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RELATIVE QUANTUM FIELD THEORY

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Abstract. We highlight the general notion of a *relative quantum field theory*, which occurs in several contexts. One is in gauge theory based on a compact Lie algebra, rather than a compact Lie group. This is relevant to the maximal superconformal theory in six dimensions.

1. Introduction

The $(0,2)$-superconformal field theory in six dimensions, which we term Theory $\mathcal{X}$ for brevity, was discovered as a limit of superstring theories [W1, S]. It is thought not to have a lagrangian description, so is difficult to access directly, yet some expectations can be deduced from the string theory description [W2, GMN]. Two features are particularly relevant: (i) it is not an ordinary quantum field theory, and (ii) the theory depends on a Lie algebra, not on a Lie group. A puzzle, emphasized by Greg Moore, is that the dimensional reduction of Theory $\mathcal{X}$ to five dimensions is usually understood to be an ordinary quantum field theory—contrary to (i)—and it is a supersymmetric gauge theory so depends on a particular choice of Lie group—contrary to (ii). In this paper we spell out the modified notion indicated in (i), which we call a *relative quantum field theory*, and use it to resolve this puzzle about Theory $\mathcal{X}$ by pointing out that the dimensional reduction is also a relative theory. Relative gauge theories are not particular to dimension five. In fact, the possibility of studying four-dimensional gauge theory as a relative theory was exploited in [VW] and [W3].

A relative quantum field theory F is related to an *extended* quantum field theory α in one higher dimension. On a compact (Euclidean) spacetime the partition function of F is a vector in the quantum Hilbert space of α, and on a compact space the quantum Hilbert space of F is also related to the value of α, which is a category. An *anomalous* quantum field theory F may be viewed as a relative theory.[1] The anomaly α is an *invertible* quantum field theory: each of its quantum Hilbert spaces is one-dimensional and the partition functions are nonzero. The invertibility of α implies that the partition functions of F are defined as numbers only up to a scalar; the field theory α controls the indeterminacy. Another well-known example is the two-dimensional chiral Wess-Zumino-Witten conformal field theory, which is a theory relative to three-dimensional topological Chern-Simons theory. In these examples, as well as the ones in this paper, the higher dimensional theory α obeys strong finiteness conditions, though the definition does not require that. For example, the quantum

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[1]That point of view may not always be useful; we give examples in §2.

Hilbert spaces are finite dimensional. In addition, in our main examples here α is a *topological* field theory. (That is not always true—for example, anomaly theories are generally not topological.)

In §§3–5 we study three examples of relative theories. The first two are constructed by quantization of a classical model, whose fields form a fibration

$$p\colon \mathcal{F} \longrightarrow \mathcal{F}'' \tag{1.1}$$

Relative fields are the fibers of p; the fiber $\mathcal{F}'$ over the basepoint of $\mathcal{F}''$ is special. In our examples the quotient $\mathcal{F}''$ is finite in the sense that the path integral over the field $\mathcal{F}''$ reduces to a finite sum.[2] The third example is Theory $\mathcal{X}$, for which classical fields are only a heuristic unless the theory is noninteracting. The theory α in each of our three examples involves a finite group π. In §3 we study a relative σ-model, for which π is an *arbitrary* finite group. There are relative gauge theories based on finite covers $G \to \overline{G}$ of compact connected Lie groups with covering group an *abelian* group π. These are discussed in §4. In §5 we turn to Theory $\mathcal{X}$, and π is restricted to be a *Pontrjagin self-dual* finite abelian group. The data which defines Theory $\mathcal{X}$ is usually taken to be a compact simple real Lie algebra of type A, D, or E. In Data 5.1 we generalize to include noninteracting theories and many other examples. Appendix A posits a brief definition of a field; it is useful for the discussions in the body of the paper.

We begin in §2 with a general discussion of a relative quantum field theory. This notion has appeared elsewhere in various guises. One of the first is Segal's discussion [S2, §5] of a "weakly" two-conformal field theory with associated "modular functor"; the modular functor is part of topological Chern-Simons theory. We already mentioned Witten's description [W2] of Theory $\mathcal{X}$. One can view a relative theory F as a *boundary theory* for the higher dimensional theory α, in which case the notion is ubiquitous; in a topological context it is embedded in Kapustin's discussion [K]. It is also the framework in which Kevin Walker [Wa] describes Chern-Simons theory, and it is a very special case of Lurie's notion [L] of a topological field theory defined on manifolds with singularities.

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2. Relative quantum theories

An m-dimensional quantum field theory (QFT) f assigns to an m-dimensional manifold X a partition function $f(X) \in \mathbb{C}$ and to an $(m-1)$-dimensional manifold Y a quantum Hilbert space $f(Y)$. We have in mind a theory defined on Riemannian manifolds—so Wick rotated from a theory on Minkowski spacetime—though it may be a conformal theory or a topological theory. The manifolds X, Y may also carry topological structure, such as an orientation, spin structure, or framing. Finally, all manifolds are assumed compact to avoid convergence issues, and above X, Y do not have boundary. The theory is also defined for compact m-manifolds X with boundary

[2] We can make the relative theory completely rigorous if $\mathcal{F}$ is also finite in this sense.

$\partial X = Y_0 \amalg Y_1$ expressed as a disjoint union of closed[3] manifolds, and viewed as a map $X\colon Y_0 \to Y_1$ (see Figure 1). Then $f(X)\colon f(Y_0) \to f(Y_1)$ is a linear map on the quantum Hilbert spaces. For example, if X is a closed m-manifold with $x_1, \dots, x_k \in X$, define X_ϵ as X with open balls of radius ϵ about each x_i omitted, and view the boundary spheres as incoming. In the limit $\epsilon \to 0$ the theory gives a map $V \times \dots \times V \to \mathbb{C}$ is the correlation function on the space V of operators attached to a point. The modern mathematical take is that a quantum field theory f is a homomorphism from a geometric bordism category to a category $\mathrm{Vect}_{\mathrm{top}}$ of topological vector spaces; the Hilbert structure emerges under special conditions.[4] We do not give a precise formulation here.[5] An *extended* quantum field theory f also assigns values to closed $(m-2)$-manifolds Z and $(m-1)$-manifolds with boundary: $f(Z)$ is a linear category whose hom-sets are topological vector spaces. The extended theories in this paper (denoted 'α') are finite dimensional—both the quantum spaces $f(Y)$ and the hom-sets in the linear category $f(Z)$ are finite dimensional vector spaces—whereas the relative theories (denoted 'F') are typically infinite dimensional.

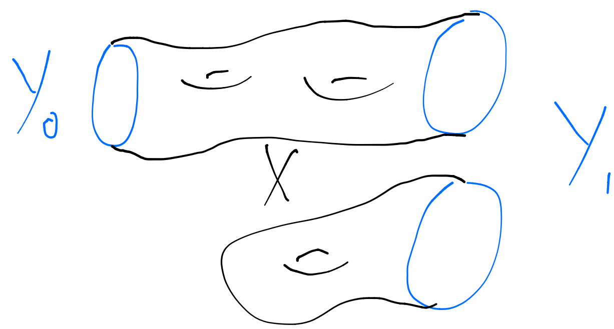

FIGURE 1. A geometric bordism $X\colon Y_0 \to Y_1$

A theory f is *invertible* if $f(X) \neq 0$ for all m-manifolds, the vector space $f(Y)$ is one-dimensional for all $(m-1)$-manifolds, and the linear category $f(Z)$ is similarly invertible for all $(m-2)$-manifolds: it is a free $\mathrm{Vect}_{\mathrm{top}}$-module of rank one. The trivial theory $\mathbf{1}$ is the constant invertible theory with values $\mathbf{1}(X) = 1$, $\mathbf{1}(Y) = \mathbb{C}$, and $\mathbf{1}(Z) = \mathrm{Vect}_{\mathrm{top}}$.

