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PHASE CONTOURS OF SCATTERING AMPLITUDES. II

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I. INTRODUCTION

Our objective in this paper is to investigate consistency conditions that are imposed by crossing symmetry and analyticity when a given high energy behavior is assumed for a scattering amplitude. These consistency conditions are obtained by the use of phase contours, which were introduced and studied in the previous paper[1] (hereafter denoted by I).

Our assumption for the high energy behavior is based on single pole dominance in a Regge model having a continuously rising Regge trajectory. This establishes the phase of the scattering amplitude in the s-channel for fixed t, or for fixed u, as $s \to \infty$. We assume a single symmetric scattering amplitude that corresponds to equal mass spinless bosons. We define a phase contour as a curve of constant phase, or more generally a surface of constant phase in the complex space of the invariant energies s, t, and u.

A consistent Regge model requires that the trajectories should be complex above threshold and that they go through integer values on the unphysical sheets at resonance poles. In addition zeros are required in residues to avoid the existence of nonphysical poles in the scattering amplitude, or in partial wave amplitudes. These zeros and the resonance poles play an essential role in establishing a consistent topology for the phase contours. This is because phase contours that correspond to different (real constant) values of the phase cannot intersect each other, except possibly at singularities or zeros of the invariant scattering amplitude.

In addition to our assumptions about Regge behavior at high energy, our main simplification is the neglect of local distortions of phase contours at low energies due to nearby resonances. These distortions were illustrated and discussed in the previous paper, I. However, they are not essential to the derivation of a consistent topology of phase contours under the conditions assumed in this paper, although they would be important if we imposed a stronger form of bootstrap consistency. We do take account of resonances in their crossed channel high energy effects, and in the resulting interference that determines the continuation between phase contours in different regions. We also take account of an indirect effect of resonances that we describe in terms of a generalized scattering length. This relates to real zeros of the amplitude on the crossed branch cuts, just as the scattering length itself may be related to real zeros below threshold.

In this paper we are working towards a consistent solution for phase contours that describes the interference pattern coming from the resonance poles and the zeros. Our arguments for the location of zeros on the physical sheet are based mainly on consistency. Although some of these zeros may be identified as different parts of the same complex surface, their occurrence on the physical sheet can be deduced from different requirements. In order to separate these requirements we start from a very simplified model and obtain its phase contours using crossing symmetry. We then introduce successive complications that lead eventually to the phase contours for our Regge model.

In Section 2 we summarize the properties of phase contours that are required for our subsequent discussion. Most of these properties were discussed in more detail in I. In Section 3 we introduce a phase model that has no poles or zeros on the physical sheet, and from it we obtain a solution for the phase contours. In Section 4, we discuss zeros below threshold that depend on the scattering length, and extend this to deduce a sequence of real zeros on the crossed branch cuts. Associated with these real zeros, there are curves of complex zeros on the physical sheet that lead to a modification of the phase contours of our first simple model. This modification can be interpreted as arising when zeros move on to the physical sheet through the crossed branch cuts at infinity.

In Section 5, we study the complex zeros on the physical sheet that come from zeros of the residues of the leading Regge terms, and we show how they modify the phase contours. These zeros may be identified as parts of the complex surfaces of zeros that disappear through the crossed branch cuts and are related to the generalized scattering length. In Section 6 we introduce the effects of the Regge model above the thresholds where the trajectories become complex. Resonance poles on the physical sheet produce striking changes in the asymptotic phase contours that are analagous to those produced by the zeros of residues below threshold. There is, however, an essential difference in the way the zeros move on the physical sheet. Below the t threshold, they move out to infinity at finite values of t where the residues are zero. Above threshold they move to infinity only when t becomes infinite. This becomes evident in Section 7, where we give the crossing symmetric phase contours for the Regge model. We also indicate in Section 7 the way resonances and zeros are related, by considering a complex section of the phase contours on the physical sheet and on neighboring unphysical sheets. In Section 8 we give a brief discussion of our results.

2. ASSUMPTIONS AND PROPERTIES OF PHASE CONTOURS

The phase $\phi(s,t)$ of a scattering amplitude $F(s,t)$ is defined by

$$\phi(s,t) = \text{Im}\,[\log\{F(s,t)\}]. \tag{2.1}$$

It is also necessary to specify the phase at an initial point (s_0,t_0). When $F(s,t)$ has zeros or poles on the physical sheet, the phase may be changed by multiples of 2π by choosing different routes from the initial point to the point (s,t). We must therefore specify the route that we use when relating the phases at two different points.

A phase contour is defined by

$$\phi(s,t) = C, \tag{2.2}$$

where C is a real constant. We will study phase contours both for real s and t, and for complex s when t is held at real values.

For fixed t and complex s $(s = s_1 + is_2)$, the phase is a harmonic function of s_1 and s_2, when F is regular. In the s plane the phase contours are orthogonal to the modulus contours, but this does not apply in other planes, like s and t real, for example.

Phase contours, for different constant values of the phase, cannot meet except at singularities or zeros of the amplitude $F(s,t)$.

$$t_1^0 < t < 4m^2, \tag{2.6}$$

where t_1^0 denotes the first zero of the residue, that occurs for negative t, when

$$\alpha(t_1^0) = -1, \tag{2.7}$$

provided $b(t)$ does not have any zeros in the range (2.6). We will discuss the effects of zeros at $\alpha(t) = -(2N + 1)$ in Section 5. Above the threshold $t = 4m^2$, $\alpha(t)$ becomes complex, and the phase of the Regge term (2.3) is no longer given by Eq. (2.5). We will consider the resulting phase in Section 6.

Our initial simplifying assumptions about the phase are based on Eq. (2.5). We assume that the phase of the amplitude has the form (2.5) as $s \to \infty$ along real $s + i0$, for any fixed real t. We also assume that $\alpha(t)$ is real for all real t, even above threshold. This is no longer a Regge model but it is useful for illustrating the first requirements of the consistency problem. Crossing symmetry is achieved by making analagous asymptotic assumptions for fixed u and fixed s.

In the forward direction, $t = 0$, the optical theorem requires that, along $s + i0$,

$$\text{Im } F(s,0) > 0, \quad \text{for } s > 4m^2, \tag{2.8}$$

in order that the total cross section shall be positive. Since our amplitude is to be symmetric, there is a similar condition in the backward direction, $u = 0$.

The relation (2.9) can be extended to any value of t in the range

$$0 \leqslant t < 4m^2. \tag{2.9}$$

Hence, using Eq. (2.5), which holds also for the Regge amplitude (2.3) in this region, we must have

$$0 < \phi(s,t) < \pi, \tag{2.10}$$

for $s > 4m^2$, in the range (2.9). There is a similar condition in

$$0 \leqslant u < 4m^2. \tag{2.11}$$

From this result we see that the phase at threshold $s = 4m^2$, reached along $t = 0$ from $s = +\infty$, must be zero or π. If there are no poles or zeros below threshold, the phase must be either 0 or π throughout the region

$$s < 4m^2, \quad t < 4m^2, \quad u < 4m^2. \tag{2.12}$$

Which of the values, $\phi = 0$ or π, is relevant will depend on the scattering length, which will be discussed in Section 4. The value of the phase ϕ, in the triangle (2.12) also depends on the route by which it is reached from our starting point given in Eq. (2.4).

3. A MODEL WITH NO ZEROS ON THE PHYSICAL SHEET

We assume an asymptotic behavior that is consistent with a symmetric amplitude and has the phase (2.5), as $s \to +\infty$,

$$F(s,t) \sim b(t)s^{\alpha(t)} \exp\left[i\pi\left\{1 - \frac{1}{2}\alpha(t)\right\}\right] . \tag{3.1}$$

We assume that $b(t)$ has no poles or zeros on the physical sheet, and that $\alpha(t)$ is real, and corresponds to a continuously rising trajectory, for all t. We make similar asymptotic assumptions for fixed real u, and s.

Our first objective is to obtain a solution for phase contours on the physical sheet, when there are no zeros or poles of the amplitude on the physical sheet. It is not evident, a priori, that such a solution will exist. Our reason for requiring no zeros (or poles), is that the phase of F will be unambiguously defined, so that it is independent of the path on the physical sheet by which it is obtained from the initial value in Eq. (2.4).

