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Abstract

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Given a punctured Riemann surface with a pair-of-pants decomposition, we compute its wrapped Fukaya category in a suitable model by reconstructing it from those of various pairs of pants. The pieces are glued together in the sense that the restrictions of the wrapped Floer complexes from two adjacent pairs of pants to their adjoining cylindrical piece agree. The A_∞-structures are given by those in the pairs of pants. The category of singularities of the mirror Landau-Ginzburg model can also be constructed in the same way from local affine pieces that are mirrors of the pairs of pants.

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Chapter 1

Introduction

Mirror symmetry is a duality between symplectic and complex geometries, and the homological mirror symmetry (HMS) conjecture was formulated by Kontsevich [**Ko94**] to capture the phenomenon by relating two triangulated categories. This first formulation of HMS is for pairs of Calabi-Yau manifolds $(X, X^\vee)$ and it predicts two equivalences: the derived Fukaya category of X (which depends only on its symplectic structure) is equivalent to the bounded derived category of coherent sheaves of $X^\vee$ (which depends only on its complex structure), and the bounded derived category of coherent sheaves of X is equivalent to the derived Fukaya category of $X^\vee$.

A non-Calabi-Yau manifold X can also belong to a mirror pair $(X, (X^\vee, W))$, where $(X^\vee, W)$ is a Landau-Ginzburg model consisting of a non-compact manifold $X^\vee$ and a holomorphic function $W : X^\vee \to \mathbb{C}$ called the superpotential. HMS has been extended to cover Fano manifolds by Kontsevich [**Ko98**] based on works by Batyrev [**Ba94**], Givental [**Gi96**], Hori-Vafa [**HV00**], and others, and more recently to cover general type manifolds [**Ka07**, **KKOY09**]. The complex side of $(X^\vee, W)$ is described by Orlov's triangulated category of singularities of the singular fiber $W^{-1}(0)$, or equivalently the category of matrix factorizations $MF(X^\vee, W)$ [**Or04**]. The symplectic side of $(X^\vee, W)$ is described by the derived Fukaya-Seidel category [**Se08**] of Lagrangian vanishing cycles associated with W.

Another recent discovery is that, in the case where X is an open manifold, the symplectic side of X needs to be described by its wrapped Fukaya category. (Similarly, when the fibers of $W : X^\vee \to \mathbb{C}$ are open, the symplectic side of $(X^\vee, W)$ is determined by its fiberwise wrapped Fukaya category [**AA**].) The wrapped Fukaya A_∞-category is an extension of the Fukaya category constructed by Abouzaid and Seidel [**AS10**] for a large class of non-compact symplectic manifolds known as Liouville manifolds. Examples of such manifolds include cotangent bundles, complex affine algebraic manifolds, and many more general properly embedded submanifolds of $\mathbb{C}^n$. A Liouville manifold X is equipped with a Liouville 1-form λ, has an exact symplectic form $\omega = d\lambda$ and a complete Liouville vector field Z determined by $\iota_Z\omega = \lambda$, and satisfies a convexity condition at infinity.

In general, the wrapped Fukaya category of a Liouville manifold, $\mathcal{W}(X)$, is very hard to compute; therefore, it is of much interest to develop sheaf-theoretic techniques to compute $\mathcal{W}(X)$ by first decomposing X into simpler standard pieces $X = \bigcup_{i\in I} S_i$ and then reconstructing $\mathcal{W}(X)$ from the wrapped Fukaya categories of the standard pieces.

Inspired by Viterbo's restriction idea for symplectic cohomology [**Vi99**], Abouzaid and Seidel [**AS10**] constructed an A_∞-restriction functor from a quasi-isomorphic full-subcategory of $\mathcal{W}(X)$ to $\mathcal{W}(N)$ for every Liouville subdomain N of X (A Liouville subdomain is a codimension-0 compact submanifold of X whose boundary is transverse to the flow of the Liouville vector field Z. Its completion is a Liouville manifold.) Suppose $X = S_1 \cup S_2$ can be decomposed into two standard Liouville subdomains, then we can hope to compute $\mathcal{W}(X)$ from $\mathcal{W}(S_i), i = 1, 2$, by gluing them along $S_1 \cap S_2$ in the sense of matching the images of the restriction functors $\rho_i : \mathcal{W}(S_i) \to \mathcal{W}(S_1 \cap S_2)$. However this procedure cannot be readily implemented due to two obstacles. First, many pseudo-holomorphic discs that contribute to the A_∞-structures of $\mathcal{W}(X)$ are not contained in any single S_i. Second, in general it is not always possible to equip $X = \bigcup_{i\in I} S_i$ with a single Liouville structure such that all S_i's are Liouville subdomains. In addition, the restriction functors could in general have higher order terms which could make the computation intractable.

We focus on punctured Riemann surfaces that have decompositions into standard pieces which are pairs of pants. A pair of pants is a sphere with three punctures; its wrapped Fukaya category is computed in [**AAEKO13**]. The intersection between two adjacent pairs of pants is a cylinder. We provide a suitable model for the wrapped Fukaya category of such a punctured Riemann surface and compute it by providing an explicit way to glue together the wrapped Fukaya categories of the pairs of pants. Thus, our results achieve something very close to the picture conjectured by Seidel [**Se12**]. (Other instances of sheaf-theoretic computation methods include calculations for cotangent bundles [**FO97**, **NZ09**], and a program proposed by Kontsevich [**Ko09**] to compute the Fukaya categories in terms of the topology of a Lagrangian skeleton on which they are conjectured to be local. See also [**Na14**], [**Ab14**] and others for recent developments.)

The category of matrix factorizations $MF(X^\vee, W)$ of the toric Landau-Ginzburg mirror can also be constructed in the same manner from a Čech cover of $(X^\vee, W)$ by local affine pieces that are mirrors of the various pairs of pants. We will demonstrate that the restriction from the category of matrix factorizations on an affine piece to that of the overlap with an adjacent piece is homotopic to the corresponding restriction functor for the wrapped Fukaya categories. In turn, we prove the HMS conjecture that the wrapped Fukaya category of a punctured Riemann surface is equivalent to $MF(X^\vee, W)$; in fact, HMS served as our guide in developing this sheaf-theoretic method for computing the wrapped Fukaya category. The HMS conjecture with the A-model being the the wrapped Fukaya category of a punctured Riemann surface has been proved for punctured spheres and their multiple covers in [**AAEKO13**], and punctured Riemann surfaces in [**Bo**]. However, our approach yields a new proof that is in some sense more natural, and the main benefit of this approach is that one can hope to extend it to higher dimensions.

Chapter 2

Review of hypersurfaces and tropical geometry

Let H be a punctured Riemann surface with a pair-of-pants decomposition. We will focus on the case where H is a hypersurface in $(\mathbb{C}^*)^2$ that is nearly tropical, in which case H always has a preferred pair-of-pants decomposition. Mikhalkin [**Mi04**] used ideas from tropical geometry to decompose hypersurfaces in projective toric varieties into higher dimensional pairs of pants. We decompose H into pairs of pants in his style because it is natural for mirror symmetry [**Ab06, AAK12**] and we hope to generalize our results to hypersurfaces in $(\mathbb{C}^*)^n$. In this section, we summarize this decomposition procedure as explained in [**Mi04, Ab06, AAK12**].

Consider a family of hypersurfaces

$$H_t = \left\{ f_t := \sum_{\alpha \in A} c_\alpha t^{-\rho(\alpha)} z^\alpha = 0 \right\} \subset (\mathbb{C}^*)^2, \quad t \to \infty, \tag{2.0.1}$$

where $z = (z_1, z_2) \in (\mathbb{C}^*)^2$, A is a finite subset of $\mathbb{Z}^2$, $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$, and $c_\alpha, t \in \mathbb{R}_{>0}$. The function $\rho : A \to \mathbb{R}$ is the restriction to A of a convex piecewise linear function $\bar{\rho} : \mathrm{Conv}(A) \to \mathbb{R}$.

The family of hypersurfaces H_t has a maximal degeneration for $t \to \infty$ if the maximal domains of linearity of $\bar{\rho} : \mathrm{Conv}(A) \to \mathbb{R}$ are exactly the cells of a lattice polyhedral decomposition $\mathcal{P}$ of the convex hull $\mathrm{Conv}(A) \subset \mathbb{R}^2$, such that the set of vertices of $\mathcal{P}$ is exactly A and every cell of $\mathcal{P}$ is congruent to a standard simplex under $GL(2, \mathbb{Z})$ action.

The logarithm map $\mathrm{Log}_t : (\mathbb{C}^*)^2 \to \mathbb{R}^2$ is defined as $\mathrm{Log}_t(z) = \frac{1}{|\log t|}(\log|z_1|, \log|z_2|)$. Due to [**Mi04**, **Ru01**], as $t \to \infty$, Log-amoebas $\mathcal{A}_t := \mathrm{Log}_t(H_t)$ converge in the Gromov-Hausdorff metric to the tropical amoeba Π, a 1-dimensional polyhedral complex which is the singular locus of the Legendre transform

$$L_\rho(\xi) = \max\{\langle \alpha, \xi \rangle - \rho(\alpha) | \alpha \in A\}. \tag{2.0.2}$$

An edge of Π is where two linear functions from the collection $\{\langle \alpha, \xi\rangle - \rho(\alpha) | \alpha \in A\}$ agree and a vertex is where three linear functions agree. In fact, Π is combinatorially the 1-skeleton of the dual cell complex of $\mathcal{P}$, and we can label the components of $\mathbb{R}^2 - \Pi$ by elements of A, which are vertices of $\mathcal{P}$, i.e.

$$\mathbb{R}^2 - \Pi = \bigsqcup_{\alpha \in A} C_\alpha - \partial C_\alpha.$$

We identify $(\mathbb{C}^*)^2$ with the cotangent bundle of $\mathbb{R}^2$ with each cotangent fiber quotiented by $2\pi\mathbb{Z}^2$, via $(\mathbb{C}^*)^2 \cong \mathbb{R}^2 \times (S^1)^2 \cong T^*\mathbb{R}^2/2\pi\mathbb{Z}^2$ given by

$$z_j = t^{u_j + i\theta_j} \in (\mathbb{C}^*)^2 \quad \mapsto \quad (u_j, \theta_j) = \frac{1}{|\log t|}(\log|z_j|, \arg(z_j)) \in T^*\mathbb{R}^2/(2\pi/|\log t|)\mathbb{Z}^2. \tag{2.0.3}$$

This gives a symplectic form on $(\mathbb{C}^*)^2$

$$\omega = \frac{i}{2|\log t|^2} \sum_{j=1}^{2} d\log z_j \wedge d\log \bar{z}_j = \sum_{j=1}^{2} du_j \wedge d\theta_j, \tag{2.0.4}$$

which is invariant under the $(S^1)^2$ action with the moment map

$$(u_1, u_2) = \frac{1}{|\log t|}(\log|z_1|, \log|z_2|).$$

By rescaling the symplectic form by $|\log t|$, all $f_t^{-1}(0) \subset (\mathbb{C}^*)^2$ are symplectomorphic.

We study a hypersurface $H = f_t^{-1}(0)$ (fix $t \gg 1$) that is nearly tropical, meaning that it is a member of a maximally degenerating family of hypersurfaces as above, with the property that the amoeba $\mathcal{A} = \mathrm{Log}_t(H) \subset \mathbb{R}^2$ is entirely contained inside an ϵ-neighborhood of the tropical hypersurface Π which retracts onto Π, for a small ϵ. Then each open component $C_{\alpha,t}$ of $\mathbb{R}^2 - \mathrm{Log}_t(H)$ is approximately $C_\alpha - \partial C_\alpha$ as $\partial C_{\alpha,t}$ is contained in an ϵ-neighborhood of ∂C_α. The monomial $c_\alpha t^{-\rho(\alpha)} z^\alpha$ dominates on $\mathrm{Log}_t^{-1}(C_{\alpha,t})$.

The SYZ mirror to H is shown in [**AAK12**] to be the Landau-Ginzburg model (Y, W), where Y is a toric variety with its moment polytope being the noncompact polyhedron

$$\Delta_Y = \{(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R} | \eta \geq L_\rho(\xi)\}.$$

The superpotential $W : Y \to \mathbb{C}$ is the toric monomial of weight $(0, 0, 1)$; it vanishes with multiplicity 1 exactly on the singular fiber $D = W^{-1}(0)$ that is a disjoint union $D = \coprod_{\alpha \in A} D_\alpha$ of irreducible toric divisors of Y. Each irreducible toric divisor D_α corresponds to a facet of Δ_Y, which corresponds to a connected component of $\mathbb{R}^2 - \Pi$, and $\mathrm{Crit}(W)$ is a union of $\mathbb{P}^1$'s and $\mathbb{C}^1$'s corresponding to bounded and unbounded edges of Π, respectively. In fact, (Y, W) is equivalent to the mirror of the blow up of $(\mathbb{C}^*)^2 \times \mathbb{C}$ along $H \times \{0\}$.

We would like to demonstrate homological mirror symmetry that is the equivalence between the wrapped Fukaya category of H and the triangulated category of the singularities $D^b_{sg}(D)$, which is defined as the Verdier quotient of the bounded derived category of coherent sheaves $D^b(\mathrm{Coh}(D))$ by the subcategory of perfect complexes $\mathfrak{Perf}(D)$. The category $D^b_{sg}(D)$ is equivalent to the triangulated category of matrix factorizations [**Or11**].

Chapter 3

The wrapped Fukaya category of the punctured Riemann surface H

3.1 Generating Lagrangians

In this section, we describe a set of Lagrangians that split-generates the wrapped Fukaya category $\mathcal{W}(H)$.

There is a projection $\pi : H \to \Pi$ from the Riemann surface to the tropical amoeba such that π is a circle fibration over the complement of the vertices of Π. The complement consists of open edges $\operatorname{int}(C_\alpha \cap C_\beta)$, whose preimage in H is an open cylinder denoted by $\tilde{e}_{\alpha\beta} = \pi^{-1}(\operatorname{int} C_\alpha \cap C_\beta)$. The preimage of π over each tripod graph at a vertex of Π is a pair of pants.

Recall the defining function for the Riemann surface H is

$$\begin{aligned} f_t &= \textstyle\sum_{\gamma\in A} c_\alpha t^{-\rho(\alpha)} z^\gamma \\ &= c_\alpha t^{-\rho(\alpha)} z^\alpha \left(1 + \tfrac{c_\beta}{c_\alpha} t^{\rho(\alpha)-\rho(\beta)} z^{\beta-\alpha} + \textstyle\sum_{\gamma\in A\setminus\{\alpha,\beta\}} \tfrac{c_\gamma}{c_\alpha} t^{\rho(\alpha)-\rho(\gamma)} z^{\gamma-\alpha}\right). \end{aligned} \tag{3.1.1}$$

Near the edge $C_\alpha \cap C_\beta$,

$$|t^{\rho(\alpha)-\rho(\gamma)} z^{\gamma-\alpha}| = t^{\rho(\alpha)-\rho(\gamma)} t^{\langle \operatorname{Log}_t z, \gamma-\alpha\rangle} = \frac{t^{\langle \operatorname{Log}_t z,\gamma\rangle - \rho(\gamma)}}{t^{\langle \operatorname{Log}_t z,\alpha\rangle - \rho(\alpha)}}$$

is very small when t is very large. Hence $\tilde{e}_{\alpha\beta}$ in the Riemann surface lies close to the cylinder with the defining equation

$$c_\alpha t^{-\rho(\alpha)} z^\alpha + c_\beta t^{-\rho(\beta)} z^\beta = 0. \tag{3.1.2}$$

Since this equation can be written in the form $z^{\alpha-\beta} = -\frac{c_\beta}{c_\alpha} t^{\rho(\alpha)-\rho(\beta)} < 0$, an approximation to $\tilde{e}_{\alpha\beta}$ is given by the complexification of $\mathrm{int}(C_\alpha \cap C_\beta)$ with the argument defined by

$$\tilde{e}_{\alpha\beta} \cong \{z \in (\mathbb{C}^*)^2 | \operatorname{Log}_t(z) \in \mathrm{int}(C_\alpha \cap C_\beta), \arg z^{\alpha-\beta} \equiv \pi\} \cong \mathrm{int}(C_\alpha \cap C_\beta) \times S^1. \tag{3.1.3}$$

We parametrize each bounded edge of Γ, i.e. $\mathrm{int}(C_\alpha \cap C_\beta)$, by the interval $(-\epsilon, 4+\epsilon)$ for $\epsilon \ll 1$. From now on, we will leave out a small neighborhood around each vertex of Π instead of taking the entire $\mathrm{int}(C_\alpha \cap C_\beta) \times S^1$, and we define the *edge* $e_{\alpha\beta}$ of the Riemann surface H to be the subset of $\tilde{e}_{\alpha\beta}$ corresponding to the model $(0,4) \times S^1$, with coordinates (τ, ψ) and symplectic form

$$\omega_{\alpha\beta} = n_{\alpha\beta} d\tau \wedge d\psi.$$

The constant $n_{\alpha\beta}$ is determined by the symplectic area of the edge $e_{\alpha\beta}$, which is $8n_{\alpha\beta}\pi$, so $n_{\alpha\beta} = |(C_\alpha \cap C_\beta) \cap \mathbb{Z}^2| - 1$ is the "length" of the edge $\tilde{e}_{\alpha\beta}$.

