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Author
Werner, Christian

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EXPLORATIONS INTO THE FORMAL STRUCTURE OF DRAINAGE BASINS

CHRISTIAN WERNER

University of California, Irvine, CA, 92717, U.S.A.

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ABSTRACT

Every basin of higher than first order is drained by a channel network composed of two subnetworks. Their basins are separated by a drainage divide line, called the basin divider, which is the primary organizing feature of the main basin. Each basin of magnitude n contains $n-1$ subnetworks of higher order, and is therefore organized by a set of $n-1$ dividers. The dividers and the basin boundary are interconnected in a graph called the divider network of the basin; in graph-theoretic terms this network forms a tree and has the same magnitude and link numbers as the channel network draining the basin. While the subbasins and subnetworks of a drainage basin form a nesting hierarchy, the corresponding dividers do not; indeed, any two dividers share at most one node in common, and whether they do so is independent of whether the corresponding subbasins are nesting or disjoint. However, the dividers of nesting basins are linked by recursive relationships which permit the derivation of a set of algebraic equations; these equations relate the dividers of a basin to other basin components; for example, their combined length is equal to half the length of all first-order basin boundaries minus the length of the main basin boundary.

The second part of the paper explores the dependence of the divider length on other basin parameters. The expected length, as predicted by the assumption of topological randomness, is clearly rejected by the data. An alternative approach (regression) is based on the observed magnitudes of the subbasins separated by each divider, and is reasonably successful in estimating divider length. The last section introduces the concept of the standardized basin defined by a boundary length of unity; the estimated lengths of the basin divider and the basin boundary permit an approximate reconstruction of the idealized basin shape and the location of the divider in it.

KEY WORDS Drainage basin Drainage divide Graph theory Ridge/channel networks

INTRODUCTION AND OUTLINE

This paper derives its purpose from Shreve's (1969) principal statement: '... to comprehend the geometry of drainage basins and channel networks... is prerequisitive to explaining their mechanics...'. Its aim is to expand the existing language of formal basin description, to deduce several algebraic functions that govern the interrelations between the descriptive concepts of basin structure, and to relate topologic and geometric parameters of drainage basins to those of the channel networks by which they are drained.

Both the geometric and the graph-theoretic analyses of channel networks have progressed steadily since the foundations were laid in the early work of Horton (1945), Strahler (1952), Melton (1957) and Shreve (1966); it has sharpened existing concepts and, very importantly, recognized and defined additional channel network features (Gardiner, 1975; Mark, 1975); it has generated theoretical explanations for a variety of observed regularities, such as some of the Horton laws of drainage composition (Shreve, 1969, 1975; Smart, 1972); and it has successfully separated the effects of formal or physical constraints from those of chance (Abrahams, 1980, 1983, 1984; Werner 1982, 1984). At least in theory, through the formulation of explicit functions relating slope development to channel growth and incision, it should be possible to '... comprehend the geometry of drainage basins...' simply as a derivation from what is known about

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channel networks. However, except for computer simulations (Howard, 1990) and the derivation of some topological duality relations (Werner, 1991), the connection between the vectors of channel network descriptors and similar vectors describing corresponding drainage basins has not yet been worked out. For that matter, even the formal description (geometry, connectivity) of the line patterns that define and subdivide drainage basins has received only limited attention (Warntz, 1972; Jarvis, 1976; Mark, 1975, 1979; Werner, 1982).

Within this broad context the present paper: (1) introduces several new concepts to capture in greater detail and make operationally accessible the graph-theoretic and geometric structure of drainage basins; (2) derives several logical relations between new and established basin parameters; (3) examines the effects of chance on the internal partition of basins and establishes empirical relations between certain channel and basin parameters; and (4) explores the possible reconstruction of the basin from selected basin variables and their functional relations.

BRIEF REVIEW OF SOME FAMILIAR CONCEPTS

A drainage basin is an area whose surface runoff is drained by a single channel network and which, in turn, is the only area drained by that network. Unless it consists of a single channel, the network has one or more nodes in which it bifurcates. Since, at any given bifurcation, a channel typically splits into no more than two branches, the network can be characterized in graph-theoretic terms as a trivalent planar rooted tree (Smart, 1972). That is to say: each node interconnects exactly three channel links; the network can be mapped unambiguously into the two-dimensional plane; it has a particular outlet link which makes it a directed graph; and any two links are connected by exactly one path. Thus, if the network is of magnitude n, i.e. has n exterior links, then it has $n - 1$ interior links as well as $n - 1$ nodes, where each node is the upstream end point of exactly one interior link. Each channel link defines a particular channel subnetwork with its own unique drainage basin and basin boundary. Thus, a channel link, its upstream end node, the channel subnetwork defined by the link, and the corresponding drainage basin and basin boundary are all one-to-one related entities. Finally, the magnitude of a drainage basin is defined as the magnitude of the channel network by which it is drained; basins of magnitude 1 are traditionally referred to as first-order basins.

DEFINITION OF THE BASIN DIVIDER

In the following analysis we will use small letters x for links; large letters X for the networks defined by the links x; A(X) and A(x) for the drainage basin and the link drainage area of X and x respectively; and B(X), B(x) for the boundaries of the basin A(X) and the link drainage area A(x).

Let P be a channel node, let w and u, v be its downstream and upstream links, and let W, U, V refer to the network and its two subnetworks defined by these links. Since every point within the drainage basin A(W) drains into exactly one link of W, A(W) is simply the sum of the drainage basin areas A(U), A(V) and the link drainage area A(w). In as much as A(w) tends to be comparatively small, the line separating the two subbasins A(U), A(V) is the primary organizing feature of the basin A(W). Morphologically it forms a ridge that starts in a point Q_1 next to the channel node P between the two links u and v, and terminates in a node Q_2 of the basin boundary B(W). This line is the intersection $B(U) \cap B(V)$ of the boundaries of the two subbasins A(U) and A(V); we call it the divider of the drainage basin A(W) and denote it by D(W) (Figure 1).

GRAPH-THEORETICAL DESCRIPTION OF DIVIDER PATTERNS

Most of the following statements are easily verified and are therefore listed without proof.

