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**THERMAL EXPANSION OF
HETEROGENEOUS MATERIALS
WITH APPLICATION TO
LOW TEMPERATURE ENVIRONMENTS**

BY

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Thermal Expansion of Heterogeneous Materials with Application to Low Temperature Environments

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Abstract

In this paper we present direct formulations of the effective thermal linear expansion (TLE) of heterogeneous materials or multi-constituent composites with temperature dependent constituent properties and with an arbitrary initial incompatible eigenstrain distribution. The effective properties are expressed in terms of the stress and strain concentrators. The effective coefficient of thermal linear expansion (CTLE) is then defined as the temperature derivative of the effective TLE. The development is based on the linear theory of uncoupled thermo-elasticity. For a special class of bi-constituent composites, the effective thermal expansion is expressed in terms of the effective elastic properties rather than the concentrators. An example is presented for niobium (Nb) fibers embedded in a copper (Cu) matrix at cryogenic temperatures. It is shown that this composite achieves negative CTLE despite the fact that the CTLE of both Nb and Cu are strictly greater than zero. In addition, it is shown that the presence of an initial field of incompatible eigenstrains is capable of causing anisotropic thermal expansion coefficients in an otherwise macroscopically isotropic material of isotropic constituents. Due to the form equivalence of the governing equations the developments which are presented are also applicable to the area of moisture swelling.

I Introduction

One of the first scientific investigations of thermal expansion concerned the keeping of accurate time through the use of pendulums. The linearized approximation to the period of a pendulum is proportional to the square root of the length of the pendulum. Thus dimensional changes in the length of the pendulum due to thermal fluctuations effect the accuracy of the instrument. The first experiments concerning these issues were performed by Musschenbrock (Desaguliers, 1745, pp. 436-46) in the early 18-th century.

The two-and-a-half centuries since the work of Musschenbrock have not lessened the demands on engineers in dealing with thermal expansion. One needs only to consider the \$2.5 billion Hubble Space Telescope (HST) deployed into Earth orbit in April of 1990. The pointing accuracy of the telescope was severely affected by thermally induced vibrations in the solar panels every time the telescope passed into or out of the Earth's shadow (Burrows *et al*, 1991). The problem was corrected during a repair mission of the Space Shuttle in December of 1993; the solar panels were replaced.

As with the pendulum, thermal expansion design issues are not exclusive to exotic systems like the HST. Thermal expansion has long been a concern of the bridge and aircraft engineers to name just two

professions. However, it is the extreme environments that many of today's high-technology systems are being asked to endure that gives rise to the research presented in this paper. Technological interest in the areas of superconductivity (Vidali, 1993) and functionally graded materials (FGMs) (Cherradi, 1994) are two such areas. Interest in FGMs, which are materials that are both microscopically and macroscopically heterogeneous, was born from experimental work that demonstrated their admirable performance when subjected to severe thermal gradients.

In the near totality of previous studies, interest in thermal response of heterogeneous or composite materials has been limited to the coefficient tensor of thermal linear expansion. This is defined as the temperature derivative of the thermal linear expansion, and is thus appropriate when the temperature interval is sufficiently small and the thermal expansion varies linearly with temperature. In general, however, thermal expansion of a material depends non-linearly on the temperature. Figure 1 illustrates this non-linear temperature dependence for polycrystalline copper and niobium. Thus it is advantageous from a theoretical standpoint to initially consider the effective thermal expansion. Specifically, this paper concerns the effective thermal *linear* expansion (TLE) since a linearized strain measure is used.

As a point of clarification we note that this paper is not concerned with efforts at predicting the thermal expansion of homogeneous materials through the use of anharmonic oscillators and lattice dynamics. Rather we assume the properties of the homogeneous materials to be known and we concern ourselves with the prediction of the effective TLE properties of heterogeneous materials or composites. We do not consider phase transformations of the constituents nor chemical interactions amongst the constituents. Expressions for the effective coefficients of thermal linear expansion (CTLE) are then obtained, by definition, by taking the temperature derivative of the effective TLE.

The earliest theory of effective CTLE appears to be that of Guertler (1922, as cited by Hughes and Brittain (1964)) for two-phase alloys. Guertler specifically proposed the volume weighted average of the constituent CTLE, i.e., the rule of mixtures. Turner (1946), alternatively, arrived at an expression that can be viewed of as the bulk modulus weighted average of the isotropic constituents' CTLE. More mechanically based theories then emerged in the works of Kerner (1956), Hughes and Brittain (1964), Levin (1967), Schapery (1968) and Rosen and Hashin (1970).

In the experimental literature on thermal expansion there exists data which are not supported by theory. Lototskaya *et al* (1989) observed negative coefficients of thermal expansion of a Cu composite with Ti-Nb filaments at cryogenic temperatures in apparant contradiction to the data supporting that the Cu and Ti-Nb individually have positive coefficients of thermal expansion at the same temperatures. Finlayson *et al* (1981) observed anisotropic thermal expansion in composites that to the best of their knowledge were isotropic in all of their properties. We do not attempt to explain fully the behavior observed in these experiments—since sufficient data is not available—but rather demonstrate that the theory of thermal expansion developed herein is consistent with the observed behavior.

The developments presented in this paper are not limited to the thermomechanical field, but actually address several phenomena that are governed by equations analogous to those of thermomechanics. One such phenomena is moisture swelling (Hashin, 1983). This area is also of technological interest. For an example we return to the HST. During its service to date, there have been numerous mirror adjustments on the HST to correct for moisture desorption in the graphite epoxy metering truss (Hasan, Burrows and Schroeder, 1993).

This paper is organized in the following manner. Section 2 presents mathematical preliminaries concerning the basic governing equations. Section 3 prepares for later sections by presenting effective elastic properties. Section 4 presents the definitions for effective thermal expansion followed by section 5 which explicitly evaluates the definitions of section 4 in terms of strain and stress concentrators. Section 6 then specializes the results of section 5 for a specific class of bi-constituent composites. Section 7 presents an example.

2 Preliminaries

The thermomechanical behavior of a physical body \mathcal{P} is modeled in this paper by application of the theory of continuum uncoupled thermoelasticity (Boley and Weiner, 1985). This assumes the identification of \mathcal{P} with a region $\mathcal{B} \subseteq \mathbb{R}^3$, with boundary $\partial\mathcal{B}$ on which there are assigned tensor fields that characterize physical properties (e.g., thermal expansion, elasticity) and field variables (e.g., temperature, strain). In general,

unless otherwise explicitly stated, all tensor fields are assumed to be spatially variable or, in other words, non-homogeneous, and in yet other words, dependent on \mathbf{x} where \mathbf{x} is an element of the domain on which the tensor field is defined. A tensor field is said to be homogeneous if it is not spatially variable. A domain is said to be homogeneous if the constitutive relations (which are tensor fields) defined on that domain are homogeneous. Boundary conditions are said to be homogenous if the same boundary conditions when applied to a homogenous domain result in homogeneous solution fields.

The governing equations of the thermomechanical model are presented below. For our purposes we shall restrict our attention to simply connected domains and to time-invariant problems with no heat sources or body forces. Since the theory is uncoupled in the manner mentioned we shall begin by presenting the theory of heat conduction followed by the theory of thermo-elasticity.

Heat conduction through the physical body is modeled by

$$\operatorname{div} \mathbf{q} = 0 \quad \text{in } \mathcal{B} \quad (1)$$

and

$$\operatorname{curl} \mathbf{H} = 0 \quad \text{in } \mathcal{B} \quad \Leftrightarrow \quad \mathbf{H} = -\operatorname{grad} \theta \quad \text{in } \mathcal{B} \quad (2)$$

where θ is the scalar temperature field, \mathbf{q} is the heat flux density and \mathbf{H} , as defined above in equ (2), is the negative of the temperature gradient. The symbols div , grad and curl denote the divergence, gradient and curl operators, respectively. The vector fields \mathbf{q} and \mathbf{H} are herein taken to be related through the Fourier law: $\mathbf{q} = \kappa \mathbf{H}$, where κ is the second-rank thermal conductivity tensor. Let the boundary $\partial \mathcal{B}$ be partitioned¹ into $\{\partial \mathcal{B}_\theta, \partial \mathcal{B}_q\}$ where the boundary conditions on $\partial \mathcal{B}_\theta$ and $\partial \mathcal{B}_q$ are of the Dirichlet and Neumann type, respectively. That is,

$$\theta = \hat{\theta} \quad \text{on } \partial \mathcal{B}_\theta \quad (3)$$

$$q_n \equiv \mathbf{q} \cdot \mathbf{n} = \hat{q}_n \quad \text{on } \partial \mathcal{B}_q \quad (4)$$

where \mathbf{n} is the outward unit normal and $\hat{\theta}$ and \hat{q}_n denote prescribed scalar fields. The boundary conditions are homogeneous if either $\hat{\theta} = \mathbf{H}^\circ \cdot \mathbf{x} + \theta^\circ$ on $\partial \mathcal{B}$ or $\hat{q}_n = \mathbf{q}^\circ \cdot \mathbf{n}$ on $\partial \mathcal{B}$ where $\mathbf{x} \in \mathcal{B}$ and where \mathbf{H}° , \mathbf{q}° and θ° are homogeneous tensor fields of rank 1, 1 and 0, respectively.

We now consider the thermo-mechanical response of the physical body. Let \mathbf{u} denote the displacement field over the domain \mathcal{B} . On the assumption of infinitesimal displacement gradients, the strain field ϵ is given by $\epsilon = \operatorname{grad}^S[\mathbf{u}]$ where $2\operatorname{grad}^S[\mathbf{u}] := \operatorname{grad}[\mathbf{u}] + \operatorname{grad}^T[\mathbf{u}]$. A superscript T denotes the transpose operator. The strain field is assumed to be additively decomposed into two parts: $\epsilon = \epsilon^e + \epsilon^*$. The second part ϵ^* is termed the eigenstrain (Mura, 1987) (or, non-elastic strain, or stress-free strain). Examples of such eigenstrains are thermal expansion strains, plastic strains, moisture swelling strains, transformation strains etc. The first part ϵ^e is termed the elastic strain and is taken to be related to the stress field σ through a constitutive relation

$$\sigma = \mathbf{C} : (\epsilon - \epsilon^*) \quad (5)$$

where the stiffness tensor $\mathbf{C} = \mathbf{C}(\theta, \mathbf{x})$ is taken to be a function of temperature. In many practical applications it is a reasonable assumption to take the elastic properties as being temperature independent. For this reason, the implications of this simplifying assumption on the theory of thermal expansion presented in this paper will be addressed. However, there are instances, as will be presented in section 7, where the effects of the temperature dependence of the elastic constitutive relation is significant.