Here is a concise formal definition of a relative QFT.

Definition 2.1. Fix an integer $n \geq 0$ and let α be an extended $(n+1)$-dimensional quantum field theory. A *quantum field theory F relative to α* is a homomorphism

$$F\colon \mathbf{1} \longrightarrow \tau_{\leq n}\alpha. \tag{2.2}$$

The relative theory ignores the partition functions and correlation functions of α on $(n+1)$-manifolds and consider only the truncation $\tau_{\leq n}\alpha$ to a theory of n- and $(n-1)$-manifolds. In some cases the relative theory is more naturally a map in the other direction:

$$\widetilde{F}\colon \tau_{\leq n}\alpha \longrightarrow \mathbf{1}. \tag{2.3}$$

[3] For manifolds 'closed'='compact without boundary'.

[4] Therefore, we use 'quantum topological vector space' in place of the usual 'quantum Hilbert space'.

[5] See [S1] for a recent discussion of geometric axioms for quantum field theory.

We now spell out what data is contained in a homomorphism $\widetilde{F}$; the story is similar for (2.2).

Let X be a closed n-manifold. Then $\alpha(X) \in \mathrm{Vect}_{\mathrm{top}}$ is a topological vector space and $\mathbf{1}(X) = \mathbb{C}$. As mentioned above, in all examples considered here $\alpha(X)$ is finite dimensional. The relative theory $\widetilde{F}$ assigns to X a linear functional $\widetilde{F}(X)\colon \alpha(X) \to \mathbb{C}$. So for each vector $\xi \in \alpha(X)$ there is a partition function

$$\widetilde{F}(X;\xi) \in \mathbb{C}. \tag{2.4}$$

Let Y be a closed $(n-1)$-manifold. Then $\alpha(Y)$ is a linear category and $\mathbf{1}(Y) = \mathrm{Vect}_{\mathrm{top}}$. The relative theory assigns to Y a homomorphism $\widetilde{F}(Y)\colon \alpha(Y) \to \mathrm{Vect}_{\mathrm{top}}$. Thus for each object μ in the category $\alpha(Y)$ there is a quantum topological vector space

$$\widetilde{F}(Y;\mu) \in \mathrm{Vect}_{\mathrm{top}}\,. \tag{2.5}$$

The situation for a compact n-manifold $X\colon Y_0 \to Y_1$ is a bit more complicated. The categories $\alpha(Y_0), \alpha(Y_1)$ and the maps $\widetilde{F}(Y_0), \widetilde{F}(Y_1)$ fit into the commutative diagram

$$\begin{array}{ccc} \alpha(Y_0) & \xrightarrow{\;\widetilde{F}(Y_0)\;} & \\ {\scriptstyle \alpha(X)}\big\downarrow & & \mathrm{Vect}_{\mathrm{top}} \\ \alpha(Y_1) & \xrightarrow{\;\widetilde{F}(Y_1)\;} & \end{array} \tag{2.6}$$

The relative theory assigns to X a homomorphism

$$\widetilde{F}(X)\colon \widetilde{F}(Y_1)\circ\alpha(X) \longrightarrow \widetilde{F}(Y_0). \tag{2.7}$$

As a check on the definition, a QFT f relative to the trivial theory $\mathbf{1}$ is an *absolute* n-dimensional QFT,[6] where we use 'absolute' to describe a usual quantum field theory as opposed to a relative one.

If α is an *invertible* $(n+1)$-dimensional theory, and $\widetilde{F}$ is a theory relative to α, then we say $\widetilde{F}$ *is anomalous with anomaly* α. In this case $\alpha(X)$ is one-dimensional and there is a single partition function (2.4) determined up to a scalar controlled by $\alpha(X)$. If $\xi \in \alpha(X)$ is nonzero, then any other vector has the form $\xi' = \lambda\xi$ for some $\lambda \in \mathbb{C}$, and then $\widetilde{F}(X;\xi') = \lambda\widetilde{F}(X;\xi)$. If α is a unitary theory, then we can choose ξ, ξ' to have unit norm, in which case the partition function is determined up to a phase. Similarly, the quantum topological vector space (2.5) is determined up to a vector space controlled by $\alpha(Y)$. In the unitary case we can restrict to "unit norm" objects $\mu \in \alpha(Y)$, which comprise a gerbe, and if $\mu' = L\otimes\mu$ for a complex line L then $\widetilde{F}(Y;\mu') = L\otimes\widetilde{F}(Y;\mu)$. In particular, the underlying projective space of $\widetilde{F}(Y;\mu)$ is independent of μ. This is the standard picture of an anomalous theory.

[6] The arrow in (2.7) is opposite to what we expect if $\alpha = \mathbf{1}$, but $\widetilde{F}(X^\vee)^\vee$ does point in the right direction, where '$\vee$' denotes the dual bordism and the dual linear map.

A concrete example is provided by the conformal anomaly of an $n = 2$ dimensional conformal field theory f. The theory f is an ordinary theory of oriented *Riemannian* manifolds of dimension 1 and 2, or more precisely an anomalous theory of oriented *conformal* manifolds of the same dimensions. The conformal anomaly theory α is 3-dimensional, but we only consider its truncation[7] to manifolds of dimension ≤ 2. On an oriented conformal surface X,

$$\alpha(X) = (\mathrm{Det}_X)^{\otimes c_L} \otimes (\overline{\mathrm{Det}_X})^{\otimes c_R}, \tag{2.8}$$

where Det_X is the determinant line of the $\bar{\partial}$-operator determined by the conformal structure and orientation, and $c_L, c_R \in \mathbb{R}$ differ by an integer; the latter condition implies that the partition function is well-defined without further tangential structure on X. See [S2, §4] for more discussion.

Here are two examples of relative field theories with α invertible. In the first case F_1 is a 4-dimensional gauge theory with chiral fermions, so the fermionic path integral has an anomaly. The partition function depends on (X, g, A), where X is a closed 4-manifold, g a Riemannian metric, and A a connection (gauge field), and it takes values in a determinant or pfaffian line $\alpha_1(X, g, A)$. One cannot treat A as a quantum field due to the anomaly, but the theory makes good sense with A as a background field. In the second case F_2 is the two-dimensional gauged WZW model [W4]. The partition function depends on (X, g, A) for a closed oriented 2-manifold X with metric g and connection A, and it takes values in the Chern-Simons line $\alpha_2(X, A)$. In both cases the invertible anomaly theory α can be regarded as *classical*, rather than quantum. Indeed, a classical field theory is an example of an invertible field theory, and the formulation in terms of geometric bordism categories does not distinguish classical from quantum. One can quantize the pair (α_2, F_2). The quantization[8] of classical Chern-Simons theory α_2 is quantum Chern-Simons theory $\hat{\alpha}_2$. The function $F_2(X, g)(A)$ of A, which is a section of the Chern-Simons line bundle $\alpha_2(X)$ over the space of gauge fields A, is a vector $F_2(X, g)$ in the vector space $\hat{\alpha}_2(X)$ of quantum Chern-Simons theory $\hat{\alpha}_2$. (There is a polarization condition—holomorphy—which is part of the quantization [W4, §2.1].) The pair of quantum theories $(\hat{\alpha}_2, \widehat{F_2})$ is mentioned in the next paragraph, where it is denoted $(\chi, \mathcal{F})$. Note that $\hat{\alpha}_2$ is almost never invertible.