We proceed similarly for unbounded edges of Π, using $(0, \infty) \times S^1$ instead of $(0,4) \times S^1$. Each unbounded edge corresponds to a neighborhood of a puncture. A punctured Riemann surface is a Liouville manifold when the neighborhood of each puncture is modelled after a cylindrical end $([1,\infty) \times S^1, \tau d\psi)$.

The Lagrangian objects under consideration are $L_\alpha(k)$, for $\alpha \in A$ and $k \in \mathbb{Z}$. Each $L_\alpha(k)$ is an embedded curve in H follows the contour of C_α running through all edges $e_{\alpha\beta}$, for each $\beta \in A$ such that C_β is adjacent to C_α. We require $L_\alpha(k)$ to be invariant under the Liouville flow everywhere on the cylindrical end.

Let $l_\alpha(0)$ be a Lagrangian that runs through each edge "straight" in a prescribed manner, e.g. with constant ψ, also see Remark 3.1.2 for an eample. Denote $d_{\alpha,\beta} = \deg \mathcal{O}(D_\alpha)|_{D_\alpha \cap D_\beta}$. Let $\delta_{\alpha,\beta}$ and $\delta_{\beta,\alpha}$ be integers satisfying $\delta_{\beta,\alpha} - \delta_{\alpha,\beta} = 1 + d_{\alpha,\beta}$; this is well defined since the Calabi-Yau property implies that $d_{\alpha,\beta} + d_{\beta,\alpha} = -2$. Each $L_\alpha(k)$ would differ from $l_\alpha(0)$ in each bounded cylindrical edge $e_{\alpha\beta}$ by the time-$(k + \delta_{\alpha,\beta}/n_{\alpha\beta})$ flow of the action given by a Hamiltonian $h_{\alpha\beta}$ whose restriction on $(1,3) \times S^1$ is $h_{\alpha\beta}(\tau,\psi) = -\frac{\pi}{2} n_{\alpha\beta}^2 (\tau - 2)^2$. The time-$(k + \delta_{\alpha,\beta}/n_{\alpha\beta})$ flow of this Hamiltonian action on $(1,3) \times S^1$ is given by $\varphi_{\alpha\beta}^{k+\delta_{\alpha,\beta}/n_{\alpha\beta}}(\tau,\psi) = (\tau, \psi - \pi k_{\alpha\beta}(\tau - 2))$, where $k_{\alpha\beta} = n_{\alpha\beta} k + \delta_{\alpha,\beta}$. So, this action rotates $l_\alpha(0)$ for $k_{\alpha\beta}$ times in the region $(1,3) \times S^1$. Lastly, we orient each $L_\alpha(k)$ counterclockwise along ∂C_α. A few examples are shown in Figure 3.1. Applying Abouzaid's generation criterion [**Ab10**] to a suitably chosen Hochschild cycle, we find that:

Lemma 3.1.1. *The wrapped Fukaya category $\mathcal{W}(H)$ is split-generated by objects $L_\alpha(k)$, $\alpha \in A, k \in \mathbb{Z}$.*

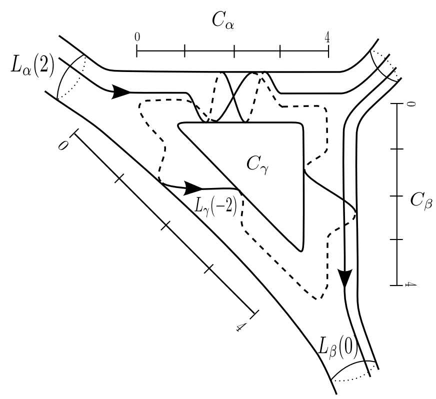


Figure 3.1: Examples of Lagrangians in H: $L_\beta(0)$, $L_\alpha(2)$, and $L_\gamma(-2)$. In this example, $d_{\alpha,\gamma} = d_{\beta,\gamma} = 0$, and we pick, $\delta_{\alpha,\gamma} = \delta_{\beta,\gamma} = 0$ and $\delta_{\gamma,\alpha} = \delta_{\gamma,\beta} = 1$.

Remark 3.1.2. We can choose the generating Lagrangians in H as boundaries of Lagrangians in $(\mathbb{C}^*)^2 - H$. Assume that for each $\alpha \in A$, there is a point $p_\alpha \in \mathbb{Z}^2 \cap C_\alpha$ with the property that $\langle p_\alpha - u, \alpha - \beta \rangle$ is an odd integer for any $u \in C_\alpha \cap C_\beta$. Note that $\alpha - \beta$ is an integral normal vector to the edge $C_\alpha \cap C_\beta$ pointing into the region C_α. Let $\mathcal{L}_\alpha$ be the zero section of the cotangent bundle $T^*\mathbb{R}^2/2\pi\mathbb{Z}^2$ with its base restricted to C_α, on which we consider the Hamiltonian function

$$H_\alpha = -\pi[(u_1 - p_{\alpha,1})^2 + (u_2 - p_{\alpha,2})^2].$$

Then for $k \in \mathbb{Z}$, the time-$(\frac{1}{2} + k)$ Hamiltonian flow is given by (written in the universal cover of $T^*\mathbb{R}^2/2\pi\mathbb{Z}^2$),

$$\phi_\alpha^{k+\frac{1}{2}}(u_1, \theta_1, u_2, \theta_2) = (u_1, \theta_1 - (\pi + 2\pi k)(u_1 - p_{\alpha,1}), u_2, \theta_2 - (\pi + 2\pi k)(u_2 - p_{\alpha,2})).$$

Let $l_\alpha(k) := \partial(\phi_\alpha^{\frac{1}{2}+k}(\mathcal{L}_\alpha))$. Lemma 3.1.3 shows that each $l_\alpha(k)$ corresponds to a Lagrangian contained in H, and the correspondence comes from the identification $\tilde{e}_{\alpha\beta} \cong e_{\alpha\beta}$ discussed earlier. We also need to modify $l_\alpha(k)$ by untwisting it so that it is invariant under Liouville flow in each cylindrical end, and further twist the bounded edges as needed to account for the $\delta_{\alpha,\beta}$'s.

Lemma 3.1.3. $l_\alpha(k) \subset \overline{\bigcup_{\beta\in A} \tilde{e}_{\alpha\beta}}$.

Proof. For any $u \in \operatorname{int} C_\alpha \cap \partial C_\beta$, let z be the preimage of u, suppose $z \in l_\alpha(k)$, and let θ be the fiber coordinate of z. Then

$$\begin{aligned} \arg(z^{\alpha-\beta}) &\equiv \textstyle\sum_{j=1}^{2} \theta_j(\alpha_j - \beta_j) \\ &= \textstyle\sum_{j=1}^{2} -(\pi + 2\pi k)(u_j - p_{\alpha,j})(\alpha_j - \beta_j) \\ &\equiv -\pi(1+2k)\langle u - p_\alpha, \alpha - \beta\rangle \\ &\equiv \pi \end{aligned} \tag{3.1.4}$$

where $k \in \mathbb{Z}$, and the last equivalence comes from $(1+2k)\langle u - p_\alpha, \alpha - \beta\rangle$ being an odd integer. By equation (3.1.3), $z \in \tilde{e}_{\alpha\beta}$. □

3.2 Hamiltonian perturbation

We now introduce Hamiltonian perturbations that will eventually enable us to define a model for $\mathcal{W}(H)$ that is suitable for pair-of-pants decompostions.

For every integer $n \geq 0$, let $H_n : H \to \mathbb{R}$ be a Hamiltonian such that for any unbounded edge $e_{\alpha\beta}$, H_n is linear on its cylindrical end (as in [**AS10**]), i.e. $H_n(\tau) = n\tau + d_n$, $\tau \in [1, \infty)$ there. On each bounded edge $e_{\alpha\beta}$, H_n agrees with the following function $H_{\alpha\beta,n}$ on $(0,4) \times S^1$ with coordinates (τ, ψ),

$$H_{\alpha\beta,n}(\tau,\psi) = \begin{cases} n_{\alpha\beta}(-n\pi\tau^2 + 2n\pi) + d_{\alpha\beta,n}, & 0 < \tau \leq 1 \\ n_{\alpha\beta} n\pi(\tau-2)^2 + d_{\alpha\beta,n}, & 1 \leq \tau \leq 3 \\ n_{\alpha\beta}(-n\pi(\tau-4)^2 + 2n\pi) + d_{\alpha\beta,n}, & 3 \leq \tau < 4 \end{cases}. \tag{3.2.1}$$

The constants d_n and $d_{\alpha\beta,n}$ above (regardless of whether the edge is bounded or unbounded)are picked so that H_n is continuous for all n and that it is constant near the vertices in the complement of all the edges. These can be satisfied by letting

$$c_n = \max\left\{2n_{\alpha\beta}n\pi \mid e_{\alpha\beta} \text{ is an bounded edge}\right\},$$

and then setting $d_{\alpha\beta,n} = c_n - 2n_{\alpha\beta}n\pi$ for each bounded edge $e_{\alpha\beta}$ and $d_n = c_n - n$ for each unbounded edge $e_{\alpha\beta}$. This way, $H_n = c_n$ in the complement of all the edges. We can check that $H_n \geq H_{n-1}$ for all n. Even though H_n is defined for $n \in \mathbb{Z}_{\geq 0}$, we can also define H_w for any real value $w \geq 0$ the same way. We will make use of it in Section 3.5.

The corresponding time-1 flow of the vector field $X_{H_{\alpha\beta,n}}$ is given by

$$\omega_{\alpha\beta}(\cdot, X_{H_{\alpha\beta,n}}) = n_{\alpha\beta} d\tau \wedge d\psi(\cdot, X_{H_{\alpha\beta,n}}) = dH_{\alpha\beta,n}.$$

The time-1 flow written in the universal cover of $E_{\alpha\beta} \cong (0,4) \times \mathbb{R}$ is

$$\phi^1_{\alpha\beta,n}(\tau,\psi) = \begin{cases} (\tau, \psi - 2n\pi\tau), & 0 < \tau < 1 \\ (\tau, \psi + 2n\pi(\tau - 2)), & 1 \leq \tau \leq 3 \\ (\tau, \phi - 2n\pi(\tau - 4)), & 3 < \tau < 4 \end{cases} . \tag{3.2.2}$$

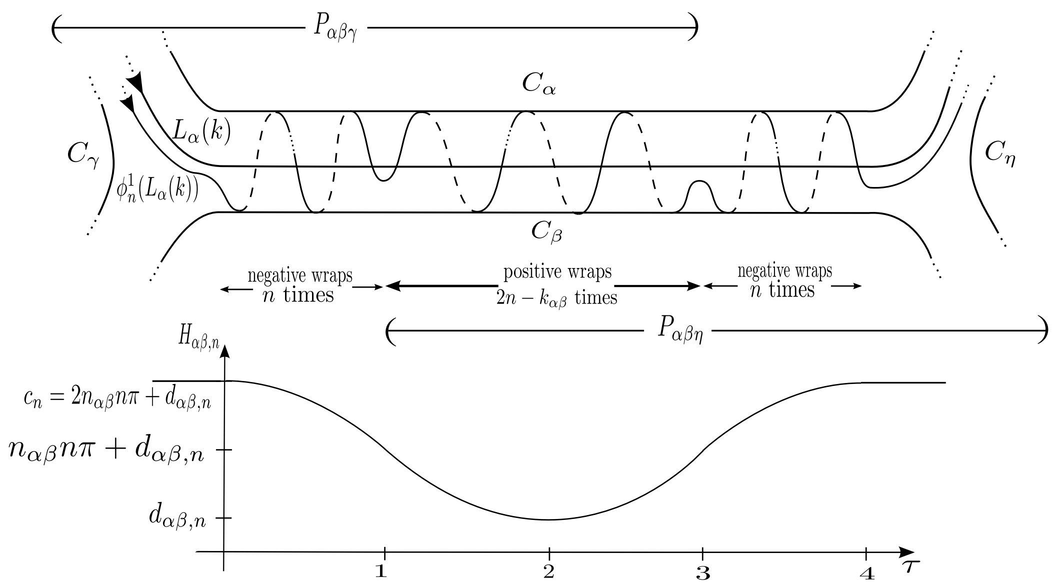


Figure 3.2: Time-1 perturbation of $L_\alpha(k)$ by Hamiltonian H_n on the edge $e_{\alpha\beta}$.

Denote by ϕ_n the flow of H_n. We can see from equation 3.2.2 that the Lagrangians $\phi^1_n(L_\alpha(k))$ and $\phi^1_n(L_\beta(k))$ wrap around the bounded cylinder $e_{\alpha\beta}$ in the "negative" direction on intervals $(0,1) \cup (3,4)$ for n times each and in the "positive" direction on $(1,3)$ for $2n - k_{\alpha\beta}$ times, as shown in figure 3.2. Note that the degree of the generators of $CF(L_\alpha(k), L_\alpha(k); H_n)$ is even in the region where H_n wraps positively and odd when it wraps negatively.

We also need to make a small modification of H_n for all $n \in \mathbb{Z}$ so that $\phi_n^1(L_\alpha(k))$ intersects $L_\alpha(k)$ transversely and to make it smooth at $\tau = 0, 1, 3, 4$ on bounded edges and at $\tau = 1$ on unbounded edges. Furthermore, the perturbations are chosen consistently inside the pair of pants regions (i.e. complements of the edges) so that $\phi_n^1(L_\alpha(k))$ intersects $L_\alpha(k)$ transversely, just once inside the pair of pants, and always at the same point which has degree 0.

3.3 Backgrounds on Floer complex and product operations

For any pairs of objects $L_i(k_1)$, $L_j(k_2)$, we get a Floer complex $CF^*(L_i(k_1), L_j(k_2); H_n)$, where H_n is the Hamiltonian defined in Section 3.2. This Floer complex is generated by the set $\mathcal{X}(L_i(k_1), L_j(k_2); H_n)$ consisting of Reeb chords that are time-1 trajectories of the Hamiltonian H_n starting in $L_i(k_1)$ and ending in $L_j(k_2)$. Equivalently, these chords correspond to intersection points of $\phi_n^1(L_i(k_1)) \cap L_j(k_2)$.

The differential $d : CF^*(L_i(k_1), L_j(k_2); H_n) \to CF^*(L_i(k_1), L_j(k_2); H_n)$ is given by the count of pseudo-holomorphic strips $u : \mathbb{R} \times [0, 1] \to H$ which are solutions to the inhomogeneous Floer's equation

$$(du - X_{H_n} \otimes dt)^{0,1}_{J_t} = 0, \text{ equivalently, } \frac{\partial u}{\partial s} + J_t\left(\frac{\partial u}{\partial t} - X_{H_n}\right) = 0 \tag{3.3.1}$$

with boundaries $u(s, 0) \in L_i$ and $u(s, 1) \in L_j$. As $s \to \pm\infty$, u converges to Hamiltonian flow lines that are generators the Floer complex involved in the differential. Such a solution $u(s, t)$ can be equivalently seen as an ordinary $\tilde{J} = (\phi_n^{1-t})_* J$-holomorphic strip $\tilde{u}(s, t) = \phi_n^{1-t}(u(s, t))$ with boundaries on $\phi_n^1(L_i(k_1))$ and $L_j(k_2)$. Indeed,

$$\frac{\partial \tilde{u}}{\partial s} = (\phi_n^{1-t})_* \left(\frac{\partial u}{\partial s}\right) \text{ and } \frac{\partial \tilde{u}}{\partial t} = (\phi_n^{1-t})_* \left(\frac{\partial u}{\partial t} - X_{H_n}\right),$$

so the Floer equation (3.3.1) becomes $\frac{\partial \tilde{u}}{\partial s} + \tilde{J}\frac{\partial \tilde{u}}{\partial t} = 0$.