For the present development we shall restrict our attention to two sources of eigenstrains. It is a relatively simple matter to include additional sources of eigenstrain in a more general analysis. For our purposes, however, it is sufficient to consider only thermal expansion and an initial eigenstrain field. The thermal expansion incurred on the domain \mathcal{B} by a change in temperature from θ_r to θ will be denoted by $\mathbf{Y} = \mathbf{Y}(\theta, \theta_r, \mathbf{x})$. The initial eigenstrain field is denoted by ϵ_r^* and it quantifies the the physical body's state of incompatible strains in its initial or reference state. In a physical body a set of incompatible strains produces what is commonly called a residual stress field. It is common in mathematical models of physical systems to make the approximation that in the initial state of the domain $\epsilon_r^* = \mathbf{0}$. This implies that the

¹Let \mathcal{I} and \mathcal{P} denote sets and suppose that for each $i \in \mathcal{I}$ there is associated a subset of \mathcal{P} denoted by \mathcal{P}_i . The family of the sets \mathcal{P}_i , denoted by $\{\mathcal{P}_i\}$, is termed a partition of \mathcal{P} iff (i) $\bigcup_{i \in \mathcal{I}} \mathcal{P}_i = \mathcal{P}$ and (ii) $\forall i, j \in \mathcal{I}, \mathcal{P}_i \cap \mathcal{P}_j = \emptyset$ for $i \neq j$. The set \mathcal{I} is termed the index set of the partition.

initial stress and strain fields are also zero. The physical body, however, may not possess such a stress-free strain-free state thus implying that the initial eigenstrain field is non-zero. As will be shown in the sections that follow the effective thermal expansion of a body is a function of this initial eigenstrain field when the temperature dependence of the elastic constitutive relations are taken into account. It will be shown that the effective thermal expansion is independent of the initial eigenstrain field when the assumption of temperature independence of the elastic constitutive relation is invoked.

The presentation of governing equations for the thermo-elastic problem is completed with the equilibrium equation

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0} \quad \text{in } \mathcal{B} \quad (6)$$

and boundary conditions. To present the boundary conditions let the boundary $\partial \mathcal{B}$ be partitioned into $\{\partial \mathcal{B}_u, \partial \mathcal{B}_t\}$ where the boundary conditions on $\partial \mathcal{B}_u$ and $\partial \mathcal{B}_t$ are of the Dirichlet and Neumann type, respectively. That is,

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \partial \mathcal{B}_u \quad (7)$$

$$\mathbf{t}_n := \boldsymbol{\sigma} \mathbf{n} = \hat{\mathbf{t}}_n \quad \text{on } \partial \mathcal{B}_t \quad (8)$$

where $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}_n$ denote prescribed tensor fields. The boundary conditions are homogeneous if either $\hat{\mathbf{u}} = \boldsymbol{\epsilon}^\circ \mathbf{x} + \mathbf{u}^\circ$ on $\partial \mathcal{B}_u$ or $\hat{\mathbf{t}}_n = \boldsymbol{\sigma}^\circ \mathbf{n}$ on $\partial \mathcal{B}_t$ where $\boldsymbol{\epsilon}^\circ$ and $\boldsymbol{\sigma}^\circ$ are both homogeneous tensor fields of rank 2. The vector field \mathbf{u}° is a vector field consistent with a rigid body translation of the domain. Since \mathbf{u}° does not result in any strains we can conveniently take $\mathbf{u}^\circ = \mathbf{0}$ without any loss in generality with respect to the resulting strain field. Therefore, in what follows, homogeneous displacement boundary conditions will be taken in the form $\hat{\mathbf{u}} = \boldsymbol{\epsilon}^\circ \mathbf{x}$.

This concludes the presentation of the governing equations. In the following section the definitions of a representative volume element and effective elastic properties are introduced. The effective elastic properties are then evaluated using the direct approach.

3 Effective Elastic Properties and the Direct Approach

Prior to discussing the effective thermal expansion we deem it convenient to introduce the concepts of a representative volume element (RVE) and effective elastic properties of an RVE. The direct approach of Hill is then presented to yield expressions for the effective elastic properties in terms of quantities called strain and stress concentrators.

Throughout this paper we concern ourselves with the effective properties of a microscopically heterogeneous or composite material. We assume that on a sufficiently large scale the composite material may be regarded as being macroscopically homogeneous. The volume of composite material required to achieve this ideal of macroscopic homogeneity is termed the representative volume element (RVE). For a further discussion of RVEs the reader is referred to Hashin (1983).

In the realm of effective property calculations the spatial average of specific tensorial quantities is required over the domain of interest. We shall denote the spatial average of an arbitrary quantity \mathbf{T} defined on \mathcal{R} by $\langle \mathbf{T} \rangle_{\mathcal{R}}$ where by definition

$$\langle \cdot \rangle_{\mathcal{R}} := \frac{1}{V_{\mathcal{R}}} \int_{\mathcal{R}} (\cdot) d\mathbf{x} \quad (9)$$

and $V_{\mathcal{R}} := \int_{\mathcal{R}} d\mathbf{x}$ is the volume occupied by the domain \mathcal{R} . When the domain of definition \mathcal{R} is unambiguous the subscript \mathcal{R} will be dropped from the notation.

Given an RVE subjected to homogeneous boundary conditions the effective elastic properties are defined in the following way. The effective stiffness \mathbf{C}^* is defined as the quantity that relates the average stress $\langle \boldsymbol{\sigma} \rangle$ to the average strain $\langle \boldsymbol{\epsilon} \rangle$ through the relation

$$\langle \boldsymbol{\sigma} \rangle = \mathbf{C}^* : \langle \boldsymbol{\epsilon} \rangle. \quad (10)$$

Similarly, the effective compliance \mathbf{S}^* is defined as the quantity that relates the average strain $\langle \boldsymbol{\epsilon} \rangle$ to the average stress $\langle \boldsymbol{\sigma} \rangle$ through the relation

$$\langle \boldsymbol{\epsilon} \rangle = \mathbf{S}^* : \langle \boldsymbol{\sigma} \rangle. \quad (11)$$

If we now assume a special structure to the elastic constitutive relation \mathbf{C} it is possible to develop explicit expressions for the effective elastic properties. We proceed therefore by restricting our attention to physical bodies comprised of a number of distinct homogeneous materials. That is to say that the constitutive relation \mathbf{C} defined on the domain \mathcal{R} is “piecewise” homogeneous. We now require a partition $\{\mathcal{R}_i\}$ with index set \mathcal{I} of the domain \mathcal{R} such that for $i \in \mathcal{I}$ the constitutive relation \mathbf{C} restricted to the domain \mathcal{R}_i , herein to be denoted by \mathbf{C}^i is homogeneous. A convenient way of denoting this statement mathematically is to write $\mathbf{Y}^i = \mathbf{Y} \mid \mathcal{R}_i$ which may be read as \mathbf{Y}^i equals \mathbf{Y} restricted to the domain \mathcal{R}_i . We shall encounter this notation again.

In general, the index set \mathcal{I} may be uncountable. In particular, there may exist situations where a sum over a subset of \mathcal{I} may be replaced by an integral. Such a situation will occur in section 6 where orientational averaging will be utilized.

The direct approach originated with Hill (1963). The features of Hill’s approach required here are those of the concept of the strain and stress concentrators. When a domain is subjected to homogeneous displacement boundary conditions—quantified by the homogeneous field ϵ° —the equilibrium strain field ϵ is uniquely determined by and linearly related to the homogeneous field ϵ° :

$$\epsilon = \tilde{\mathbf{A}} : \epsilon^\circ \quad (12)$$

where $\tilde{\mathbf{A}}$ is termed the strain concentrator. Since $\langle \epsilon \rangle = \epsilon^\circ$ it follows from eqn (12) that $\langle \tilde{\mathbf{A}} \rangle = \mathbf{I}$ where \mathbf{I} is the fourth rank symmetric identity tensor. Similarly, when a domain is subjected to homogeneous traction boundary conditions—quantified by the homogeneous field σ° —the resulting stress field σ is uniquely determined by and linearly related to the homogeneous field σ° :

$$\sigma = \tilde{\mathbf{B}} : \sigma^\circ \quad (13)$$

where $\tilde{\mathbf{B}}$ is termed the stress concentrator. Since $\langle \sigma \rangle = \sigma^\circ$ it follows from eqn (13) that $\langle \tilde{\mathbf{B}} \rangle = \mathbf{I}$. For notational convenience define $\mathbf{A}^i := \langle \tilde{\mathbf{A}} \rangle_{\mathcal{R}_i}$ and $\mathbf{B}^i := \langle \tilde{\mathbf{B}} \rangle_{\mathcal{R}_i}$ which are the volume averages of the concentrators over the subdomain \mathcal{R}_i of the RVE \mathcal{R} .

The identities $\langle \tilde{\mathbf{A}} \rangle = \mathbf{I}$ and $\langle \tilde{\mathbf{B}} \rangle = \mathbf{I}$ can be rewritten as

$$\sum_{i \in \mathcal{I}} c^i \mathbf{A}^i = \mathbf{I} \quad (14)$$

and

$$\sum_{i \in \mathcal{I}} c^i \mathbf{B}^i = \mathbf{I}, \quad (15)$$

respectively, where $c^i := V_{\mathcal{R}_i}/V_{\mathcal{R}}$ is the volume fraction of the i -th subdomain \mathcal{R}_i . Imposing homogeneous displacement boundary conditions and using the volume averaged strain concentrators, it follows from the defining relation (10) that the effective stiffness of the RVE can be expressed as

$$\mathbf{C}^* = \sum_{i \in \mathcal{I}} c^i \mathbf{C}^i : \mathbf{A}^i. \quad (16)$$

Similarly, imposing homogeneous traction boundary conditions and using the volume averaged stress concentrators it follows that the effective compliance of the RVE can be expressed as

$$\mathbf{S}^* = \sum_{i \in \mathcal{I}} c^i \mathbf{S}^i : \mathbf{B}^i. \quad (17)$$

From the defining relations (10) and (11) it is clear that the effective properties are inverses of one another: $\mathbf{C}^* : \mathbf{S}^* = \mathbf{I}$.

In section 5 we shall return to the strain and stress concentrators presented here to evaluate explicit expressions for the effective thermal expansion of an RVE. Before doing so we must define the effective thermal expansion. We do this in the following section.

