory enjoys $\mathcal{N} = 2$ supersymmetry, Type A theory might naively seem like another construction of the same Type C theory in disguise, up to a simple change of coordinates. However, recall that when we introduced our supercharges, we selected once and for all Q (and not $\overline{Q}$, or any other linear superposition of them) to be our BRST charge of topological symmetry. This selection lifts the $\mathcal{N}=2$ democracy between the two chiralities, and makes Type A theory *a priori* distinct from Type C.

In more detail, in Type A theory we covariantize the time derivative using gauge superfield N^i, which transforms according to (2.9), now with an antichiral Ξ^i. And we covariantize the superderivative $\overline{\mathrm{D}}$ to $\overline{\mathcal{D}}$ by introducing the odd gauge superfield $\overline{S}^i$. The other superderivative D is already covariant, and no S^i superfield is introduced or needed. The relation between N^i and $\overline{S}^i$ in Type A theory is

$$N^i = -\mathrm{D}\overline{S}^i, \tag{2.22}$$

which makes N^i automatically antichiral. This relation is again an expression of a covariant constraint, which ensures that

$$\{\mathrm{D}, \overline{\mathcal{D}}\} T_{ij} = -\nabla_t T_{ij}, \tag{2.23}$$

on any symmetric 2-tensor T_{ij}.

Note that $\overline{S}^i$, if further unconstrained, would have two gauge-invariant components, for which we have no use. A constraint should again be imposed to eliminate them, but it cannot be simply the chirality condition on $\overline{S}^i$, which is inconsistent with the transformations of $\overline{S}^i$ under antichiral superdiffeomorphisms. The correct constraint takes the form of a nonlinear improvement of the naive antichirality constraint,

$$\overline{\mathrm{D}S}^i = \overline{S}^k \partial_k \overline{S}^i. \tag{2.24}$$

2.1.3 Type B: The balanced theory

While Theories C and A appear to be the minimal theories with $\mathcal{N}=2$ supersymmetry and gauge superdiffeomorphism symmetry, they each break the symmetry between ghosts and antighosts, due to the (anti)chirality condition on the superdiffeomorphism parameters Ξ^i. Now we will construct a theory with Ξ^i fully unconstrained, which will restore the ghost-antighost symmetry. In the literature, topological theories with such a ghost-antighost symmetry are sometimes referred to as "balanced" [50]. We will adopt this terminology for our case here as well, and will call this theory "Type B" (with "B" naturally standing for "balanced").

In order to allow for unconstrained Ξ^i supergauge transformations, we must covariantize the time derivative and both superderivatives D, $\overline{\mathrm{D}}$, by introducing gauge superfields N^i, S^i and $\overline{S}^i$, which transform according to (2.9), (2.12) and (2.13). They correctly covariantize all our derivatives, but carry way too many independent components and must therefore be subjected to a series of natural constraints. First of all, N^i can be algebraically expressed in terms of S^j, $\overline{S}^j$ and their various derivatives, by imposing

$$\left\{\mathcal{D},\overline{\mathcal{D}}\right\}T_{ij}=-\nabla_t T_{ij} \tag{2.25}$$

on symmetric two-tensors. This condition gives

$$N^i=-\overline{\mathrm{D}}S^i-\mathrm{D}\overline{S}^i+S^k\partial_k\overline{S}^i+\overline{S}^k\partial_k S^i. \tag{2.26}$$

Note several interesting facts about this formula: First of all, in the absence of $\overline{S}^k$ (or S^k), it reduces to the expressions for N^i in Type C (or Type A) theory, respectively. Secondly, in the Type B theory, the expression for N^i also contains important *nonlinear cross-terms* between S^i and $\overline{S}^k$, which had no analog in Type C and Type A theories.

Our relation (2.26) uniquely expresses N^i in terms of S^i and $\overline{S}^i$. Thus, we expect that Wess-Zumino gauge exists, in which we keep only the leading component n^i and the bosonic diffeomorphisms ξ^i as symmetries, using the remaining three components of Ξ^i (for each i) to eliminate the remaining three components of N^i. However, this still leaves us with too many components of the *a priori* unrelated S^i and $\overline{S}^i$, a problem which we already noticed in Type C and Type A theories. We therefore return to the geometric interpretation of all our constraints, in the "umbrella" case of Type B theory.

2.2 Geometric interpretation I: Superconnections, constraints and flatness

To find suitable constraints that should be imposed on S^i and $\overline{S}^i$, we can calculate the appropriate graded commutators of our covariant derivatives, and define "supercovariant field strengths" W^i_{MN}, $M,N\in\{t,\theta,\overline{\theta}\}$, in a way reminiscent of more traditional supersymmetric gauge theories (such as super Yang-Mills), as obstructions against the closure of the algebra of derivatives isomorphic to the algebra of $\mathrm{D},\overline{\mathrm{D}}$ and ∂_t. Evaluating the graded commutators of $\mathcal{D},\overline{\mathcal{D}}$ and ∇_t on our spatial metric superfield G_{ij} gives:

$$
\begin{aligned}
\{\mathcal{D},\mathcal{D}\}G_{ij} &= -\partial_i W^k_{\theta\theta}G_{kj} - \partial_j W^k_{\theta\theta}G_{ik} - W^k_{\theta\theta}\partial_k G_{ij}, &(2.27)\\
\{\overline{\mathcal{D}},\overline{\mathcal{D}}\}G_{ij} &= -\partial_i W^k_{\overline{\theta\theta}}G_{kj} - \partial_j W^k_{\overline{\theta\theta}}G_{ik} - W^k_{\overline{\theta\theta}}\partial_k G_{ij}, &(2.28)\\
\{\mathcal{D},\overline{\mathcal{D}}\}G_{ij} &= -\nabla_t G_{ij} - \partial_i W^k_{\theta\overline{\theta}}G_{kj} - \partial_j W^k_{\theta\overline{\theta}}G_{ik} - W^k_{\theta\overline{\theta}}\partial_k G_{ij}, &(2.29)\\
\left[\nabla_t,\mathcal{D}\right]G_{ij} &= -\partial_i W^k_{t\theta}G_{kj} - \partial_j W^k_{t\theta}G_{ik} - W^k_{t\theta}\partial_k G_{ij}, &(2.30)\\
\left[\nabla_t,\overline{\mathcal{D}}\right]G_{ij} &= -\partial_i W^k_{t\overline{\theta}}G_{kj} - \partial_j W^k_{t\overline{\theta}}G_{ik} - W^k_{t\overline{\theta}}\partial_k G_{ij}, &(2.31)
\end{aligned}
$$

where

$$
\begin{aligned}
W^i_{\theta\theta} &= 2(\mathrm{D}S^i - S^k\partial_k S^i), &(2.32)\\
W^i_{\overline{\theta\theta}} &= 2(\overline{\mathrm{D}S}^i - \overline{S}^k\partial_k\overline{S}^i), &(2.33)\\
W^i_{\theta\overline{\theta}} &= N^i + \overline{\mathrm{D}}S^i + \mathrm{D}\overline{S}^i - \overline{S}^k\partial_k S^i - S^k\partial_k\overline{S}^i, &(2.34)\\
W^i_{t\theta} &= \dot{S}^i - N^k\partial_k S^i - \mathrm{D}N^i + S^k\partial_k N^i, &(2.35)\\
W^i_{t\overline{\theta}} &= \dot{\overline{S}}^i - N^k\partial_k\overline{S}^i - \overline{\mathrm{D}}N^i + \overline{S}^k\partial_k N^i. &(2.36)
\end{aligned}
$$

Note that our constraint (2.26) is simply indicating the vanishing of the field strength

$$
W^i_{\theta\overline{\theta}} = 0. \tag{2.37}
$$

This suggests that we impose the rest of the "flatness conditions" on supertime,

$$
W^i_{\theta\theta} = 0, \qquad W^i_{\overline{\theta\theta}} = 0, \tag{2.38}
$$

which requires

$$
\mathrm{D}S^i = S^k\partial_k S^i, \qquad \overline{\mathrm{D}S}^i = \overline{S}^k\partial_k\overline{S}^i. \tag{2.39}
$$

In fact, our set of constraints (2.26) and (A.2) is the minimal set that implies the vanishing of *all* the field strengths W^i_{MN} by Bianchi identities.

In our construction of Type C and A theories, we pointed out that S^i and $\overline{S}^i$ should each satisfy a chirality-like constraint, but showed the inconsistency of constraining S^i and $\overline{S}^i$ by the naive linear (anti)chirality conditions. The two constraints (A.2) are the required consistent nonlinear extensions of the naive (anti)chirality constraints. Note that these two conditions remain nonlinear in S^i (or $\overline{S}^i$) when reduced to Type C (or Type A) theory.

Any potential worry that the nonlinear nature of the constraints (A.2) could lead to over-constraining is eliminated by finding the explicit solutions of the constraints in components. Constraints (A.2) are solved by

$$S^i = \sigma^i + \theta\sigma^k\partial_k\sigma^i + \overline{\theta}Y^i + \theta\overline{\theta}(\dot{\sigma}^i + Y^k\partial_k\sigma^i - \sigma^k\partial_k Y^i), \tag{2.40}$$

$$\overline{S}^i = \overline{\sigma}^i + \theta X^i + \overline{\theta}\overline{\sigma}^k\partial_k\overline{\sigma}^i + \theta\overline{\theta}(\overline{\sigma}^k\partial_k X^i - X^k\partial_k\overline{\sigma}^i). \tag{2.41}$$

Then (2.26) is solved by setting

$$Y^i = -n^i - X^i + \sigma^k\partial_k\overline{\sigma}^i + \overline{\sigma}^k\partial_k\sigma^i, \tag{2.42}$$

and expressing the three remaining components in the N^i superfield (2.8) as follows,

$$\psi^i = \dot{\sigma}^i + \sigma^k\partial_k n^i - \partial_k\sigma^i n^k, \tag{2.43}$$

$$\chi^i = \dot{\overline{\sigma}}^i + \overline{\sigma}^k\partial_k n^i - \partial_k\overline{\sigma}^i n^k, \tag{2.44}$$

$$B^i = -\dot{X}^i + n^k\partial_k X^i - X^k\partial_k n^i + \dot{\sigma}^k\partial_k\overline{\sigma}^i + \sigma^j\partial_j n^k\partial_k\overline{\sigma}^i - n^j\partial_j\sigma^k\partial_k\overline{\sigma}^i + \overline{\sigma}^k\partial_k\dot{\sigma}^i + \overline{\sigma}^k\partial_k\sigma^j\partial_j n^i + \overline{\sigma}^k\sigma^j\partial_k\partial_j n^i - \overline{\sigma}^k\partial_k n^j\partial_j\sigma^i - \overline{\sigma}^k n^j\partial_k\partial_j\sigma^i \tag{2.45}$$

All the constraints have been solved, and the shift superfields are all expressed in terms of the bosonic component fields n^i, X^i, and the fermionic components σ^i and $\overline{\sigma}^i$.

This concludes our construction of the self-consistent covariantization under unconstrained diffeomorphisms Ξ^i. To summarize, imposing the flatness conditions (2.37) and (2.38) leads to the set of constraints (2.26) and (A.2) on N^i, S^i and $\overline{S}^i$. This reduces the number of independent components in the gauge superfields so that we can eliminate all the components besides n^i by an analog of the Wess-Zumino gauge.

2.3 Geometric interpretation II: Supersymmetric Diff(Σ) Yang-Mills theory

The set of constraints that we just identified in terms of the superfield strengths has another intriguing geometric interpretation, which sheds some additional light on our covariantization construction, and which can also be of independent interest.

In this new interpretation, we take a different perspective on the structure of spacetime: We view the theory as a supersymmetric gauge theory on supertime $\mathscr{M}_0$, *i.e.*, a theory in (1|2) dimensions. The spatial slices Σ will be interpreted as an internal space, not as dimensions of spacetime. Thus, a field such as $N^i(t,\theta,\overline{\theta},x^j)$

is interpreted as a field on $(t, \theta, \overline{\theta})$, with (i, x^i) interpreted as a continuous internal (multi)index.

Recall that for any (typically compact, and finite-dimensional) internal Yang-Mills symmetry group $\mathcal{G}$, there are standard rules for constructing the corresponding supersymmetric Yang-Mills theory in a superspace of dimension $(d|d')$: One postulates the existence of superconnections Γ^α_M on superspace, where the index M goes over all $d+d'$ values, and α indicates the adjoint representation of the bosonic gauge group $\mathcal{G}$. The derivatives on superspace have thus been covariantized to D_M. Next one defines the supersymmetric field strengths W^α_{MN} via

$$[D_M, D_N\} = T_{MN}{}^P D_P + W_{MN}, \qquad W^\alpha_{MN} T_\alpha, \tag{2.46}$$

with $T_{MN}{}^P$ the torsion of the flat superspace, and T_α the generators of $\mathcal{G}$. Finally, one imposes a set of constraints sometimes referred to as "conventional" , [59, 60]:

$$W^\alpha_{MN} = 0, \qquad \text{whenever both indices } M, N \text{ are odd.} \tag{2.47}$$

The rest of the constraints is implied by the Bianchi identities.

With the full list of constraints identified, one can then construct various candidate Lagrangians in superspace, typically by invoking an invariant metric $g_{\alpha\beta}$ on the Lie algebra of $\mathcal{G}$ in order to contract the pairs of internal indices on expressions quadratic in W^α. This is the standard way in which supersymmetric Yang-Mills gauge theories in various spacetime dimensions are constructed in superspace [59, 60].

Note the remarkable fact that our construction of the shift-vector sector in our topological gravity theory in Section 2.2 in terms of the fields N^i, S^i and $\overline{S}^I$, their superfield strengths $\mathcal{W}$, and the corresponding constraints, takes precisely the form of the just reviewed standard supersymmetric Yang-Mills theory construction, with the following identifications:

- The underlying spacetime is the supertime, of dimension $(1|2)$, with coordinates $(t, \theta, \overline{\theta})$.

- The connections Γ^α_M are N^i for $M = t$, S^i for $M = \theta$, and $\overline{S}^i$ for $M = \overline{\theta}$, and with the adjoint index α being the multi-index (i, x^k). More precisely, in a language independent of the choice of coordinates on Σ, the Lie algebra $\mathcal{G}$ is the infinite-dimensional algebra of vector fields on Σ. Thus, the Lie algebra of this Yang-Mills theory is the Lie algebra of spatial diffeomorphisms $\mathcal{G} = \mathrm{Diff}(\Sigma)$!

- One can check directly that the definitions of superfield strengths $\mathcal{W}$ (2.32–2.36) indeed correspond precisely to the Yang-Mills field strength definition in (2.46). We recognize the structure constants of $\mathcal{G} = \text{Diff}(\Sigma)$ in the expressions for $\mathcal{W}$'s, and we also see that the first term in (2.29) is the torsion term anticipated in (2.46).

- Our collection of constraints (2.37) and (2.38) is equivalent to the "conventional constraints" (2.47) of the standard superspace construction of supersymmetric Yang-Mills gauge theory.

Thus we reach a perhaps surprising conclusion: The construction of the shift superfield sector in our topological quantum gravity on the superspace $\mathscr{M}$ of dimension $(D + 1|2)$ is precisely equivalent to the construction of supersymmetric Yang-Mills gauge theory on supertime of dimension $(1|2)$, and with the internal Lie algebra of gauge symmetries being the algebra of spatial diffeomorphisms $\text{Diff}(\Sigma)$ of the spatial slices Σ!

This intimate connection between conventional supersymmetric Yang-Mills theory with an internal symmetry $\mathcal{G}$ on one hand, and the gauging of spatial diffeomorphisms in gravity on the other, can potentially be of some broader interest. One is reminded of the BCJ color-kinematics duality [61–63], which relates amplitudes in Yang-Mills theories to amplitudes in gravity by replacing internal symmetry factors with kinematic factors. It has been quite mysterious so far what kind of algebraic structure can underlie this procedure on the kinematic side. Perhaps our relation between gravity and Yang-Mills may be useful in identifying the hidden algebraic structure on the side of kinematics, at the cost of singling out the role of time and making the description not manifestly relativistic.

While the parallel between our shift superfield sector and supersymmetric Yang-Mills is quite precise, there is one instance where the similarity stops: Unlike compact finite Lie algebras $\mathcal{G}$, our Lie algebra of spatial diffeomorphisms $\text{Diff}(\Sigma)$ does not have a constant invariant metric, and therefore one cannot construct standard quadratic kinetic terms for the action. This is of course consistent with the prior knowledge that no such kinetic terms for the shift vector should exist. In one wishes to construct an invariant metric on the Lie algebra of $\text{Diff}(\Sigma)$, it can only be done in a field-dependent way, by invoking a spatial metric $g_{ij}(x)$. For two generators $\xi^i(x)$ and $\zeta^i(x)$ of $\text{Diff}(\Sigma)$, we define their inner product by

$$(\xi, \zeta) = \int d^D x \sqrt{g}\, g_{ij}\, \xi^i \zeta^j. \tag{2.48}$$

This metric on $\text{Diff}(\Sigma)$ is thus field-dependent, and cannot be used to construct kinetic terms for the superfield strengths of the shift sector.

2.4 The action

Using the covariant derivatives $\nabla_t G_{ij}, \mathcal{D}G_{ij}$ and $\overline{\mathcal{D}}G_{ij}$ constructed in the previous paragraphs, the action for Type B theory can be easily constructed in the manifestly supersymmetric form in our $\mathcal{N}=2$ superspace.

The kinetic term is covariantized to

$$S_K = \int dt\, d^2\theta\, d^D x\, \sqrt{G}\,\left\{\left(\lambda_\perp G^{ik}G^{j\ell} - \lambda G^{ij}G^{k\ell}\right)\overline{\mathcal{D}}G_{ij}\,\mathcal{D}G_{k\ell} + \dots\,\right\}. \tag{2.49}$$

(This kinetic term is valid in Type B theory; the corresponding kinetic terms in Type A and C cases are simply obtained by reducing $\mathcal{D}$ or $\overline{\mathcal{D}}$ to D or $\overline{\mathrm{D}}$ as appropriate.) The superpotential stays the same as in our primitive theory, (1.20).

The path integral for this theory is, in superspace language,[10]

$$\mathcal{Z} = \int \mathscr{D}\mu[G_{ij}, N^i, S^i, \overline{S}^i]\, \exp\left\{-\frac{1}{\hbar\kappa^2}(S_K - S_{\mathcal{W}})\right\}. \tag{2.50}$$

This path integral requires further gauge fixing of the newly introduced spacetime Diff(Σ) gauge symmetry, which we will discuss briefly in Section 2.5. In addition, the same points that we presented in our brief discussion of the path integral of the primitive theory in Section 1.5 apply here as well.

The main improvement compared to the primitive theory is that the flow equations for the metric g_{ij} are now covariant under time-dependent spatial diffeomorphisms, with the path integral localizing to the solutions of

$$\nabla_t g_{ij} = -\alpha_R R_{ij} + \frac{\alpha_R}{2}\left[1-\widetilde{\lambda}(D-2)\right] g_{ij}R + \frac{\alpha_\Lambda}{2} g_{ij}. \tag{2.51}$$

The possibility of modifying the Ricci flow by the time-dependent spatial diffeomorphism generated by n^i has been very useful in the mathematical theory, where it is known as "DeTurck's trick". We are now in a position to see how these techniques

[10] In the construction of the path-integral measure $\mathcal{D}\mu$, one must keep in mind that N^i, S^i and $\overline{S}^i$ are not independent, but related to each other by our constraints (A.2) and (2.26). At this stage, it might be better to switch from the superspace to the component formulation, in which the definition of the $\mathcal{N}=2$ supersymmetric measure for the component fields is more straightforward. Alternatively, Appendix B solves the constraints and expresses our superfields in terms of unconstrained prepotential superfields.

emerge in our quantum gravity, as a part of the process of gauge fixing the spatial diffeomorphism symmetry.

2.5 Wess-Zumino gauge

In order to construct a theory with spatial diffeomorphism gauge symmetry generated by bosonic generators $\xi^i(t, x^k)$ in a way manifestly consistent with $\mathcal{N} = 2$ global supersymmetry of supertime, we first promoted ξ^i to a superfield of symmetry generators Ξ^i, and used superspace techniques to find a theory invariant under the much larger symmetry generated by Ξ^i. Now we need to decide how to interpret – or eliminate – the additional gauge symmetries contained in Ξ^i of (2.3), *i.e.*, symmetries generated by

$$\zeta^i, \qquad \eta^i, \quad \text{and} \quad \alpha^i, \tag{2.52}$$

so that we reduce the gauge group back to the desired Diff(Σ). Until this point, our strategy for the gauging process has closely parelleled the construction of supersymmetric relativistic gauge theories (see, *e.g.*, [59] for a review and introduction), which also offers a natural way of reducing the gauge symmetries to the bosonic ones, known as Wess-Zumino gauge [58]: One simply sets all the higher components of the gauge superfield, in our case the shift superfield N^i, to zero. This is algebraically possible, leads to no additional constraints, and leaves only the bosonic ξ^i symmetry unfixed.

We will adopt this Wess-Zumino gauge for the spatial diffeomorphism symmetry in our theory. Thus, the action that appears in (2.50) in Wess-Zumino gauge remains gauge invariant only under the bosonic Diff(Σ) symmetry, but still in a way consistent with the $\mathcal{N} = 2$ supersymmetry. The path-integral measure is also correspondingly reduced in Wess-Zumino gauge.

Having disposed of the higher gauge symmetries (2.52), we must next decide how to treat the remaining bosonic gauge symmetries Diff(Σ). There are several useful options. First, we can leave the theory in its manifestly Diff(Σ) invariant form for as long as possible, and introduce its gauge fixing by the standard Faddeev-Popov ghosts when necessary (for example, for developing Feynman diagrams around a given background). This "equivariant" approach is the strategy often preferred in topological field theories. It is followed for example for relativistic topological Yang-Mills, where the Yang-Mills gauge symmetry is typically left unfixed.

Altenatively, there might be reasons why one may want to fix, fully or partially, the Diff(Σ) symmetry. For the topological gravity of the Ricci flow, this option turns out to be very useful for the comparison to the mathematical literature. In fact, we will see three different natural gauge choices, each corresponding to an operation performed in the mathematical theory of the Ricci flow. We refer to them as "DeTurck gauge," "Perelman gauge", and "Hamilton gauge".

- *Perelman gauge:* In this gauge, one simply sets the shift vector to zero,

$$n^i(t, x^j) = 0. \tag{2.53}$$

  In the context of gravity and spatial diffeomorphism symmetry, this is the analog of temporal gauge. Adopting this gauge choice, the covariant time derivative $\nabla_t g_{ij}$ in the localization equation (2.51) is reduced to the ordinary time derivative $\dot{g}_{ij}$, as in the original form (0.1) of Hamilton's Ricci flow (which was not invariant under time-dependent spatial diffeomorphisms).

- *Hamilton gauge:* If the theory contains another field $h(t, x^i)$, which transforms as a scalar under spatial diffeomorphisms, one can replace (2.53) with

$$n^i(t, x^j) = g^{ik}\partial_k h. \tag{2.54}$$

  We do not have any such scalars in the theory yet, but will see that this type of gauge will be useful when we extend the gauge symmetries to foliation-preserving diffeomorphisms of spacetime. The gauge-fixing condition (2.54) played an important role in Perelman's original approach to Ricci flow, in particular in re-establishing the relation to the original Hamilton-Ricci flow (0.1); the role of h was played by Perelman's dilaton field ϕ.

- *DeTurck gauge:* This is the context in which the original DeTurck trick first appeared [64]. To define this gauge, one first chooses a fixed fiducial metric $\widetilde{g}_{ij}$ on Σ, and sets

$$n^i = g^{jk}\left(\Gamma^i_{jk} - \widetilde{\Gamma}^i_{jk}\right), \tag{2.55}$$

  where Γ^i_{jk} and $\widetilde{\Gamma}^i_{jk}$ are the Christoffel symbols representing the torsion-free Levi-Cività connections of g_{ij} and $\widetilde{g}_{ij}$ respectively (see, *e.g.*, Ch. 3.3 of [11] or Ch. 2.6 of [23] for additional mathematical context and motivation). This choice obviously breaks spatial diffeomorphism invariance. The usefulness of this gauge choice stems from the fact that the Ricci flow equation in this gauge is found to be manifestly parabolic, a property not obvious in other gauges, and definitely untrue for the gauge-unfixed flow equation (0.1) (which is parabolic only modulo spatial diffeomorphisms, or "weakly" parabolic). In turn, this manifest parabolicity leads to a simple proof of the existence and uniqueness theorem, stating that a solution of the initial value problem for the flow equation exists for some amount of time $\varepsilon > 0$, and that on that time interval the solution is unique.

The first two of these gauge choices are going to be particularly useful once we extend the gauge symmetries to foliation-preserving spacetime diffeomorphisms, especially in the theory with the nonprojectable lapse.

3 The gauge theory: Gauging time translations

Next, we wish to gauge time translations, or at least those that preserve the preferred foliation of spacetime. In the bosonic theory, such foliation-preserving time diffeomorphisms are generated by

$$\delta t = f(t). \tag{3.1}$$

To promote them to a gauge symmetry, we introduce a new field, the lapse function $n(t)$, which transforms as

$$\delta n = f\dot{n} + \dot{f}n. \tag{3.2}$$

Multiplying the covariant time derivative $\nabla_t g_{ij}$ with the inverse lapse function, we obtain $(1/n)\nabla_t g_{ij}$, which transforms as a scalar under time diffeomorphisms. Such scalars can then be used to build invariant Lagrangians, which take the form of the covariant spacetime volume element

$$d\mathcal{V}(g,n) \equiv d^D x\, dt\, n\sqrt{g}, \tag{3.3}$$

mutliplied by any scalar function made out of the available ingredients.

We will now generalize this gauging procedure to our supersymmetric case. In order to make $f(t)$ consistent with supersymmetry, it must first be promoted into a superfield,

$$F(t,\theta,\overline{\theta}) = f(t) + \theta\varphi(t) + \overline{\theta}\overline{\varphi}(t) + \theta\overline{\theta}\gamma(t). \tag{3.4}$$

In addition, we may choose this superfield to be further constrained, for example by a chirality condition. Note that the superfield of time reparametrizations is independent of x^i, reflecting the fact that our gauge symmetries preserve the structure of the spacetime foliation $\mathscr{M}_{\mathcal{F}}$ by spatial manifolds Σ of constant $t, \theta, \overline{\theta}$.

3.1 The projectable case

In this section, we will construct the minimal theory consistent with the gauge symmetries of (3.4). This theory will have a projectable lapse $n(t)$, promoted to a superfield $N(t)$. As in the case of spatial diffeomorphisms, there are three versions of the theory, depending on whether we impose a chirality or antichirality condition on N, or keep the superfield unconstrained.

3.1.1 Type C theory

We begin with our chiral Type C theory of Section 2.1.1, and we extend the gauge symmetry of the chiral spatial diffeomorphisms $\Xi^i(t,\theta)$ to also include the chiral version of time reparametrization symmetry generated by F which satisfies $\overline{\mathrm{D}}F = 0$.

The gauge transformations of the previously introduced superfields G_{ij}, N^i and S^i are:

$$\begin{aligned} \delta G_{ij} &= F\dot{G}_{ij} + \Xi^k\partial_k G_{ij} + \partial_i\Xi^k G_{kj} + \partial_j\Xi^k G_{ik}, && (3.5)\\ \delta N^i &= F\dot{N}^i - \dot{F}N^i + \dot{\Xi}^i + \Xi^k\partial_k N^i - \partial_k\Xi^i N^k, && (3.6)\\ \delta S^i &= F\dot{S}^i - \mathrm{D}F\overline{\mathrm{D}}S^i + \mathrm{D}\Xi^i + \Xi^k\partial_k S^i - \partial_k\Xi^i S^k. && (3.7)\end{aligned}$$

The first two of these rules follow straightforwardly from the requirement that the bosonic component fields g_{ij} and n^i transform under $f(t)$ as in the bosonic theory. The third rule follows from the requirement that the constraint (2.17) that relates N^i to S^i be preserved under the time reparametrizations.

In order to construct the theory with F gauge symmetry, we could introduce a supervielbein on $(t,\theta,\overline{\theta})$ (which would be a 3×3 matrix of superfields) and impose enough constraints on it so that we reduce the number of independent component fields to the bosonic lapse function and its superpartner under Q. Here we will follow a much more straightforward "bottom-up" strategy, and will return to the supervielbein interpretation below once our construction is complete.