(1) A channel network W of magnitude n has $n - 1$ bifurcation nodes; each node interconnects three links which define a network and its two subnetworks. The drainage basins of the two subnetworks are separated by a divider defined as the intersection of their boundaries. As such, the divider is unique

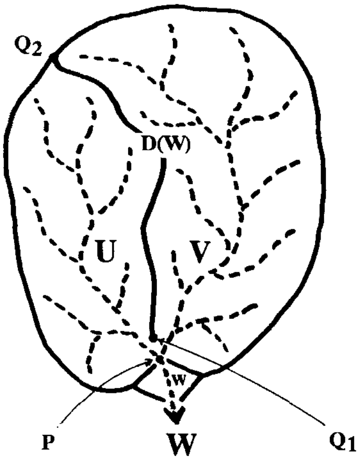

Figure 1. A drainage network W and its two subnetworks U and V. The intersection of the basin boundaries of U and V defines the basin divider D(W). The divider starts in a point Q_1 next to the first channel node P upstream of the outlet of W and terminates in a point Q_2 of the basin boundary of W

to that pair of subbasins and therefore also to the associated bifurcation node. It follows that a basin of magnitude n contains exactly $n - 1$ dividers (Figure 2).

(2) Within a drainage basin no divider is separated by channels from both the basin boundary and the other dividers, as that would mean the existence of circuits in the channel network draining the basin, in contradiction to the assumption that graph theoretically the network forms a tree. Similarly, no part of the boundary of the basin and the dividers in it can form a circuit, as that would constitute an area of internal drainage, which we rule out by assuming a humid, non-karstic environment. Thus, the boundary

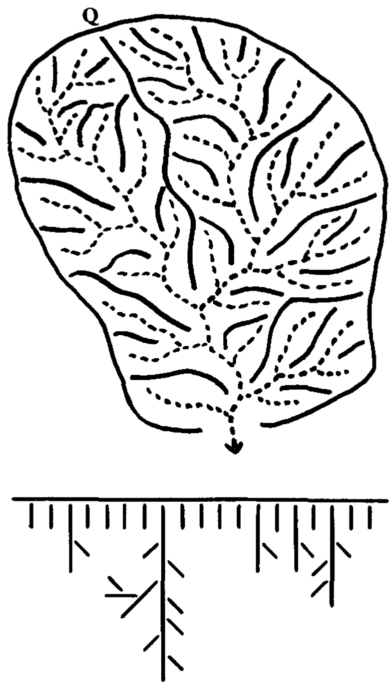

Figure 2. A channel network, the boundary of its basin, and the dividers of the basin and its subbasins. The nodes connecting dividers to each other and to the basin boundary have been omitted so as to explicitly show the extent of each divider. The lower part of the figure shows the (abstracted) graph composed of the boundary of the main basin and the divider trees in it, each tree consisting of one or more dividers

of a basin and the dividers it contains merge into an interconnected line pattern which, like the corresponding channel network, forms a tree; this tree we call the divider network R of the basin (Figures 2, 4; in Figure 2 connecting nodes have been intentionally omitted).

(3) Since both natural ridges and dividers are lines separating adjacent drainage basins it is not surprising that, by and large, natural divider networks coincide with patterns of ridges. To trace the lines of a divider network in the field or on a topographic map is similar to tracing the corresponding channel network, with one exception, however: while the degree of a divider node is usually three ('trivalency') if it is clearly discernible, one occasionally encounters situations that make it difficult, if not impossible, to assign a particular degree value. Fortunately, by allowing for links of a very small length we can approximate the actual divider network layout by assuming that, as in a channel network, all nodes are of degree three.

(4) Since each divider starts with an exterior link positioned between the two upstream links of a channel bifurcation, and since the basin boundary terminates in two exterior links, the total exterior link number of a divider network is $n + 1$, which is equal to the number of exterior links of the channel network if we include its outlet. It is therefore reasonable to define the magnitude of the divider network as equal to that of the corresponding channel network, that is, n. With this convention the magnitudes of divider networks possess the same additivity as their channel networks: if two channel networks U, V merge to form a network W then the magnitude of the corresponding divider network R(W) is equal to the sum of the magnitudes of the divider networks R(U) and R(V).

(5) Each divider D defines a set consisting of all dividers that are interlinked and include D. Such a set forms a subnetwork which, in light of statement (2) above, is linked to the basin boundary in a single boundary node Q. We denote their number with k; for an example see Figure 4 where these nodes are labelled Q_i, $i = 1, 2, \ldots, k$ and $k = 11$. In as much as the divider network is a tree, these subnetworks must be trees as well; we therefore call them the divider trees of the basin A(W) and denote the set of all k divider trees in A(W) by T(W). Since the k divider trees are composed of $n - 1$ dividers their combined magnitude is also $n - 1$, and their combined link number is therefore $2(n - 1) - k$. Thus, the average link number L(D) of dividers in a basin of magnitude n containing k divider trees is $(2n - 2 - k)/(n - 1) = 2 - k/(n - 1)$. The ratio on the right side of the equation approaches 1 for the value of k approaching $n - 1$, and zero for small values of k; hence the average link number of a divider is limited by the inequality $1 \leq \mathrm{L(D)} < 2$. The extremal values of N(D) are 1 and $n - 1$ respectively. Figure 3a shows examples of the two cases for magnitude 7 networks.

(6) It is interesting to note that basin dividers do not possess the nesting quality of channel networks or their basins: while these may contain, and are usually contained in, other channel networks (basins), no

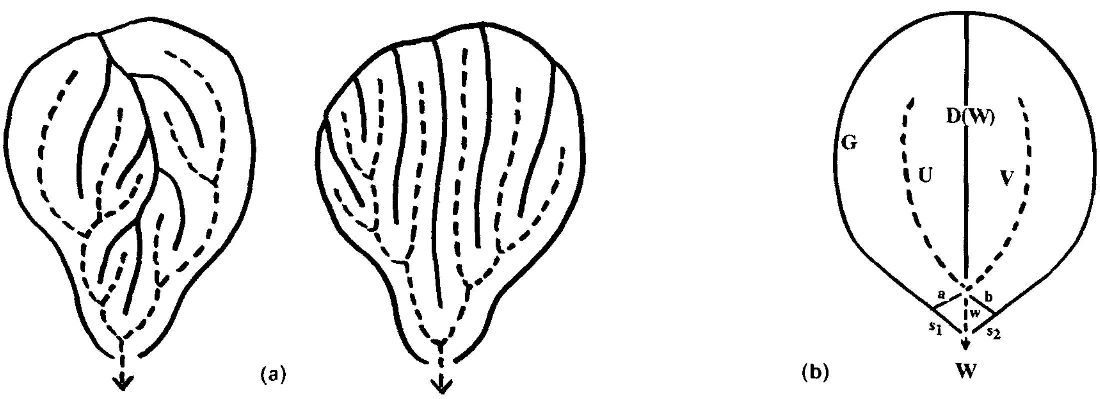

Figure 3. (a) Two channel networks of magnitude 7 together with their divider networks. While in the left basin the basin divider has the maximum possible link number $n - 1$, the divider of the right basin consists of a single link. Note also that the two channel networks have the same topological configuration, in contrast to their divider networks. (b) Simplified graph of a channel network W, its two subnetworks U and V, the basin boundaries of W, U, V, and hence, the basin divider D(W) and the boundary of the area drained by the outlet link w. Note that for $a + b = s_1 + 2_2$ the basin boundary of W is the sum of the boundaries of U and V minus twice the basin divider D(W)