4 Definitions of Effective Thermal Expansion

Thermal expansion of a homogeneous material is by definition related to the difference in the two states of strain after the material is subjected to a uniform change in temperature. Thermal expansion may be qualitatively described in the following manner for the majority of materials. An increase in temperature typically has the effect of causing the material to expand (a net “positive” strain) while a decrease in temperature causes a contraction (a net “negative” strain) of the material. This is not a universal behavior since some materials contract along particular directions when subjected to an *increase* in temperature. Graphite is such a material. These two states will be referred to as the reference and current states. Since the reference state can be construed as a previous current state it will be convenient to discuss some elements common to both states first. The particular differences between the two states will then be discussed.

The following construction is common to both the current and reference states. Consider a RVE \mathcal{R} with boundary $\partial\mathcal{R}$. In regards to the presentation of the previous section we can take the governing equations to apply to the RVE by taking $\mathcal{B} = \mathcal{R}$. We assume there to be no heat sources or body forces. The thermal boundary condition is homogeneous with a uniform temperature θ° , (i.e., $\hat{\theta} = \theta^\circ$ on $\partial\mathcal{R}$). The domain \mathcal{R} is loaded by a general eigenstrain field ϵ^* . The mechanical boundary conditions are also homogeneous. As was alluded to above two different but equivalent definitions for the effective thermal expansion will be presented. These two definitions result from considering the two possible types of homogeneous mechanical boundary conditions:

- **Case A** Zero-displacement boundary condition: $\hat{\mathbf{u}} = \mathbf{0}$ on $\partial\mathcal{B}$.
- **Case B** Zero-traction boundary condition: $\hat{\mathbf{t}}_n = \mathbf{0}$ on $\partial\mathcal{B}$.

The equilibrium solution to both of the above boundary value problems will be denoted by the ordered pair (θ, \mathbf{u}) . Notation will not be introduced to distinguish between the equilibrium solutions of Cases A and B since it will be clear from the context in what follows which solution is intended.

We are now interested in the equilibrium solution to the thermo-mechanical problem of the RVE. Since the temperature boundary condition is uniform, the temperature field is given by $\theta = \theta^\circ$ regardless of the thermal conductivity field κ . The solution of the thermo-elastic problem, i.e. \mathbf{u} , is, in general, non-trivial due to the eigenstrain field ϵ^* . If, however, $\epsilon^* = \mathbf{0}$ then $\mathbf{u} = \mathbf{0}$. Thus, $\epsilon = \mathbf{0}$ and $\sigma = \mathbf{0}$ and the state is strain- and stress-free. Conversely, if the state is both strain- and stress-free then $\epsilon^* = \mathbf{0}$ due to the uniqueness of solution. We shall return to this point when we discuss the reference state below.

The reference or initial state will be quantified by the ordered pair of equilibrium solutions (θ_r, \mathbf{u}_r) when the uniform temperature boundary condition is $\theta^\circ = \theta_r^\circ$ and the eigenstrain field is $\epsilon^* = \epsilon_r^*$. The initial eigenstrain field ϵ_r^* models the initial state of incompatible strains in the physical body. This field is assumed to be known and is independent of the current temperature. The reference state is typically taken to be strain- and stress-free which, as stated above, implies that $\epsilon_r^* = \mathbf{0}$. In the remainder of this paper a subscript r on any quantity will denote that quantity in the reference state. For example, σ_r and ϵ_r denote the stress and strain fields, respectively, in the reference state.

The current state will be quantified by the ordered pair of equilibrium solutions (θ, \mathbf{u}) when the uniform temperature boundary condition is $\hat{\theta} = \theta^\circ$ and the eigenstrain field is $\epsilon^* = \mathbf{Y} + \epsilon_r^*$ where \mathbf{Y} is the field of local thermal expansion for the change in temperature from θ_r° to θ° . In other words, upon altering the temperature from θ_r° to θ° it is assumed that the only eigenstrains induced during the change in temperature result from thermal expansion.

In the following two subsections, the definitions for effective thermal expansion are presented for the two cases of homogeneous mechanical boundary conditions presented above. These cases have been labeled above by Case A and Case B. In both cases the definitions for effective thermal expansion are guided by those definitions for thermal expansion of homogeneous materials.

Case A: Zero-displacement boundary condition

We consider here the case of homogeneous, zero-displacement mechanical boundary conditions, i.e. $\hat{\mathbf{u}} = \mathbf{0}$ on $\partial\mathcal{R}$. The effective thermal expansion \mathbf{Y}^* of the RVE as its uniform temperature is altered from θ_r° to θ°

is defined as

$$\mathbf{Y}^* := \langle -\boldsymbol{\sigma} \rangle : \mathbf{S}^* - \langle -\boldsymbol{\sigma}_r \rangle : \mathbf{S}_r^* \quad (18)$$

where \mathbf{S}^* (\mathbf{S}_r^*) is the effective compliance of the RVE in the current (reference) state. To motivate this definition recall that a strain- and stress-free homogeneous domain with homogeneous zero-displacement boundary conditions and temperature independent elastic constitutive relation when subjected to a temperature change from θ_r° to θ° , thus incurring a homogeneous thermal expansion strain of \mathbf{Y} , also incurs a homogeneous stress field $\boldsymbol{\sigma} = -\mathbf{C} : \mathbf{Y} = -\mathbf{Y} : \mathbf{C}$. Solving this last equation for the thermal expansion yields $\mathbf{Y} = -\boldsymbol{\sigma} : \mathbf{S}$. If we now assume that the thermal and mechanical constitutive relations are non-homogeneous then this last equation is still valid with the difference that all three quantities are non-homogeneous. We define the effective thermal expansion by the volume average of the field \mathbf{Y} . This leads to $\mathbf{Y}^* := \langle \mathbf{Y} \rangle = \langle -\boldsymbol{\sigma} : \mathbf{S} \rangle = \langle -\boldsymbol{\sigma} \rangle : \mathbf{S}^*$. Equation (18) follows by allowing for a non-zero initial eigenstrain field and temperature dependence of the elastic constitutive relation.

The effective thermal expansion (18) of a RVE is dependent on the initial eigenstrain field $\boldsymbol{\epsilon}_r^*$ since both $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_r$ are implicitly functions of the initial eigenstrain field. Due to the linearity of the governing equations, the effective elastic properties are independent of the initial eigenstrain field. There are, however, at least two sufficient conditions which render the current theory for the effective thermal expansion independent of the initial eigenstrain field. These two sufficient conditions are discussed next.

The first sufficient condition is homogeneity of the elastic properties. If \mathbf{S} is homogeneous then $\mathbf{S}^* = \mathbf{S}$ and the definition (18) reduces as

$$\mathbf{Y}^* := \langle -\boldsymbol{\sigma} \rangle : \mathbf{S}^* - \langle -\boldsymbol{\sigma}_r \rangle : \mathbf{S}_r^* \quad (19)$$

$$= -\langle (\boldsymbol{\epsilon} - \mathbf{Y} - \boldsymbol{\epsilon}_r^*) : \mathbf{C} \rangle : \mathbf{S} + \langle (\boldsymbol{\epsilon}_r - \boldsymbol{\epsilon}_r^*) : \mathbf{C}_r \rangle : \mathbf{S}_r \quad (20)$$

$$= -\langle \boldsymbol{\epsilon} - \mathbf{Y} - \boldsymbol{\epsilon}_r^* \rangle + \langle \boldsymbol{\epsilon}_r - \boldsymbol{\epsilon}_r^* \rangle \quad (21)$$

$$= -\langle \boldsymbol{\epsilon} \rangle + \langle \mathbf{Y} \rangle + \langle \boldsymbol{\epsilon}_r^* \rangle + \langle \boldsymbol{\epsilon}_r \rangle - \langle \boldsymbol{\epsilon}_r^* \rangle \quad (22)$$

$$= \langle \mathbf{Y} \rangle. \quad (23)$$

To obtain eqn (23) from eqn (22) recall that the homogeneous displacement boundary condition is $\hat{\mathbf{u}} = \mathbf{0}$, i.e. $\boldsymbol{\epsilon}^\circ = \mathbf{0}$, thus $\langle \boldsymbol{\epsilon} \rangle = \boldsymbol{\epsilon}^\circ = \mathbf{0}$ and, likewise, $\langle \boldsymbol{\epsilon}_r \rangle = \mathbf{0}$. Thus, if the elastic properties of the RVE are homogeneous then the effective thermal expansion is independent of the initial eigenstrain field. Furthermore, if the thermal expansion \mathbf{Y} of the RVE is also homogeneous (in addition to the homogeneity of \mathbf{C}) then $\mathbf{Y}^* = \mathbf{Y}$. Note that homogeneity of \mathbf{Y} does *not* in and of itself imply that $\mathbf{Y}^* = \mathbf{Y}$.

The second sufficient condition to ensure independence of the effective thermal expansion with respect to the initial eigenstrain field is temperature independence of both the elastic properties \mathbf{C} and the effective elastic properties \mathbf{C}^* . By temperature independence of the elastic properties we mean that $\dot{\mathbf{C}} = \mathbf{0}$ and $\dot{\mathbf{C}}^* = \mathbf{0}$ for all possible current temperatures θ° where the over-dot notation denotes differentiation with respect to the current temperature θ° . That is, $(\dot{\cdot}) \equiv (\cdot) := d(\cdot)/d\theta^\circ$. We shall frequently encounter occasions, as in the previous sentence, when the over-dot notation must be applied to a quantity which has superscripts and/or subscripts, i.e., $\dot{\mathbf{C}}^*$. For notational simplicity we note that $\dot{\mathbf{C}}^* \equiv (\mathbf{C}^*)$. Note that if $\dot{\mathbf{S}} = \mathbf{0}$ for all θ° then $\mathbf{S}_r = \mathbf{S}$; similarly, if $\dot{\mathbf{S}}^* = \mathbf{0}$ for all θ° then $\mathbf{S}_r^* = \mathbf{S}^*$. The proof proceeds from definition (18) as follows:

$$\mathbf{Y}^* := \langle -\boldsymbol{\sigma} \rangle : \mathbf{S}^* - \langle -\boldsymbol{\sigma}_r \rangle : \mathbf{S}_r^* \quad (24)$$

$$= \langle \boldsymbol{\sigma}_r - \boldsymbol{\sigma} \rangle : \mathbf{S}^* \quad (25)$$

$$= \langle (\boldsymbol{\epsilon}_r - \boldsymbol{\epsilon}_r^*) : \mathbf{C} - (\boldsymbol{\epsilon} - \mathbf{Y} - \boldsymbol{\epsilon}_r^*) : \mathbf{C} \rangle : \mathbf{S}^* \quad (26)$$

$$= \langle (\boldsymbol{\epsilon}_r - \boldsymbol{\epsilon} + \mathbf{Y}) : \mathbf{C} \rangle : \mathbf{S}^*. \quad (27)$$