For Lagrangians $L_{i_0}(k_0), L_{i_1}(k_1), L_{i_2}(k_2)$, $n \gg |k_0|, |k_1|, |k_2|$, the product

$$\begin{aligned}\mu^2(H_n) : CF^*(L_{i_1}(k_1), L_{i_2}(k_2); H_n) \otimes CF^*(L_{i_0}(k_0), L_{i_1}(k_1); H_n)& \\ \to CF^*(L_{i_0}(k_0), L_{i_2}(k_2); 2H_n)&\end{aligned}$$

is given by the count of solutions $u : D \to H$ of the perturbed Floer equation

$$(du - X_{H_n} \otimes \beta)^{0,1}_{J_t} = 0 \tag{3.3.2}$$

where D is a disc with three strip-like ends, and the images of the three components of ∂D are contained in the respective Lagrangians $L_{i_0}(k_0), L_{i_1}(k_1)$, and $L_{i_2}(k_2)$. The 1-form β on D satisfies $d\beta = 0$ and it pulls back to dt on the input strip-like ends and to $2dt$ on the output strip-like end. Again, by changing to a domain dependent almost-complex structure, this is equivalent to counting standard holomorphic discs with boundaries on $\phi_n^2(L_{i_0}(k_0)), \phi_n^1(L_{i_1}(k_1))$, and $L_{i_2}(k_2)$ instead. The higher products

$$\mu^d(H_n) : CF^*(L_{i_{d-1}}(k_{d-1}), L_{i_d}(k_d); H_n) \otimes \cdots \otimes CF^*(L_{i_0}(k_0), L_{i_1}(k_1); H_n) \\ \to CF^*(L_{i_0}(k_0), L_{i_d}(k_d); dH_n)$$

are constructed in a similar way.

3.4 Orientation and grading

Let's orient each $L_\alpha(k)$ counterclockwise along ∂C_α. This will give each Floer complex a $\mathbb{Z}_2$-grading. For each transverse intersection point x of L_0 and L_1, we can identify $T_xL_0 \cong \mathbb{R}$ and $T_xL_1 \cong i\mathbb{R}$ via a linear symplectic transformation. Consider the path l_t of Lagrangian lines with $l_0 = \mathbb{R}$ and $l_1 = i\mathbb{R}$, and $l_t = e^{-i\pi t/2}\mathbb{R}$. If the path l_t maps the orientation of T_xL_0 to the orientation of T_xL_1, then $\deg(x) = 0$; otherwise, $\deg(x) = 1$.

It is also possible to define a $\mathbb{Z}$-grading on Floer complexes of the Lagrangians in consideration. To do that, we need to pick a trivialization of $T^*H^{1,0}$. There's no canonical way to choose this trivialization. We can just choose a global nonvanishing section Ω of $T^*H^{1,0}$ to be any meromorphic form allowed to have zeros or poles at each puncture. Once we make that choice, for any Lagrangian plane $l \subset T_xH$, $\Omega|_l = \alpha \mathrm{vol}_l$ where $\alpha \in \mathbb{C}^*$ and vol_l is a real volume form. And we can define $\arg(l) := \arg(\Omega|_l) = \arg(\alpha) \in \mathbb{R}/\pi\mathbb{Z}$ (or $\mathbb{R}/2\pi\mathbb{Z}$ if Lagrangians are oriented already). The Lagrangian Grassmannian $LGr(TH)$ can then be lifted to a fiberwise universal cover $\widetilde{LGr}(T_pH) = \{(l, \theta) | l \in LGr(T_pH), \theta \in \mathbb{R}, \theta \equiv \arg(\Omega|_l) \bmod \pi\}$.

The tangent lines along a Lagrangian L form a path of Lagrangian planes, which are mapped by the above phase function to S^1. Because of the Lagrangians we consider are simply connected, the homotopy class of this map is trivial, i.e. the Lagrangians have vanishing Maslov class. Hence for each Lagrangian L, we can equip it with a grading, which is a consistent choise of a graded lift to $\widetilde{LGr}(TH)$ of the section $p \mapsto T_pL$ of $LGr(TH)$ over L. Then we can assign a degree to a transverse intersection $p \in L_0 \cap L_1$. In the case of Riemann surface H, $\deg(p) = \left\lceil \frac{\theta_1 - \theta_0}{\pi} \right\rceil$. (Ref. [**Se08**, **Au13**].)

Once this $\mathbb{Z}$-grading is available, the index of a rigid holomorphic polygon in H corresponding to a higher product μ^k is deg(output) $- \sum$ deg(inputs) $= 2 - k$.

3.5 Linear continuation map

For any two Lagrangians $L_1 = L_i(k_1), L_2 = L_j(k_2)$, we would like to define the wrapped Floer complex $CW^*(L_1, L_2)$ as a direct limit of the perturbed Floer complexes $CF^*(L_1, L_2; H_n)$ as $n \to \infty$. This definition relies on the existence of continuation maps

$$\kappa : CF^*(L_1, L_2; H_n) \to CF^*(L_1, L_2; H_N),$$

whenever $n \leq N$ (recall, by construction, $H_n \leq H_N$ whenever $n \leq N$). In general, the direct limit construction would not be compatible with A_∞ structures, hence, $CW^*(L_1, L_2)$ is defined as the homotopy direct limit in [**AS10**]. However, we will show in this section that it turns out in our case, each continuation map κ above is just an inclusion for n sufficiently large depending on L_1, L_2. Furthermore, we will show in the next section that higher order continuation maps (i.e. those with $d \geq 2$ inputs) are trivial when mapping from sufficiently perturbed Floer complexes. Consequently, the wrapped Floer complex can be defined as

$$CW^*(L_i(k_1), L_j(k_2)) = \bigcup_{n=n_0}^{\infty} CF^*(L_i(k_1), L_j(k_2); H_n)/\sim \tag{3.5.1}$$

where the equivalence relation is given by continuation maps which are inclusions. Moreover, we have well defined differential and A_∞ products

$$\mu^d : CW^*(L_{d-1}, L_d) \otimes \cdots \otimes CW^*(L_0, L_1) \to CW^*(L_0, L_d)$$

which are given by ordinary Floer differential and products.

In the usual definition of the continuation map, for an input $p \in \mathcal{X}(L_1, L_2; nH_1) = \mathcal{X}(L_1, L_2; H_n)$, the coefficient of $q \in \mathcal{X}(L_1, L_2; NH_1) = \mathcal{X}(L_1, L_2; H_N)$ in $\kappa(p)$ is given by the count of index zero solutions to the perturbed Floer equation (the continuation equation)

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - \lambda(s) X_{H_1}\right) = 0, \tag{3.5.2}$$

where $\lambda(s)$ is a smooth function which equals N for $s \ll 0$ and n for $s \gg 0$, and such that $\lambda'(s) \leq 0$. Such a solution also needs to satisfy the boundary conditions $u(s, 0) \in L_1$ and $u(s, 1) \in L_2$, and it converges to the generator p as $s \to \infty$ and the generator q as $s \to -\infty$. Instead of considering moduli spaces of perturbed Floer solutions, we will introduce the related moduli spaces of cascades of pseudoholomorphic discs by taking the limit where the derivative of λ tends to zero. Counting cascades of pseudoholomorphic discs with appropriate indices give equivalent definitions of continuation maps. We will only discuss the construction of cascades briefly following Appendix A of [**Au10**], [**AS10**], and Section 10(e) of [**Se08**], and we refer to them for details.

First of all, we would like to go to the universal cover to make visualization and discussion easier. For any given input data $(p; L_1, L_2)$, we can obtain a lift $(p'; L_1', L_2')$ in the universal cover E_H of the Riemann surface by picking an arbitrary lift L_1' of L_1 and then tracing the lift of L_2 through p' to determine L_2'. Rather than counting pseudo-holomorphic strips in H with the above conditions, it is equivalent to counting pseudo-holomorphic strips in the universal cover E_H with boundaries in the lifts L_1' and L_2'. The Lagrangians $L_1' = L_i(k_1)'$ and $L_2' = L_j(k_2)'$ satisfy some nice properties which are necessary for defining cascades:

(P1) For all integer values of $w \geq n_0$ large enough, $\phi_w^1(L_1)$ is transverse to L_2, so $\phi_w(L_1')$ is transverse to L_2'.

(P2) For $w \geq n_0$ large enough, each point $x' \in \phi_w^1(L_1') \cap L_2'$ lies on a unique maximal smooth arc $\gamma : [n_0, \infty) \to E_H$ given by $t \mapsto \gamma(t)$, where $\gamma(t)$ is a transverse intersection point of $\phi_t^1(L_1')$ and L_2' for all t. In other words, as w increases from n_0 to ∞, no new intersection between $\phi_w(L_1')$ and L_2' are created, and the existing intersections remain transverse.

Both (P1) and (P2) can be achieved by choosing $n_0 > \max\{1, |k_1| + |k_2|\}$. For Lagrangians $L_1, L_2 \subset H$, property (P2) for their corresponding lifts in E_H implies that as w increases from n_0 to ∞, any newly created intersection of $\phi_w(L_1)$ and L_2 will not be the output of any J-holomorphic disc. For any $x \in \phi_w^1(L_1) \cap L_2$ and its lift $x' \in \phi_w^1(L_1') \cap L_2'$ which lies on the arc γ, we can identify x with a unique point $\vartheta_w^{w'}(x)$, $w' \geq w$, determined by requiring the lift of $\vartheta_w^{w'}(x)$ to be $\gamma(w')$.

For Lagrangians $L_1 = L_i(k_1)$, $L_2 = L_j(k_2)$, and $n \geq n_0$, we define the continuation map $\kappa : CF^*(L_1, L_2; H_n) \to CF^*(L_1, L_2; H_N)$, $N \geq n$, as follows. Given $p \in \mathcal{X}(L_1, L_2; H_n)$ and $q \in \phi_N^1(L_1) \cap L_2$, the coefficient of q in $\kappa(p)$ is given by the count of linear cascades from p to q of Maslov index zero. A *k-step linear cascade* from p to q is a sequence of k ordinary pseudo-holomorphic strips $u_1, \ldots u_k : \mathbb{R} \times [0,1] \to H$ satisfying:

- $u_i(\mathbb{R} \times 0) \subset \phi_{w_i}^1(L_1)$, $u_i(\mathbb{R} \times 1) \subset L_2$ for some $w_1 \leq \cdots \leq w_k$ in the interval $[n, N]$;
- u_i has finite energy, and we denote by $p_i^{\pm} \in \phi_{w_i H}(L_1) \cap L_2$ the intersection points to which u_i converge at $\pm\infty$;
- $p_{i+1}^+ = \vartheta_{w_i}^{w_{i+1}}(p_i^-)$, $p_1^+ = \vartheta_n^{w_1}(p)$, and $q = \vartheta_{w_k}^N(p_k^-)$.

When $q = \vartheta_n^N(p)$, we allow the special case of $k = 0$.

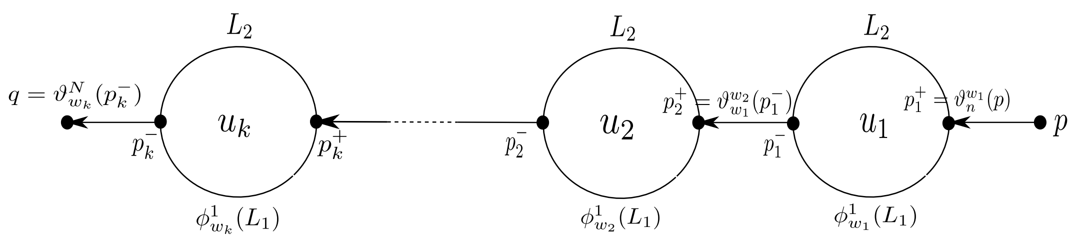


Figure 3.3: a k-step linear cascade from p to q.

Lemma 3.5.1. *Given any two Lagrangians, $L_1 = L_i(k_1)$ and $L_2 = L_j(k_2)$, and $n_0 \leq N$, the linear continuation map $\kappa : CF^*(L_1, L_2; H_n) \to CF^*(L_1, L_2; H_N)$ is just the inclusion.*

Proof. The components of an index zero linear cascade are holomoprhic strips whose Maslov indices sum to zero. The only holomorphic strip of index zero on a Riemann surface (with an arbitrary complex structure) is the constant disc, and there are no holomoprhic strips of negative index. Hence the continuation map has to be the inclusion induced by identifying intersection points using ϑ_n^N. $\square$

As expected, κ also has to be an inclusion in the usual definition of the continuation map which is defined by counting inhomogeneous holomorphic strips that are solutions to (3.5.2). Otherwise, there are nontrivial index zero inhomogeneous strips from p to a different $q \neq \vartheta_n^N(p)$. Taking the limit where $d\lambda(s)/ds \to 0$, the index zero inhomogeneous holomorphic strips from p to q converge in the sense of Gromov to a nontrivial index zero linear cascade, thus contradicting Lemma 3.5.1.

3.6 Higher continuation maps

A given set of boundary data consists of Lagrangians $L_0, \ldots, L_d$ with each $L_i = L_{\alpha_i}(k_i), \alpha_i \in A, k_i \in \mathbb{Z}$, inputs $p_1 \in \mathcal{X}(L_0, L_1; H_n), \ldots, p_d \in \mathcal{X}(L_{d-1}, L_d; H_n)$, and the output $q \in \mathcal{X}(L_1, L_2; H_N)$ with $N \geq dn$. The coefficient of q in the continuation map

$$CF^*(L_{d-1}, L_d; nH) \otimes \cdots \otimes CF^*(L_0, L_1; nH) \to CF^*(L_0, L_d; NH)$$

can be defined by counting exceptional solutions, $u : D \to H$ where D is a disc with $d+1$ strip-like ends, to the perturbed Floer equation $(du - X_{\tilde{H}} \otimes \beta)^{0,1} = 0$ with boundaries on Lagrangians $L_0, \ldots, L_d$. Here, β is a closed 1-form, and $\tilde{H}(s)$ is a domain dependent Hamiltonian. If we represent D as a strip-like disc illustrated in Fig. 3.4, then $\tilde{H} = \lambda(s)H$ interpolates between $\tilde{H}(s) = nH$ as $s \to \infty$ and $\tilde{H}(s) = \frac{N}{d}H$ as $s \to -\infty$.

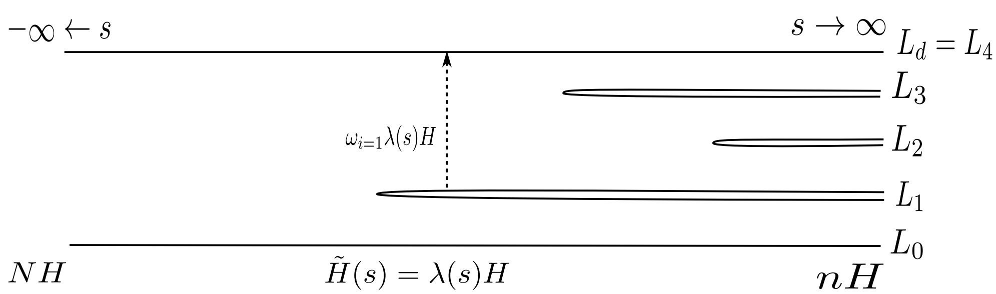


Figure 3.4: Higher continuation maps with $d = 4$ inputs can be defined by counting perturbed Floer equations $u : D \to H$ with boundaries on $L_0, \ldots, L_{d=4}$. The disc D with $4+1$ strip-like ends is equivalent to the strip $\{(s,t) \in (\mathbb{R} \times [0,d]) \backslash ((d-1) \text{ slits})\}$.

As with linear continuations, such a definition is hard to use for explicit computations, hence we use the equivalent definition of counting cascades. Intuitively, perturbed Floer solutions are homotopic to cascades as we take the limit by stretching the strip so that the interpolating Hamiltonian $\tilde{H}(s)$ changes infinitely slowly, i.e. $\lambda'(s) \to 0$. Gromov compactness tells us that there are finitely many places where the energy concentrates and we get unperturbed pseudo-holomorphic discs as pieces of a cascade. A cascade is a collection of unperturbed pseudo-holomorphic discs with boundary on any $(r+1)$-tuple

$$\phi_{\omega_{i_0}\lambda(s)H}(L_{i_0}), \dots, \phi_{\omega_{i_r}\lambda(s)H}(L_{i_r})$$

of perturbed Lagrangians with $i_0 < \cdots < i_r \in \{0, \dots, d\}$, $n \le \lambda \le N/d$, $\omega_j = d - j$ for each $j = 0, \dots, d$, and such that at least one of the pseudo-holomorphic discs in this collection must be exceptional, i.e. of index less than (2-#inputs). Again, refer [**Au10**],[**AS10**], [**Se08**] for details. In this section, we will show that there are no such exceptional discs, hence higher continuation maps are all trivial.

Stability of intersection points and crossing changes.

Given a set of boundary data as above, let $L'_0, \dots, L'_d$ be lifts of $L_0, \dots, L_d$ in the universal cover E_H obtained from choosing an arbitrary lift L'_1 of L_1 and then determining $L'_2, \dots, L'_d$ by tracing through $p_1, \dots, p_d$. By taking $n_0 > \max\{1, |k_1| + \cdots + |k_d|\}$, any pair of L'_i, L'_j of Lagrangians in this collection of lifts satisfy property (P1) and (P2) mentioned in Section 3.5.