divider is part of any other divider. In particular, the dividers of a basin's subbasins are not part of, and are indeed unrelated to, the divider of the main basin. What makes this observation all the more noteworthy is the very definition of dividers, as it is a direct derivative of the concept of subbasins. In fact, except for the nodes in which they interconnect, dividers are always pairwise disjoint: no two share a divider link in common (Figure 2). We verify this statement through an indirect proof: let r be a link of the divider network R and D(X) and D(Y) be two dividers of R which share the link r. Like all drainage basins the corresponding basins A(X) and A(Y) are either nesting or disjoint. We first assume that they are nesting—say, $A(X) \subset A(Y)$. Let $A(Y_1)$ and $A(Y_2)$ denote the two subbasins of A(Y) whose boundaries define D(Y): $D(Y) = B(Y_1) \cap B(Y_2)$. Since drainage basins never overlap, A(X) must be fully contained in either $A(Y_1)$ or $A(Y_2)$. In either case, its divider D(X) cannot share a link with D(Y) as it is located inside A(X). If, on the other hand, then basins A(X) and A(Y) are disjoint, so must be their dividers D(X), D(Y), again because by virtue of their definition, dividers are always located inside their respective basins.

(7) As a graph, each divider network has its particular topological configuration, and for any given magnitude the number of possible configurations is equal to that of the corresponding channel network. But here the equivalence ends. Unlike channel networks, divider networks are the aggregation of n specific, well defined network paths, namely, the $n-1$ dividers and the basin boundary (Figure 2). For any given topology of a divider network the number of ways in which it can be constituted as an aggregation of a boundary and $n-1$ divider path is 2^{n-k-1}, where k is the number of divider trees in the basin and n is again the magnitude of the divider network. However, once the topology of the corresponding channel network is also fixed, the divider network's breakdown into dividers is unique (interestingly, the reverse statement does not hold). Figure 3a shows two magnitude 7 channel networks as well as their divider networks. While the former are topologically equivalent, the same does not hold for the latter.

ALGEBRA OF BASIN STRUCTURE

It might be useful to briefly review the different components of the divider network using Figures 2 and 4 as examples. A basin of magnitude $n(n = 35$ and 15, respectively, in Figures 2 and 4) contains $n-1$ individual

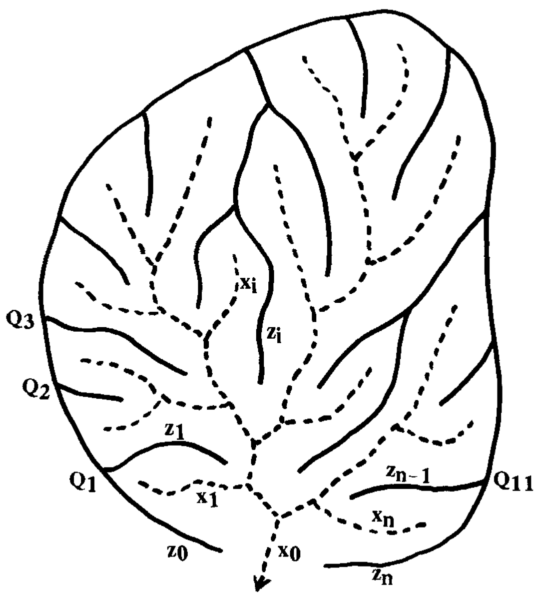

Figure 4. A channel network and the corresponding divider network. Including the outlet link each network has $n+1$ exterior links which are strictly alternating. The basin contains a total of 11 divider trees which are linked to the boundary in boundary nodes Q_i, where $i = 1, \ldots, 11$

dividers D; in Figure 2 they are the 34 solid lines located within the basin boundary. These dividers are either connected to the basin boundary or else to each other. Again in Figure 2, the interconnecting nodes have been omitted so as to clearly show the end points of each divider (and thus the fact that dividers do not share links in common). Each set of one or more interconnected dividers that includes all dividers linked to them is called a divider tree; the magnitude of such a tree is equal to the number of dividers forming it. Each divider tree is connected to the basin boundary. The number of divider trees in a basin is denoted by k; in Figure 2 the total number of divider trees is $k = 19$; the largest of these is connected to the basin boundary in node Q; it consists of 10 dividers and is therefore of magnitude 10; in Figure 4 the number of divider trees is $k = 11$; two are of magnitudes 3 and 2 respectively, the rest being of magnitude one. The set of all divider trees contained in the basin A(W) of a channel network W is labelled T(W). The union of the boundary B(W) and the divider trees T(W) of the basin A(W) is called the divider network corresponding to the channel network W, and is denoted by R(W); an example is the set of interconnected solid lines in Figure 4.

The various components of the structure of a basin and its subbasins—the dividers D, the set T of divider trees, the basin boundary B and the divider network R—are interlinked by several identities. Let w, u, v be the three links interconnected in a channel network node and let W, U, V, and A, B, D, T, R of W, U, V, refer to the corresponding channel networks and their basins, basin boundaries, dividers, divider trees and divider networks. To denote the graph theoretic aggregation, deletion and intersection of sets we use the symbols Σ or $+$; $-$; $\cap$. Evidently, the following identities hold (Figures 3b, 4):

$$T(W) = T(U) + T(V) + D(W) \tag{1}$$

$$R(W) = T(W) + B(W) \tag{2}$$

$$B(W) = B(U) + B(V) + B(w) - 2\{[B(U) \cap B(w)] + [B(V) \cap B(w)] + [B(U) \cap B(V)]\} \tag{3}$$

where B(w) refers to the boundary of the area drained by the channel link w. For the following investigation we will restrict our consideration to the length measures L of T, R, B and D, where L might be defined as either the number of links or the planimetric length. Evidently, Equations 1, 2, 3 remain valid if we substitute each component by its length measure. In particular, Equations 1 and 2 translate into:

$$L[T(W)] = L[T(U)] + L[T(V)] + L[D(W)] \tag{4}$$

$$L[R(W)] = L[T(W)] + L[B(W)] \tag{5}$$

The relation between the boundaries B(W), B(U), B(V) of the main basin and its two subbasins, as stated in Equation 3, can be substantially simplified if we restrict our consideration to their length values and assume that, again in terms of length, the intersections of the boundaries B(U) and B(V) with the boundary B(w) are equal to the intersection of the boundary B(W) with B(w), or, using the labels of Figure 3b, $a + b = s_1 + s_2$:

$$L[B(U) \cap B(w)] + L[B(V) \cap B(w)] = L[B(W) \cap B(w)] \tag{6}$$

i.e.

$$L[B(W)] = L[B(U)] + L[B(V)] - 2L[D(W)] \tag{7}$$

or

$$L[D(W)] = 1/2\{L[B(U)] + L[B(V)] - L[B(W)]\} \tag{8}$$

The assumption formulated in Equation 6 has the sole purpose to facilitate the exploration of the inherent formal structure of drainage basins and is justified in part because it constitutes, on average, a fairly reasonable approximation of reality, and in part because the impact of any error in Equation 6 on 8 becomes negligible for increasing network magnitude.

