Branching of Hitchin's Prym cover for SL(2)

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Abstract

It is shown that the map from the Jacobian of the spectral curve to the moduli of stable bundles of rank 2 is generically simply branched along an irreducible divisor. This observation falsifies the key step in the "abelianization of the SU(2) WZW connection" presented in a recent paper [Y].

1. Statement

Let Σ be a smooth complex projective curve of genus $g \geq 3$ and B a reduced divisor in $|K^2|$. The square root r of a section of K^2 vanishing on B defines a double cover $p : \tilde{\Sigma} \to \Sigma$ embedded in the total space of K, branched along B. It is a smooth curve of genus $\tilde{g} = 4g-3$, with Galois involution ι, the sign change on K. $\tilde{\Sigma}$ is the simplest example of a *spectral curve* [H], for rank 2 bundles on Σ. More precisely, for a line bundle L on $\tilde{\Sigma}$, the direct image $E = p_*L$ is a vector bundle on Σ, and multiplication by r on sections of L defines the *Higgs field* $\phi : E \to E \otimes K$.

It is know that E is stable, if L avoids a sub-variety V of co-dimension $\geq g-1$ in the Jacobian of $\tilde{\Sigma}$ [H, BNR]. The construction works in families, so it defines a morphism π from the Jacobian (minus V) to the moduli space of stable vector bundles on Σ. Moreover, π is generically finite, of degree 2^{3g-3}. We chose $g \geq 3$ so that singularities of the moduli spaces, as well as the stable/semi-stable distinction can be ignored.

Let us concentrate on the critical Jacobian $\tilde{J}$ of degree $\tilde{g}-1$, which maps to the moduli space M of semi-stable rank 2 bundles of slope $g-1$; the story is similar for all even degrees. Call K_M the canonical bundle of M. In this note, I verify the following (known) fact:

Theorem 1. *$\pi : \tilde{J} \setminus V \to M$ is étale away form an irreducible divisor D, and is generically simply branched along D. Moreover, $\mathcal{O}(D) = \pi^* K_M^\vee$.*

Up to isogeny, $\tilde{J}$ factors as $J \times P$ (see 1.2) and D comes from an ample divisor on the Prym factor P. The important part is the simple branching; it implies the second statement, because the canonical bundle of $\tilde{J}$ is trivial and the Jacobian determinant of π gives a section of $\pi^* K_M^\vee$ with simple vanishing along D.

In a recent paper [Y], Yoshida proposed a solution of a long-standing problem, a reduction of the flat connection in the WZW model for SU(2) to abelian Theta-functions. The key ingredient in the construction is a distinguished Theta-fuction Π, living in a *square root* of the anti-canonical pull-back $\pi^* K_M^\vee$ and vanishing along D. Both properties of Π are essential for the constructions that follow. However, the theorem shows that such Π does not exist.[1]

The interesting part of the story concerns SL(2) bundles and an associated Prym variety P; but their relation to GL(2) is straightforward, because π is compatible with the tensor action (on $\tilde{J}$ and M) of the degree zero Jacobian J of Σ. More precisely, let $\tilde{K}$ be the canonical bundle

[1] Yoshida constructs Π on an isogenous cover of $\tilde{J}$, but the distinction is unimportant. Page 2 of *loc. cit.* explicitly claims that Π^2 is the Jacobian determinant.

of $\tilde\Sigma$ and call $\tilde B$ the branch divisor; note the isomorphism $\tilde K \cong p^*K(\tilde B)$. For a line bundle L on $\tilde\Sigma$, the exact sequence

$$0 \to L \to p^*p_*L \to \iota^*L(-\tilde B) \to 0$$

shows the equivalence of the conditions

$$L \otimes \iota^*L \cong \tilde K \quad \text{and} \quad \det(p_*L) \cong K. \tag{1.1}$$

They define the Prym variety $P \subset \tilde J$. Mind, however, that the first isomorphism is always *anti-invariant* for ι, which changes the sign on the fibres of $\tilde K$ over $\tilde B$. With M_K denoting the moduli space of semi-stable bundles on Σ with determinant K and $\Gamma \subset J$ its 2-torsion subgroup, we have

$$\tilde J = J \times_\Gamma P \quad \text{and} \quad M = J \times_\Gamma M_K, \tag{1.2}$$

compatibly with the map π. Up to translation, the restricted morphism $P \setminus V \to M_K$ is equivalent to the Prym covering of the moduli space of SL(2)-bundles.