This does not conclude the proof since both $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}_r$ are implicitly functions of the initial eigenstrain field. We now show, however, that the right-hand-side of eqn (27) as a whole is independent of $\boldsymbol{\epsilon}_r^*$. To proceed it is convenient to introduce some auxiliary notation. Let $\bar{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^*)$ denote the equilibrium strain field of the domain loaded by the eigenstrain field $\boldsymbol{\epsilon}^*$. Our usual notation may therefore be equivalently expressed as $\boldsymbol{\epsilon} \equiv \bar{\boldsymbol{\epsilon}}(\mathbf{Y} + \boldsymbol{\epsilon}_r^*)$ and $\boldsymbol{\epsilon}_r \equiv \bar{\boldsymbol{\epsilon}}_r(\boldsymbol{\epsilon}_r^*)$. Returning to eqn (27) we have

$$\mathbf{Y}^* = \langle [\bar{\boldsymbol{\epsilon}}_r(\boldsymbol{\epsilon}_r^*) - \bar{\boldsymbol{\epsilon}}(\mathbf{Y} + \boldsymbol{\epsilon}_r^*) + \mathbf{Y}] : \mathbf{C} \rangle : \mathbf{S}^* \quad (28)$$

$$= \langle [\bar{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}_r^*) - \bar{\boldsymbol{\epsilon}}(\mathbf{Y} + \boldsymbol{\epsilon}_r^*) + \mathbf{Y}] : \mathbf{C} \rangle : \mathbf{S}^* \quad (29)$$

$$= \langle [\mathbf{Y} - \bar{\boldsymbol{\epsilon}}(\mathbf{Y})] : \mathbf{C} \rangle : \mathbf{S}^*. \quad (30)$$

From eqn (30) it is clear that the effective thermal expansion is independent of the initial eigenstrain field. In obtaining eqn (29) from eqn (28) temperature independence of the elastic properties was utilized. Linearity of the governing equations provided the means of obtaining eqn (30) from eqn (29).

This concludes the presentation of the definition of effective thermal expansion utilizing homogeneous zero-displacement boundary conditions. In the following subsection effective thermal expansion is defined utilizing homogeneous traction free boundary conditions.

Case B: Zero-traction boundary condition

We consider here the case when the mechanical boundary conditions on the RVE are traction free, i.e., $\mathbf{t}_n = \hat{\mathbf{t}}_n = \mathbf{0}$ on $\partial\mathcal{R}$. The effective thermal expansion \mathbf{Y}^* of the RVE as its uniform temperature is altered from θ_r° to θ° is defined as

$$\mathbf{Y}^* := \langle \boldsymbol{\epsilon} \rangle - \langle \boldsymbol{\epsilon}_r \rangle. \quad (31)$$

Equation (31) is the difference in the volume averaged strain fields of the current and reference states. Definition (31) simplifies to the more familiar definition $\mathbf{Y}^* = \langle \boldsymbol{\epsilon} \rangle$ under the assumption that the reference state is strain- and stress-free, i.e. $\boldsymbol{\epsilon}_r^* = \mathbf{0}$.

As with Case A presented above, this definition (31) of effective thermal expansion is a function of the initial eigenstrain field. Proofs similar to those given for Case A can be constructed for Case B to yield the results that (i) homogeneous elastic properties and (ii) temperature independence of the elastic properties are both, independently, sufficient for the effective thermal expansion to be independent of the initial eigenstrain field. The proofs are omitted here for brevity.

We now define the effective coefficient of thermal expansion. Recall that if \mathbf{Y} is homogeneous thermal expansion then the homogeneous coefficient of thermal expansion $\boldsymbol{\alpha}$ is by definition the temperature derivative of the thermal expansion \mathbf{Y} . That is, $\boldsymbol{\alpha} := \dot{\mathbf{Y}}$. In an analogous manner, the effective thermal expansion $\boldsymbol{\alpha}^*$ is then defined as

$$\boldsymbol{\alpha}^* := \dot{\mathbf{Y}}^*. \quad (32)$$

In the next section we utilize the concentrators of section 3 to evaluate explicit expressions for the definitions of effective thermal expansion given above.

5 Effective Thermal Expansion

Prior to developing expressions for the effective thermal expansion we present a useful identity. Consider a domain B endowed with a stiffness tensor \mathbf{C} and no body forces. Let $\boldsymbol{\sigma}^I$ be a statically admissible stress field and let $\boldsymbol{\epsilon}^{II}$ be a kinematically admissible strain field. The fields $\boldsymbol{\sigma}^I$ and $\boldsymbol{\epsilon}^{II}$ are not necessarily related through a constitutive relation. If either $\boldsymbol{\sigma}^I$ or $\boldsymbol{\epsilon}^{II}$ is compatible with homogeneous boundary conditions then the following relation exists between volume averages of the admissible fields

$$\langle \boldsymbol{\sigma}^I : \boldsymbol{\epsilon}^{II} \rangle = \langle \boldsymbol{\sigma}^I \rangle : \langle \boldsymbol{\epsilon}^{II} \rangle. \quad (33)$$

If $\boldsymbol{\sigma}^I$ and $\boldsymbol{\epsilon}^{II}$ are related through the linear constitutive relation \mathbf{C} , then $\boldsymbol{\sigma}^I = \boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}^{II} = \boldsymbol{\epsilon}$ where $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are the solutions to the elasticity problem. The relation (33) then reads $\langle \boldsymbol{\sigma} : \boldsymbol{\epsilon} \rangle = \langle \boldsymbol{\sigma} \rangle : \langle \boldsymbol{\epsilon} \rangle$, which is sometimes known as the Hill condition (Bishop and Hill, 1951; Hill, 1963).

We now develop expressions for the effective thermal expansion for the mechanical boundary conditions of Case A and Case B presented above. To proceed, however, we assume that the partition $\{\mathcal{R}_i\}$ is such that for $i \in \mathcal{I}$ not only is $\mathbf{C}^i := \mathbf{C} | \mathcal{R}_i$ homogeneous but so too are $\mathbf{Y}^i := \mathbf{Y} | \mathcal{R}_i$ and $\boldsymbol{\epsilon}_r^{*i} := \boldsymbol{\epsilon}_r^* | \mathcal{R}_i$.

Case A: Zero-displacement boundary condition

Consider a typical RVE \mathcal{R} with no body forces or heat sources subjected to a uniform temperature boundary condition $\hat{\theta} = \theta^\circ$ where it is recalled that θ° is a homogeneous field. Let the ordered double $(\boldsymbol{\epsilon}^\circ, \boldsymbol{\epsilon}^*)$ whose elements denote the fields which fully quantify the homogeneous mechanical boundary conditions and loading

of the domain \mathcal{R} , respectively. Our goal is to develop an expression for $\langle -\sigma \rangle$ when the mechanical boundary condition is $\hat{\mathbf{u}} = \mathbf{0}$ and the domain is loaded by a piecewise homogeneous eigenstrain field ϵ^* . To this end, we can consider two problems: $A' \equiv (\epsilon^\circ, \mathbf{0})$ and $A'' \equiv (\mathbf{0}, \epsilon^*)$. In problem A' the tensor ϵ° is arbitrary. Note that the negative of the spatial average of the stress field associated with problem A'' is the quantity we desire. An expression for such a spatial average can be obtained using eqn (33) twice. First use $I = A'$ and $II = A''$ to deduce that $\langle \epsilon^{A'} : \mathbf{C} : \epsilon^{A''} \rangle = \mathbf{0}$. Then use $I = A''$ and $II = A'$, together with the previously obtained result, to arrive at the desired result:

$$\langle -\sigma \rangle = \left[\sum_{i \in \mathcal{I}} c^i \epsilon^{*i} : \mathbf{C}^i : \mathbf{A}^i \right]. \quad (34)$$

In deducing the above equation, the arbitrariness of ϵ° was utilized. Using the eqn (34) to evaluate eqn (18) yields

$$\mathbf{Y}^* = \left[\sum_{i \in \mathcal{I}} c^i (\mathbf{Y}^i + \epsilon_r^{*i}) : \mathbf{C}^i : \mathbf{A}^i \right] : \mathbf{S}^* - \left[\sum_{i \in \mathcal{I}} c^i \epsilon_r^{*i} : \mathbf{C}_r^i : \mathbf{A}_r^i \right] : \mathbf{S}_r^*. \quad (35)$$

The effective coefficient of thermal expansion is given by

$$\begin{aligned} \alpha^* &= \left[\sum_{i \in \mathcal{I}} c^i \left\{ \alpha^i : \mathbf{C}^i : \mathbf{A}^i + (\mathbf{Y}^i + \epsilon_r^{*i}) : (\mathbf{C}^i : \dot{\mathbf{A}}^i + \dot{\mathbf{C}}^i : \mathbf{A}^i) \right\} \right] : \mathbf{S}^* \\ &\quad - \left[\sum_{i \in \mathcal{I}} c^i (\mathbf{Y}^i + \epsilon_r^{*i}) : \mathbf{C}^i : \mathbf{A}^i \right] : \mathbf{S}^* : \dot{\mathbf{C}}^* : \mathbf{S}^* \end{aligned} \quad (36)$$

upon substituting eqn (35) into the definition (32).

There are many situations when it is valid to assume that the elastic properties \mathbf{C}^i are temperature independent. For this reason we now present the implications of such an assumption of the form of eqns (35) and (36). Temperature independence of the elastic properties implies that $\mathbf{C}_r^i = \mathbf{C}^i$ and $\dot{\mathbf{C}}^i = \mathbf{0}$. As a result of such an assumption, eqns (35) and (36) reduce, respectively, to

$$\mathbf{Y}^* = \left[\sum_{i \in \mathcal{I}} c^i \mathbf{Y}^i : \mathbf{C}^i : \mathbf{A}^i \right] : \mathbf{S}^* \quad (37)$$

$$\alpha^* = \left[\sum_{i \in \mathcal{I}} c^i \alpha^i : \mathbf{C}^i : \mathbf{A}^i \right] : \mathbf{S}^*. \quad (38)$$

Note that eqns (35)–(38) are expressed in terms of *both* strain and stress concentrators because of the presence of the effective compliance in these equations which is expressed in terms of the stress concentrators by eqn (17). In the exact theory this does not pose a problem since, as we stated above, the effective elastic properties are inverses of one another. However, in practice the exact concentrators, and thus the exact effective elastic properties, are not known and so recourse is made to approximate concentrators. This is where the issue of consistent approximate presents itself. For a choice of both strain and stress concentrators it is generally true that the corresponding effective properties are not inverses of one another. Therefore, what we mean by a consistent approximation is evaluating an effective property on the choice of either a set of strain concentrators or a set of stress concentrators but not both.