Combining Equations 4, 5 and 8 provides a similar formulation for the length of divider networks:

$$L[R(W)] = L[R(U)] + L[R(V)] - L[D(W)] \tag{9}$$

Equations 4, 7 and 9 reveal a noteworthy structural quality of the lengths of the divider trees T, the boundary B and the divider network R of any given drainage basin A(W). Each of these basin parameters can be expressed as a function of the same parameters of the two subbasins and the basin divider, thus creating a recursive relationship that interrelates these parameters over the entire hierarchy of nesting basins and subbasins. In particular, through a process of iterative substitution, each parameter can be formulated as a function of the same parameters of the n first-order basins and the set of dividers contained in the main basin. For an example, consider Equation 7. Since it applies to any basin boundary we can apply it, in particular, to B(U) and B(V) on its right side. Hence, if U_1, U_2 and V_1, V_2 are the subnetworks constituting U and V respectively, then:

$$\begin{aligned} L[B(W)] = \{&L[B(U_1)] + L[B(U_2)] - 2L[D(U)]\} \\ &+ \{L[B(V_1)] + L[B(V_2)] - 2L[D(V)]\} \\ &- 2L[D(W)]\} \end{aligned} \tag{10}$$

Through repeated application of Equation 7 we continue to substitute the lengths of the basin boundaries on the right side of Equation 10 until no further substitutions are feasible and only the lengths of the basin dividers and the first-order basin boundaries remain:

$$L[B(W)] = \sum_{i=1}^{n} L[B(X_i)] - 2\sum_{j=1}^{n-1} L[D(Y_j)] \tag{11}$$

where n is the magnitude of the network W, $\{X_i|i = 1, \ldots, n\}$ is the set of all magnitude 1 subnetworks of W, and $\{Y_j|j = 1, \ldots, n-1\}$ is the set of all magnitude > 1 subnetworks of W. In words: the length of a basin boundary is equal to the combined lengths of all first-order basin boundaries minus twice the sum of the lengths of all dividers in the basin.

Similarly, Equations 4 and 9 are recursive, permitting the representation of the length of the divider trees and the divider network of a given basin in terms of the lengths of its first-order basin boundaries and the dividers in the basin: Since first-order basins do not contain dividers (and, hence, divider trees) their divider networks consist of their basin boundaries only: $L[D(X)] = 0$; $L[T(X)] = 0$; $L[R(X)] = L[B(X)]$, and therefore:

$$L[T(W)] = \sum L[T(X_i)] + \sum L[D(Y_j)] = \sum L[D(Y_j)] \tag{12}$$

$$L[R(W)] = \sum L[R(X_i)] - \sum L[D(Y_j)] = \sum L[B(X_i)] - \sum L[D(Y_j)] \tag{13}$$

where $\{X_i|i = 1, \ldots, n-1\}$ and $\{Y_j|j = 1, \ldots, n-1\}$ are defined as before. Equation 12 simply verifies the (trivial) statement that the combined length of the divider trees in any basin is the sum of the lengths of all dividers in that basin, and Equation 13 expresses the length of the divider network of a basin in terms of the lengths of the first-order basin boundaries and the dividers contained in the basin.

In a later section of this paper we will estimate the length of a basin's boundary as a power function of the basin area which, in turn, is approximated by the basin magnitude: Est $L[B(Z)] = K(n^a)$, where Z refers to a channel network, n to its magnitude, Est L[B(Z)] to the estimated boundary length L of the basin drained by Z, and where K and a are numerical constants to be determined empirically (for an example see Equation 16). Substituting this estimation into Equation 11 gives us a simple estimate of the combined length of all dividers in a basin of magnitude n:

$$\text{Est} \sum[L(D_j)] = (K/2)(n - n^a)$$

where $j = 1, \ldots, n-1$. Observed values of the exponent a are always well below 1. Hence, for large basins the length of the line pattern which organizes a basin into a hierarchy of subbasins tends to be proportional to the basin magnitude and therefore proportional to the length of the channel network. Of course, this conclusion is not surprising; what the argument presented here adds is an algebraic derivation of an intuitive expectation.

The formal proofs of Equations 11, 12 and 13 by mathematical induction are straightforward and have therefore been omitted.

THE EFFECT OF CHANCE ON DIVIDER LENGTH

We will first calculate the expected length of the basis divider on strictly theoretical grounds. The only model of network design that permits the derivation of the expected link number of the divider without resorting to empirical calibration is the random model (Shreve, 1966; Smart, 1974). While that model has been developed for natural channel networks, there is no principal reason that prevents its application to divider networks, as they too form trivalent planar trees; whether such application is any more successful than in the case of channel networks is, of course, a matter of testing.

Let X be a channel network of magnitude n. Since X is rooted and planar its exterior links form a sequence $S(x) = \{x_i | i = 1, \ldots, n\}$ where by established convention x_1 refers to the first exterior link to the left of the outlet link. Within the sequence S(x) we call the link x_p the kth neighbour of the link x_q if $|p - q| = k$ (Figure 4).

Each of the $n-1$ pairs of consecutive links x_i, x_{i+1} in S(x) define a particular channel node P_i, namely the point in which the two paths connecting these links with the outlet merge. Since the network has $n-1$ nodes, this correspondence is a one-to-one mapping operation. As was shown earlier, there exists for each channel node exactly one divider, and therefore one exterior link of the divider network R(X). Thus, exterior channel links and exterior divider links form an alternating sequence. To reflect this particular relation between the networks X and R(X), we label the exterior divider link associated with the channel node P_i (and therefore, via transitivity, with the exterior channel links x_i, x_{i+1}) with z_i; since the links z_i alternate with the links x_i they too form a sequence which we label S(z). Finally, we add the channel outlet to the beginning of S(x) as x_0, and the end links of the basin boundary B(X) to the beginning and the end of S(z) as z_0 and z_n. As a result the two sequences now consist of $n+1$ links each and can be mapped onto each other such that $x_i \Leftrightarrow z_i$ (Figure 4). Note that in S(x) the last link x_n is the nth neighbour of the first link x_0; the mapping operation translates this relation into a corresponding statement for S(z), namely that z_n is the nth neighbour of z_0. In words: in a drainage basin of magnitude n each end link of the basin boundary is the nth neighbour of the other end link.