Consider the derivatives $\nabla_t G_{ij}$, $\mathcal{D}G_{ij}$ and $\overline{\mathrm{D}}G_{ij}$, which serve as ingredients for building our Lagrangian in superspace. Under F, some of these derivatives do not transform as scalars. The task is to modify them minimally so that the modified derivatives do transform as scalars under F, and can then be again used as simple ingredients for constructing gauge invariant Lagrangians.

Start with the time derivative ∇_t. In the bosonic theory, its covariantization under $f(t)$ is simply accomplished by multiplying it with the inverse lapse function $1/n$. In the supersymmetric case, we introduce superfield $E(t,\theta,\overline{\theta})$ whose lowest component is $1/n$, and observe that $E\nabla_t G_{ij}$ transforms as a scalar under F,

$$\delta(E\nabla_t G_{ij}) = F\partial_t(E\nabla_t G_{ij}), \tag{3.8}$$

if we postulate that E transform as

$$\delta E = F\dot{E} - \dot{F}E. \tag{3.9}$$

Our next step is to covariantize similarly the remaining derivatives $\mathcal{D}G_{ij}$ and $\overline{\mathrm{D}}G_{ij}$. Since in Theory C, $\overline{\mathrm{D}}G_{ij}$ does not contain a gauge field, it transforms as a scalar under F. On the other hand, $\mathcal{D}G_{ij}$ contains S^i terms and does not transform as a scalar, and therefore requires a modification. As in the case of the time derivative, the first step is to introduce a new superfield $\mathcal{E}(t,\theta,\overline{\theta})$ and replace $\mathcal{D}G_{ij}$ with $\mathcal{E}\mathcal{D}G_{ij}$. This by itself is not sufficient, since the transformation of $\mathcal{D}G_{ij}$ under F will also contain terms proportional to $\nabla_t G_{ij}$. One must introduce one additional, odd superfield Θ, and shift $\mathcal{E}\mathcal{D}G_{ij}$ by an additive term $\Theta\nabla_t G_{ij}$. Postulating the transformation rules

$$\delta\mathcal{E} = F\dot{\mathcal{E}}, \tag{3.10}$$
$$\delta\Theta = -\mathcal{E}\,\mathrm{D}F + F\dot{\Theta} - \dot{F}\Theta \tag{3.11}$$

then ensures that

$$\mathcal{E}\,\mathcal{D}G_{ij} + \Theta\nabla_t G_{ij} \tag{3.12}$$

transforms as a scalar under F.

Thus, the covariantization of the derivatives consistently with supersymmetry requires the introduction of three superfields E,Θ and $\mathcal{E}$, which play the role which in the bosonic theory was played by the (inverse) lapse function. Clearly, these three superfields must be further constrained, so that they do not lead to a proliferation of gauge-invariant component fields for which we have no interpretation or desire.

The first such constraint is easy to propose: Since $\mathcal{E}$ transforms under F covariantly as a scalar, it is consistent to set

$$\mathcal{E} = 1. \tag{3.13}$$

Then there must be a constraint that relates E to Θ. A closer examination of the transformation properties reveals that $1 - \overline{\mathrm{D}}\Theta$ transforms the same way as E, and we therefore impose the constraint

$$E = 1 - \overline{\mathrm{D}}\Theta. \tag{3.14}$$

Note that this constraint implies that E is chiral, and therefore only contains two components, as it should: The inverse lapse function and its superpartner under Q.

Finally, Θ also must satisfy a constraint which reduces its components to two. Much like the gauge field S^i already present in Type C theory, Θ should satisfy some covariantized version of the antichirality constraint; the unique combination that transforms correctly is

$$\mathrm{D}\Theta = -\Theta\dot{\Theta}. \tag{3.15}$$

This completes our construction of the projectable version of Theory C with chiral $F(t,\theta)$ and $\Xi^i(t,\theta)$ gauge symmetries.

3.1.2 Geometric interpretation of constraints: Flatness of supertime

Before considering the lift to the nonprojectable case, we make one additional observation, which will be useful in the more complicated cases below. The constraints postulated on E and Θ above have a very natural geometric interpretation – they simply state that the vielbein geometry of supertime is flat! Indeed, it is natural to consider the graded commutators of the covariantized derivatives

$$\mathscr{D}_t \equiv E\nabla_t, \qquad \mathscr{D}_\theta \equiv \mathcal{E}\mathcal{D} + \Theta\nabla_t, \qquad \mathscr{D}_{\overline{\theta}} \equiv \overline{\mathrm{D}} \tag{3.16}$$

acting on G_{ij}. The (super)curvature of the geometry on supertime represented by our superfielbein fields $E, \mathcal{E}$ and Θ is then defined as the deviation of such graded commutators from the standard graded commutation relations satisfied before the gauging of time translations by ∇_t, $\mathcal{D}$ and $\overline{\mathrm{D}}$. (Recall that the latter three operators already represent a flat Yang-Mills connection of Diff(Σ), as we found out in Section 2.1.3.)

A straightforward evaluation of all the graded commutators of (3.16) shows that our constraints (3.14), (3.15) (together with (3.13)) imply the vanishing of all the curvature terms.

3.2 The nonprojectable case

In bosonic nonrelativistic gravity of the Lifshitz type, the more interesting and useful theory is obtained when the lapse function is allowed to be nonprojectable, $n(t, x^i)$. It is then natural to ask whether the vielbein superfields E, $\mathcal{E}$ and Θ can be promoted into spacetime fields, *i.e.*, allowed to depend on x^i.[11] Note that the gauge symmetries will stay the same foliation-preserving diffeomorphisms of spacetime as in our previous construction with the projectable lapse superfields; in particular, the generator of time reparametrizations F is only a function of t and/or $\theta, \overline{\theta}$.

3.2.1 Type C theory

In this section, we will show that such a nonprojectable version of our theory does indeed exist, first in the Type C case.

When our lapse-sector superfields $E, \mathcal{E}$ and Θ are extended to be nonprojectable superfields on spacetime, they transform as scalars under Ξ^i. Thus, their full transformation rules are

$$\begin{aligned} \delta E &= F\dot{E} - \dot{F}E + \Xi^k \partial_k E, & (3.17)\\ \delta\Theta &= -\mathcal{E}\,\mathrm{D}F + F\dot{\Theta} - \dot{F}\Theta + \Xi^k\partial_k\Theta, & (3.18)\\ \delta\mathcal{E} &= F\dot{\mathcal{E}} + \Xi^k\partial_k\mathcal{E}. & (3.19)\end{aligned}$$

What is the nonprojectable version of the constraints? Consider first the superfield $\mathcal{E}$. Since it transforms as a scalar under both F and Ξ^i, it is again consistent to set it equal to a constant, which we choose without any loss of generality to be $\mathcal{E} = 1$. We will impose this constraint from now on, and return to the more general case of arbitrary $\mathcal{E}$ later, in Section 3.2.3.

The constraint relating Θ and E stays the same as in the projectable case,

$$E = 1 - \overline{\mathrm{D}}\Theta, \tag{3.20}$$

but the constraint on Θ is modified to

$$\mathrm{D}\Theta - S^k\partial_k\Theta = -\Theta(\dot{\Theta} - N^k\partial_k\Theta). \tag{3.21}$$

The easiest way to derive these constraints is to evaluate again all the conditions for the vanishing of the supertime curvatures of the nonprojectable fields E and Θ, as we discussed in the projectable version above.

[11] In the geometry of foliations, and in the literature on the bosonic version of nonrelativistic gravity, such fields are commonly referred to as “nonprojectable”, in contrast to the “projectable” fields which are functions of only the leaves of the foliation, and whose lift to all of spacetime is simply via the pull-back by the natural projection of the foliation.

3.2.2 Type B theory

Now we extend the gauging of time translations to our balanced Type B theory, in which F is an unconstrained superfield. We jump directly to the nonprojectable case; the projectable one results by simply restricting the lapse superfields to be independent of x^i. Similarly, the gauging of time translations in the antichiral Type A theory will follow by restricting F to be antichiral, and the lapse superfields correspondingly constrained as well; see Section 3.2.4.

The gauge parameter $F(t,\theta,\overline{\theta})$ is of course independent of x^i, but otherwise unconstrained. We introduce superfields $E,\mathcal{E},\overline{\mathcal{E}},\Theta$ and $\overline{\Theta}$ to covariantize all derivatives. The transformation rules of all fields under the spacetime gauge symmetries are:

$$\begin{aligned}
\delta G_{ij} &= F\dot{G}_{ij}+\Xi^k\partial_k G_{ij}+\partial_i\Xi^k G_{kj}+\partial_j\Xi^k G_{ik},\\
\delta N^i &= F\dot{N}^i-\dot{F}N^i+\dot{\Xi}^i+\Xi^k\partial_k N^i-\partial_k\Xi^i N^k,\\
\delta S^i &= F\dot{S}^i+\mathrm{D}F\,N^i+\mathrm{D}\Xi^i+\Xi^k\partial_k S^i-\partial_k\Xi^i S^k,\\
\delta\overline{S}^i &= F\dot{\overline{S}}^i+\overline{\mathrm{D}}F\,N^i+\overline{\mathrm{D}}\Xi^i+\Xi^k\partial_k\overline{S}^i-\partial_k\Xi^i\overline{S}^k,\\
\delta E &= F\dot{E}-\dot{F}E+\Xi^k\partial_k E,\\
\delta\Theta &= -\mathcal{E}\,\mathrm{D}F+F\dot{\Theta}-\dot{F}\Theta+\Xi^k\partial_k\Theta,\\
\delta\overline{\Theta} &= -\overline{\mathcal{E}}\,\overline{\mathrm{D}}F+F\dot{\overline{\Theta}}-\dot{F}\overline{\Theta}+\Xi^k\partial_k\overline{\Theta},\\
\delta\mathcal{E} &= F\dot{\mathcal{E}}+\Xi^k\partial_k\mathcal{E},\\
\delta\overline{\mathcal{E}} &= F\dot{\overline{\mathcal{E}}}+\Xi^k\partial_k\overline{\mathcal{E}}.
\end{aligned}\tag{3.22}$$

With these rules, the following derivatives transform as scalars:

$$\mathscr{D}_t G_{ij}\equiv E\nabla_t G_{ij},\qquad \mathscr{D}_\theta G_{ij}\equiv\mathcal{E}\mathcal{D}G_{ij}+\Theta\nabla_t G_{ij},\qquad \mathscr{D}_{\overline{\theta}}G_{ij}\equiv\overline{\mathcal{E}}\overline{\mathcal{D}}G_{ij}+\overline{\Theta}\nabla_t G_{ij}.\tag{3.23}$$

As in the simpler Type C case, the supervielbein fields $E,\mathcal{E},\overline{\mathcal{E}},\Theta$ and $\overline{\Theta}$ must satisfy a number of constraints. First, we will follow our strategy from Type C theory and set

$$\mathcal{E}=1,\qquad \overline{\mathcal{E}}=1.\tag{3.24}$$

E is then constrained to be expressed in terms of Θ and $\overline{\Theta}$ and their derivatives,

$$E=1-\mathrm{D}\overline{\Theta}+S^k\partial_k\overline{\Theta}-\overline{\mathrm{D}}\Theta+\overline{S}^k\partial_k\Theta-\Theta(\dot{\overline{\Theta}}-N^k\partial_k\overline{\Theta})-\overline{\Theta}(\dot{\Theta}-N^k\partial_k\Theta).\tag{3.25}$$

Finally, Θ and $\overline{\Theta}$ are constrained to satisfy

$$\mathrm{D}\Theta - S^k\partial_k\Theta = -\Theta(\dot{\Theta} - N^k\partial_k\Theta), \tag{3.26}$$

$$\overline{\mathrm{D}\Theta} - \overline{S}^k\partial_k\overline{\Theta} = -\overline{\Theta}(\dot{\overline{\Theta}} - N^k\partial_k\overline{\Theta}). \tag{3.27}$$

These constraints again leave the desired number of four independent component fields: the inverse lapse function and its superpartners under Q and $\overline{Q}$.

3.2.3 Constraints as the flatness of supertime

It is instructive to check the geometric origin of our constraints in the nonprojectable Type B theory, which we will again interpret simply as the statement of the flatness of our supervielbein fields on supertime. We also take this opportunity to address our earlier somewhat *ad hoc* step of setting $\mathcal{E} = \overline{\mathcal{E}} = 1$, and will allow these superfields now to be unconstrained. Thus, our covariant derivatives are those we constructed in (3.23), before imposing any *ad hoc* constraints.

Now we evaluate their graded commutators to evaluate the conditions for flatness. We begin by evaluating

$$\{\mathscr{D}_\theta, \mathscr{D}_\theta\}\, G_{ij} = 2(\mathcal{E}\mathcal{D} + \Theta\nabla_t)^2 G_{ij} = 2\left[\mathcal{E}(\mathrm{D}\mathcal{E} - S^k\partial_k\mathcal{E}) + \Theta(\dot{\mathcal{E}} - N^k\partial_k\mathcal{E})\right]\mathcal{D}G_{ij} \tag{3.28}$$

$$+\ 2\left[\mathcal{E}(\mathrm{D}\Theta - S^k\partial_k\Theta) + \Theta(\dot{\Theta} - N^k\partial_k\Theta)\right]\nabla_t G_{ij} \tag{3.29}$$

The vanishing of the corresponding curvature requres that the right-hand side be zero, which implies the constraints

$$\mathcal{E}(\mathrm{D}\mathcal{E} - S^k\partial_k\mathcal{E}) + \Theta(\dot{\mathcal{E}} - N^k\partial_k\mathcal{E}) = 0, \tag{3.30}$$

$$\mathcal{E}(\mathrm{D}\Theta - S^k\partial_k\Theta) + \Theta(\dot{\Theta} - N^k\partial_k\Theta) = 0. \tag{3.31}$$

Similarly, the anticommutator $\{\mathscr{D}_{\overline{\theta}}, \mathscr{D}_{\overline{\theta}}\}\, G_{ij}$ gives the analogous condition for the barred quantities,

$$\overline{\mathcal{E}}(\overline{\mathrm{D}\mathcal{E}} - \overline{S}^k\partial_k\overline{\mathcal{E}}) + \overline{\Theta}(\dot{\overline{\mathcal{E}}} - N^k\partial_k\overline{\mathcal{E}}) = 0, \tag{3.32}$$

$$\overline{\mathcal{E}}(\overline{\mathrm{D}\Theta} - \overline{S}^k\partial_k\overline{\Theta}) + \overline{\Theta}(\dot{\overline{\Theta}} - N^k\partial_k\overline{\Theta}) = 0. \tag{3.33}$$

Next, we evaluate the anticommutator

$$\{\mathscr{D}_\theta, \mathscr{D}_{\overline{\theta}}\}\, G_{ij} = \Big[\mathcal{E}(\mathrm{D}\overline{\mathcal{E}} - S^k\partial_k\overline{\mathcal{E}}) + \Theta(\dot{\overline{\mathcal{E}}} - N^k\partial_k\overline{\mathcal{E}})\Big]\,\overline{\mathcal{D}}G_{ij} \tag{3.34}$$
$$+ \Big[\overline{\mathcal{E}}(\overline{\mathrm{D}}\mathcal{E} - \overline{S}^k\partial_k\mathcal{E}) + \overline{\Theta}(\dot{\mathcal{E}} - N^k\partial_k\mathcal{E})\Big]\,\mathcal{D}G_{ij}$$
$$+ \Big[-\mathcal{E}\overline{\mathcal{E}} + \mathcal{E}(\mathrm{D}\overline{\Theta} - S^k\partial_k\overline{\Theta}) + \overline{\mathcal{E}}(\overline{\mathrm{D}}\Theta - \overline{S}^k\partial_k\Theta) + \Theta(\dot{\overline{\Theta}} - N^k\partial_k\overline{\Theta}) + \overline{\Theta}(\dot{\Theta} - N^k\partial_k\Theta)\Big]\,\nabla_t G_{ij}.$$

The flatness condition then requires that this anticommutator be equal to $-E\nabla_t G_{ij}$, implying the following constraints:

$$\mathcal{E}(\mathrm{D}\overline{\mathcal{E}} - S^k\partial_k\overline{\mathcal{E}}) + \Theta(\dot{\overline{\mathcal{E}}} - N^k\partial_k\overline{\mathcal{E}}) = 0, \tag{3.35}$$
$$\overline{\mathcal{E}}(\overline{\mathrm{D}}\mathcal{E} - \overline{S}^k\partial_k\mathcal{E}) + \overline{\Theta}(\dot{\mathcal{E}} - N^k\partial_k\mathcal{E}) = 0, \tag{3.36}$$
$$E = \mathcal{E}\overline{\mathcal{E}} - \mathcal{E}(\mathrm{D}\overline{\Theta} - S^k\partial_k\overline{\Theta}) - \overline{\mathcal{E}}(\overline{\mathrm{D}}\Theta - \overline{S}^k\partial_k\Theta) - \Theta(\dot{\overline{\Theta}} - N^k\partial_k\overline{\Theta}) - \overline{\Theta}(\dot{\Theta} - N^k\partial_k\Theta) \tag{3.37}$$

Finally, the commutators of the odd superderivatives with $E\nabla_t$ imply that the remaining conditions of vanishing curvature are

$$E(\dot{\mathcal{E}} - N^k\partial_k\mathcal{E}) = 0, \tag{3.38}$$
$$E(\dot{\Theta} - N^k\partial_k\Theta) = \mathcal{E}(\mathrm{D}E - S^k\partial_k E) + \Theta(\dot{E} - N^k\partial_k E), \tag{3.39}$$

and

$$E(\dot{\overline{\mathcal{E}}} - N^k\partial_k\overline{\mathcal{E}}) = 0, \tag{3.40}$$
$$E(\dot{\overline{\Theta}} - N^k\partial_k\overline{\Theta}) = \overline{\mathcal{E}}(\overline{\mathrm{D}}E - \overline{S}^k\partial_k E) + \overline{\Theta}(\dot{E} - N^k\partial_k E). \tag{3.41}$$

Note that in the projectable case, assuming that $E, \mathcal{E}$ and $\overline{\mathcal{E}}$ are invertible, the constraints imply that $\mathcal{E}$ and $\overline{\mathcal{E}}$ are constants, which in retrospect justifies our choice of setting them equal to 1 from the outset. In the nonprojectable case, the constraints are solved by $\mathcal{E}$ and $\overline{\mathcal{E}}$ whose covariant time derivative is zero.

It is easy to check that our constraints (3.24–3.27) represent the minimal set of constraints which imply all the other conditions of flatness of supertime by Bianchi identities. The constraints can be solved explicitly by finding the component expressions for the lapse-sector superfields, repeating the steps we took in Section 2.2 when we solved the analogous constraints in the shift sector.

3.2.4 Type A theory

The antichiral Type A theory results by restricting F and Ξ^i to be antichiral:

$$\mathrm{D}F = 0, \qquad \mathrm{D}\Xi^i = 0, \tag{3.42}$$

and by setting

$$\mathcal{E} = 1, \qquad \Theta = 0, \quad \text{and} \quad S^i = 0 \tag{3.43}$$

in the rules specified in the "umbrella" theory of Type B presented above. We will not require any further details about this Type A theory in the rest of this thesis.

3.3 The supervielbein approach

Now we are ready to compare our approach to the top-down construction using supervielbeins. (We present the construction only for Type B theory, and for simplicity for its projectable version, with the projectable Type A and C cases following by a simple reduction.)

In the supervielbein approach, one postulates the existence of a 3×3 supervielbein matrix of superfields

$$e_M{}^A, \qquad M \in \{t, \theta, \overline{\theta}\}, \quad A \in \{0, \vartheta, \overline{\vartheta}\}, \tag{3.44}$$

where M is the coordinate index and A is the internal tangent-space index on supertime, and find enough constraints on $e_M{}^A$ to reduce them drastically to just four component fields: the lapse function and its superpartners.

Here we will establish the connection between our bottom-up construction involving superfields $E, \mathcal{E}, \overline{\mathcal{E}}, \Theta$ and $\overline{\Theta}$, and the full top-down supervielbein construction. It will be more convenient for us to work with the inverse supervielbein $e_A{}^M$, which is simply defined to interpolate between the coordinate basis ∂_M in the tangent space to supertime, and the moving-frame basis D_A whose three elements are labeled by the internal index A:

$$D_A = e_A{}^M \partial_M. \tag{3.45}$$

On the rigid supertime before the introduction of E, we have $D_A = (\partial_t, \mathrm{D}, \overline{\mathrm{D}})$, and the standard flat (inverse) supervielbein is given by

$$e_A^{(0)\,M} = \begin{pmatrix} 1 & 0 & 0 \\ -\overline{\theta} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.46}$$

Once we gauge time translations, the D_A's are given by the covariantized derivatives $E\partial_t, \mathcal{E}\mathrm{D} + \Theta\partial_t$ and $\overline{\mathcal{E}\mathcal{D}} + \overline{\Theta}\partial_t$, and the inverse supervielbein becomes

$$e_A{}^M = \begin{pmatrix} E & 0 & 0 \\ \Theta - \overline{\theta}\mathcal{E} & \mathcal{E} & 0 \\ \overline{\Theta} & 0 & \overline{\mathcal{E}} \end{pmatrix}. \tag{3.47}$$

Consider the generic superdiffeomorphism of supertime, generated in our coordinates $\tau^M \equiv (t, \theta, \overline{\theta})$ by some

$$\delta\tau^M = F^M(\tau^N). \tag{3.48}$$

Under this superdiffeomorphism, the supervielbein transforms geometrically, as

$$\delta e_M{}^A = F^N \partial_N e_M{}^A + \partial_M F^N e_N{}^A, \tag{3.49}$$

and analogously for the inverse supervielbein $e_A{}^M$.

Our gauge symmetry of gauged time translations is a specific subalgebra of this, consisting only of supertime-dependent time reparametrizations: $F^t = F(t, \theta, \overline{\theta})$, and $F^\theta = F^{\overline{\theta}} = 0$. Therefore, we should verify that our constrained supervielbein (3.47) which we derived in our bottom-up approach indeed transforms under F according to (3.49). It is a pleasing check that with our transformation rules for $E, \Theta, \overline{\Theta}, \mathcal{E}$ and $\overline{\mathcal{E}}$ established above, the vielbein indeed transforms geometrically as anticipated. For example, the variation $\delta e_\vartheta{}^t$ implied by (3.49) should be

$$\delta e_\vartheta{}^t = F\,\dot{e}_\vartheta{}^t - \dot{F}\,e_\vartheta{}^t - e_\vartheta{}^\theta \partial_\theta F. \tag{3.50}$$

Substituting from (3.47), this predicts

$$\begin{aligned} \delta(\Theta - \overline{\theta}\mathcal{E}) &= F(\dot{\Theta} - \overline{\theta}\dot{\mathcal{E}}) - \dot{F}(\Theta - \overline{\theta}\mathcal{E}) - \mathcal{E}\partial_\theta F \\ &= F(\dot{\Theta} - \overline{\theta}\dot{\mathcal{E}}) - \dot{F}\Theta - \mathcal{E}\,\mathrm{D}F, \end{aligned} \tag{3.51}$$

which exactly matches the result obtained directly by using the projectable version of the transformation rules (3.22).

We note that geometrically, the gauge symmetries that we have implemented on our system are those of spacetime diffeomorphisms that preserve the structure of a nested double foliation of the spacetime supermanifold,

$$\mathscr{M} \to \mathscr{M}_0^{1|2} \to \mathscr{M}_0^{0|2}. \tag{3.52}$$

In particular, the supertime $\mathscr{M}_0^{1|2}$ itself is naturally foliated by leaves of constant $(\theta, \overline{\theta})$, with the leaves parametrized by t.

3.4 The action

The covariant volume element on $\mathscr{M}$ is now

$$d\mathcal{V}(G, E) = dt\, d^2\theta\, d^D x\, \frac{\sqrt{G}}{E}. \tag{3.53}$$

In the projectable theory, the lowest-dimension kinetic term invariant under our full spacetime gauge symmetry is given by

$$S_K = \int dt\, d^2\theta\, d^D x\, \frac{\sqrt{G}}{E} \left\{ (\lambda_\perp G^{ik} G^{j\ell} - \lambda G^{ij} G^{k\ell}) \mathscr{D}_{\overline{\theta}} G_{ij}\, \mathscr{D}_\theta G_{k\ell} + \dots \right\}; \tag{3.54}$$

as in the primitive theory, we again set $\lambda_\perp = 1$ for simplicity. On the other hand, the superpotential now allows for a more refined structure.

3.4.1 The superpotential and Perelman's $\mathcal{F}$-functional

In the bosonic nonprojectable gravity of the Lifshitz type, it is well appreciated that new ingredients appear and can be used to construct new terms in the action. In particular, the spatial derivatives $\partial_i n$ of the nonprojectable lapse transform as a spatial one-form, and it can give rise to new invariant Lagrangian terms. In our $\mathcal{N} = 2$ supersymmetric theory, we similarly find new ingredients, which give rise to new invariants that can appear in the superpotential. In particular,

$$A_i \equiv \frac{\partial_i E}{E} \tag{3.55}$$

transforms as a spatial one-form and a time scalar,

$$\delta A_i = F\dot{A}_i + \Xi^k \partial_k A_i + \partial_i \Xi^k A_k. \tag{3.56}$$

We can form new invariants in the action, made of the appropriate contractions of A_i. In terms of the superfield Φ defined via

$$\Phi \equiv \log E, \tag{3.57}$$

we simply have $A_i = \partial_i \Phi$. The superpotential part of the action is now

$$S_{\mathcal{W}} = \int dt\, d^2\theta\, d^D x\, e^{-\Phi}\sqrt{G}\left\{\alpha_R R^{(G)} + \alpha_\Phi\, G^{ij}\partial_i\Phi\partial_j\Phi + \alpha_\Lambda\right\}, \tag{3.58}$$

for some coupling constants α_R, α_Φ and α_Λ. We recognize $S_{\mathcal{W}}$ as a superfield version of Perelman's $\mathcal{F}$-functional (0.2), simply generalized to include the cosmological constant term! Note that the role of Perelman's "dilaton" is played in our theory by the logarithm of the nonprojectable lapse function. These two results are the central results of the present thesis.

3.4.2 Localization equations and generalizations of Perelman's Ricci flow

This picture can be fleshed out even more by switching to the component formulation. As in Section 1.6, we will again suppress all the fermionic terms which are uniquely determined from supersymmetry, and focus only on the bosonic fields. In addition, for reasons of simplicity, we present the results only for Type C or Type A theory.

The bosonic component action corresponding to the superspace action (3.54) and (3.58) is:

$$\begin{aligned} S_{\text{bose}} = & -\frac{1}{\kappa^2}\int dt\, d^D x\,\sqrt{g}n\left(g^{ik}g^{j\ell} - \lambda g^{ij}g^{k\ell}\right)B_{ij}B_{k\ell} \\ & + \frac{1}{\kappa^2}\int dt\, d^D x\,\sqrt{g}\left(g^{ik}g^{j\ell} - \lambda g^{ij}g^{k\ell}\right)B_{ij}\,\nabla_t g_{k\ell} \\ & -\frac{1}{\kappa^2}\int dt\, d^D x\,\sqrt{g}n\,B_{ij}\left\{\alpha_R\left(\frac{1}{2}Rg^{ij} - R^{ij}\right) + \left(\frac{1}{2}\alpha_\Phi - \alpha_R\right)g^{ij}(\partial\phi)^2\right. \\ & \left. + \alpha_R\, g^{ij}\,\Delta\phi + (\alpha_R - \alpha_\Phi)g^{ik}\,g^{j\ell}\,\partial_k\phi\,\partial_\ell\phi - \alpha_R\, g^{ik}\,g^{j\ell}\,\nabla_k\partial_\ell\phi + \frac{1}{2}\alpha_\Lambda g^{ij}\right\} \end{aligned} \tag{3.59}$$

The saddle points of the action correspond to the spatial metric g_{ij} satisfying the appropriate flow equation, governed by the variation of a functional which is a direct generalization of Perelman's $\mathcal{F}$-functional. Integrating out the bosonic auxiliary field B_{ij} we obtain the localization equations, in the form of a flow equation covariantized with respect to foliation-preserving spacetime diffeomorphisms,

$$\begin{aligned}\frac{1}{n}\left(\dot{g}_{ij}-\nabla_i n_j-\nabla_j n_i\right) &= -\alpha_R\, R_{ij}+\frac{\alpha_R}{2}\left[1-\widetilde{\lambda}(D-2)\right] g_{ij}R \\ &+ (\alpha_R-\alpha_\Phi)\partial_i\phi\partial_j\phi-\left[\left(\alpha_R-\frac{\alpha_\Phi}{2}\right)(1-\widetilde{\lambda}D)+(\alpha_R-\alpha_\Phi)\widetilde{\lambda}\right] g_{ij}(\partial\phi)^2 \\ &+\alpha_R\left[1-\widetilde{\lambda}(D-1)\right] g_{ij}\Delta\phi-\alpha_R\nabla_j\partial_j\phi+\frac{1}{2}\alpha_\Lambda\, g_{ij}. \end{aligned} \tag{3.60}$$

This is a multi-parameter family of generalized Ricci-type flow equations for the spatial metric g_{ij}.