It is important to specify the limit in which the boundary of the physical sheet is approached, since this will affect the phase. Thus Eq. (3.1) holds in the limit $(s + i0)$ with s real. In the limit $(s + i0)$ as $s \to -\infty$ along the real axis the phase will be

$$\phi(s \to -\infty,\ t) \sim \pi\left[1 + \frac{1}{2}\alpha(t)\right] . \tag{3.2}$$

We begin by obtaining phase contours in the physical s channel (in the limit $s + i0$), on the assumption that the amplitude has the form

$$F(s,t) = b(t)\, s^{\alpha(t)} \exp\left[i\pi\left\{1 - \frac{1}{2}\alpha(t)\right\}\right] + b(u)\, s^{\alpha(u)} \exp\left[i\pi\left\{1 - \frac{1}{2}\alpha(u)\right\}\right] + \text{background} \qquad (3.3)$$

Instead of the variable s in Eq. (3.3), we could use the variable z_t in the first term and z_u in the second term, where

$$z_t = 1 + \frac{2s}{t - 4m^2}, \quad z_u = 1 + \frac{2s}{u - 4m^2}. \qquad (3.3a)$$

However in the region $t < 0$, $u < 0$, these would lead to the same topology as we obtain from Eq. (3.3). We take $\alpha(0) = 1$, so as to give a constant total cross section, and we take the background to have only a slowly varying phase. At high energies the background is neglected at all angles in the physical regions. We will find that we cannot neglect the effects of the background in all unphysical regions.

The phase contours for real s and t in the s-channel are shown in Fig. (3.1). We have neglected small oscillations of the type discussed in I. It is not evident at this stage, whether the phase contours $\phi = \frac{1}{2}\pi$ bend away from the physical region as shown, or whether they join through the physical region like the other contours shown. We will see that the contours shown are the simplest ones that are compatible with our requirement that our solution is to have no zeros on the physical sheet.

From Fig. 3.1 we can obtain phase contours in other real regions in two essentially different ways. These depends on whether we require the phase in the limit $(s + i0, t + i0)$, or in the limit $(s + i0, t - i0)$. Other limits give contours topologically similar to one of these, for the model considered here.

The phases in the limits $(s + i0, t + i0, u + i0)$ for each relevant variable, or pair of variables, have the same form in each physical region as Fig. (3.1). In the unphysical regions, for example $s > 4m^2$, $t > 4m^2$, we write the amplitude

$$F(s,t) = b(t)\, z_t^{\alpha(t)} \exp\left[i\pi \left\{ 1 - \frac{1}{2}\alpha(t) \right\} \right] + b(s)\, z_s^{\alpha(s)} \exp\left[i\pi \left\{ 1 - \frac{1}{2}\alpha(s) \right\} \right] + \text{background}, \qquad (3.4)$$

where z_t is given by Eq. (3.3a) and z_s is defined similarly. With assumptions about the smoothness of the background similar to those made in the physical region, the asymptotic contours above the s threshold join smoothly to those above the t threshold. The resulting phase contours in the real (s,t,u) plane in the limits from the upper half planes are shown in Fig. 3.2.

The analagous diagram showing phase contours in the limits $(s + i0, t - i0, u - i0)$ is more interesting. The phases in the t-channel and the u-channel are obtained by analytic continuation in $\mathrm{Im}\, s > 0$ along

$$s = K \exp(i\theta), \qquad 0 \leqslant \theta \leqslant \pi, \tag{3.5}$$

where K is large. In this simple case with no zeros, one obtains phase contours in the u-channel $(u - i0)$, which are complex conjugate to those in the s-channel $(s + i0)$. In the regions of crossed branch cuts, we replace (3.4) by

$$F(s,t) = b(t)\, z_t^{\alpha(t)} \exp\left[i\pi\left\{1 + \tfrac{1}{2}\alpha(t)\right\}\right] + b(s)\, z_s^{\alpha(s)} \exp\left[i\pi\left\{1 - \tfrac{1}{2}\alpha(s)\right\}\right] + \text{background}. \tag{3.6}$$

This is appropriate to the limit $(s + i0, t - i0)$. Along $s = t$, the background must be real; so the phase will be π, since it has this value asymptotically and there are no zeros by assumption.

The resulting phase contours are shown in Fig. 3.3. We see that along $t = 0$, the phase is $\frac{1}{2}\pi$ as $s \to +\infty$ but is $\frac{3}{2}\pi$ as $s \to -\infty$ (keeping on $s + i0$). In the region of crossed cuts, say $(t - i0, s + i0)$, the $\frac{1}{2}\pi$ phase contour is required to separate the

π contour from the 0 contour. Similarly the $\frac{3}{2}\pi$ contour must lie between the π and 2π contours in this region. This determines that these contours must bend away from the physical regions in this diagram, and also in Fig. 3.2 since below $t = 4m^2$, the $\frac{1}{2}\pi$ contour follows the same path as in Fig. 3.3. We see also that the phase in the triangle below threshold must be equal to π; this is a consequence of our assumption that there are no zeros on the physical sheet. We relax this assumption in the next section.

In Fig. 3.4 we show complex sections of the phase contour surfaces for three real values of t, in the complex s plane. The values of t are chosen with one well below $t = 0$, the second just above $t = 0$, and the third well above $t = 4m^2$. The phase contours in Fig. 3.4 indicate the asymptotic phase as $s \to \infty$,

$$\phi(s,t) \sim \pi\left[1 - \frac{1}{2}\alpha(t)\right] + \theta\alpha(t) , \qquad (3.7)$$

where $s = |s| \exp i\theta$. The phase lines $\phi = \pi$ meet at stagnation points, so that $\phi = \pi - \epsilon$ and $\phi = \pi + \epsilon$ diverge away from these points as indicated in Fig. 3.4(b).

4. ZEROS AT FINITE REAL POINTS

In this section we will extend the model described in Section 3, so that there are zeros at finite real points. Attached to these zeros are curves of complex zeros that lie on the physical sheet.

We begin by stating the results that have been established by Jin and Martin[2] for a symmetric scattering amplitude below threshold. These give an indication of where we may expect to find a set of real or complex zeros of the scattering amplitude on the physical sheet. We will extrapolate heuristically from the rigorous results of Jin and Martin to deduce the effects on phase contours of the first real zero. These lead us to obtain a consistent solution for phase contours with an infinite sequence of real zeros on the crossed branch cuts in the limits from opposite half planes, for example $(s + i0, u - i0)$. This solution can be continuously varied so that it goes over to the solution obtained in the previous section when the zeros move through infinity to unphysical sheets.

(a) The Amplitude Below Threshold

Define the variable z by

$$z = \frac{1}{4}(s - u)^2 = (s - 2m^2 + \frac{1}{2}t)^2. \qquad (4.1)$$

When t is in the range $(-4m^2, +4m^2)$, the amplitude can be expressed by a dispersion relation in z, with one subtraction.[2,3] Let $G(z,t)$

denote the amplitude $F(s,t)$ expressed in terms of the variable z. Then

$$G(z,t) = C(t) + \frac{z - x_0}{\pi} \int_{x_0}^{\infty} dx \frac{\text{Im}\, G(x,t)}{(x - x_0)(x - z)} \quad . \tag{4.2}$$

In the region $0 \leqslant t < 4m^2$, ImG is positive for $x > x_0$, hence

$$\left(\frac{d}{dz}\right)^n G(z,t) > 0, \quad \text{for } z \text{ real } < x_0, \tag{4.3}$$

where

$$x_0 = \left(2m^2 + \frac{1}{2} t\right)^2 . \tag{4.4}$$

When z is real and less than x_0, the function $G(z,t)$ will be real for $0 \leqslant t < 4m^2$. Hence for t in this range

(i) if $C(t) < 0$, $G(x,t)$ will have no zeros when $x < x_0$;

(ii) if $C(t) > 0$, $G(x,t)$ will have at most one zero when $x < x_0$.

Using crossing symmetry, this result can be extended to give information about the amplitude $F(s,t)$, within the triangle where it is real, namely

$$s < 4m^2, \quad t < 4m^2, \quad u < 4m^2. \tag{4.5}$$

It has been shown by Jin and Martin[2] that $F(s,t)$ has an absolute minimum at the symmetry point

$$s = \frac{4m^2}{3}, \qquad t = \frac{4m^2}{3} . \tag{4.6}$$

The amplitude F increases (inside the triangle) along any straight line originating at this symmetry point.

If we assume asymptotic power behavior as indicated in Section 2, there will be one real zero of $G(x,0)$ for $x < x_0$, when the scattering length $C(0)$ is positive. More generally for t in the range,

$$-4m^2 < t < 4m^2, \tag{4.7}$$

there will be one real zero of $G(x,t)$, when the subtraction term in (4.2) is positive, $C(t) > 0$. Let this zero be at $z_0(t,f)$,

$$G[z_0(t,f), t] = 0, \tag{4.8}$$

where f denotes a parameter that permits us to vary the scattering length $C(0)$ and other values of the subtraction term $C(t)$. For example, let f denote the value of F at the symmetry point

$$f = F(4m^2/3, 4m^2/3). \tag{4.9}$$

If f is positive, there will be no zeros of F inside the triangle (4.5), but there will be a real zero of $G(x, 4m^2/3)$ in $x < 0$. This real zero z_0 corresponds to two complex conjugate zeros of F,

$$s_0 = \frac{4m^2}{3} \pm i|z_0|^{\frac{1}{2}}. \tag{4.10}$$

If f is decreased and becomes negative, the two zeros (4.10) become real and separate inside the triangle (4.5). If f is sufficiently negative, the zeros reach the threshold branch points and move through them on to unphysical sheets.