Divider networks are usually subnetworks of much larger networks, so that it is appropriate to apply the random model under the assumption that the magnitude of the embedding network approaches infinity. Under the assumption of topological randomness the expected link number of a path connecting an exterior link with its kth neighbour in a network of infinite magnitude is $8\pi^{-1/2}k^{1/2}$ (Werner, 1984). Let W, U, V refer to a channel network and its two subnetworks, with magnitudes n, i, j where $i + j = n$; and let E[B(W)] denote the expected link number of the basin boundary B(W). We assume that the corresponding divider network R(W) is embedded in a network of infinite magnitude. Applying the result of the preceding paragraph it is

$$E[B(W)] = 8\pi^{-1/2}n^{1/2} \tag{14}$$

Equivalent equations give the expected link numbers E[B(U)] and E[B(V)] of the boundaries of the two subbasins A(U), A(V) of the main basin A(W). Substituting these three equations into Equation 8 produces the expected link number of the divider D(W):

$$E[D(W)] = 4\pi^{-1/2}[i^{1/2} + j^{1/2} - (i+j)^{1/2}] \tag{15}$$

To evaluate the effectiveness of the random model in predicting the link number of basin dividers, we have sampled a set of natural drainage basin data from the 1:24,000 USGS topographic map, Varney, Kentucky, quadrangle. This study area is considered to be a good example of a maturely eroded landscape without significant geologic controls and a reasonably uniform link destiny (Krumbein and Shreve, 1970; Smart, 1978). Both network channels and basin dividers were identified following the contour crenulation method as described by Krumbein and Shreve (1970) and later modified and extended (Werner, 1988). Specifically, for each of 50 basins the data consist of the planimetric length of the basin boundary broken down into its two subbasin components, the length of the basin divider, and the magnitudes of the two subbasins. In addition, the link numbers of the basin and subbasin boundaries and the basin divider were determined for 32 of the 50 basins. The basins were selected in such a way that the sample contained a reasonably even distribution for different basin and subbasin magnitudes; that approach was virtually unavoidable in light of the fact that the smallest basins, those of magnitudes 1 to 4, constitute over 90 per cent of all natural basins. Otherwise the selection process was essentially random.

Test results

While a number of parameters of natural networks show a surprising degree of control by chance (Smart, 1972), the link number of the basin divider is not one of them. That is clearly demonstrated by Figure 5, which shows the observed link numbers of 32 basin dividers plotted against those expected by the random model. Moreover, since planimetric path length and path link number are highly correlated (Werner, 1982), it follows that the model would not be successful for metric length measures of dividers either. To check for any size-related discrepancies the observed data have been ordered by link number. Apparently the deviation of the expected values is systematic, which would permit correction of the discrepancy by appropriate adjustment of the model parameters on the basis of empirical data; however, after such a corrective measure the model would no longer be strictly random or deductive.

Discussion

Given the considerable chance effect in the design of channel networks (Shreve, 1975) it is at least initially puzzling to find such major and persistent deviation from randomness in the configuration of divider networks. We offer two comments, the first of which provides a plausible explanation for the last part of the above statement, and the second examines the seeming discrepancy by comparing channel and divider networks with regard to one of the critical parameters.

I. As stated previously, there is ample empirical evidence that in homogeneous terrain the magnitude of a channel network and the link number of a basin boundary are highly correlated to the metric measures of the area drained by the network and the physical length of the boundary in question. In its most compact form

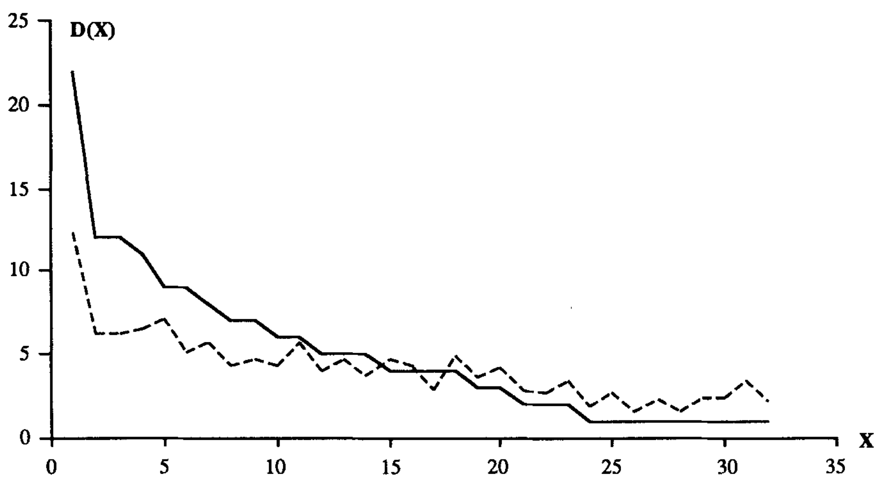

Figure 5. Observed (solid line) and expected divider link numbers for 32 natural drainage basins. The observed values are ordered horizontally by size; the expected values result from the assumption of random network topology

the shape of a basin is a circle, and the length of the circle perimeter is the minimum length of any possible boundary delineating the basin. This limitation is not a physical constraint but rather a mathematical one, dictated by the axioms of Euclidean geometry. In light of the correlations referred to above, this conclusion translates into an equivalent topologic statement regarding basins and their boundaries: for any given channel network magnitude there is a tight stochastic limit to the minimum possible number of links of the boundary delineating the area drained by that network. Consequently, the link numbers of drainage basin boundaries cannot be randomly distributed. Furthermore, since the divider length is a function of the length of basin boundaries (Equation 8) a similar conclusion applies to it. To repeat this argument in general terms: in a world of random topology the constraint referred to above would not be applicable. But since the topologic parameters in question are empirically dependent on their geometric counterparts they also become dependent on the laws of Euclidean geometry and thus lose their purely random nature.

Interestingly, the length of basin boundaries also has an upper limitation for any given basin magnitude; but it is in the nature of an upper bound and not a matter of logical necessity; rather, it is a matter of empirical observation. Irrespective of their size, drainage basins do not deviate from a certain degree of compactness which limits the length of their boundaries. In the absence of a mathematical constraint any explanations for it must be of a physical nature (e.g. river piracy) which goes beyond the scope of this paper.

