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Publication Date
1965-10-18

UCRL-16530

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DISCLAIMER

Three introductory lectures in S-matrix theory delivered at the National Reactor Testing Station, Idaho Falls, Idaho, November 3-5, 1965.

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AEC Contract No. W-7405-eng-48

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Richard C. Slansky

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Many optimists believe we stand on the verge of the third great revolution in physics of this century. To clarify the cause of all this excitement, let us take a quick but sweeping tour of the previous two revolutions, making special note of the results most applicable to a theory of strong interactions. This will more than set the scene of action since the present revolution is attempting to unite the insights of the previous two into a single theory. The first revolution, inspired by Albert Einstein, did more than teach us that the form of a law of nature must be invariant under Lorentz transformations. However, the key word is given here--invariance. The great contribution was the insight that symmetries exist, and physical laws will not change when the system (or observer) is transformed by one of the symmetry operations. Lorentz transformations are a typical example. If the system is rotated, translated in space or time, or its uniform velocity with respect to the observer is boosted, the form of the laws governing the system is unchanged. A memorable result is: For every group of symmetry operations, there exist conserved quantities. The operations of translations in space-time of the Lorentz group yield the conservation of linear momentum and energy. Hence,

*This work performed under the auspices of the U.S. Atomic Energy Commission.

by finding conserved quantities, we learn something about the symmetries of the physical system. However, there are many symmetries of strong interactions whose origins are not well understood. Ultimately a complete theory should show that these symmetries are implied by its postulates. The first revolution is summed up in our awareness of symmetries in nature.

The second revolution occurred in the twenties and was headed by such physicists as Schrödinger, Heisenberg, and Dirac. Quantum mechanics has had a profound influence on our understanding of natural phenomena in the small. For example, understanding the phenomenon of electron diffraction could be very complicated, since this is an example of a particle behaving like a wave. The formal result of quantum mechanics is to consider the <u>probability amplitude</u> as the basic quantity. By postulating that it belonged to some sort of linear vector space, it became a simple matter to account for the particle-wave duality exhibited in atomic phenomena. Then if we take a measurement, the probability of a certain result is given by the absolute square of the amplitude. Perhaps the most memorable result of quantum mechanics is this concept of a probability amplitude, and its relation to the process of making an observation. As a postulate, we accept the existence of the amplitude in the new theory.

Quantum mechanics is a nonrelativistic theory, and the attempts to generalize it to the relativistic domain cause trouble. A very natural extention of quantum mechanics to the relativistic domain is called "relativistic quantum field theory." Although this theory has met with much success in understanding electromagnetic phenomena, in its useable forms it has been rather unreliable for strong interactions. However, field theory is a strong candidate for being the correct theory. We shall say more about this later.

Before exposing the essence of the third revolution, let us note a basic characteristic of all three of these theories. The theory of relativity as a generalization of Newtonian mechanics requires that we specify the particles (or mass distribution) of the system before we can write down the complete field equations. Quantum mechanics also requires knowledge of the particles in the system. Also we must know the potential before we can write down the Schrödinger equation for the probability amplitude. Even in the standard formulations of field theory, the particle-fields and their interaction must be given before calculation can be started. It is in just this way that the new theory may differ drastically. The content of the third great revolution may be stated as this: In the domain of strong interactions, it may be possible to construct a theory in which we need not give the particles or their interaction. Within the postulates of the theory there may be only one self-consistent set of particles. The masses, coupling constants and quantum numbers of the particles will be determined by the theory and will not have to be inserted into the theory from experiment. This set of particles will be just that set we know from experiment. The only fundamental constants we need will relate our convention of measurement to nature's. (Two of the three necessary constants are already known--$\hbar$ and c.) Thus, the revolutionary claims there are no "elementary" particles. The existence of any strongly interacting particle is due to the existence of all other strongly interacting particles and the forces by which they interact. The amazing prospect is that we may be able to build a theory without any arbitrary degrees of freedom. Consequently, we begin with "nothing", and pull ourselves up by our bootstraps. So the theories are dubbed "bootstrap theories," and the formulation the bootstrap theories have been the object of studying the analytically continued S-matrix.

Obviously, one can be more conservative! To be more in line with traditional quantum field theory, some physicists believe that some particles are elementary (i.e., no matter how strong the mathematical methods, the masses, and coupling constants of these particles will have to be inserted into the theory). Then other particles may be "bound states" of the more elementary particles. For the very conservative, all the particles are aristocrats, and there exists a quantum field operator corresponding to the particle which can be inserted into the theory at the physicist's will.

In these lectures, we shall favor the first idea, that of nuclear democracy. Although such a revolutionary idea might fail, the preliminary calculations have been quite successful and the ideas behind the bootstrap theory are sufficiently exciting to warrant closer examination. Since no background in particle physics is assumed in these lectures, we shall begin from the beginning--a survey of experiment. Then a quasi-historical review of the ideas that have lead to the bootstrap idea will precede a simple example given in the third lecture. To avoid confusion as to the direction of these lectures, we now give a rather complete outline of the lecture topics.

Part I discusses phenomenology. After describing a typical experiment, we shall consider some of the physical quantities the experimentalist might measure (cross sections, branching ratios, etc.). In accord with quantum mechanics, we then define the S-matrix. Since S-matrix theory must be relativistic, some relativistic kinematics will be necessary. From the kinematics and the description of scattering processes, we learn how short lived particles (10^{-24} sec.) can be detected.

In Part II, the dynamics of strongly interacting particle processes will be given more serious consideration. Memorable results of quantum mechanics and field theory will be used to motivate S-matrix theory. The principle of maximal analyticity will be explained, and with the concept of crossing, the basis of the bootstrap theory should become quite plausible.

In Part III, we use just analyticity and unitarity to derive equations for the amplitude. In deriving the N/D equations, the ideas of Part II will be emphasized.

Before leaping into the problem, let me make a couple of pedagogical notes.

Particle names: I don't want to spend a lot of time naming particles. I will rarely refer to specific particles, except for the pion (π) and the nucleon (N) . If you don't know the other particle names, then just consider the words I use as labels to describe objects with a fairly well-defined mass and other quantum numbers. The π has been called the "glue" that holds the nucleons in a nucleus together. Its mass is about 140 MeV and it exists in three charge states (π^+, π^0, π^-) . The spin of the π is 0 and its parity is negative. Everyone knows about the proton and neutron (N). The nucleon exists in 2 charge states (n and p), has spin $\frac{1}{2}$, parity even, and mass of about 940 MeV.

Units: Instead of using human measuring units, we shall use the natural system in which $\hbar = c = 1$, both in magnitude and dimensionally. Hence $E = mc^2 = m$. In ordinary units

$$\hbar = 1.054 \times 10^{-27} \text{erg sec} = 6.58 \times 10^{-22} \text{ MeV-sec} .$$

If $\hbar = c = 1$ then

$$\hbar c = 1 = (6.58 \times 10^{-22} \text{ MeV-sec})(3 \times 10^{23} \text{ fermi/sec}) = 197 \text{ Mev-fermi} .$$

This tells us how to recover the proper dimensions of any experimental quantity. For example

$$\frac{1}{m_\pi^2} = \frac{1}{(140\text{ MeV})^2} = \left(\frac{197}{140}\right)^2 f^2 \approx 20\text{ millibarns} .$$

Hence, we can "ignore" $\hbar$'s and c's in our equations.

PART I - A VISTA OF STRONG INTERACTION PHENOMENOLOGY

Before discussing theoretical ideas on how strong interactions take place, we should develop some intuition about what is happening. Such discussions lead to basic ideas as to the requirements to be placed on the theory. The fact is that only one kind of experiment can be used to probe the nuclear force: scattering experiments. Hence, the theory need be concerned only with calculation of the scattering amplitude. Before introducing this quantum mechanical quantity, consider the following "typical" experiment. The scene is the Bevatron at the Lawrence Radiation Laboratory in Berkeley.

An arc in H_2 gas produces positive ions (protons) which are transported to a static accelerating field of 500,000 volts d.c. Before injection into the Bevatron itself, these $\frac{1}{2}$ MeV protons are accelerated to 10 MeV (lab kinetic energy) in a linear accelerator. In the Bevatron, small accelerating boosts are supplied by electric fields, and the protons are kept on the circular track by an increasing magnetic field. As the bunch of protons becomes more energetic, the magnetic field necessary to keep them on the track increases. The final energy is limited by the strength of the magnets. The wave packet of about 2×10^{11} protons is only several inches long when the protons reach 6.2 BeV. (The protons have now gone around the Bevatron about 400,000 times--or to the moon and half way back.)

The experiment now begins. A target is dropped into the beam path, and a great spray of nuclear matter emerges: Protons, antiprotons, neutrons,

π mesons, K mesons, and strange baryons, such as Λ and Ξ. One of the major tasks that the experimentalist has is to "design a beam" from this conglomeration. By using separators, focusing magnets and other electro-magnetic devices, he can pick out particles of some type and energy which are then transported to a second target.

The second target may be (for example) a bubble chamber where the tracks of the incoming and outgoing charged particles are recorded on a photograph. The bubble chamber is a big tank in which liquid hydrogen (a proton target) is placed under high pressure. The pressure is suddenly released just before the particles enter the chamber. In the superheated liquid, the charged particles leave tracks which a camera photographs. The pressure is then reapplied and the bubbles are squeezed out of existence. A few seconds later, after the next packet of protons is accelerated, the cycle is repeated.

The particle tracks are curved due to high magnetic fields. The amount of curvature depends on the charge and momentum of the particle. Hence, from the picture of the particle event, the experimenter can identify the resultant particles and their momenta.

Another interesting detector is a spark chamber. After hitting a target, the scattered particles pass through a set of parallel plates. Because of the high voltage across the plates, the particles ionize the air enough that a spark leaps between plates. Sonic detectors then locate the spark and the information is put on a magnetic tape or sent directly into an on-line computer.