The various situations of real zeros, that we wish to consider are shown in Fig. 4.1. The first diagram (a) corresponds to the situation when $f > 0$ and there are no real zeros, but there will be complex zeros. It is due to these complex zeros that we can choose different phases indicated in the diagram by $\phi = 0$, or 2π. In Fig. 4.1 (b) we have decreased f so that it is negative and there is a loop of real zeros in the triangle. Reducing f further gives (c), in which the broken lines indicate zeros that have moved on to the second sheet. In Fig. 4.1 (d) some of the second sheet zeros have become complex, as $z_0(t,f)$ decreases past zero on the second sheet. In Fig. 4.1 (e), all second sheet zeros are complex except for the black circles where the complex zeros move through its real boundary $(s + i0, u - i0)$ etc. on to the physical sheet. The complex zeros on the physical sheet are indicated by dotted lines in diagram (e). There are also complex zeros on the physical sheet for all the other diagrams shown, They rise out from the curves of real zeros, except in Fig. 4.1 (a) when they are at complex parts of the physical sheet and have no intersection with the real triangle.

(b) Complex Sections of Phase Contours

The location of zeros and their relation to the phase contours becomes clearer by considering complex sections. For illustration, we consider two complex sections when the real zeros have the form of Fig. 4.1 (b). These show the phase contours and zeros in the complex s plane when $t = 0$, in Fig. 4.2 (a), and when $t < 0$, so that there are no real zeros, in Fig. 4.2 (b) and (c).

Since there are zeros of the amplitude $F(s,t)$, the value of the phase $\phi(s,t)$ will depend on the route taken from our initial point, $s \to \infty$ along $s + i0$, and $t = 0$, when the phase is $\frac{1}{2}\pi$. In Fig. 4.2(a) we define the phase by keeping in the upper half s plane, so we always go above the real zeros at s_0 and s_1. In Fig. 4.2 (b), the zeros have become complex, and only s_1 is in $\mathrm{Im}s > 0$. The path by which the phases have the values shown are indicated by arrows. The Fig. 4.2 (c) is an identical section to Fig. 4.2 (b) but we obtain different phases by passing below the complex zero s_1, as indicated by the arrows. The phases in diagram (b) are relevant if we use a route through asymptotic values in $\mathrm{Im}s > 0$, but those in (c) are relevant if we proceed along $s + i0$.

(c) Crossing Symmetric Phase Contours

We now extrapolate from the location of the zeros shown in Fig. 4.1 (e), and assume an infinite sequence of real zeros on the crossed cuts in the limits

$$(s + i0,\ u - i0) \quad \text{along} \quad s = u, \tag{4.11}$$

$$(s + i0,\ t - i0) \quad \text{along} \quad s = t. \tag{4.12}$$

Only the leading zero may go below threshold, when it may have the form shown in Fig. 4.1 (b), (c), (d). However, we have chosen to take it on the crossed cuts as in Fig. 4.1 (e), since the resulting phase contours are slightly more simple than in the other cases.

The phase contours with pairs of variables in the limits $(s + i0,\ t + i0,\ u + i0)$ can be taken to be the same as those in Fig. 3.2, since they are consistent without the introduction of any real zeros on the physical sheet in these limits.

The phase contours in the limits $(s + i0,\ u - i0,\ t - i0)$ are shown in Fig. 4.3. The labeling of phases is obtained by going from the physical region for the s-channel near $t = 0$, or $u = 0$, through asymptotic values in $\text{Im}s > 0$ to the physical regions for other channels. Then we use continuity out of these physical regions to their neighboring unphysical regions on the indicated sides of the branch cuts.

In Fig. 4.3 zeros are shown as small black circles, and the attached complex zeros as dotted lines. The direction in the complex space taken by these zeros depends on whether we vary t and consider complex s, or vary u and consider complex s, etc. The heavy line through the zeros has a different phase on either side. It is part of the complex surface of branch cuts of $\log F(s,t)$. Similar cuts should be drawn through the complex zeros along the dotted lines. However, we will find it convenient to discuss various routes for defining the phase so we will not normally consider such branch cuts, which specify the phase in a less flexible manner.

The intermediate phase lines, on the (u,s) and (t,s) crossed cuts in Fig. 4.3, do not cross the heavy phase contour $\phi = n\pi$ that goes through the zeros. The detailed form of these contours is shown in Fig. 4.4, which is an enlargement of the region labeled $(u - i0, s + i0)$ in Fig. 4.3. This figure indicates more intermediate contours, but omits the symmetric $\phi = n\pi$ contour.

Complex sections, for real t and complex s, of Figs. 4.3 and 4.4 are similar to the sections given in Fig. 4.2 for small values of $(-t)$. For large values of $\pm t$, they are more complicated but we will proceed to a more realistic version of the Regge model before considering further complex sections.

5. ZEROS OF REGGE RESIDUES

The asymptotic Regge amplitude (2.3) is zero for t below threshold, whenever the residue vanishes,[4] namely at

$$\alpha(t) = -(2n + 1), \; n = 0,1,2,\cdots \tag{5.1}$$

In order to obtain the effects of this zero on phase contours, we must consider more than one term in the Regge asymptotic expansion. We will study the first two terms and will assume that the zeros of their residues do not coincide. We write them in the form

$$F(s,t) \sim \beta_1(t)\, s^{\alpha_1(t)} \exp\left[i\pi\left\{1 - \frac{1}{2}\alpha_1(t)\right\}\right] + \beta_2(t)\, s^{\alpha_2(t)} \exp\left[i\pi\left\{1 - \frac{1}{2}\alpha_2(t)\right\}\right] \tag{5.2}$$

where

$$\beta_i(t) = \frac{b_i(t)}{\sin\left[\frac{1}{2}\pi\alpha_i(t)\right]\,\Gamma[\alpha_i(t)]}, \quad i = 1,2. \tag{5.3}$$

We assume that α_1 and α_2 are real for $t < 4m^2$, and that

$$\alpha_1(t) > \alpha_2(t). \tag{5.4}$$

For simplicity we will assume also that their difference is constant, and

$$\alpha_1(t) = \alpha_2(t) + 1. \tag{5.5}$$

There would be no significant change in our results if we took any constant difference between 0 and 2. For a larger difference, the results would be more complicated.

The optical theorem requires

$$\alpha_1(0) = 1, \quad b_1(0) > 0. \tag{5.6}$$

The residue β could have additional zeros in $t < 0$, due to zeros in $b_1(t)$. However, we will limit the possibilities that we need to consider by taking

$$b_1(t) > 0, \quad \text{for } t \text{ real}. \tag{5.7}$$

An important aspect of the phase contours depends on whether $b_2(t)$ has the same sign or a different sign from $b_1(t)$, when the residue $\beta_1(t)$ vanishes at values of t satisfying Eq. (5.1). Although $b_2(t)$ should be positive at resonance poles above threshold in our model, it could change sign in $t < 4m^2$, before we reach the first zero of the leading Regge residue $\beta_1(t)$. We will therefore consider the two situations (a) $b_1(t) > 0$, $b_2(t) > 0$, and (b) $b_1(t) > 0$, $b_2(t) < 0$. In the general case we could have situation (a) holding at some zeros of $\beta_1(t)$, and situation (b) holding at other zeros. However, we will limit our discussion by assuming that either (a) or (b) holds for all $t < 0$. The former leads to an oscillating phase in the physical regions, the latter leads to an increasing phase.

(a) An Oscillating Phase; $b_1(t) > 0$, $b_2(t) > 0$, in $t < 0$

We consider firstly the phase change of $F(s,t)$, given by Eq. (5.2), as t decreases from zero at a fixed positive real value of s along ABC in Fig. 5.1 (a) (on $s + i0$). Since $\beta_i(t)$ is zero, when $\alpha_i(t) = -(2n + 1)$, the term

$$F_i = \beta_i(t)\, s^{\alpha_i(t)} \exp\left[i\pi\left\{1-\tfrac{1}{2}\alpha_i(t)\right\}\right], \qquad (5.8)$$

will describe a spiral in the half plane $\mathrm{Re}F_i \leqslant 0$. This spiral will touch the imaginary F_i axis whenever t takes a value so that α_i is a negative odd integer. On account of our assumption (5.5) about the trajectory difference, the spirals for F_1 and F_2 will be out of phase in general, but they will be in the same half-plane. The spirals for F_1 and F_2 are shown in Fig. 5.1 (b) and (c), together with the path of their sum in Fig. 5.1 (d), as t decreases for fixed real s. Note that the relative size of the spiral (c) will decrease if s is increased.