II. the number of different topologic network parameters is only limited by our imagination. Some display fairly random distributions in nature (e.g. stream numbers) while others do not (e.g. proportion of *cis* links). To compare the topologic behaviour of channel and divider networks we have to use the same parameters, and we have to keep in mind that any answer is only an answer with regard to the comparison parameters selected. The empirical data presented in this paper demonstrate that, for any given network magnitude, the link number of basin dividers is non-random, one reason being that dividers are defined by basin boundaries whose link numbers cannot be randomly distributed because of their high correlation with their metric counterpart—see the previous argument. Turning now to channel networks we have to ask what parameter in a channel network corresponds to the link number of a basin boundary of a divider network. The answer is provided by the concept of network duality (Werner, 1991). In particular, a basin boundary is:

> any path in a divider network whose end links are separated by a single channel link (i.e. the outlet of the basin).

The dual counterpart to a basin boundary is:

> any path in a channel network whose end links are separated by a single divider link.

Thus, the dual concept to a basin drained by a channel network is simply an area delineated by a channel path and containing a divider network, the root of which separates the end links of the path. This particular type of channel path has been investigated in a previous study, which has indeed rendered similar results, including the finding that the link number of such channel paths is consistently larger than the number derived from the assumption of randomness (Werner, 1984). Thus, with regard to the particular parameter examined here—the link number of network paths forming quasi-closed curves—channel networks and divider networks do not principally differ in as much as the random model fails to fit the observed values of both.

Since a basin of magnitude n is organized by $n-1$ dividers, the failure of the random model affects the entire basin geometry. We therefore finish this part of the analysis with the conclusion that the random model, while providing a good approximation for some other channel network parameters (Shreve, 1975), is not compatible with the internal partition as observed in our sample of natural drainage basins.

MODELLING DIVIDER LENGTH BY REGRESSION

In the absence of any axiomatic theory that derives the length of dividers from other basin parameters we have, of course, the less satisfactory option of trying to establish such a relation inductively. Equation 8, which expresses the length of the divider as a function of the boundaries of the basin and its two subbasins, is not quite, but almost, a logical identity (see Figure 3b and associated text) and is therefore devoid

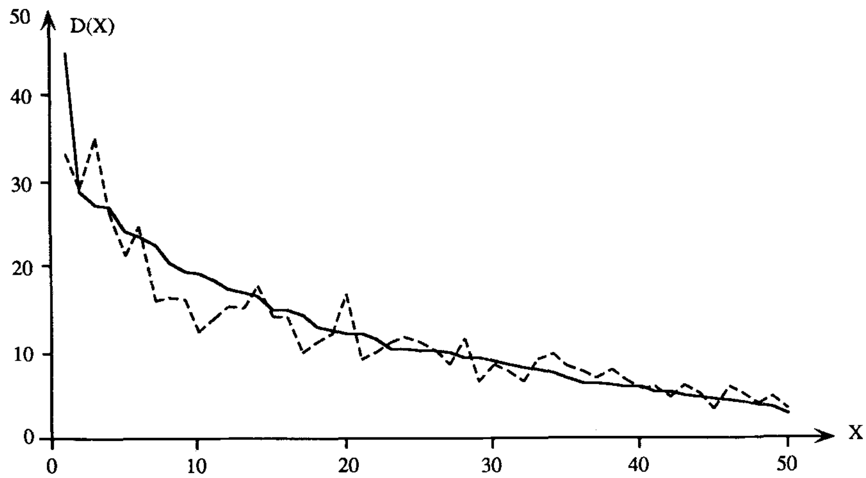

Figure 6. Observed lengths of dividers (solid line) of 50 natural drainage basins, and the corresponding length values as estimated by regression from the magnitudes of the subbasins separated by the dividers. The observed values are again ordered by size; the unit of the vertical scale is 127·4 m

of much substantive information. Instead, we will examine the dependence of the length of the divider on the magnitudes of the two basins it separates. To this end we first determine the approximate functional relationship between network magnitude and basin boundary length on the basis of sampled data; that function will then be used in conjunction with Equation 8 to estimate the length of the divider for any given basin A(W) composed of subbasins A(U) and A(V).

There is a considerable body of evidence indicating that basin area and one-dimensional basin features tend to be related by a power function (see, for example, Hack, 1957; Leopold *et al.*, 1964; Montgomery and Dietrich, 1992). In areas approximating uniform environmental conditions, channel link density tends to be roughly uniform, making the magnitude of a channel network a convenient surrogate for the area of the basin drained by that network (Smart, 1978); we therefore use basin magnitude n as the independent variable to estimate the length of the basin boundary, Est L[B(X)]. As one would expect from these considerations and the research referred to above, the plot of the (metric) boundary lengths and the magnitudes of 127 basins in the study area revealed a close correspondence; the regression line of the respective logarithms translates into the power function

$$\text{Est}\,\text{L}[\text{B}(\text{X})] = 10{\cdot}1265 n^{0{\cdot}5695} \tag{16}$$

with a correlation coefficient of $r = 0{\cdot}9877$ and a mean deviation of the expected from the observed values of 9·34 per cent. Combining Equations 16 and 8 permits the estimation of the length of the basin divider:

$$\text{Est}\,\text{L}[\text{D}(\text{W})] = (10{\cdot}1265/2)[i^{0{\cdot}5695} + j^{0{\cdot}5695} - (i + j)^{0{\cdot}5695}] \tag{17}$$

where i, j are the magnitudes of the two subnetworks U, V of W.

Both estimations and observations for a sample of 50 drainage basins are shown in Figure 6; the observed divider values have once again been ordered by size. Although the deviation of the estimated from the observed values is on average 16·2 per cent, the close dependence of the length of the divider on the magnitudes of the two subbasins it divides is nevertheless quite apparent.