1.3 Remark. $\tilde K \cong \mathcal{O}(2\tilde B)$, so one can use $L = \mathcal{O}(\tilde B)$ to identify $\tilde J$ with the degree zero Jacobian; ι^* becomes an automorphism.

2. Proof

Let us abusively call the points in $\tilde J \setminus V$ where π fails to be étale the 'branch points', even though π may not be everywhere finite; the contraction locus has co-dimension $\geq g-2$ (because the Theta-polarisations of the two spaces are compatible, Remark 2.5.i below). I describe the branching locus in terms of a ramified cover of a projective space and show its irreducibility. Finally, I show that the branching is simple by studying linearised deformations.

(2.1) The branch locus. Let us compare first-order deformations of L and of $E = p_*L$. The tangent space to P is the (-1)-eigenspace for ι on $H^1(\tilde\Sigma; \mathcal{O})$, while the tangent space to M_K at E is $H^1(\Sigma; \mathcal{E}nd^0(E))$, the traceless endomorphism bundle. Note that $p_*\mathcal{O}$ splits into the $+/-$ eigenspaces of ι as $\mathcal{O} \oplus K^\vee$, so that TP is identified with $H^1(\Sigma; K^\vee)$. Unravelling the definition shows that the differential of π at L is the map induced by the Higgs field $\phi \in \mathcal{E}nd^0(E) \otimes K$:

$$\phi : H^1(\Sigma; K^\vee) \to H^1\left(\Sigma; \mathcal{E}nd^0(E)\right).$$

(For $\tilde J$ and GL(2), one adds the $H^1(\Sigma; \mathcal{O})$ summands to both sides.) When E is stable, both spaces have the same dimension $3g-3$, and the short exact sequence on Σ,

$$0 \to K^\vee \xrightarrow{\phi} \mathcal{E}nd^0(E) \to \mathcal{Q} \to 0,$$

shows that π is not étale iff the quotient $\mathcal{Q}$ has $h^1 \neq 0$. In terms of L, $\mathcal{Q} = p_*\left(\iota^*L^{-1}L(\tilde B)\right)$, and is a rank 2 vector bundle with determinant K. It follows from Serre duality that $h^0(\mathcal{Q}) = h^1(\mathcal{Q})$. Thus, L is a branch point iff $\iota^*L^{-1}L(\tilde B)$ has sections over $\tilde\Sigma$, in other words, the last line bundle lies in the Theta-divisor Θ of $\tilde\Sigma$.

(2.2) The Prym Theta-divisor. Consider the endomorphism $\sigma : L \mapsto \iota^*(L)^{-1}L(\tilde B)$ of $\tilde J$. It factors via the projection to $\tilde J/J$ and lands in P. Restricted to P, $\sigma(L) = L^2(-\tilde B)$ (or just the square, if we use $\mathcal{O}(\tilde B)$ as base-point). We now show that Θ meets P transversely in an irreducible (and locally unibranch) divisor. Its pre-image $\sigma^*(\Theta \cap P)$ will be the branching divisor D of π, and we will relate transversality to simple branching.

Theta is the Abel-Jacobi image of $\mathrm{Sym}^{\tilde{g}-1}\tilde{\Sigma}$, and the condition $L \otimes \iota^* L \cong \tilde{K}$ defining P says that each divisor $S \in |L|$ satisfies $S + \iota(S) \in |\tilde{K}|$: multiply the matching sections of L and $\iota^* L$. The resulting section of $\tilde{K}$ is anti-invariant under ι, as was the isomorphism in (1.1). The anti-invariant p_*-image of $\tilde{K}$ is K^2, and we obtain a bijection between divisors $S + \iota(S) \in |\tilde{K}|$ and points of $|K^2|$ (on Σ).