In contrast to g_{ij}, the lapse field $n=\exp(-\phi)$ does not yet receive any nontrivial time evolution from localization. In Type A or Type C theory, this is because the chirality condition on N eliminates the auxiliary field associated with n, and the topological symmetries of the theory are not yet fully gauge-fixed. Even in Type B theory, however, the required lowest-dimension kinetic term for n (or ϕ) cannot appear. This is simply because our spacetime foliation-preserving gauge invariance, which has so far been unfixed, prevents such terms from being gauge invariant. This is as far as the gauge-invariant theory can take us, and to make a closer contact with the exact form of Perelman's flow, additional gauge fixing steps will be necessary.

3.4.3 Physical versus topological theory

We return to the possibility of analytically continuing the topological theory from imaginary time to real time, raised briefly in our comments on the path integral (1.28) of the primitive theory.

In the case of relativistic quantum field theories, such a direct continuation of a topological field theory to real time would have little sense: In real time, the fermions would violate the spin-statistics theorem, and the field theory could not be interpreted as a unitary theory of propagating degrees of freedom, at least not without some additional difficult "untwisting" steps. In contrast, in the case of topological nonrelativistic gravity, one can at least entertain the possibility of continuing the theory to real time and interpreting it as a theory with propagating degrees of freedom. This would require an analytic continuation of our superspace, such that $\overline{\theta}$ and θ would now be complex, and conjugates of each other. This is needed so that the component fields could have physically sensible dispersion relations at least in some portions of the space of the coupling constants λ and α, and their quanta could be interpreted as physical particles. Since in nonrelativistic field theory, there is no spin-statistics theorem, this continuation could in principle lead to a consistent nonrelativistic gravity with gravitons and their superpartners with $\mathcal{N} = 2$ supersymmetry. The absence of the spin-statistics theorem in nonrelativistic systems makes the boundary between Faddeev-Popov ghosts and propagating physical fields interestingly fuzzy, and the appealing direct relation between a topological and a physical theory possible in principle.

However, before making sense of this rotation to real time and a nonrelativistic gravity with propagating degrees of freedom, another serious obstacle would have to be addressed. The process of Wick rotation between real and imaginary time is relatively well controlled in theories with a static, eternal vacuum (such as the vacuum of a relativistic field theory). In theories far from equilibrium, where the "vacuum" may not be eternal and static, the continuation would be much more subtle. In the topological gravity of the Ricci flow, the saddle-point solutions to which the path integral localizes are the "vacua" of the theory, and they are often cosmologies with substantial time dependence, and even with singularities (recall Figs. 0.1 and 0.2). They inherently represent systems very far from equilibrium, and one therefore would not expect that a simple analytic continuation interpolates between the real- and imaginary-time versions of the theory. The full machinery of the Schwinger-Keldysh formalism for quantum systems far from equilibrium[12] may be needed in order to settle this intriguing question.

4 Summary

In this part, we have established contact between the mathematics of Ricci flow and topological quantum field theory. It takes the form of a nonrelativistic topological quantum gravity, of the Lifshitz type.

Even though this theory would perhaps be most interesting in $3+1$ dimensions, for most of this part we presented our results in an arbitrary spatial dimension D. This was possible primarily because we spent most of our work on constructing the action of the classical theory, with the correct gauge symmetries and BRST supersymmetry

[12]For a recent discussion of the Schwinger-Keldysh formalism in the context of string theory, and for extensive references on the formalism, see [65].

structure. We expect the quantum properties of the theory to be more sensitive to D. Note that the special case of $D = 2$ would require some additional treatment already at the classical level, because of the well-known degeneracies that occur in Riemannian geometry in two spatial dimensions. On the mathematical side, the $D = 2$ analog of the Ricci flow is well-covered in the literature [66] (see also Ch. 5 of [11]), and leads to a novel proof of the uniformization theorem for Riemann surfaces. It should be possible to adjust the details of our construction to accommodate the special features of $2 + 1$ spacetime dimensions, which also happens to be the critical dimension in which quantum gravity of the Ricci flow is power-counting renormalizable.

With the identification of Perelman's dilaton as our nonprojectable lapse function, and his $\mathcal{F}$-functional as our superpotential, the localization equations in our topological quantum gravity represent a multi-parameter family of cousins to Perelman's original Ricci flow, parametrized by several coupling constants. Yet, it might be difficult to see, in this forest of the many couplings in (3.60), where exactly the original Perelman Ricci flow equations are precisely reproduced. In fact, since the localization equations (3.60) of the theory constructed in Section 3 are by design gauge invariant under foliation-preserving time reparametrizations – a symmetry not shared by Perelman's equations – they cannot reduce precisely to Perelman's flow equations for *any* values of the couplings. The precise embedding of Perelman's original flow into our theory requires a few additional steps, including a partial gauge fixing of our gauge symmetries, and will be presented in Part III.

One natural generalization that is accessible by our methods, but has not been discussed in the present part, is the construction of topological gravity associated with the Kähler-Ricci flows, on spacetimes whose spatial slices Σ carry a complex structure and whose dynamical spatial metric is Kähler. This is an active area of current mathematical research, in particular in dimension $4 + 1$ (see [67] or Ch. 2 of [12]). It would be very interesting to see what novel features the complex structure on space induces on the quantum gravity path integral, and the physical structure of the theory.

Another intriguing connection, not explored in this thesis, is the possible relation to quantum information theory. In the mathematical context, Perelman's theory of the Ricci flow contains various quantities deservedly referred to as entropy. In particular, the $\mathcal{F}$-functional (and its close cousins the $\mathcal{W}$- and $\mathcal{W}_+$-functionals) belong to this category, and exhibit precise monotonicity properties, crucial for the proofs of various theorems about the behavior of the flow. Their proper interpretation in the context of our topological quantum gravity is likely to be intimately connected to concepts of quantum information theory [68, 69], which have started playing a more dominant role in quantum field theory and quantum gravity in recent years.

We fully expect that further study of topological quantum gravity associated with the Ricci flow should be beneficial both for physics and for mathematics: The wealth of mathematical results, generated especially in the past two decades, can teach us new lessons about quantum gravity, at least in the topological setting. In turn, the methods of topological quantum field theory, which have proven so instrumental in influencing modern geometry in the past few decades, can now be extended to topological quantum gravity, and applied to the original mathematical theory of the Ricci flow. In this context, it will be particularly interesting to study topological observables of the quantum theory. While the BRST cohomology of our supermultiplets appears quite simple, and the "moduli spaces" of solutions are often elementary, it will be natural to probe the Ricci-flow spacetimes by *extended* topological observables, such as topological strings and topological membranes. Much of the mathematical ground for such observables has already been prepared, since extended spacetime probes of Perelman's flow have been studied extensively. The mathematical results reviewed in [25] appear particularly promising, and suggest strongly that the topological quantum gravity introduced in this thesis should naturally couple to topological brane excitations.

Part II
BRST Refinement

Recall that in Part I, a "primitive" topological gravity – whose only dynamical field is the *spatial* metric g_{ij} – was constructed, using the standard BRST techniques. The existence of an extended $\mathcal{N} = 2$ BRST supersymmetry algebra was required, and there was no secondary gauge symmetry besides the topological deformations of g_{ij}. The most convenient formulation of this theory is in terms of superfields on the appropriate $\mathcal{N} = 2$ superspace, and with global supercharges Q and $\overline{Q}$. In the next stage, the foliation-preserving spacetime diffeomorphisms were gauged, leading to new ingredients that bring the theory closer to the form that can accommodate the mathematical structure of Perelman's Ricci flow [2–4].

We chose such a two-stage construction of the theory in $\mathcal{N} = 2$ superspace because of its efficiency in identifying a suitable Lagrangian that serves our needs and makes contact with the Ricci flow. However, this construction may also have left the reader somewhat mystified about several points. First, the theory proposed has both topological gauge symmetries and spacetime gauge symmetries; yet, the two-stage construction uses a strange hybrid between a BRST gauge-fixed topological symmetry, and an unfixed gauge symmetry of foliation-preserving diffeomorphisms of spacetime. It may be natural to seek a one-step construction, in which *all* the gauge symmetries are handled uniformly, by BRST quantization. Second of all, if the topological theory has a secondary gauge symmetry of spacetime diffeomorphisms, why were there no ghost-for-ghost fields required? Such secondary ghost-for-ghosts are well-known to be essential in making sense of gauge theories with redundant gauge symmetries [30]. And finally, another puzzling feature emerges in the lapse sector: The lapse superfields were introduced in the process of making the theory invariant under foliation-preserving time reparametrizations, and they were chosen to be nonprojectable superfields. This nonprojectability was in turn important for making contact with Perelman's theory of the Ricci flow, since the role of the Perelman dilaton was found to be played by our nonprojectable lapse. One might then question the number of local propagating degrees of freedom in such a theory: The projectable time reparametrizations are not sufficient to remove the propagating polarization in the lapse, so is this theory even properly topological?

The purpose of the present part is to provide clarifying answers to all of these questions. In particular, we show how the theory constructed in Part I can be consistently interpreted as a one-stage BRST gauge fixing of a topological theory whose original Lagrangian is zero. In the process, we will be led to study in more detail the possible symmetries acting on the time dimension in the foliated spacetime, and on its $\mathcal{N}=2$ supersymmetric extension to "supertime." We will also clarify the notion of scaling dimensions in quantum gravity in situations when one cannot rely on a preferred, highly symmetric, background reference geometry. Some of the lessons learned here may be of more general interest in relativistic and nonrelativistic quantum gravity, beyond the scope of the topological theory that we focus on in this thesis.

At the center of our investigations in this thesis will be the symmetries of time, and their interplay with the geometry and symmetries of the D-dimensional spatial slices of the $D+1$ dimensional spacetime. Perhaps the more appropriate label for the topics studied here would be the "chronometry of time and supertime". Indeed, already more than six decades ago [70], one of the pioneers of the modern geometric approach to general relativity J.L. Synge advocated convincingly that measurements of space in general relativity are all secondary to the measurements of time, suggesting that spacetime "geometry"[13] should be more properly referred to as spacetime "chronometry"[14] (see also [71]). This suggestion appears particularly suitable in the context of nonrelativistic gravity [32, 33], where time indeed plays a privileged role, quite different from that of spatial dimensions. Moreover, in the context of topological quantum gravity associated with the nonrelativistic Ricci flow on Riemannian manifolds, time is far from being simply a real parameter t taking values in $\mathbf{R}$. In this nonrelativistic gravity, the "geometry" associated with time can be quite sophisticated, including a supersymmetrization of time – but not of space – to a supermanifold $\mathscr{M}_0$ of dimension $(1|2)$, which we will naturally refer to as "supertime." In this thesis, we will find further geometric structure associated with time in nonrelativistic topological gravity, in particular a rather unusual form of local time reparametrization gauge symmetry that turns out to underlie the theory presented in Part I. One can thus say that the present part represents an investigation into this geometry of time, or "chronometry," in topological quantum gravity of the Ricci flow.

5 Gauge symmetries in topological quantum gravity of the Ricci flow

As we pointed out above, the superspace construction of the theory with gauged $\mathrm{Diff}(\Sigma)$ in Part I leaves a few questions and loose ends. It is a hybrid construction, in which the topological symmetry appears to have already been gauge fixed, but the secondary spatial diffeomorphism symmetry does not. Thus, on one hand Q is interpreted as the BRST charge for the topological symmetry, but from the perspective of $\mathrm{Diff}(\Sigma)$ gauge symmetry, it appears to be treated as a supercharge of rigid supersymmetry. It would be sensible to ask for a one-step interpretation of the resulting theory as a BRST gauge fixing of a clear list of gauge symmetries specified at the outset. If such a one-step interpretation of our theory is possible, is the supercharge Q the full BRST charge *after* the gauging of spatial diffeomorphisms? What are the underlying gauge symmetries that are being gauged? Do they exhibit a redundancy, and if so, why did the procedure of Part I not require the presence of secondary ghost-for-ghosts?

To clarify these important questions, let us review how the standard process of constructing a topological field theory of the cohomological type, in the traditional

[13] Geometry: From the Greek $\gamma\varepsilon\omega\mu\varepsilon\tau\rho\acute{\iota}\alpha$, "measurement of earth or land".
[14] Chronometry: From the Greek $\chi\rho\acute{o}\nu o\varsigma$, "time"; thus "measurement of time".

component-field formulation [31], would work for nonrelativistic gravity.

5.1 Spatial diffeomorphisms and the shift vector n^i

In the standard process of constructing a gauge theory, we first specify the fields, and then the gauge symmetries acting on those fields. Then we apply the algorithm of BRST gauge fixing, which requires additional choices of gauge-fixing conditions. In this section, we choose our fields to be the spatial metric g_{ij} and the shift vector n^i. We postpone the addition of the lapse function n and time-reparametrization symmetries until Section 7 below, and focus first on clarifying the gauging of the spatial diffeomorphisms $\mathrm{Diff}(\Sigma)$.

In order to combine the topological gauge symmetries with the spatial diffeomorphisms, one could naturally start with a redundant system of symmetries,

$$\delta n^i = f^i(t,x^j) + \dot{\xi}^i + \xi^k\partial_k n^i - \partial_k\xi^i n^k, \tag{5.1}$$

$$\delta g_{ij} = f_{ij}(t,x^k) + \xi^k\partial_k g_{ij} + \partial_i\xi^k g_{kj} + \partial_j\xi^k g_{ik}. \tag{5.2}$$

Here f_{ij} is the topological gauge symmetry acting by arbitrary local deformations of g_{ij}, f^i generates a similar topological symmetry on the shift vector n^i, and ξ^i is the standard generator of $\mathrm{Diff}(\Sigma)$. These symmetries are indeed redundant: A given diffeomorphism transformation ξ can be compensated by the choice of f_{ij} and f^i so that the combined transformation is zero. In such cases, it is well-known that the proper BRST gauge fixing requires not only the fermionic ghosts Ψ_{ij}, Ψ^i and c^i associated with the gauge symmetries generated by f_{ij}, f^i and ξ^i (all of ghost number one), but also a bosonic ghost-for-ghost ϕ^i of ghost number two [30,31], which accounts for the redundancy of the original gauge symmetries (5.1) and (5.2). These fields form a BRST multiplet of the BRST charge Q_B (with ${Q_B}^2 = 0$),

$$\begin{aligned}
Q_B\, n^i &= \Psi^i + \dot{c}^i + c^k\partial_k n^i - \partial_k c^i n^k,\\
Q_B\, g_{ij} &= \Psi_{ij} + c^k\partial_k g_{ij} + \partial_i c^k g_{kj} + \partial_j c^k g_{ik},\\
Q_B\, c^i &= c^k\partial_k c^i + \phi^i,\\
Q_B\, \Psi^i &= -\dot{\phi}^i + n^k\partial_k\phi^i - \phi^k\partial_k n^i + c^k\partial_k\Psi^i + \Psi^k\partial_k c^i,\\
Q_B\, \Psi_{ij} &= -\phi^k\partial_k g_{ij} - \partial_i\phi^k g_{jk} - \partial_j\phi^k g_{ik} + c^k\partial_k\Psi_{ij} + \partial_i c^k\Psi_{jk} + \partial_j c^k\Psi_{ik},\\
Q_B\, \phi^i &= \phi^k\partial_k c^i - c^k\partial_k\phi^i.
\end{aligned} \tag{5.3}$$

In contrast, in the hybrid two-step construction of Part I, no such ghost-for-ghost fields ϕ^i of ghost number two appeared to be necessary, and we wish to understand why.

To clarify this, we note first that this redundancy between topological and spacetime symmetries is almost inevitable in relativistic theories, especially in higher than the lowest few dimensions, if one wishes to maintain their relativistic symmetries. In the case of our nonrelativistic gravity, however, there is a more elementary theory, which does not require ghost-for-ghosts. The reasoning is reminiscent of the effects observed in the context of relativistic topological gravity in two dimensions in [72]. Our non-redundant gauge symmetries will be

$$\begin{aligned} \delta n^i &= \dot{\xi}^i + \xi^k\partial_k n^i - \partial_k\xi^i n^k, && (5.4)\\ \delta g_{ij} &= f_{ij} + \xi^k\partial_k g_{ij} + \partial_i\xi^k g_{kj} + \partial_j\xi^k g_{ik}. && (5.5) \end{aligned}$$

We have simply left out from (5.1) the topological gauge symmetry generated by f^i and acting on n^i. Indeed, these symmetries are now non-redundant: While the action of the spatial diffeomorphism ξ^i on the spatial metric can be compensated for by a shift in the topological symmetry f_{ij}, no such compensation is possible in the action on n^i, and the two symmetries generated by f_{ij} and ξ^i are mutually independent. Yet, clearly, these symmetries are powerful enough to eliminate all local propagating degrees of freedom, and the resulting theory will therefore still be "topological" in this sense. The BRST charge acts as in (5.3), but with the fields Ψ^i and ϕ^i omitted:

$$\begin{aligned} Q_B\, n^i &= \dot{c}^i + c^k\partial_k n^i - \partial_k c^i n^k,\\ Q_B\, g_{ij} &= \Psi_{ij} + c^k\partial_k g_{ij} + \partial_i c^k g_{kj} + \partial_j c^k g_{ik},\\ Q_B\, c^i &= c^k\partial_k c^i,\\ Q_B\, \Psi_{ij} &= c^k\partial_k\Psi_{ij} + \partial_i c^k\Psi_{jk} + \partial_j c^k\Psi_{ik}. \end{aligned} \qquad (5.6)$$

Note one additional useful fact: The symmetries are not only non-redundant, their Lie algebra decomposes into a direct sum of a symmetry acting solely on the shift n^i, and a topological symmetry acting only on g_{ij}. This follows from a simple change of basis in the symmetry algebra, from ξ^i and f_{ij} to ξ^i and $\hat{f}_{ij}$, with

$$\hat{f}_{ij} \equiv f_{ij} + \xi^k\partial_k g_{ij} + \partial_i\xi^k g_{kj} + \partial_j\xi^k g_{ik}. \qquad (5.7)$$

This is a change of coordinates on the symmetry algebra, whose Jacobian is equal to one. Calling by ψ_{ij} the ghost associated with the shifted topological symmetry $\hat{f}_{ij}$, the BRST transformations now simplify to

$$\begin{aligned} Q_B\, n^i &= \dot{c}^i + c^k\partial_k n^i - \partial_k c^i n^k, \\ Q_B\, c^i &= c^k\partial_k c^i, \\ Q_B\, g_{ij} &= \psi_{ij}, \\ Q_B\, \psi_{ij} &= 0. \end{aligned} \tag{5.8}$$

The fact that such a shift exists, and that in the new basis the Diff(Σ) symmetry acts trivially on g_{ij}, is important: The supermultiplets of Q introduced in Part I in the process of gauging Diff(Σ) exhibit the same type of decoupling between the two symmetries. It is this form (5.8) of the BRST transformations that will be best suited for the comparison against our topological gravity constructed in Section 3 of Part I.

We claim that the $\mathcal{N}=2$ supersymmetry multiplets found in Section 3 of Part I are indeed equivalent to the standard multiplets associated with the BRST gauge fixing of (5.4) and (5.5), and that the supercharge Q is simply the standard BRST charge of this one-step construction. This will be best seen when we rewrite the theory developed in Part I in the component formalism. In order to determine the independent component fields and their properties, we must first solve the constraints relating N^i, S^i and $\overline{S}^i$:

$$\mathrm{D}S^i = S^k\partial_k S^i, \tag{5.9}$$

$$\overline{\mathrm{D}}\overline{S}^i = \overline{S}^k\partial_k\overline{S}^i, \tag{5.10}$$

$$N^i = -\overline{\mathrm{D}}S^i - \mathrm{D}\overline{S}^i + S^k\partial_k\overline{S}^i + \overline{S}^k\partial_k S^i. \tag{5.11}$$

Since the constraints (5.9) and (5.10) involve only S^i and $\overline{S}^i$ respectively, they can be solved first, yielding

$$S^i = \sigma^i + \theta\sigma^k\partial_k\sigma^i + \overline{\theta}Y^i + \theta\overline{\theta}(\dot{\sigma}^i + Y^k\partial_k\sigma^i - \sigma^k\partial_k Y^i), \tag{5.12}$$

$$\overline{S}^i = \overline{\sigma}^i + \theta X^i + \overline{\theta}\overline{\sigma}^k\partial_k\overline{\sigma}^i + \theta\overline{\theta}(\overline{\sigma}^k\partial_k X^i - X^k\partial_k\overline{\sigma}^i). \tag{5.13}$$

Thus, the independent components of S^i and $\overline{S}^i$ have been reduced to $\sigma^i, \overline{\sigma}^i, X^i$ and

Y^i, transforming as follows under Q:

$$Q\,\sigma^i = \sigma^k \partial_k \sigma^i, \tag{5.14}$$
$$Q\,Y^i = -\dot{\sigma}^i - Y^k \partial_k \sigma^i + \sigma^k \partial_k Y^i, \tag{5.15}$$
$$Q\,\overline{\sigma}^i = X^i, \tag{5.16}$$
$$Q\,X^i = 0. \tag{5.17}$$

These are not yet the appropriate component fields of our desired supersymmetry multiplets, as we still need to solve the constraint (5.11) that expresses the components of N^i in terms of the independent components in S^i and $\overline{S}^i$. First, solving (5.11) at the lowest component gives

$$n^i = -Y^i - X^i + \sigma^k \partial_k \overline{\sigma}^i + \overline{\sigma}^k \partial_k \sigma^i. \tag{5.18}$$

Note that because X^i is Q invariant, it is our candidate for the auxiliary field in a trivial BRST multiplet, paired up with $\overline{\sigma}^i$ which will play the role of the antighost. This means that the correct way of reading (5.18) is to interpret Y^i as a composite field, determined by this equation in terms of the four independent component fields n^i, X^i, σ^i and $\overline{\sigma}^i$,

$$Y^i = -n^i - X^i + \sigma^k \partial_k \overline{\sigma}^i + \overline{\sigma}^k \partial_k \sigma^i. \tag{5.19}$$

This is the sense in which Y^i should be understood as a composite field when it appears in expressions such as (5.12).

The remaining components of N^i can now be determined in terms of the independent fields n^i, X^i, σ^i and $\overline{\sigma}^i$ by evaluating the higher-order terms in (5.18). They are found to be

$$\psi^i = \dot{\sigma}^i + \sigma^k \partial_k n^i - \partial_k \sigma^i n^k, \tag{5.20}$$
$$\chi^i = \dot{\overline{\sigma}}^i + \overline{\sigma}^k \partial_k n^i - \partial_k \overline{\sigma}^i n^k, \tag{5.21}$$
$$B^i = -\dot{X}^i + n^k \partial_k X^i - X^k \partial_k n^i + \dot{\sigma}^k \partial_k \overline{\sigma}^i + \sigma^j \partial_j n^k \partial_k \overline{\sigma}^i - n^j \partial_j \sigma^k \partial_k \overline{\sigma}^i$$
$$+ \overline{\sigma}^k \partial_k \dot{\sigma}^i + \overline{\sigma}^k \partial_k \sigma^j \partial_j n^i + \overline{\sigma}^k \sigma^j \partial_k \partial_j n^i - \overline{\sigma}^k \partial_k n^j \partial_j \sigma^i - \overline{\sigma}^k n^j \partial_k \partial_j \sigma^i. \tag{5.22}$$

Thus, the entire sector of superfields N^i, S^i and $\overline{S}^i$ reduces in components to four independent fields, belonging to two BRST multiplets of the BRST charge Q, with

BRST transformations

$$Q\,n^i \;=\; \dot\sigma^i + \sigma^k\partial_k n^i - n^k\partial_k\sigma^i, \tag{5.23}$$

$$Q\,\sigma^i \;=\; \sigma^k\partial_k\sigma^i, \tag{5.24}$$

and

$$Q\,\overline{\sigma}^i \;=\; X^i, \tag{5.25}$$

$$Q\,X^i \;=\; 0. \tag{5.26}$$

Clearly, σ^i is the ghost of the spatial diffeomorphism symmetry, and should be identified with c^i in (5.3), while $\overline{\sigma}^i$ and X^i form the trivial BRST multiplet consisting of an antighost and an auxiliary field. In addition to these two BRST multiplets, the theory that was constructed in Section 3 of Part I by gauging spatial diffeomorphisms of the primitive topological gravity contains also the two BRST multiplets in the metric superfield of (**??**),

$$Q\,g_{ij} = \psi_{ij}, \qquad Q\,\psi_{ij} = 0, \tag{5.27}$$

$$Q\,\chi_{ij} = -B_{ij}, \qquad Q\,B_{ij} = 0. \tag{5.28}$$

We see that these component supermultiplets (5.23), (5.24) and (5.27) of Q match exactly the multiplets (5.8) of Q_B, which we obtained by the one-step BRST gauge fixing of the non-redundant gauge symmetries (5.4) and (5.5). And the remaining multiplets of Q, listed in (5.25), (5.26) and (5.28), are the standard antighost-auxiliary BRST multiplets ready for the implementation of our gauge fixing choice. In this way, the two-step construction in Section 3 of Part I can indeed be consistently interpreted as the standard one-step gauge fixing of a gauge theory with non-redundant gauge symmetries, with Q identified as the BRST charge.

5.2 Time reparametrizations and the lapse function n

In Section 4 of Part I, the theory was further extended by gauging foliation-preserving time reparametrizations, and adding the appropriate $\mathcal{N}=2$ supersymmetrization of the lapse function n. In order to clarify whether this extended theory can also be interpreted as a one-step BRST gauge fixing of appropriate non-redundant gauge symmetries, we need to overcome additional challenges which were absent in the theory with Diff(Σ) gauge symmetry discussed above. We begin by briefly reviewing the symmetries and fields introduced in Part I.

The gauge symmetries of spacetime-dependent spatial diffeomorphisms Diff(Σ) are extended to all foliation-preserving diffeomorphisms of the $D+1$ dimensional spacetime $\mathcal{M}_{\mathcal{F}}$. Besides the generators $\xi^i(t,x^k)$ that were considered in the previous paragraph, this extended symmetry also contains time-dependent time reparametrizations, generated by $f(t)$:

$$\begin{aligned} \delta n &= \dot{f}n + f\dot{n}, \\ \delta n^i &= \dot{f}n^i + f\dot{n}^i, \\ \delta g_{ij} &= f\dot{g}_{ij}. \end{aligned} \tag{5.29}$$

Here we have introduced besides the spatial metric g_{ij} and the shift vector n^i also the lapse function n, which plays the role of the gauge field associated with time reparametrizations: More precisely, it is $\log n$ that transforms as a gauge field: $\delta \log n = \dot{f} + \ldots$. While it is mathematically consistent in nonrelativistic gravity to declare the field n to be a projectable field $n(t)$, it is physically more interesting to allow the lapse to be a nonprojectable field $n(t,x^i)$, *i.e.*, to promoting it to a spacetime-dependent field. Allowing the lapse to be nonprojectable was indeed crucial for us in establishing contact with Perelman's theory of the Ricci flow, since we found that the role of Perelman's "dilaton" field is played by the nonprojectable lapse field.