THE STANDARDIZED DRAINAGE BASIN

Equation 17 expresses the length of the basin divider as a power function of two independent variables, namely the magnitudes i and $j = n - i$ of the two subbasins it separates. We now take advantage of the

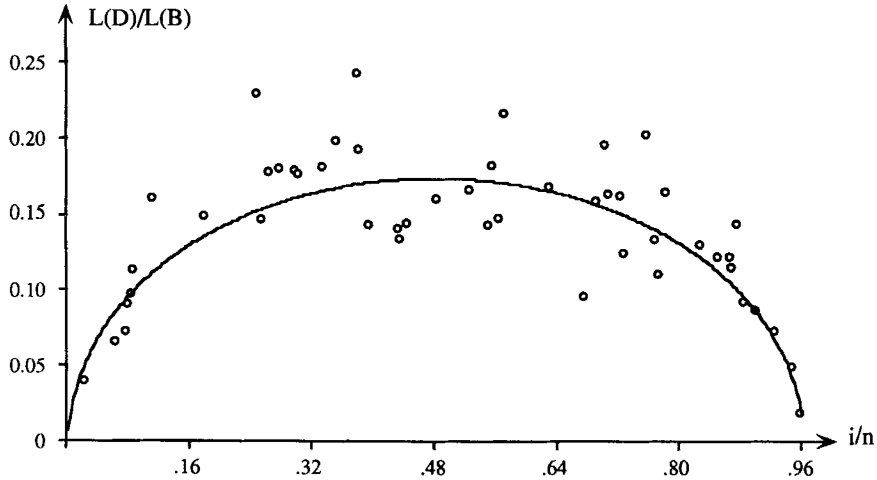

Figure 7. Distribution of observed (circles) and estimated divider length values for 50 natural drainage basins, after division by their respective basin boundary lengths, thus permitting direct comparison across differing basin size. By definition each divider belongs to one and only one basin, within which it constitutes the intersection of the two subbasin boundaries. The data are ordered horizontally by the ratio of the magnitudes of their left subbasin and their main basin

fact that, for any fixed ratio $t = j/i$ and variable $n = i + j$ the function describes homothetic (similar) figures. In particular, if measured as a proportion of the basin boundary the divider is constant and hence independent of the basin magnitude. That becomes immediately clear from the following consideration. Let $\text{Est}\,\text{L}[\text{B}(\text{X})] = \text{Kn}^a$ be the regression equation estimating the length L of the boundary B of the basin drained by the network X as a function of the network magnitude n. Substituting this equation into Equation 8 and setting $j/i = t$ it follows that:

$$\frac{\text{Est}\,\text{L}[\text{D}(\text{X})]}{\text{Est}\,\text{L}[\text{B}(\text{X})]} = \frac{1}{2}K\left\{\frac{i^a + (ti)^a - (i + ti)^a}{K(i + ti)^a}\right\} = \frac{1}{2}\left\{\frac{1 + t^a}{(1 + t)^a} - 1\right\} \tag{18}$$

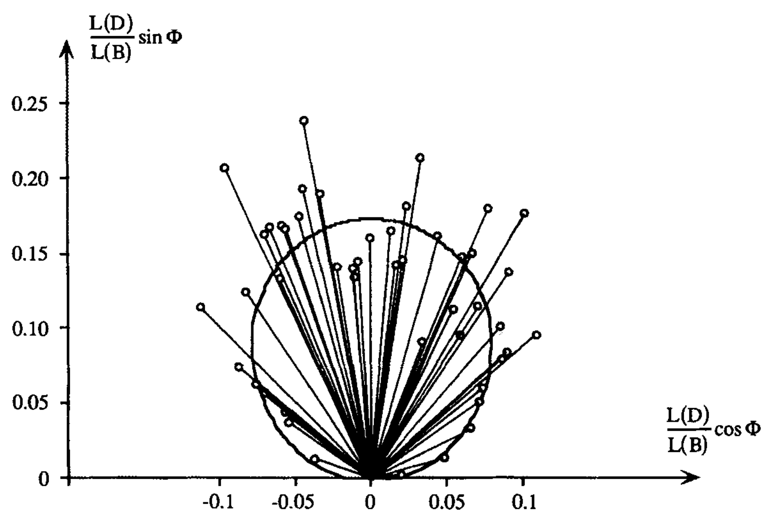

Figure 8. Estimated and observed distribution of standardized length and location of basin dividers sampled from 50 natural drainage basins. The number of estimated values has been enlarged to 299 so as to allow better visualization of the underlying theoretical basin shape

Thus, the ratio of the estimated basin divider and the estimated basin boundary is only a function of the ratio t of the magnitudes of the two subbasins, but not of the basin magnitude n. This permits us to standardize the estimated and observed divider values by division through their respective basin boundary values, which allows direct comparison of estimated and observed divider length across basins of different magnitudes.

Figure 7 shows the standardized divider values of the observed set consisting of 50 basin dividers (dots); superimposed is the curve representing the standardized theoretical values estimated by the regression model. The data are ordered by the relative size of the left subbasin, i.e. as a function of the ratio $q = i/(i+j) = 1/(t+1)$. As anticipated, all values are small if one of the subbasins is much larger than the other ($q \approx 0$ or $q \approx 1$) and reach their maximum when the two subbasins are of equal magnitude ($q = 1/2$). It should be noted, however, that the curve in Figure 7 represents the ratio of estimates, not the estimate of ratios (i.e. it is not the regression line for the observed data shown in the figure). A similar comment applies to Figure 8.

RELATING BASIN DIVIDER TO BASIN SHAPE

While the previous sections investigated the length of the basin divider, this section focuses on the interrelated topics of the theoretical shape of the basin and the location of its divider. Let A(W) be a drainage basin of magnitude n composed of the two subbasins A(U) and A(V) with magnitudes i and $j = n - i$ (Figure 9). For the following investigation we will keep n fixed and allow i to vary from 1 to $n - 1$ (and hence j from $n - 1$ to 1). Furthermore, let D_i be the divider of the basin and $L(D_i)$ its length; we label with Q_i the node in which D_i is connected to the basin boundary. Finally we denote with G_i the part of the basin boundary B(W) left of Q_i, and its length with $L(G_i)$; evidently, G_i and D_i are interconnected in Q_i and their combined length is equal to the length of the basin boundary B(U). By construction both G_i and D_i start near the basin outlet, and for the present purpose of determining the geometric shape of the basin A(W) and the position of the divider in it we will assume that, as an approximation, both G_i and D_i originate in the basin outlet (Figure 9).

Briefly, our plan is as follows. We first determine the estimated length of G_i, Est $L(G_i)$; building on that result we than use the estimated length Est $L(D_i)$ of the divider D_i (Equation 17) as a second coordinate to fix the position of Q_i. With i running from 1 to $n - 1$ we obtain $n - 1$ positions of Q_i and thus a discrete but representative dot pattern for the planimetric shape of the basin boundary B(W); together with the outlet the dots form a polygon with n edges fitted into the boundary. The construction of Q_i will be based on the (admittedly crude) assumption that the basin divider D is a straight line.