Chiral, or holomorphic, conformal field theories in $n = 2$ dimensions are a motivating example for Definition 2.1. We use Riemannian manifolds to avoid the conformal anomaly. The manifolds must be oriented and carry an additional topological structure which has several alternative descriptions: a 2-framing [A], a p_1-structure [BHMV], or a rigging [S2]. For concreteness let us consider the chiral WZW model $\widetilde{F}$ based on a compact Lie group G and a level k. Then there is an associated 3-dimensional *topological* field theory χ, the Chern-Simons theory associated to (G, k). It assigns a vector space $\chi(X)$ to an oriented rigged surface. In the chiral WZW model, $\chi(X)$ is interpreted as the space of conformal blocks and there is a partition function (2.4) for each conformal block. The linear category $\chi(S^1)$ attached to the circle $Y = S^1$ is a modular tensor category, which in many cases has a combinatorial description. For example, if G is 1-connected it can be described in terms of a quantum group. But the description of $\chi(S^1)$ as the category of positive energy representations

[7] On a closed 3-manifold the partition function is the exponential of a multiple of an η-invariant. It does not play a role in the 2-dimensional truncation

[8] There is a well-known framing anomaly, and quantum Chern-Simons is defined on a bordism category which includes framings, as described in the next paragraph.

of the loop group of G at level k is more adapted to the WZW model: the topological vector space $\widetilde{F}(S^1;\mu)$ *is* the underlying space of the representation $\mu \in \chi(S^1)$. The truncation $\tau_{\le 2}\chi$ is called a *modular functor* in [S2, §5] and the relative theory $\widetilde{F}$ is called a *weakly conformal field theory*. (It is formulated on conformal surfaces, rather than Riemannian surfaces, and there is an additional holomorphy condition.)

The topological Chern-Simons theory determines a modular tensor category $A = \chi(S^1)$ (up to equivalence of categories). The WZW model specifies a weak braided tensor functor $A \to \mathrm{Vect}_{\mathrm{top}}$.

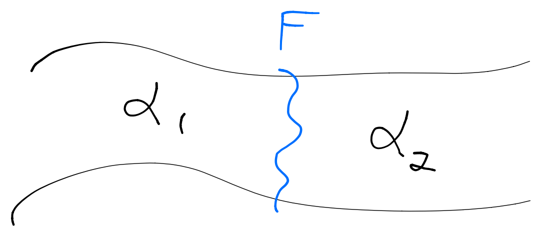

Figure 2. A domain wall

We conclude by briefly indicating the relationship of a relative quantum field theory to other variations of standard quantum field theories. First, if α_1, α_2 are $(n+1)$-dimensional theories, then a *domain wall* is an n-dimensional theory which lives on a codimension one submanifold of an $(n+1)$-manifold on which α_1 and α_2 are defined, as depicted in Figure 2. See [K] for a discussion in the context of extended topological theories. To match Definition 2.1 put $\alpha_1 = \alpha$ and $\alpha_2 = \mathbf{1}$. Ignoring the trivial theory we obtain from Figure 2 an $(n+1)$-manifold with boundary. The theory α lives in the bulk and the theory $\widetilde{F}$ on the boundary. So a relative theory $\widetilde{F}$ may be regarded as a *boundary theory* for α.

Finally, in the context of topological field theories there is a vast generalization based on bordism categories of manifolds with singularities [L, §4.3]. The appropriate "singular" manifold for a relative theory is the cone on a point. The cobordism hypothesis asserts that a fully extended[9] topological theory is determined by its value on a point, and the extension to manifolds with singularities implies the same for a fully extended relative theory in which both α and $\widetilde{F}$ are topological.

In rest of the paper α is a topological theory, the relative theory F more naturally maps in the direction (2.2), and F is not topological.

3. Warmup: relative σ-models

In this section we interpret the familiar example of a σ-model as a relative quantum field theory. We show how formal quantization of classical fields leads to the relative QFT structure. The classical theory is defined in any dimension n.

[9] A fully extended n-dimensional theory is defined for all manifolds of dimension $\le n$. The values increase in category number as the dimension decreases.

Data 3.1.

(i) π a finite group
(ii) M a smooth manifold with a free left π-action and quotient $\overline{M}$.

We study the σ-model into $\overline{M}$, or equivalently the gauged σ-model into M.

Let $B\pi(X)$ denote the collection of principal π-bundles over X. Recall that a principal π-bundle, or Galois covering space with Galois group π, is a covering space $P \to X$ and a free π-action on P such that $P \to X$ is a quotient map for the π-action. The collection of π-bundles over X is a *groupoid*, not a space. This is to account for symmetries of fields: a symmetry

$$\varphi\colon (P \to X) \longrightarrow (P' \to X) \tag{3.2}$$

is a diffeomorphism $\varphi\colon P \to P'$ which commutes with the π-action and covers id_X. The automorphism group of $(P \to X)$ is the group of gauge transformations. The path integral over $B\pi(X)$ is an integral over the equivalence classes of π-bundles. Canonical quantization over $B\pi(Y)$ for an $(n-1)$-manifold Y also remembers the gauge symmetry (Gauss law).

The space of fields of the σ-model into $\overline{M}$ on a manifold X is the space $\mathrm{Map}(X, \overline{M})$ of smooth maps $f\colon X \to \overline{M}$. A σ-model field induces a gauge field: define

$$\begin{aligned} p\colon\ \mathrm{Map}(X,\overline{M}) &\longrightarrow B\pi(X) \\ f &\longmapsto f^*(M \to \overline{M}) \end{aligned} \tag{3.3}$$

That is, to a map $f\colon X \to \overline{M}$ the map p assigns the pullback of the π-bundle $M \to \overline{M}$. This pullback is the obstruction to lifting f to a map $X \to M$: a lift is precisely a trivialization of the pullback $f^*(M \to \overline{M})$. The map p need not be surjective. For example, in the extreme case $M = \overline{M} \times \pi$ the image of p contains only the trivial π-bundle over X.

Relative fields are the fibers of the map p in (3.3).

Definition 3.4. Fix $(P \to X) \in B\pi(X)$. A *relative field* over $(P \to X)$ is a pair (f, θ) consisting of a smooth map $f\colon X \to \overline{M}$ and an isomorphism $\theta\colon (P \to X) \longrightarrow f^*(M \to \overline{M})$ of π-bundles over X.

Equivalently, a relative field (f, θ) is a π-equivariant map $P \to M$. In particular, relative fields are rigid—there are no automorphisms—so form a space, not a groupoid. Notice that the fiber of p over the trivial bundle $(X \times \pi \to X)$ is the mapping space $\mathrm{Map}(X, M)$.