By counting the different types of events, the experimentalist can measure various experimental quantities. It is the job of some theoreticians to calculate the quantities from their theories. So we indicate several of the most important experimental parameters.

The cross section for event X is the probability that X will happen per unit time per unit volume divided by the incident flux or total incoming probability current per unit volume. For example, the total cross section is

$$\sigma^{total} = \frac{\text{Probability that anything happens/sec/unit volume}}{\text{incident flux}} .$$

We estimate the "size" of the cross section by the following dimensional argument. The range of the nuclear force is approximately equal to the inverse mass of the pion. Hence

$$\sigma \lesssim \pi r^2 = \pi \frac{1}{m_\pi^2} = \pi\left(\frac{197}{140}\right)^2 (\text{fermi})^2 \approx 60 \text{ millibarns.}$$

So a typical cross section is on the order of millibarns.

The elastic differential cross section is

$$d\sigma^{e\ell} = \frac{\text{Probability of an elastic scattering going into } d\Omega\text{/sec/volume}}{\text{incident flux}} .$$

Note that we must sum over all those states which are not specifically observed. For example, we must sum over the spins and momenta of the final state to find the differential cross section.

Another experimental quantity is a branching ratio R .

$$R = \frac{\sigma(A + B \to \alpha_1 + \alpha_2 + \cdots)}{\sigma(A + B \to \beta_1 + \beta_2 + \cdots)} .$$

There are polarization cross sections, etc., but these would only serve as further examples of the same idea.

A general description of scattering processes is given by quantum mechanics. In following quantum mechanics this far, we assume the existence of a probability amplitude whose absolute value squared gives the probability for a system to go from some initial state i to some final state f. The final state need not have the same particles as the initial state. If we assume that the set of all possible initial states spans the same space as the final states, then there exists a unitary operator that takes us from one "basis" to the other. The operator is the S-matrix S. If $\phi^{(in)}$ represents the set of all possible initial states and $\phi^{(out)}$ the set of all possible final states then:

$$\phi^{(out)} = S^{\dagger} \phi^{(in)} .$$

We define matrix elements of S as follows:

$$S_{fi} = (\phi_f^{(in)}, S \phi_i^{(in)}) = (\phi_f^{(out)}, \phi_i^{(in)}) = (\phi_f^{(out)}, S \phi_i^{(out)}) .$$

In these definitions, we have used the unitarity of S

$$S S^{\dagger} = S^{\dagger} S I$$

where I is the identity operator. If the states $\phi^{(in)}$ and $\phi^{(out)}$ really form a basis for the same space, then the scalar product must be preserved under the transformation from the (in) to the (out) states. Thus S is unitary:

$$(\phi_f^{(in)}, \phi_i^{(in)}) = (S \phi_f^{(out)}, S \phi_i^{(out)}) \qquad \text{(defn of S)}$$

$$= (\phi_f^{(out)}, S^{\dagger} S \phi_i^{(out)}) \qquad \text{(defn of unitary)}$$

$$= (\phi_f^{(out)}, \phi_i^{(out)}) \qquad \text{(requirement of preservation of scalar product)} .$$

Hence

$$S^{\dagger} S = I .$$

The unitarity of S is one of the most basic ideas exploited in S-matrix theory.

For convenience and other reasons that will become apparent, we split S into two pieces. First, nothing might happen (no scattering); second, if something does happen energy and momentum must be conserved. We write:

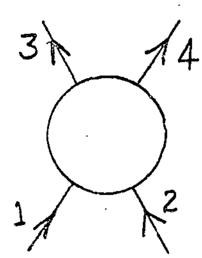
$$S_{fi} = \delta_{fi} + i\,\delta^4(P_f - P_i)A_{fi}$$

where $\delta_{fi} = 1$ if $f = i$ and 0 otherwise, and $\delta^4(P_f - P_i)$ is nonzero only if energy and linear momentum are both conserved. We have factored out these two unanalytic pieces so that A will be a smooth function of its arguments.

We will characterize the process

$$i \rightarrow f$$

by the amplitude A. For the scattering $1 + 2 \rightarrow 3 + 4$, we draw a picture of A.

$A_{1+2\rightarrow 3+4}$ = 

By well-defined quantum mechanical procedures, A can be related to the various experimental quantities. If particles 1 through 4 are all spinless, then the differential cross section is given by:

$$\frac{d\sigma}{d\Omega} = \frac{\pi^2}{4E^2}\,\frac{|\underset{\sim}{p}_1|}{|\underset{\sim}{p}_3|}\,|A|^2$$

where E is the total center-of-mass energy and $|\underset{\sim}{p}_1|$ and $|\underset{\sim}{p}_3|$ are the magnitudes of the three momenta of particles 1 and 3 in the center-of-mass frame. The matrix elements of A are functions of scalar variables.

By the relativistic invariance of the theory, A transforms like a scalar under Lorentz transformations. Hence A is a function of the scalar invariants that can be formed out of the momenta in the process. If p is a four vector whose components are $(\underset{\sim}{p}, E)$ $(c = 1)$, then the scalar product between p_1 and p_2 is defined by

$$p_1 \cdot p_2 = E_1 E_2 - \underset{\sim}{p}_1 \cdot \underset{\sim}{p}_2$$

and

$$p \cdot p = E^2 - \underset{\sim}{p}^2 = m^2$$

where m^2 is a constant according to special relativity. If $\underset{\sim}{p} = 0$, then m is just the rest mass of the particle. Hence p·p gives a trivial invariant. Now consider the nontrivial invariant:

$$s = (p_1 + p_2)^2 ;$$

s is necessarily a Lorentz invariant--it has the same value in any frame of reference. In the center-of-mass frame, where $\underset{\sim}{p}_1 = -\underset{\sim}{p}_2 = \underset{\sim}{p}$, then

$$s = (E_1 + E_2)^2 = \left(\sqrt{\underset{\sim}{p}^2 + m_1^2} + \sqrt{\underset{\sim}{p}^2 + m_2^2}\right)^2 = (\text{total CM energy})^2 .$$

For reactions like $1 + 2 \rightarrow 3 + 4$, there exist two scalar invariants which we call s and t. Hence $A = A(s,t)$(see Part III). The main result of this kinematical analysis is that we perform our calculations in any convenient frame of reference. Then we can redefine the scalar invariants in terms of the lab system, and hence compare with experiment.

Armed with the knowledge of the existence and Lorentz invariance of A, we now re-examine a typical scattering experiment while keeping two questions in mind: (1) How are short lived particles detected? (2) Do the experimental results yield any hints about the structure of A ?

There has been a great proliferation of the strongly interacting particles, but most are unstable under strong interactions. We can estimate their life time by an argument which we give later. The result is that for a typical width of $\Gamma = 100$ MeV

$$\tau = \frac{1}{\Gamma} = \frac{1}{100\ \text{MeV}} = \frac{\hbar}{100\ \text{MeV}} = \frac{6.58\times 10^{-22}\ \text{MeV-sec}}{100\ \text{MeV}} \simeq 6\times 10^{-24}\ \text{sec}\ .$$

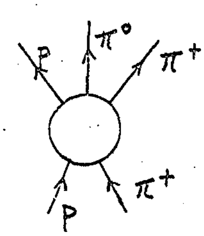
Hence, if a particle is produced at near the speed of light, it travels about a fermi before it decays. We could never see such a particle as a track in a bubble chamber. How, then, are these "resonances" detected? The detection of resonances follows from the analysis of the momenta of the final particles.

Consider the reaction

$$p + \pi^+ \rightarrow p + \pi^0 + \pi^+\ .$$

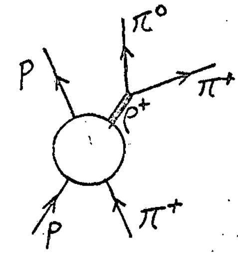
We might guess that

$A_{(2\pi)}$ =



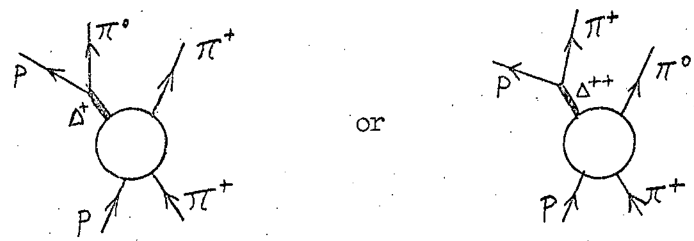
However the $\pi^0\ \pi^+$ might form a short lived particle before breaking into two pions:

$A_{(\rho)}$ =



or the p and the π^o or π^+ might resonate:

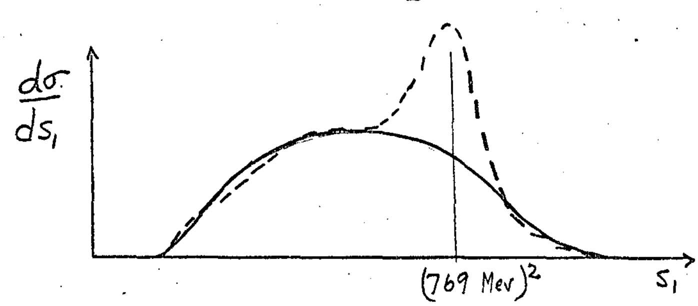
$A_{(\Delta)}$ =



.

There are other resonances that can be studied by this reaction, but let's forget them. We note that in all three of these cases, the final particles are the same. As the instructive example, we shall look for the ρ meson. It is a well known particle with quantum numbers $I = 1$, $J = 1$, $P = (-)$, and $G = (+)$; or in shorthand $1(1^{-+})$. Its mass and width are $m_\rho = 769$ MeV and $\Gamma_\rho = 112$ MeV.

If we define the scalar invariant $s_1 = (p_{\pi^+} + p_{\pi^o})^2$ then $A = A(s_1 \cdots)$ where the dots stand for the other independent scalar variables of the problem. Now suppose that $A(s_1 \cdots) =$ constant. We can calculate the cross section assuming $A(s_1 \cdots)$ constant for a given incident energy of the incoming pion. We find the plot of $\frac{d\sigma}{ds_1}$ to look like the solid line:



The cross section assuming A is constant is called the <u>phase space cross-section</u>.