In Section 4 of Part I, we constructed a nonrelativistic gravity theory which is gauge invariant under $f(t)$ and consistent with our requirement of rigid $\mathcal{N}=2$ supersymmetry of supertime, using the standard techniques for gauging symmetries in superspace. We promoted $n(t,x^i)$ to a nonchirial, nonprojectable superfield, whose lowest component is the nonprojectable lapse, $N = n + \ldots$. In fact, in Part I we found it convenient to introduce the *inverse* lapse superfield, $E = 1/N$, whose lowest component is the nonprojectable field $e \equiv 1/n$,

$$E = e + \theta\psi + \overline{\theta}\chi + \theta\overline{\theta}B. \tag{5.30}$$

This E superfield then covariantizes the time derivative under (the supersymmetric extensions of) the $f(t)$ symmetries, in the way explained in detail in Part I.

Since the odd derivatives also need to be covariantized, two more superfields Θ and $\overline{\Theta}$ had to be introduced in addition to E; they are also nonprojectable, nonchiral, but odd, and we introduce the following notation for their components:[15]

$$\Theta = -\nu - \theta\overline{w} - \overline{\theta}z - \theta\overline{\theta}\rho, \quad (5.31)$$
$$\overline{\Theta} = -\overline{\nu} - \theta w - \overline{\theta}\overline{z} - \theta\overline{\theta}\overline{\rho}. \quad (5.32)$$

As in the case of spatial diffeomorphisms discussed in the previous section of the present part, this proliferation of component fields is reduced to the minimal independent set by suitable constraints, which now involve both the shift sector and the lapse sector superfields:

$$\mathrm{D}\Theta - S^k\partial_k\Theta = -\Theta(\dot{\Theta} - N^k\partial_k\Theta), \quad (5.33)$$
$$\overline{\mathrm{D}}\overline{\Theta} - \overline{S}^k\partial_k\overline{\Theta} = -\overline{\Theta}(\dot{\overline{\Theta}} - N^k\partial_k\overline{\Theta}), \quad (5.34)$$

and

$$E = 1 - \mathrm{D}\overline{\Theta} + S^k\partial_k\overline{\Theta} - \overline{\mathrm{D}}\Theta + \overline{S}^k\partial_k\Theta - \Theta(\dot{\overline{\Theta}} - N^k\partial_k\overline{\Theta}) - \overline{\Theta}(\dot{\Theta} - N^k\partial_k\Theta). \quad (5.35)$$

In addition to these mixed constraints, the shift sector superfields $S^i, \overline{S}^i$ and N^i still satisfy constraints (5.9), (5.10) and (5.11). These constraints appear somewhat more involved than in the case of the shift sector alone, and we postpone solving them until Section 7.1.

Once the nonprojectable lapse sector has been introduced, the pattern that we uncovered in the case with Diff(Σ) gauge symmetry will continue to be valid if the BRST multiplets in the lapse sector can be interpreted as having originated in BRST gauge fixing of a hidden symmetry associated with time. Note that since the lapse sector contains only nonprojectable fields, this hidden symmetry cannot be simply the projectable time reparametrizations generated by $f(t)$. This hidden symmetry will have to be nonprojectable, generated by some space and time dependent parameter $\zeta(t, x^i)$. Is there such a hidden symmetry, and if so, what is its geometric interpretation?

The first guess might be that $\zeta(t, x^i)$ should perhaps be some kind of nonprojectable time reparametrization symmetry, which would extend the symmetry of spatial diffeomorphisms generated by $\xi^i(t, x^k)$. This symmetry would have to satisfy several stringent requirements. For example, it would have to act on n^i (and also on g_{ij})

[15] The minus signs introduced here in the process of naming the component fields will simplify some future component formulas below, and they also make the parallel between the odd superfields $\Theta, \overline{\Theta}$ of the lapse sector and the odd superfields $S^i, \overline{S}^i$ of the shift sector more explicit.

trivially, or more accurately, in a way which can be absorbed into a redefinition of the spatial diffeomorphisms ξ^i (and the topological gauge transformations $\hat{f}_{ij}$). This phenomenon is analogous to what happened above, when we found the basis of symmetry generators in which ξ^i acts trivially on g_{ij}, and therefore $Q_B\, g_{ij}$ is independent of c^i (see (5.8)). Is it possible to construct such a symmetry ζ? We could try to read off the rules directly from the transformation properties of the component fields under Q, assuming that they will conform to the interpretation as having come from the BRST fixing of a symmetry, with Q being the BRST charge. However, we find it more instructive to look first for available symmetries in the context of previously studied symmetries of spacetime, which we will do in the next section. In the process, we will learn a few surprising facts which might be of more general interest for quantum gravity, beyond the applications to the main subject of this thesis. The reader who is solely interested in our topological nonrelativistic gravity construction can proceed directly to Section 7.

6 Spacetime diffeomorphisms in relativistic and nonrelativistic gravity

Consider the spacetime diffeomorphism generated by ξ^μ, which we decompose into $\zeta(t,x^i)\equiv\xi^0$ and $\xi^i(t,x^k)$. On the ADM decomposition [43] of the relativistic spacetime metric into the spatial metric plus the lapse and shift, the relativistic spacetime diffeomorphisms act via

$$\begin{aligned}\delta n &= \xi^k\partial_k n+\zeta\dot{n}+\dot{\zeta}n, &(6.1)\\ \delta n^i &= \dot{\xi}^i+\xi^k\partial_k n^i-\partial_k\xi^i n^k+\zeta\dot{n}^i+\dot{\zeta}n^i-\partial_k\zeta n^k n^i-\partial_k\zeta g^{ik}n^2, &(6.2)\\ \delta g_{ij} &= \xi^k\partial_k g_{ij}+\partial_i\xi^k g_{kj}+\partial_j\xi^k g_{ik}+\zeta\dot{g}_{ij}+(\partial_i\zeta g_{jk}+\partial_j g_{ik})n^k. &(6.3)\end{aligned}$$

Can the standard action of the time diffeomorphism $\zeta(t,x^i)$ on n^i be absorbed into the spatial diffeomorphism ξ^i? Almost: If the last term in (6.2), proportional to $g^{ij}n^2$, weren't there, it would be possible to absorb the action of ζ into that of the spatial diffeomorphism by changing the basis in the Lie algebra by first shifting $\xi^i=\hat{\xi}^i-\zeta n^i$ and then using $\hat{\xi}^i$ and $\hat{\zeta}\equiv\zeta$ as the new basis in the space of symmetry generators. Note that this is again a change of variables whose Jacobian is equal to one. The resulting algebra would then satisfy the three stringent requirements necessary for the possible matching with the superspace construction of our nonrelativistic gravity from Section 4 of Part I:

- The shifted time reparametrization generators $\hat{\zeta}$ should act trivially on n^i;
- The shifted spatial diffeomorphism generators $\hat{\xi}^i$ should form the gauge algebra of Diff(Σ);
- The commutator of a shifted spatial diffeomorphism $\hat{\xi}^i$ and a shifted time reparametrization $\hat{\zeta}$ should yield another shifted time reparametrization, and therefore act trivially on n^i.

In the standard spacetime diffeomorphism symmetry algebra, the last term on the right-hand side of (6.2) represents an obstruction to these three requirements. In order to clarify the status of this obstruction term, and to see under what circumstances it can be set to zero, we will revisit the nonrelativistic decomposition of the spacetime diffeomorphism symmetries, and their nonrelativistic limits.

6.1 Scales and scaling dimensions

In order to discuss the nonrelativistic decomposition of a relativistic spacetime, we wish to re-introduce the measurement of time and space in unrelated units: L and T. First, we recall a few elementary facts about scaling and dimensions in quantum field theory, and propose a simple refinement to the case when the system contains a dynamical spacetime metric.

In quantum field theory in a Minkowski spacetime, scaling dimensions of various objects (quantum fields, spacetime derivatives, composite operators built out of them) play a central role, controlled by the concept of the renormalization group (RG). In relativistic field theories, it is natural to set the speed of light $c = 1$, and all dimensions are then expressed in terms of one dimensionful scale. By convention, largely for historical reasons, in particle physics this scale is typically selected to be the momentum scale, or equivalently the inverse length scale when one sets $\hbar = 1$. In momentum units, Cartesian spacetime coordinates x^μ of the Minkowski spacetime are of dimension -1, and the derivatives ∂_μ are of dimension 1. Dimensions of quantum fields and various composite operators then depend on the type of theory in question, typically controlled by an RG fixed point. Sometimes, however, there is a strong geometric reason to assign a given field a particular scaling dimension. For example, in general relativity, the components g_{ij} of the spacetime metric relate the proper length element ds to the coordinate length elements dx^i via $ds^2 = g_{ij}dx^i dx^j$. It is only sensible to consider ds^2 to be of dimension -2 in momentum units, hence implying the often-quoted fact that for the purposes of power counting, the components g_{ij} of the metric are dimensionless. This picture is of course well-known, and almost universally accepted, but as we explain below, in need of a slight conceptual modification in theories with dynamical inhomogeneous geometries.

Nonrelativistic gravity of the Lifshitz type [32, 33], and its topological counterpart studied here, belong to the class of quantum field theories with two distinct, *a priori* unrelated scales: a length scale L, and a time scale T, or their inverses: an energy scale $E = 1/T$ and a momentum scale $M = 1/L$ (having again set $\hbar = 1$). The relativistic strategy of assigning dimensions needs to be refined: We can declare that the time coordinate carries dimension $T \equiv E^{-1}$, and the spatial coordinates x^i are of dimension $L \equiv M^{-1}$. (This was also the convention we used for assigning classical scaling dimensions to various objects in topological gravity of the Ricci flow in Section 2 of Part I.) In addition, one simplifying convention seems appropriate: Instead of writing the dimension $[\mathcal{O}]$ of an object $\mathcal{O}$ multiplicatively in terms of powers of T and L (or E and M), we will simply write $[\mathcal{O}] = (n, m)$ when $\mathcal{O}$ is of dimension $E^n M^m$ in energy and momentum units. With this convention we have, for instance,

$$[\partial/\partial t] = (1, 0), \qquad [\partial_i] = (0, 1). \tag{6.4}$$

Applying this analysis to the fields of nonrelativistic gravity, we thus obtain the "standard" dimension assignments:

$$[g_{ij}] = (0, 0), \qquad [n] = (0, 0), \qquad [n^i] = (1, -1). \tag{6.5}$$

They simply follow from the fact that in nonrelativistic gravity, the geometry of spacetime is characterized by an invariant line element $d\sigma$ of spatial distance, defined via

$$d\sigma^2 = g_{ij}(dx^i - n^i dt)(dx^j - n^j dt), \tag{6.6}$$

which should naturally be of dimension $[d\sigma] = (0, -1)$, and an independent invariant element of time,

$$d\tau = n\, dt, \tag{6.7}$$

naturally of dimension $(-1, 0)$.

In theories with two (or more) *a priori* unrelated scales, the "one-scale" physics of the renormalization group scaling emerges in the vicinity of an RG fixed point, where the two scales become locked together by a scaling relation

$$E = M^z, \tag{6.8}$$

involving the so-called dynamical critical exponent z. This exponent is characteristic of the specific RG fixed point, not of the theory itself – it is indeed possible to have a theory with more than one RG fixed points, with distinct values of z. In the vicinity of a given fixed point, one is then free to use (6.8) to express the RG scaling properties of the system in terms of just one independent scale (say E).

In the present context of topological gravity of the Ricci flow, and more generally in nonrelativistic gravity with a dynamical metric, it has become increasingly apparent that a slightly modified strategy for assigning dimensions would be more logical. There are several reasons for that. In the flat Minkowski spacetime, or in a theory with a preferred and highly symmetric background, it may be natural to continue with the picture developed in quantum field theory, and see the spacetime coordinates x^μ or the derivatives ∂_μ as the carriers of a scaling dimension: After all, the idea of scaling is often phrased as a study of the properties of the system under the rescalings $x^\mu \mapsto bx^\mu$ for constant b. However, this intuition is increasingly problematic in a theory whose background geometries are often highly inhomogeneous, such as our topological theory of the Ricci flow, whose path integral localizes to the solutions of generalized Ricci flow equations with arbitrarily inhomogeneous initial conditions. On such backgrounds, there is no analog of a "preferred coordinate system" $x^\mu = (t, x^i)$. Instead, we prefer to treat all coordinate systems equally; this will make sense only if we assign the scaling dimension $(0,0)$ to both t, x^i and the derivatives ∂_t, ∂_i. Only then the elementary covariance of our rules will be restored, and the concept of scaling dimensions will not have to rely on the existence of a preferred system of coordinates associated with a maximally symmetric ground-state geometry.

In this modified picture, the dimensions are now carried by the physical spatial length element $d\sigma^2$ and time element $d\tau$,

$$[d\sigma^2]' = (0,-2), \qquad [d\tau]' = (-1,0), \tag{6.9}$$

just as we declared above. However, since now dx^i and dt are declared dimensionless, this implies that the old-fashioned rules (6.4) and (6.5) are modified to

$$[g_{ij}]' = (0,-2), \qquad [n]' = (-1,0), \qquad [n^i]' = (0,0), \tag{6.10}$$

and

$$[\partial/\partial t]' = (0,0), \qquad [\partial_i]' = (0,0). \tag{6.11}$$

We have denoted the dimension in the modified counting system by $[\ ]'$, to distinguish it from the dimension $[\ ]$ in the old-fashioned system.

One can see that both systems of assigning dimensions are for all physical purposes equivalent to each other, for instance they lead to the same dimension counting rules for the terms that can appear in the action. However, the modified system in which the underlying spacetime coordinates are naturally dimensionless is mathematically more accurate and conceptually cleaner that the old-fashioned one, since it is manifestly consistent with nonlinear changes of coordinates, and does not lead to apparent violations of the additivity properties of the classical scaling dimensions during such nonlinear coordinate transformations.

After this detour, we can now return to the study of the relativistic diffeomorphism symmetry in the ADM decomposition, in a theory with two scales L and T. Restoring the dimensions of various objects in the symmetry algebra (6.1), (6.2) and (6.3), we see that the last term on the right-hand side of (6.2) is the only one whose dimension does not match the rest of the terms, and therefore a dimensionful constant is needed to provide the conversion. This constant is of course the speed of light c, of dimension $[c] = [c]' = (1, -1)$, which was set equal to one in the relativistic theory. Restoring c, the algebra is now

$$\begin{aligned} \delta n &= \xi^k \partial_k n + \zeta \dot{n} + \dot{\zeta} n, \\ \delta n^i &= \dot{\xi}^i + \xi^k \partial_k n^i - \partial_k \xi^i n^k + \zeta \dot{n}^i + \dot{\zeta} n^i - \partial_k \zeta n^k n^i - c^2 \partial_k \zeta g^{ik} n^2, \\ \delta g_{ij} &= \xi^k \partial_k g_{ij} + \partial_i \xi^k g_{kj} + \partial_j \xi^k g_{ik} + \zeta \dot{g}_{ij} + (\partial_i \zeta g_{jk} + \partial_j g_{ik}) n^k. \end{aligned} \tag{6.12}$$

It is an elementary but valuable exercise to verify why this insertion of c^2 makes the dimensions right in both of our dimension-counting systems.

Now we can return to our quest for a symmetry that would match the ingredients required by the topological nonrelativistic gravity of Part I. As we saw at the beginning of Section 2 above, it is precisely this c^2 term that represents an obstruction against what we need. This term can be simply eliminated by taking the $c \to 0$ limit, well-known in the literature [73,74] (see also [42] for the nonrelativistic context) as the "ultralocal" limit of the relativistic diffeomorphism symmetries of general relativity.

6.2 The $c \to 0$ limit

Taking $c \to 0$ in (6.12) will allow us to follow the strategy outlined at the beginning of Section 6, and to make the time transformation generated by ζ act trivially on the shift vector n^i. Before taking these steps, let us take a closer look at this "ultralocal" limit of $c \to 0$, especially in the light of our improved prescription for assigning scaling dimensions. According to this prescription, our coordinates x^i, t

are dimensionless, and therefore the generators ξ^i, ζ of the corresponding spacetime symmetries are dimensionless as well. Yet, the algebra defined by the transformation rules (6.12) clearly depends on c. Indeed, we are planning to take the contraction of the symmetries by sending $c \to 0$. One might therefore think that the structure constants of this symmetry algebra, which one could obtain from the commutation relations of the transformations in (6.12), will depend on c. However, that clearly represents a puzzle: If the generators ξ^i and ζ, and all the derivatives ∂_t and ∂_i that can appear in the commutation relations are all dimensionless, how could the commutator of two such transformations possibly depend on a dimensionful constant such as c?

The resolution is simple, yet perhaps a little surprising. A direct calculation reveals that regardless of the value of c, the commutation relations of the transformations in (6.12) are independent of c: The commutator of a transformation δ_1 generated by ξ_1^i and ζ_1 with the transformation generated by ξ_2^i and ζ_2,

$$[\delta_1, \delta_2] = \delta_3, \tag{6.13}$$

is the transformation δ_3 generated by the following ξ_3^i and ζ_3,

$$\begin{aligned} \zeta_3 &= \zeta_1\dot{\zeta}_2 - \zeta_2\dot{\zeta}_1 + \xi_1^k\partial_k\zeta_2 - \xi_2^k\partial_k\zeta_1, & (6.14)\\ \xi_3^i &= \xi_1^k\partial_k\xi_2^i - \xi_1^k\partial_k\xi_2^i + \zeta_1\dot{\xi}_2^i - \zeta_2\dot{\xi}_1^i. & (6.15)\end{aligned}$$

These are indeed the commutation relations of the general spacetime diffeomorphisms, familiar from the relativistic theory of gravity.

Consider now the theory of bosonic gravity which would be invariant under the $c \to 0$ limit of the symmetries specified in (6.12). This is of course the theory known in the literature as the "ultralocal" theory of gravity [73, 74], proposed originally as a possible strong-coupling limit of general relativity. Do the relativistic commutation relations of the symmetry generators mean that this theory is somehow relativistic? Not in the standard sense of full relativistic general covariance: While the commutation relations appear relativistic, the realization of the symmetries on the ADM decomposition of the metric does depend on c. Consequently, Lagrangians that are invariant under this realization of the symmetries will be contractions of the standard Lagrangians of general relativity. In particular, the lowest-derivative

Lagrangian invariant under these symmetries,

$$S = \frac{1}{\kappa^2} \int dt\, d^D x \left\{ \frac{\sqrt{g}}{n} \left(g^{ik} g^{j\ell} - g^{ij} g^{k\ell} \right) \left(\dot{g}_{ij} - \nabla_i n_j - \nabla_j n_i \right) \left(\dot{g}_{k\ell} - \nabla_k n_\ell - \nabla_\ell n_k \right) - 2n\sqrt{g}\Lambda \right\}, \tag{6.16}$$

contains the standard kinetic term for the spatial metric with two time derivatives, but no spatial-derivative term consistent with it. This of course is the standard result known from [73, 74]. It is intriguing, however, that the dependence of this theory on the preferred foliation by constant time slices appears here at the level of the dynamical metric fields, and not at the level of the underlying symmetries of the differentiable structure of spacetime with dimensionless generators ξ^i and ζ.

6.3 Decoupling time reparametrizations from the shift vector

We can now finally propose a meaningful candidate for the spacetime gauge symmetry that underlies the construction of the topological nonrelativistic theory in Section 4 of Part I.

After performing the change of basis $\hat{\xi}^i = \xi^i - \zeta n^i$ and $\hat{\zeta} = \zeta$ in the generators of the $c = 0$ transformation rules, which is again a change of basis whose Jacobian is one, we get

$$\hat{\delta} n = \hat{\xi}^k \partial_k n + \hat{\zeta}(\dot{n} - n^k \partial_k n) + (\partial_t \hat{\zeta} - n^k \partial_k \hat{\zeta}) n, \tag{6.17}$$

$$\hat{\delta} n^i = \partial_t \hat{\xi}^i + \hat{\xi}^k \partial_k n^i - \partial_k \hat{\xi}^i n^k, \tag{6.18}$$

$$\hat{\delta} g_{ij} = \hat{\xi}^k \partial_k g_{ij} + \partial_i \hat{\xi}^k g_{kj} + \partial_j \hat{\xi}^k g_{ik} + \hat{\zeta}(\dot{g}_{ij} - n^k \partial_k g_{ij} - \partial_i n^k g_{kj} - \partial_j n^k g_{ik}). \tag{6.19}$$

These transformations satisfy the following commutation relations: The commutator of a transformation $\hat{\delta}_1$ generated by $\hat{\xi}^i_1$ and $\hat{\zeta}_1$ with a transformation $\hat{\delta}_2$ generated by $\hat{\xi}^i_2$ and $\hat{\zeta}_2$,

$$[\hat{\delta}_1, \hat{\delta}_2] = \hat{\delta}_3, \tag{6.20}$$

is the transformation $\hat{\delta}_3$ generated by

$$\hat{\zeta}_3 = \hat{\zeta}_1 \left(\partial_t \hat{\zeta}_2 - n^k \partial_k \hat{\zeta}_2 \right) - \hat{\zeta}_2 \left(\partial_t \hat{\zeta}_1 - n^k \partial_k \hat{\zeta}_1 \right) + \hat{\xi}^k_1 \partial_k \hat{\zeta}_2 - \hat{\xi}^k_2 \partial_k \hat{\zeta}_1, \tag{6.21}$$

$$\hat{\xi}^i_3 = \hat{\xi}^k_1\, \partial_k \hat{\xi}^i_2 - \hat{\xi}^k_1\, \partial_k \hat{\xi}^i_2. \tag{6.22}$$

We will simply refer to this symmetry algebra as $\mathcal{G}$. Note that the structure "constants" of $\mathcal{G}$ in this basis are field-dependent, due to the explicit appearance of n^k. This field dependence of the structure constants could of course be undone by "unshifting" the basis and representing the transformations using the original generators ξ^i and ζ. However, in the context of the BRST gauge-fixing in our topological gravity from Part I, it is the shifted representation in terms of $\hat{\xi}^i$ and $\hat{\zeta}$ that will make a natural appearance. Note also that the action of the algebra $\mathcal{G}$ on n and n^i satisfies all the necessary requirements listed in our three bullet points at the beginning of Section 6, needed to be a candidate for our underlying spacetime symmetry for the topological gravity of Part I: The subalgebra in $\mathcal{G}$ generated by $\hat{\xi}^i$ is the standard Lie algebra of spatial diffeomorphisms, while the subalgebra generated by $\hat{\zeta}$ is an ideal in $\mathcal{G}$, and this ideal acts trivially on the shift vector n^i.

Besides the symmetries generated by $\hat{\zeta}$ and $\hat{\xi}^i$, we also include the original topological symmetries f_{ij} acting on g_{ij}, as we did in (5.2). Putting all the symmetries together, we reach the following algebra:

$$\hat{\delta} n = \hat{\xi}^k\partial_k n + \hat{\zeta}(\dot{n} - n^k\partial_k n) + (\partial_t\hat{\zeta} - n^k\partial_k\hat{\zeta})n, \tag{6.23}$$

$$\hat{\delta} n^i = \partial_t\hat{\xi}^i + \hat{\xi}^k\partial_k n^i - \partial_k\hat{\xi}^i n^k, \tag{6.24}$$

$$\hat{\delta} g_{ij} = f_{ij} + \hat{\xi}^k\partial_k g_{ij} + \partial_i\hat{\xi}^k g_{kj} + \partial_j\hat{\xi}^k g_{ik} + \hat{\zeta}(\dot{g}_{ij} - n^k\partial_k g_{ij} - \partial_i n^k g_{kj} - \partial_j n^k g_{ik}). \tag{6.25}$$

Finally, redefining the topological transformation to

$$\hat{f}_{ij} \equiv f_{ij} + \hat{\xi}^k\partial_k g_{ij} + \partial_i\hat{\xi}^k g_{kj} + \partial_j\hat{\xi}^k g_{ik} + \hat{\zeta}(\dot{g}_{ij} - n^k\partial_k g_{ij} - \partial_i n^k g_{kj} - \partial_j n^k g_{ik}), \tag{6.26}$$

which is again a change of basis whose Jacobian is equal to one, we bring the symmetries to the following simple form:

$$\begin{aligned} \hat{\delta} n &= \hat{\xi}^k\partial_k n + \hat{\zeta}(\dot{n} - n^k\partial_k n) + (\partial_t\hat{\zeta} - n^k\partial_k\hat{\zeta})n, \\ \hat{\delta} n^i &= \partial_t\hat{\xi}^i + \hat{\xi}^k\partial_k n^i - \partial_k\hat{\xi}^i n^k, \\ \hat{\delta} g_{ij} &= \hat{f}_{ij}. \end{aligned} \tag{6.27}$$

Note that the symmetries generated by $\hat{\zeta}$, $\hat{\xi}^i$ and $\hat{f}_{ij}$ are non-redundant. When we treat these gauge symmetries using the BRST formalism, only first-generation ghosts will be needed, and no ghost-for-ghosts. Introducing the ghosts c, c^i and ψ_{ij} associated with our symmetry generators $\hat{\zeta}$, $\hat{\xi}^i$ and $\hat{f}_{ij}$, the BRST transformations of this BRST multiplet are

$$\begin{aligned}
Q_B\, n^i &= \dot{c}^i + c^k\partial_k n^i - \partial_k c^i n^k, \\
Q_B\, c^i &= c^k\partial_k c^i, \\
Q_B\, n &= c(\dot{n} - n^k\partial_k n) + (\dot{c} - n^k\partial_k c)\, n + c^k\partial_k n, \\
Q_B\, c &= c(\dot{c} - n^k\partial_k c) + c^k\partial_k c, \\
Q_B\, g_{ij} &= \psi_{ij}, \\
Q_B\, \psi_{ij} &= 0.
\end{aligned} \tag{6.28}$$

It is this BRST algebra that we now wish to compare to the structure of the Q supermultiplets of the topological nonrelativistic gravity from Section 4 of Part I.

7 Superfields in topological quantum gravity of the Ricci flow

With the improved understanding of the relevant aspects of relativistic and nonrelativistic diffeomorphisms developed in the previous section, we are now ready to examine the supersymmetric theory of gravity with nonprojectable lapse constructed in Part I, and see if it comes from a simple BRST gauge fixing of an underlying symmetry acting on the ADM metric variables g_{ij}, n^i and n. First, we need to solve the superfield constraints (5.33), (5.34) and (5.35) of the lapse sector in terms of the component fields.

7.1 Solving the constraints

The process of solving the constraints for the lapse sector superfields follows the steps parallel to the steps we took in solving for the components of the shift superfield N^i, S^i and $\overline{S}^i$ in Section 5.1.

First, we solve (5.33) and find

$$\begin{aligned}
\Theta &= -\nu + \theta(-\nu\dot{\nu} - \sigma^k\partial_k\nu + n^k\nu\partial_k\nu) - \overline{\theta}z \\
&\quad + \theta\overline{\theta}(-\dot{\nu} - z\dot{\nu} + \nu\dot{z} - Y^k\partial_k\nu + \sigma^k\partial_k z + \chi^k\nu\partial_k\nu + n^k z\partial_k\nu - n^k\nu\partial_k z).
\end{aligned} \tag{7.1}$$

and then solving (5.34) gives

$$\overline{\Theta} = -\overline{\nu} - \theta w + \overline{\theta}(-\overline{\nu}\dot{\overline{\nu}} - \overline{\sigma}^k\partial_k\overline{\nu} + n^k\overline{\nu}\partial_k\overline{\nu}) + \theta\overline{\theta}(w\dot{\overline{\nu}} - \overline{\nu}\dot{w} + \overline{\sigma}^i\partial_i\overline{\sigma}^k\partial_k\overline{\nu} - \overline{\sigma}^k\partial_k w - \psi^k\overline{\nu}\partial_k\overline{\nu} - n^k w\partial_k\overline{\nu} + n^k\overline{\nu}\partial_k w). \tag{7.2}$$

Here we must remember that Y^i and χ^i appearing in the top component of Θ are composite fields, expressed in terms of the independent component fields n^i, X^i, σ^i and $\overline{\sigma}^i$ of the shift sector as

$$Y^i = -n^i - X^i + \sigma^k\partial_k\overline{\sigma}^i + \overline{\sigma}^k\partial_k\sigma^i, \tag{7.3}$$

$$\chi^i = \dot{\overline{\sigma}}^i + \overline{\sigma}^k\partial_k n^i - \partial_k\overline{\sigma}^i n^k. \tag{7.4}$$

Then we can simply evaluate the components of the E superfield by evaluating the right-hand side of (5.35). For the inverse lapse e we get

$$e = 1 + w + z - \nu(\dot{\overline{\nu}} - n^k\partial_k\overline{\nu}) - \overline{\nu}(\dot{\nu} - n^k\partial_k\nu) - \sigma^k\partial_k\overline{\nu} - \overline{\sigma}^k\partial_k\nu. \tag{7.5}$$

This relation is very similar to (5.18), and we will handle it in the same way: Our independent bosonic component fields will be the inverse lapse e, plus the field w which satisfies $Q\,w = 0$ and therefore is our candidate auxiliary field of the antighost-auxiliary multiplet. Thus, (7.5) serves to express z as a composite field in terms of the independent bosonic and fermionic component fields e, w, ν and $\overline{\nu}$.