We now indicate a pair (α, F) of theories in which α is topological, defined on all manifolds, and F is a Riemannian theory. We use our knowledge of the σ-model F to predict the structure of α. To define the classical σ-model, we assume $\overline{M}$ carries geometric data—a metric, B-field, etc.—which is lifted to π-invariant geometric data on M. Formally, the *relative path integral* $F(X)$ on a closed n-manifold X is an integral over relative fields, so a function $F(X)\colon B\pi(X) \to \mathbb{C}$. The formal structure implies that $F(X)$ is invariant under symmetries in $B\pi(X)$, so passes to a function on equivalence classes. Let $H^1(X;\pi)$ denote the set of equivalence classes, which are isomorphism classes of principal π-bundles over X. Then

$$F(X)\colon H^1(X;\pi) \longrightarrow \mathbb{C}. \tag{3.5}$$

If F is to be a QFT relative to an $(n+1)$-dimensional theory α in the sense that $F\colon \mathbf{1} \to \tau_{\leq n}\alpha$, then $F(X)\colon \mathbb{C} \to \alpha(X)$. In other words, $F(X)$ can be identified as an element of the vector space $\alpha(X)$. This leads to the prediction that $\alpha(X)$ is the free vector space

$$\alpha(X) = \mathbb{C}\left\{H^1(X;\pi)\right\} \tag{3.6}$$

on the finite set $H^1(X;\pi)$.

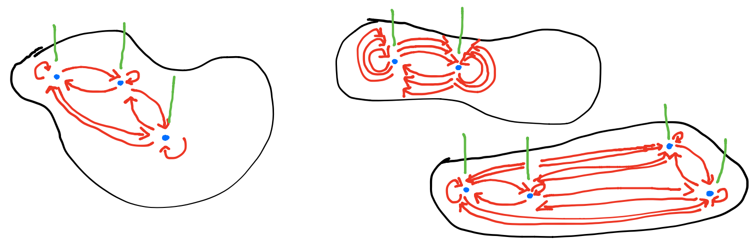

FIGURE 3. The vector bundle $F(Y) \to B\pi(Y)$

Now let Y be a closed $(n-1)$-manifold. The *relative canonical quantization* $F(Y)$ is obtained by carrying out canonical quantization on the fibers of $\mathrm{Map}(Y, \overline{M}) \to B\pi(Y)$. Thus $F(Y) \to B\pi(Y)$ is a vector bundle (whose fibers are typically infinite dimensional topological vector spaces). Furthermore, it is an *equivariant* vector bundle: symmetries of fields in $B\pi(Y)$ come with a lift to the vector bundle. We depict it in Figure 3. The blue dots represent π-bundles $Q \to Y$ and the red arrows represent isomorphisms of π-bundles. The bundles are grouped into isomorphism classes. The groupoid $B\pi(Y)$ is equivalent[10] to a much simpler groupoid which has a finite set of objects $H^1(Y;\pi)$ and in which there are no arrows between distinct objects; see Figure 4. If $m \in H^1(Y;\pi)$ is the class of a π-bundle $Q \to Y$, then the automorphism group of m is the group $\mathrm{Aut}(Q \to Y)$ of gauge transformations of $Q \to Y$. The equivariant bundle

$$F(Y) \longrightarrow B\pi(Y) \tag{3.7}$$

therefore decomposes into topological vector spaces indexed by pairs (m, e) in which $m \in H^1(Y;\pi)$ and e is an irreducible complex representation of the automorphism group of m. Now $F(Y)\colon \mathbf{1}(Y) \to \alpha(Y)$ may be identified with an object of the linear category $\alpha(Y)$. This leads to the prediction that $\alpha(Y)$ is the free $\mathrm{Vect}_{\mathrm{top}}$-module with basis pairs (m, e).

FIGURE 4. A groupoid equivalent to $B\pi(Y)$

[10] To make the equivalence, which is noncanonical, we choose a representative bundle $Q \to Y$ in each isomorphism class.

The special case $n = 1$ is most familiar. (It has the added advantage that the quantum theory makes sense.) If $X = S^1$ in (3.5), then $H^1(S^1;\pi)$ is the set of conjugacy classes in π. If $Y = \mathrm{pt}$ is a single point, then $H^1(\mathrm{pt};\pi)$ has a single element which represents the trivial bundle and has automorphism group π. We identify $\alpha(\mathrm{pt})$ as the category of representations of π. The entire field theory α is familiar: it is the finite 2-dimensional gauge theory with gauge group π. It may be defined On *all* manifolds of dimension ≤ 2 by a *finite path integral* [F, FHLT] as we now briefly review.

The lagrangian of the theory vanishes, so the exponentiated action function is constant. On a compact 2-manifold W the constant is 1, whence the path integral is a weighted count of π-bundles:

$$\alpha(W) = \sum_{[R\to W]} \frac{1}{\#\,\mathrm{Aut}(R\to W)}. \tag{3.8}$$

The sum is over equivalence classes. The finite path integral over $X = S^1$ is a sum on π-bundles over S^1 of the exponentiated action, which is the constant function that assigns to each π-bundle $Q \to S^1$ the trivial complex line $\mathbb{C}$. The result $\alpha(S^1)$ is the space of invariant sections of the trivial equivariant line bundle over $B\pi(S^1)$. Since $B\pi(S^1)$ is equivalent to the quotient groupoid $G/\!/G$ of G acting on itself by conjugation, this is the space of central functions on π. Finally, the exponentiated action on the codimension two[11] manifold $Y = \mathrm{pt}$ has constant value the linear category $\mathrm{Vect}_{\mathrm{top}}$. In this case the finite path integral returns the subcategory of "invariants" under π, which is the category $\alpha(\mathrm{pt})$ of representations of the group π, as predicted above. The finite gauge theory can be defined in any dimension $n+1$. Also, there is a twisted version defined using a group cocycle, which can be incorporated into the σ-model as a topological term in the action.

The picture of F as a *boundary* theory for α is manifest in terms of classical fields. If W is an $(n+1)$-manifold with boundary, and $R \to W$ a principal π-bundle—a field in the bulk theory α—then the boundary field f is a π-equivariant function $P \to M$, where $P \to \partial W$ is the restriction of $R \to W$ to the boundary. The relative theory on an n-manifold X has a bulk field $R \to [0,1]\times X$, a boundary field f on $\{1\}\times X$, and no additional field at $\{0\}\times X$. (At $\{0\}\times X$ lies the trivial theory $\mathbf{1}$, whose space of classical fields consists of a single point.)

We remark that the relative theory can be defined in terms of topological disorder operators in the σ-model to M. For example, to define the partition function (3.5) represent a class in $H^1(X;\pi)$ by a π-bundle $P \to X$ and endow it with a trivialization away from a normally oriented codimension one "defect" submanifold $D \subset X$. The trivialization determines a locally constant function $j\colon D \to \pi$, the jump across D. Then a relative field determines a function $X \setminus D \to M$ which obeys the jump j. Notice that this function is only defined away from D, so does not contain all of the information of the relative field.

We can use the relative theory to recover absolute theories. Namely, if $\pi' < \pi$ is a subgroup, then we can recover the σ-model into M/π' by "integrating over $B\pi'$". So for a closed n-manifold X we define

$$f_{\pi'}(X) = \sum_{m'\in H^1(X;\pi')} \frac{1}{\#Z_{\pi'}(m')}\, F(X;m'), \tag{3.9}$$

[11] The zero-dimensional manifold $Y = \mathrm{pt}$ has codimension two in a two-dimensional theory.

where the sum is over equivalence classes of π'-bundles $P' \to X$ and $Z_{\pi'}(m')$ is the automorphism group of a representative of the equivalence class m'. (While the group $Z_{\pi'}(m')$ depends on the choice of representative, its cardinality does not.) Similarly, the quantum topological vector space on a closed $(n-1)$-manifold is

$$f_{\pi'}(Y) = \bigoplus_{m'=[Q'\to Y]\in H^1(Y;\pi')} F(Y; Q \to Y)^{\mathrm{Aut}(Q'\to Y)}. \tag{3.10}$$

The sum is over equivalence classes of π'-bundles with chosen representatives. A π'-bundle $Q' \to Y$ has an associated π-bundle $Q \to Y$, and automorphisms of $Q' \to Y$ induce automorphisms of $Q \to Y$. The summand in (3.10) is the subspace of invariants of the fiber of (3.7) at $Q \to Y$. The extreme cases $\pi' = \{e\}$ and $\pi' = \pi$ give the σ-models into M and $\overline{M}$, respectively. Clearly the relative theory encodes more information than the absolute theories.