An experiment is now performed at the same incident energy for which the phase space plot was calculated. But experimentally we find that $\frac{d\sigma}{ds_1}$ does not follow the phase-space cross-section. Experimentally the results look like the dotted line.

At $s_1 = (769\text{ MeV})^2$, there is a bump in the cross section with respect to phase space. There are two major points to be made about this plot.

First, the lab-bound experimenter can make such a plot. The reason is that we can evaluate s_1 in both the lab frame and in the center of mass system of the two pions:

$$s_1 = E_{CM}^2$$

where E_{CM} is the total energy of the $\pi^o \pi^+$ system in its center of mass. In the lab,

$$s_1 = \left(\sqrt{|\underset{\sim}{p}_{\pi^+}^{lab}|^2 + \mu^2} + \sqrt{|\underset{\sim}{p}_{\pi^o}^{lab}|^2 + \mu^2}\right)^2 - \left(\underset{\sim}{p}_{\pi^+}^{lab} + \underset{\sim}{p}_{\pi^o}^{lab}\right)^2 ,$$

where μ is the pion mass. Hence the lab bound experimenter can measure $\frac{d\sigma}{ds_1}$.

Secondly, the existence of the bump is very suggestive. The cross section can be fairly well duplicated if we assume that A is dominated by a pole for values of s_1 near $(769\text{ MeV})^2$:

$$A(s_1 \cdots) \simeq A(s_1) \simeq \frac{g^2}{s_1 - (M_\rho - i\frac{\Gamma_\rho}{2})^2} \simeq \frac{g^2}{s_1 - M_\rho^2 + i M_\rho \Gamma_\rho} .$$

Consequently:

$$\frac{d\sigma}{ds_1} \sim |A|^2 = \frac{g^4}{(s_1 - M_\rho)^2 + \Gamma_\rho^2 M_\rho^2} .$$

With this experimental motivation, we shall postulate that single particles correspond to poles in the scattering amplitude. The pole-particle association is basic in S-matrix theory.

We now derive the formula that $\tau = 1/\Gamma \simeq$ lifetime of particle. This derivation shows that the particle pole association and the concept of complex mass do have a quantum mechanical correspondence. A ρ meson at rest can be approximated by a plane wave:

$$\psi = e^{-iEt} ,$$

$$|\psi|^2 \sim e^{iE^*t} e^{-iEt} = e^{-it(E-E^*)} = e^{2t \operatorname{Im} E} .$$

But

$$s_1 = (M_\rho - i\frac{\Gamma}{2})^2 = E^2 \quad \text{or} \quad \operatorname{Im} E = -\frac{\Gamma}{2} .$$

Thus

$$|\psi|^2 \sim e^{-\Gamma t} = e^{-t/\tau} .$$

Hence, in nonrelativistic quantum mechanics, the pole does correspond to a decaying particle with a life time of $\tau = \frac{1}{\Gamma}$.

PART II.. THE RISE OF DYNAMICAL THEORIES

Before physicists began to worry about atomic and sub-atomic phenomena, all (known) nature was described by the classical theories of mechanics and electromagnetism. However, the idealized mass points, charged or uncharged, have little resemblance to the particles of high energy interest. In fact these theories have nothing to say about the interactions of "real" particles. Consequently, the strong interaction theory need not have a correspondence principle to classical theory.

The first theory that was primarily concerned with physics in the small was quantum mechanics. As is well known, quantum mechanics has had much success in explaining low energy data. For example, the scattering of 1 ev neutrons on complicated gas molecules can be explained strictly within the framework of quantum mechanics. The deuteron, considered as a bound state of two elementary nucleons, is partially understood in terms of a particular potential in a quantum mechanical theory. However, when we go to high energies, there are two conditions of quantum mechanics that are not well satisfied.

(1) In high energy physics, the velocities of the particles are not small compared to the speed of light. A 6 BeV proton is traveling at velocity

$$v = \frac{p}{E} = \left[\frac{E^2 - m^2}{E^2}\right]^{\frac{1}{2}} = \sqrt{\frac{35}{36}} \cong 0.986$$

the speed of light. At these velocities, the potentials of quantum mechanics become ambiguous, since they imply "action at a distance." To avoid violating our notions of causality (the message can't be received until after it is sent), we must throw out the approximation of action at a distance.... unfortunately potentials go also.

But this isn't the only problem with quantum mechanics:

(2) Standard formulations of quantum mechanics, such as the Schrödinger equation, cannot handle the creation and destruction of particles in a physical process. At low energies, the Schrödinger equation does a good job describing the process

$$p + p \rightarrow p + p \quad .$$

However, if the center of mass kinetic energy is greater than 140 MeV, then the reaction

$$p + p \rightarrow p + p + \pi^{o}$$

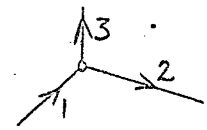
is possible and we are outside the framework of Schrödinger theory. In most interesting elementary particle processes, we need dynamical equations that can easily describe creation or destruction phenomena during the process.

Probably relativity and creation and destruction are closely related. In any event, the attempts to generalize the Schrödinger equation by relativistic equations such as the Klein-Gordon or Dirac equation (as a quantum mechanical theory) fail because they lead to negative energy densities which we are unable to interpret. Hence we must find a more radical departure from quantum mechanics.

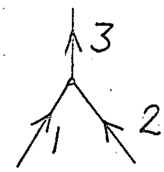
By a time worn path of reasoning, one introduces fields to avoid violating causality (the reason is exactly the same in classical electrodynamics.) And just as in the classical theory, the field carries the energy, momentum, and other observables. In a quantum mechanical theory, the observables correspond to Hermitian operators. Since certain functions of the fields are these observables, the fields must be operators. Next we write the field operators as superpositions of plane waves. Using the Fourier transform, one then notes that the Fourier components of the relativistic field equations look like the harmonic oscillator in momentum space. Since we know how to quantize the harmonic oscillator in quantum mechanics, we can quantize the fields, and hence derive the commutation relations for the field operators.

This is the most pedestrian approach to field theory and the construction of canonical field theory is of no interest here. In fact the resultant quantized fields described free particles, and their interactions must be introduced in a semiphenomenological fashion. However, there are several important and thought provoking results that are playing an important role in the development of more modern theories. Much of the language is still couched in field-theoretic language. Also there is the possibility that field theory and the newer theories may be equivalent in many ways. We first make a few comments on more sophisticated formulations of field theory than the one we just gave. Then we discuss one solution of field theory--perturbation theory and Feynman graphs. Feynman graphs will give us a dramatic way to talk about particle processes.

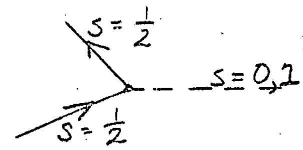
Just as in classical electrodynamics, we have to specify the interaction in our canonical construction. In local field theory the fields propagate continuously from one point to the next and interact with one another only at a point of interaction, the interaction is described by a product of field operators called an interaction Lagrangian. Since the quantized field operators are composed of creation and annihilation operators (remember the quantization of the harmonic oscillator), we can picture such a point as a vertex where various particles are created or destroyed. For example if the three fields $\phi_1(x), \phi_2(x)$ and $\phi_3(x)$ interact, the interaction Lagrangian is given by $g\,\phi_1(x)\,\phi_2(x)\,\phi_3(x)$ where g is the strength or coupling constant of the interaction. We represent some of the possibilities pictorially:



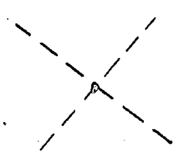
describes the destruction of 1 and the creation of 2 and 3 ; or the same graph describes the destruction of 1 and the antiparticle of 3 and the creation of 2 ; and



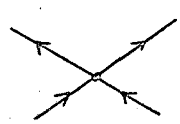
1 and 2 are annihilated and 3 created. However, surprisingly enough, only several interactions lead to a self consistent theory, and this is true only after we have renormalized masses and coupling constants by subtracting ∞ from ∞. Only the



vertex among two spin $\frac{1}{2}$ particles and a spin zero or a massless spin one one field, and



vertex among four spin zero fields lead to finite results for physical quantities after the renormalization procedure. The interaction



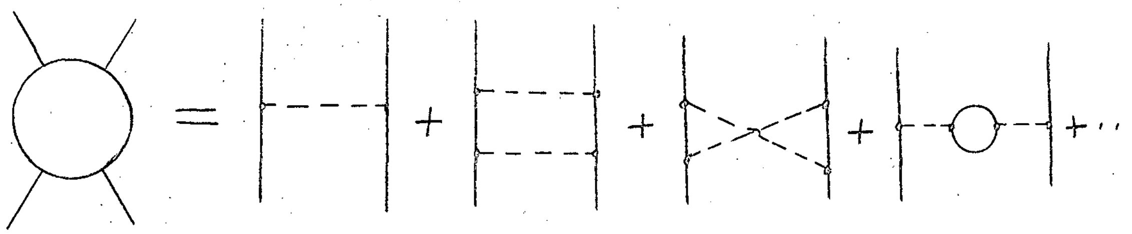
(four fermions) does <u>not</u> lead to a finite theory in higher order perturbation theory, although the first order theory has been very important in understanding β decay. Moreover, the attempts to quantize massive fields of spin one and all fields of higher spin have encountered great difficulty. Such particles do exist.

The fact that only two interactions lead to a good theory shows the restrictiveness of the quantum field idea. Although when we constructed quantum field theory in analogy to classical field theories we specified the interaction, the interaction Lagrangian may be superfluous to quantum field theory. It may be that the axioms of quantum field theory (the field idea and commutation relations, etc.) are so restrictive that we can construct a field theory without Lagrangians. This has lead to the study of non-Lagrangian or Axiomatic Field Theory.