For any finite real s, it is evident that the phase of the amplitude F, given by Eq. (5.2), will oscillate between $\left(\frac{1}{2}\pi + \epsilon\right)$ and $\left(\frac{3}{2}\pi - \epsilon\right)$, where $\epsilon \to 0$ as $s \to \infty$,

$$\phi(s,t) \sim \eta(t), \quad \frac{1}{2}\pi \leq \eta(t) \leq \frac{3}{2}\pi. \tag{5.9}$$

The phase, for finite large s, is shown in Fig. 5.2 (a) as a function of $\alpha_1(t)$.

Before discussing the location of the zeros on the physical sheet that come from the zeros of residues, we consider the phase in case (b).

(b) An Increasing Phase; $b_1(t) > 0$, $b_2(t) < 0$, in $t < 0$

In this case, as we follow the path ABC in Fig. 5.1 (a) for fixed s and decreasing t, the term F_1 given by Eq. (5.8) with $i = 1$, follows the spiral shown as (b) in Fig. 5.1. However, F_2 will follow the spiral shown as (e) in Fig. (5.1) in $\mathrm{Re}F_1 > 0$. The resulting sum, that gives the asymptotic phase of $F(s,t)$ will follow the path indicated by diagram (f) in Fig. (5.1). For a larger fixed value of s, there will be a smaller part of the curve in $\mathrm{Re}F > 0$, but it will always loop round the origin for any finite s (no matter how large).

The asymptotic phase as $s \to +\infty$, in this case, is given by

$$\phi(s,t) \sim \pi[1 - \alpha_1(t)] + \zeta(t), \tag{5.10}$$

where

$$-\frac{1}{2}\pi \leq \zeta(t) \leq \frac{1}{2}\pi. \tag{5.11}$$

This phase, for finite large s, is shown in Fig. 5.2 (b) as a function of $-\alpha_1(t)$, (note that $-\alpha_1$ increases as t decreases).

(c) Zeros on the Physical Sheet

We consider now, how the zeros from the residues in case (a) move in the finite regions of the physical sheet as t is decreased through negative values. We begin from the fixed value of t that gave point B in Fig. 5.1 (b), (c) and (d), for some fixed real s, which we denote by s_0. We then follow the path in $\text{Im}\,s > 0$,

$$s = s_0 \exp(i\theta), \quad 0 \leqslant \theta \leqslant \pi. \tag{5.12}$$

Along this path,

$$F(s,t) \sim F_1 + F_2 = F_2 \left[\frac{F_1}{F_2} + 1\right], \tag{5.13}$$

where

$$\frac{F_1}{F_2} = -\left|\frac{\beta_1(t)}{\beta_2(t)}\right| s_0^{(\alpha_1-\alpha_2)} \exp\left[i(\alpha_2 - \alpha_1)\left(\frac{1}{2}\pi - \theta\right)\right]. \tag{5.14}$$

This ratio is real and negative when $\theta = \frac{1}{2}\pi$, that is, when s is pure imaginary. We can now hold t fixed and choose s_0 so that at $\theta = \frac{1}{2}\pi$,

$$\frac{F_1}{F_2} = -\left|\frac{\beta_1(t)}{\beta_2(t)}\right| s_0^{(\alpha_1-\alpha_2)} = -1. \tag{5.15}$$

If we have chosen the point B in Fig. 5.1 (a) - (d), so that $\beta_1(t)$ is very small, then the solution s_0 of Eq. (5.15) will be large. As t moves further above the value t_1^n at which

$$\alpha_1(t_1^n) = -(2n + 1), \tag{5.16}$$

the ratio β_1/β_2 increases and $s_0 = s_0(t)$ decreases, until $s_0 = 0$, when $t = t_2^n$, where

$$\alpha_2(t_2^n) = -(2n + 1). \tag{5.17}$$

Before we reach the value t_2^n, we must of course replace F_2 in Eq. (5.13) by another correction term or a sum of such terms. We should also use the variable $(s - u)$, instead of s, in order to preserve crossing symmetry. The zero associated with (5.16) then moves in from infinity along a curve in the plane, t real $(s - u)$ pure imaginary, as t increases from t_1^n given by (5.16). We will see that after these modifications, it is still consistent to assume the zeros become real, although this will no longer occur at $t = t_2^n$. We denote the real zeros by

$$t = a_1, \quad t = a_2, \cdots \tag{5.18}$$

and since they move in along $(s - u)$ pure imaginary, we will assume that they are real along the symmetry line $\mathrm{Re}(s) = \mathrm{Re}(u)$.

We have already, in Section 4, established a need for zeros of the amplitude $F(s,t)$ that are real along $s = u$, and are at complex values on the physical sheet along curves that go through the real zeros. It seems natural to identify those curves of zeros with the curves of zeros that are asymptotic to $t = t_1^n$ satisfying Eq. (5.16), along $(s - u)$ pure imaginary. The resulting complex section containing these zeros is shown in Fig. 5.3 (a), where the axes are t (real) and $(s - u)$ (pure imaginary). In order to establish the consistency of this figure, we should also consider the complex s plane (or $(s - u)$ plane) for $t < t_1^n$. The ratio in Eq. (5.14) becomes modified because $\beta_1(t)$ is now negative. If

$$0 < (\alpha_2 - \alpha_1) < 2,$$

the ratio F_1/F_2 does not become real and negative for any value of θ. Hence there are no zeros of F in the asymptotic region for case (a) with $t < t_1^0$, when $|t - t_1^0|$ is small.

In case (b), considered in subsection (b) above, the situation is reversed. For $t > t_1^0$ there are no zeros in the asymptotic region, but for $t < t_1^0$ there will be one zero. The resulting curves of zeros in the complex section, t real and $(s - u)$ pure imaginary, are shown in Fig. 5.3 (b).

The shape of the phase contours in the real region $s > 4m^2$, $u > 4m^2$, in the limit $(s + i0,\ u - i0)$ in both cases (a) and (b), obliges us to draw the attached complex curves of zeros as shown in

Fig. 5.3 (a) and (b). The zeros are on the intersection of phase contour surfaces. In case (b) they will normally remain in the finite part of the complex section shown in Fig. 5.3 (b) even when $t \to -\infty$. In this case (b) we are unable to identify directly the zeros coming from the real symmetry points $s = u$, with the zeros coming from the vanishing of the Regge residues. However, by a variation of the parameters t_1^0 and a_1, for example, we can cross over from situation (a) to situation (b) for the first zero. It is then evident that the zero at a_1 connects to the zero coming in from t_1^0 on the physical sheet in case (a), but on an unphysical sheet in case (b).

More complicated situations can occur if we relax the condition $0 < (\alpha_2 - \alpha_1) < 2$. This would permit more than one zero to come from each vanishing residue. Apart from these possibilities, there will in general be local distortions of phase contours, and hence of the curves of zeros, due to resonances. These would have their greatest effect on the physical sheet near the real axes.

(d) Phase Contours in $t < 0$

With so many zeros on the physical sheet it is necessary to specify the routes by which the phase is defined. Unfortunately the route that gives the most natural phase labeling for one section of the surfaces of constant phase becomes rather unnatural for other sections. We will therefore sometimes change the routes used for defining the phase, when we change to a different section of the phase surfaces.

We assume that the zeros at real points in $t < 0$, are located in position similar to those shown in Fig. 4.3, on the overlapping branch cuts $u > 4m^2$, $t > 4m^2$. Taking account of the zeros of residues, we find that Fig. 4.3 for $t < 0$, in case (a) is replaced by Fig. 5.4. Here we have used a phase labeling beginning from $t = 0$, $s \to \infty$ along $s + i0$, where the phase is $\frac{1}{2}\pi$. The phases in the s-channel are found by continuity along t real. The phase in the u-channel is found by crossing along $s = |s| \exp i\theta$, for large s with $t = 0$, giving a phase $\frac{3}{2}\pi$. Then we proceed by continuity along t real. In the physical regions of the s- and u-channels the phase is never equal to a half integer multiple of π. For a more realistic model that had resonance distortions of phase contours at finite energies, one would expect this result to continue to hold for large s and for large u.

In Fig. 5.5 we show for case (a) some complex sections of the phase contour, for several fixed real values of t, in $\text{Im}s > 0$ in the complex s plane. The labeling in Fig. 5.5 (a) corresponds to that in Fig. 5.4 for small negative t. In Fig. 5.5 (b) t has become more negative. The labeling in brackets corresponds to a route above the zero (in agreement with Fig. 5.4), the other corresponds to a route below the zero. The latter is the most natural labeling to use in Fig. 5.5 (c), when t has decreased just below a_1. At the value t_1^0 of t (see Fig. 5.3 (a)), the zero shown in Fig. 5.5 (b) has moved upwards to infinity. As t decreases below the value t_1^0, the contours $\frac{1}{2}\pi$ and $\left(\frac{3}{2}\pi\right)$ in Fig. 5.5 (b) have stretched to $i\infty$ and separated as shown in Fig. 5.5 (c). In the latter figure, t has decreased below a_2 (see Fig. 5.5 (a)), so there is a new zero on the imaginary (s - u) axis, which connects the $-\frac{1}{2}\pi$ and $\left(\frac{5}{2}\pi\right)$ contours. In Fig. 5.5 (d), we have taken a value of t in the range

$$a_3 > t > t_1{}^2, \tag{5.19}$$

and have used the phase labels in the complex (s - u) plane that are appropriate for this value of t, given that the phase of the right most contour is π, as in the s-channel of Fig. 5.4.