We use the language of Appendix A to describe the relative fields.[12] Namely, (3.3) is a map

$$p\colon \overline{M} \longrightarrow B\pi \tag{3.11}$$

with fiber M. In fact, p is the classifying map of the principal π-bundle $M \to \overline{M}$. Quantization in the relative theory is integration over the fibers of p (on a particular manifold X). The inclusion $i\colon \pi' \to \pi$ of a subgroup induces a pullback diagram

$$\begin{array}{ccc} M/\pi' & \longrightarrow & \overline{M} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B\pi' & \xrightarrow{\;Bi\;} & B\pi \end{array} \tag{3.12}$$

On a manifold X integration over the fibers of p' followed by integration over $B\pi'(X)$ is equivalent to integration over M/π'. This explains the formulas in the previous paragraph. In these terms the topological vector space $f_{\pi'}(Y)$ of (3.10) is the space of invariant sections of $(Bi)^*\bigl(F(Y) \to B\pi(Y)\bigr)$.

Our hypothesis in Data 3.1 is that π acts *freely* on M. We can generalize to arbitrary π-actions if we interpret $\overline{M} = M/\!/\pi$ as the stack quotient, so the σ-model into $\overline{M}$ as the gauged σ-model on M.

4. Relative gauge theories

The relative theory in this section is an exact analog of the relative σ-model of §3 with all of the fields bumped up one categorical level—there is an extra layer of symmetry.

Data 4.1. A covering homomorphism $G \to \overline{G}$ of compact connected Lie groups with kernel π

[12] So (3.11) is a map of simplicial sheaves (or sheaves of groupoids or stacks) on the category of smooth manifolds.

So π is a finite central subgroup of G, necessarily abelian. We denote the common Lie algebra of G and $\overline{G}$ as $\mathfrak{g}$.

Example 4.2. If $\mathfrak{g}$ is a real algebra with negative definite Killing form, then there is a canonically associated compact 1-connected Lie group G. Let $\pi \subset G$ be the center and $\overline{G} = G/\pi$ the adjoint group. This gives a canonical choice of Data 4.1 associated to a compact semisimple Lie algebra. The more general data allows for torus factors as well. Simple representative examples are the covers $SU(2) \to SO(3)$, $U(2) \to U(2)/\{\pm 1\}$, and $\mathbb{T} \xrightarrow{\lambda \mapsto \lambda^2} \mathbb{T}$, each with π cyclic of order two. Here $\mathbb{T} \subset \mathbb{C}$ is the circle group of unit norm complex numbers.

Let $\overline{P} \to X$ be a principal $\overline{G}$-bundle. The obstruction to lifting to a principal G-bundle is measured by a π-*gerbe* $\mathcal{G}(\overline{P}) \to X$. For $\mathrm{Spin}(n) \to SO(n)$ this π-gerbe is a geometric manifestation of the second Stiefel-Whitney class. There is a tautological construction of $\mathcal{G}(\overline{P}) \to X$ as a sheaf. (The reader may wish as a warmup to construct directly from f the sheaf associated to the pullback π-bundle in (3.3).) Let $U \subset X$ be an open set. The value of $\mathcal{G}(\overline{P})$ on U is the collection of lifts of $(\overline{P} \to X)\big|_U$ to a principal G-bundle. For small contractible $U \subset X$ such lifts always exist, but for general U there may be no lifts, in which case $\mathcal{G}(\overline{P})(U)$ is empty. More formally, an object in $\mathcal{G}(\overline{P})(U)$ is a pair $(P_U \to U, \varphi)$ consisting of a principal G-bundle $P_U \to U$ and an isomorphism

$$\begin{array}{ccc} \overline{P}\big|_U & \xrightarrow[\cong]{\varphi} & P_U/\pi \\ & \searrow \quad \swarrow & \\ & U & \end{array} \tag{4.3}$$

We leave the reader to define the notion of an isomorphism $(P_U \to U, \varphi) \to (P'_U \to U, \varphi')$. Thus $\mathcal{G}(\overline{P})(U)$ is a groupoid and $\mathcal{G}(\overline{P})$ a sheaf of groupoids. A global section is a lift of $\overline{P} \to X$ to a G-bundle.

Let $B^2\pi(X)$ denote the collection of π-gerbes over X. It is a 2-groupoid: there are isomorphisms of objects and isomorphisms of isomorphisms. For example, the groupoid of automorphisms of any π-gerbe is the groupoid $B\pi(X)$ of principal π-bundles over X. The set of equivalence classes in $B^2\pi(X)$ is the cohomology group $H^2(X;\pi)$. (Since π is abelian, this cohomology group is well-defined.) The group of equivalence classes of automorphisms of any object is $H^1(X;\pi)$ and the group of automorphisms of automorphisms is $H^0(X;\pi)$. Homotopy groups are defined for (higher) groupoids, and here

$$\begin{aligned} \pi_0\big(B^2\pi(X)\big) &\cong H^2(X;\pi) \\ \pi_1\big(B^2\pi(X)\big) &\cong H^1(X;\pi) \\ \pi_2\big(B^2\pi(X)\big) &\cong H^0(X;\pi) \end{aligned} \tag{4.4}$$

Let $B_\nabla\overline{G}(X)$ denote the groupoid of $\overline{G}$-connections on X. An object is a principal $\overline{G}$-bundle $\overline{P} \to X$ with connection $\overline{\Theta} \in \Omega^1(\overline{P};\mathfrak{g})$, which we simply denote as $\overline{\Theta}$. An isomorphism $\overline{\Theta} \to \overline{\Theta}'$ is an isomorphism $\varphi\colon \overline{P} \to \overline{P}'$ of the underlying $\overline{G}$-bundles which satisfies $\varphi^*\overline{\Theta}' = \overline{\Theta}$. The construction of the previous paragraph defines a map

$$p\colon B_\nabla \overline{G}(X) \longrightarrow B^2\pi(X) \qquad \overline{\Theta} \longmapsto \bigl(\mathcal{G}(\overline{\Theta}) \to X\bigr) \tag{4.5}$$

in which $\mathcal{G}(\overline{\Theta})$ denotes the π-gerbe associated to the $\overline{G}$-bundle carrying the connection $\overline{\Theta}$.

Definition 4.6. Fix $(\mathcal{G} \to X) \in B^2\pi(X)$. A *relative field* over $(\mathcal{G} \to X)$ is a pair $(\overline{\Theta}, \theta)$ consisting of a $\overline{G}$-connection $\overline{\Theta}$ and an isomorphism $\theta\colon \mathcal{G} \to \mathcal{G}(\overline{\Theta})$ of π-gerbes.

The fiber of p over the trivial π-gerbe on X is the groupoid $B_\nabla G(X)$ of G-connections on X.