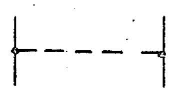
Axiomatic quantum field theory is a very careful mathematical inquiry into the field idea. It is very far from calculating cross-sections and for our purposes, it is still a long ways from physics. We shall sacrifice the firmness of mathematical basis for a theory which tells us something about physics. The decision to do this is personal, and it may be that axiomatic field theory will lead to a very reasonable explanation of strong interaction phenomena. But let's leave it at that and return to Feynman graphs.

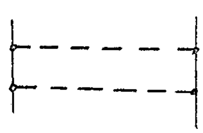
Field theories have not been solved exactly, but only through approximation techniques. The most famous of these is the Feynman-Dyson perturbation expansion of the S-matrix (Of course the S-matrix exists in field theory, just as it does in quantum mechanics.) Perhaps the nicest feature of the theory is the ease with which we can interpret the pictures that represent the terms of the expansion. Such intuitions obtained from the graphs permeate all of strong interaction physics, and in fact, most of modern physics.

The result is derivable in several fashions. The answer is that the amplitude for a process is the sum of all possible (connected) graphs. For example, the amplitude for N - N scattering is given by:

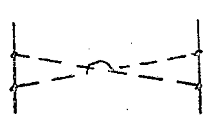


where the ---- lines represent pions, the solid lines are nucleons, and overall momentum conservation is implied. The theory gives a well defined set of rules for calculating the contribution of each graph to the series. These rules tell how to associate each vertex, internal line and external line to some mathematical operation. The mere existence of the rules is sufficient here and we emphasize the interpretation of the graphs.

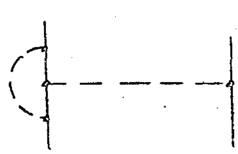
The first graph,  , can be interpreted to say that the N-N scattering proceeds by one nucleon creating a pion which the other nucleon annihilates. This is called the one pion exchange diagram. The basic idea is that the scattering is due to the exchange of a particle. The strong interaction force is due to exchanges of particles in which all quantum numbers are conserved at each vertex. Examples of other graphs are:



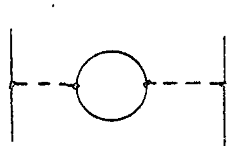
Two pion exchange.



Crossed two pion exchange.



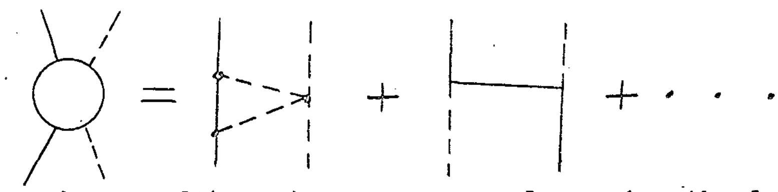
One π exchange with "radiative" correction.



One π exchange with "vacuum polarization."

etc.

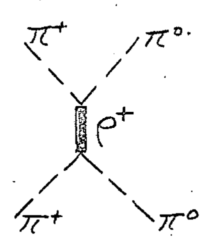
(The names come from quantum electrodynamics and not from strong interaction physics.) You should note that the particles that are exchanged (the forces) can be the same type of particles as those that are being scattered. The first few terms of the $\pi + N \rightarrow \pi + N$ series are given by



Here the exchange of two pions or one nucleon give the lowest order terms. The four line vertex is necessary because the pi is a pseudoscaler particle, and parity would not be conserved at a three pion vertex.

Three absolutely essential results of Feynman graphs will lead us to a theory of the S-Matrix, where we shall generalize these statements accordingly.

(1) Suppose there exists a stable or unstable particle that communicates with (ie, has the same quantum numbers as) the initial or final state. Then for $\pi - \pi$ scattering, there exists a graph like

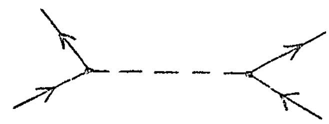


From the Feynman rules, the amplitude has a pole at the complex mass of the ρ. (There exist Feynman rules for any spin.) Hence we see that the Feynman graphs confirm our suspicion about the correspondence of particles to poles in the scattering amplitude.

(2) Except for certain definite singularities (poles and branch points), the individual graphs are analytic functions of the external momenta. We mean analytic in the sense of complex variables and the Cauchy Riemann conditions.

(3) The graphs are sufficiently analytic that they can be read both

upward and sideways. The same function describes the different processes. For example, the one pion exchange graph,



can be read from left to right. Then we have $N + \overline{N} \rightarrow \pi \rightarrow N + \overline{N}$. The pion pole exists in this channel. (A channel is a set of quantum numbers belonging to a state of any number of particles.) We call this reaction a <u>crossed reaction</u>.

This is nearly all we can say from a field theory in which Feynman graphs are basic. Although many of the conjectures we will now make might be provable from field theory, the perturbation series itself does not converge in strong interactions. So as things stand now, mathematically we are on very shaky ground. Instead of waiting for field theory to complete some very difficult proofs, we can formulate a new theory just to be on the safe side.

Although the S-Matrix Theory of Strong Interactions is not truly a complete theory, successful calculations have been done within the context of certain models. We begin by discussing the basics of this theory.

Perhaps the most basic postulate is the unitarity of the S-Matrix. Recall that

$$\phi^{(in)} = S\, \phi^{(out)}$$

and that the matrix elements of S can be written.

$$S_{fi} = \delta_{fi} + i\, \delta^4(P_f - P_i)\, A_{fi}$$

where

$$S^+ S = I \quad \text{or} \quad \Sigma_n S_{nf}^* S_{ni} = \delta_{fi}$$

Moreover, we demand that the S-Matrix be Lorentz invariant so A is a function of the invariants.

The very fundamental postulate that is suggested by Feynman graphs is that A is an analytic function. How analytic should it be? We make the conjecture of maximal analyticity. The amplitude has only those singularities required by unitarity. Once we establish the analyticity properties of A, we will be able to see that crossing has physical content and A does describe the main reaction and the two crossed reactions.

We have good reason to discuss the singularities required by unitarity. If A is an analytic function, its value is determined by its singularities. By using the Cauchy residue theorem, we can find integral equations for A . We do this for a simple case in part III. In other words, the analytic structure of the S-Matrix contains the dynamics of strong interactions.

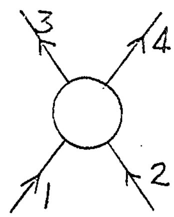
Before discussing details, we generalize the results from Feynman graphs to S-Matrix Theory. These postulates will be the basis of the rest of our discussion.

(1) A communicating particle corresponded to a pole in the Feynman graph. Postulate: If a particle communicates with the channels of A, then A contains a pole corresponding to the particle. Hence finding the poles of A is equivalent to identifying all the particles with the same quantum numbers as the initial or final states that A connects. Of course the particle pole cannot be in the physical region, for if it were, A would have an infinite value which could not be interpreted experimentally.

(2) The analyticity of the Feynman graphs is extended to the amplitude A . Postulate: A is a Lorentz invariant function of the momentum invariants with only those singularities required by unitarity.

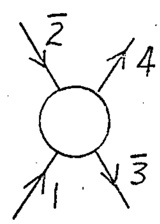
(3) The crossing property of diagrams is extended to the full amplitude. A is sufficiently analytic that (for example) the amplitude for $1 + 2 \rightarrow 3 + 4$:

A =



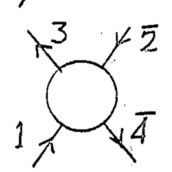
is also the amplitude for the two crossed reactions

$1 + \bar{3} \rightarrow \bar{2} + 4$



and

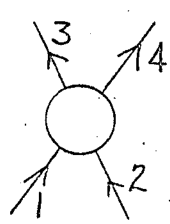
$1 + \bar{4} \rightarrow 3 + \bar{2}$



The same function is the amplitude for all three reactions. Again this is called crossing. (The other possible reactions are reached by symmetries such as TCP.)

To avoid too abstract a discussion, we restrict ourselves to

A =



where 1 through 4 are all spinless particles of equal mass m . As mentioned before, there exists two independent invariants. But we define three invariants--each will correspond to the center of mass energy of each reaction.

$$s = (p_1 + p_2)^2$$

$$t = (p_1 - p_3)^2$$

$$u = (p_1 - p_4)^2$$

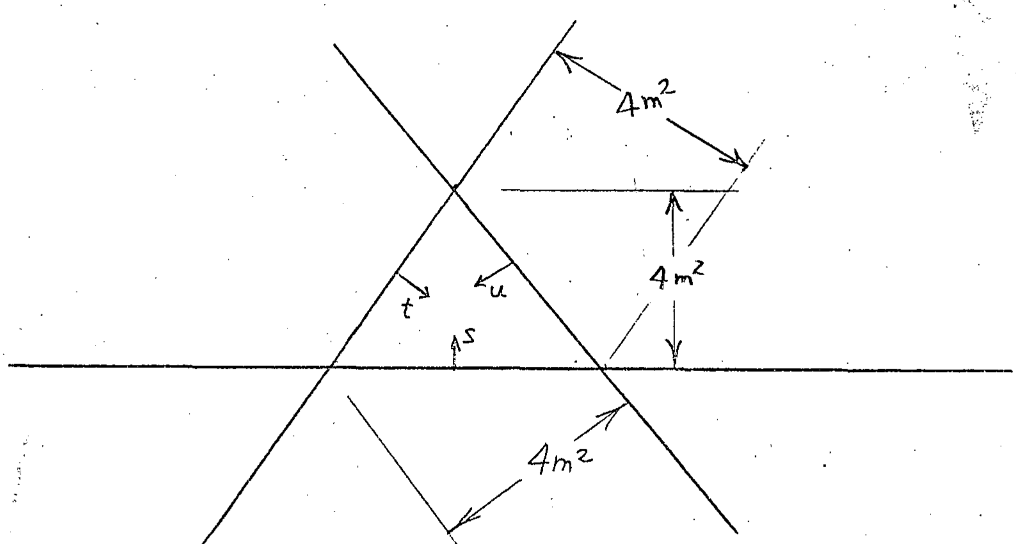
where

$$s + t + u = 4m^2$$

since

$$p_1 + p_2 = p_3 + p_4 \quad \text{and} \quad p_i^{\,2} = m^2 .$$

s, t, and u are called the Mandelstam variables and they can be visualized on a triangular plot called a Mandelstam diagram:



At every point on the diagram, $s + t + u = 4m^2$.