The two fermionic components of the inverse lapse superfield E can also be easily evaluated by applying (5.35),

$$\psi = \nu(\dot{e} - n_k\partial_k e) - (\dot{\nu} - n^k\partial_k\nu)e + \sigma^k\partial_k e, \tag{7.6}$$

$$\chi = \overline{\nu}(\dot{e} - n_k\partial_k e) - (\dot{\overline{\nu}} - n^k\partial_k\overline{\nu})e + \overline{\sigma}^k\partial_k e, \tag{7.7}$$

and so can the top bosonic component B, although its explicit expression is not particularly illuminating and will not be useful in the rest of the thesis.

7.2 BRST interpretation of the component fields and gauge symmetries

Here we review all the Q multiplets, in the component language, which we obtained in Sections 5.1 and 7.1 above by solving the superfield constraints for the lapse and shift superfields derived in Part I. We can then check that these multiplets of the nonrelativistic gravity can indeed be interpreted as BRST multiplets of a one-step gauge-fixing of a non-redundant gauge symmetry which acts by a combination of topological transformations and spacetime diffeomorphisms.

First, we have the original Q multiplet containing the spatial metric,

$$Q\,g_{ij} = \psi_{ij}, \qquad Q\,\psi_{ij} = 0. \tag{7.8}$$

ψ_{ij} is indeed the topological BRST ghost, associated with the topological symmetry generated by $\delta g_{ij} = \hat{f}_{ij}$. The spatial metric supefield G_{ij} also contains the multiplet

$$Q\,\chi_{ij} = -B_{ij} \qquad Q\,B_{ij} = 0. \tag{7.9}$$

Here χ_{ij} is the BRST antighost, and $-B_{ij}$ its associated auxiliary field.

Next we summarize the structure of the Q multiplets in the lapse and shift sectors. The component solutions of the superspace constraints that we found in Section 5.1 show that the shift vector n^i forms a multiplet with σ^i,

$$\begin{aligned} Q\,n^i &= \dot{\sigma}^i + \sigma^k\partial_k n^i - \partial_k\sigma^i n^k, & (7.10)\\ Q\,\sigma^i &= \sigma^k\partial_k\sigma^i. & (7.11)\end{aligned}$$

This is indeed the standard BRST multiplet for gauge fixing spatial diffeomorphisms $\hat{\xi}^i$ acting on the shift vector, when we identify σ^i as the ghost associated with the diffeomorphism generator $\hat{\xi}^i$. The shift sector also contains the trivial Q multiplet

$$Q\,\overline{\sigma}^i = X^i, \qquad Q\,X^i = 0. \tag{7.12}$$

Here $\overline{\sigma}^i$ is naturally interpreted as the antighost, and X^i its associated auxiliary field.

In the lapse sector, we also find two multiplets, but since the constraints have involved also the shift-vector superfields, the Q transformations of the components are a bit more intricate,

$$\begin{aligned} Q\,e &= \nu(\dot{e} - n^k\partial_k e) - (\dot{\nu} - n^k\partial_k\nu)e + \sigma^k\partial_k e, & (7.13)\\ Q\,\nu &= \nu(\dot{\nu} - n^k\partial_k\nu) + \sigma^k\partial_k\nu, & (7.14)\end{aligned}$$

as one can see from the component solutions of the lapse-sector constraints found in Section 7.1. Happily, on closer inspection (and recalling that $e = 1/n$), these happen to be the BRST transformation rules (6.28) associated with the $\hat{\zeta}$ time-reparametrization symmetry, if we identify ν as the ghost field c.

Thus, we conclude that the Q transformation rules of the multiplets containing g_{ij}, n^i and e are exactly the standard BRST tranformation rules (6.28) we obtained in the process of gauge fixing of the spacetime reparametrization and topological symmetries (6.27) that we discussed in the previous section! In addition, among the components of the lapse-sector superfields we have also found the antighost-auxiliary multiplet

$$Q\,\overline{\nu} = w, \qquad Q\,w = 0. \tag{7.15}$$

Together with the antighost-auxiliary multiplets (7.9) and (7.12) found in the spatial metric sector and the shift sector, these multiplets are ready to be used in the standard way for implementing the BRST gauge fixing conditions.

We see that the multiplet structure of the topological nonrelativistic gravity constructed in Part I precisely reproduces the standard BRST multiplets associated with our gauge symmetries (6.27), and gives the correct number of the antighost-auxiliary multiplets needed for the gauge fixing. The supercharge Q is precisely the standard BRST charge associated with the symmetries (6.27). Since these symmetries are non-redundant, the BRST construction closes after the first step, and no ghost-for-ghost fields are needed. This clarifies the meaning of the two-step superspace construction of our topological gravity, and answers fully all the questions about this theory that we raised at the beginning of this part.

7.3 Action functionals and Wess-Zumino gauge

Having clarified the structure of the BRST multiplets, and in particular having understood that the supercharge Q is the standard BRST charge associated with our gauge symmetries (6.27), it is time to use the BRST machinery and construct the appropriate gauge-fixed action S; or, more precisely, a family of such actions, parametrized by various coupling constants.

The only action $S_0(g_{ij}, n^i, n)$ that is invariant under our gauge symmetries is zero, or more precisely, a sum of available topological invariants (whose precise list can depend on the spacetime dimension). No local dynamics is induced by this gauge-invariant action S_0; as usual in topological field theories of the cohomological type [31], the entire dynamics will be generated by the judicious choice of the gauge-fixing part of the action, which should take the form

$$S = \int dt\, d^D x\, \{Q, \Psi\}\,, \tag{7.16}$$

with the gauge-fixing fermion Ψ being a local functional built out of the available BRST multiplets. Since we insist on the $\mathcal{N} = 2$ superalgebra (1.13) with supercharges

Q and $\overline{Q}$, the most efficient method for constructing an action consistent with this extended BRST supersymmetry is the superspace approach used in Part I. The action will thus be written as

$$S = \int dt\, d^D x\, d^2\theta\, \mathcal{L}, \tag{7.17}$$

where the superspace Lagrangian $\mathcal{L}$ is a local functional of all the superfields of Part I. The superspace geometry then automatically implies that every such superspace action is of the form (7.16) for some Ψ.

We are dealing with a nonrelativistic theory, which has two *a priori* unrelated scales L and T as discussed in Section 6.1, so the next step is to decide which values of the dynamical exponent z are appropriate for our goals. Since we wish to make contact with the Ricci flow equations, which happen to select the value of $z = 2$, we will be interested in writing down the action that respects this $z = 2$ scaling at short distances, and contains the minimal number of time derivatives on the component fields. As we did in Part I, it is useful to write such an action in superspace as a sum of two terms,

$$S = \frac{1}{\kappa^2}(S_K - S_{\mathcal{W}}), \tag{7.18}$$

where the "kinetic" term S_K contains at least one of the supertime derivatives D, $\overline{\mathrm{D}}$, ∂_t, while the "superpotential" term $S_{\mathcal{W}}$ contains no such supertime derivatives. As we showed in Part I, in order the achieve the desired $z = 2$ scaling at short distances, S_K will contain terms with one D and one $\overline{\mathrm{D}}$, while the superpotential $S_{\mathcal{W}}$ will contain terms with up to two spatial derivatives. Interestingly, this two-derivative superpotential is essentially playing the role of Perelman's $\mathcal{F}$-functional in our theory.

Finally, we need to decide whether we wish to keep any of the underlying gauge symmetries unfixed, or whether we prefer the fully gauge-fixed version of the theory. The appropriate choice may be different depending on which of the three symmetries generated by $\hat{f}_{ij}$, $\hat{\xi}^i$ and $\hat{\zeta}$ we consider. Keeping some of the gauge symmetries unfixed would lead to the so-called "equivariant" theory: Roughly speaking, the BRST cohomology is then defined as the cohomology of the BRST charge on objects that are gauge invariant under the unfixed symmetry. This is the strategy often followed for example in topological Yang-Mills gauge theories in four dimensions, where only the topological symmetry is gauge-fixed while the unfixed "equivariant" symmetry consists of the ordinary Yang-Mills gauge transformations.

Does it make sense to consider a similar equivariant theory in the context of our topological nonrelativistic gravity of the Ricci flow? In Section 4.4 of Part I, we constructed an action which was invariant under the (supersymmetric extension of) the foliation-preserving spacetime diffeomorphisms. Going to Wess-Zumino gauge as discussed in Part I, the resulting theory is still equivariant with respect to the bosonic foliation-preserving spacetime diffeomorphisms, generated by spacetime-dependent spatial diffeomorphisms $\xi^i(t, x^k)$ and projectable time reparametrizations $f(t)$. This unfixed gauge symmetry is strictly larger that the gauge symmetries exhibited by Perelman's Ricci flow.

In the present context, to achieve a closer connection to the Ricci flow, it makes more sense to consider a hybrid theory, in which the gauge-fixing fermion Ψ in (7.16) (or the Lagrangian $\mathcal{L}$ in (7.17)) is chosen such that not only the topological symmetries $\hat{f}_{ij}$ but also the time-reparametrization gauge symmetries $\hat{\zeta}$ are fully gauge-fixed. On the other hand, it is meaningful to treat the spatial diffeomorphisms generated by ξ^i as an unfixed, equivariant symmetry: This symmetry can be gauge-fixed later, as outlined in Section 3.5 of Part I. Several gauge choices for the spatial diffeomorphism gauge symmetry – which we referred to as Perelman gauge, Hamilton gauge, and DeTurck gauge Part I – are naturally available, and they match standard manipulations known in the mathematical theory of the Ricci flow.

In the superfield language of Part I, keeping the bosonic symmetries generated by ξ^i unfixed is accomplished by first choosing $\mathcal{L}$ that is invariant under the superymmetric generalization of the spatial diffeomorphism, and then going to Wess-Zumino gauge, setting the shift superfield equal to its lowest, bosonic component:

$$N^i = n^i. \tag{7.19}$$

In the language of component fields, this amounts to setting the ghost field σ^i, as well as the antighost $\overline{\sigma}^i$ and its auxiliary X^i all equal to zero. Indeed, this makes sense: By throwing away the antighost and the auxiliary, we prevent ourselves from gauge-fixing the associated bosonic symmetry generated by ξ^i. In this Wess-Zumino gauge, the action can then be gauge invariant under the remaining bosonic symmetry ξ^i.

In contrast, it would not be very interesting to treat the nonprojectable symmetry of time reparametrizations $\hat{\zeta}$ equivariantly: As we pointed out in (6.16), if we insist on the unfixed *nonprojectable* bosonic symmetry $\hat{\zeta}(t, x^i)$, the only lowest-derivative Lagrangians that respect this symmetry are ultralocal in space, with the superpotential given by just the cosmological constant term, and leading to a rather trivial theory and no $z = 2$ scaling.

As we shall see in Part III, in which we establish the precise contact with Perelman's equations of the Ricci flow, it will indeed be vital to use this hybrid strategy: Gauge-fixing the topological symmetries in the g_{ij} and n sector, while keeping the spatial diffeomorphism symmetry manifest.

7.4 Dual gauge symmetries

As an aside remark, we note the following interesting curiosity: Our theory has an alternative dual interpretation. Indeed, it is possible to interpret the same component fields studied above as having originated from the gauge fixing of a *dual* copy of spacetime diffeomorphisms, with the role of the BRST charge in this dual picture played by the original anti-BRST charge $\overline{Q}$. This is a consequence of the extended $\mathcal{N}=2$ supersymmetry, together with the "balanced" property of the theory, which implies a symmetry between the ghosts and antighosts.

The existence of such a dual interpretation of the theory can be directly verified by taking a closer look at the $\overline{Q}$ transformations of the component fields. In the shift sector, we have

$$\overline{Q}\,n^i = \dot{\overline{\sigma}}^i - n^k\partial_k\overline{\sigma}^i + \overline{\sigma}^k\partial_k\overline{\sigma}^i, \tag{7.20}$$

$$\overline{Q}\,\overline{\sigma}^i = \overline{\sigma}^k\partial_k\overline{\sigma}^i. \tag{7.21}$$

This is indeed the BRST structure obtained by gauge fixing spatial diffeomorphisms, with the antighost $\overline{\sigma}^i$ of the original intepretation now playing the role of the ghost. Similarly, in the lapse sector, we find

$$\overline{Q}\,e = \overline{\nu}(\dot{e} - n^k\partial_k e) - (\dot{\overline{\nu}} - n^k\partial_k\overline{\nu})e + \overline{\sigma}^k\partial_k e, \tag{7.22}$$

$$\overline{Q}\,\overline{\nu} = \overline{\nu}(\dot{\overline{\nu}} - n^k\partial_k\overline{\nu}) + \overline{\sigma}^k\partial_k\overline{\nu}. \tag{7.23}$$

Again, this is the BRST multiplet obtained from our ultralocal time reparametrization gauge symmetry, with the original antighost $\overline{\nu}$ now playing the role of the ghost associated with the dual symmetry.

Similarly, the original ghosts and auxiliary fields of Q give rise to the standard antighost-auxiliary multiplets of $\overline{Q}$ in the dual interpretation. First, in the shift sector one finds

$$\overline{Q}\,\sigma^i = Y^i, \qquad \overline{Q}\,Y^i = 0. \tag{7.24}$$

Note that the correct identification of the auxiliary field requires that we solve (5.18) in a different way than we did in (5.19), now expressing X^i in terms of the independent components n^i, Y^i, σ^i and $\overline{\sigma}^i$:

$$X^i = -n^i - Y^i + \sigma^k \partial_k \overline{\sigma}^i + \overline{\sigma}^k \partial_k \sigma^i. \tag{7.25}$$

Thus, the auxiliaries X^i and Y^i of the two dual interpretations are related by a nonlinear field redefinition.

Similarly, the antighost-auxiliary $\overline{Q}$ multiplet in the lapse sector is found to be

$$\overline{Q}\,\nu = z, \qquad \overline{Q}\,z = 0. \tag{7.26}$$

This also requires that we solve (7.5) differently than how we solved it in Section 7.1, with w expressed in terms of the independent component fields that now include z:

$$w = -1 + e - z + \nu(\dot{\overline{\nu}} - n^k \partial_k \overline{\nu}) + \overline{\nu}(\dot{\nu} - n^k \partial_k \nu) + \sigma^k \partial_k \overline{\nu} + \overline{\sigma}^k \partial_k \nu. \tag{7.27}$$

As in the shift sector, the auxiliaries w and z in the two dual interpretations are related by this nonlinear redefinition. The role of the ghosts and antighosts is simply exchanged between the two dual pictures.

8 Summary

In this part, we have analyzed the ingredients of topological nonrelativistic gravity associated with the Ricci flow, presented in Part I. In particular, we substantially clarified the structure of the underlying gauge symmetries of this theory, especially in the time sector. The construction of Part I used a two-step procedure, starting with the rigid $\mathcal{N} = 2$ nonrelativistic BRST superspace, and then gauging the symmetries of foliation-preserving spacetime diffeomorphisms. Here we have demonstrated that this theory can be understood in the more traditional way, as a standard one-step BRST gauge fixing of a theory whose dynamical fields are simply the ADM variables of bosonic gravity describing the spatial metric g_{ij}, the shift vector n^i and the lapse function n, and whose gauge symmetries consist of the topological symmetries acting on the spatial metric, combined with the "ultralocal" nonrelativistic limit of spacetime diffeomorphisms. These gauge symmetries appear most naturally in their "shifted" form (6.27). These findings explain the origin of the ingredients obtained in the superspace formulation in Part I, and the origin of the superfield constraints derived from geometric superspace arguments. In particular, since the underlying gauge symmetries of the one-step construction presented in this part are non-redundant, it is clear why no ghost-for-ghost fields were needed in the two-step construction.

It is now also clear that the theory is a cohomological quantum field theory of the standard type [31], with the supercharge Q exactly playing the role of the standard BRST charge in the full theory. Moreover, we have also shown why this theory is topological, in the sense of containing no local propagating excitations: Since all of our gauge symmetries, including the time reparametrizations, are nonprojectable, the number of gauge symmetries matches locally the number of dynamical field components, and the number of local propagating degrees of freedom is zero. We have also pointed out an intriguing dual interpretation of this theory, as having originated from the gauge fixing of a dual copy of spacetime diffeomorphisms, and with the second supercharge $\overline{Q}$ now playing the role of the BRST charge.

With this improved understanding of the supermultiplets and the underlying gauge symmetries in the topological quantum gravity of the Ricci flow, it is now possible to study the precise relation between this topological quantum theory and the mathematical theory of Perelman's Ricci flow.

Part III

Localizing to Perelman's Ricci Flow

Since its inception in the 1980's [1], the mathematical theory of the Ricci flow on Riemannian manifolds has undergone several stages of development. The first stage, lasting for about two decades, was dominated by the study of Hamilton's Ricci flow equation,

$$\frac{\partial \hat{g}_{ij}}{\partial t} = -2\hat{R}_{ij}, \tag{8.1}$$

and its geometrical consequences. Many important achievements highlight this era [26]. The next stage was reached with Perelman's Ricci flow equations [2–4] (see [11–15, 22–24] for extensive reviews), which are designed so that the flow of the metric is coupled to the flow of another field, which Perelman called the "dilaton." (This field is traditionally denoted by f, but we will call it $\hat{\phi}$ in this part.)[16]

$$\frac{\partial \hat{g}_{ij}}{\partial t} = -2\hat{R}_{ij} - 2\hat{\nabla}_i \partial_j \hat{\phi}, \tag{8.2}$$

$$\frac{\partial \hat{\phi}}{\partial t} = -\hat{R} - \hat{\Delta}\hat{\phi}. \tag{8.3}$$

The major advance in Perelman's formulation stems from the fact that the right-hand side of these coupled flow equations is given by the gradient of a functional, Perelman's "$\mathcal{F}$-functional"

$$\hat{\mathcal{F}} = 2\int d^D x \sqrt{\hat{g}} e^{-\hat{\phi}} \left\{ \hat{R} + \hat{g}^{ij} \partial_i \hat{\phi} \partial_j \hat{\phi} \right\}, \tag{8.4}$$

assuming that the variations of $\hat{g}_{ij}$ and $\hat{\phi}$ are subjected to the constraint requiring that the volume element

$$e^{-\hat{\phi}} \sqrt{\hat{g}}\, d^D x = dm(x^i) \tag{8.5}$$

be held fixed in time and equal to a fixed measure $dm(x^i)$ on the spatial manifold Σ.

Perelman's equations are further simplified by an application of what has become known in the mathematical literature as "DeTurck's trick": a specific spatial diffeomorphism is applied to the original equations, with its generating vector field ξ^i given by the gradient of the dilaton, $\xi^i = \hat{g}^{ij}\partial_j \hat{\phi}$. After this diffeomorphism, Perelman's Ricci flow equations simplify to

$$\frac{\partial \hat{g}_{ij}}{\partial t} = -2\hat{R}_{ij}, \tag{8.6}$$

$$\frac{\partial \hat{\phi}}{\partial t} = -\hat{R} - \hat{\Delta}\hat{\phi} + \hat{g}^{ij}\partial_i \hat{\phi} \partial_j \hat{\phi}, \tag{8.7}$$

In this form, Hamilton's original metric flow equation is now nicely separated from the flow equation for the dilaton.

Perelman's Ricci flow was of course instrumental in his proof of the Poincaré conjecture [4,18–21], and since then in the proofs of many other important results: the Thurston geometrization conjecture [4,19], the generalized Smale conjecture [6–10], and even a new proof of the uniformization theorem in two spatial dimensions [5].

[16] Throughout this part, we will systematically denote Perelman's variables by hats, $\hat{}$. This includes both the geometric fields $\hat{g}_{ij}$, $\hat{\phi}$, … and the various geometric quantities such as the covariant derivative $\hat{\nabla}_i$. We reserve the notation without hats for our variables g_{ij}, ϕ, …, ∇_i. These two sets of variables will be related by a nonlinear transformation which involves a change of frame for the metric.

The topological quantum gravity theory presented in Part I has been designed around a family of generalized Ricci flow equations similar to Perelman's. These equations appear in the topological gravity in a central role, as localization equations: With the appropriate initial or boundary conditions, the path integral is reduced by standard topological arguments to an integral over the space of classical solutions to the flow equations.

The localization equations depend on many physical coupling constants available in this theory. One can naturally ask whether Perelman's equations can be precisely reproduced in some regime of this topological quantum gravity, and if so, what is the precise mapping of the variables between the mathematical and the physical picture. In the present part, we address these questions in the semiclassical limit of the theory.

That such a direct embedding of Perelman's Ricci flow equations into our topological gravity should even exist is not immediately obvious, for several reasons. First, note that Perelman's equations have less spacetime symmetry than the localization equations of the theory constructed in Part I, at least before we fix a part of the secondary gauge symmetry of foliation-preserving spacetime diffeomorphisms. At best, we can find a covariantized version of Perelman's equations which respects time-reparametrization invariance (this will be done in Section 9); or we need to propose an appropriate gauge fixing of time reparametrizations in our theory to match the symmetries of Perelman's equations (this will be the subject of Section 10). Secondly, our version of the flow equations is schematically of the form

$$e^{\phi}\dot{g}_{ij} = -\alpha_R R_{ij} + \dots, \tag{8.8}$$

with an extra multiplicative e^{ϕ} factor between the two sides. This suggests the need for a nonlinear "reframing" field redefinition between Perelman's variables and ours. Thirdly, the question is how to interpret the additional volume-fixing condition (8.5), which was postulated by Perelman in order to derive the flow equations. This condition cannot be a part of gauge fixing of the residual gauge symmetries in our theory, since (i) the theory presented in Part I only exhibits *spatially-independent* time reparametrizations, and (ii) we wish to reserve all the spatial diffeomorphism symmetry in the unfixed form, so that we can implement the DeTurck trick as a gauge fixing condition, as we did in Part I. Finally, note also that in our approach, *both* sides of the flow equations come from a variational principle, which leads not only to more couplings but also to additional restrictions on the form of the equations of motion one can obtain.

In this part, we resolve these issues in two steps. First, in Section 9, we consider the covariant theory of Part I, with secondary gauge symmetry of foliation-preserving spacetime diffeomorphisms. We identify the regime where the localization equations represent the covariant version of Perelman's equations. In the process, we learn how our fields are related to Perelman's by a change-of-frame transformation, and how Perelman's fixed-volume condition emerges dynamically in our topological gravity. Then, in Section 10, we perform the gauge fixing that leads directly to the localization of the path integral on Perelman's Ricci flow equations (8.2) and (8.3), and the fixed-volume condition (8.5). Those readers who are interested only in our final product – the topological gravity of Perelman's Ricci flow – can go directly to Section 10, and return to Section 9 for the motivation and logical derivations as needed. In Section 11 we extend our construction to include the $\mathcal{W}$ and $\mathcal{W}_+$ entropy functionals associated with the shrinking and expanding Ricci solitons; the main new ingredient that we will need to introduce is going to be the Goldstone superfield T asssociated with spontaneously broken time translations.

9 Covariantized Perelman-Ricci flow equations from topological gravity

The theory presented in Parts I and II is a theory of the spacetime metric expressed in the ADM variables [43], consisting of the spatial metric g_{ij}, the shift vector n^i, and the lapse function n. The theory is designed to be topologically invariant, and nonrelativistic – it is sensitive to a preferred foliation $\mathscr{F}$ of spacetime $\mathcal{M}$ by leaves Σ of constant time. Given the required $\mathcal{N}=2$ extension (1.13) of the BRST symmetry, this topological theory is most concisely formulated in an $\mathcal{N}=2$ superspace extension Part I of the nonrelativistic spacetime manifold $\mathcal{M}$. Spacetime is thus extended to a supermanifold $\mathscr{M}$ of superdimension $(D+1|2)$, which also inherits a foliation by the spatial leaves Σ.

Throughout this section, we will consider the theory which enjoys – besides the topological symmetry – a secondary gauge symmetry of foliation-preserving diffeomorphisms of spacetime,

$$\mathcal{G} \equiv \mathrm{Diff}_{\mathscr{F}}(\mathcal{M}). \tag{9.1}$$

This symmetry is locally generated by infinitesimal spacetime-dependent spatial diffeomorphisms $\delta x^i = \xi^i(t, x^j)$, and time-dependent time reparametrizations $\delta t = f(t)$. We intend to treat this secondary symmetry equivariantly: In particular, we will construct our action functionals in this section to be manifestly invariant under

$\mathcal{G}$.

As is often done in supersymmetric gauge-theory constructions, we impose this secondary gauge symmetry $\mathcal{G}$ by first extending it to a gauge symmetry in superspace, making $\mathcal{G}$ manifestly consistent with the underlying $\mathcal{N} = 2$ supersymmetry. The full list of superfields in this theory consists of the unconstrained metric superfield G_{ij}, whose lowest component is g_{ij}, the superfields N^i, S^i and $\overline{S}^i$ in the shift-vector sector, with the bosonic shift vector n^i appearing as the lowest component of N^i; and the superfields E, Θ and $\overline{\Theta}$ in the lapse function sector, with the lapse function n appearing as the inverse of the lowest component e of E, $n = 1/e$. In components, each of the three sectors contains the original ADM bosonic field, its ghost, its antighost, and an auxiliary. In the spatial metric sector, these component fields are

$$g_{ij}, \quad \psi_{ij}, \quad \chi_{ij}, \quad B_{ij}. \tag{9.2}$$

In the lapse and shift sectors, they are

$$e \equiv 1/n, \quad \nu, \quad \overline{\nu}, \quad w \tag{9.3}$$

and

$$n^i, \quad \sigma^i, \quad \overline{\sigma}^i, \quad X^i, \tag{9.4}$$

with B_{ij}, w and X^i the bosonic auxiliaries in the corresponding sectors. See Parts I and II for all additional details, including the Q and $\overline{Q}$ transformation rules for these component multiplets, and for our conventions.

9.1 Localization equations for g_{ij} in topological quantum gravity

The action can be written as

$$S = \frac{1}{\kappa^2}\left(S_K - S_{\mathcal{W}}\right), \tag{9.5}$$

where the kinetic term S_K is defined as the part of the action containing at least one supertime derivative, and the term $S_{\mathcal{W}}$ – which we call the superpotential – contains all the remaining terms, with no supertime derivatives. The minimal kinetic term is

$$S_K = \int d^2\theta\, dt\, d^D x\, \sqrt{G} N \left(G^{ik}G^{j\ell} - \lambda G^{ij}G^{k\ell}\right) \mathscr{D}_{\overline{\theta}} G_{ij}\, \mathscr{D}_\theta G_{k\ell}. \tag{9.6}$$

Here $\mathscr{D}_{\overline{\theta}}$ and $\mathscr{D}_\theta$ are the covariant superspace derivatives associated with the gauge group $\mathcal{G}$.

The superpotential terms can be organized by the increasing dimension of the operators. In the case of interest, relevant to Ricci flow, we focus on all the terms up to second order in spatial derivatives, which gives – up to integration by parts – the following superpotential:

$$S_{\mathcal{W}} = \int d^2\theta\, dt\, d^D x\, \sqrt{G} N \left\{ \alpha_R R^{(G)} + \alpha_\Phi G^{ij} \partial_i \Phi \partial_j \Phi + \alpha_\Lambda \right\}, \tag{9.7}$$

where α_R, α_Φ and α_Λ are real coupling constants, and the superfield $R^{(G)}$ is the Ricci scalar superfield constructed from the metric superfield G_{ij}. The supefield Φ is simply related to the lapse superfield $N \equiv 1/E$,

$$\Phi \equiv -\log N \equiv \log E. \tag{9.8}$$

Since we have restricted our attention to the terms in $S_{\mathcal{W}}$ with up to two spatial derivatives, we are anticipating that the short-distance behavior in this theory will exhibit anisotropy between time and space characterized by the dynamical exponent $z = 2$. With this scaling, the two-derivative terms in $S_{\mathcal{W}}$ are of the same classical scaling dimension as the kinetic term. The cosmological constant term α_Λ is the only available relevant term.