Relative fields form an ordinary groupoid—there are no automorphisms of automorphisms. We describe the automorphism group $\mathrm{Aut}_{\mathrm{rel}}(\overline{\Theta})$ of $(\overline{\Theta}, \theta)$ in elementary terms. As the notation suggests, this automorphism group is independent of θ. Suppose $\overline{P} \to X$ is the $\overline{G}$-bundle which carries $\overline{\Theta}$. Conjugation $G \to \mathrm{Aut}(G)$ in G drops to a group homomorphism $\overline{G} \to \mathrm{Aut}(G)$, since $\pi \subset G$ is central. There is an associated bundle of groups $\overline{P} \times_{\overline{G}} G \to X$ associated to the conjugation action. Sections of this bundle act on $\overline{P} \to X$, and $\mathrm{Aut}_{\mathrm{rel}}(\overline{\Theta})$ is the stabilizer subgroup of $\overline{\Theta} \in \Omega^1(\overline{P}; \mathfrak{g})$.

Remark 4.7. Heuristically, the fields are $\overline{G}$-connections with G-gauge transformations. Definition 4.6 gives a precise formulation in terms of *local* fields. We do not know a precise formulation in terms of absolute fields.

We turn to the quantum theories α and F built from these fields. Fix a dimension n. The theory α is an $(n+1)$-dimensional topological theory and F is a relative n-dimensional theory of Riemannian[13] manifolds. The topological theory α is defined by a finite path integral over π-gerbes; see [Q, T, FHLT] for general discussions of homotopy finite quantum theories. If W is a closed $(n+1)$-manifold, then

$$\alpha(W) = \frac{\#H^2(X;\pi)\cdot \#H^0(X;\pi)}{\#H^1(X;\pi)}. \tag{4.8}$$

Of course, this partition function is ignored in the truncation $\tau_{\le n}\alpha$, so too in $F\colon \mathbf{1} \to \tau_{\le n}\alpha$.

Let X be a closed n-manifold. Then $\alpha(X)$ is the vector space of complex-valued functions on $B^2\pi(X)$. These are invariant functions on the collection of π-gerbes, so they factor down to functions on equivalence classes $\pi_0\bigl(B^2\pi(X)\bigr)$. The relative theory F gives an element of $\alpha(X)$, so a function

$$F(X)\colon H^2(X;\pi) \longrightarrow \mathbb{C}. \tag{4.9}$$

Let Y be a closed $(n-1)$-manifold. Then $\alpha(Y)$ is the linear category of vector bundles (of infinite rank) over $B^2\pi(Y)$. The relative theory F determines a particular vector bundle

$$F(Y) \longrightarrow B^2\pi(Y). \tag{4.10}$$

[13] There may be spin structures and the theory may be conformal, depending on particulars.

A complex vector bundle over a 2-groupoid only senses π_0 and π_1, not π_2 (nor higher homotopy groups if they were present). If we choose a basepoint in each component, then the fibers at the basepoints are complex representations of π_1, so can be decomposed according to the irreducible representations. For any finite abelian group A, let

$$A^\vee = \mathrm{Hom}(A, \mathbb{T}) \tag{4.11}$$

denote the *Pontrjagin dual group* of characters. Then, after choosing basepoints, (4.10) determines topological vector spaces $F(Y; m, e)$ for

$$m \in H^2(Y;\pi), \qquad e \in H^1(Y;\pi)^\vee. \tag{4.12}$$

The class m is a *discrete magnetic flux* and e is a *discrete electric flux* [W3]. Note that if Y is oriented, then Poincaré-Pontrjagin duality is an isomorphism

$$H^1(Y;\pi)^\vee \xrightarrow{\;\cong\;} H^{n-2}(Y;\pi^\vee). \tag{4.13}$$

As in §3 the picture of F as a *boundary* theory for α is manifest in terms of classical fields. If W is an $(n+1)$-manifold with boundary, and $\mathcal{G}\to W$ a π-gerbe—a field in the bulk theory α—then the boundary field is a G-connection twisted by the restriction of the π-gerbe to the boundary. In other words, it is exactly $(\overline{\Theta},\theta)$ in Definition 4.6 relative to $\mathcal{G}\big|_{\partial W}\to \partial W$. The relative theory on an n-manifold X has a bulk field $\mathcal{G}\to[0,1]\times X$, a boundary field $(\overline{\Theta},\theta)$ on $\{1\}\times X$, and no additional field at $\{0\}\times X$.

There are disorder operators in the gauge theory with gauge group G associated to classes in $H^2(X;\pi)$. Suppose such a class is represented by a normally oriented submanifold $D\subset X$ of codimension two, together with a locally constant function $j\colon D\to\pi$. Then, roughly speaking, the value of $F(X)$ in (4.9) on the class represented by (D,j) is the path integral over G-connections on $X\setminus D$ with limiting holonomy prescribed by j, where the limiting holonomy is computed around an oriented circle surrounding D in the normal space as the radius of the circle shrinks to zero. As in §3 this field defined on the complement of D does not contain all the information that the relative field defined on all of X does.

As in §3 we can recover absolute (not relative) gauge theories with gauge group G/π' for subgroups $\pi'<\pi$ from the relative theory F. For example, the partition function on a closed n-manifold X is

$$f_{\pi'}(X) = \sum_{m'\in H^2(X;\pi')} \frac{\#H^0(X;\pi')}{\#H^1(X;\pi')}\,F(X;m'); \tag{4.14}$$

analogous to (3.9). There is a formula similar to (3.10) for the quantum topological vector space.

We use the language of Appendix A to describe the relative fields. Thus (4.5) is a map

$$p\colon B_\nabla\overline{G}\longrightarrow B^2\pi \tag{4.15}$$

with fiber $B_\nabla G$. It is the classifying map for the principal $B\pi$-bundle $B_\nabla G \to B_\nabla \overline{G}$. The inclusion $i\colon \pi' \to \pi$ of a subgroup induces a pullback diagram

$$\begin{array}{ccc} B_\nabla(G/\pi') & \longrightarrow & B_\nabla\overline{G} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B^2\pi' & \xrightarrow{B^2 i} & B^2\pi \end{array} \tag{4.16}$$

On a manifold X integration over the fibers of p' followed by integration over $B^2\pi'(X)$ is equivalent to integration over the groupoid $B_\nabla(G/\pi')(X)$ of G/π'-connections on X.

Finally, we briefly comment on the case when both α and F are homotopy finite quantum theories. In this paragraph each topological space is finite in the sense that it has only finitely many nonzero homotopy groups and each is a finite group. Let B be a connected example of such a space and α the $(n+1)$-dimensional theory defined by counting maps into B up to homotopy, as in (4.8). A fibration $E \xrightarrow{\pi} B$ defines a relative n-dimensional theory $F_\pi\colon \mathbf{1} \to \tau_{\le n}\alpha$ using relative σ-model fields up to homotopy. Equivalently, it defines a relative theory $\widetilde{F}_\pi\colon \tau_{\le n}\alpha \to \mathbf{1}$. Given a second fibration $E' \xrightarrow{\pi'} B$, the composition

$$\widetilde{F}_{\pi'} \circ F_\pi\colon \mathbf{1} \to \mathbf{1} \tag{4.17}$$

is an (absolute) n-dimensional theory. It can be described as the σ-model into the total space of the fiber product of π and π'. In terms of boundary conditions, the fibrations give an extra boundary field in the $(n+1)$-dimensional σ-model with field $f\colon W \to B$, namely a lift $g\colon \partial W \to E$ of the restriction of f to the boundary. The fields in the n-dimensional theory (4.17) on an n-manifold X are a map $f\colon [0,1] \to B$, a lift $g\colon \{0\}\times X \to E$ of $f\big|_{\{0\}\times X}$, and a lift $g'\colon \{1\}\times X \to E'$ of $f\big|_{\{1\}\times X}$. The canonical choices $B \xrightarrow{\pi'=\mathrm{id}} B$ and $* \xrightarrow{\pi'} B$ correspond roughly to Neumann and Dirichlet boundary conditions, and give the absolute n-dimensional σ-models into E and the homotopy fiber of π, respectively. Here $*$ denotes the contractible space of paths in B which begin at a fixed point. There is also an interpretation in terms of gauge theory with gauge group ΩB. Then the first boundary condition is no condition at all and the second fixes the gauge at the boundary.