Using the Mandelstam diagram, we now find the physical regions for the three reactions. In the s channel ($1 + 2 \to 3 + 4$) and in the center of mass system:

$$s = (p_1 + p_2)^2 = 4(|\underset{\sim}{p}|^2 + m^2) \Rightarrow s > 4m^2$$

$$t = (p_1 - p_3)^2 = -(\underset{\sim}{p}_1 - \underset{\sim}{p}_3)^2 = -[\underset{\sim}{p}_1{}^2 + \underset{\sim}{p}_3{}^2 - 2\underset{\sim}{p}_1 \cdot \underset{\sim}{p}_3]$$

$$= -2\,|\underset{\sim}{p}|^2\,(1 - \cos\Theta) \Rightarrow t < 0$$

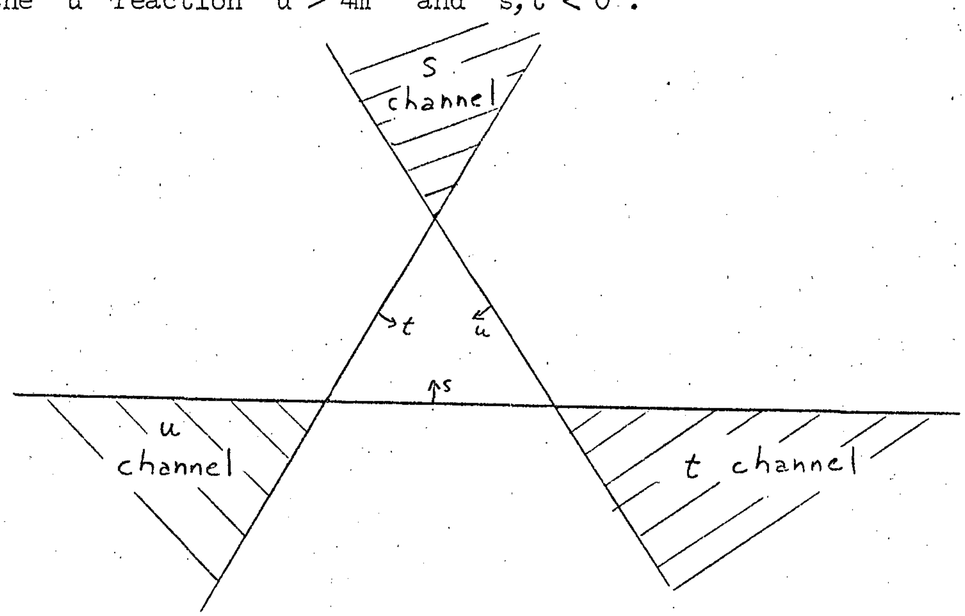
$$u = -2\,|\underset{\sim}{p}|^2\,(1 + \cos\Theta) \Rightarrow u < 0$$

The physical region for the s-channel is $s > 4m^2$, $t < 0$, and $u < 0$.

(For unequal masses the physical region is slightly modified and the algebra is more complicated.)

For the t-channel, $p_3 \rightarrow -p_3$ and $p_2 \rightarrow -p_2$. Hence t is the square of the center of mass energy $(t > 4m^2)$ and s and u are less than zero.

In the u reaction $u > 4m^2$ and $s, t < 0$.



The shaded portions of the diagram are the physical regions of the three reactions. The physical regions never overlap (this is general). Consequently, if crossing is to have physical content, there can be no barriers to prevent analytic continuation between physical regions. But this analyticity is guaranteed by the principle of maximal analyticity, as we shall show. So we finally ask about the singularities that are required by unitarity.

The unitarity relation $S^\dagger S = I$ can be written as

$$\Sigma_n S_{nf}^* S_{ni} = \delta_{fi}$$

where the sum on n goes over all possible intermediate states. Factoring out the non-analytic pieces:

$$S_{k\ell} = \delta_{k\ell} + i\, \delta^4(P_k - P_\ell)\, A_{k\ell}(s,t) \;;$$

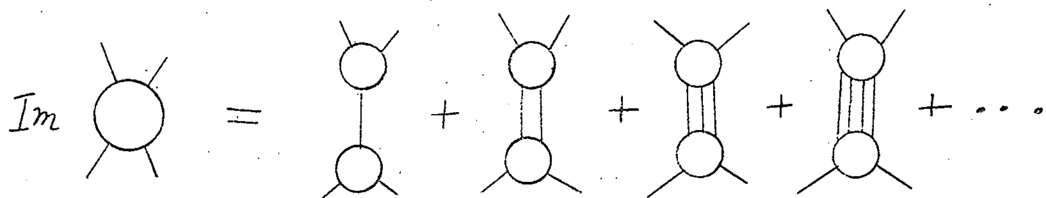
and $S^\dagger S = I$ becomes

$$i\,[\,A_{if}^*(s,t) - A_{fi}(s,t)] = \Sigma_n A_{nf}^*(s,t)\, A_{ni}(s,t)\, \delta(P_n - P_i)\;.$$

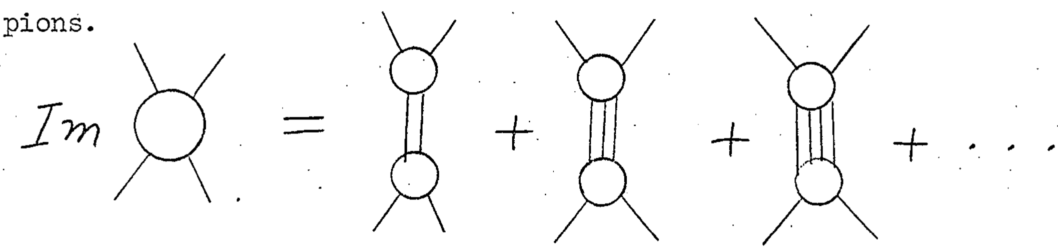
(This derivation is carried out in detail in Part III.) The f and i label the final and initial states. For elastic scattering, $i = f$, and $A_{fi}^* = A_{ii}^*$. Hence,

$$\mathrm{Im}\, A_{ii}(s,t) = \tfrac{1}{2}\, \Sigma_n\, |A_{in}|^2\;\; \delta^4(P_n - P_i) \quad .$$

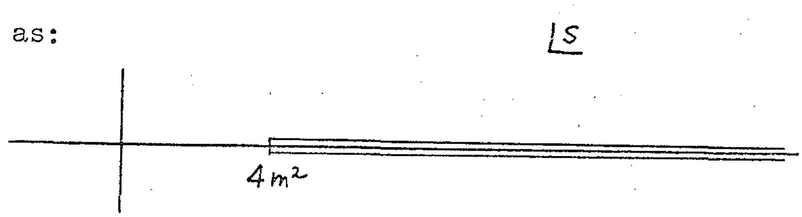
These results are very important. To make them more graphic, we can construct a pictorial scheme for $\mathrm{Im}\, A$. Assume there is only one kind of particle, and the single particle communicates with two particles. Then,



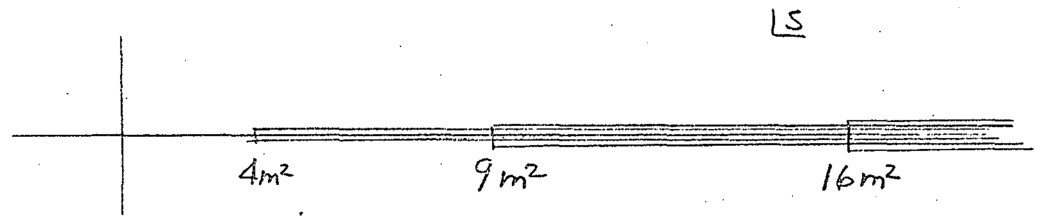
The sum over intermediate states is broken up into 2,3,4, etc. particle intermediate states and each two bubble diagram implies a sum over all intermediate momenta (and spins). The first term is the pole term. For pseudo scalar particles (such as pions) the pole term doesn't exist because the pion doesn't communicate with the two pion system without violating parity. To simplify the algebra, we shall assume the four external particles are pions.



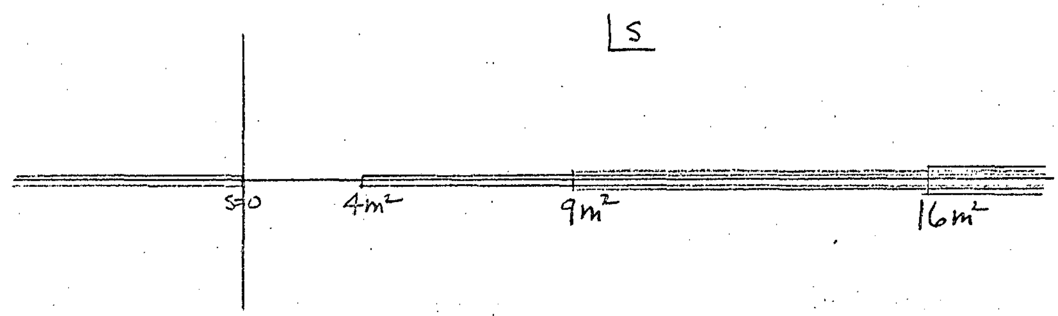
The singularities of A can be found from this equation. At the threshold of the first intermediate state $(s = 4m^2)$, the amplitude suddenly develops an imaginary part. Below this threshold A is real for real s because unitarity doesn't force A to have an imaginary part. But at threshold there must be a singularity for the analytic function to suddenly develop an imaginary part for real s. The singularity is a branch point, and the cut is drawn as:



As s is increased to $9m^2$, suddenly a new channel opens up, the three body channel. Hence there is another branch point at $9m^2$, and so on. So far, A has the following analytic structure in the complex s plane.