Now we wish to study the localization equations, and their possible relation to Perelman's Ricci flow. We begin with a simple theory, in which the lapse superfield E is constrained to be chiral, and no chirality conditions are imposed on either G_{ij} or N^i; as before, we shall refer to this theory as Type C (for "chiral" lapse). First, we will focus on the localization equation for the spatial metric g_{ij} in this Type C theory. In terms of the component fields listed in (9.3), the chirality condition amounts to setting the antighost $\overline{\sigma}^i$ and the auxiliary w to zero.

When focusing on the localization equations, we will often – for clarity of discussion – keep track of only the bosonic fields, setting all the fermionic component fields to zero.[17] When we write the type C theory in components, it will be consistent with our equivariant treatment of spatial diffeomorphisms to adopt Wess-Zumino gauge in the superspace shift sector, as discussed in Part I. This choice is equivalent to setting σ^i, $\overline{\sigma}^i$ and X^i in (9.4) to zero. In this gauge, the bosonic gauge symmetry $\mathcal{G}$ of the

[17] We use the symbol "$\approx$" to denote the evaluation of any quantity by keeping its full dependence on the bosonic component fields while setting all the fermionic components to zero.

action is still manifest. In particular, the time derivative of g_{ij} is still covariantized to

$$\nabla_t g_{ij} \equiv \partial_t g_{ij} - \nabla_i n_j - \nabla_j n_i. \tag{9.9}$$

With this notation, the bosonic part of the action is

$$S_K \approx -\int dt\, d^D x\, \sqrt{g} n\, (g^{ik}g^{j\ell} - \lambda g^{ij}g^{k\ell})\, B_{ij}B_{k\ell} + \int dt\, d^D x\, \sqrt{g}\, (g^{ik}g^{j\ell} - \lambda g^{ij}g^{k\ell})\, B_{ij}\nabla_t g_{k\ell} \tag{9.10}$$

and

$$S_{\mathcal{W}} \approx \int dt\, d^D x \sqrt{g} n\, B_{ij}\mathcal{E}^{ij}, \tag{9.11}$$

with $\mathcal{E}^{ij}$ given by

$$\begin{aligned} \mathcal{E}^{ij} \equiv \frac{1}{\sqrt{g}n}\frac{\delta\mathcal{F}}{\delta g_{ij}} &= \alpha_R\left(\frac{1}{2}Rg^{ij} - R^{ij}\right) + \alpha_R\left(g^{ik}g^{j\ell} - g^{ij}g^{k\ell}\right)\frac{1}{n}\nabla_k\partial_\ell n \\ &\quad + \alpha_\Phi\left(\frac{1}{2}g^{ij}g^{k\ell} - g^{ik}g^{j\ell}\right)\partial_k\phi\partial_\ell\phi + \frac{1}{2}\alpha_\Lambda g^{ij}. \end{aligned} \tag{9.12}$$

Here we denoted by $\mathcal{F}$ the spacetime integral of the lowest component of the superpotential Lagrangian density in superspace,

$$\mathcal{F} = \int dt\, d^D x\, \sqrt{g}n\left(\alpha_R R^{(g)} + \alpha_\Phi g^{ij}\partial_i\phi\partial_j\phi + \alpha_\Lambda\right), \tag{9.13}$$

with $R^{(g)}$ in (9.13) now being the bosonic Ricci scalar of g_{ij}. We have also introduced a slight change of notation in the lapse sector: From now on, we will use

$$\phi = -\log n, \tag{9.14}$$

which is often a more convenient variable than the lapse field n itself. Also, it is this field ϕ which will turn out to be simply proportional to Perelman's dilaton $\hat{\phi}$.

The equation of motion obtained from varying B_{ij} in the full action S is

$$2B_{ij} = \frac{1}{n}\nabla_t g_{ij} - (g_{ik}g_{j\ell} - \widetilde{\lambda}g_{ij}g_{k\ell})\mathcal{E}^{k\ell} \equiv E_{ij}. \tag{9.15}$$

Here the tensor

$$g_{ik}g_{j\ell} - \widetilde{\lambda}g_{ij}g_{k\ell} \tag{9.16}$$

is the inverse to the DeWitt metric on the space of metrics

$$g^{ik}g^{j\ell} - \lambda g^{ij}g^{k\ell}, \tag{9.17}$$

with $\widetilde{\lambda}$ given in terms of λ by [33]

$$\widetilde{\lambda} = \frac{\lambda}{\lambda D - 1}. \tag{9.18}$$

By solving for B_{ij} algebraically, the action now becomes

$$S \approx \frac{1}{4}\int dt\, d^D x \sqrt{g} n\, E_{ij}\left(g^{ik}g^{j\ell} - \lambda g^{ij}g^{k\ell}\right)E_{k\ell}. \tag{9.19}$$

Assuming that the DeWitt metric (9.17) is weakly positive definite (*i.e.*, positive definite modulo spatial diffeomorphisms), standard arguments of topological quantum field theory will localize the path integral to the minima of the action, which thus requires the validity of the localization equation

$$E_{ij} = 0. \tag{9.20}$$

For our specific superpotential (9.7), this equation yields the metric flow

$$\begin{aligned} e^{\phi}\nabla_t g_{ij} \;=\; & -\alpha_R R_{ij} + \frac{\alpha_R}{2}\left[1 + (2-D)\widetilde{\lambda}\right] g_{ij}R - \alpha_R \nabla_i\partial_j\phi \\ & + \alpha_R\left[1 + (1-D)\widetilde{\lambda}\right] g_{ij}\Delta\phi + (\alpha_R - \alpha_\Phi)\partial_i\phi\partial_j\phi \\ & + \left\{\frac{\alpha_\Phi}{2}\left[1 + (2-D)\widetilde{\lambda}\right] - \alpha_R\left[1 + (1-D)\widetilde{\lambda}\right]\right\} g_{ij}(\partial\phi)^2 + \frac{\alpha_\Lambda}{2}(1 - \widetilde{\lambda}D)g_{ij}. \end{aligned} \tag{9.21}$$

Our next challenge is to identify for which values of the couplings, if any, these localization equations are related to Perelman's Ricci flow.

9.2 Finding Perelman's equations: The $\mathrm{Diff}_{\mathscr{F}}(\mathcal{M})$ equivariant case

Our first step is to propose a change of variables, with the metric rescaled via

$$\hat{g}_{ij} = e^{\phi}g_{ij}. \tag{9.22}$$

This change of frames of the spatial metric is designed to eliminate the extra factor of e^{ϕ} between the two sides of (8.8) or (9.21), so that the leading terms on both sides match those of the flow equation (8.2). Given the importance of such reframing transformations throughout this part, we have collected the relevant change-of-frame formulas for various geometric objects in Appendix B for convenience and completeness.

Next we need to determine how Perelman's dilaton $\hat{\phi}$ should be related to our ϕ. Rewriting our superpotential $S_{\mathcal{W}}$ in the mixed variables $\hat{g}_{ij}$ and ϕ and comparing to Perelman's $\hat{\mathcal{F}}$-functional then suggests that we need to set

$$\hat{\phi} = \frac{D}{2}\phi. \tag{9.23}$$

Finally we also identify $\hat{n}^i = n^i$.

It is natural to introduce the couplings $\hat{\alpha}_R$ and $\hat{\alpha}_\Phi$ in Perelman's functional,

$$\hat{\mathcal{F}} = 2\int d^D x\sqrt{\hat{g}}e^{-\hat{\phi}}\left\{\hat{\alpha}_R\,\hat{R} + \hat{\alpha}_\Phi\,\hat{g}^{ij}\partial_i\hat{\phi}\partial_j\hat{\phi}\right\}, \tag{9.24}$$

noting that the original $\hat{\mathcal{F}}$ functional (8.4) corresponds to the specific choice of

$$\hat{\alpha}_R = \hat{\alpha}_\Phi = 2. \tag{9.25}$$

Using the reframing formulas from Appendix B shows that in our original variables, this choice translates into

$$\alpha_R = 2, \qquad \alpha_\Phi = \frac{2-D}{2}. \tag{9.26}$$

These are the predictions for the regime of our topological gravity where we may expect contact with Perelman's flow equations.

With this proposed relation (9.22) and (9.23) between Perelman's variables and ours, it is now illuminating to rewrite the covariant form of his volume-fixing condition (8.5) in our variables. It is pleasing to see that this condition becomes simply

$$\nabla_t\sqrt{g} = 0, \tag{9.27}$$

the condition of the spatial volume element being covariantly constant in time. This observation is very suggestive: It is indeed well-known in nonrelativistic quantum gravity of the Lifshitz type [32–34, 42] how to realize such a spatial "unimodularity" condition dynamically! This condition is the result of the equations of motion when we take the limit of

$$|\lambda| \to \infty. \tag{9.28}$$

This regime of nonrelativistic quantum gravity is particularly interesting for a number of reasons [32–34,42], and it has also been studied in the context of physical cosmology [75]. Leaving such physical motivations aside, it is intriguing to see that this same regime of "gravity at the farpoint" $\lambda = \pm\infty$ in the kinetic coupling λ makes an independent appearance in the topological quantum gravity of Perelman's Ricci flow!

We can now verify how our localization equation (9.21) is precisely related to Perelman's equations. Taking the values of the couplings in (9.26), setting $\lambda = \pm\infty$, and rewriting our localization equations in Perelman's variables using the reframing formulas from Appendix B, our (9.21) becomes

$$\hat{\nabla}_t \hat{g}_{ij} - \frac{2}{D}\hat{g}_{ij}\hat{\nabla}_t\hat{\phi} = -2\hat{R}_{ij} - 2\hat{\nabla}_i\partial_j\hat{\phi} + \frac{2}{D}\hat{g}_{ij}\hat{R} + \frac{2}{D}\hat{g}_{ij}\hat{\Delta}\hat{\phi}, \tag{9.29}$$

where $\hat{\nabla}_t\hat{\phi} \equiv \partial_t\hat{\phi} - \hat{n}^i\partial_i\hat{\phi}$. This is just the sum of Perelman's first equation (8.2) and $-(D/2)\hat{g}_{ij}$ times Perelman's second equation (8.3)! Note that we did not have to take the cosmological constant α_Λ to zero: This is one of the benefits of being at the "farpoint" $|\lambda| = \infty$ in the kinetic coupling λ. Moreover, by taking the trace of (9.29), we obtain

$$\hat{g}^{ij}\hat{\nabla}_t\hat{g}_{ij} - 2\hat{\nabla}_t\hat{\phi} = 0, \tag{9.30}$$

which can be usefully rewritten as

$$\hat{\nabla}_t\left(e^{-\hat{\phi}}\sqrt{\hat{g}}\right) = 0. \tag{9.31}$$

This confirms that in accord with our anticipation, our theory in the $|\lambda| = \infty$ limit indeed dynamically imposes Perelman's fixed-volume condition (8.5), in its covariant form, and in a form in which the rather awkward and unnecessary reference to an arbitrary constant measure $dm(x^i)$ on Σ in (8.5) is nicely absent.

This is as close as we can get to Perelman's original equations in the covariant theory studied in this section. The two equations cannot be separated in a covariant way: Such a split would be inconsistent with the fact that the localization equations in our theory with the $\mathcal{G}$ gauge symmetry realized equivariantly must be gauge invariant under time-dependent time reparametrization, a symmetry not respected by the individual equations (8.2) and (8.3) but respected by their appropriate sum. Thus, our system (9.29) represents a covariantized version of Perelman's equations, made consistent with time reparametrizations.

9.3 Gauge fixing of spatial diffeomorphisms and DeTurck's trick

The theory is still gauge invariant under the symmetry of spacetime-dependent spatial diffeomorphisms. We can perform a gauge fixing of this symmetry to make contact with Perelman's equations before and after the DeTurck trick. The simplest natural gauge choice, which we referred to in Part I as *Perelman gauge*, is to set $n^i = 0$. In this case, we simply obtain Perelman's original flow equations (8.2) and (8.3).

Another natural gauge fixing, which we referred to in Part I as *Hamilton gauge*, is given in Perelman's variables by

$$\hat{n}_i = \partial_i \hat{\phi}, \tag{9.32}$$

where we define $\hat{n}_i \equiv \hat{g}_{ij} n^j$. In our original variables, this condition can be rewritten as

$$e^{\phi} n_i = \frac{D}{2} \partial_j \phi, \quad \text{or} \quad n_i = -\frac{D}{2} \partial_i n. \tag{9.33}$$

With this choice of n_i, our localization equations in Perelman's variables become

$$\dot{\hat{g}}_{ij} - \frac{2}{D} \hat{g}_{ij} \dot{\hat{\phi}} = -2\hat{R}_{ij} + \frac{2}{D} \hat{g}_{ij} \left(\hat{R} + \hat{\Delta} \hat{\phi} - \hat{g}^{k\ell} \partial_k \hat{\phi} \partial_\ell \hat{\phi} \right), \tag{9.34}$$

which nicely reproduces the sum of Perelman's equations after the DeTurck trick, (8.6) and (8.7).

Having shown how to use spatial diffeomorphisms to perform DeTurck's trick as their gauge fixing, one might still be concerned that our theory contains $D(D+1)/2$ localization equations (9.21) for $[D(D+1)/2] + 1$ field variables g_{ij} and ϕ, given the fact that our lapse sector is nonprojectable and ϕ therefore spacetime-dependent. Where is the missing equation, which would provide the localization condition in the ϕ sector?

In Type C theory, on which we have focused in this section so far, the fact that we are missing one localization equation is consistent: By imposing the chirality condition on the lapse superfield E, we effectively set the antighost and the auxiliary field associated with ϕ to zero. Thus, our Type C theory is not yet fully gauge fixed: For example, the ghost field in the lapse sector does not have a non-degenerate kinetic term. Choosing a BRST-trivial antighost-auxiliary multiplet and adding another gauge-fixing term to the action would be required in order to complete the construction of the theory.

In Type B theory on the other hand, these remaining gauge-fixing ingredients are already present. Since one does not impose any chirality condition on the lapse superfields, the component content is then "balanced" (in the sense reviewed in Part I): ν and $\overline{\nu}$ are the ghost and the antighost, and w is a bosonic auxiliary. Note that the secondary gauge symmetry $\mathcal{G}$ that we require throughout this section only contains *spatially-independent* time reparametrizations, a symmetry clearly not large enough to mimic what we did in the shift sector and set these three component fields to zero by some Wess-Zumino-type gauge. If we choose the same action (9.6) and (9.7) in the Type B theory, the w auxiliary field will yield the missing scalar flow equation. It is intriguing that it does so in a time-reparametrization covariant way.

This last observation also clearly indicates that the Type B theory, with the gauge symmetry $\mathcal{G}$ treated equivariantly, will *not* be the correct setting if our goal is to obtain Perelman's Ricci flow as the exact localization equations. In order to achieve that goal, we will have to revisit the original gauge symmetries and their gauge fixing, following the arguments developed in Part II. This will be the main focus of Section 10 below. However, since the Type B theory with the gauge symmetry $\mathcal{G}$ may still be of independent interest, we present the structure of its localization equations now, before we proceed with our main task in Section 10.

9.4 Type B theory

In Type B theory, the bosonic part of the action is

$$S_K \approx -\int dt\, d^D x\, \sqrt{g} n\, (g^{ik} g^{j\ell} - \lambda g^{ij} g^{k\ell})\, \mathcal{B}_{ij} \mathcal{B}_{k\ell} + \int dt\, d^D x\, \sqrt{g}\, (g^{ik} g^{j\ell} - \lambda g^{ij} g^{k\ell})\, \mathcal{B}_{ij} \nabla_t g_{k\ell}, \tag{9.35}$$

and

$$S_{\mathcal{W}} \approx \int dt\, d^D x \sqrt{g} n\, \mathcal{B}_{ij} \mathcal{E}^{ij} + \int dt\, d^D x\, w\pi, \tag{9.36}$$

where $\mathcal{E}^{ij}$ is again given by (9.12), and

$$\begin{aligned} \pi \;\; = \;\; & \alpha_R \sqrt{g} \left(g^{ik} g^{j\ell} - g^{ij} g^{k\ell} \right) \nabla_i \left(\nabla_j n \nabla_t g_{k\ell} - n \nabla_j \nabla_t g_{k\ell} \right) \\ & - 2\alpha_\Phi \partial_i \left[n \nabla_t \left(\sqrt{g} g^{ij} \partial_j \phi \right) \right] . \end{aligned} \tag{9.37}$$

In order to get the component action into such a nice diagonalized form in the auxiliaries, we had to perform a simple redefinition of the auxiliary fields B_{ij} and w,

$$\mathcal{B}_{ij} = B_{ij} - w\nabla_t g_{ij}. \tag{9.38}$$

This field transformation from B_{ij}, w to $\mathcal{B}_{ij}, w$ has the unit Jacobian and therefore it is an allowed change of variables in the path integral, including its component field measure.

The equations of motion that are obtained from varying $\mathcal{B}_{ij}$ and w in the full action S are:

$$2\mathcal{B}_{ij} = \frac{1}{n}\nabla_t g_{ij} - (g_{ik}g_{j\ell} - \widetilde{\lambda} g_{ij}g_{k\ell})\mathcal{E}^{k\ell} \equiv E_{ij}, \tag{9.39}$$

$$\pi = 0. \tag{9.40}$$

Here E_{ij} is again the same as in Type C theory above. The path integral over w yields a delta function and imposes the constraint $\pi = 0$, which represents the "missing" scalar equation. Unfortunately, we have not been able to decode the geometric meaning of this $\pi = 0$ constraint in terms relevant to the mathematical theory of the Ricci flow equations.

10 Perelman's equations and gauge fixing of time reparametrizations

We are now ready to combine together the pieces of the puzzle that we learned in this thesis so far, and finally to write down the precise topological quantum gravity theory whose localization equations are equivalent to Perelman's Ricci flow, after the appropriate change of frames.

10.1 The theory

The key is to choose the suitable underlying gauge symmetry, and to gauge fix it such that the residual symmetries match the symmetries of Perelman's equations. The crucial lesson was learned in Part II, where we showed how the superspace construction of Part I can be interpreted as the one-step BRST gauge fixing of a theory of the ADM metric variables, with a non-redundant gauge symmetry. In Part II, this symmetry was a combination of topological deformations of the spatial metric g_{ij} and the ultralocal limit of all spacetime diffeomorphisms acting on the lapse and shift n and n^i. It will be useful to change the gauge symmetry to a closely related, also non-redundant symmetry $\mathscr{G}$, generated by

$$\begin{aligned} \delta n &= f(t,x^i), && (10.1)\\ \delta n^i &= \dot{\xi}^i+\xi^k\partial_k n^i - n^k\partial_k\xi^i, && (10.2)\\ \delta g_{ij} &= f_{ij}(t,x^k). && (10.3)\end{aligned}$$

The interpretation of this symmetry structure is as follows: ξ^i acts via standard spacetime-dependent spatial diffeomorphisms on n^i as indicated, and on n and g_{ij} in the standard way as well. In addition, we have arbitrary topological deformations f_{ij} of the spatial metric g_{ij}, as well as arbitrary topological deformations f of the lapse n; much like in Part II, we have absorbed the action by ξ^i on g_{ij} and n into a shift in the definition of f and f_{ij}, certainly a change of variables whose Jacobian is equal to one. This explains the simplicity of the gauge transformations, and makes it clear that the gauge symmetries are non-redundant. Note that this symmetry structure leads to zero local propagating degrees of freedom, and thus a topological theory.

We choose to realize only the spacetime-dependent spatial diffeomoprhisms equivariantly. The symmetries generated by f and f_{ij} need to be gauge fixed. Thus, the $\mathcal{N}=2$ BRST superfields that we will use are as follows. In the metric sector, we use the same unconstrained metric superfield G_{ij} that we have used so far,

$$G_{ij}=g_{ij}+\theta\psi_{ij}+\overline{\theta}\chi_{ij}+\theta\overline{\theta}\,B_{ij}. \quad (10.4)$$

The superfields N^i, S^i and $\overline{S}^i$ in the shift sector will be identical to those used thus far; their component fields are n^i, σ^i, $\overline{\sigma}^i$ and X^i, interpreted as the shift vector, its ghost, its antighost, and its auxiliary associated with the gauge symmetries generated by ξ^i. We will again adopt the Wess-Zumino gauge, setting σ^i, $\overline{\sigma}^i$ and X^i to zero. Finally, in the lapse sector we will use an unconstrained lapse superfield N, or better yet, the superfield $\Phi\equiv-\log N$, whose lowest component is our ϕ:[18]

$$\Phi=\phi+\theta\psi+\overline{\theta}\chi+\theta\overline{\theta}\mathscr{B}. \quad (10.5)$$

Given these component fields, the action of the gauge-fixed theory will be of the

[18]Note that this is the only place where our conventions in this part differ from those of Part II, where ψ and χ were used to denote the component fermions of the E superfield, $E=\log\Phi$.

form

$$S=\int dt\,d^Dx\{Q,\Psi\},\qquad (10.6)$$

with an appropriately chosen gauge-fixing fermion Ψ. Writing this action as a superspace integral

$$S=\int dt\,d^Dx\,d^2\theta\,\mathcal{L}\qquad (10.7)$$

with some superspace Lagrangian $\mathcal{L}$ will make sure that the action is of the form (10.6) for some gauge-fixing fermion Ψ, and it will also ensure the extension to the $\mathcal{N}=2$ BRST supersymmetry. Now, everything rests on the choice of Ψ.

It is the entire idea of gauge fixing to make sure that the choice of the gauge-fixing fermion Ψ fixes the part of the gauge group that we wish to gauge-fix. In our case, we decided to keep the spatial diffeomorphisms generated by ξ^i unfixed, so that we can make contact with the symmetries of the Ricci flow. Thus, Ψ (or, equivalently, $\mathcal{L}$) must be chosen such that it fixes the topological symmetries generated by f and f_{ij}, while still being invariant under ξ^i. This is where our construction will be different from that in Section 9 above: The gauge-fixing fermion of the $\mathrm{Diff}_{\mathscr{F}}(\mathcal{M})$ covariant theory studied in Section 9 was chosen such that the time-dependent time diffeomorphisms were unfixed; here we choose Ψ such that even these residual time reparametrizations are gauge-fixed.

The appropriate modification will come from the kinetic sector. First, instead of the minimal kinetic term (9.6) with time-reparametrization gauge invariance, we now begin with

$$S_K^{(0)}=\int d^2\theta\,dt\,d^Dx\,\sqrt{G}\,(G^{ik}G^{j\ell}-\lambda G^{ij}G^{k\ell})\,\overline{\mathcal{D}}G_{ij}\,\mathcal{D}G_{k\ell}.\qquad (10.8)$$

The superderivatives $\mathcal{D}$ and $\overline{\mathcal{D}}$ are covariant with respect to spatial diffeomorphisms, but not with respect to any time reparametrizations. Their definition is the same as in Part I,

$$\mathcal{D}G_{ij}\;\equiv\;\mathrm{D}G_{ij}-S^k\partial_kG_{ij}-G_{kj}\partial_iS^k-G_{ik}\partial_jS^k,\qquad (10.9)$$

$$\overline{\mathcal{D}}G_{ij}\;\equiv\;\overline{\mathrm{D}}G_{ij}-\overline{S}^k\partial_kG_{ij}-G_{kj}\partial_i\overline{S}^k-G_{ik}\partial_j\overline{S}^k.\qquad (10.10)$$

By design, this kinetic term fixes not just the topological gauge symmetry of g_{ij} but also the gauge symmetries acting on n, and leaves no local time reparametrizations unfixed. To this kinetic term, we are free to add other terms of the same classical scaling dimension that are of the minimal form in derivatives and respect the same symmetries: We can add terms of the form $\mathcal{D}\Phi\,\overline{\mathcal{D}}\Phi$, $G^{ij}\mathcal{D}G_{ij}\,\overline{\mathcal{D}}\Phi$ and $G^{ij}\overline{\mathcal{D}}G_{ij}\,\mathcal{D}\Phi$, with independent couplings. For our purposes, the first one – the kinetic term for Φ – will be sufficient, and the couplings of the off-diagonal terms mixing Φ with the metric will be set to zero. Thus, our full kinetic term will be

$$S_K = \int d^2\theta\, dt\, d^Dx\, \sqrt{G}\, \left\{ (G^{ik}G^{j\ell} - \lambda G^{ij}G^{k\ell})\, \overline{\mathcal{D}}G_{ij}\, \mathcal{D}G_{k\ell} + \lambda_\Phi\, \overline{\mathcal{D}}\Phi\, \mathcal{D}\Phi \right\}. \qquad (10.11)$$

As we will see below, the value of the coupling constant λ_Φ for which the match to Perelman's equations will be accomplished is $\lambda_\Phi = D$. The superpotential (9.7) stays unchanged,

$$S_{\mathcal{W}} = \int d^2\theta\, dt\, d^Dx\, \sqrt{G}e^{-\Phi} \left\{ \alpha_R R^{(G)} + \alpha_\Phi G^{ij}\partial_i\Phi\partial_j\Phi + \alpha_\Lambda \right\}, \qquad (10.12)$$

except that now Φ is an unconstrained superfield, with the lapse also unconstrained and given by $N = \exp(-\Phi)$.

In components, the bosonic part of the action is

$$\begin{aligned} S \;\approx\; & -\int dt\, d^Dx\, \sqrt{g}\, \left\{ B_{ij} \left(g^{ik}g^{j\ell} - \lambda g^{ij}g^{k\ell} \right) B_{k\ell} + \lambda_\Phi \mathscr{B}^2 \right\} \\ & + \int dt\, d^Dx\, \sqrt{g}\, B_{ij} \left\{ \left(g^{ik}g^{j\ell} - \lambda g^{ij}g^{k\ell} \right) \nabla_t g_{k\ell} - \mathscr{E}^{ij} \right\} \\ & - \int dt\, d^Dx\, \sqrt{g}\, \mathscr{B} \left(\lambda_\Phi \nabla_t \phi - \mathscr{E} \right), \end{aligned} \qquad (10.13)$$

where ∇_t continues to denote the time derivative covariantized with respect to spatial diffeomorphisms,

$$\nabla_t g_{ij} = \dot{g}_{ij} - \nabla_i n_j - \nabla_j n_i, \qquad (10.14)$$

$$\nabla_t \phi = \dot{\phi} - n^k \partial_k \phi, \qquad (10.15)$$

and where

$$\mathscr{E}^{ij} \equiv \frac{1}{\sqrt{g}}\frac{\delta\mathcal{F}}{\delta g_{ij}} = e^{-\phi}\left\{\alpha_R\left(-R^{ij}+\frac{1}{2}Rg^{ij}\right)+\left(\frac{1}{2}\alpha_\Phi-\alpha_R\right)g^{ij}(\partial\phi)^2+\alpha_R g^{ij}\Delta\phi \right.$$
$$\left. +(\alpha_R-\alpha_\Phi)g^{im}g^{js}\partial_m\phi\partial_s\phi-\alpha_R g^{im}g^{js}\nabla_m\partial_s\phi+\frac{\alpha_\Lambda}{2}g^{ij}\right\}, \tag{10.16}$$

$$\mathscr{E} \equiv \frac{1}{\sqrt{g}}\frac{\delta\mathcal{F}}{\delta\phi} = e^{-\phi}\left\{-\alpha_R R+\alpha_\Phi\left[g^{ij}\,\partial_i\phi\,\partial_j\phi-2\Delta\phi\right]-\alpha_\Lambda\right\}, \tag{10.17}$$

with $\mathcal{F}$ again given by

$$\mathcal{F}=\int dt\,d^Dx\,\sqrt{g}e^{-\phi}\left\{\alpha_R R^{(g)}+\alpha_\Phi g^{ij}\partial_i\phi\partial_j\phi+\alpha_\Lambda\right\}. \tag{10.18}$$

In the process of deriving $\delta\mathcal{F}/\delta g_{ij}$, it is important to recall the correct formula for the variation of the Einstein-Hilbert scalar curvature term,

$$\delta(\sqrt{g}R)=-\sqrt{g}\left(R^{ij}-\frac{1}{2}Rg^{ij}\right)\delta g_{ij}+\sqrt{g}\left(g^{ik}g^{j\ell}-g^{ij}g^{k\ell}\right)\nabla_i\nabla_j\delta g_{k\ell}. \tag{10.19}$$

It then follows from the form of our component action that the localization equations are

$$e^\phi\nabla_t g_{ij} = e^\phi\left(g_{ik}g_{j\ell}-\widetilde{\lambda}g_{ij}g_{k\ell}\right)\mathscr{E}^{k\ell}\equiv-\alpha_R R_{ij}+\frac{\alpha_R}{2}\left[1-\widetilde{\lambda}(D-2)\right]g_{ij}R$$
$$+(\alpha_R-\alpha_\Phi)\partial_i\phi\partial_j\phi+\left[\left(\frac{\alpha_\Phi}{2}-\alpha_R\right)(1-\widetilde{\lambda}D)+(\alpha_\Phi-\alpha_R)\widetilde{\lambda}\right]g_{ij}(\partial\phi)^2$$
$$+\alpha_R\left[1-\widetilde{\lambda}(D-1)\right]g_{ij}\Delta\phi-\alpha_R\nabla_i\partial_j\phi+\frac{\alpha_\Lambda}{2}(1-\widetilde{\lambda}D)g_{ij}, \tag{10.20}$$

$$\lambda_\Phi e^\phi\nabla_t\phi = -\alpha_R R+\alpha_\Phi\left[g^{ij}\,\partial_i\phi\,\partial_j\phi-2\Delta\phi\right]-\alpha_\Lambda. \tag{10.21}$$

We see that in this theory, the previously "missing" localization equation for ϕ has been supplied by our choice of the gauge-fixing fermion leading to the action (10.11) and (10.12).