Besides pairwise intersections, in order to obtain a vanishing result for cascades, we also want to make sure that intersection points of three or more Lagrangians also stabilize. That is, for any $(r+1)$-tuple $l_0 = \phi_{\omega_{i_0}\lambda}(L'_{i_0}), \dots, l_r = \phi_{\omega_{i_r}\lambda}(L'_{i_r})$ with with $i_0 < \cdots < i_r \in \{0, \dots, d\}$, $n \le \lambda \le N/d$, $\omega_j = d - j$ for each $j = 0, \dots, d$ as above, we would like to show that no intersection points in $l_0 \cap \cdots \cap l_r$ are created or canceled as we vary $\lambda(s)$ so long as n is large enough. This is equivalent to saying that there are no crossing changes (i.e. Reidemeister moves) between multiple Lagrangians in $\{\phi_{\omega_0\lambda}(L'_0), \dots, \phi_{\omega_d\lambda}(L'_d)\}$ as $\lambda(s)$ varies so long as n is large enough.

First, suppose all of $l_0, \dots, l_r$ run through the universal cover $E_{\alpha\beta} \cong (0, 4) \times \mathbb{R}$ of the edge $e_{\alpha\beta}$, we can write down the coordinates for each $l_j = (\tau, \psi_j(\tau))$ from Equation (3.2.2)

$$\psi_j(\tau) = \begin{cases} c_j(\tau) - 2\pi(d-i_j)\lambda(s)\tau, & 0 < \tau < 1 \\ c_j(\tau) + 2\pi(d-i_j)\lambda(s)(\tau-2), & 1 \leq \tau \leq 3 \\ c_j(\tau) - 2\pi(d-i_j)\lambda(s)(\tau-4), & 3 < \tau < 4 \end{cases}, \tag{3.6.1}$$

where $c_j(\tau)$ is a continuous function that can be arranged to be constant in the τ-interval $(0,1) \cup (5/4,4)$ and $c_j(5/4) - c_j(1) = -2\pi k_{i_j}$. To arrange $c_j(\tau)$ to be constant in $(0,1) \cup (5/4,4)$, we need to adjust the construction of each $L_\alpha(k)$ in Section 3.1 so that $L_\alpha(k)$ twists k times in the interval $(1,5/4)$ instead of in the interval $(1,3)$. All the results in this paper are valid (with some obvious changes in the arguments) when we use this modified construction for $L_\alpha(k)$, which is homotopic to the construction given in Section 3.1. We will use this modified construction only in this section.

As n increases, the τ-coordinate of any intersection point, p, between any pair of Lagrangians l_j and $l_{j'}$ in $\{l_0,\ldots,l_r\}$ moves toward 0, 2, or 4. Indeed, it follows from Equation 3.6.1 that one of $\tau(p)$, $\tau(p)-2$, or $\tau(p)-4$ is equal to $\frac{c_j(\tau)-c_{j'}(\tau)}{\lambda(s)2\pi(i_{j'}-i_j)} \to 0$ as $\lambda(s) \to \infty$. Consequently, given inputs $p_1,\ldots,p_d$, we can always choose n to be large enough so that none of the τ-coordinates of the given p_i's will be in the the region $(1,5/4)$ after replacing $p_1,\ldots,p_d$ by their corresponding generators via the linear continuation map. Furthermore, by choosing n to be large enough, the appropriate lifts $l_0,\ldots,l_r$ to the universal cover of the cylinder will no longer have any intersections in the $(1,5/4)$ region.

In each of the intervals $(0,1),(5/4,3)$, and $(3,4)$, l_j is a line segment with the same slope of $2\pi(d-i_j)\lambda(s)$ in absolute value. In each interval, these line segments belong to $r+1$ lines that may or may not intersect at a single point. Whether they intersect in a single point or not depends on the ψ-intercepts $c_j(\tau)$'s and ratios between the differences in the slopes, which are independent of $\lambda(s)$. Take the interval $(0,1)$, if the corresponding $(r+1)$ lines do intersect at a single point, then for n large enough, the line segments in $l_0,\ldots,l_r$ will either always intersect in a single point in the interval $(0,1)$ or they will never intersect in that interval. The same can be said for the other intervals. We can then apply the same procedure to all edges and for all subsets of multiple Lagrangians in $\{\phi_{d\lambda}(L_0'),\ldots,L_d'\}$. We can obtain a large enough n for each of these cases and take the maximum value. The argument for intersections inside the unbounded cylinders is the same. There will be constant multiple intersections inside each pair of pants (in the complement of the edges), but because of the small perturbations we chose at the end of Section 3.2, they will be always be of degree zero (so, not exceptional), and they will never move or undergo crossing changes.

Exceptional discs are constant.

A pseudoholomorphic disc u bounded by Lagrangians

$$l_0 = \phi_{\omega_{i_0}\lambda H}(L'_{i_0}), \ldots, l_r = \phi_{\omega_{i_r}\lambda H}(L'_{i_r})$$

is either a nondegenerate polygon, a constant disc at the intersection of all $r+1$ Lagrangians, or a polygon with some of its corners being intersection points of multiple Lagrangians. We want to investigate when such a pseudo-holomorphic disc is exceptional, i.e. of index less than $2-r$. A nondegenerate polygon has index

$$2 - r + 2 \cdot \#(\text{interior branch points}) + \#(\text{boundary branch points}) \geq 2 - r,$$

hence it's not exceptional.

Next, we analyze the degenerate cases involving multiple intersection points. At such a point, note that even though $l_0, \ldots, l_r$ may come from different components and hence have different orientations, we can change the orientation of some of them so that $l_0, \ldots, l_r$ have the same orientation on every edge in trying to compute the index of the disc. This is because changing the orientation of a Lagrangian l_j will change the degree of intersections between $l_{j-1} \cap l_j$ and $l_j \cap l_{j+1}$ in opposite ways leaving the overall index of the disc unchanged. For the rest of this section, let us assume that $l_0, \ldots, l_r$ have the same orientation on every edge.

For any two Lagrangians l_j, l_{j+1}, then the degree of an intersection point $p \in l_j \cap l_{j+1}$ inside a cylindrical edge is either 0 or 1 depending on the location of p. If p is in the positively wrapped region $(1,3) \times \mathbb{R}$ of some edge, then $\deg(p) = 0$ because the slope of l_i is positive and larger than that of l_{i+1} which is also positive. On the other hand, if p is in the negatively wrapped region $((0,1) \cup (3,4)) \times \mathbb{R}$ of some edge, then $\deg(p) = 1$.

Case 1: Suppose u is a constant disc with its image in a cylindrical end or in the positively wrapped region of a bounded edge (i.e. in $(1,3) \times S^1$). In this case, all the input and output intersection points have degree zero, hence $\operatorname{ind} u = 0$. When $r \geq 2$, $\operatorname{ind} u \geq 2 - r$, so it is not an exceptional disc that contribute to the higher continuation map.

Case 2: Suppose u is a constant disc with its image in the negatively wrapped region of a bounded edge (i.e. in $((0,1)\cup(1,3))\times S^1$). In this case, all the input and output intersection points have degree 1, hence $\operatorname{ind} u = 1-r < (2-r)$ and u is an exceptional disc. We have $\dim\ker D_{\bar\partial,u} = r-2$ coming from the freedom to move the $r+1$ marked points. (Note that $\dim\ker D_{\bar\partial,u}=0$ if we fix the marked points because u is constant with image p and $u^*TH = T_pH \cong \mathbb{C}$. By the open mapping theorem, the only holomorphic disc with boundary in the union of the lines $T_pl_0,\dots,T_pl_s$ is the constant map at the origin of the tangent space.) Hence $\dim\operatorname{coker} D_{\bar\partial,u} = r-1>0$ and u is not regular. This analysis shows that we need to perform a deformation to achieve transversality, and this will be our topic of the next section.

Case 3: Suppose u is a polygon with $\tilde r+1$ geometrically distinct vertices $p_0,\dots,p_{\tilde r}$ ($\tilde r\geq 1$ since we assume u is not constant) with some of its vertices being intersection points of multiple Lagrangians. We know that the index of a nondegenerate polygon with $\tilde r+1$ vertices (i.e. bounded by $\tilde r+1$ Lagrangians) is at least $2-\tilde r$. If a vertex of u which is an intersection point of multiple Lagrangians is located in the positively wrapped region, then its contribution to $\operatorname{ind} u$ is zero. If a vertex of u is an intersection point of $v+2$ Lagrangians ($v>0$) located in the negatively wrapped region, then the extra degenerate edges and vertices add to the contribution of this vertex to $\operatorname{ind} u$ by $-v$. That is, each of the $(r-\tilde r)$ extra Lagrangians contributes at least -1 to $\operatorname{ind} u$. Hence, $\operatorname{ind} u \geq (2-\tilde r)-(r-\tilde r) = 2-r$, i.e. u is not exceptional.

Deformation of constant discs.

At each intersection of $r+1$ ($r>1$) Lagrangians

$$l_0 = \phi_{\omega_{i_0}\lambda(s)H}(L'_{i_0}),\dots,l_r = \phi_{\omega_{i_r}\lambda(s)H}(L'_{i_r}),$$

in the negatively wrapped portion of a cylinder, we want to pick Hamiltonian perturbations $ch_0(s),\dots,ch_r(s)$ that perturb these Lagrangians locally so that as soon as we turn on the Hamiltonian perturbation, the constant disc u at the intersection of these Lagrangians is removed and we don't introduce any new holomorphic discs with boundary on $l_0,\dots,l_r$ in that order. Recall that each weight $\omega_{i_j} = d-i_j$, and the Lagrangians are ordered with $i_0<\dots<i_r$.

Let $h(s)$ be a Hamiltonian supported near the moving intersection point (this is the only way in which it depends on s) such that locally $h(s)(\tau,\psi) = \psi - \psi_0$, so $X_h \sim -\frac{\partial}{\partial\tau}$ points along the negative τ-axis. We use $ch_j(s) = c\omega_{i_j}h(s)$ to perturb l_j. This pushes the Lagrangians into the desired positions as shown in Fig. 3.5. Indeed, up to rescaling of both axes, each l_j was initially the line $\psi = -\omega_{i_j}\tau$, and after translation, it becomes $\psi = -\omega_{i_j}(\tau+\omega_{i_j})$. Lagrangians l_j and l_k intersect where $\psi = -\omega_{i_j}(\tau+\omega_{i_j}) = -\omega_{i_k}(\tau+\omega_{i_k})$, so $\tau = -(\omega_{i_j}+\omega_{i_k})$. For a fixed j, these are all distinct and in the correct order. These local perturbations for the Lagrangians are automatically consistent because they are built from a single $ch(s)$ and the existing weights.

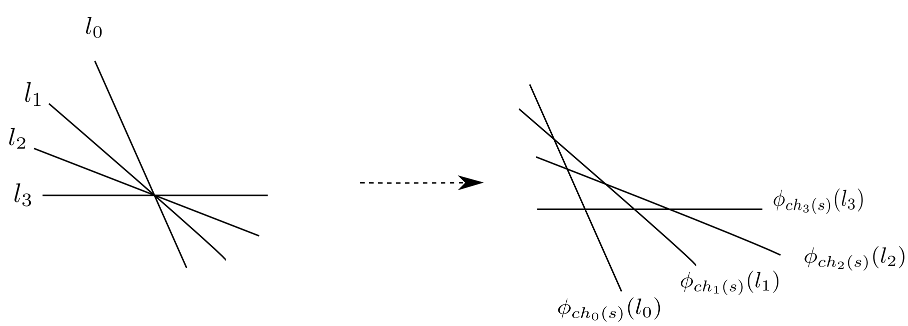


Figure 3.5:

The constant map u, with its image being the point p, is a non-regular solution to the perturbed Floer equation $(du+cX_{h(s)}\otimes\beta)^{0,1} = 0$ for $c=0$; we study its deformations among solutions to this equation for small c. Let us consider the first order variation

$$(d(u+cv) + cX_h(u+cv)\otimes\beta)^{0,1} = 0,$$

where $v : D \to T_pH = \mathbb{C}$. From the above equation, v satisfies the linearized equation $\bar\partial v = (X_h(p)\otimes\beta)^{0,1}$ with boundary conditions on the real line T_pl_j. We claim that, no matter the position of the boundary marked points on the domain D, this linearized equation has no solutions. Indeed, we rewrite the linearized equation for $\tilde v = v - tX_h(p)$, where t is a coordinate on D with $dt = \beta$, and $t = -\omega_{i_j}$ on the j-th piece of the boundary of D. Then a solution $\tilde v$ is a holomorphic map with boundary conditions in $T_pl_j + \frac{\partial l_j}{\partial c} = T_pl_j + \omega_{i_j}X_h(p)$, which are lines in $\mathbb{C}$ as in Fig. 3.5. Hence this linearized equation has no solution; that is, the projection to $\mathrm{coker}_{D_{\bar{\partial}_J}}$ of the perturbation term yields a nonvanishing section of the obstruction boundle over the moduli space of solutions. This means that there is no way of deforming the constant map u to a solution with the perturbation term added, i.e. cascades with such a constant component do not contribute to the continuation map.

3.7 Wrapped Fukaya category of a pair of pants: some notations.

We can view H as a union of pairs of pants, $H = \bigcup_{\alpha,\beta,\gamma} P_{\alpha\beta\gamma}$, where each $P_{\alpha\beta\gamma}$ is a pair of pants whose image $\mathrm{Log}_t(P_{\alpha\beta\gamma})$ is adjacent to all three components C_α, C_β, and C_γ. Also, if $e_{\alpha\beta}$ is bounded, we require that the leg $P_{\alpha\beta\gamma} \cap e_{\alpha\beta} \cong (0,3) \times S^1$, which comes from the above model of $e_{\alpha\beta}$. This way, if $e_{\alpha\beta}$ is a bounded edge connecting two pairs of pants $P_{\alpha\beta\gamma}$ and $P_{\alpha\beta\eta}$, then these two pairs of pants will overlap on the positive wrapping part of $e_{\alpha\beta}$ (fig.3.2). Denote $S_{\alpha\beta} = P_{\alpha\beta\gamma} \cup P_{\alpha\beta\eta}$. When we are considering a Lagrangian restricted to a pair of pants, $L_\alpha(k) \cap P_{\alpha\beta\gamma}$, we still call it $L_\alpha(k)$ for convenience. Note that $L_\alpha(k)$ are actually equivalent to L_α in the wrapped Fukaya category of the pair of pants. The objects $L_i(k)$, $i \in \{\alpha, \beta, \gamma\}$, $k \in \mathbb{Z}$, generate the wrapped Fukaya category of $P_{\alpha\beta\gamma}$.

The definitions for Floer complex and product structures introduced in Section 3.3 apply to a pair of pants $P_{\alpha\beta\gamma}$ as well. (In fact, if we extend each bounded leg of a pair of pants to an infinite cylinderical end, then the quadratic Hamiltonian H_n on $(1,3) \times S^1$ is equivalent to a linear Hamiltonian on that cylindrical end.) We can define the Floer complex $CF^*_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2); H_n)$ and its set of generators $\mathcal{X}_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2); H_n)$ for any pairs of objects $L_i(k_1)$, $L_i(k_2)$ in $\mathcal{W}(P_{\alpha\beta\gamma})$. Similarly, we can define the differential

$$d : CF^*_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2); H_n) \to CF^*_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2); H_n)$$

and products

$$\begin{aligned}\mu^d_{P_{\alpha\beta\gamma}}(H_n) : CF^*_{P_{\alpha\beta\gamma}}(L_{i_{d-1}}(k_{d-1}), L_{i_d}(k_d); H_n) \otimes \cdots \otimes CF^*_{P_{\alpha\beta\gamma}}(L_{i_0}(k_0), L_{i_1}(k_1); H_n)\\ \to CF^*_{P_{\alpha\beta\gamma}}(L_{i_0}(k_0), L_{i_d}(k_d); dH_n)\end{aligned}$$

by counting perturbed pseudo-holomorphic discs $u : D \to P_{\alpha\beta\gamma}$ with strip-like ends satisfying appropriate boundary and limiting conditions. For the same reason as in the previous section, the continuation maps

$$\kappa : CF^*_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2); H_n) \to CF^*_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2); H_N), \quad n < N,$$

are inclusions and compatible with A_∞ structures if $n > n_0$ for a large n_0. We then have the wrapped Floer complex

$$CW^*_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2)) = \bigcup_{n=n_0}^{\infty} CF^*_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2); H_n)/\sim$$

and the product

$$\mu^d_{P_{\alpha\beta\gamma}} : CW^*_{P_{\alpha\beta\gamma}}(L_{d-1}, L_d) \otimes \cdots \otimes CW^*_{P_{\alpha\beta\gamma}}(L_0, L_1) \to CW^*_{P_{\alpha\beta\gamma}}(L_0, L_d).$$

3.8 Pair of pants decomposition

We can split the generators $\mathcal{X}_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2); H_n)$ into two parts

$$\mathcal{X}_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2); H_n) = \mathcal{J}^\gamma_{\alpha\beta}(L_i(k_1), L_j(k_2); H_n) \cup \mathcal{C}_{\alpha\beta}(L_i(k_1), L_j(k_2); H_n), \qquad (3.8.1)$$

consisting of generators in $P_{\alpha\beta\gamma} \setminus ((1,3) \times S^1)$ and $(1,3) \times S^1 \subset e_{\alpha\beta}$, respectively. Denote

$$\mathcal{J}^\gamma_{\alpha\beta}(H_n) = \bigcup_{i,j,k_1,k_2} \mathcal{J}^\gamma_{\alpha\beta}(L_i(k_1), L_j(k_2); H_n),$$

$$\mathcal{C}_{\alpha\beta}(H_n) = \bigcup_{i,j,k_1,k_2} \mathcal{C}_{\alpha\beta}(L_i(k_1), L_j(k_2); H_n).$$

Let $CF^*_{\mathcal{J}^\gamma_{\alpha\beta}}(L_i(k_1), L_j(k_2); H_n)$ and $CF^*_{\mathcal{C}_{\alpha\beta}}(L_i(k_1), L_j(k_2); H_n)$ be subspaces generated by $\mathcal{J}^\gamma_{\alpha\beta}(L_i(k_1), L_j(k_2); H_n)$ and $\mathcal{C}_{\alpha\beta}(L_i(k_1), L_j(k_2); H_n)$, respectively.