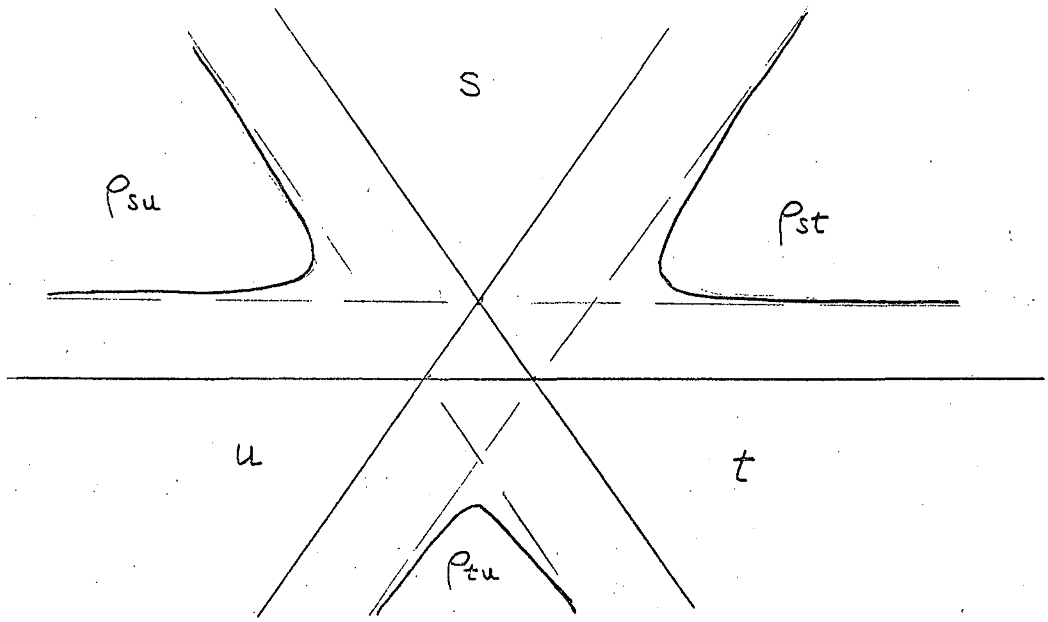


But this isn't all-there are the crossed channel reactions and they also contribute to the analytic structure of A because of the unitarity relations for these reactions. In both the t and the u reactions, s is physical for $s < 0$. Hence there are branch points along the negative s axis. The full analytic structure of A in the s plane is then



Due to the simplicity of this particular problem, the t and u planes will look the same. Moreover the right hand cuts in the s plane contribute to the left hand cuts in the t or u plane.

We can find the region of these singularities in the Mandelstam diagram. This is because all the singularities lie along the real axis. From the unitarity relation, Im $A(s,t,u) = 0$ in the region $0 < s < 4m^2$, $0 < t < 4m^2$, and $0 < u < 4m^2$. There can be singularities only if: $s > 4m^2$ and $t > 4m^2$; $s > 4m^2$ and $u > 4m^2$; or $t > 4m^2$ and $u > 4m^2$.



The actual boundary of the region of singularities are the regions of ρ_{st}, ρ_{su} and ρ_{tu}, and they are called the double spectral regions. This representation of the analytic structure of A is called the Mandelstam Representation. Among other things, this proves that crossing is always possible.

What physics have we done? We appeal to the correspondence principle with nonrelativistic quantum mechanics. In quantum mechanics, the left hand singularities correspond to the potential. In the nonrelativistic theory, the crossed channels don't exist so we have to put in a potential, but in the relativistic theory, unitarity relations replace the potentials. The physics is: The existence of the crossed channels gives rise to the forces by which strongly interacting particles scatter. To emphasize this result, a simple calculation is given in Part III.

PART III. AN EXPLORATORY CALCULATION

Before we submerge ourselves in the minutiae of a lengthy calculation, a quick review is in order. In Part II, maximal analyticity was used to show two startling facts about the scattering amplitude: (1) The amplitude was sufficiently analytic that it not only described the main reaction, but also the two crossed reactions. We derived the possibility of crossing (no barriers between the physical region--remember the Mandelstam representation) from the existence of the "gap" $\mathrm{Im}\, A(s,t,u) = 0$ for $0 \leqslant s,t,u < 4m^2$. We shall make good use of this "gap". (2) The discontinuity across the cuts could be calculated from unitarity, thus determining the scattering amplitude.

There are two famous paths we could follow. The first is to use the mathematical statement of the Mandelstam representation. By working between the double spectral functions and the unitarity formulae, one can generate the scattering amplitude by the Mandelstam interation procedure. Although simple in principle, this procedure is difficult in practice, and we won't attempt to clarify the details here.

As for the second path, the gap from 0 to $4m^2$ makes it possible to write down the partial wave amplitude as a ratio

$$A_\ell(s) = \frac{N_\ell(s)}{D_\ell(s)}$$

where $N_\ell(s)$ has only the left hand cuts and $D_\ell(s)$ the right hand cuts of the s-plane. (We shall derive partial wave unitarity with some simplifications.) Then using the Cauchy integral representation of an analytic function, we can find a pair of coupled integral equations for $N_\ell(s)$ and $D_\ell(s)$.

We follow the N/D method because it will reveal some interesting physics. In particular, the postulates of S-Matrix theory that we discussed in Part II were simply generalizations of Feynman graphs. It is very possible they are equivalent to field theory. Thus it is possible to introduce "elementary" particles and arbitrary parameters into the theory. But then how do we distinguish elementary particles from "composite" particles? And how do we reject elementary particles from the theory? This This question is briefly considered after the N/D derivation.

It is the intention of the following calculation to (1) Show how analyticity can be used to set up equations for the scattering amplitude in a very simple model; (2) To discuss the rejection of "elementary" particles; (3) To serve as an introduction to the typical style of calculation found in S-Matrix theory. The calculation is spelled out in great detail.

We shall begin with unitarity, make the elastic unitarity approximation and then calculate the partial wave unitarity relation. By writing $A = A(s,t)$, we restrict ourselves to the scattering of two spinless particles into two spinless particles. Unitarity is given by

$$\Sigma_n \, S_{nf}^{*} \, S_{ni} = \delta_{fi} \qquad (1)$$

where

$$S_{k\ell} = \delta_{k\ell} + i\,\delta^4(P_k - P_\ell)\, A_{k\ell}(s,t) \quad . \qquad (2)$$

Substituting (2) into (1) :

$$\delta_{fi} = \Sigma_n S_{nf}^* S_{ni} = \Sigma_n [\delta_{nf} - i\, \delta^4(P_n - P_f)\, A_{nf}^*(s,t)]$$

$$[\delta_{ni} + i\, \delta^4(P_n - P_i)\, A_{ni}(s,t)]$$

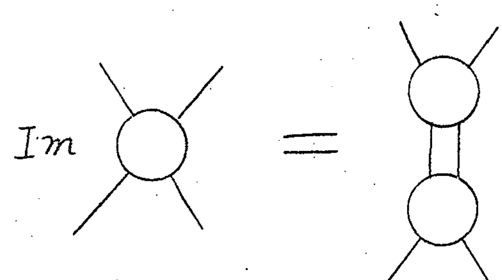
$$= \delta_{fi} + i\, \delta^4(P_f - P_i)\, A_{fi}(s,t) - i\, \delta^4(P_i - P_f)\, A_{if}^*(s,t)$$

$$+ \delta^4(P_i - P_f)\, \Sigma_n\, \delta^4(P_i - P_n)\, A_{nf}^*(s,t)\, A_{ni}(s,t)$$

or

$$i[A_{if}^*(s,t) - A_{fi}(s,t)] = \Sigma_n A_{nf}^*(s,t)\, A_{ni}(s,t)\, \delta^4(P_i - P_n) \ .$$

We make two simplifications: (1) The final state contains the same two particles as the initial state (elastic scattering.) (2) Only the elastic unitarity diagram need be considered in the sum on n . In pictures



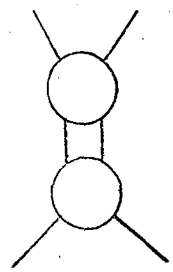
Simplifying the notation

$$A_{fi}(s,t) = A(s,t)$$

and using

$$A^*(s,t) - A(s,t) = -2\, i\, \mathrm{Im}\, A(s,t) \quad .$$

In the unitarity integral



we must still sume over the momenta of the intermediate state:

$$\text{Im}\, A(s,t) = \frac{1}{2} \sum_{\substack{\text{intermediate}\\ \text{momenta}}} A^*(s,t') \, A(s,t'') \, \delta^4(P_n - P_i)$$

where t' is the momentum transfer from the initial to the intermediate state and t'' the momentum transfer from the intermediate to the final state.