We see that in case (a), defined in Section 5 (a) above, the phase remains near to the value π in the s-channel, as shown in Fig. 5.4. The phase contours that are relevant to the high energy behavior are those in the region of overlapping branch cuts. This is illustrated by Fig. 5.5 (d), where the power behavior s^{α} has $\alpha \approx -4$. Thus the oscillations of ImF are ineffective in the physical region. An approximation to a superconvergence relation that included only the physical regions would give a completely wrong result in this case (a), where the region of crossed cuts plays a vital role.

The situation is different in case (b) described in subsection 5 (b), and giving zeros as shown in Fig. 5.3 (b). The phase has the asymptotic value given by Eq. (5.10) as $s \to +\infty$. If we defined the phase in the u-channel by crossing asymptotically in Ims > 0 near t = 0, the phase there as $s \to -\infty$, would satisfy

$$\phi(s,t) \sim \pi\,[1 + \alpha_1(t)] - \zeta(t). \tag{5.20}$$

Thus the net phase change is $2\alpha_1\pi$, which is twice as much as that obtained at each fixed negative t from the asymptotic behavior $s^{\alpha(t)}$. The discrepancy is taken up by the zeros that move in from infinity when $\alpha(t) = -(2n + 1)$. In this case, however, the zeros that come in from infinity do not leave the physical sheet. They are in addition to the zeros that enter the physical sheet through the real points $t = a_1, a_2, a_3, \cdots$ along $s = u$. The two types of zeros are separated on the physical sheet by the phase contour $\phi(s,t) = \pi$, in this case.

The phase contours for case (b) are shown for real s and t in Fig. 5.6, which is analagous to Fig. 5.4 (which applies to case (a)). In Fig. 5.7, we show a complex section that corresponds to fixed real t in Fig. 5.6 just below the first real zero,

$$a_2 < t_1^1 < t < a_1 < t_1^0. \tag{5.21}$$

The phase labeling in Fig. 5.7 is obtained by continuity in the s plane for this particular value of t, so it does not correspond to that on the left of Fig. 5.6. The upper zero in Fig. 5.7 comes from the zero at the residue, whereas the lower zero comes from the unphysical sheet through the real zero $t = a_1$, $s = u$, in Fig. 5.6. If we begin from case (b) and increase a_1 until $a_1 = t_1^0$, we obtain case (a). The flip of the contours when $a_1 = t_1^0$, will occur at infinity where the two zeros become coincident. For $a_1 > t_1^Q$, they are separated again but one of them is on the unphysical sheet. The other is the zero discussed in case (a). More complicated situations can be obtained by varying parameters so that the two zeros in Fig. 5.7, meet along $Re(s) = Re(u)$, at finite $In(s)$. They could then separate again on opposite sides of the line $Re(s) = Re(u)$, and could move down towards the physical regions.

6. RESONANCE POLES AND ASYMPTOTIC PHASES

In this section we investigate the asymptotic phase above threshold in the Regge model related to Eq. (2.3), and find the associated phase contours. Above threshold the Regge trajectory becomes complex, and it is important to distinguish whether we have t above or below the branch cut along the real axis. We will find that the presence of nearby resonance poles on the second sheet produces an important change in phase from the simple model that we used in Sections 3 and 4. It is necessary to consider the phase for different limits before we can study the associated phase contours.

(a) $s + i0$, $t + i0$, With $s \to +\infty$ and $t > 4m^2$

The phase can be obtained from the asymptotic expression (2.3) for F in the Regge model, namely

$$F(s,t) \sim \frac{b(t)\, s^{\alpha(t)} \exp\left[i\pi\left\{1 - \frac{1}{2}\alpha(t)\right\}\right]}{\sin\left[\frac{1}{2}\pi\,\alpha(t)\right]\, \Gamma[\alpha(t)]} . \tag{6.1}$$

We write for $t > 4m^2$,

$$\alpha(t) = \alpha_1 + i\alpha_2, \quad \text{with} \quad \alpha_2 > 0. \tag{6.2}$$

The residue $b(t)$ is assumed to be nearly real and to have a slowly changing phase in $t > 4m^2$, and the gamma function is almost real, since $\alpha_1 > 1$ and we assume α_2 is much less than α_1. The power of s leads to a factor

$$\exp[i\alpha_2 \log s]. \tag{6.3}$$

The phase of this term is a slowly varying function of s, so we will ignore it in our present discussion of phases and phase contours. It may become important when considering more detailed questions of consistency, but it does not appear to be relevant for our work in this section.

However, the phase of the $\sin\left(\frac{1}{2}\pi\alpha\right)$ term in the denominator of the expression (6.1) is important. This term can be written,

$$\sin\left(\frac{1}{2}\pi\alpha_1\right)\cosh\left(\frac{1}{2}\pi\alpha_2\right) + i\cos\left(\frac{1}{2}\pi\alpha_1\right)\sinh\left(\frac{1}{2}\pi\alpha_2\right). \tag{6.4}$$

Its phase lies in the same quadrant as the phase of

$$\exp\left[i\pi\left(\frac{1}{2} - \frac{1}{2}\alpha_1\right)\right]. \tag{6.5}$$

Hence neglecting the factor (6.3), the asymptotic phase $\phi(s,t)$ of the scattering amplitude F, given by (6.1) will satisfy

$$\phi(s + i0, t + i0) \sim \frac{1}{2}\pi + \chi_a(t), \quad \text{as} \quad s \to +\infty, \tag{6.6}$$

where $\chi_a(t)$ depends on α_1 and α_2;

$$\chi_a(t) = 0, \quad \text{when} \quad \alpha_1 = n, \quad n = 1,2,3,\cdots \tag{6.7a}$$

$$0 < \chi_a < \frac{1}{2}\pi, \quad \text{when} \quad 2n < \frac{1}{2}\alpha_1 < 2n + 1, \tag{6.7b}$$

$$-\frac{1}{2}\pi < \chi_a < 0, \quad \text{when} \quad 2n + 1 < \frac{1}{2}\alpha_1 < 2n + 2, \tag{6.7c}$$

$$\chi_a(t) \to 0, \quad \text{as} \quad \alpha_2(t) \to \infty. \tag{6.7d}$$

In this limit, the phase oscillates about the value $\frac{1}{2}\pi$ so that $\text{ImF} \geqslant 0$. The corresponding complex section of the phase contours in the t plane is shown in Fig. 6.1 (a). This diagram shows part of the physical sheet in $\text{Imt} > 0$, and also part of the unphysical sheet reached through the t -branch cut along the real axis. There are many such unphysical sheets that depend on how many threshold branch cuts are crossed. On all these unphysical sheets there will be poles or shadow poles corresponding to resonances.[5] In Fig. 6.1 (a) we have considered only one such sheet. We have indicated zeros on this sheet, in addition to the resonance poles, since they are required for a consistent pattern of phase contours. There are no such zeros from the term (6.1) alone, but there will be zeros when a correction term is added, that has a slowly varying phase.

The resonance poles occur at the usual values for an amplitude of even signature, namely at the zeros of $\sin\left[\frac{1}{2}\pi\alpha(t)\right]$,

$$\alpha(t) = 2n, \quad n = 1,2,3,\cdots \tag{6.8a}$$

$$t = t_2, t_4, t_6, \cdots \tag{6.8b}$$

The S-state pole will lie below threshold on the real axis of the unphysical sheet, since we have assumed that there are no bound state poles on the physical sheet.

(b) $s + i0$, $t - i0$, with $s \to +\infty$ and $t > 4m^2$

The phase contours near and on the boundary of the physical sheet in this limit are quite different from those considered above in case (a). Now we have $\alpha_2 < 0$, and the phase of (6.4) will be in the same quadrant as that of

$$\exp\left[i\pi\left(-\frac{1}{2} + \frac{1}{2}\alpha_1\right)\right], \qquad (6.9)$$

instead of (6.5). From (6.1) the asymptotic phase of F will satisfy

$$\phi(s + i0, t - i0) \sim \frac{3}{2}\pi - \alpha_1\pi - \chi_b(t), \qquad (6.10)$$

as $s \to \infty$, where

$$-\frac{1}{2}\pi < \chi_b(t) < \frac{1}{2}\pi, \qquad (6.11a)$$

$$\chi_b(t) \to 0, \text{ when } \alpha_1 = n. \qquad (6.11b)$$

There is some ambiguity in choosing the phase of $\left(-\frac{1}{2}\pi\right)$ in (6.9). We determine it by continuity of ϕ from the region $0 < t < 4m^2$, where the asymptotic phase must satisfy $0 < \phi < \frac{1}{2}\pi$.