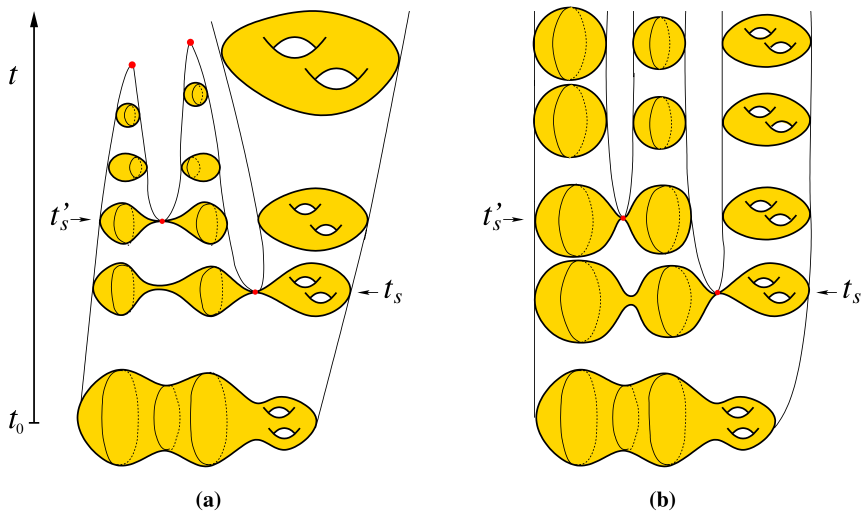

Figure 10.1: A typical qualitative example of a solution of Perelman's Ricci flow equations in $3+1$ dimensions, which includes both topology-changing transitions at time instants t_s and t'_s and Perelman's extinction of positively curved spatial regions. **(a):** The evolution in the Perelman frame, and **(b):** in our frame.

10.2 Reframing to Perelman's variables

The main conclusion from our investigations of the covariant theory in Section 9 was that our variables g_{ij} and ϕ are related to Perelman's variables $\hat{g}_{ij}$ and $\hat{\phi}$ by

$$\hat{g}_{ij} = e^{\phi} g_{ij}, \tag{10.22}$$

$$\hat{\phi} = \frac{D}{2}\phi. \tag{10.23}$$

In addition, the shift vector transforms trivially, $\hat{n}^i = n^i$. We will accept these relations here as well. Its strongest motivation comes from our desire to see Perelman's volume condition take the simple form of a covariant constancy of $\sqrt{g}$ in our variables, so that we can dynamically impose it by taking the $|\lambda| \to \infty$ limit.

As an aside remark, we note that the reframing transformation (10.22) from Perelman's metric to ours will have an interesting effect on the solutions of Perelman's Ricci flow equations. Consider the case of 3+1 dimensions, for which the most detailed information is available in the mathematical literature. As we briefly reviewed in the beginning of Part I, under the influence of Perelman' Ricci flow, spatial geometries with positive sectional curvatures round themselves out with time and shrink to an extinguishing singularity in finite time. In contrast, hyperbolic spatial geometries with negative sectional curvatures expand forever. Besides the extinguishing singularities of positively-curved regions, the geometries can also go through topology-changing "neckpinch" singularities. We have illustrated such a generic evolution of an initial spatial geometry in Figure 10.1(a). All these features are found when the spatial metric is in Perelman's frame, $\hat{g}_{ij}$.

The reframing to our frame g_{ij} has an interesting effect on the qualitative behavior of the solutions. Viewed in our frame, the positively-curved regions round themselves up, but approach a constant radius limit asymptotically, as $t \to \infty$. Similarly, the metric of the hyperbolic regions also approaches a stationary limit as $t \to \infty$. In contrast, the topology-changing "neckpinch" singularities still happen at finite time. We illustrate this qualitative behavior in our frame in Figure 10.1(b).

This contrast between the spacetime geometry viewed from different frames does not mean that the spacetime configuration is somehow different between the two frames: It just shows, in the context of the Ricci flow, that viewing the same geometric solution in different frames may reveal new features, often difficult to see if one insists on one preferred frame. This phenomenon is well-understood in string theory, where the same spacetime geometry can be probed by different probes (such as strings, or branes of various dimensions), revealing complementary information about the same solution of the theory. It is pleasing to see a similar behavior in the topological quantum gravity of the Ricci flow.

10.3 Gravity at farpoint: Taking the $|\lambda| \to \infty$ limit

We know that the desired fixed-volume condition in our variables will be dynamically imposed when we take the "farpoint" limit of the kinetic coupling λ, taking

$$\lambda \to \pm\infty. \tag{10.24}$$

At the level of the localization equations, either of these two limits is permissible: In fact, they both lead to the same localization equations. Indeed, in terms of the dual coupling $\widetilde{\lambda}$, both cases correspond to the same value,

$$\widetilde{\lambda} = \frac{1}{D}. \tag{10.25}$$

While the localization equations are formally identical in both limits $\lambda \to \pm\infty$, the two cases differ from the perspective of the path integral. To see that, let us integrate out the auxiliary fields B_{ij} and $\mathscr{B}$ in the path integral. The reduced action is now a sum of squares of the localization equations,

$$S \approx \int dt\, d^D x \sqrt{g} \left\{ \frac{1}{4} \left(g^{ik} g^{j\ell} - \lambda g^{ij} g^{k\ell} \right) (\nabla_t g_{ij} - \dots)(\nabla_t g_{k\ell} - \dots) + \frac{\lambda_\Phi}{4} (\nabla_t \phi - \dots)^2 \right\}, \tag{10.26}$$

where the "..." stand for the right-hand sides of the localization equations determining $\nabla_t g_{ij}$ and $\nabla_t \phi$. We see that for $\lambda \to -\infty$, and with $\lambda_\Phi > 0$, the action is manifestly $S \geq 0$, with the equality saturated exactly for those bosonic configurations that satisfy the flow equations. This is then the preferred value to take, in order to make the path integral well-defined. Taking the other limit, $\lambda \to \infty$, would put us in a situation similar to the one encountered in general relativity, where the Euclidean action is not bounded from below and the path integral requires a subtle analytic continuation.

When the $|\lambda| \to \infty$ limit is taken, our localization equations (10.20) and (10.21) reduce to the more manageable system

$$\begin{aligned} e^\phi \nabla_t g_{ij} &= -\alpha_R R_{ij} + \frac{\alpha_R}{2} g_{ij} R + \frac{\alpha_R}{D} g_{ij} \Delta\phi - \alpha_R \nabla_i \partial_j \phi \\ &\quad + (\alpha_R - \alpha_\Phi) \partial_i \phi \partial_j \phi + \frac{1}{D} (\alpha_\Phi - \alpha_R) g_{ij} (\partial\phi)^2, \end{aligned} \tag{10.27}$$

$$\lambda_\Phi e^\phi \nabla_t \phi = -\alpha_R R + \alpha_\Phi \left[g^{ij}\, \partial_i \phi\, \partial_j \phi - 2\Delta\phi \right] - \alpha_\Lambda. \tag{10.28}$$

Having determined our full list of localization equations in the "farpoint" limit of $|\lambda| \to \infty$, and having committed to the change-of-frame relation between our variables and Perelman's, we can now determine whether Perelman's Ricci flow corresponds to a particular choice of our couplings α_R, α_Φ and α_Λ after reframing.

10.4 Localization and Perelman's Ricci flow

We consider the covariantized form of Perelman's flow equations,

$$\hat{\nabla}_t \hat{g}_{ij} = -2\hat{R}_{ij} - 2\hat{\nabla}_i \partial_j \hat{\phi}, \tag{10.29}$$

$$\hat{\nabla}_t \hat{\phi} = -\hat{R} - \hat{\Delta}\hat{\phi}. \tag{10.30}$$

In Perelman gauge $\hat{n}^i = 0$, these reduce back to Perelman's original system (8.2) and (8.3). We begin with the scalar flow equation (10.30) for $\hat{\phi}$. Does it match our scalar localization equation (10.28) after reframing? This is a nontrivial check: Notice that both the $\Delta\phi$ term and the $(\partial\phi)^2$ term on the right-hand side of our (10.28) are controlled by the *same* coupling α_Φ. If the reframing of the right-hand side of Perelman's scalar flow equation (10.30) yields the two terms $\Delta\phi$ and $(\partial\phi)^2$ with a relative coefficient different than -2, our program would fail. Happily, applying our reframing formulas from Appendix B gives

$$\hat{R} + \hat{\Delta}\hat{\phi} = e^{-\phi}\left\{R + \frac{D-2}{4}\left[(\partial\phi)^2 - 2\Delta\phi\right]\right\}. \tag{10.31}$$

Comparing this to our scalar flow (10.28), we obtain the ratio of α_R and α_Φ in our frame's Lagrangian,

$$\alpha_\Phi = \frac{2-D}{4}\alpha_R. \tag{10.32}$$

This is indeed the same ratio of the couplings which was predicted in (9.26)! In fact, we can set $\alpha_R = 2$ by convention, and use (10.32) to determine that $\alpha_\Phi = (2-D)/2$, as given in (9.26). We must also set the cosmological constant $\alpha_\Lambda = 0$. Finally, we see that the scalar flows will match exactly if we set $\lambda_\Phi = D$.

With these values of the couplings, it remains to see whether our metric flow equation matches Perelman's after reframing. There is no more freedom of choice of any couplings left, so this highly nontrivial check in the metric sector must work identically, for our goal to succeed. The right-hand side of our metric flow equation (10.27) now simplifies to

$$e^\phi\nabla_t g_{ij} = -2R_{ij} + \frac{2}{D}g_{ij}R - \frac{2+D}{2D}g_{ij}(\partial\phi)^2 + \frac{2+D}{2}\partial_i\phi\partial_j\phi + \frac{2}{D}g_{ij}\Delta\phi - 2\nabla_i\partial_j\phi. \tag{10.33}$$

Note that this expression is correctly traceless, as it must, since it corresponds to the $\lambda = \pm\infty$ limit which imposes dynamically the unimodularity condition on g_{ij}. This is a good check of self-consistency of our framework.

The question is whether (10.33) is a reframing of the right-hand side of the Perelman flow equation for the metric. Let us begin with the right-hand side of Perelman's flow equation for the metric:

$$2\hat{R}_{ij} + 2\hat{\nabla}_i\partial_j\hat{\phi}. \tag{10.34}$$

In order to write this expression in our frame, we need to invoke the reframing equation (B.8) from the Appendix. Using this relation, we find that the reframing of

(10.34) is

$$2R_{ij} + 2\nabla_i\partial_j\phi - g_{ij}\Delta\phi - \frac{D+2}{2}\partial_i\phi\partial_j\phi + g_{ij}(\partial\phi)^2. \qquad (10.35)$$

This does not look at all like the right-hand side of (10.33). However, it should not yet look like it, because (10.34) is the right-hand side of Perelman's equation for *his* metric $\hat{g}_{ij}$, while (10.33) is the right-hand side of the flow equation for *our* metric g_{ij}. Since they are related by $\hat{g}_{ij} = e^\phi g_{ij}$, we have

$$\hat{\nabla}_t \hat{g}_{ij} = e^\phi \left(\nabla_t g_{ij} + g_{ij} \nabla_t \phi \right). \qquad (10.36)$$

Hence, to compare (10.34) to (10.33), we must subtract from it $e^\phi g_{ij}\dot{\phi}$, and substitute for $e^\phi \dot{\phi}$ using our scalar flow equation whose right-hand side is in (10.31). This gives

$$\begin{aligned} 2R_{ij} &+ 2\nabla_i\partial_j\phi - g_{ij}\Delta\phi - \frac{D+2}{2}\partial_i\phi\partial_j\phi + g_{ij}(\partial\phi)^2 \\ &- \frac{2}{D} g_{ij} \left\{ R + \frac{D-2}{4}\left[(\partial\phi)^2 - 2\Delta\phi\right] \right\}. \end{aligned} \qquad (10.37)$$

Simple algebra then shows that this expression is

$$2R_{ij} - \frac{2}{D}g_{ij}R + 2\nabla_i\partial_j\phi - \frac{2}{D}g_{ij}\Delta\phi - \frac{D+2}{2}\partial_i\phi\partial_j\phi + \frac{D+2}{2D}g_{ij}(\partial\phi)^2. \qquad (10.38)$$

Happily, this indeed coincides with our $-\mathscr{E}_{ij}$ in (10.33). This concludes the proof that the localization equations of our topological quantum gravity, for the values of the couplings

$$\alpha_R = 2, \quad \alpha_\Phi = \frac{2-D}{2}, \quad \alpha_\Lambda = 0, \quad \lambda = \pm\infty, \quad \lambda_\Phi = D, \qquad (10.39)$$

given by

$$e^\phi \nabla_t g_{ij} = -2R_{ij} + \frac{2}{D}Rg_{ij} - 2\nabla_i\partial_j\phi + \frac{2}{D}\Delta\phi g_{ij} + \frac{D+2}{2}\partial_i\phi\partial_j\phi - \frac{D+2}{2D}(\partial\phi)^2 g_{ij}, \qquad (10.40)$$

$$e^\phi \nabla_t \phi = -\frac{2}{D}R - \frac{D-2}{2D}\left[(\partial\phi)^2 - 2\Delta\phi\right], \qquad (10.41)$$

are exactly equivalent to Perelman's Ricci flow equations (10.29) and (10.30), after the appropriate change of frames. This is the central result of the thesis.

11 Shrinking and expanding solitons: Perelman's $\mathcal{W}$ entropy functional

While Perelman's $\hat{\mathcal{F}}$-functional is at the core of the modern theory of the Ricci flow, it turns out to be best suited for the study of static solitons. The important cases of shrinking and expanding solitons are associated with refined versions of the $\hat{\mathcal{F}}$ functional, known as the $\mathcal{W}$ and $\mathcal{W}_+$ entropy functionals. These satisfy important monotonicity properties along the appropriate Ricci flow.

11.1 Shrinking solitons and the $\mathcal{W}$ entropy functional

In his analysis of shrinking solitons [2], Perelman introduced the entropy $\mathcal{W}$-functional,

$$\mathcal{W} = \int d^D x \sqrt{\hat{g}} \frac{1}{(4\pi\tau)^{D/2}} e^{-\hat{\phi}} \left\{ \tau \left(\hat{R} + \hat{g}^{ij} \partial_i \hat{\phi} \partial_j \hat{\phi} \right) + \hat{\phi} - D \right\}, \tag{11.1}$$

which depends – besides the fields $\hat{g}_{ij}$ and $\hat{\phi}$ – on a projectable field $\tau(t)$. In the original context of [2], this field was playing the role of a spatial scale, related to Perelman's original inspiration from the renormalization group behavior of non-linear sigma models in string theory. We shall comment on the interpretation of this field τ in our topological gravity below.

Using the $\mathcal{W}$-functional instead of the $\mathcal{F}$-functional in the variational principle, and imposing a modified fixed-volume condition which requires that

$$\frac{1}{(4\pi\tau)^{D/2}} e^{-\hat{\phi}} \sqrt{\hat{g}}\, d^D x \tag{11.2}$$

be held fixed with time, one obtains the modified gradient flow equations (see Ch. 6.1 of [12] for detailed derivation),

$$\begin{aligned} \frac{\partial \hat{g}_{ij}}{\partial t} &= -2\hat{R}_{ij} - 2\hat{\nabla}_i \partial_j \hat{\phi}, \\ \frac{\partial \hat{\phi}}{\partial t} &= -\hat{R} - \hat{\Delta}\hat{\phi} + \frac{D}{2\tau}, \\ \frac{\partial \tau}{\partial t} &= -1. \end{aligned} \tag{11.3}$$

The last of these equations is usually solved by setting $\tau = t_0 - t$ for some positive constant t_0, and then τ is usually interpreted in the mathematical literature of the Ricci flow as such a linear function of time.

Just as in the case of the original equations (8.2) and (8.3), these modified equations can be further simplified by DeTurck's trick to

$$\begin{aligned} \frac{\partial \hat{g}_{ij}}{\partial t} &= -2\hat{R}_{ij}, \\ \frac{\partial \hat{\phi}}{\partial t} &= -\hat{R} - \hat{\Delta}\hat{\phi} + \hat{g}^{ij}\partial_i\hat{\phi}\partial_j\hat{\phi} + \frac{D}{2\tau}, \\ \frac{\partial \tau}{\partial t} &= -1, \end{aligned} \tag{11.4}$$

which again separates Hamilton's original metric flow from the flow of the dilaton.

It is natural to ask whether our construction from Section 9 can be extended to accommodate the $\mathcal{W}$-functional and its modified flow equations in our topological quantum gravity.

11.2 Adding the Goldstone superfield T

In the context of quantum gravity, we must first find an interpretation of Perelman's τ function as a dynamical field. In fact, this field can be interpreted in several seemingly distinct ways,[19] which all stem from the simple idea of spontaneous symmetry breaking. The symmetry in question is the symmetry of global time translations. While global time translations may be an isometry of the static Ricci flow solitons most suitable for the $\mathcal{F}$ functional, shrinking or expanding Ricci solitons (or any other time-dependent background solution) will break this symmetry spontaneously. On general grounds, it is natural to expect the presence of a gapless Goldstone mode associated with this global symmetry breaking. A very similar field has played a prominent role in modern cosmology, in the effective field theory of inflation [78–83] (see also [84] for further geometric clarifications).

In order to accommodate Perelman's τ field consistently with our $\mathcal{N}=2$ supersymmetry, we first promote it to a superfield. We choose to introduce a *projectable*

[19]The field τ can be intepreted as the dilaton for anisotropic conformal transformations [33, 44] of spacetime (see Ch. 13.2 of [76] for a particularly lucid discussion of the relation between the dilaton and scale invariance); or it can be interpreted as a compensator field similar to those that appear in relativistic supergravity [77]. It can also be interpreted as the Goldstone field associated with spontaneous breaking of time translation symmetries in the background, analogous to a very similar Goldstone field π that appears prominently in the effective field theory approach to cosmological inflation [78–83].

superfield $T(t,\theta,\overline{\theta})$, otherwise unconstrained, and require its dynamics to be consistent with the condition of a constant shift symmetry,

$$T(t,\theta,\overline{\theta}) \mapsto T(t,\theta,\overline{\theta}) + c. \tag{11.5}$$

This shift symmetry is indeed a hallmark of T being a Goldstone field. While in nonrelativistic systems, shift symmetries allow an intriguing refinement [85, 86] on symmetric backgrounds, for our purposes it is sufficient to consider this simplest case, which is background independent.

Next, we choose the action of the T sector to be of the minimal form. The kinetic term is augmented to

$$S_K = \int d^2\theta\, dt\, d^D x\, \sqrt{G}\, \big\{ (G^{ik}G^{j\ell} - \lambda G^{ij}G^{k\ell})\, \overline{\mathcal{D}}G_{ij}\, \mathcal{D}G_{k\ell} + \lambda_\Phi\, \overline{\mathcal{D}}\Phi\, \mathcal{D}\Phi \big\} + \int dt\, d^2\theta\, \overline{\mathrm{D}}T\, \mathrm{D}T. \tag{11.6}$$

The lowest-derivative kinetic term for T is indeed consistent with the constant shift symmetry. For a projectable T, we also find that no T-dependent terms can be added to the superpotential, which stays the same as before,

$$S_{\mathcal{W}} = \int d^2\theta\, dt\, d^D x\, \sqrt{G} N \left\{ \alpha_R R^{(G)} + \alpha_\Phi G^{ij}\partial_i\Phi\partial_j\Phi + \alpha_\Lambda \right\}. \tag{11.7}$$

Any appearance of T without derivatives would violate the shift symmetry, and spatial derivative terms are not available because T is projectable. Note that we have also required the absence of mixing terms between the T sector and the g_{ij}, n^i and n sector of the theory. Thus, the T sector is trivial and decoupled from the metric geometry. The localization equation for the lowest component field in T simply states that this component field is constant in time.

Using the insights from effective field theory of inflation about the treatment of the Goldstone field associated with spontaneously broken time translations [78, 80, 84], we choose to define the component fields of T as follows,

$$T(t,\theta,\overline{\theta}) = t + \tau(t) + \theta\,\eta(t) + \overline{\theta}\,\overline{\eta}(t) + \theta\overline{\theta}\, b(t). \tag{11.8}$$

Note that we have defined the component field $\tau(t)$ such that the lowest component of the T superfield is $t + \tau(t)$. The pair of real projectable fermions η, $\overline{\eta}$ are the ghost and the antighost, and b is a projectable auxiliary field. Since the BRST charge Q acts on τ as $Q\tau = \eta$, this BRST multiplet clearly originates from an underlying projectable topological symmetry acting on τ,

$$\delta\tau(t) = \mathrm{f}(t), \tag{11.9}$$

with $\mathrm{f}(t)$ an arbitrary real function of t. Together with the underlying gauge symmetries of our topological theory of the metric multiplets, the full gauge symmetry of the theory including the τ sector continues to be non-redundant, and Q continues to play the role of the standard BRST charge.

With this choice of variables, the localization equation in the T sector now gives

$$\frac{\partial}{\partial t}\left[t + \tau(t)\right] = 0, \tag{11.10}$$

which we will rewrite in the following suggestive form,

$$\frac{\partial}{\partial t}\tau = -1. \tag{11.11}$$

This is indeed the last of Perelman's flow equations (11.3) for the shrinking solitons in the context of the $\mathcal{W}$ functional.

In the effective field theory of cosmological inflation [78,80,84], the Goldstone field that corresponds to our τ is traditionally called π; more importantly, in inflationary cosmology this field π is nonprojectable. It is interesting to note that in our construction of topological quantum gravity of the Ricci flow, it is also possible to promote T to a nonprojectable field. This would lead to two modifications of the action: First, the kinetic term needs to be covariantized under spatial diffeomorphisms, and integrated over the entire spacetime:

$$\int dt\, d^D x\, d^2\theta\, \sqrt{G}\, \overline{\mathcal{D}}T\, \mathcal{D}T, \tag{11.12}$$

with the covariant derivatives $\mathcal{D}T$ and $\overline{\mathcal{D}}T$ given by

$$\mathcal{D}T = \mathrm{D}T - S^i\partial_i T, \tag{11.13}$$

$$\overline{\mathcal{D}}T = \overline{\mathrm{D}}T - \overline{S}^i\partial_i T. \tag{11.14}$$

Secondly, when we still keep the constant shift symmetry (11.5), it is now possible to add new terms to the superpotential $S_{\mathcal{W}}$,

$$\int dt\, d^D x\, d^2\theta\, \sqrt{G}\, \{\alpha_T\, \partial_i T \partial_i T + \dots\}\,. \tag{11.15}$$

Here we have indicated just the simplest, lowest-derivative term quadratic in T and consistent with the shift symmetry. In principle, one should also consider the possibility of mixing between the T sector and the metric sector, both in the kinetic and the superpotential terms. The localization equation that corresponds to (11.12) and (11.15) is

$$\nabla_t \tau = -1 + \alpha_T\, \Delta\tau. \tag{11.16}$$

Writing $\tau = -t + \mathscr{T}(t, x^i)$, (11.16) becomes simply the covariant heat equation $\nabla_t \mathscr{T} = \alpha_T \Delta \mathscr{T}$ on Σ: Under the spatial diffeomorphism of Σ, $\mathscr{T}$ transforms as a scalar, and Δ is thus the Laplacian of g_{ij} on scalars. While such nonprojectable extensions of our theory are indeed possible, for the purposes of this thesis we see no advantage in extending the theory to nonprojectable T, and therefore we will consider only the case of projectable T from now on.

11.3 Perelman's equations for shrinking solitons from topological gravity

The transformation between the fields g_{ij} and ϕ of topological gravity and Perelman's variables $\hat{g}_{ij}$ and $\hat{\phi}$ will now have to involve factors of τ. On the topological gravity side, our equations are the same as in Section 9, schematically of the form (8.8). No τ has been introduced yet – the entire dependence on τ will come from rewriting the theory in Perelman's variables. Therefore, in order to match the leading terms on the two sides of (8.8), we must again set

$$\hat{g}_{ij} = e^{\phi} g_{ij}, \tag{11.17}$$

as we did in Section 9.2. We need another relation to determine the change of variables uniquely. The key is again to look at Perelman's fixed-volume condition, (11.2), which suggests that we identify

$$g_{ij} = \frac{1}{4\pi\tau} \hat{g}_{ij} e^{-2\hat{\phi}/D} \tag{11.18}$$

and interpret (11.2) as the condition of time independence of the spatial volume element $\sqrt{g}$ in topological gravity – a condition we know how to ensure dynamically, by going to the $|\lambda| \to \infty$ limit.

Putting these two conditions together, we obtain our transformation rules between the two sets of fields,

$$\hat{g}_{ij} = e^{\phi} g_{ij}, \tag{11.19}$$

$$\hat{\phi} = \frac{D}{2}\left[\phi - \log(4\pi\tau)\right]. \tag{11.20}$$

In addition, the shift vector and the τ field stay unchanged: $\hat{n}^i = n^i$, $\hat{\tau} = \tau$. Note that in the inverse of this transformation, g_{ij} depends explicitly on τ:

$$g_{ij} = \frac{1}{4\pi\tau} e^{-2\hat{\phi}/D} \hat{g}_{ij}, \tag{11.21}$$

$$n = \frac{1}{4\pi\tau} e^{-2\hat{\phi}/D}, \tag{11.22}$$

with our lapse function as always given by $n \equiv e^{-\phi}$.

With the transformation properties determined, we are in the position to rewrite our localization equations (10.40), (10.41) and (11.11) of topological gravity in Perelman's variables. A direct calculation yields

$$\begin{aligned} \hat{\nabla}_t \hat{g}_{ij} &= -2\hat{R}_{ij} - 2\hat{\nabla}_i \partial_j \hat{\phi}, \\ \hat{\nabla}_t \hat{\phi} &= -\hat{R} - \hat{\Delta}\hat{\phi} + \frac{D}{2\tau}, \\ \dot{\tau} &= -1. \end{aligned} \tag{11.23}$$

These equations are indeed the covariantized version of the flow equations (11.3) that Perelman derived from his $\mathcal{W}$ entropy functional! They reduce to (11.3) in Perelman gauge $\hat{n}^i = 0$. Thus, the conclusion is the same as in the case of the $\mathcal{F}$-functional: By taking the $|\lambda| \to \infty$ limit of our topological quantum gravity – augmented now by the decoupled Goldstone superfield T – we get a covariantized version of Perelman's equations (11.3) associated with the $\mathcal{W}$ entropy functional and the shrinking solitons.