5. Expectations for Theory $\mathcal{X}$

In this section we fit the expectations [W2, §4] for Theory $\mathcal{X}$—the $(0,2)$-superconformal theory in six dimensions—into a relative quantum field theory.

A reductive real Lie algebra $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ of its center $\mathfrak{z}$ and its semisimple subalgebra $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$. A Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{g}'$ determines a coroot lattice, which is a full sublattice $\Gamma' \subset \mathfrak{h}'$. Any two Cartan subalgebras are conjugate by an element of $\mathfrak{g}'$; the conjugation preserves the coroot lattices. A real Lie algebra with an invariant inner product is reductive.

Data 5.1.

(i) A real Lie algebra $\mathfrak{g}$ with an invariant inner product $\langle -,-\rangle$ such that all coroots have square length 2
(ii) A full lattice $\Gamma\supset\Gamma'$ in $\mathfrak{h}=\mathfrak{z}\oplus\mathfrak{h}'$ such that the inner product is integral and even on Γ

The condition on the inner product implies that the semisimple subalgebra $\mathfrak{g}'$ is a sum of simple Lie algebras of ADE type, i.e., $\mathfrak{g}'$ is *simply laced*. The lattice Γ may be specified by choosing any Cartan subalgebra $\mathfrak{h}'\subset\mathfrak{g}'$. Any other choice is conjugate by some $\xi'\in\mathfrak{g}'$, and we then conjugate Γ (and $\mathfrak{h}$) by the same element ξ', viewed as an element of $\mathfrak{g}$. A special case is $\Gamma=\Gamma_{\mathfrak{z}}\oplus\Gamma'$ for a chosen full lattice $\Gamma_{\mathfrak{z}}\subset\mathfrak{z}$.

Given Data 5.1, define $\Lambda\supset\Gamma$ as the dual lattice to Γ in $\mathfrak{h}$, the subset of vectors $\eta\in\mathfrak{h}$ such that $\langle\eta,\Gamma\rangle\subset\mathbb{Z}$. The quotient $\pi=\Lambda/\Gamma$ is a finite abelian group equipped with a perfect pairing

$$\pi\times\pi\longrightarrow\mathbb{Q}/\mathbb{Z}\subset\mathbb{T} \tag{5.2}$$

induced from $\langle -,-\rangle$. The pairing induces an isomorphism $\pi\cong\pi^\vee$. In other words, π is *Pontrjagin self-dual*. There is also a quadratic form refining (5.2) induced from $\langle -,-\rangle$ which plays a role. The data determine a covering $G\to\overline{G}$ of compact connected Lie groups with kernel π. The lattice Γ is the fundamental group of a maximal torus of G and Λ is the fundamental group of a maximal torus of $\overline{G}$.

There are two extreme cases worth noting. If $\mathfrak{g}=\mathfrak{z}$ is abelian, then Data 5.1 reduces to a choice of lattice $\Gamma_{\mathfrak{z}}$ with a positive definite even[14] integral form $\langle -,-\rangle$. In that case $G=\overline{G}$ is the torus $\mathfrak{z}/\Gamma_{\mathfrak{z}}$. The resulting theory $\mathcal{X}_{(\Gamma_{\mathfrak{z}},\langle -,-\rangle)}$ is meant to be noninteracting. At the other extreme, if $\mathfrak{g}=\mathfrak{g}'$ is semisimple, so a sum of ADE Lie algebras, then G is the 1-connected compact Lie group with Lie algebra $\mathfrak{g}$, the finite group $\pi\in G$ is its center, and $\overline{G}$ is the adjoint group. We remark that Data 5.1 allows many intermediate cases, for example a lattice in $\mathfrak{so}(4)$ for which $G=\overline{G}=SO(4)$.

Now we can state the expected formal structure of Theory $\mathcal{X}$.

Expectation 5.3. Given $(\mathfrak{g},\langle -,-\rangle,\Gamma)$ in Data 5.1 there exists a finite 7-dimensional topological quantum field theory $\alpha_{\mathfrak{g}}=\alpha_{(\mathfrak{g},\langle -,-\rangle,\Gamma)}$ and a 6-dimensional quantum field theory $\mathcal{X}_{\mathfrak{g}}=\mathcal{X}_{(\mathfrak{g},\langle -,-\rangle,\Gamma)}$ relative to $\alpha_{\mathfrak{g}}$.

The topological theory $\alpha_{\mathfrak{g}}$ should be defined as an extended theory on a bordism multi-category of manifolds which have a tangential structure to be determined, roughly some sort of framing.[15] It is a 7-dimensional analog of 3-dimensional Chern-Simons theory for torus groups [BM, FHLT, St]. The 6-dimensional theory $\mathcal{X}_{\mathfrak{g}}$ is meant to be defined on a geometric bordism category of manifolds with a conformal structure as well as some (topological) tangential structure which includes a spin structure, since the theory has spinor fields. There is a conformal anomaly, and as for the chiral WZW theory mentioned in §2, we treat it as a theory of Riemannian manifolds and ignore the conformal anomaly.

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[14] There are 3-dimensional torus Chern-Simons theories for which form $\langle -,-\rangle$ is not even; they are defined on spin manifolds. We similarly expect that the evenness in Data 5.1(ii) can be omitted to define a larger class of theories.

[15] It will certainly include an orientation and, at least in many cases, an integral Wu structure [HS].

A useful formal picture is to imagine $\mathfrak{X}_{\mathfrak{g}}$ as constructed from classical relative fields[16]

$$p\colon \text{“}B^2_{\nabla}\overline{G}\text{”} \longrightarrow B^3\pi, \tag{5.4}$$

analogous to the quantization of relative σ-models and relative gauge fields in the previous sections. But while $B^3\pi$ exists—a field $X \to B^3\pi$ is a 2-gerbe with structure group π over X—we do not know an applicable notion of gerbes with connection to define $B^2_{\nabla}\overline{G}$ unless the Lie algebra $\mathfrak{g}$ is abelian. So (5.4) is in general only a heuristic, whence the quotation marks.

An important point is that both fields in (5.4) are to be treated as *self-dual*. This is what restricts the dimensions of $\mathfrak{X}_{\mathfrak{g}}$ and $\alpha_{\mathfrak{g}}$ to 6 and 7, respectively. The quantization procedure for self-dual fields is not the standard Feynman functional integral, but rather involves a sort of square root. This is a well-defined and interesting story for abelian fields, such as[17] $B^3\pi$, and we hope to develop it more elsewhere. Here we indicate what such a development will give for the finite topological theory $\alpha_{\mathfrak{g}}$.

Let X be a closed oriented 6-manifold. The cup product and pairing (5.2) combine to a nondegenerate skew-symmetric pairing

$$H^3(X;\pi) \times H^3(X;\pi) \longrightarrow \mathbb{T} \tag{5.5}$$

on the finite group $H^3(X;\pi)$. This determines, up to noncanonical isomorphism,[18] a Heisenberg representation of a Heisenberg central extension of $H^3(X;\pi)$. The finite dimensional vector space $\alpha_{\mathfrak{g}}(X)$ is the underlying vector space of this representation.