Let $\mathcal{X}_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2))$ be the generators of the wrapped Floer complex $CW^*_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2))$ given in equation (3.5.1). We can similarly define subsets of generators $\mathcal{J}^\gamma_{\alpha\beta} = \bigcup_{n=n_0}^{\infty} \mathcal{J}^\gamma_{\alpha\beta}(H_n)/\sim$ and $\mathcal{C}_{\alpha\beta} = \bigcup_{n=n_0}^{\infty} \mathcal{C}_{\alpha\beta}(H_n)/\sim$. We can also define $CW^*_{\mathcal{J}^\gamma_{\alpha\beta}}(L_i(k_1), L_j(k_2))$ and $CW^*_{\mathcal{C}_{\alpha\beta}}(L_i(k_1), L_j(k_2))$ as subspaces of $CW^*_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2))$ generated by $\mathcal{J}^\gamma_{\alpha\beta}$ and $\mathcal{C}_{\alpha\beta}$, respectively.

Note that $\mathcal{X}_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2); H_n) = \mathcal{X}_{P_{\alpha\beta\gamma}}(L_i(k_1), L_j(k_2); nH_1)$ because nH_1 only differs from H_n by a constant on each edge and both Hamiltonians only act on the edges (and the small perturbations inside the pants yield the same generators as well).

Lemma 3.8.1. *Given*

$$x_1 \in \mathcal{X}_{P_{\alpha\beta\gamma}}(L_{i_0}(k_0), L_{i_1}(k_1)), \ldots, x_d \in \mathcal{X}_{P_{\alpha\beta\gamma}}(L_{i_{d-1}}(k_{d-1}), L_{i_d}(k_d)),$$

where at least one of $x_j, j = 1, ..., d$ is in $\mathcal{J}^{\gamma}_{\alpha\beta}$, then the output $y = \mu^d_{P_{\alpha\beta\gamma}}(x_1, \ldots, x_d)$ is in $CW^*_{\mathcal{J}^{\gamma}_{\alpha\beta}}(L_{i_0}(k_0), L_{i_d}(k_d))$.

Proof. From the definition of the wrapped Floer complex, there is a sufficiently large N, dependent on $x_1, \ldots, x_d$, such that for all $n \geq N$, $x_j \in \mathcal{X}_{P_{\alpha\beta\gamma}}(L_{i_{j-1}}(k_{j-1}), L_{i_j}(k_j))$ has a representative $x_j \in \mathcal{X}_{P_{\alpha\beta\gamma}}(L_{i_{j-1}}(k_{j-1}), L_{i_j}(k_j); H_n)$, for all $j = 1, \ldots, d$ (we use the same notation for the representative x_j for convenience). Also at least one of x_j is in $\mathcal{J}^{\gamma}_{\alpha\beta}(H_n)$.

To prove this lemma, we would like to show that there is a sufficiently large integer N, dependent on $x_1, \ldots, x_d$, such that for all $n \geq N$, any generator y appearing in the output of the product

$$\mu^d_{P_{\alpha\beta\gamma}}(x_d, \ldots, x_1; H_n) \in CF^*_{P_{\alpha\beta\gamma}}(L_{i_0}(k_0), L_{i_d}(k_d); dH_n)$$

is in $\mathcal{J}^{\gamma}_{\alpha\beta}(dH_n)$.

We prove by contradiction. Suppose the output y can be in $\mathcal{C}_{\alpha\beta}(L_{i_0}(k_0), L_{i_d}(k_d); dH_n)$ for infinitely many values of n. Due to this assumption, $i_0, i_d \in \{\alpha, \beta\}$. This output y is given by the count of index $(2-d)$ pseudo-holomorphic discs (with a modified almost-complex structure) with boundaries on $\phi^d_n(L_{i_0}(k_0)), \phi^{d-1}_n(L_{i_1}(k_1)), \ldots,$ and $L_{i_d}(k_d)$ and with strip-like ends converging to intersection points $\phi^{d-1}_n(x_1) \in \phi^d_n(L_{i_0}(k_0)) \cap \phi^{d-1}_n(L_{i_1}(k_1))$, $\ldots$, $x_d \in \phi^1_n(L_{i_{d-1}}(k_{d-1})) \cap L_{i_d}(k_d)$, and $y \in \phi^d_n(L_{i_0}(k_0)) \cap L_{i_d}(k_d)$.

From now on we will only consider the universal cover of $e_{\alpha\beta}$ and lifts of all Lagrangians in $e_{\alpha\beta}$ to this universal cover. We keep the same notation for convenience.

The boundary of a holomorphic disc satisfying the above traces out two curves on $e_{\alpha\beta}$ starting at y, each of which is connected and piecewise smooth. One curve C_1 consists of boundary arcs in $\phi^d_n(L_{i_0}(k_0))$, $\phi^{d-1}_n(L_{i_1}(k_1))$, $\ldots$, $\phi^{d-c_1}_n(L_{i_{c_1}}(k_{c_1}))$. The other curve C_2 consists of boundary arcs in $L_{i_d}(k_d)$, $\phi^1_n(L_{i_{d-1}}(k_{d-1}))$, $\ldots$, $\phi^{c_2}_n(L_{i_{d-c_2}}(k_{d-c_2}))$. We choose the largest possible c_1 and c_2 so that the four conditions below are satisfied. We list the first three now and the fourth later:

1. all of $i_0, \ldots, i_{c_1}, i_d, i_{d-1}, \ldots, i_{d-c_2} \in \{\alpha, \beta\}$;

2. C_1 and C_2 do not contain any input intersection points that are not in $e_{\alpha\beta}$;

3. $d - c_2 > c_1$.

Note that if the boundary of the holomorphic disc leaves $e_{\alpha\beta}$ and enters back into $e_{\alpha\beta}$ again, then it must go through an intersection point outside $e_{\alpha\beta}$. This is because if it doesn't go through an intersection point outside $e_{\alpha\beta}$, then it must create a boundary branch point by backtracking along a Lagrangian, but a rigid holomorphic disc does not have any boundary branch points. Hence, C_1 and C_2 are connected, and they don't leave $e_{\alpha\beta}$ and then enter back.

A fourth condition for choosing c_1 and c_2 becomes necessary when C_1 and C_2 satisfying conditions (1)-(3) intersect at an input point, which happens if and only if the entire holomorphic disc is contained in $e_{\alpha\beta}$ with its boundary being the closed loop $C_1 \cup C_2$. In this case, $d - c_2 - 1 = c_1$. Let $\tau_c = \min\{\tau | (\tau, \psi) \in C_1 \cup C_2)\}$. Every Lagrangian involved has the property that its lift intersects each fiber of the universal cover of $e_{\alpha\beta}$ at only one point. Hence τ_c must be the τ coordinate of an intersection point $\tilde{x}_c = \phi_n^{d-c}(x_c) \in \phi_n^{d-(c-1)}(L_{i_{c-1}}(k_{c-1})) \cap \phi_n^{d-c}(L_{i_c}(k_c))$, i.e. $\tau_c = \tau(\tilde{x}_c)$. To summarize, we require that

(4) if the holomorphic disc is contained in $e_{\alpha\beta}$, then choose $d - c_2 - 1 = c_1 = c$ where $\tau(\tilde{x}_c) = \min\{\tau | (\tau, \psi) \in C_1 \cup C_2\}$.

Figure 3.6 illustrates the subset of the holomorphic disc which lies inside $e_{\alpha\beta}$ with boundary C_1 and C_2.

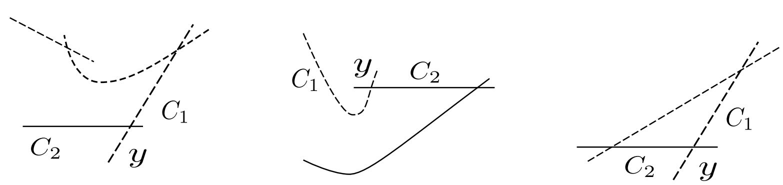


Figure 3.6: The subset of the holomorphic disc which lies inside $e_{\alpha\beta}$ has boundary on $C_1 \cup C_2$. The dashed curves represent C_1. The right-most picture illustrates the situation where the curves C_1 and C_2 intersect at an input point.

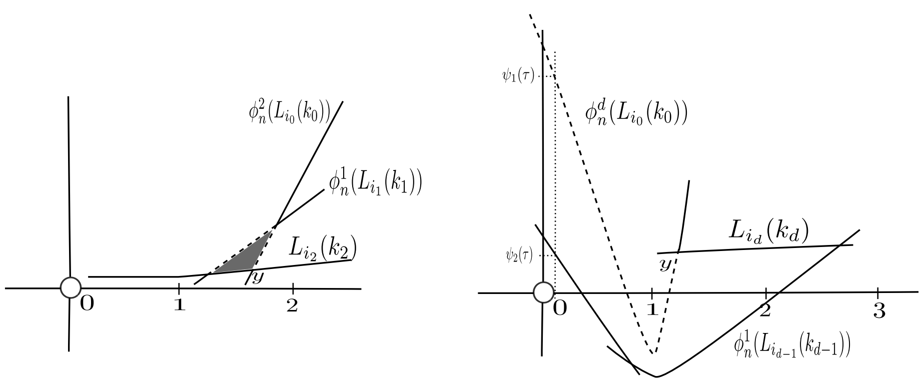
Observe for each $\tau < \tau(y)$, the fiber over τ intersects C_1 and C_2 at no more than one unique point $(\tau, \psi_1(\tau)) \in C_1$ and one unique point $(\tau, \psi_2(\tau)) \in C_2$. We show this observation is true by contradiction. If the fiber of the universal cover over some $\tau < \tau(y)$ intersects C_1 at more than one point, then an interior point of C_1 must be an intersection point $\tilde{x}_b = \phi_n^{d-b}(x_b) \in \phi_n^{d-(b-1)}(L_{i_{b-1}}(k_{b-1})) \cap \phi_n^{d-b}(L_{i_b}(k_b))$ at which the τ coordinate of C_1 backtracks, meaning that $\tau(\tilde{x}_b)$ is the minimum value of τ in an open neighborhood of $\tilde{x}_b$ in C_1. The lift of each Lagrangian intersects each fiber (level of τ) of the universal cover of $e_{\alpha\beta}$ at only one point, the rigid holomorphic disc has no branch point, and each corner of the disc is convex. For these reasons, the τ coordinate cannot backtrack more than once along $C_1 \cup C_2$, and where it backtracks, $\tau(\tilde{x}_b)$ is actually the minimum value of τ achieved by the holomorphic disc, i.e. the holomorphic disc is contained in $\{(\tau, \psi) \in e_{\alpha\beta} | \tau \geq \tau(\tilde{x}_b)\}$. From property (4), $b = c$ and $\tilde{x}_b$ is the end point of C_1, contradicting $\tilde{x}_b$ being an interior point of C_1. The same reasoning can be applied for C_2.

Let's choose $N \gg |k_0|, \ldots, |k_d|$, then for every $\tau \in (0, \tau(y))$, the absolute value of the slope of the tangent line to C_1 at $(\tau, \psi_1(\tau))$ is greater than that of C_2 at $(\tau, \psi_2(\tau))$. Also note that from inside the holomorphic disk, the angle between tangent lines $T_y\phi_n^d(L_{i_0}(k_0))$ and $T_yL_{i_d}(k_d)$ must be greater than $\pi/2$ due to the ordering of the boundary Lagrangians and the assumption that $n > |k_0|, |k_d|$.

We want to show for $n > N$ sufficiently large, the holomorphic disc under consideration cannot have any input in $\mathcal{J}^{\gamma}_{\alpha\beta}$. We analyse two cases.

Case 1: the holomorphic disc is contained in $e_{\alpha\beta}$. Use the same notation that we used when explaining property (4). The holomorphic disc is contained in $\{\tau \geq \tau(\tilde{x}_c)\}$ for an input $\tilde{x}_c$. The slope of the tangent lines $T_{\tilde{x}_c}\phi_n^{d-(c-1)}(L_{i_{c-1}}(k_{c-1}))$ and $T_{\tilde{x}_c}\phi_n^{d-c}(L_{i_c}(k_c))$ have the same sign, i.e. negative if $\tau(\tilde{x}_c) \in (0, 1)$ and positive if $\tau(\tilde{x}_c) \in (1, 3)$. Thus the angle between these tangent lines, from inside the holomorphic disc, must be less than $\pi/2$. The input $\tilde{x}_c$ must be in $\mathcal{C}_{\alpha\beta}(H_n)$, i.e. $\tau(\tilde{x}_c) > 1$, because the angle at any input in the positively wrapped region is less than $\pi/2$ and in the negatively wrapped region is greater than $\pi/2$. This is due to the ordering of the boundary Lagrangians and the assumption that $n \gg |k_0|, \ldots, |k_d|$. So we just have a triangle in $\{\tau > 1\} \subset e_{\alpha\beta}$ with no inputs in $\mathcal{J}^{\gamma}_{\alpha\beta}$. (See Fig. 3.7a.)

Case 2: the holomorphic disc is not contained in $e_{\alpha\beta}$. As explained before, for every $\tau < \tau(y)$, the fiber over τ intersects C_1 once and C_2 once, with $\psi_1(\tau) > \psi_2(\tau)$ as illustrated in Figure 3.7b. Note that all boundary Lagrangians $\phi_n^{d-j}(L_{i_j}(k_j))$ are dependent on n, so are ψ_1, ψ_2. The value of $\psi_1(0)$ and $\psi_2(0)$ are dictated by inputs outside of $e_{\alpha\beta}$, so they will stay almost constant as a function of n. Indeed, $\psi_1(0)$ and $\psi_2(0)$ are determined by the remaining portion of the boundary of the disc (other than $C_1 \cup C_2$). As n varies and the inputs $\tilde{x}_{c_1}, \ldots, \tilde{x}_{d-c_2}$ move by continuation, this boundary curve varies by a homotopy inside the cylindrical ends and remains constant inside the pants, and in particular $\psi_1(0)$ and $\psi_2(0)$ remain constant. Hence $\psi_1(0) > \psi_2(0)$ always, but their difference remains bounded no matter how large n gets. However, as n gets large enough, $\psi_1(1) < \psi_2(1)$ because the absolute value of the slope of the tangent line to C_1 at each $(\tau, \psi_1(\tau))$ is greater than that of C_2 at $(\tau, \psi_2(\tau))$ and the slopes increase with n. Hence C_1 crosses C_2 before reaching $\tau = 1$, which contradicts an assumption that $y \in \mathcal{C}_{\alpha\beta}(dH_n)$. Hence, we can pick a sufficiently large N, so that for all $n \geq N$, there does not exist any rigid holomorphic disc with the given inputs $x_1, \ldots, x_d$, at least one of which lies in $\mathcal{J}^\gamma_{\alpha\beta}(H_n)$ and its output in $\mathcal{C}_{\alpha\beta}(dH_n)$.



(a) (b)

Figure 3.7: dashed line is C_1

□

Corollary 3.8.2. *There is a restriction map*

$$\rho^\gamma_{\alpha\beta} : \bigoplus_{L_i, L_j} CW^*_{P_{\alpha\beta\gamma}}(L_i, L_j) \to \bigoplus_{L_i, L_j} CW^*_{\mathcal{C}_{\alpha\beta}}(L_i, L_j)$$

which is a quotient by $CW^*_{\mathcal{J}^\gamma_{\alpha\beta}}(L_i, L_j)$, and it is compatible with A_∞ structures with no higher order terms.