What remains to do is to perform the alternate gauge fixing of spatial diffeomorphisms and to go to Hamilton gauge, in order to establish the relation with Perelman's equations (11.4) after the DeTurck trick has been performed on them. The gauge fixing condition turns out to be the same (in either set of variables) as in Section 9.3: In Perelman's variables, Hamilton gauge is

$$\hat{n}_i = \partial_i \hat{\phi}, \tag{11.24}$$

while in our variables it can be equivalently written as

$$n_i = -\frac{D}{2}\partial_i n. \tag{11.25}$$

Note that these relations imply that

$$n_i = \frac{1}{4\pi\tau} e^{-2\hat{\phi}/D}\partial_i\hat{\phi}, \tag{11.26}$$

which together with (11.21) and (11.22) gives the full list of our ADM fields describing the spacetime geometry of topological quantum gravity in Hamilton gauge in terms of Perelman's geometric data.

11.4 Expanding solitons and the $\mathcal{W}_+$ entropy functional

Shortly after Perelman's work, Feldman, Ilmanen and Ni [87] modified the $\mathcal{W}$ entropy functional to a form suitable for expanding Ricci solitons. Their $\mathcal{W}_+$ entropy functional depends on $\hat{g}_{ij}$, $\hat{\phi}$ and instead of $\tau(t)$ a new projectable field $\sigma(t)$ which will now be a *growing* linear function of time. It takes the form

$$\mathcal{W}_+ = \int d^D x\sqrt{\hat{g}}\frac{1}{(4\pi\sigma)^{D/2}}e^{-\hat{\phi}}\left\{\sigma\left(\hat{R} + \hat{g}^{ij}\partial_i\hat{\phi}\partial_j\hat{\phi}\right) - \hat{\phi} + D\right\}. \tag{11.27}$$

The volume-fixing condition is to hold the following measure,

$$\frac{1}{(4\pi\sigma)^{D/2}}e^{-\hat{\phi}}\sqrt{\hat{g}}\, d^D x, \tag{11.28}$$

fixed in time, and the corresponding gradient flow equations are

$$\begin{aligned} \frac{\partial\hat{g}_{ij}}{\partial t} &= -2\hat{R}_{ij} - 2\hat{\nabla}_i\partial_j\hat{\phi}, \\ \frac{\partial\hat{\phi}}{\partial t} &= -\hat{R} - \hat{\Delta}\hat{\phi} - \frac{D}{2\sigma}, \\ \frac{\partial\sigma}{\partial t} &= +1. \end{aligned} \tag{11.29}$$

The last of these equations is solved by setting $\sigma = t - t_0$ for some constant t_0, and that is how σ is often interpreted in the mathematical literature. After DeTurck's trick, these equations are simplified to

$$\begin{aligned}
\frac{\partial \hat{g}_{ij}}{\partial t} &= -2\hat{R}_{ij}, \\
\frac{\partial \hat{\phi}}{\partial t} &= -\hat{R} - \hat{\Delta}\hat{\phi} + \hat{g}^{ij}\partial_i\hat{\phi}\partial_j\hat{\phi} - \frac{D}{2\sigma}, \\
\frac{\partial \sigma}{\partial t} &= +1.
\end{aligned} \tag{11.30}$$

It is easy to see how this set of equations can be reproduced in our topological quantum gravity. Using the same action (11.6) and (11.7) as in the case of shrinking solitons, we now define the components of the Goldstone superfield T as

$$T(t,\theta,\overline{\theta}) = t - \sigma(t) + \theta\,\eta(t) + \overline{\theta}\,\overline{\eta}(t) + \theta\overline{\theta}\,b(t). \tag{11.31}$$

In terms of these components, the localization equation for σ reads

$$\frac{\partial \sigma}{\partial t} = +1, \tag{11.32}$$

which reproduces the last equation in (11.30). Next, we propose the following change of variables from our fields to those of the $\mathcal{W}_+$ functional,

$$\hat{g}_{ij} = e^{\phi} g_{ij}, \tag{11.33}$$

$$\hat{\phi} = \frac{D}{2}\left[\phi - \log(4\pi\sigma)\right], \tag{11.34}$$

again keeping the lapse n^i and the projectable field σ unchanged: $\hat{n}^i = n^i$, $\hat{\sigma} = \sigma$. In these Perelman-like variables, our localization equations (10.40), (10.41) and (11.32) are found to be

$$\begin{aligned}
\hat{\nabla}_t \hat{g}_{ij} &= -2\hat{R}_{ij} - 2\hat{\nabla}_i\partial_j\hat{\phi}, \\
\hat{\nabla}_t \hat{\phi} &= -\hat{R} - \hat{\Delta}\hat{\phi} - \frac{D}{2\sigma}, \\
\dot{\sigma} &= +1.
\end{aligned} \tag{11.35}$$

Thus, we again find the perfect match between the localization equations of topological quantum gravity, and the covariantized flow equations for the expanding solitons associated with the $\mathcal{W}_+$ functional. Going to Perelman or Hamilton gauge will again establish the isomorphism with the $\mathcal{W}_+$ flow equations (11.29) before or (11.30) after

DeTurck's trick.

12 Summary

In this part, we identified the precise regime of nonrelativistic topological quantum gravity of Part I, in which the localization equations in the path integral are identical to Perelman's celebrated Ricci flow equations. This map involves an interesting change of frames between Perelman's geometric variables and ours. Perelman's fixed-volume condition is implemented by taking the "farpoint" limit $\lambda \to -\infty$ in the kinetic coupling of our nonrelativistic topological gravity.

When the localization equations correspond to the Ricci flow equations, the quantum theory exhibits anisotropic scaling between time and space, characterized by dynamical exponent $z = 2$, for any spacetime dimension $D + 1$. Such a theory would be power-counting renormalizable for $D = 2$, but its mathematical structure is richer and more relevant to deep questions of topology and geometry when studied for $D = 3$. This raises an intriguing question of a short-distance completeness, and possibly renormalizability of this quantum field theory that goes beyond naive perturbative power counting. In which dimensions is our topological quantum gravity UV complete? Does its topological BRST symmetry play a role in improving the short-distance behavior of the path integral? Such questions remain open for closer examination.

Perhaps the main importance of the precise embedding of Perelman's Ricci flow theory into topological nonrelativistic quantum gravity that we found stems from the fact that it sets the stage for importing the remarkable wealth of mathematical results accumulated in the theory of Perelman's Ricci flow over the past two decades into the physical setting of quantum gravity, at least in the relatively well-controlled situation of a topological theory with no local propagating degrees of freedom. The structure of solutions of Perelman's equations is well-studied, and exhibits many fascinating dynamical features, including topology-changing processes and other deeply nonequilibrium phenomena, not only in the most analyzed case of $3 + 1$ spacetime dimensions. The frequent appearance of various "entropy functionals" with precise monotonicity properties along the general flow is also begging for an explanation more directly grounded in the physical arguments of quantum gravity and quantum field theory. For instance, in this thesis we found an intimate relation between the $\mathcal{W}$ and $\mathcal{W}_+$ entropy functionals associated with the mathematical theory of the shrinking and expanding Ricci solitons, and the important physical concept of spontaneous symmetry breaking. We expect that future investigations will continue this process of mutual illumination between the physical and mathematical aspects of the theory.

References

[1] R. Hamilton, *Three-manifolds with positive Ricci curvature*, *J. Diff. Geom.* **17** (1982) 255.

[2] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159.

[3] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math/0303109.

[4] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math/0307245.

[5] X. Chen, P. Lu and G. Tian, *A note on uniformization of Riemann surfaces by Ricci flow*, *Proc. Amer. Math. Soc.* **134** (2006) 3391 [arXiv:math/0505163].

[6] S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, *Ann. Math.* **74** (1961) 391.

[7] R. Bamler and B. Kleiner, *Uniqueness and stability of Ricci flow through singularities*, arXiv:1709.04122.

[8] R. Bamler and B. Kleiner, *Ricci flow and diffeomorphism groups of 3-manifolds*, arXiv:1712.06197.

[9] B. Kleiner and J. Lott, *Singular Ricci flows I*, arXiv:1408.2271.

[10] B. Kleiner and J. Lott, *Singular Ricci flows II*, arXiv:1804.03265.

[11] B. Chow and D. Knopf, *The Ricci Flow: An Introduction*, Mathematical Surveys and Monographs **110**. American Mathematical Society (Providence, RI), 2004.

[12] B. Chow et al, *The Ricci Flow: Techniques and Applications. Part I: Geometric Aspects*, Mathematical Surveys and Monographs **135**. American Mathematical Society (Providence, RI), 2007.

[13] B. Chow et al, *The Ricci Flow: Techniques and Applications. Part II: Analytic Aspects*, Mathematical Surveys and Monographs **144**. American Mathematical Society (Providence, RI), 2008.

[14] B. Chow et al, *The Ricci Flow: Techniques and Applications. Part III: Geometric-Analytic Aspects*, Mathematical Surveys and Monographs **163**. American Mathematical Society (Providence, RI), 2010.

[15] B. Chow et al, *The Ricci Flow: Techniques and Applications. Part IV: Long-Time Solutions and Related Topics*, Mathematical Surveys and Monographs **206**. American Mathematical Society (Providence, RI), 2015.

[16] R. S. Hamilton, *The formation of singularities in the Ricci flow*, *Surv. Diff. Geom.* **2** (1995) 7.

[17] B. Kleiner and J. Lott, *Notes on Perelman's papers*, *Geom. Topol.* **12** (2008) 2587 [arXiv:math/0605667].

[18] J. W. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*, arXiv:math/0607607.

[19] J. Morgan and G. Tian, *Ricci Flow and the Poincaré Conjecture*, Clay Mathematics Monographs **3**. American Mathematical Society, Clay Mathematics Institute, 2007.

[20] J. W. Morgan and G. Tian, *Completion of the proof of the geometrization conjecture*, arXiv:0809.4040.

[21] J. W. Morgan and G. Tian, *Correction to Section 19.2 of Ricci flow and the Poincaré conjecture*, arXiv:1512.00699.

[22] P. Topping, *Lectures on the Ricci flow*, LMS Lecture Note Series **325**. Cambridge University Press, 2006.

[23] B. Chow, P. Lu and L. Ni, *Hamilton's Ricci Flow*, Graduate Studies in Mathematics **77**. American Mathematical Society, Science Press, 2006.

[24] R. Müller, *Differential Harnack inequalities and the Ricci flow*, EMS Series of Lectures in Mathematics. European Mathematical Society, 2006.

[25] T. Tao, *Perelman's proof of the Poincaré Conjecture: A nonlinear PDE perspective*, arXiv:math/0610903.

[26] Editors:, H. Cao, B. Chow, S. Chu and S. Yau, *Collected Papers on Ricci Flow*, Series in Geometry and Topology **37**. International Press, 2003.

[27] E. Witten, *Topological quantum field theory*, *Commun. Math. Phys.* **117** (1988) 353.

[28] E. Witten, *Topological sigma models*, *Commun. Math. Phys.* **118** (1988) 411.

[29] E. Witten, *Topological gravity*, *Phys. Lett.* **B206** (1988) 601.

[30] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*. Princeton University Press, 1992.

[31] E. Witten, *Introduction to cohomological field theories*, *Int. J. Mod. Phys.* **A6** (1991) 2775.

[32] P. Hořava, *Membranes at quantum criticality*, *JHEP* **03** (2009) 020 [arXiv:0812.4287].

[33] P. Hořava, *Quantum gravity at a Lifshitz point*, *Phys. Rev.* **D79** (2009) 084008 [arXiv:0901.3775].

[34] P. Hořava, *General covariance in gravity at a Lifshitz point*, *Class. Quant. Grav.* **28** (2011) 114012 [arXiv:1101.1081].

[35] J. Ambjørn, A. Görlich, S. Jordan, J. Jurkiewicz and R. Loll, *CDT meets Hořava-Lifshitz gravity*, *Phys. Lett.* **B690** ((2010)) 413 [arXiv:1002.3298].

[36] J. Ambjørn, A. Görlich, J. Jurkiewicz and R. Loll, *Quantum gravity via causal dynamical triangulations*, in *Springer Handbook of Spacetime* (A. Ashtekar and V. Petkov, eds.), pp. 723–741. Springer, (2014). arXiv:1302.2173.

[37] J. Ambjørn, A. Görlich, J. Jurkiewicz and R. Loll, *Causal dynamical triangulations and the search for a theory of quantum gravity*, in *Proceedings, 13th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories (MG13): Stockholm, Sweden, July 1-7, 2012*, pp. 120–137, (2015), arXiv:1305.6680.

[38] S. Mukohyama, *Hořava-Lifshitz Cosmology: A Review*, Class. Quant. Grav. **27** (2010) 223101 [arXiv:1007.5199].

[39] C. T. C. Wall, *Surgery on compact manifolds*, Mathematical surveys and monographs; No. 69. American Mathematical Society, 1999.

[40] A. A. Tseytlin, *On sigma model RG flow, "central charge" action and Perelman's entropy*, Phys. Rev. **D75** (2007) 064024 [hep-th/0612296].

[41] M. Headrick and T. Wiseman, *Ricci flow and black holes*, Class. Quant. Grav. **23** (2006) 6683 [arXiv:hep-th/0606086].

[42] P. Hořava and C. M. Melby-Thompson, *General covariance in quantum gravity at a Lifshitz point*, Phys. Rev. **D82** (2010) 064027 [arXiv:1007.2410].

[43] R. L. Arnowitt, S. Deser and C. W. Misner, *The dynamics of general relativity;* in *Gravitation: An Introduction to Current Research,* ed: L. Witten (Wiley, 1962)*;* reprinted, Gen. Rel. Grav. **40** (2008) 1997 [arXiv:gr-qc/0405109].

[44] P. Hořava and C. M. Melby-Thompson, *Anisotropic conformal infinity*, Gen. Rel. Grav. **43** (2011) 1391 [arXiv:0909.3841].

[45] P. Hořava, *Topological strings and QCD in two dimensions*, in *NATO Advanced Research Workshop on New Developments in String Theory, Conformal Models and Topological Field Theory; Cargèse, France*, 1993, arXiv:hep-th/9311156.

[46] P. Hořava, *Topological rigid string theory and two-dimensional QCD*, Nucl. Phys. **B463** (1996) 238 [arXiv:hep-th/9507060].

[47] S. Cordes, G. W. Moore and S. Ramgoolam, *Large N 2D Yang-Mills theory and topological string theory*, Commun. Math. Phys. **185** (1997) 543 [arXiv:hep-th/9402107].

[48] P. Hořava, *On QCD string theory and AdS dynamics*, JHEP **01** (1999) 016 [arXiv:hep-th/9811028].

[49] M. Blau and G. Thompson, *N=2 topological gauge theory, the Euler characteristic of moduli spaces, and the Casson invariant*, Commun. Math. Phys. **152** (1993) 41 [arXiv:hep-th/9112012].

[50] R. Dijkgraaf and G. W. Moore, *Balanced topological field theories*, *Commun. Math. Phys.* **185** (1997) 411 [arXiv:hep-th/9608169].

[51] G. Curci and R. Ferrari, *Slavnov transformations and supersymmetry*, *Phys. Lett. B* **63** (1976) 91.

[52] I. Ojima, *Another BRS Transformation*, *Prog. Theor. Phys.* **64** (1980) 625.

[53] R. Dijkgraaf, D. Orlando and S. Reffert, *Relating field theories via stochastic quantization*, *Nucl. Phys.* **B824** (2010) 365 [arXiv:0903.0732].

[54] D. Orlando and S. Reffert, *On the renormalizability of Hořava-Lifshitz-type gravities*, *Class. Quant. Grav.* **26** (2009) 155021 [arXiv:0905.0301].

[55] A. U. O. Kisisel, O. Sarioglu and B. Tekin, *Cotton flow*, *Class. Quant. Grav.* **25** (2008) 165019 [arXiv:0803.1603].

[56] N. Lashkari and A. Maloney, *Topologically Massive Gravity and Ricci-Cotton Flow*, *Class. Quant. Grav.* **28** (2011) 105007 [arXiv:1007.1661].

[57] G. Gibbons and S. Hawking, eds., *Euclidean quantum gravity*. World Scientific, Singapore, 1994.

[58] J. Wess and B. Zumino, *Supergauge invariant extension of quantum electrodynamics*, *Nucl. Phys.* **B78** (1974) 1.

[59] P. C. West, *Introduction to supersymmetry and supergravity*. World Scientific, 1990.

[60] J. Wess and J. Bagger, *Supersymmetry and supergravity*. Princeton University Press, Princeton, NJ, USA, 1992.

[61] Z. Bern, J. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev. D* **78** (2008) 085011 [arXiv:0805.3993].

[62] Z. Bern, J. J. M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [arXiv:1004.0476].

[63] Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, *The duality between color and kinematics and its applications*, arXiv:1909.01358.

[64] D. M. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, *J. Diff. Geom.* **18** (1983) 157.

[65] P. Hořava and C. J. Mogni, *Large-N expansion and string theory out of equilibrium*, arXiv:2008.11685.

[66] J. Isenberg, R. Mazzeo and N. Sesum, *Ricci flow in two dimensions*, arXiv:1103.4669.

[67] J. Song and B. Weinkove, *Lecture notes on the Kähler-Ricci flow*, arXiv:1212.3653.

[68] J. Preskill, *Lecture notes on quantum computation,* available at http://theory.caltech.edu/ preskill/ph229/, 2020.

[69] E. Witten, *A mini-introduction to information theory*, *Riv. Nuovo Cim.* **43** (2020) 187 [arXiv:1805.11965].

[70] J. L. Synge, *A plea for chronometry*, *The New Scientist* **5** (1959) 410.

[71] M. Jammer, *Concepts of time in physics: A synopsis* (the 2007 Abraham Pais Prize Lecture), *Phys. Perspect.* **9** (2007) 266.

[72] M. Henneaux and C. Teitelboim, *Comment on Lagrangians and symmetries*, *Phys. Lett. B* **234** (1990) 487.

[73] C. Teitelboim, *The Hamiltonian structure of space-time*, in *General Gravity and Gravitation, Ed: A. Held, Vol. 1*, 1981.

[74] M. Henneaux, *Geometry of zero signature space-times*, *Bull. Soc. Math. Belg.* **31** (1979) 47.

[75] A. Gumrukcuoglu and S. Mukohyama, *Horava-Lifshitz gravity with $\lambda \to \infty$*, *Phys. Rev. D* **83** (2011) 124033 [arXiv:1104.2087].

[76] M. B. Green, J. Schwarz and E. Witten, *Superstring Theory. Vol. 2: Lool Amplitudes, Anomalies and Phenomenology*. Cambridge University Press, 1988.

[77] W. Siegel, *Fields*, arXiv:hep-th/9912205.

[78] D. Baumann and L. McAllister, *Inflation and String Theory*, Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2015, [arXiv:1404.2601].

[79] P. Creminelli, M. A. Luty, A. Nicolis and L. Senatore, *Starting the universe: Stable violation of the null energy condition and non-standard cosmologies*, *JHEP* **12** (2006) 080 [arXiv:hep-th/0606090].

[80] C. Cheung, P. Creminelli, A. Fitzpatrick, J. Kaplan and L. Senatore, *The effective field theory of inflation*, *JHEP* **03** (2008) 014 [arXiv:0709.0293].

[81] S. Weinberg, *Quantum contributions to cosmological correlations*, *Phys. Rev. D* **72** (2005) 043514 [arXiv:hep-th/0506236].

[82] S. Weinberg, *Quantum contributions to cosmological correlations. II. Can these corrections become large?*, *Phys. Rev. D* **74** (2006) 023508 [arXiv:hep-th/0605244].

[83] S. Weinberg, *Effective field theory for inflation*, *Phys. Rev. D* **77** (2008) 123541 [arXiv:0804.4291].

[84] Y. Hidaka, T. Noumi and G. Shiu, *Effective field theory for spacetime symmetry breaking*, *Phys. Rev. D* **92** (2015) 045020 [arXiv:1412.5601].

[85] T. Griffin, K. T. Grosvenor, P. Hořava and Z. Yan, *Scalar field theories with polynomial shift symmetries*, *Commun. Math. Phys.* **340** (2015) 985 [arXiv:1412.1046].

[86] P. Hořava, *Surprises with nonrelativistic naturalness*, *Int. J. Mod. Phys. D* **25** (2016) 1645007 [arXiv:1608.06287].

[87] M. Feldman, T. Ilmanen and L. Ni, *Entropy and reduced distance for Ricci expanders*, *J. Geom. Anal.* **15** (2005) 49 [arXiv:math/0405036].

Appendices

A Prepotentials for the lapse and shift superfields

In order to gauge spatial diffeomorphisms and time reparametrizations, we introduced superfields $N^i, S^i, \overline{S}^i$ and $E, \Theta, \overline{\Theta}$ respectively. These superfields satisfy a complicated set of mutual constraints. In order to make the superspace formulation simpler, especially in the quantum case, it would be beneficial to solve the constraints and express these constrained superfields in terms of unconstrained prepotential superfields. The purpose of this Appendix is to identify such prepopotentials, both for the lapse and for the shift sector.

A.1 Prepotential for the supervielbein

Consider first the projectable Type B theory. Introduce an unconstrained projectable superfield $U(t, \theta, \overline{\theta})$, the *prepotential* for the projectable supervielbein. Θ and $\overline{\Theta}$ are given by

$$\Theta = -\frac{\mathrm{D}U}{1+\dot{U}}, \qquad \overline{\Theta} = -\frac{\overline{\mathrm{D}}U}{1+\dot{U}}. \tag{A.1}$$

Such Θ and $\overline{\Theta}$ satisfy their nonlinear constraints. E then follows by plugging these expressions into the constraint that expresses E in terms of $\Theta, \overline{\Theta}$ and their derivatives:

$$E = \frac{1}{1+\dot{U}}. \tag{A.2}$$

It seems appropriate to refer to the prepotential U of the lapse sector as "prelapse."

The extension to the nonprojectable Type B case is straightforward. $U(t, \theta, \overline{\theta}, x^k)$ is now an unconstrained nonprojectable superfield, and

$$\Theta = -\frac{\mathrm{D}U - S^k\partial_k U}{1+\dot{U} - N^j\partial_j U}, \qquad \overline{\Theta} = -\frac{\overline{\mathrm{D}}U - \overline{S}^k\partial_k U}{1+\dot{U} - N^j\partial_j U}. \tag{A.3}$$

These expressions satisfy the full nonprojectable constraints (3.25-3.27), and give E in terms of U.

The gauge transformations of the prepotential are

$$\delta U = F + F\dot{U} + \Xi^k\partial_k U, \tag{A.4}$$

and they correctly imply the standard gauge transformations for $\Theta, \overline{\Theta}$ and E.

Note that in (A.3), the constrained superfields $S^i, \overline{S}^i$ and N^i of the shift sector appear explicitly. In order to get an expression for the nonprojectable lapse superfieds in terms of only unconstrained superfields, we now have to find the prepotentials V for the shift sector, express $S^i, \overline{S}^i$ and N^i in terms of V, and substitute back in (A.3).

A.2 Prepotential for the shift superfields

Consider the shift superfields N^i, S^i and $\overline{S}^i$ of Type B theory. They can be expressed in terms of an unconstrained superfield prepotential V^i as follows. Denote by ∂V the matrix $\partial_k V^i$, and by $\mathbb{I}$ the unit matrix δ^i_k. Write

$$S^i = \mathrm{D}V^k \left(\frac{1}{\mathbb{I}+\partial V}\right)_k^{\ i}, \qquad \overline{S}^i = \overline{\mathrm{D}}V^k \left(\frac{1}{\mathbb{I}+\partial V}\right)_k^{\ i}. \tag{A.5}$$

These expressions again imply that the constraints on S^i and $\overline{S}^i$ are satisfied, and N^i is then expressed in terms of V^i via the constraints that gives N^i in terms of $S^i, \overline{S}^i$ and their derivatives. The vector prepotential transforms under the gauge symmetries as

$$\delta V^i = \Xi^i + F\dot{V}^i + \Xi^k \partial_k V^i. \tag{A.6}$$

While these expressions for the gauge superfields in terms of the prepotential superfields look quite simple, they are rather nonlocal and perhaps of limited practical use.

B Collection of the reframing formulas

The map between Perelman's metric $\hat{g}_{ij}$ and dilaton $\hat{\phi}$ on one hand, and our fields g_{ij} and ϕ on the other is given by a nonlinear transformation of the fields, which involves a change of frame of the metric. Such changes of frame are common in theories of gravity coupled to scalar fields, in particular in string theory. It is well-known that different geometric probes (such as branes of diverse dimensions) may be naturally probing the spacetime geometry in distinct frames.

In the present context of nonrelativistic quantum gravity, the slight novelty of the change of frames stems from the fact that we are reframing the *spatial* metric g_{ij}, and that the role of the spatial scalar field is played by the (logarithm of the) lapse function n. Of course, the formulas for the transformation properties of various geometric objects under such a change of frame are standard and well-understood. We collect them in this Appendix simply for convenience and completeness, given the prominent role that they play in the precise comparison between our theory and Perelman's equations in the bulk of the thesis.

We begin with a spatial metric g_{ij} on a D-dimensional manifold Σ, and a field ϕ which transforms under the spatial diffeomorphims of Σ as a scalar. A "change of frame" from g_{ij} to $\widetilde{g}_{ij}$ is defined as the transformation

$$\widetilde{g}_{ij} = e^{\alpha\phi} g_{ij}, \tag{B.1}$$

for some real constant α. We will systematically denote all the geometric objects in the new frame $\widetilde{g}_{ij}$ by $\widetilde{\ }$. The tilde and un-tilde quantities are related as follows. The volume element is given by

$$\sqrt{\widetilde{g}} = e^{\alpha D\phi/2}\sqrt{g}. \tag{B.2}$$

The Christoffel symbols of the Levi-Cività connections of $\widetilde{g}_{ij}$ and g_{ij} are related by

$$\widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \frac{\alpha}{2}\left(\delta^k_j\,\partial_i\phi + \delta^k_i\,\partial_j\phi - g_{ij}g^{k\ell}\partial_\ell\phi\right). \tag{B.3}$$

The Riemann tensor is given by

$$\begin{aligned}\widetilde{R}^i{}_{jk\ell} &= R^i{}_{jk\ell} + \frac{\alpha}{2}\left(\delta^i_\ell\nabla_k\partial_j\phi - \delta^i_k\nabla_\ell\partial_j\phi + g_{jk}g^{im}\nabla_\ell\partial_m\phi - g_{j\ell}g^{im}\nabla_k\partial_m\phi\right)\\ &\quad + \frac{\alpha^2}{4}\left(\delta^i_k\,\partial_\ell\phi\,\partial_j\phi - \delta^i_\ell\,\partial_k\phi\,\partial_j\phi + \delta^i_\ell\,g_{jk}g^{ms}\partial_m\phi\,\partial_s\phi\right.\\ &\quad \left.-\delta^i_k\,g_{j\ell}g^{ms}\,\partial_m\phi\,\partial_s\phi + g_{j\ell}g^{is}\,\partial_s\phi\,\partial_k\phi - g_{jk}g^{is}\,\partial_s\phi\,\partial_\ell\phi\right),\end{aligned} \tag{B.4}$$

the Ricci tensor by

$$\begin{aligned}\widetilde{R}_{ij} &= R_{ij} + \frac{\alpha}{2}\left\{(2-D)\nabla_i\partial_j\phi - g_{ij}\Delta\phi\right\}\\ &\quad + \frac{\alpha^2}{4}\left\{(D-2)\partial_i\phi\,\partial_j\phi + (2-D)g_{ij}(g^{k\ell}\partial_k\phi\,\partial_\ell\phi)\right\},\end{aligned} \tag{B.5}$$

and the Ricci scalar is

$$\widetilde{R} = e^{-\alpha\phi}\left\{R + \alpha(1-D)\Delta\phi - \frac{\alpha^2(D-2)(D-1)}{4}g^{ij}\partial_i\phi\,\partial_j\phi\right\}. \tag{B.6}$$

The Laplace operator on scalars transforms as

$$\widetilde{\Delta} f = e^{-\alpha\phi} \left\{ \Delta f + \frac{\alpha(D-2)}{2} g^{ij} \partial_i \phi \, \partial_j f \right\} . \tag{B.7}$$

Finally, in the bulk of the thesis we also need the relation between the uncontracted second derivatives on a scalar,

$$\widetilde{\nabla}_i \partial_j f = \nabla_i \partial_j f + \frac{\alpha}{2} g_{ij} (g^{k\ell} \partial_k \phi \, \partial_\ell f) - \frac{\alpha}{2} (\partial_i \phi \, \partial_j f + \partial_j \phi \, \partial_i f). \tag{B.8}$$