Let Y be a closed oriented 5-manifold. Then $H^3(Y;\pi)$ is the group of self-dual fluxes. To that end, notice from (5.2) and Poincaré-Pontrjagin duality the isomorphisms

$$H^3(Y;\pi) \cong H^3(Y;\pi^\vee) \cong H^2(Y;\pi)^\vee. \tag{5.6}$$

The first group may be thought of a magnetic and the last as electric, as in (4.12); the self-duality identifies them. The linear category $\alpha_{\mathfrak{g}}(Y)$ is the free $\mathrm{Vect}_{\mathrm{top}}$-module with basis $H^3(Y;\pi)$, and for each self-dual flux $\sigma \in H^3(Y;\pi)$ we expect a quantum topological vector space $\mathfrak{X}_{\mathfrak{g}}(Y;\sigma)$.

We turn now to dimensional reductions of Theory $\mathfrak{X}$.

Claim 5.7. *The dimensional reduction of $\mathfrak{X}_{\mathfrak{g}}$ to five dimensions is a relative gauge theory based on $G \to \overline{G}$ with kernel π.*

The particular gauge theory we obtain has maximal supersymmetry. Our focus here is on the assertion that it is a *relative* quantum field theory, so on the statement that the dimensional reduction of $\alpha_{\mathfrak{g}}$ is the finite π-gerbe theory α (in five dimensions) discussed in §4. To see this, observe that if $X = S^1 \times \widetilde{X}$ for a closed oriented 5-manifold $\widetilde{X}$, then

$$H^3(X;\pi) \cong H^2(\widetilde{X};\pi) \times H^3(\widetilde{X};\pi) \tag{5.8}$$

[16] $\mathfrak{X}_{\mathfrak{g}}$ has many other fields, which are well-defined. We focus here on the self-dual 2-form field.

[17] The Pontrjagin self-duality (5.2) of π is needed to treat $B^3\pi$ as a self-dual field.

[18] One role for the tangential structure in $\alpha_{\mathfrak{g}}$ will be to make this construction canonical so that diffeomorphisms of X act linearly, not just projectively.

has a canonical polarization, as written. We can realize the Heisenberg representation as functions on $H^2(\widetilde{X};\pi)$, so identify $\mathcal{X}_{\mathfrak{g}}(S^1\times\widetilde{X})$ as a function on $H^2(\widetilde{X};\pi)$, which matches (4.9). Similarly, if $Y=S^1\times\widetilde{Y}$ is a closed oriented 5-manifold, then

$$H^3(Y;\pi)\cong H^2(\widetilde{Y};\pi)\times H^3(\widetilde{Y};\pi)\cong H^2(\widetilde{Y};\pi)\times H^1(\widetilde{Y};\pi)^\vee. \tag{5.9}$$

We therefore obtain a quantum topological vector space $\mathcal{X}_{\mathfrak{g}}(\widetilde{Y};m,e)$ for each $\sigma=(m,e)\in H^2(\widetilde{Y};\pi)\times H^1(\widetilde{Y};\pi)^\vee$, which is consistent with (4.12).

We conclude with three brief remarks. The first is in [W1, §4.1]. The latter two explain why the two ways we can imagine extracting an absolute quantum field theory from the relative theory $\mathcal{X}_{\mathfrak{g}}$ fail.

Remark 5.10 (Reduction to four dimensions)*.* Compactify on $S^1\times S^1$ to obtain a four-dimensional relative gauge theory. For a closed oriented 4-manifold there are natural polarizations of $H^3(S^1\times S^1\times\widetilde{X})$ corresponding to primitive elements of $H^1(S^1\times S^1;\mathbb{Z})$. The obvious basis elements are exchanged by $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ acting on the torus, which is implemented on the Heisenberg representation as Fourier transform. Using (4.14) to obtain ordinary gauge theories for groups G/π', this Fourier transform exchanges π' and its orthogonal complement $\pi'^{\perp}$ relative to the perfect pairing (5.2). The groups G/π' and $G/\pi'^{\perp}$ are Langlands dual.

Remark 5.11*.* We passed from relative σ-models and relative gauge theories to absolute theories by summing over a subgroup $\pi'<\pi$. The analog in the self-dual situation involves a *maximal* isotropic subgroup ($\pi'^{\perp}=\pi'$), and these generally do not exist.

Remark 5.12*.* In two dimensions one tensors chiral and anti-chiral relative conformal field theories to obtain an absolute conformal field theory. This works with supersymmetry since the vector representation is reducible in two dimensions. Thus the tensor product of a theory with $(p,0)$-supersymmetry and one with $(0,q)$-supersymmetry has (p,q)-supersymmetry. But in six dimensions the vector representation is irreducible, and the tensor product of theories with $(p,0)$-supersymmetry and $(0,q)$-supersymmetry has a symmetry group which contains two copies of the ordinary Poincaré group. The diagonal subgroup does not admit an extension to a super Poincaré group which is a symmetry of the theory. In short, the tensor product has no supersymmetry.

Appendix A. What is a classical field?

A “scalar field” or “gauge field” is not specific to a particular manifold, but rather is defined for all manifolds X. Furthermore, fields pull back under maps $X'\to X$ and they are *local*: compatible fields on open sets glue uniquely into a field on the union. The geometric object which encodes this locality is a *sheaf*. However, it is not a sheaf defined only on open subsets of a fixed space, but rather a sheaf $\mathcal{F}$ which one evaluates on smooth manifolds. The value $\mathcal{F}(X)$ on a smooth manifold X is a *set* for a scalar field—the set of real-valued functions on X—but for a gauge field

$\mathcal{F}(X)$ is the *groupoid* of connections on X. A leisurely introduction to sheaves in this context may be found in [FH].

Some fields—scalar fields, gauge fields—pull back under arbitrary smooth maps of manifolds, and the manifolds can have any dimension. Others, such as metrics, require both that the manifolds be of the same dimension and that the map be a local diffeomorphism.[19] Therefore, we are led to the following definition of the domain category for classical fields in an n-dimensional field theory.

Definition A.1. For each integer $n \geq 0$ define $\mathbf{Man}_n$ as the category whose objects are smooth n-manifolds and morphisms are local diffeomorphisms.

We have already mentioned that G-connections on a manifold X form a groupoid $B_\nabla G(X)$. In this paper we encounter fields which form a higher groupoid: $B^2\pi(X)$ in §4 is a 2-groupoid and $B^3\pi(X)$ in §5 is a 3-groupoid. It is convenient to view all of these as having values in the category $\mathbf{Set}_\Delta$ of simplicial sets. For the purposes of this paper the reader need only be aware that we need some mathematical object which tracks internal symmetries of fields, symmetries of symmetries, etc.

This discussion is summarized in a succinct statement.

Definition A.2. A classical field, or collection of classical fields, in an n-dimensional field theory is a simplicial sheaf $\mathcal{F}\colon \mathbf{Man}_n^{op} \to \mathbf{Set}_\Delta$.

Any homomorphism $\mathcal{F}\colon \mathbf{Man}_n^{op} \to \mathbf{Set}_\Delta$ is called a *presheaf*; the sheaf condition, which we do not spell out here (see [FH, §5]), expresses the locality with respect to gluing on open covers. As mentioned, some fields—including all those encountered in this paper—extend to a sheaf on the category of all smooth manifolds and smooth maps. A map of fields, such as (3.11) and (4.15), is defined as a natural transformation of functors.

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