Proof. It follows from Lemma 3.8.1. □

Lemma 3.8.3. *Given inputs $x_1 \in \mathcal{X}(L_{i_0}(k_0), L_{i_1}(k_1)), \dots, x_d \in \mathcal{X}(L_{i_{d-1}}(k_{d-1}), L_{i_d}(k_d))$ and output $y = \mu^d(x_1, \dots, x_d) \in \mathcal{X}(L_{i_0}(k_0), L_{i_d}(k_d))$, there is a single pair of pants $P_0 \subset H$ and a sufficiently large N, dependent on $x_1, \dots, x_d, y$, such that for all $n \geq N$, there are representatives $x_j \in \mathcal{X}_{P_0}(L_{i_{j-1}}(k_{j-1}), L_{i_j}(k_j); H_n), y \in \mathcal{X}_{P_0}(L_{i_0}(k_0), L_{i_d}(k_d); dH_n)$ for all $j = 1, \dots, d$. Hence, $\mu^d(x_1, \dots, x_d) = \mu^d_{P_0}(x_1, \dots, x_d)$.*

Proof. From the assumptions, there is a sufficiently large N_0, dependent on $x_1, \dots, x_d, y$, such that for all $n \geq N_0$, x_j has a representative $x_j \in \mathcal{X}(L_{i_{j-1}}(k_{j-1}), L_{i_j}(k_j); H_n)$, for all $j = 1, \dots, d$, and $y \in \mathcal{X}_{P_0}(L_{i_0}(k_0), L_{i_d}(k_d); dH_n)$. Suppose the holomorphic disc has at least one of its vertices is contained in a pair of pants $P_{\alpha\beta\gamma}$, but not in an adjacent pair of pants $P_{\alpha\beta\eta}$, and at least one of its vertices is contained in $P_{\alpha\beta\eta}$ but is not contained in $P_{\alpha\beta\gamma}$. We will show that such a holomorphic disc cannot exist. The same arguments also excludes discs in which an edge goes into $P_{\alpha\beta\eta}$ to reach an vertex lying in another pair of pants even further away from $P_{\alpha\beta\gamma}$. Hence, all intersections points are actually in a single pair of pants. Note that $P_{\alpha\beta\gamma}$ and $P_{\alpha\beta\eta}$ overlap on the cylinder $e_{\alpha\beta}$. We focus on the portion of the disc that lies in $e_{\alpha\beta}$, and the manner in which it escapes into both ends of the cylinder. Again, we view perturbed pseudo-holomorphic discs as ordinary peudo-holomorphic discs with perturbed boundaries as explained in Section 3.3.

None of the inputs can be in $\mathcal{C}_{\alpha\beta}(H_n)$, the positively wrapped region shared by both pairs of pants. This is due to what we noticed before (in the proof of Lemma 3.8.1) that the angle at any input in the positively wrapped region is less than $\pi/2$. Then by convexity, the holomorphic disc will not go beyond that input point, i.e. it does not escape into both $P_{\alpha\beta\gamma} \backslash e_{\alpha\beta}$ and $P_{\alpha\beta\eta} \backslash e_{\alpha\beta}$. Hence, in the region $[1, 3] \times S^1 \subset e_{\alpha\beta}$, either there's no vertex at all, or there is one output.

First, assume the output is not in the positively wrapped region, then there's no intersection point at all in this region. Consider the universal cover of the cylinder $e_{\alpha\beta}$ and lifts of all Lagrangians in $e_{\alpha\beta}$ to this universal cover. We can find two boundary arcs in Lagrangians $\phi^s_n(L_{i_{d-s}}(k_{d-s}))$ and $\phi^t_n(L_{i_{d-t}}(k_{d-t}))$ that are non-parallel lines and that they intersect the fiber of the universal cover over $\tau = 2$ at $\psi_1(2)$ and $\psi_2(2)$, repsectively. The Hamiltonian perturbations we use actually keep $\psi_1(2)$ and $\psi_1(2)$ constant as n varies, and the input marked points move along by continuation. So, these two lines will cross each other for all $n > N = |\frac{\psi_1(2)-\psi_2(2)}{2\pi(t-s)}|$. So there are no holomorphic discs with the assumed inputs and outputs for sufficiently large n.

Now, suppose the output y is in the positively wrapped region. In the universal cover of $e_{\alpha\beta}$, the disc will look like one of the configurations in Figure 3.8. The output point is bounded by Lagrangians $L_{i_d}(k_d)$ and $\phi_n^d(L_{i_0}(k_0))$, which have the biggest and smallest slopes. When we increase n to be large enough, $\psi_1(2)$ and $\psi_2(2)$ remain constant as before, and the boundary arcs will cross each other again inside the positively wrapped region, which rules out the existence of such holomorphic discs.

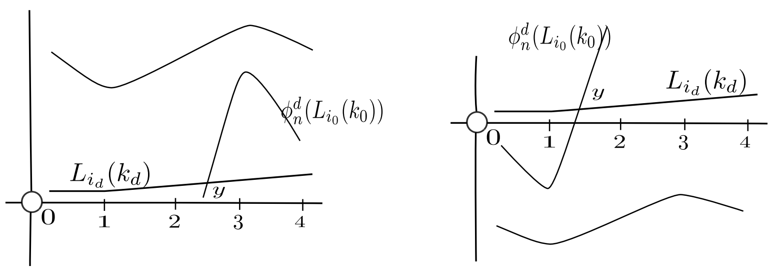


Figure 3.8:

$\square$

Theorem 3.8.4. *The wrapped Fukaya category $\mathcal{W}(H)$ is split-generated by the Lagrangian objects $L_\alpha(k)$ where $\alpha \in A$ and $k \in \mathbb{Z}$. In a suitable model for $\mathcal{W}(H)$, the morphism complex between any two objects, $L_\alpha(k)$ and $L_\beta(l)$, is generated by*

$$\mathcal{X}(L_\alpha(k), L_\beta(l)) = \left(\bigcup \mathcal{X}_{P_{\alpha\beta\gamma}}(L_\alpha(k), L_\beta(l))\right) / \sim \tag{3.8.2}$$

where $\mathcal{X}_{P_{\alpha\beta\gamma}}(L_\alpha(k), L_\beta(l))$ is the set of generators of the morphism complexes in $\mathcal{W}(P_{\alpha\beta\gamma})$, and the equivalence relation identifies $x \in \mathcal{X}_{P_{\alpha\beta\gamma}\backslash \mathcal{J}^\gamma_{\alpha\beta}}$ with $y \in \mathcal{X}_{P_{\alpha\beta\eta}\backslash \mathcal{J}^\eta_{\alpha\beta}}$ whenever $\rho^\gamma_{\alpha\beta}(x) = \rho^\eta_{\alpha\beta}(y)$, where $\rho^\gamma_{\alpha\beta}$, $\rho^\eta_{\alpha\beta}$ are restriction maps to the cylinder $\mathcal{C}_{\alpha\beta} = P_{\alpha\beta\gamma} \cap P_{\alpha\beta\eta}$. Moreover, in this model, the A_∞-products in $\mathcal{W}(H)$ are given by those in the pairs of pants.

To clarify the last statement in Theorem 3.8.4, any product $\mu^d(x_1, \ldots, x_d)$ vanishes unless $x_1, \ldots, x_d$ all live in a single pair of pants $P_{\alpha\beta\gamma}$ (by Lemma 3.8.3), in which case we take the product inside $P_{\alpha\beta\gamma}$. If in fact the generators $x_1, \dots, x_d$ all live inside the same bounded cylinder $\mathcal{C}_{\alpha\beta}$, then so do the discs with these inputs (by the preceding arguments in Lemma 3.8.1), and the calculations A_∞-products in $P_{\alpha\beta\gamma}$ and $P_{\alpha\beta\eta}$ give the same answer, i.e. the product is compatible with the equivalence relation. On the other hand, if at least one of the given inputs $x_1, \dots, x_d$ lives in $\mathcal{J}^\gamma_{\alpha\beta}$, then so does the output by Lemma 3.8.1, and so there is no question of compatibility with the equivalence relation.

Proof. Follows from Lemma 3.8.3. □

We label the $2n - n_{\alpha\beta}(k-l)+1$ generators in $\mathcal{C}_{\alpha\beta}(L_\alpha(k), L_\alpha(l); H_n)$ by $x^{-n}_{\alpha\beta}, \dots, x^{-1}_{\alpha\beta}, x^0_{\alpha\beta}$, $x^1_{\alpha\beta}, \dots, x^{n-n_{\alpha\beta}(l-k)}_{\alpha\beta}$, successively with the minimum of the Hamiltonian labelled as $x^0_{\alpha\beta}$ as illustrated in Fig. 3.9. Each Floer complex $CW^*_{P_{\alpha\beta\gamma}}(L_\alpha(k), L_\beta(l))$ is a $CW^*_{P_{\alpha\beta\gamma}}(L_\alpha(k), L_\alpha(k))$–$CW^*_{P_{\alpha\beta\gamma}}(L_\beta(l), L_\beta(l))$-bimodule. We label the $2n - n_{\alpha\beta}(l-k) - d_{\alpha,\beta} - 1$ generators in $\mathcal{C}_{\alpha\beta}(L_\alpha(k), L_\beta(l); H_n)$ by $x^{-n}_{\alpha;\beta}, \dots, x^{-1}_{\alpha;\beta}$, $x^0_{\alpha;\beta}, \dots, x^{n-n_{\alpha\beta}(l-k)-d_{\alpha,\beta}-2}_{\alpha;\beta}$ (Fig. 3.9).

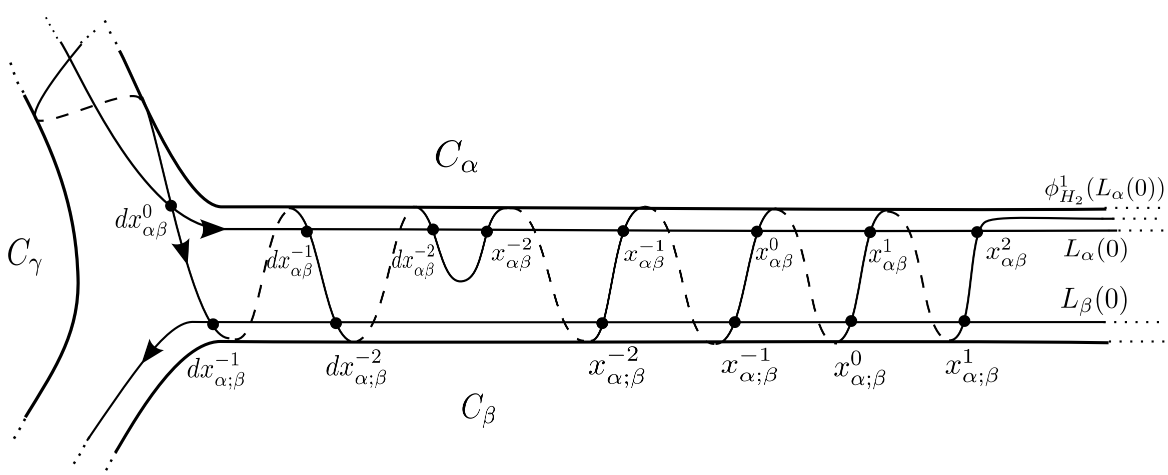


Figure 3.9: Generators in $\mathcal{C}_{\alpha\beta}(L_\alpha(0), L_\alpha(0); H_2)$ and $\mathcal{C}_{\alpha\beta}(L_\alpha(0), L_\beta(0); H_2)$; assuming $d_{\alpha,\beta} = -1$.

The object $L_\alpha(k)$ is equivalent to $L_\alpha(l)$ in the category $\mathcal{W}(P_{\alpha\beta\gamma})$, and similarly in $\mathcal{W}(P_{\alpha\beta\eta})$. However, they can be distinguished as the object for which the generator $x^i \in \mathcal{C}_{\alpha\beta}(L_\alpha(k), L_\alpha(l))$ in the image of the restriction functor $\rho^\gamma_{\alpha\beta}$ is identified with the generator $\tilde{x}^{-(i+n_{\alpha\beta}(l-k))} \in \mathcal{C}_{\alpha\beta}(L_\alpha(k), L_\alpha(l))$ in the image of $\rho^\eta_{\alpha\beta}$. In addition,$x^i_{\alpha;\beta}$ is identified with $\tilde{x}^{-(i+n_{\alpha\beta}(l-k)+d_{\alpha,\beta}+2)}_{\alpha;\beta}$.

Chapter 4

The Landau-Ginzburg mirror

4.1 Generating objects.

The category of matrix factorizations $MF(X,W)$ is defined to be the Verdier quotient of $MF^{naive}(X,W)$ by the subcategory $Ac(X,W)$ of acyclic elements [**AAEKO13**, **LP11**, **Or11**]. The objects of MF^{naive} are

$$T := T_1 \underset{t_0}{\overset{t_1}{\rightleftarrows}} T_0 ,$$

where T_1, T_0 are locally free sheaves of finite rank on X, and t_1, t_0 are morphisms satisfying $t_{i+1} \circ t_i = W \cdot id_{T_i}$. The morphism complex

$$\mathcal{H}om(S,T) = \oplus_{i,j}\mathcal{H}om(S_i,T_j)$$

is graded by $(i+j)$mod 2, i.e. even and odd, that is

$$\mathcal{H}om^{even}(S,T) = \mathcal{H}om(S_0,T_0) \oplus \mathcal{H}om(S_1,T_1),$$

$$\mathcal{H}om^{odd}(S,T) = \mathcal{H}om(S_0,T_1) \oplus \mathcal{H}om(S_1,T_0).$$

The differential on this complex is $d : f \mapsto t \circ f - (-1)^{|f|} f \circ s$. The equivalence $MF(X,W) \xrightarrow{\sim} D^b_{sg}(D)$ is given by $T \mapsto \operatorname{coker}(t_1)$, which is a sheaf on D because it is annihilated by W.

For each irreducible divisor $D_\alpha = \{t_{\alpha;1} = 0\}$, $t_{\alpha;1} : \mathcal{O}(-D_\alpha) \to \mathcal{O}$ is an injective sheaf homomorphism. It also induces a map $t_{\alpha;1} : \mathcal{O}(-D_\alpha)(k) \to \mathcal{O}(k)$ for any $k \in \mathbb{Z}$ (with the notation $\mathcal{O}(k) := \mathcal{O}(1)^{\otimes k}$, where $\mathcal{O}(1)$ is the polarization (ample line bundle) on X determined by the polytope Δ_X). For each $\alpha \in A, k \in \mathbb{Z}$, we consider

$$T_\alpha(k) := \mathcal{O}(-D_\alpha)(k) \underset{t_{\alpha;0}}{\overset{t_{\alpha;1}}{\rightleftarrows}} \mathcal{O}(k) . \tag{4.1.1}$$

in $MF(X,W)$. The object $T_\alpha(k)$ in $D^b_{sg}(D)$ is $\operatorname{coker}(t_{\alpha;1}) = \mathcal{O}_{D_\alpha}(k)$. An argument similar to that in Section 6 of [**AAEKO13**] implies that:

Lemma 4.1.1. *$MF(X,W)$ is split-generated by objects $T_\alpha(k), k \in \mathbb{Z}, \alpha \in A$.*

4.2 Čech model and homotopic restriction functors

Let

$$\mathfrak{U} = \{U_{\alpha\beta\gamma} = \mathbb{C}[x_{\alpha\beta}, x_{\alpha\gamma}, x_{\beta\gamma}], W = x_{\alpha\beta}x_{\alpha\gamma}x_{\beta\gamma}\}_{\alpha,\beta,\gamma\in A \text{ adjacent}}$$

be a finite covering of (X,W) by affine toric subsets. The moment polytope of each $U_{\alpha\beta\gamma}$ corresponds to a corner of Δ_X given by $C_\alpha, C_\beta, C_\gamma$. The divisor D restricted to $U_{\alpha\beta\gamma}$ is equal to the restriction $(D_\alpha \cup D_\beta \cup D_\gamma)|_{U_{\alpha\beta\gamma}}$, with $D_\alpha|_{U_{\alpha\beta\gamma}} = \{x_{\beta\gamma} = 0\}$, $D_\beta|_{U_{\alpha\beta\gamma}} = \{x_{\alpha\gamma} = 0\}$, $D_\gamma|_{U_{\alpha\beta\gamma}} = \{x_{\alpha\beta} = 0\}$, and W is locally given by the product of these affine coordinates. The object $T_\alpha(k)$ restricted to this local chart is

$$T_\alpha(k)(U_{\alpha\beta\gamma}) = \mathcal{O}(-D_\alpha)(k)(U_{\alpha\beta\gamma}) \underset{x_{\alpha\beta}x_{\alpha\gamma}}{\overset{x_{\beta\gamma}}{\rightleftarrows}} \mathcal{O}(k)(U_{\alpha\beta\gamma}) ,$$

and similarly for $T_\beta(k)(U_{\alpha\beta\gamma})$ and $T_\gamma(k)(U_{\alpha\beta\gamma})$. Note that $T_\alpha(k)(U_{\alpha\beta\gamma})$ is actually equivalent to $T_\alpha(0)(U_{\alpha\beta\gamma})$ in the category of matrix factorizations on the local chart $U_{\alpha\beta\gamma}$.