The phase contours for complex t in this limit relate to poles that are reached from the physical sheet below the real branch cut. They are shown in Fig. 6.1 (b), where $\mathrm{Im}\,t < 0$ corresponds to the physical sheet. We have also indicated zeros on the physical sheet in Fig. 6.1 (b). These occur when the Regge term (6.1) is combined with a suitable background. Asymptotically in s, these zeros will recede to infinity on the physical sheet, so the phases have been labeled along the real axis corresponding to this asymptotic situation. We will identify these zeros with those deduced for other reasons on and near the crossed branch cuts in Sections 4 and 5. As indicated in Fig. 6.1 (b), the zeros are expected to become real and then go into the unphysical sheet when $\mathrm{Re}\,t$ is increased so that $\mathrm{Re}\,t = s$. It should be noted that our previous discussion, in Section 5, of zeros of this type was for real negative t on the s,u crossed branch cuts. The analogue of that discussion here would be for fixed real u on the s,t crossed branch cuts. However, although we are now concerned with the same complex surface of zeros, we are varying t (and not u) through real values. The complex path of these zeros is therefore different from that found in Section 5. The simplest behavior, consistent with Eq. (6.1), is that the zeros move steadily towards infinity in the complex s plane as t increases through real values to plus infinity.

The phase $\phi(s,t)$ given asymptotically by (6.10) will take the value,

$$\phi(s,t) = \left(-2n + \frac{1}{2}\right)\pi, \tag{6.12}$$

when $\alpha_1(t) = (2n + 1)$. This value of α_1 does not correspond to a resonance pole since the amplitude has even signature. The phase contours (6.12) go through the zeros of F, while the phase contours (6.13) related to $\alpha_1(t) = 2n$, go through the resonance pole,

$$\phi(s,t) = -\left(2n - \frac{3}{2}\right)\pi. \tag{6.13}$$

Before obtaining the phase contours in the full s,t,u plane, we require information about the asymptotic phase in three more types of limits on to the boundary of the physical sheet.

(c) $s + i0$, $t + i0$, With $t \to +\infty$, $s > 4m^2$

By symmetry, this limit gives phases that are exactly analagous to those in (a) above

$$\phi(s + i0,\ t + i0) \sim \frac{1}{2}\pi + \chi_a(t), \quad \text{as} \quad t \to +\infty, \tag{6.14}$$

where χ_a satisfies the conditions (6.7).

(d) $s - i0$, $t + i0$, With $s \to +\infty$, $t > 4m^2$

The Regge term in the amplitude on this boundary, that is analagous to (6.1), is

$$\frac{b(t)\ s^{\alpha} \exp\left[i\pi\left\{1 + \frac{1}{2}\alpha(t)\right\}\right]}{\sin\left[\frac{1}{2}\pi\alpha(t)\right]\ \Gamma[\alpha(t)]} . \tag{6.15}$$

Hence the asymptotic phase will be

$$\phi(s - i0, t + i0) \sim \frac{1}{2}\pi + \alpha_1\pi + \chi_d(t), \quad \text{as} \quad s \to +\infty, \tag{6.16}$$

where $\chi_d(t) = 0$ when $\alpha_1 = n$, and satisfies conditions analagous to (6.7).

(e) $s - i0$, $t - i0$, With $s \to +\infty$, $t > 4m^2$

The phase in this case follows from (6.15) and (6.4), giving

$$\phi(s - i0, t - i0) \sim \frac{3}{2}\pi + \chi_e(t), \quad \text{as} \quad s \to +\infty, \tag{6.17}$$

where $\chi_e(t) = 0$, when $\alpha_1 = n$, and satisfies conditions analagous to (6.7).

7. CROSSING SYMMETRIC PHASE CONTOURS

Using the results of Section 6 we can obtain the simplest family of phase contours for the Regge model in the region t real above threshold. Combining these with the contours obtained in Section 5 for t real below threshold, we obtain the phase contour diagram shown in Fig. 7.1. In this figure we have assumed that, in the physical s-channel, the conditions of Section 5 subsection (b) hold. Thus the phase contours in this region are the same as those shown for the s-channel in Fig. 5.6. The phase in the u-channel can be obtained by crossing symmetry near $t = 0$ and then by continuity along $(u - i0)$ for decreasing real t. This gives the phase labels shown in Fig. 5.6. The labeling in Fig. 7.1 in the u-channel corresponds to that obtained through asymptotic values of s in $\text{Im}s > 0$ from the s-channel for each fixed $t < 0$. The dotted lines from the zeros on the u and s overlapping branch cuts are complex in $\text{Im}s > 0$ along $\text{Re}(s) = \text{Re}(u)$ for decreasing real t. For case (b) of Section 5, these zeros remain on the physical sheet as t decreases indefinitely. In addition there are complex zeros along $\text{Re}(s) = \text{Re}(u)$ in $\text{Im}s > 0$, that come from the zeros of residues. In this case the two kinds of zeros do not identify with each other on the physical sheet. This contrasts with case (a) considered in Section 5.

Above $t = 4m^2$, as t increases, complex zeros come out of the s and t overlapping branch cuts from real points along $\text{Re}(s) = \text{Re}(t)$. These zeros remain on the physical sheet as t increases and go to infinity as $t \to +\infty$. For finite t we have the s plane analogue of Fig. 6.1 (b), which shows the t plane for real s. Note that the complex path of these zeros for increasing real t is different from the complex path for decreasing real u. The latter is the analogue, on the s,t crossed cuts, of our discussion in Section 5 on the s,u crossed cuts. In the present case we do not expect the zeros to go to infinity for finite real t above threshold.

A complex section, based on Fig. 7.1 is shown in Fig. 7.2. This section shows the complex s plane for real t at a value above $4m^2$, when two of the zeros are complex and the third is nearly real but still on the unphysical sheet. The right hand and left hand branch cuts $(s > 4m^2$ and $t > 4m^2)$ have been pulled down to show part of the unphysical sheets. The lack of symmetry is due to the fact that we are above the threshold in t, at a real point $t - i0$ approached from the t physical sheet $\text{Im}t < 0$.

If, instead of case (b) of Section 5, we had taken case (a), the lower half of Fig. 7.1 would change to the pattern indicated in Fig. 5.4, with complex sections in $t < 0$ as shown in Fig. 5.5. However, the contours for $t > 0$ will remain the same as those shown in Fig. 7.1. In case (a) the $\pi,\pi,\pi,$ pattern applies in all physical channels. The same phases are obtained in case (a) above threshold in t, either by crossing near $t = 0$ (or $u = 0$) through asymptotic values of s, or by crossing at each fixed value of t (or u), through asymptotic values of s. This shows that the number of zeros encircled is the same either way, and confirms our remark above that the zeros, emerging from the symmetry points $s = t$, do not leave the $(\mathrm{Im}\, s > 0)$ physical sheet as t increases through positive real values (along $t - i0$). Thus, as t increases, the zeros on the right of Fig. 7.2 will move upwards in $\mathrm{Im}\, s > 0$, and more zeros will emerge from the unphysical sheet.

8. DISCUSSION

The method of phase contours has been used to study crossing symmetry in a Regge model based on rising Regge trajectories. Families of solutions have been obtained that show how zeros and poles of scattering amplitudes can be related by means of phase contours. Zeros of the amplitude were shown to arise from three initially independent sources. The first source, discussed in Section 4, may be called symmetry zeros, since they occur along $\mathrm{Re}(s) = \mathrm{Re}(u)$ and are deduced from crossing symmetry arguments. The symmetry zeros may move on to the unphysical sheet if "scattering length" parameters could be varied sufficiently. The resulting phase contours would be those considered in Section 3.

The second source of zeros comes from the zeros of Regge residues $\beta(t)$ in $t < 0$. Two main possibilities were considered in subsection (a) and (b) of Section 5. In the first one (a), as t is decreased through negative real values, the residue zeros move in from infinity along $\mathrm{Re}(s) = \mathrm{Re}(u)$ in $\mathrm{Im}\, s > 0$, and leave the physical sheet at the real symmetry zeros. In this case the phase in the physical regions for fixed $t < 0$ does not cycle as s moves along the real axis, but only oscillates about the value π. The high energy behavior is directly related to the zeros and the oscillations of the phase in the region where the s and u branch cuts overlap. This is indicated by the phase contours in Fig. 5.5 (d), for example.

In case (b) of subsection 5, the residue zeros and the symmetry zeros cannot be identified by connecting curves of zeros on the physical sheet. Both types remain on the physical sheet after they have entered it, but their paths do not meet. Presumably they will meet on an unphysical sheet since a continuous variation is possible from case (a) to case (b) in which the complex curves of zeros flip at certain critical values of the parameters.

The third type of zero is deduced as a consequence of interference between resonance poles. For $t \to +\infty$, these zeros will move along $\text{Im}s \to +\infty$. As t is decreased the interference zeros successively leave the physical sheet through the symmetry zeros. A typical section of the complex s plane is shown in Fig. 7.2, for a real value of t $(t - i0)$ such that only three of the interference zeros remain complex.