Now, S involves, in addition, a choice of point within each mirror pair in $S + \iota(S)$. The collection of choices defines a finite cover $\tilde{\mathbb{P}}$ of $|K^2|$, simply branched over the hyperplanes of sections which vanish at some point of B. The monodromy around a hyperplane defined by $b \in B$ switches the point of S which is near b with its ι-mirror. It follows that the monodromies act transitively on the fibres of $\tilde{\mathbb{P}} \to |K^2|$, so that $\tilde{\mathbb{P}}$ is irreducible. The same follows then for the intersection $\Theta \cap P$, which is set-theoretically the Abel-Jacobi image of $\tilde{\mathbb{P}}$. Finally, the fibres of the Abel-Jacobi map are connected, so the image is locally unibranch.

(2.3) Simple branching. First, observe that P contains smooth points of Θ. Indeed, over a singular point $L \in \Theta$, $\mathrm{Sym}^{\tilde{g}-1}\tilde{\Sigma}$ has positive-dimensional fibre; but this is also the fibre of the map $\tilde{\mathbb{P}} \to \Theta \cap P$, which is generically finite for dimensional reasons. Next, at any smooth $L \in \Theta$ which lies in P, I claim that the normal to Θ is a (-1)-vector for ι. For this, observe that the tangent space $T_L\Theta$ comprises the $\xi \in H^1(\tilde{\Sigma}; \mathcal{O})$ which induce the zero map $H^0(L) \to H^1(L)$, these ξ being the first-order variations of L which carry sections. Equivalently, the co-normal line to Θ is the image in $T^\vee \tilde{J} = H^0(\tilde{K})$ of the cup-product $H^0(L) \otimes H^0(\tilde{K}L^{-1})$. For $L \in P$, $\tilde{K}L^{-1} \cong \iota^* L$, so the image contains the product of a section with its ι-transform; but we saw earlier that this is *anti-invariant* under ι. This proves transversality.

In terms of π, this shows that $h^0(\Sigma; \mathcal{Q}) = 1$ generically on D, and that the section fails to extend over the first-order neighbourhood of D (which surjects to that of $\Theta \cap P$ in P). Since a first variation makes ϕ an isomorphism, the branching is simple.

(2.4) Irreducibility. Recall that in an Abelian variety of rank 2 or more, any ample divisor is connected. As a connected étale cover of a locally unibranch divisor, D is irreducible itself.

2.5 Remark.

(i) The moduli space M is polarised by the inverse determinant of cohomology, which lifts to $\mathcal{O}(\Theta)$ on $\tilde{J}$: this is because $H^*(\tilde{\Sigma}; L) = H^*(\Sigma; p_* L)$. However, $\mathcal{O}(\Theta)$ is *not* principal on P. One way to normalise line bundles on P is to relate them to M_K, whose Picard group is $\mathbb{Z}$. The bundle $K_M^\vee$, which has Chern class 4, lifts to $\sigma^*\mathcal{O}(\Theta)$ over P. (This is the *level* 8 *line bundle* in [Y].)

(ii) The sign in §2.3 is meaningful, as the opposite would make Θ tangent to P. Now, the Jacobian determinant of π is the $\bar{\partial}$-determinant of $\mathcal{Q}$. There is a perfect pairing $\mathcal{Q} \otimes \mathcal{Q} \to K$, the determinant; in terms of ϕ, $q_1 \wedge q_2 \mapsto \frac{1}{2}\mathrm{Tr}\left([\phi, q_1] \cdot q_2\right)$. The sign is in the skew-symmetry of the pairing; in the symmetric case, $\det \bar{\partial}$ would have a Pfaffian square root.

References

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