There is a restriction functor, simply to the complement of the coordinate plane $\{x_{\alpha\beta} = 0\}$,

$$\sigma^\gamma_{\alpha\beta} : MF\left(U_{\alpha\beta\gamma}, x_{\alpha\beta}x_{\alpha\gamma}x_{\beta\gamma}\right) \to MF\left(\mathbb{C}^*[x_{\alpha\beta}] \times \mathbb{C}[x_{\alpha\gamma}, x_{\beta\gamma}], x_{\alpha\beta}x_{\alpha\gamma}x_{\beta\gamma}\right) \cong D^b(\mathrm{Coh}(\mathbb{C}^*[x_{\alpha\beta}])),$$

where $\mathbb{C}^*[x_{\alpha\beta}]$ is the $\mathbb{C}^*$ with ring of functions $\mathbb{C}[x^\pm_{\alpha\beta}]$. The equivalence

$$MF\left(\mathbb{C}^*[x_{\alpha\beta}] \times \mathbb{C}[x_{\alpha\gamma}, x_{\beta\gamma}], x_{\alpha\beta}x_{\alpha\gamma}x_{\beta\gamma}\right) \cong D^b(\mathrm{Coh}(\mathbb{C}^*[x_{\alpha\beta}]))$$

is due to the Knörrer periodicity theorem [**Or04**, **Kn87**]. In this section, we will check the commutativity of the following diagram up to the first order

$$
\begin{tikzcd}
\mathcal{A} := \mathcal{W}\left(\coprod P_{\alpha\beta\gamma}\right) \arrow[r, "\rho"] \arrow[d, "q_a"'] & \mathcal{B} := \mathcal{W}\left(\coprod \mathcal{C}_{\alpha\beta}\right) \arrow[d, "q_b"'] \\
\mathcal{A}' := MF\left(\coprod (U_{\alpha\beta\gamma}, x_{\alpha\beta}x_{\alpha\gamma}x_{\beta\gamma})\right) \arrow[r, "\sigma"] & \mathcal{B}' := D^b(\mathrm{Coh}(\coprod \mathbb{C}^*[x_{\alpha\beta}])).
\end{tikzcd}
$$

In this diagram, ρ is the restriction functor from Section 3.8. Due to HMS for pairs of pants (as shown in [**AAEKO13**]), the category $\mathcal{A}$ is quasi-equivalent to $\mathcal{A}'$ via the A_∞-functor q_a. The categories B and B' are quasi-equivalent due to HMS for cylinders, and the functor q_b has no higher order terms. We want to show that the two A_∞ restriction functors

$$\mathcal{F} = q_b \circ \rho, \quad \mathcal{G} = \sigma \circ q_a : \mathcal{A} \to \mathcal{B}'$$

are equal up to the first order.

The following diagram shows the morphisms $(f_1, f_0) \in \mathcal{H}om^0(T_\alpha(k)(U_{\alpha\beta\gamma}), T_\alpha(l)(U_{\alpha\beta\gamma}))$ and $(h_0, h_1) \in \mathcal{H}om^1(T_\alpha(k)(U_{\alpha\beta\gamma}), T_\alpha(l)(U_{\alpha\beta\gamma}))$,

$$
\begin{tikzcd}
\mathcal{O}(-D_\alpha)(k)(U_{\alpha\beta\gamma}) \arrow[r, shift left, "x_{\beta\gamma}"] \arrow[d, "f_1"'] \arrow[dr, "h_1"] & \mathcal{O}(k)(U_{\alpha\beta\gamma}) \arrow[l, shift left, "x_{\alpha\beta}x_{\alpha\gamma}"] \arrow[d, "f_0"] \arrow[dl, "h_0"'] \\
\mathcal{O}(-D_\alpha)(l)(U_{\alpha\beta\gamma}) \arrow[r, shift left, "x_{\beta\gamma}"] & \mathcal{O}(l)(U_{\alpha\beta\gamma}) \arrow[l, shift left, "x_{\alpha\beta}x_{\alpha\gamma}"]
\end{tikzcd}
.
$$

The differential maps

$$(f_1, f_0) \mapsto (x_{\beta\gamma}(f_1 - f_0), x_{\alpha\beta}x_{\alpha\gamma}(f_1 - f_0)),$$

$$(h_0, h_1) \mapsto (x_{\alpha\beta}x_{\alpha\gamma}h_1 - x_{\beta\gamma}h_0, x_{\alpha\beta}x_{\alpha\gamma}h_1 - x_{\beta\gamma}h_0).$$

Hence in cohomology

$$\mathrm{Hom}^0(T_\alpha(k)(U_{\alpha\beta\gamma}), T_\alpha(l)(U_{\alpha\beta\gamma})) \cong \mathcal{O}(l-k)(U_{\alpha\beta\gamma})/(x_{\beta\gamma}, x_{\alpha\beta}x_{\alpha\gamma}),$$

$$\mathrm{Hom}^1(T_\alpha(k)(U_{\alpha\beta\gamma}), T_\alpha(l)(U_{\alpha\beta\gamma})) = 0.$$

Restricting via $\sigma^\gamma_{\alpha\beta}$ gives

$$\mathrm{Hom}^0(T_\alpha(k)(U_{\alpha\beta\gamma}), T_\alpha(l)(U_{\alpha\beta\gamma})) \cong \mathcal{O}(l-k)(U_{\alpha\beta\gamma})/(x_{\beta\gamma}, x_{\alpha\gamma}) \cong \mathcal{O}_{D_{\alpha\beta}|U\alpha\beta\gamma}(l-k),$$

where $D_{\alpha\beta}|U_{\alpha\beta\gamma} = (D_\alpha \cap D_\beta)|U_{\alpha\beta\gamma} \cong \mathbb{C}^*[x_{\alpha\beta}]$. Hence $T_\alpha(k)(U_{\alpha\beta\gamma})$, $T_\alpha(l)(U_{\alpha\beta\gamma})$ and $T_\alpha(k)(U_{\alpha\beta\eta})$, $T_\alpha(l)(U_{\alpha\beta\eta})$ are respective objects in $MF(U_{\alpha\beta\gamma}, x_{\alpha\beta}x_{\gamma\beta}x_{\alpha\gamma})$ and $MF(U_{\alpha\beta\eta}, x_{\alpha\beta}x_{\eta\beta}x_{\alpha\eta})$ for which the generator $x^i_{\alpha\beta} \in \mathcal{O}_{D_{\alpha\beta}|U\alpha\beta\gamma}(l-k)$ in the image of the restriction function $\sigma^\gamma_{\alpha\beta}$ is identified with $\tilde{x}_{\alpha\beta}^{-(i+n_{\alpha\beta}(l-k))} \in \mathcal{O}_{D_{\alpha\beta}|U\alpha\beta\eta}(l-k)$ in the image of $\sigma^\eta_{\alpha\beta}$. Indeed, the restriction of $\mathcal{O}(1)$ to $D_{\alpha\beta}$ has degree given by the length of the corresponding edge of Δ_X, i.e. $n_{\alpha\beta}$, so $\mathcal{O}(l-k)_{D_{\alpha\beta}}$ has degree $n_{\alpha\beta}(l-k)$.

A similar calculation can be carried out for $\mathrm{Hom}(T_\alpha(k)(U_{\alpha\beta\gamma}), T_\beta(l)(U_{\alpha\beta\gamma}))$. One finds that the restriction functors from adjacent affine charts to their common overlap now identify generators whose degrees add up to the degree of $\mathcal{O}(-D_\alpha)(l-k)_{|D_{\alpha\beta}}$, namely $n_{\alpha\beta}(l-k)+d_{\alpha\beta}$. This matches with the behavior described at the end of Section 3 for the restriction functors in Floer theory, via the natural identification between cohomology-level morphisms on two sides of mirror symmetry as suggested by our notations. Hence the functors $\mathcal{F}$ and $\mathcal{G}$ agree on cohomology as claimed.

The restriction functor makes $\mathcal{B}'$ a module over $\mathcal{A}$, and the Hochschild cohomology $HH^1(\mathcal{A}, \mathcal{B}')$ determines the classification of A_∞-functors from $\mathcal{A}$ to $\mathcal{B}'$ which induce a a given cohomology-level functor; see Section 1h of [**Se08**]. A Hochschild cohomology calculation similar to that in Section 3 of [**AAEKO13**] shows that:

Lemma 4.2.1. *The Hochschild cohomology $HH^1(\mathcal{A}, \mathcal{B}')$ vanishes in all pieces which have length filtration index greater than one.*

Corollary 4.2.2. *Two A_∞-functors from $\mathcal{A}$ to $\mathcal{B}'$ which agree to first order are homotopic.*

Appendix A

A global cohomology level computation

We compute the wrapped Fukaya category $\mathcal{W}(H)$ and the category of singularity $D^b_{sg}(W^{-1}(0))$ at the level of cohomology, using methods very similar to those in [**AAEKO13**]. Aside from using a few earlier notations, this appendix is self contained, independent from the sheaf theoretic computation in the rest of the paper. We show it because it is a straightforward demonstration of HMS, though we do not know how to extend this to compute the higher A_∞-structures. We will list the generators of the morphism complexes for both categories, but we will be very brief in discussing the product structures on the morphism complexes because this computation is not the main point of this paper.

A.1 The wrapped Fukaya category.

We use Abouzaid's model of the wrapped Fukaya category [**Ab10**], with only a Hamiltonian perturbation $H : M \to \mathbb{R}$ that is quadratic in the cylindrical ends. Then the wrapped Floer complex $CW^*(L_\alpha(k), L_\beta(l))$ is generated by the Reeb chords that are the time-1 trajectories of the Hamiltonian flow from $L_\alpha(k)$ to $L_\beta(l)$. Equivalently, up to a change of almost-complex structure, $CW^*(L_\alpha(k), L_\beta(l)) = \langle \phi^1(L_\alpha(k)) \cap L_\beta(l)\rangle$. The product $\mu^2 : CW^*(L_\beta(j), L_\gamma(l)) \otimes CW^*(L_\alpha(j), L_\beta(k)) \to CW^*(L_\alpha(j), L_\gamma(l))$ is equivalent to the usual Floer product $CF^*(\phi^1(L_\beta(j)), L_\gamma(l)) \otimes CF^*(\phi^2(L_\alpha(j)), \phi^1(L_\beta(k)) \to CF^*(\phi^2(L_\alpha(j)), L_\gamma(l)),$

which counts holomorphic triangles with boundaries on $\phi^2(L_\alpha(j))$, $\phi^1(L_\alpha(k))$, and $L_\alpha(l)$. The inputs are $\phi^1(p_1) \in \phi^2(L_\alpha(j)) \cap \phi^1(L_\beta(k))$ and $p_2 \in \phi^1(L_\beta(k)) \cap L_\alpha(l)$, and the output is $\tilde{q} \in \phi^2(L_\alpha(j)) \cap L_\alpha(l)$, which corresponds to $q \in \phi^1(L_\alpha(j)) \cap L_\alpha(l)$ by the rescaling method explained in [**Ab10**].

Lagrangians from the same component.

First, we list the generators of $CW^*(L_\alpha(k), L_\alpha(l))$; see Fig. A.1 and A.2. There are two cases.

Case 1: on a bounded edge $e_{\alpha\beta}$. For $k < l$, $L_\alpha(k)$ intersects $L_\alpha(l)$ at $n_{\alpha\beta}(l-k)+1$ points on $e_{\alpha\beta}$, all of which have degree 0. We label each intersection point sequentially by $x_{\alpha\beta}^{n_{\alpha\beta}(l-k)-j} y_{\alpha\beta}^j$, for $j = 0, \ldots, n_{\alpha\beta}(l-k)$. If $e_{\alpha\beta}$ is adjacent to another bounded edge $e_{\alpha\eta}$, then there is a generator that is an interior intersection point at the joint of these two edges labelled by $x_{\alpha\beta}^{n_{\alpha\beta}(l-k)}$ on one side and $y_{\alpha\eta}^{n_{\alpha\eta}(l-k)}$ on the other side; we identify them $x_{\alpha\beta}^{n_{\alpha\beta}(l-k)} = y_{\alpha\eta}^{n_{\alpha\eta}(l-k)}$. For $k > l$, $L_\alpha(k)$ intersects $L_\alpha(l)$ at $n_{\alpha\beta}(k-l)-1$ points of degree 1, and we label them sequentially by $\left(x_{\alpha\beta}^{n_{\alpha\beta}(k-l)-2-j} y_{\alpha\beta}^j\right)^*$, for $j = 0, \ldots, n_{\alpha\beta}(k-l)-2$. When $e_{\alpha\beta}$ is adjacent to another bounded edge $e_{\alpha\eta}$, we get an extra degree 1 generator that is an interior intersection point at the joint of these two edges. Again, we identify the two labels coming from both sides.

Case 2: on an unbounded edge $e_{\alpha\gamma}$. Let $e_{\alpha\eta}$ be the adjacent edge. If $k < l$, then $\phi^1(L_\alpha(k))$ and $L_\alpha(l)$ have one interior intersection point at the joint of $e_{\alpha\gamma}$ and $e_{\alpha\eta}$, which we label by $x_{\alpha\gamma}^0$, and we label the infinitely many purturbed intersections points on $e_{\alpha\gamma}$ successively by $x_{\alpha\gamma}^j$ for $j = 1, 2, \ldots$. Like before, we need to set $x_{\alpha\gamma}^0$ to equal the label for the same generator coming from the edge $e_{\alpha\eta}$, i.e. $x_{\alpha\gamma}^0 = x_{\alpha\eta}^0$ if $e_{\alpha\eta}$ is unbounded, and $x_{\alpha\gamma}^0 = y_{\alpha\eta}^{n_{\alpha\eta}(l-k)}$ (or $= x_{\alpha\eta}^{n_{\alpha\beta}(l-k)}$ depending on which side $e_{\alpha\gamma}$ is attached to $e_{\alpha\beta}$) if $e_{\alpha\eta}$ is bounded. If $k > l$, then there is no interior intersection point, only infinitely many perturbed intersection points labeled by $x_{\alpha\gamma}^j$ for $j = 1, 2, \ldots$.

As for the products $CW^*(L_\alpha(k), L_\alpha(l)) \otimes CW^*(L_\alpha(j), L_\alpha(k)) \to CW^*(L_\alpha(j), L_\alpha(l))$, there are no holomorphic triangles with vertices in more than one edge. The product

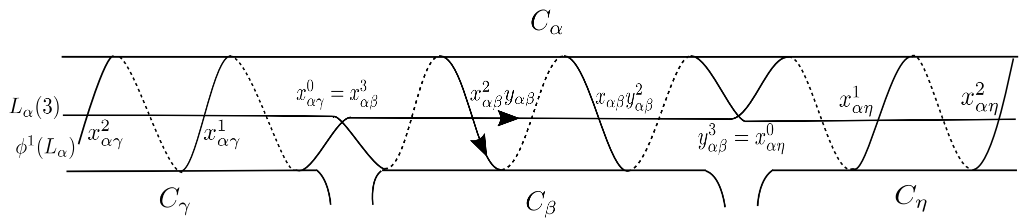


Figure A.1: Generators of $CW^*(L_\alpha, L_\alpha(3))$ assuming $n_{\alpha\beta} = 1$.

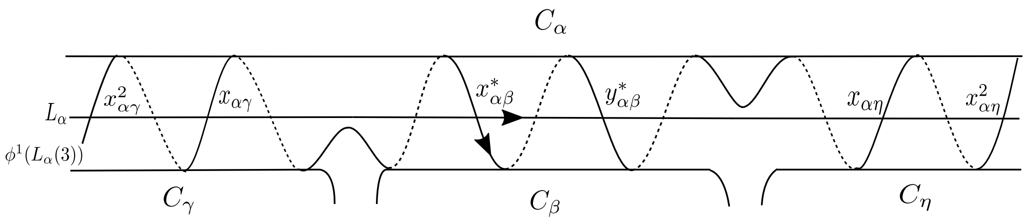


Figure A.2: Generators of $CW^*(L_\alpha(3), L_\alpha)$ assuming $n_{\alpha\beta} = 1$. The generators $x^*_{\alpha\beta}$ and $y^*_{\alpha\beta}$ are of degree 1. Intersection points on unbounded edges are always of degree 0.

$p_2 \cdot p_1 = q$ needs to satisfy $\deg(p_1) + \deg(p_2) = \deg(q) (\mathrm{mod}\ 2)$ where the grading is by $\mathbb{Z}_2$, so there are just the three possibilities listed below. The formulas for the products are obtained by counting holomorphic triangles.

• $CW^0(L_\alpha(k), L_\alpha(l)) \otimes CW^0(L_\alpha(j), L_\alpha(k)) \to CW^0(L_\alpha(j), L_\alpha(l))$, with $x^{p'}_{\alpha\beta} y^{q'}_{\alpha\beta} \cdot x^p_{\alpha\beta} y^q_{\alpha\beta} = x^{p+p'}_{\alpha\beta} y^{q+q'}_{\alpha\beta}$. For generators on a bounded edge, such a product is only possible when $j \leq k \leq l$.