The equations given above, namely (35)–(38) are appropriate for evaluation for a given set of strain concentrators. In order to conform with a consistent approximation it is necessary to define the effective compliance in the following manner $\mathbf{S}^* := (\mathbf{C}^*)^{-1}$ where \mathbf{C}^* is given in terms of the strain concentrators \mathbf{A}^i by eqn (16). In the next subsection a similar set of equations to those of this subsection are developed which are applicable when a choice for a set of stress concentrators is employed. In that case it so happens that the issue of consistent approximate does not arise.

Equation (38) includes the theory of Turner (1946) as a sub case. Restrict all constitutive relations α^i and \mathbf{C}^i to be isotropic. The result of Turner (1946) then follows from the the approximation that $\mathbf{A}^i = \mathbf{I}$ for $\forall i \in \mathcal{I}$.

Case B: Zero-traction boundary condition

As with Case A above, consider a RVE \mathcal{R} with no body forces or heat sources subjected to a uniform temperature boundary condition $\hat{\theta} = \theta^\circ$. Let the ordered double $(\sigma^\circ, \epsilon^*)$ denote the fields which fully quantify both the homogeneous traction boundary condition and loading of the domain \mathcal{R} . The goal is to develop an expression for $\langle \epsilon \rangle$ when the boundary condition is traction free (i.e., $\hat{t}_n = \mathbf{0}$) and the domain is loaded by a piecewise homogeneous eigenstrain field ϵ^* . To this end, we consider two problems: $B' \equiv (\sigma^\circ, \mathbf{0})$ and $B'' \equiv (\mathbf{0}, \epsilon^*)$. In problem B' the tensor σ° is arbitrary. Note that the spatial average of the strain field associated with problem B'' is the quantity we desire. An expression for such a spatial average can be obtained using eqn (33) twice. First take $I = B''$ and $II = B'$ to deduce that $\langle \sigma^{B''} : \mathbf{S} : \sigma^{B'} \rangle = \mathbf{0}$. Then use $I = B'$ and $II = B''$, together with the previously obtained result, to arrive at the desired result:

$$\langle \epsilon \rangle = \sum_{i \in \mathcal{I}} c^i \epsilon^{*i} : \mathbf{B}^i. \quad (39)$$

In deducing the above equation, the arbitrariness of σ° was utilized. Using the above equation to evaluate definition (31) for the effective thermal expansion of the RVE \mathcal{R} yields

$$\mathbf{Y}^* = \sum_{i \in \mathcal{I}} c^i [(\mathbf{Y}^i + \epsilon_r^{*i}) : \mathbf{B}^i - \epsilon_r^{*i} : \mathbf{B}_r^i]. \quad (40)$$

The effective coefficient of thermal expansion is given by

$$\alpha^* = \sum_{i \in \mathcal{I}} c^i [\alpha^i : \mathbf{B}^i + \{\mathbf{Y}^i + \epsilon_r^{*i}\} : \dot{\mathbf{B}}^i] \quad (41)$$

upon substituting eqn (40) into the definition (32).

If the elastic properties are independent of temperature it follows that

$$\mathbf{Y}^* = \sum_{i \in \mathcal{I}} c^i \mathbf{Y}^i : \mathbf{B}^i \quad (42)$$

$$\alpha^* = \sum_{i \in \mathcal{I}} c^i \alpha^i : \mathbf{B}^i. \quad (43)$$

Equation (43) has been previously obtained by Rosen and Hashin (1970). The approximation of Guertler (1922, p.118) can be obtained from (43) by taking $\mathbf{B}^i = \mathbf{I}, \forall i \in \mathcal{I}$.

6 A Special Class of Bi-constituent Composites

In this section, the results of the previous section are specialized for a specific class of bi-constituent composites. The specific class being quantified by the sufficient conditions necessary to express the effective properties of the previous section in terms of the effective elastic properties rather than the concentrators. This procedure has been previously considered by Levin (1967) and Rosen and Hashin (1970) for a unidirectional bi-constituent composite. Rosen and Hashin's (1970) result for the effective coefficient of thermal expansion was conveniently expressed as

$$\alpha^* = \alpha^m + (\alpha^f - \alpha^m) : (\mathbf{S}^f - \mathbf{S}^m)^{-1} : (\mathbf{S}^* - \mathbf{S}^m). \quad (44)$$

A word of clarification is in order here so that the applicability of the above equation (44) is not misconstrued. When in a later review article (Hashin, 1983) it is stated that the above equation (44) is valid for the "most general kind of thermoelastic two-phase composite" the reader must *not* interpret "phase" as meaning "constituent." In short, the phrase "two-phase composite" is equivalent to what we have herein been referring to as a "uni-directional bi-constituent" composite. The reader is referred to Nadeau and Ferrari (1995) for a discussion on phases and their distinction from constituents.

Returning to the objective of this section, we shall implement the methodology which Rosen and Hashin (1970) used to arrive at eqn (44) to find similar results for the expressions presented in the previous section.

In doing so we shall show that the expression (44) is applicable to a wider class of bi-constituent composites other than uni-directional (or two-phase). To emphasize that the results of this section are *not* applicable to all bi-constituent composites we shall present a proof by counter-example.

Consider a bi-constituent composite in the form of identically shaped ellipsoidal inhomogeneities (or short-fibers; or fibers, for short) embedded in a matrix. The constitutive relations for the matrix material are represented by the usual symbology but with a superscript m to denote matrix, i.e. \mathbf{C}^m and \mathbf{Y}^m . A typical fiber is given a local coordinate frame \mathcal{K}' . The constitutive relations for the fiber material—distinguished by a superscript f —are such that their components with respect to the local fiber-fixed frame are the same for all fibers. The orientation of a specific fiber relative to the global or specimen fixed frame \mathcal{K} is given by the Euler triad $g := (\psi_1, \phi, \psi_2) \in SO(3)$ where ψ_1 , ϕ and ψ_2 are Euler angles. The probability of a single fiber having an orientation within the “interval” $[g, g + dg)$ is given by $f(g) dg$. The function $f(g)$ is the orientation distribution function (ODF).

Let $\mathcal{R}^m \subseteq \mathcal{R}$ denote the domain of the matrix material in the RVE \mathcal{R} . The domain occupied by the fiber material is thus given by $\mathcal{R}^f = \mathcal{R} \setminus \mathcal{R}^m$. Let $\mathcal{R}_g^f \subseteq \mathcal{R}^f$ denote the domain occupied by all fibers with orientation $g \in SO(3)$. Thus, $\{\mathcal{R}_g^f\}$ is a partition of \mathcal{R}^f where $\mathcal{I}_g \subseteq SO(3)$ is the index set. The index set may or may not be countable. We now choose $\{\mathcal{R}^m, \{\mathcal{R}_g^f\}\}$ as the partition of the RVE \mathcal{R} . That is, we have taken $\mathcal{I} = \{m, \mathcal{I}_g\}$. The partition $\{\mathcal{R}^m, \mathcal{R}^f\}$ is not appropriate since the constitutive relations are not homogeneous on \mathcal{R}^f due to the ODF and the general anisotropy permitted to the constitutive relations.

It is now assumed that the components of the concentrators \mathbf{A}_g^f and \mathbf{B}_g^f with respect to their local basis are independent of the orientation of the local basis with respect to the global basis. Recognizing that it is an abuse of notation, we will write $\mathbf{A}_g^f = \mathbf{A}^f$ and $\mathbf{B}_g^f = \mathbf{B}^f$ for all $g \in \mathcal{I}_g$ where the components of \mathbf{A}^f and \mathbf{B}^f with respect to the local fiber fixed frame are independent of g .

For future reference we now present two conditions:

- **C1** Unidirectional ODF.
- **C2** Isotropic constitutive relations (mechanical and thermal) and isotropic initial eigenstrain field.

Each of these two conditions will be shown below to be independently sufficient to allow for the effective properties of the previous section for a bi-constituent composite to be expressed in terms of the effective elastic properties.

For reasons of brevity and clarity, the proof of the sufficiency of the two conditions **C1** and **C2** is only presented for Case B below. We begin, however, by presenting the results for Case A.

Case A: Zero-displacement boundary condition

In a procedure similar to that outlined below for Case B, it may be shown that when either or both of **C1** and **C2** hold the effective thermal expansion and effective coefficient of thermal expansion can be expressed in terms of the effective stiffness as

$$\begin{aligned} \mathbf{Y}^\star &= [(\mathbf{Y}^m + \epsilon_r^{\star m}) : \mathbf{C}^m + \{(\mathbf{Y}^f + \epsilon_r^{\star f}) : \mathbf{C}^f - (\mathbf{Y}^m + \epsilon_r^{\star m}) : \mathbf{C}^m\} : \mathbf{C}^\dagger] : \mathbf{S}^\star \\ &\quad - [\epsilon_r^{\star m} : \mathbf{C}_r^m + \{\epsilon_r^{\star f} : \mathbf{C}_r^f - \epsilon_r^{\star m} : \mathbf{C}_r^m\} : \mathbf{C}_r^\dagger] : \mathbf{S}_r^\star \end{aligned} \quad (45)$$

$$\begin{aligned} \alpha^\star &= [\alpha^m : \mathbf{C}^m + (\mathbf{Y}^m + \epsilon_r^{\star m}) : \dot{\mathbf{C}}^m \\ &\quad + \{\alpha^f : \mathbf{C}^f - \alpha^m : \mathbf{C}^m + (\mathbf{Y}^f + \epsilon_r^{\star f}) : \dot{\mathbf{C}}^f - (\mathbf{Y}^m + \epsilon_r^{\star m}) : \dot{\mathbf{C}}^m\} : \mathbf{C}^\dagger \\ &\quad + \{(\mathbf{Y}^f + \epsilon_r^{\star f}) : \mathbf{C}^f - (\mathbf{Y}^m + \epsilon_r^{\star m}) : \mathbf{C}^m\} : \dot{\mathbf{C}}^\dagger] : \mathbf{S}^\star \\ &\quad - [(\mathbf{Y}^m + \epsilon_r^{\star m}) : \mathbf{C}^m + \{(\mathbf{Y}^f + \epsilon_r^{\star f}) : \mathbf{C}^f - (\mathbf{Y}^m + \epsilon_r^{\star m}) : \mathbf{C}^m\} : \mathbf{C}^\dagger] : \mathbf{S}^\star : \dot{\mathbf{C}}^\star : \mathbf{S}^\star \end{aligned} \quad (46)$$

where

$$\mathbf{C}^\dagger := (\mathbf{C}^f - \mathbf{C}^m)^{-1} : (\mathbf{C}^\star - \mathbf{C}^m) \quad (47)$$

$$\dot{\mathbf{C}}^\dagger = (\mathbf{C}^f - \mathbf{C}^m)^{-1} : [\dot{\mathbf{C}}^\star - \dot{\mathbf{C}}^m - (\dot{\mathbf{C}}^f - \dot{\mathbf{C}}^m) : (\mathbf{C}^f - \mathbf{C}^m)^{-1} : (\mathbf{C}^\star - \mathbf{C}^m)]. \quad (48)$$

The definition for \mathbf{C}_r^\dagger follows from the definition (47) in a logical manner. It was assumed in the derivation that the difference $\mathbf{C}^f - \mathbf{C}^m$ is invertible.