However, there still remains a major obstacle to be overcome. The length values of G_i and D_i are conditions for the location of Q_i by specifying its two distances from the outlet, $L(G_i)$ measuring distance

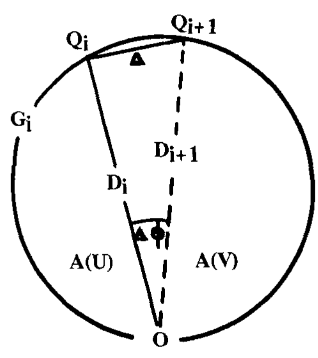

Figure 9. Idealized sketch of basin with two subbasins A(U), A(V) of magnitude i and j and the basin divider D_i. If the magnitude of A(U) increases by one, the node connecting the basin divider with the basin boundary will move from Q_i to Q_{i+1} and the new position of the divider will be D_{i+1}

along the basin boundary B(W), and $L(D_i)$ measuring the straight line distance (by assumption; see above). But since we do not yet know the shape of the boundary we have no way of measuring the length of G_i along its curve. This difficulty we will overcome as follows. Let us assume that we have constructed the position of Q_i for particular subbasin magnitudes i and $j = n - i$. The next boundary point to be determined is Q_{i+1} which connects the basin divider to the basin boundary when the magnitudes of the two subbasins are $i + 1$ and $j = n - i - 1$ respectively. If n is 'sufficiently' large, the polygon defined by the $n - 1$ boundary points $\{Q_i | i = 1, \ldots, n - 1\}$ will be a close approximation of B, so that we can substitute the boundary segment connecting Q_i and Q_{i+1} by a straight line. That, however, gives us an unequivocal fix on the position of Q_{i+1} as the point located at a distance $L(D_{i+1})$ (Equation 17) from the basin outlet, and at distance $L(G_{i+1}) - L(G_i)$ from Q_i. Starting with Q_0 as the outlet and $i = 1$ we can now iteratively construct the entire polygon $\{Q_i | i = 1, \ldots, n - 1\}$ representing the basin boundary B. In our computations we have chosen a slightly different strategy, calculating the angular increment $\Delta\Phi$ of the direction of D as Q moves from the ith to the $(i + 1)$th position along the basin boundary B (Figure 9).

To estimate the length of the boundary segment G we substitute Equation 8 into the equation $L(G) = L[B(U)] - L(D)$ and obtain

$$L(G) = 1/2\{L[B(U)] - L[B(V)] + L[B(W)]\} \tag{19}$$

which is simply a permutation of the terms on the right side of Equation (8). Combining Equations 16 and 19 and adapting the result to our present notation produces an estimate of the length of G_i:

$$\text{Est } L(G_i) = (10{\cdot}1265/2)[j^{0{\cdot}5695} + (i + j)^{0{\cdot}5695} - i^{0{\cdot}5695}] \tag{20}$$

where $j = n - i$ and n constant. As we increase the magnitude i of the left subbasin by 1, the node in which the divider connects with the basin boundary B(W) migrates along that boundary clockwise by an amount equal to the difference $\Delta = \text{Est } L(G_{i+1}) - \text{Est } L(G_i)$. Within the triangle formed by D_i, D_{i+1} and Δ, the cosine law gives us the angle between the sides D_i and D_{i+1} (Figure 9). Thus, starting with $i = 1$ we can calculate incrementally, for each value of i, the direction $\Phi(i)$ of the basin divider D_i as a function of i by summing over the angular increments. The process terminates when i is maximal, i.e. $i = n - 1$:

$$\Phi(i) = \sum_{k=1}^{i-1} \arccos \frac{[\text{Est } L(D_i)]^2 + [\text{Est } L(D_{i+1})]^2 - [\text{Est } L(G_{i+1}) - \text{Est } L(G_i)]^2}{2[\text{Est } L(D_i)][\text{Est } L(D_{i+1})]} \tag{21}$$

To establish a common base for our observed divider values we again standardize them as well as their regression estimates. Figure 8 is based on the above construction and displays the theoretical shape of the standardized drainage basin together with the observed dividers, this time represented as straight lines radiating from the outlet of the basin.

Figures 7 and 8 also demonstrate the generalization stated in an earlier section, that the length values of dividers have an upper bound which is a function of basin size. It indicates that basin boundaries and the dividers they form do not engage in wild gyrations (at least at the macro-level represented on topographic sheets) like, say, an unconstrained random walk would produce. Rather, drainage basins typically cover most of the area of the convex hull corresponding to them. To phrase it differently (and loosely): channel networks do not tend to deeply penetrate the areas drained by other networks. However, whether advancing channels of a single network send out tributaries into the areas between them (Horton, 1945) before other networks have reached those areas, or whether channel networks may occupy oddly shaped basins with very long boundaries before the process of competition and piracy leads to areal consolidation and increasing compactness is not clear, and can certainly not be answered on the basis of our data.

The theoretical construction of the standardized basin and its divider for differing proportions j/i as presented here is, at best, a promising start; in light of the assumption stated at the beginning of this section, the outcome cannot be considered as satisfactory. While it should be possible to find more appropriate formulations using empirical data for guidance, it is not obvious how to expand this approach to achieve a complete reconstruction of the basin and its internal organization from the estimated length values of the $n-1$ dividers. However, if successful, such a reconstruction would be a two-fold accomplishment: (1) since these estimates are based on channel network parameters it would establish a formal relation between channel networks and the shape and structure of their basins; and (2) it would translate the topologic values of basin magnitudes into the planimetric values of basin geometry.

CONCLUDING REMARKS

With regard to basin scale at least one cautionary remark is in order. The magnitudes of the sample basins range from 1 to 210, and while this limitation is not explicitly acknowledged in each of the conclusions drawn from the observed data, it should be emphasized that their extrapolation to basins of larger magnitude is a matter of speculation.

It might conceptually be useful to classify relations among descriptors as either logical or causal or random, i.e. independent. Of these, the first class functions as constraint on the second class of relations (while the third class determines their existence). An example of the first class would be the condition that the planimetric minimum length of a basin boundary is given by the circumference of a circle of equal area. It is this category of principles superimposing logical restrictions on causal relations among basin and network parameters to which the present paper contributes several constraining identities.

Whenever research in fluvial geomorphology pursues a comprehensive investigation it tends to emphasize the drainage basin as a 'geomorphic unit' (Leopold *et al.*, 1964), a 'geomorphic system' (Chorley *et al.*, 1984), or 'the fundamental unit in geomorphology' (Selby, 1985). While there is overwhelming empirical and theoretical support for these statements it is equally true that basins are delineated by networks of basin dividers which result from the erosional work performed in two or three adjacent basins. Thus, their study requires a different research domain. In particular, any investigation of the causes responsible for location and shape of dividers has to examine the joint effect of denudational processes on intersecting slopes of adjacent basins, and the impact of channel erosion on slope development in these basins. This paper has explored some of the formal concepts and the formal framework within which such a research agenda could be embedded.

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