The value of an analysis of families of phase contours in a crossing symmetric model lies in the insight they give about the relation between low energy and high energy behavior. For example in case (a) of Section 5, one would obtain a completely wrong evaluation of high energy behavior for large negative t, if one considered only the behavior of the amplitude in the physical channels. In this case the phases on the crossed cuts essentially control the high energy behavior at fixed negative t.

The consistency conditions that we have studied are not a bootstrap method in any complete sense, since we have not included unitarity except weakly, in that it is not violated by our model. We expect that full unitarity will provide strong additional conditions that further limit the types of phase contours that may occur. This problem could be studied in the limited case of coupled two body channels that took into account the resonances on which our model is based. The method of phase contours could also be extended to more general collision amplitudes, although the many variables involved would make their discussion somewhat elaborate.

The most important feature that has been neglected, apart from unitarity, is the local distortion that comes from resonances at low energies. It is not at all obvious how to take account of direct channel resonances as well as the asymptotic behavior from crossed channel resonances. However, our study of phase contours provides a new method of approaching this problem, which can certainly be developed much further.

The solutions for phase contours, described in this paper, give some information about fixed angle behavior. More generally, the study of phase contours and zeros permits a new formulation of the problem of relating asymptotic behavior at fixed momentum transfer and asymptotic behavior at fixed angle. We will consider this in a later paper.

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FOOTNOTES AND REFERENCES

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+ At the Cavendish Laboratory, Cambridge, England, after January 1, 1968.

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FIGURE CAPTIONS

Fig. 3.1. Phase contours in the physical region for the s-channel, based on the simplified form of a Regge model given by Eq. (3.4). The continuous curves correspond to $\mathrm{Im}F = 0$, and the broken curves to $\mathrm{Re}F = 0$.

Fig. 3.2. Crossing symmetric phase contours in the limits $(s + i0, t + i0, u + i0)$ taken in pairs, with s,t,u real on the physical sheet, when there are no zeros on the physical sheet.

Fig. 3.3. Phase contours for a symmetric amplitude in the limit $(s + i0, t - i0, u - i0)$ on the boundary of the physical sheet, for a simplified Regge model, with no zeros on the physical sheet.

Fig. 3.4. Complex sections of phase contours of Fig. 3.3 in the complex s plane for real t, (a) t negative, (b) t small and positive, (c) t well above $t = 4m^2$.

Fig. 4.1. Curves of zeros of a symmetric amplitude in the triangle below threshold, shown in the real (s,t) plane: (a) there are no real zeros but since $\phi = 0$ or 2π, there will be

complex zeros, (b) real zeros along the closed curve, (c) the real zeros indicated by a broken line lie on the unphysical sheet, (d) some of the unphysical sheet zeros have become complex, (e) all zeros are complex on the physical and unphysical sheets except for isolated points shown as small black circles, the attached dotted lines denote complex zeros on the physical sheet.

Fig. 4.2. Complex sections based on Fig. 4.1 (b). In diagram (a) we show phase contours for complex s when there are two real zeros when $t = 0$. Diagram (b) shows the phase contours when t has become negative so that the zeros are complex. Diagram (c) shows alternative routes that lead to different phase values from (b).

Fig. 4.3. Phase contours for a crossing symmetric amplitude in the limit $(s + i0, t - i0, u - i0)$. The small black circles denote real zeros, and the attached dotted lines denote complex zeros on the physical sheet.

Fig. 4.4. The $(s + i0, u - i0)$ phase contours. This is an enlarged version of the neighborhood of some zeros in Fig. 4.3. It shows phase contours for intermediate values of the phase, to indicate how they cross the symmetric line $s = u$, only at zeros of the amplitude.

Fig. 5.1. (a) The real (s,t) plane.

(b) to (f) The complex plane for various Regge amplitudes showing how they vary along ABC in Fig. (a), when the residues have zeros.

Fig. 5.2. The variation of the phase as a function of the leading Regge trajectory, (a) when the first and second Regge terms have negative real parts, (b) when they have real parts of opposite sign.

Fig. 5.3. The dotted lines show curves of zeros in the section $\mathrm{Re}(s) = \mathrm{Re}(u)$, with $\mathrm{Im}(s)$ and real (t) as coordinates. The residues have zeros at t_1^n. Fig. (a) corresponds to Section 5 (a) and Fig. (b) to Section 5 (b). The points $a_1, a_2, \cdots$ denote real zeros.

Fig. 5.4. Phase contours in the limit $(s + i0, u - i0)$ for case (a), corresponding to Fig. 5.4 (a). The dotted lines denote complex zeros that go to infinity for finite t, at $t_1^0, t_1^1, \cdots,$ and are real at $t = a_1, a_2, a_3, \cdots$

Fig. 5.5. Phase contours in the complex $(s - u)$ plane for successively decreasing values of t real, corresponding to case (a) and the real section shown in Fig. 5.5. The s and u branch cuts overlap in each of these figures.

Fig. 5.6. Phase contours for real values of the variables, in case (b), corresponding to Fig. 5.4 (b). The complex zeros from the residue zeros are not shown here. The dotted lines are complex zeros coming from the real symmetry zeros.

Fig. 5.7. The complex $(s - u)$ plane showing phase contours for fixed negative t in case (b), corresponding to Fig. 5.4 (b) and to Fig. 5.6.

Fig. 6.1. Phase contours in the complex t plane for real s, showing part of the unphysical sheet. Crosses denote resonance poles and small black circles denote zeros of the amplitude. Fig. (a) shows the sheet relevant to $(s + i0, t + i0)$, and (b) shows the sheet relevant to $(s + i0, t - i0)$.

Fig. 7.1. Crossing symmetric phase contours in the real limit $(s + i0, t - i0, u - i0)$, for case (b) of Section 5. Large black dots indicate real zeros and dotted curves indicate complex zeros.

Fig. 7.2. The complex s plane for fixed real t above threshold, showing parts of the unphysical sheets above the s and u thresholds. These phase contours correspond to the real

section given in Fig. 7.1. Poles are denoted by crosses and zeros by large black dots.