If the elastic properties are independent of temperature then the above eqns (45) and (46) reduce to

$$\mathbf{Y}^* = [\mathbf{Y}^m : \mathbf{C}^m + (\mathbf{Y}^f : \mathbf{C}^f - \mathbf{Y}^m : \mathbf{C}^m) : \mathbf{C}^\dagger] : \mathbf{S}^* \quad (49)$$

$$\boldsymbol{\alpha}^* = [\boldsymbol{\alpha}^m : \mathbf{C}^m + (\boldsymbol{\alpha}^f : \mathbf{C}^f - \boldsymbol{\alpha}^m : \mathbf{C}^m) : \mathbf{C}^\dagger] : \mathbf{S}^*. \quad (50)$$

In the following subsection we illustrate the procedure used in developing the results presented in this subsection.

Case B: Zero-traction boundary condition

Expanding eqn (40) for the effective thermal expansion of a bi-constituent composite yields

$$\mathbf{Y}^* = (1 - \nu)(\mathbf{Y}^m + \boldsymbol{\epsilon}_r^{*m}) : \mathbf{B}^m + \nu \langle (\mathbf{Y}^f + \boldsymbol{\epsilon}_r^{*f}) : \mathbf{B}^f \rangle_g - (1 - \nu) \boldsymbol{\epsilon}_r^{*m} : \mathbf{B}_r^m - \nu \langle \boldsymbol{\epsilon}_r^{*f} : \mathbf{B}_r^f \rangle_g \quad (51)$$

where $\langle \cdot \rangle_g$ denotes orientational averaging (Ferrari and Johnson, 1989) with respect to an arbitrary ODF and $\nu = V_{\mathcal{R}_f}/V_{\mathcal{R}}$ is the volume fraction of fiber material in the RVE. The effective compliance (17) (at the temperature θ°) reads

$$\mathbf{S}^* = (1 - \nu)\mathbf{S}^m : \mathbf{B}^m + \nu \langle \mathbf{S}^f : \mathbf{B}^f \rangle_g \quad (52)$$

and the identity (15) takes the form

$$(1 - \nu)\mathbf{B}^m + \nu \langle \mathbf{B}^f \rangle_g = \mathbf{I}. \quad (53)$$

Equations similar to (52) and (53) may be written for the domain in the reference state at the uniform temperature θ_r° . These quantities will be distinguished, as above, by a subscript r .

We now explore sufficient conditions for when the effective thermal expansion \mathbf{Y}^* of eqn (51) can be expressed in terms of the effective compliances \mathbf{S}^* and \mathbf{S}_r^* rather than the stress concentrators, \mathbf{B}^m , \mathbf{B}^f , \mathbf{B}_r^m and \mathbf{B}_r^f . It is sufficient that if either **C1** or **C2** holds then the eqns (52) and (53) can be solved for the quantities \mathbf{B}^m and $\langle \mathbf{B}^f \rangle_g$. Likewise, similar equations will exist for \mathbf{B}_r^m and $\langle \mathbf{B}_r^f \rangle_g$. The proofs follow from the following properties. If the ODF is unidirectional then there is no orientational averaging required and $\langle \mathbf{T} \rangle_g = \mathbf{T}$. If, say, \mathbf{T} is isotropic then $\langle \mathbf{T} : \mathbf{U} \rangle_g = \mathbf{T} : \langle \mathbf{U} \rangle_g$ since \mathbf{T} is independent of orientation and may be taken outside of the orientational averaging brackets.

In order to substitute the expressions for $\langle \mathbf{B}^f \rangle_g$ and $\langle \mathbf{B}_r^f \rangle_g$ into eqn (51) it is sufficient that either **C1** or **C2** hold. It follows that if either **C1** or **C2** holds then the effective properties \mathbf{Y}^* and $\boldsymbol{\alpha}^*$ can be expressed in the following form

$$\mathbf{Y}^* = \mathbf{Y}^m + (\mathbf{Y}^f + \boldsymbol{\epsilon}_r^{*f} - \mathbf{Y}^m - \boldsymbol{\epsilon}_r^{*m}) : \mathbf{S}^\dagger - (\boldsymbol{\epsilon}_r^{*f} - \boldsymbol{\epsilon}_r^{*m}) : \mathbf{S}_r^\dagger \quad (54)$$

$$\boldsymbol{\alpha}^* = \boldsymbol{\alpha}^m + (\boldsymbol{\alpha}^f - \boldsymbol{\alpha}^m) : \mathbf{S}^\dagger + (\mathbf{Y}^f + \boldsymbol{\epsilon}_r^{*f} - \mathbf{Y}^m - \boldsymbol{\epsilon}_r^{*m}) : \dot{\mathbf{S}}^\dagger \quad (55)$$

where

$$\mathbf{S}^\dagger := (\mathbf{S}^f - \mathbf{S}^m)^{-1} : (\mathbf{S}^* - \mathbf{S}^m) \quad (56)$$

$$\dot{\mathbf{S}}^\dagger = (\mathbf{S}^f - \mathbf{S}^m)^{-1} : [\dot{\mathbf{S}}^* - \dot{\mathbf{S}}^m - (\dot{\mathbf{S}}^f - \dot{\mathbf{S}}^m) : (\mathbf{S}^f - \mathbf{S}^m)^{-1} : (\mathbf{S}^* - \mathbf{S}^m)] \quad (57)$$

with a similar expression for \mathbf{S}_r^\dagger . In the derivation of the above equations it was assumed that the difference $(\mathbf{S}^f - \mathbf{S}^m)$ is invertible.

If the elastic properties are independent of temperature then eqns (54) and (55) take the following form

$$\mathbf{Y}^* = \mathbf{Y}^m + (\mathbf{Y}^f - \mathbf{Y}^m) : \mathbf{S}^\dagger \quad (58)$$

$$\boldsymbol{\alpha}^* = \boldsymbol{\alpha}^m + (\boldsymbol{\alpha}^f - \boldsymbol{\alpha}^m) : \mathbf{S}^\dagger. \quad (59)$$

The form of eqn (59) was originally obtained by Rosen and Hashin (1970) in the context of unidirectional composites. Equation (59), however, also holds for an arbitrary ODF when the constitutive relations and the initial eigenstrain fields are isotropic. More importantly, it should be recognized that the results of

this section have only been proven to be valid for bi-constituent composites for which condition **C1** and/or **C2** holds. To emphasize that these results are not necessarily applicable in more general circumstances we present a proof by means of a counter example.

Let us consider the effective coefficients of thermal expansion for a bi-constituent composite consisting of transversely isotropic cylindrical fibers embedded in an isotropic matrix. We shall further consider the elastic properties of both matrix and fiber material to be temperature independent. Our objective here is to show that there exists a composite not satisfying either **C1** nor **C2** such that the effective coefficient of thermal expansion α^* as evaluated from eqn (59) is *not* identical to the “exact” result from eqn (43).

The components of the coefficient of thermal expansion tensors are taken as

$$[\alpha^m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad [\alpha^f] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}. \quad (60)$$

while the components of the elastic properties are taken as

$$[C^m] = \begin{bmatrix} 3 & 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad [C^f] = \begin{bmatrix} 5 & 1 & 1 & 0 & 0 & 0 \\ 1 & 5 & 1 & 0 & 0 & 0 \\ 1 & 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}. \quad (61)$$

The components of the matrix material properties are with respect to the global frame while the components of the fiber material properties are with respect to the local fiber-fixed frame. The fibers are taken to occupy 20% of the total volume (i.e., $\nu = 0.20$) and are assumed to be uniformly distributed (i.e., $f(g) = 1$). The components of Eshelby’s tensor **E** with respect to the local fiber-fixed frame for a circular-cylindrical inclusion for the elastic constitution of the matrix material given in eqn (61) (i.e., Poisson’s ratio is 1/4) are given by (Mura, 1987)

$$[E] = \begin{bmatrix} 2/3 & 0 & 1/6 & 0 & 0 & 0 \\ 0 & 2/3 & 1/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 \end{bmatrix}. \quad (62)$$

Evaluating the “exact” expression (43) yields

$$[\alpha^*] = \begin{bmatrix} 1.7800 & 0.0000 & 0.0000 \\ 0.0000 & 1.7800 & 0.0000 \\ 0.0000 & 0.0000 & 1.7800 \end{bmatrix} \quad (63)$$

while, as anticipated, eqn (59) yields a different result:

$$[\alpha^*] = \begin{bmatrix} 0.8720 & 0.0000 & 0.0000 \\ 0.0000 & 0.8720 & 0.0000 \\ 0.0000 & 0.0000 & 3.3560 \end{bmatrix}. \quad (64)$$

This concludes the proof.

In the following section a niobium reinforced copper matrix composite is considered.

7 Discussion: Cu-Nb Composite

In this section we begin by recalling some of the observed phenomenon cited in the literature concerning the CTLE of heterogeneous materials at cryogenic temperatures. An example is then presented which illustrates that the present theory does not exclude the apparently anomalous behavior found in the literature.

Fitting Parameter	Cu	Nb
a	10.033465	1.0014257
Q_o	5.2663378×10^5	9.7822321×10^5
k	3.8132719	1.5624603
T	264.35295	435.77836

Table 1: Fitting parameters for thermal expansion of copper (Cu) and niobium (Nb)

The approximations of Guertler (1922) and Turner (1946) have been shown by Schapery (1968) to be upper and lower bounds, respectively, for a bi-constituent composite of temperature independent isotropic constituents. It follows that if each of the two constituents have CTLE which are greater than zero then the effective thermal expansion of the composite must be also be greater than zero. In the experimental investigations of Lototskaya, Popov *et al* (1989) a superconducting wire of Ti-Nb filaments in a Cu matrix, negative coefficients of thermal expansion were observed below 7–8 K. Additional experimental work confirmed that the Ti-Nb filaments possessed positive thermal expansion coefficients at these temperatures. Previous investigations were cited to support the positiveness of copper’s thermal expansion coefficients in the temperature range. The authors explain this phenomena by considering that plastic strains are developed in the Cu matrix as well as a bimetallic effect. We do not deny these explanations. Rather, what we shall illustrate below is that the present theory of thermal thermal expansion can predict this type of behavior when the temperature dependence of the elastic properties is considered.