• $CW^0(L_\alpha(k), L_\alpha(l)) \otimes CW^1(L_\alpha(j), L_\alpha(k)) \to CW^1(L_\alpha(j), L_\alpha(l))$, $k < l < j$, with $x^{p'}_{\alpha\beta} y^{q'}_{\alpha\beta} \cdot (x^p_{\alpha\beta} y^q_{\alpha\beta})^* = (x^{p-p'}_{\alpha\beta} y^{p-p'}_{\alpha\beta})^*$.

• $CW^1(L_\alpha(k), L_\alpha(l)) \otimes CW^0(L_\alpha(j), L_\alpha(k)) \to CW^1(L_\alpha(j), L_\alpha(l))$, $l < j < k$, with $(x^{p'}_{\alpha\beta} y^{q'}_{\alpha\beta})^* \cdot x^p_{\alpha\beta} y^q_{\alpha\beta} = (x^{p'-p}_{\alpha\beta} y^{p'-p}_{\alpha\beta})^*$.

Lagrangians from two adjacent components.

In this section, we focus on two adjacent components C_α and C_β. Recall that on the edge $e_{\alpha\beta}$, $L_\alpha(k)$ and $L_\beta(l)$ have the opposite orientations. When the edge $e_{\alpha\beta}$ is unbounded, we do the same as in [**AAEKO13**]; we label the infinite sequence of odd degree intersection points by $x^j_{\alpha;\beta}$, for $j = 0, 1, 2 \ldots$.

On a bounded edge $e_{\alpha\beta}$, if $k < l$, then $L_\alpha(k)$ and $L_\beta(l)$ intersect at $n_{\alpha\beta}(l-k) + d_{\alpha,\beta} + 1$

points of degree 1, and we label them by $x_{\alpha;\beta}^{n_{\alpha\beta}(l-k)+d_{\alpha,\beta}-j}y_{\alpha;\beta}^{j}$ for $j=0,\dots n_{\alpha\beta}(l-k)+d_{\alpha,\beta}$. For $k>l$, $L_\alpha(k)$ and $L_\beta(l)$ intersect at $n_{\alpha\beta}(k-l)+d_{\alpha,\beta}+1$ points of degree 0, and we label them by $\left(x_{\alpha;\beta}^{n_{\alpha\beta}(k-l)+2+d_{\alpha,\beta}-j}y_{\alpha;\beta}^{j}\right)^*$ for $j=0,\dots,n_{\alpha\beta}(k-l)+d_{\alpha,\beta}$. See Figure A.3.

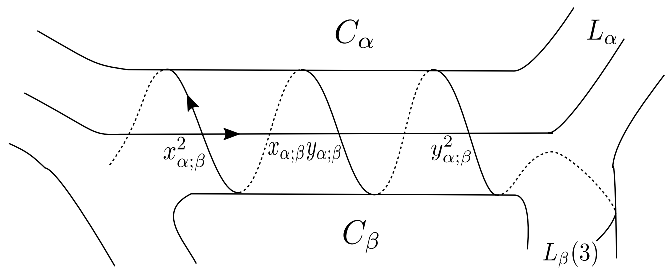


Figure A.3: Intersections of L_α and $L_\beta(3)$ on a bounded edge $e_{\alpha\beta}$; assuming $n_{\alpha\beta}=1$, $d_{\alpha,\beta}=-1$.

Assuming $j<k<l$, we have the following products:

- $CW^1(L_\alpha(k),L_\beta(l))\otimes CW^0(L_\alpha(j),L_\alpha(k))\to CW^1(L_\alpha(j),L_\beta(l))$, with $x_{\alpha;\beta}^{p'}y_{\alpha;\beta}^{q'}\cdot x_{\alpha\beta}^{p}y_{\alpha\beta}^{q}=x_{\alpha;\beta}^{p+p'}y_{\alpha;\beta}^{q+q'}$.
- $CW^0(L_\beta(k),L_\beta(l))\otimes CW^1(L_\alpha(j),L_\beta(k))\to CW^1(L_\alpha(j),L_\beta(l))$, with $x_{\alpha\beta}^{p'}y_{\alpha\beta}^{q'}\cdot x_{\alpha;\beta}^{p}y_{\alpha\beta}^{q}=x_{\alpha;\beta}^{p+p'}y_{\alpha;\beta}^{q+q'}$.
- $CW^1(L_\beta(k),L_\alpha(l))\otimes CW^1(L_\alpha(j),L_\beta(k))\to CW^0(L_\alpha(j),L_\alpha(l))$, with $x_{\beta;\alpha}^{p'}y_{\beta;\alpha}^{q'}\circ x_{\alpha;\beta}^{p}y_{\alpha;\beta}^{q}=x_{\alpha\beta}^{p+p'+1}y_{\alpha\beta}^{q+q'+1}$.

There are three other products by interchanging α and β above. There are additional products of a similar nature when we change the assumption of $j<k<l$ to other orders.

Lagrangians from three adjacent components.

Suppose a holomorphic triangle is bounded by Lagrangians from three different components $C_\alpha,C_\beta,C_\gamma$. The boundary of this triangle can be decomposed into three arcs, each arc only winding around one leg of the pair of pants. Pick a point in the center of the triangle, then joining these three arcs to this center point form three loops γ_1, γ_2, and γ_3. The fundamental group of the pair of pants is a free group on two generators a and b, where a is a loop around one leg of the pair of pants, b is a loop around another leg, and ab is a loop around the third leg. Hence $\gamma_1=a^p$, $\gamma_2=b^q$, and $\gamma_3=(ab)^r$. However $\gamma_1\cdot\gamma_2=\gamma_3$, hence we must have

$p = q = r = 1$ or $p = q = r = 0$. Hence, the vertices of such a holomorphic triangle in a pair-of-pants cannot be an intersection point that is further down any edge than the first one.

A.2 Category of singularities of the Landau-Ginzburg mirror

We describe the triangulated category of singularities $D_{sg}(D)$ of $D = W^{-1}(0)$. As in [**Or04**],

$$\mathrm{Hom}_{D_{sg}}(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\beta}(l)[n]) \cong \mathrm{Ext}^n_D(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\beta}(l)),$$

for $n > \dim D = 2$. The morphisms only depend on whether the degree n is even or odd, so calculating Ext's for $n > 2$ is enough to determine the morphisms.

Assuming $D_{\alpha\beta} := D_\alpha \cap D_\beta \neq \emptyset$, we get a 2-periodic resolution of $\mathcal{O}_{D_\alpha}(k)$ on D by locally free sheaves,

$$\{\cdots \to \mathcal{O}_D(k) \to \mathcal{O}_D(-D_\alpha)(k) \to \mathcal{O}_D(k)\} \to \mathcal{O}_{D_\alpha}(k) \to 0.$$

We can replace $\mathcal{O}_{D_\alpha}(k)$ with the above 2-periodic complex, and $\mathcal{H}om(-, \mathcal{O}_{D_\beta}(l))$ maps this complex to

$$0 \to \mathcal{O}_{D_\beta}(l-k) \overset{\phi_{\alpha\beta}}{\to} \mathcal{O}_{D_\beta}(D_\alpha)(l-k) \overset{\psi_{\alpha\beta}}{\to} \mathcal{O}_{D_\beta}(l-k) \to \cdots.$$

We get $\mathrm{Ext}^n_D(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\beta}(l))$ as the hypercohomology groups of this complex.

Morphisms $\mathrm{Hom}_{D_{sg}}(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\alpha}(l)[n])$.

In this case $\phi_{\alpha\alpha} = 0$, and $\psi_{\alpha\alpha}$ is isomorphic to the injective map $\mathcal{O}_{D_\alpha}(-H_\alpha)(l-k) \to \mathcal{O}_{D_\alpha}(l-k)$ with its cokernel being $\mathcal{O}_{H_\alpha}(l-k)$, where $H_\alpha = \bigcup_\beta D_{\alpha\beta}$. Hence for $n > 2$, $\mathrm{Ext}^{even}_D(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\alpha}(l)) = H^0(\mathcal{O}_{H_\alpha}(l-k))$ and $\mathrm{Ext}^{odd}_D(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\alpha}(l)) = H^1(\mathcal{O}_{H_\alpha}(l-k))$. So

$$\mathrm{Hom}_{D_{sg}}(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\alpha}(l)[*]) \cong H^*(\mathcal{O}_{H_\alpha}(l-k)).$$

First, suppose D_α is unbounded. If $H_\alpha = \mathbb{C} \cup \mathbb{C}$, then $H^0(\mathcal{O}_{H_\alpha}(l-k)) = \mathbb{C}[x] \oplus \mathbb{C}[y]/x^0 = y^0$ and $H^1(\mathcal{O}_{H_\alpha}(l-k)) = 0$. Otherwise, H_α consists of two $\mathbb{C}^1$'s connected by one or more

$\mathbb{P}^1$'s, as illustrated in Fig. A.4. We denote the two $\mathbb{C}^1$'s by C' and C'', and the connecting sequence of $\mathbb{P}^1$'s by $C_1, ..., C_m$, and $C = \bigcup_{j=1}^m C_j$. Then $H_\alpha = C \cup C' \cup C''$ and there is an exact sequence

$$0 \to H^0(\mathcal{O}_{C\cup C'\cup C''}(l-k)) \xrightarrow{\cong} H^0(\mathcal{O}_C(l-k)) \oplus H^0(\mathcal{O}_{C'\cup C''}) \twoheadrightarrow H^0(\mathcal{O}_{p\cup p''})$$
$$\xrightarrow{0} H^1(\mathcal{O}_{C\cup C'\cup C''}(l-k)) \xrightarrow{\cong} H^1(\mathcal{O}_C(l-k)) \to 0.$$

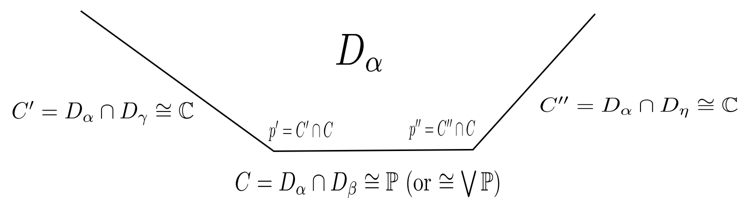


Figure A.4: $H_\alpha = C \cup C' \cup C''$

When $l - k \geq 0$,

$$H^0(\mathcal{O}_C(l-k)) \cong \bigoplus_{j=1}^m H^0(\mathcal{O}_{C_j}(l-k))/\sim\ \cong \bigoplus_{j=1}^m \mathbb{C}[x_j : y_j]^{n_j(l-k)}/\sim,$$
$$H^1(\mathcal{O}_C(l-k)) = 0,$$

where $(x_j : y_j)$ are homogenous coordinates. The equivalence relation above identifies $y_j^{n_j(l-k)} \in H^0(\mathcal{O}_{C_j}(l-k))$ and $x_{j+1}^{n_{j+1}(l-k)} \in H^0(\mathcal{O}_{C_{j+1}}(l-k))$ with each other and with the section of $\mathcal{O}_C(l-k)$ obtained by considering these two polynomials together on $C_j \cup C_{j+1}$. In this notation, if $C_j = D_\alpha \cap D_{\beta_j}$, then we are identifying $x_j \sim x_{\alpha\beta_j}$, $y_j \sim y_{\alpha\beta_j}$, and $n_j \sim n_{\alpha\beta_j}$. Suppose $C' = D_\alpha \cap D_\gamma$ and $C'' = D_\alpha \cap D_\eta$, then $H^0(\mathcal{O}_{C'\cup C''}) = \mathbb{C}[x_{\alpha\gamma}] \oplus \mathbb{C}[x_{\alpha\eta}]$.
When $l - k < 0$, $H^0(\mathcal{O}_C(l-k)) = 0$, $H^0(\mathcal{O}_{C'\cup C''}) = \mathbb{C}[x_{\alpha\gamma}] \oplus \mathbb{C}[x_{\alpha\eta}]$, and

$$H^1(\mathcal{O}_{C_j}(l-k)) = \left(\mathbb{C}[x_j : y_j]^{n_j(k-l)-2}\right)^*.$$

(When $n_j(l-k) = -1$, $H^1(\mathcal{O}_{C_j}(l-k)) = 0$.) Then $H^1(\mathcal{O}_C(l-k))$ is a direct sum of all the $H^1(\mathcal{O}_{C_j}(l-k))$ for $j = 1, \ldots, m$ plus $m-1$ extra generators associated to each vertex $C_j \cap C_{j+1}$. Note by Serre duality, $H^1(\mathbb{P}^1, \mathcal{O}(n(l-k))) \cong H^0(\mathbb{P}^1, \mathcal{O}(n(k-l)-2))^*$, with $(x_j^p y_j^q)^* = \frac{dx}{x_j^{p+1} y_j^q} = -\frac{dy}{x^p y^{q+1}}$, where $p + q = n(k-l) - 2$.

Comparing this with the wrapped Floer cohomology, we get

$$\mathrm{Hom}^*(L_\alpha(k), L_\alpha(l)) \cong \mathrm{Hom}_{D_{sg}}(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\alpha}(l)[*]).$$

Morphisms $\mathrm{Hom}_{D_{sg}}(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\beta}(l)[n])$, $\alpha \neq \beta$.

In this case $\psi_{\alpha\beta} = 0$, and $\phi_{\alpha\beta}$ is isomorphic to the injective map $\mathcal{O}_{D_\beta}(l-k) \to \mathcal{O}_{D_\beta}(D_\alpha)(l-k)$ with its cokernel being $\mathcal{O}_{D_{\alpha\beta}}(D_\alpha)(l-k)$. Hence for $n > 2$, $\mathrm{Ext}_D^{even}(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\beta}(l)) = H^1(\mathcal{O}_{D_{\alpha\beta}}(D_\alpha)(l-k))$ and $\mathrm{Ext}_D^{odd}(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\beta}(l)) = H^0(\mathcal{O}_{D_{\alpha\beta}}(D_\alpha)(l-k))$. So

$$\mathrm{Hom}_{D_{sg}}(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\beta}(l)[*+1]) \cong H^*(\mathcal{O}_{D_{\alpha\beta}}(D_\alpha)(l-k)).$$

If $D_{\alpha\beta}$ is unbounded, then $D_{\alpha\beta} = \mathbb{C}[x]$ and $H^*(\mathcal{O}_{D_{\alpha\beta}}(D_\alpha)(l-k)) \cong H^0(\mathcal{O}_\mathbb{C}) \cong \mathbb{C}$. If $D_{\alpha\beta}$ is bounded and $D_{\alpha\beta} \cong \mathbb{P}^1$, then $\mathcal{O}_{D_{\alpha\beta}}(D_\alpha) \cong \mathcal{O}_{\mathbb{P}^1}(d_{\alpha,\beta})$, where $d_{\alpha,\beta} = \deg \mathcal{O}(D_\alpha)|_{D_{\alpha\beta}}$, and

$$H^*(\mathcal{O}_{D_{\alpha\beta}}(D_\alpha)(l-k)) = H^*(\mathcal{O}_{\mathbb{P}^1}(n_{\alpha\beta}(l-k) + d_{\alpha,\beta})).$$

For $n_{\alpha\beta}(l-k) \geq -d_{\alpha,\beta}$,

$$H^0(\mathcal{O}_{\mathbb{P}^1}(n_{\alpha\beta}(l-k) + d_{\alpha,\beta})) \cong \mathbb{C}[x_{\alpha;\beta} : y_{\alpha;\beta}]^{n_{\alpha\beta}(l-k)+d_{\alpha,\beta}},$$

$$H^1(\mathcal{O}_{\mathbb{P}^1}(n_{\alpha\beta}(l-k) + d_{\alpha,\beta})) = 0.$$

For $n_{\alpha\beta}(l-k) = -d_{\alpha,\beta} - 1$,

$$H^0(\mathcal{O}_{\mathbb{P}^1}(n_{\alpha\beta}(l-k) + d_{\alpha,\beta})) = H^1(\mathcal{O}_{\mathbb{P}^1}(n_{\alpha\beta}(l-k) + d_{\alpha,\beta})) = 0.$$

For $n_{\alpha\beta}(l-k) < -d_{\alpha,\beta} - 1$,

$$H^0(\mathcal{O}_{\mathbb{P}^1}(n_{\alpha\beta}(l-k) + d_{\alpha,\beta})) = 0,$$

$$H^1(\mathcal{O}_{\mathbb{P}^1}(n_{\alpha\beta}(l-k) + d_{\alpha,\beta})) = \left(\mathbb{C}[x_{\alpha;\beta} : y_{\alpha;\beta}]^{n_{\alpha\beta}(k-l)+d_{\alpha,\beta}+2}\right)^*.$$

Comparing with wrapped Floer cohomology, we get

$$\mathrm{Hom}^*(L_\alpha(k), L_\beta(l)) \cong \mathrm{Hom}_{D_{sg}}(\mathcal{O}_{D_\alpha}(k), \mathcal{O}_{D_\beta}(l)[*]).$$

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