We sum over intermediate momentum states by integrating. As A is a scalar, we define the invariant integration measure

$$\frac{d^3 \underset{\sim}{p}}{2\, E} = d^4\, p \, \delta_+(p^2 - m^2)$$

where

$$d^3 \underset{\sim}{p}' = |\underset{\sim}{p}'|^2 \, d|\underset{\sim}{p}'| \, d\,\Omega' \quad :$$

$$\text{Im}\, A(s,t) = \frac{1}{2} \int \frac{d^3 \underset{\sim}{p}_1{}'}{2\, E_1{}'} \, \frac{d^3 \underset{\sim}{p}_2{}'}{2\, E_2{}'} \, A^*(s,t') \, A(s,t'') \, \delta^4(p_1{}' + p_2{}' - P_i) \, .$$

The momenta of the two intermediate particles are $\underset{\sim}{p}_1{}'$ and $\underset{\sim}{p}_2{}'$. This integral is most easily performed in the center of mass system where $\underset{\sim}{P}_i = 0$. The spatial part of the δ-function then says $\underset{\sim}{p}_1{}' = -\underset{\sim}{p}_2{}'$ so $E_1{}' = E_2{}' = E$ where E is the center of mass energy of one of the incoming pions. The $d^3 \underset{\sim}{p}_2{}'$ integration is done on the spatial part of the δ-function:

$$\text{Im}\, A(s,t) = \frac{1}{8} \int \frac{|\underset{\sim}{p}_1{}'|^2 \, d\, |\underset{\sim}{p}_1{}'| \, d\, \Omega'}{|\underset{\sim}{p}_1{}'|^2 + \mu^2} \; A^*(s,t') \, A(s,t'') \, \delta(2E_1{}' - E_i).$$

The $d|\underset{\sim}{p}_1{}'|$ integration is done with the δ-function by noting:

$$d|\underset{\sim}{p}_1{}'| = dE' \left(\frac{dE_1{}'}{d|\underset{\sim}{p}_1{}'|} \right)^{-1} \quad \text{and} \quad \frac{dE_1{}'}{d|\underset{\sim}{p}_1{}'|} = \frac{|\underset{\sim}{p}_1{}'|}{E_1{}'} \quad ;$$

The δ-function requires $E_1{}' = E$. Thus :

$$\boxed{\text{Im}\, A(s,t) = \frac{p}{16E} \int d\, \Omega' \; A^*(s,t') \, A(s,t'') \quad .} \qquad (3)$$

This is the statement of total elastic unitarity. To simplify its form, we use the partial wave decomposition of A .

$$A(s,t) = \Sigma_\ell (2\ell + 1) \, A_\ell(s) \, P_\ell(\cos\theta)$$

where recalling our kinematics

$$t = -2 \, |\underset{\sim}{p}|^2 \, (1 - \cos \theta)$$

or

$$\cos \theta = 1 + \frac{t}{2\underset{\sim}{p}^2} \quad .$$

Thus

$$\text{Im}\left[\Sigma_\ell (2\ell+1)\, A_\ell(s)\, P_\ell(\cos\Theta)\right] =$$

$$\frac{P}{18E} \int d\Omega' \sum_{\ell,\ell'} (2\ell+1)\,(2\ell'+1)\, A_\ell^*(s)\, A_{\ell'}(s)$$

$$P_\ell(\cos\Theta')\, P_{\ell'}(\cos\Theta'')$$

The integral can be done by applying the addition theorem for Legendre polynomials:

$$P_{\ell'}(\cos\Theta'') = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell,m}^*(\Theta',\phi')\, Y_{\ell,m}(\Theta,\phi)$$

By the orthogonality of the $Y_{\ell,m}$ only the $m = 0$ term will survive the integration

$$\sum_{\ell'} d\Omega'\, P_\ell(\cos\Theta')\, P_{\ell'}(\cos\Theta'')$$

$$= \Sigma_{\ell'} \frac{4\pi}{2\ell'+1}\, Y_{\ell',0}(\Theta,\phi) \int d\Omega'\, P_\ell(\cos\Theta)\, Y_{\ell',0}(\Theta',\phi')$$

$$= \frac{4\pi}{2\ell+1}\, P_\ell(\cos\Theta)$$

where the normalization conventions have been used. Thus

$$\text{Im}\left[\Sigma_\ell (2\ell+1)\, A_\ell(s)\, P_\ell(\cos\Theta)\right] = \frac{P}{16E}\, \Sigma_\ell\, (2\ell+1)^2\, \frac{4\pi}{(2\ell+1)}\, |A_\ell(s)|^2$$

or:

$$\boxed{\text{Im } A_\ell(s) = \frac{\pi P}{4E} |A_\ell(s)|^2} \qquad (4)$$

Defining

$$\rho(s) = \frac{\pi P}{4E} = \frac{\pi}{4}\sqrt{\frac{s - 4m^2}{s}} \qquad (5)$$

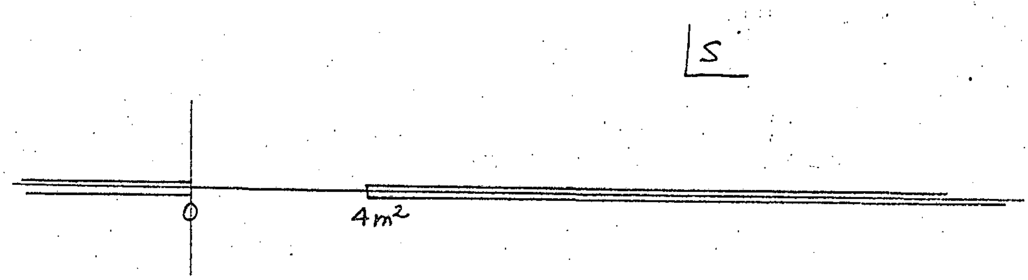
Eq. (4) can be written more conveniently:

$$\text{Im } A_\ell = \frac{1}{2i}[A_\ell - A_\ell^*] = \rho A_\ell A_\ell^*$$

or

$$\text{Im } \frac{1}{A_\ell(s)} = - \rho(s) \quad . \qquad (6)$$

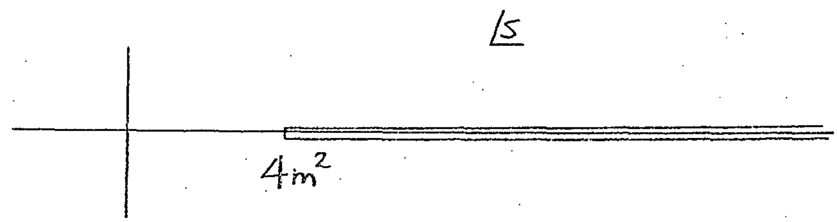
If we assume that no poles communicate with the crossed channels, then the partial waves have the same singularity structure as the total amplitude. The exactly similiar singularity structure is an accident of the problem. With elastic unitarity, there is a cut from $s = 4m^2$ to ∞, and a cut from 0 to $-\infty$.



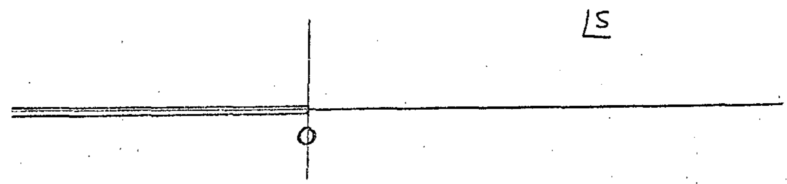
The next step is to appeal to a result of the Wiener-Hopf theory of integral equations: If $A_\ell(s)$ has the singularity structure shown above, then $A_\ell(s)$ can be written

$$A_\ell(s) = \frac{N_\ell(s)}{D_\ell(s)}$$

where $N_\ell(s)$ contains only the left hand cut and $D_\ell(s)$ contains only the right hand cut:



Singularity structure of $D_\ell(s)$



Singularity structure of $N_\ell(s)$.

Thus:

$$\text{Im } N_\ell(s) = 0 \qquad s > 0$$

$$\text{Im } N_\ell(s) = \text{Im } [A_\ell(s) D_\ell(s)] = D_\ell(s) \text{ Im } A_\ell(s) =$$

$$= D_\ell(s) \; f_\ell(s), \quad s < 0$$

where $f_\ell(s)$ is defined by:

$$\text{Im } A_\ell(s) = f_\ell(s) \quad s < 0$$

and must be determined from the knowledge of the crossed channels. Also:

$$\text{Im } D_\ell(s) = 0 \quad s < 4m^2$$

$$\text{Im } D_\ell(s) = \text{Im } \frac{N_\ell(s)}{D_\ell(s)} = N_\ell(s) \text{ Im } \frac{1}{A_\ell(s)} = -\rho(s) N_\ell(s) \qquad s > 4m^2.$$

The N/D equations are formulated by using the Cauchy integral representation

$$g(z) = \frac{1}{2\pi i} \int_C dz' \frac{g(z')}{z' - z}$$

where $g(z')$ is analytic in and on C and C encloses the point z. We assume:

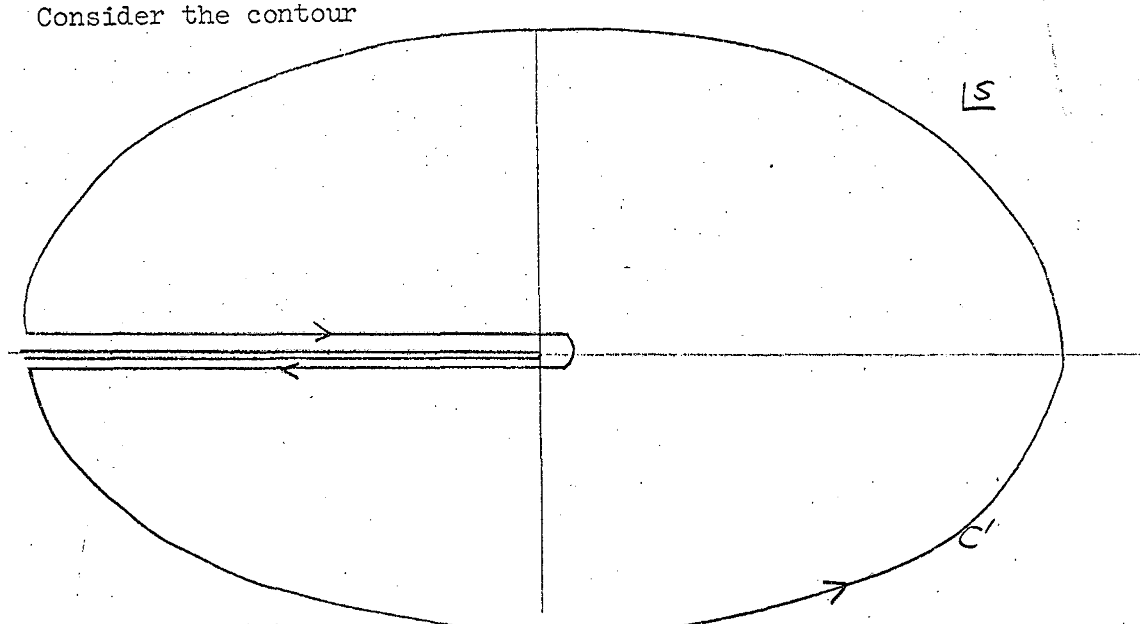
$$N_\ell(s) \xrightarrow[s \to \infty]{} 0$$

$$D_\ell(s = 0) = 1$$

$$D_\ell(s) \xrightarrow[s \to \infty]{} \text{constant} \quad .$$

Note that we can pick a value for D at some point and still not introduce arbitrary parameters into the theory.