In the early 1980’s, Finlayson, Gibbs and Smith (1981) performed experiments on the coefficients of thermal expansion of a polycrystalline sample of cubic V_3Ge in three mutually perpendicular directions. Their results indicated anisotropic thermal expansion below 90 K. To the best of their knowledge the sample was macroscopically isotropic and the single crystals of V_3Ge did not experience a distortion away from cubic symmetry. As a result the anisotropy observed in the polycrystalline sample was “most surprising.” Finlayson *et al* speculated that the observed behavior was the result of an anisotropic internal stress (Finlayson, 1981; also see Gibbs, Finlayson and Smith (1981)). In the example below we show that the initial eigenstrain field—which is capable of giving rise to an anisotropic internal stress—can engender such anisotropic behavior in an otherwise isotropic composite.

To illustrate the behaviors that were mentioned above we consider a bi-constituent composite as described in section 6. In addition we take the composite to conform with both conditions **C1** and **C2**. The specific composite considered is that of niobium (Nb) fibers embedded in a copper (Cu) matrix. The physical properties of Nb and Cu were extracted from experimental data available in the literature, and then curve-fit with a least squares procedure.

Thermal expansion data for Cu were obtained from Touloukian *et al* (1970, curves 16 and 24, pp. 77–91) and that for Nb also from Touloukian *et al* (1970, curves 7 and 33, pp. 236–243). The data for both Cu and Nb were then curve-fit with a function of the following form (Wachtman, 1962)

$$Y = \frac{1}{3a} \left(\frac{E}{Q_o - kE} \right) + \frac{1-a}{a} \quad (65)$$

where

$$E = \frac{3nRT}{4} \left[\frac{2}{e^{T/\theta} - 1} + \frac{1}{e^{T/(2\theta)} - 1} \right] \quad (66)$$

and $n = 1$, $R = 8.31434$, θ is the absolute temperature and a , Q_o , k and T are constants to be determined by non-linear least-squares curve-fitting. The results of the least squares curve-fitting are presented in Table 1. The experimental data and curve-fit functions are displayed graphically in Figure 1.

Single crystal elastic properties within the temperature interval [0, 300] Kelvin (K) were obtained for Cu from Overton and Gaffney (1955) and for Nb from Carroll (1965). The effective, isotropic polycrystalline elastic properties of both Cu and Nb were then estimated by the self-consistent theory of effective elastic properties (Hershey, 1954; Kroner, 1958). The estimates to the effective isotropic bulk and shear moduli

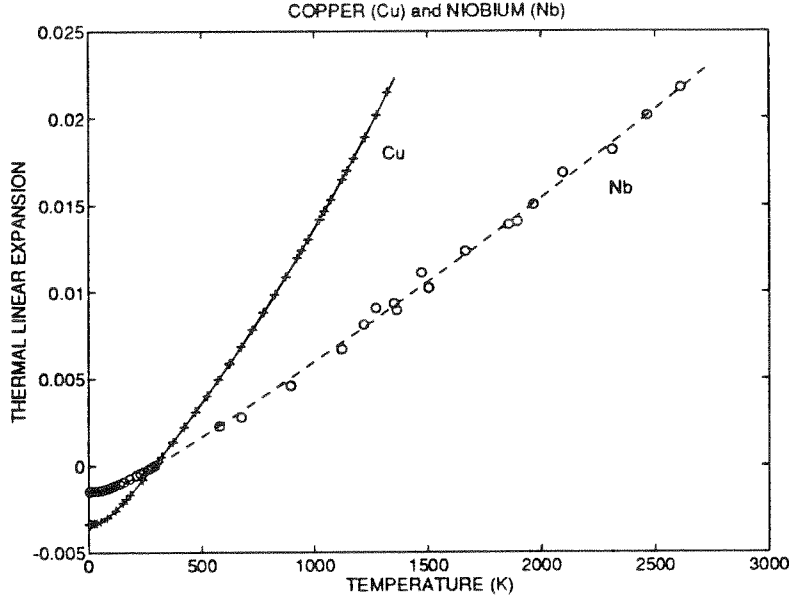


Figure 1: Thermal linear expansion of polycrystalline copper (Cu) and niobium (Nb). Experimental data are depicted by + for Cu and o for Nb. The results of the least-squares curve-fit to the data are also presented: Cu, solid line; Nb, dashed line.

Fitting parameter	Bulk modulus	Shear modulus
A	137.87700	48.313947
B	0.29207180	-0.051245974
C	2.9207612	7.8698101

Table 2: Fitting parameters in eqn (67) for elastic properties of copper (Cu)

were then curve fit. In the case of Cu, the bulk modulus and shear modulus were fitted by functions of the form (Sutton, 1953)

$$f = A e^{-3B\alpha\theta} (1 + 3Y)^{-C} \quad (67)$$

where θ is the absolute temperature, α is the coefficient of linear thermal expansion at the temperature θ , Y is the TLE given by eqn (65) and A , B and C are constants to be determined by non-linear least-squares curve-fitting. The units of f are GPa. The results of the curve-fitting are presented in Table 2. As an indication of goodness of fit, the root-mean-square (RMS) error is 0.0219 GPa and 0.0117 GPa for the curve fits to the bulk and shear data, respectively.

The isotropic bulk and shear moduli of Nb were fit by polynomials in $\theta/1000$. The linear term of the polynomial was removed prior to curve-fitting so that the slope of the polynomial curve-fit at absolute zero is zero. This constraint is noted by Huntington (1958, p.320). The results of least-squares curve-fitting for a 5-th order polynomial is presented in Table 3. The RMS values for goodness of fit are 0.1848 GPa and 0.0649 GPa for the bulk and shear modulus, respectively. For comparison purposes a 4-th order polynomial fit was also performed. Those results are presented in Table 4. The RMS values for goodness of fit are 0.3407 GPa and 0.0925 GPa for the bulk and shear modulus, respectively. The data and the least-squares curve fit functions to the bulk and shear properties of both Cu and Nb are presented in Figures 2 and 3, respectively.

The theory to be adopted for the stress concentrator is that of Mori and Tanaka (1973) as presented by Benveniste (1987). For the special case of a unidirectional, bi-constituent composite the Mori-Tanaka

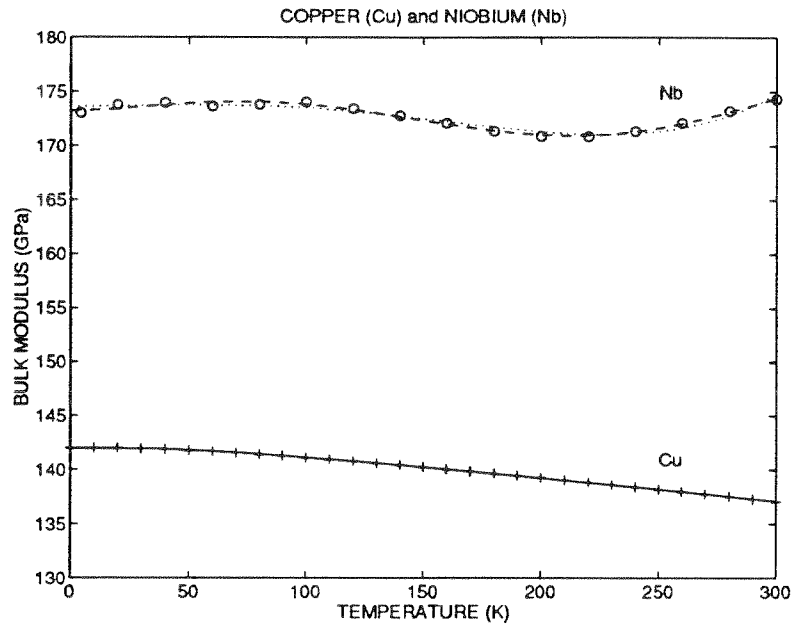


Figure 2: Bulk modulus for polycrystalline copper (Cu) and niobium (Nb). Self-consistent estimates calculated from experimental data are depicted by + for Cu and o for Nb. The results of the least-squares curve-fit to the data are also presented: Cu, solid line; Nb, dashed line = 5th order polynomial; Nb, dotted line = 4th order polynomial.

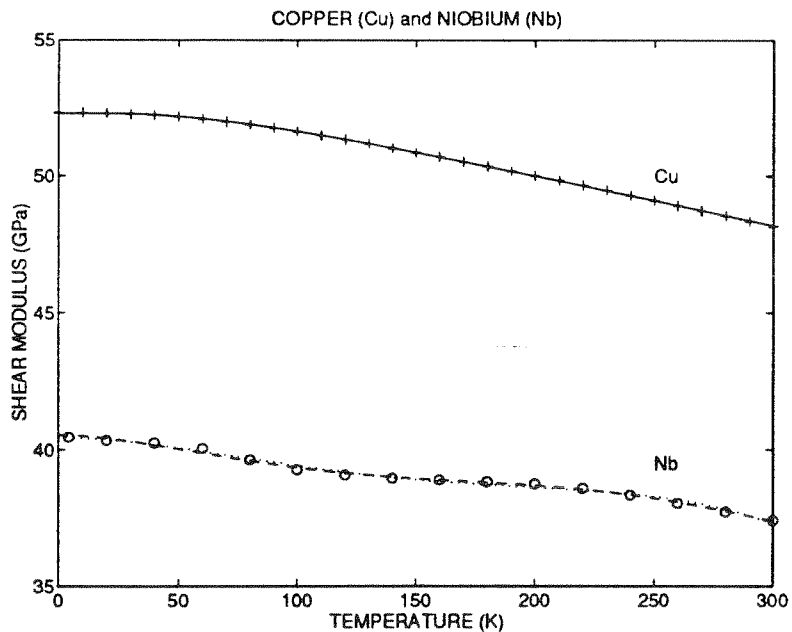


Figure 3: Shear modulus for polycrystalline copper (Cu) and niobium (Nb). Self-consistent estimates calculated from experimental data are depicted by + for Cu and o for Nb. The results of the least-squares curve-fit to the data are also presented: Cu, solid line; Nb, dashed line = 5th order polynomial; Nb, dotted line = 4th order polynomial.

Order	Bulk modulus	Shear modulus
0	1.732278×10^2	4.052584×10^1
1	-	-
2	6.531043×10^2	-3.228984×10^2
3	-9.317574×10^3	2.919407×10^3
4	3.877096×10^4	-9.948849×10^3
5	-4.941617×10^4	1.139161×10^4

Table 3: Coefficients of the 5-th order polynomial curve-fit to the elastic properties of Niobium (Nb)

Order	Bulk modulus	Shear modulus
0	1.736223×10^2	4.043489×10^1
1	-	-
2	1.460733×10^2	-2.060157×10^2
3	-2.162619×10^3	1.270019×10^3
4	5.730420×10^3	-2.332215×10^3

Table 4: Coefficients of the 4-th order polynomial curve-fit to the elastic properties of Niobium (Nb)

concentrator is given by

$$\mathbf{B}^f = [\mathbf{I} + (1 - \nu)\mathbf{C}^m : (\mathbf{I} - \mathbf{E}) : (\mathbf{S}^f - \mathbf{S}^m)]^{-1}. \quad (68)$$

where \mathbf{E} is Eshelby's tensor. This tensor is a function of the shape of the ellipsoid as well as the elastic properties of the matrix material. Explicit forms for \mathbf{E} can be found in Mura (1987). The stress concentrator is exact in the dilute limit, i.e., as $\nu \rightarrow 0$, since in this limit the exact result of Eshelby (1957) is obtained.

We now show that negative coefficients of thermal expansion are possible for a heterogeneous material even if the coefficients of thermal expansion of each constituent is greater than or equal to zero. This behavior is possible due to the temperature dependence of the elastic properties. Consider a 50% volume fraction of spheroidal Nb fibers embedded in a Cu matrix. The initial eigenstrain field is taken to be zero, i.e., $\epsilon_r^* = 0$ at the reference temperature of 300K. The effective coefficient of thermal expansion in the direction of the fibers for aspect ratios of 0, 1/2, 1, 2, 5 and ∞ are given in Figure 4. Clearly negative coefficients of thermal expansion are achieved.

We now show that the initial eigenstrain field ϵ_r^* is capable of inducing anisotropic thermal expansion coefficients in a heterogeneous material that is macroscopically isotropic and comprised of materials which themselves are isotropic. Consider a 50% volume fraction of spherical Nb fibers embedded in a Cu matrix. The difference in initial eigenstrain field is taken to be

$$[\epsilon_r^{*f} - \epsilon_r^{*m}] = 10^{-3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (69)$$

at a reference temperature of 300K. The non-zero coefficients of thermal expansion are plotted in Figure 5 over the temperature interval (0, 20]. Clearly the effective CTE is anisotropic over this interval.

Finally, we illustrate the sensitivity of the effective coefficients of thermal expansion to the curve-fitting of the elastic properties. We discuss this by considering the effect of a different polynomial curve-fit to the elastic moduli of Nb. The results presented in Figure 4 were calculated using 5-th order polynomial curve-fits to the elastic data. These results are presented again in Figure 6. For comparison the same analysis was performed but using 4-th order curve-fits to the elastic properties of Nb. These results are presented in Figure 6 with the dash-dot curves. From these results we may observe the sensitivity to the gradient of the elastic moduli with respect to temperature. Note that the magnitude of the CTE is typically larger for the 5-th order curve-fit than for the 4-th order curve-fit. This indicates that as the curve-fitting functions

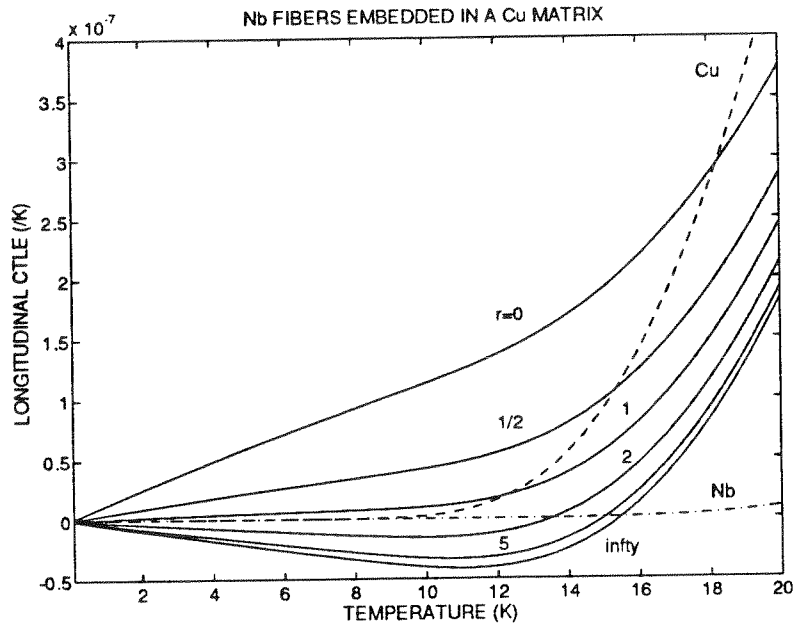


Figure 4: Effective longitudinal CTLE for fiber aspect ratios, r . $\nu = 0.50$. The reference state at 300K is strain- and stress-free.

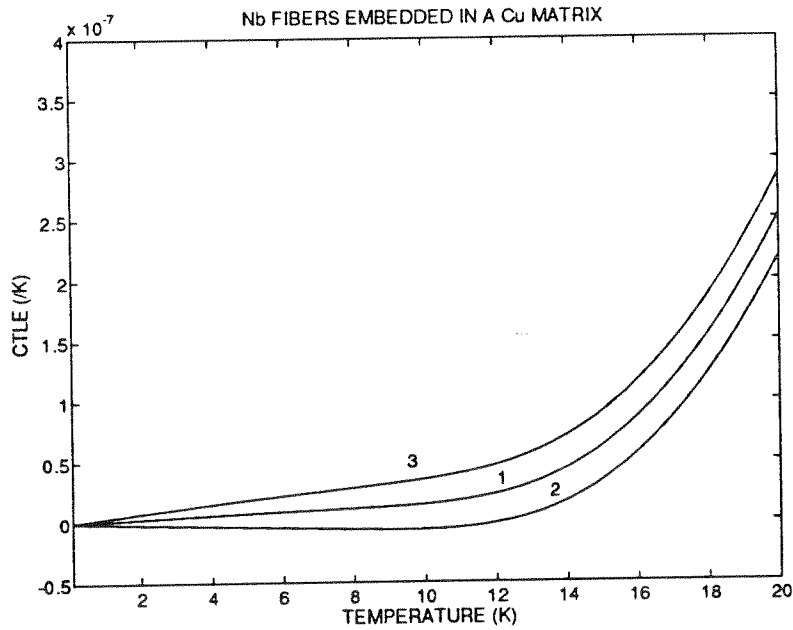


Figure 5: Effective CTLE for a spherically reinforced composite with isotropic constituents in the x_1 (1), x_2 (2) and x_3 (3) directions. $\nu = 0.50$; $\theta_r^0 = 300K$. The relevant portion of the initial eigenstrain field is given by eqn (69).

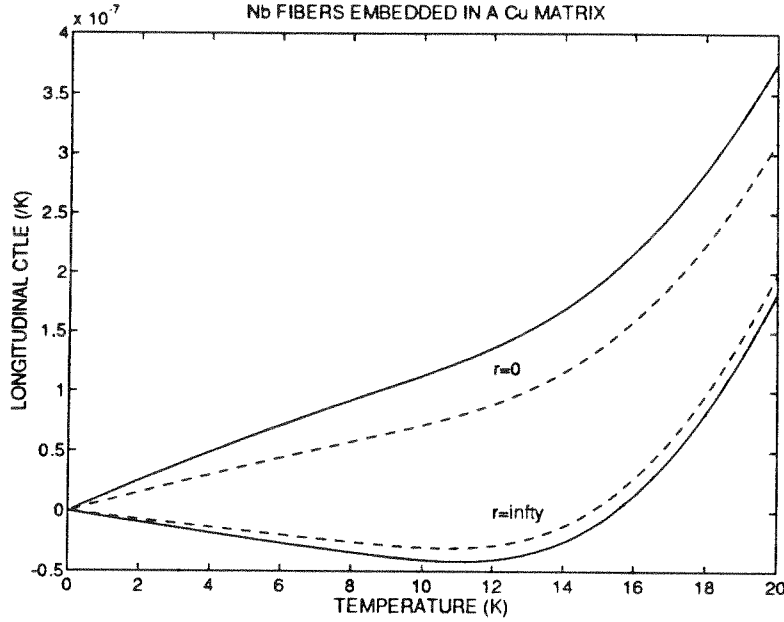


Figure 6: Effective longitudinal CTLE for fiber aspect ratios, $r = 0$ and ∞ using 5-th order (solid) and 4-th order (dashed) polynomial curve fits to the elastic properties of niobium. $\nu = 0.50$; $\theta_r^0 = 300\text{K}$; reference state is strain- and stress-free.

become “more accurate” the magnitude of the thermal expansion coefficients increases. Thus, the predicted behavior can not be construed as an artifice of curve fitting. Though it is observed that the results are sensitive to the temperature dependence of the elastic properties.

8 Conclusion

Equivalent expressions for the effective thermal expansion of a heterogeneous material or composite have been developed in terms of stress and strain concentrators. The effects of temperature dependence of the elastic properties and of an initial eigenstrain field are included. For special classes of bi-constituent materials the effective thermal expansion has been presented in terms of the effective elastic properties of the material thus removing the explicit dependence on the concentrators. Those special classes of bi-constituent materials being (i) unidirectional ODF or (ii) isotropic physical properties and isotropic initial eigenstrain field. For emphasis, it was proven by counter-example that the resulting expression is *not* valid for arbitrary bi-constituent composites.

It has been shown that, in general, the effective thermal expansion, and thus also the coefficients of thermal expansion, of a macroscopically homogeneous material are functions of the initial eigenstrain field ϵ_r^0 . Under conditions of either homogeneity of the elastic properties or temperature independence of the elastic properties and effective elastic properties it was proven that the effective thermal expansion was independent of the initial eigenstrain field. It was demonstrated by considering a Nb-Cu composite that negative coefficients of thermal expansion are achievable even when the coefficients of thermal expansion of each of the constituents are positive. Negative CTEs are possible due to the temperature dependence of the elastic properties. It was also demonstrated that the initial eigenstrain field can give rise to anisotropic thermal expansion coefficients in a heterogeneous material that is macroscopically isotropic in its effective elastic properties and comprised of constituents which are isotropic in all of their physical properties.

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