Consider the contour



for representing $N_\ell(s)$. Then

$$N_\ell(s) = \frac{1}{2\pi i} \int_C \frac{ds'}{s' - s}$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{ds'}{s' - s} N_\ell(s'^{+}) + \frac{1}{2\pi i} \int_{0}^{-\infty} \frac{ds'}{s' - s} N_\ell(s'^{-}) + \int_{C'} .$$

The integral on the circle at ∞ is zero and $N(s'^{+}) = N^{*}(s'^{-})$

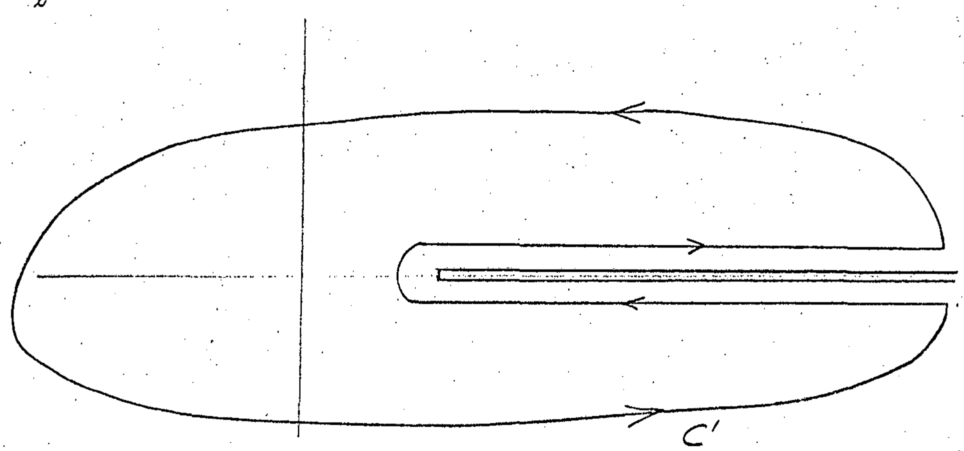
$$N_\ell(s) = \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{ds'}{s' - s} \quad N_\ell(s') - N_\ell^{*}(s')$$

$$= -\frac{1}{\pi} \int_{0}^{-\infty} \frac{ds'}{s' - s} \quad \text{Im}\, N_\ell(s')$$

or

$$\boxed{N_\ell(s) = -\frac{1}{\pi} \int_{0}^{-\infty} \frac{ds'}{s' - s} \; D_\ell(s')\, f_\ell(s')} \qquad (7)$$

For $D_\ell(s)$, we consider the contour:



Making a <u>subtraction</u> the non-rigorous way:

$$D_\ell(s) - D_\ell(0) = \frac{1}{2\pi i} \int_C \frac{ds'}{s' - s} D_\ell(s') - \frac{1}{2\pi i} \int_C \frac{ds'}{s' - 0} D_\ell(s')$$

$$= \frac{s}{2\pi i} \int_C \frac{ds'}{s'(s' - s)} D_\ell(s') = \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'(s' - s)} \operatorname{Im} D_\ell(s')$$

or

$$\boxed{D_\ell(s) = 1 - \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'(s' - s)} \rho(s') N_\ell(s')} . \qquad (8)$$

Equations (7) and (8) are coupled integral equations for N and D. They can be decoupled by substituting 7 into 8 or 8 into 7. Substituting (7) into (8) yields an equation for $D_\ell(s)$.

$$D_\ell(s) = 1 - \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{ds' \rho(s')}{s'(s' - s)} \left(- \frac{1}{\pi} \int_0^{\infty} \frac{ds''}{s'' - s'} D_\ell(s'') f_\ell(s'') \right)$$

$$= 1 - \frac{s}{\pi} \int_0^{\infty} ds' D_\ell(s') f_\ell(s') \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds''}{s''} \frac{\rho(s'')}{(s''-s)(s''-s')} .$$

Define

$$R(s) = \frac{1}{\pi} \int_{4m^2}^{\infty} ds'' \frac{\rho(s'')}{s''(s'' - s)}$$

$$R(s') = \frac{1}{\pi} \int_{4m^2}^{\infty} ds'' \frac{s - s'}{s''(s'' - s')} ,$$

then

$$R(s) - R(s') = \frac{1}{\pi} \int_{4m^2}^{\infty} ds'' \frac{\rho(s'')}{s''} \frac{s - s'}{(s'' - s)(s'' - s')} .$$

Thus

$$D_\ell(s) = 1 + \frac{s}{\pi} \int_0^{-\infty} ds' \left[\frac{R(s) - R(s')}{s' - s} f_\ell(s') \right] D_\ell(s') \quad . \tag{9}$$

Since $R(s)$ is known and $f_\ell(s)$ is presumed know, let

$$K(s,s') = \frac{s}{\pi} \left[\frac{R(s) - R(s')}{s' - s} \right] f_\ell(s') \tag{10}$$

be the kernel of the integral equation

$$D_\ell(s) = 1 + \int_0^{-\infty} ds' \, K(s,s') \, D_\ell(s') \quad . \tag{11}$$

The condition that (9) be a Fredholm equation is that the kernel be square integrable.

$$\int_{-\infty}^{0} ds \int_{-\infty}^{0} ds' \, |K(s,s')|^2 < \infty \quad .$$

By means of a rather lengthy integration, it is possible to show that $f_\ell(s)$ must be bounded by s^α where α is less than zero. (To do this, transform the integral equation so that $K(s,s')$ is symmetric; then integrate.)

$$f_\ell(s) \sim s^\alpha \qquad \alpha < 0 \quad .$$

Only then can we be sure that (9) has a solution.

We shall indicate one form of Bootstrap hypothesis. (A bootstrap hypothesis is the postulate to eliminate elementary particles from the theory.) The final result of the N/D calculation was the integral equation for the denominator function, Eq.(11) . The left-hand cut appears in the kernel, $K(s,s')$ in Eq. (10). We also showed that Eq. (11) is a Fredholm equation (and hence a solution is guaranteed) only if

$$f_\ell(s) \xrightarrow[-s \to \infty]{} s^\alpha \quad \underline{\underline{\alpha < 0}} \ .$$

However a pole in the t channel of spin ℓ' contributes a term of the form

$$\frac{P_\ell(\cos \theta_t)}{t - M^2} \ .$$

Moreover, from the kinematics

$$s = -2\, \underset{\sim}{p}_t^{\,2}(1 + \cos \theta_t)$$

or

$$\cos \theta_t = -1 + \frac{s}{2\underset{\sim}{p}_t^{\,2}} \ .$$

Fixing t and allowing s to become large, $\cos \theta_t$ becomes unphysical, and we can use the asymptotic form of $P_{\ell'}(\cos \theta_t)$

$$f_\ell(s) \sim s^{\ell'} \ .$$

Hence the integral equation is singular and arbitrary parameters are necessary.

A solution to this difficulty can be found by analytically continuing the partial wave amplitude into the complex angular momentum plane. In the ℓ-plane, one finds poles that move as a function of s. However we already know how to interpret poles--they correspond to particles when they cross a physical value of angular momentum at physical s. If a particle lies on such a "Regge" pole, then

$$\ell' = \ell'(s) \quad .$$

If $\ell'(s)$ decreases to a value less than zero as s increases, then the equation becomes Fredholm. The Chew-Frautschi-Mandelstam hypothesis is that all particles lie on Regge poles. This conjecture not only makes it possible to solve Eq. 11, but it also eliminates elementary particles from S Matrix theory, since for an elementary particle,

$$\ell = \ell' \text{, a constant} \quad .$$

ACKNOWLEDGMENTS

It is a pleasure to thank Professor Stanley Mandelstam for his helpful suggestions. I would also like to thank Dr. H. L. McMurry and Phillips Petroleum Company for the invitation to deliver these lectures under their Distinguished Lectureship Program.

<u>Some ideas for further reading</u>.

G. Chew, M. Gell-Mann and A. Rosenfeld, <u>Strongly Interacting Particles</u>, Scientific American, Feb. 1964.

An excellent survey of the particles, their quantum numbers, and various theories about their interactions is contained in this article.

M. Mandelstam, <u>Dispersion Relations in Strong Coupling Physics</u>, Reports on Progress in Physics, 25, 99-162 (1962).

This is a very readable review of dispersion relations, proofs of analyticity from field theory, and applications.

G. Källén, <u>Elementary Particle Physics</u>, Addision Wesley, Reading, Mass. (1964).

This is probably the best introduction to particle physics that has been written.

J. Bjorken and S. Drell, <u>Relativistic Quantum Mechanics</u> (1964) and <u>Relativistic Quantum Fields</u> (1965) McGraw-Hill, San Francisco, Calif.

The second volume is a pedagogically tractable introduction to field theory.

The W. A. Benjamin Inc., New York, has published a number of very relevant paperbacks in the field of S-Matrix theory at the cost of about

$5.00 apiece. They are listed below in approximate order of increasing difficulty.

R. Omnes and M. Froissart, Mandelstam Theory and Regge Poles (1963).

G. F. Chew, S-Matrix Theory of Strong Interactions, (1962).

S. Frautschi, Regge Poles and S-Matrix Theory (1963).

M. Jacob and G. Chew, Strong Interaction Physics, (1964).

E. Squires, Complex Angular Momenta and Particle Physics, (1964).