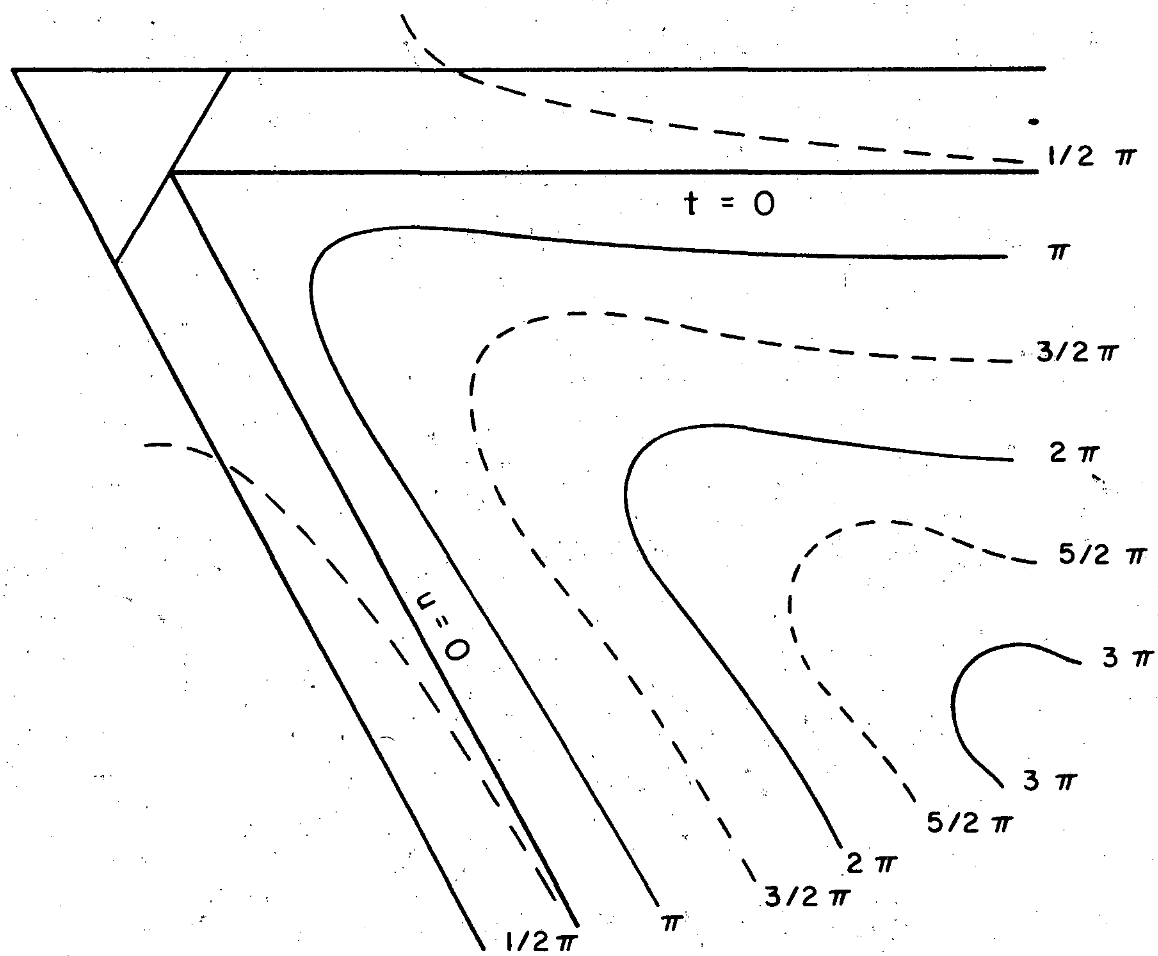
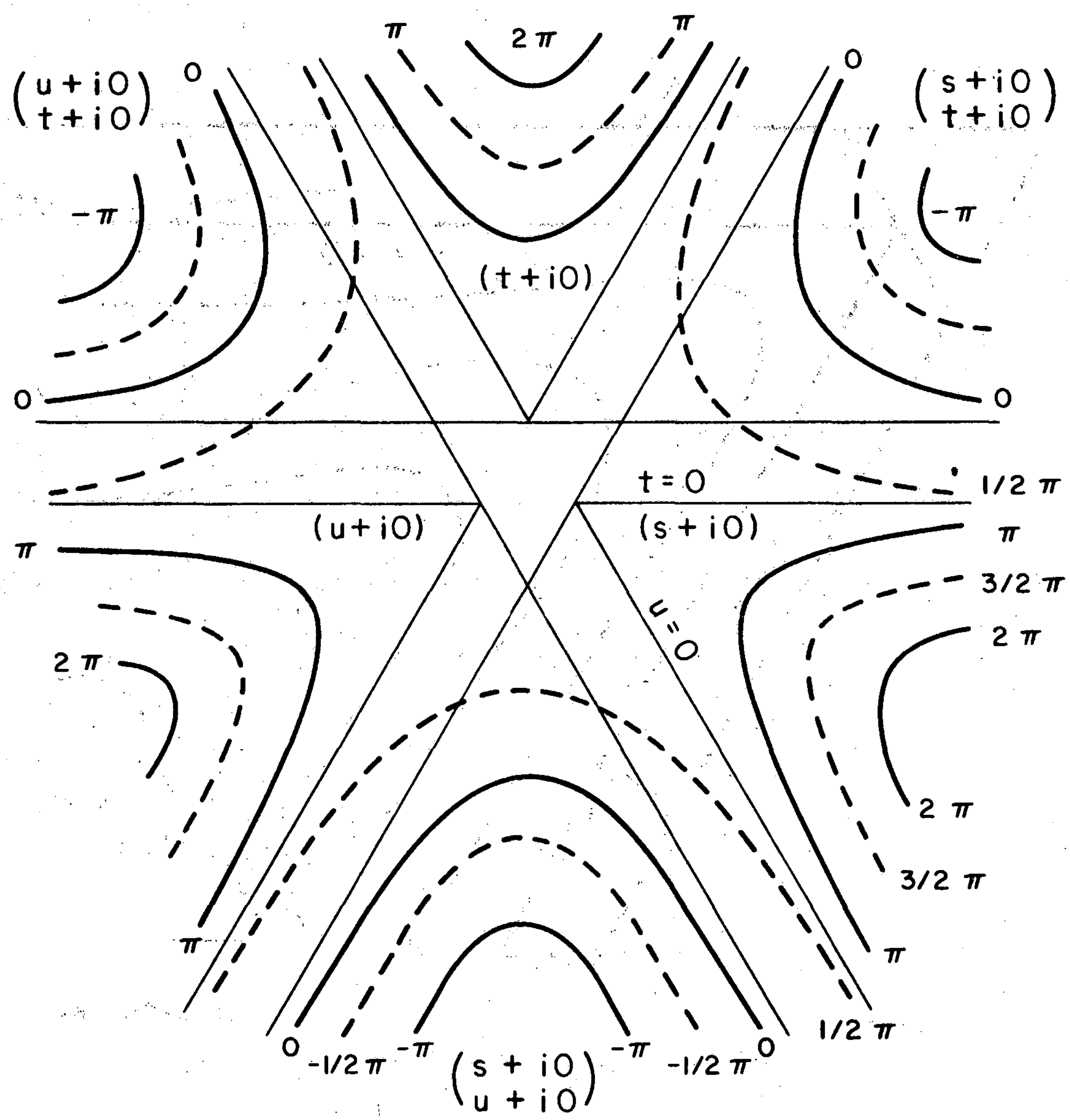


Fig. 3.1



(u+i0 / t+i0)
(s+i0 / t+i0)
(t+i0)
(u+i0)
(s+i0)
(s + i0 / u + i0)
t=0
u=0
2 π
π
-π
0
1/2 π
3/2 π
-1/2 π
XBL6711-5646

Fig. 3.2

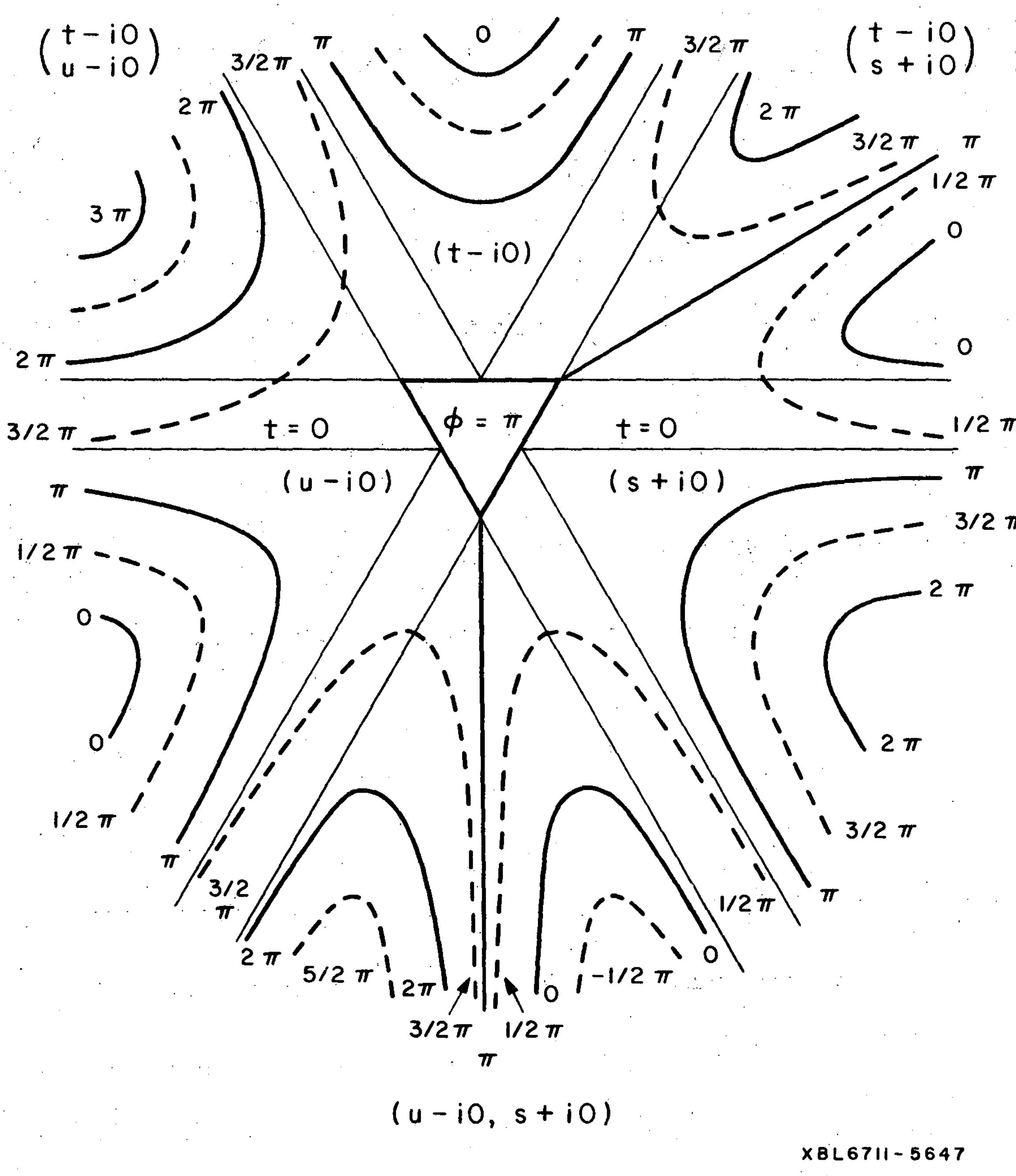
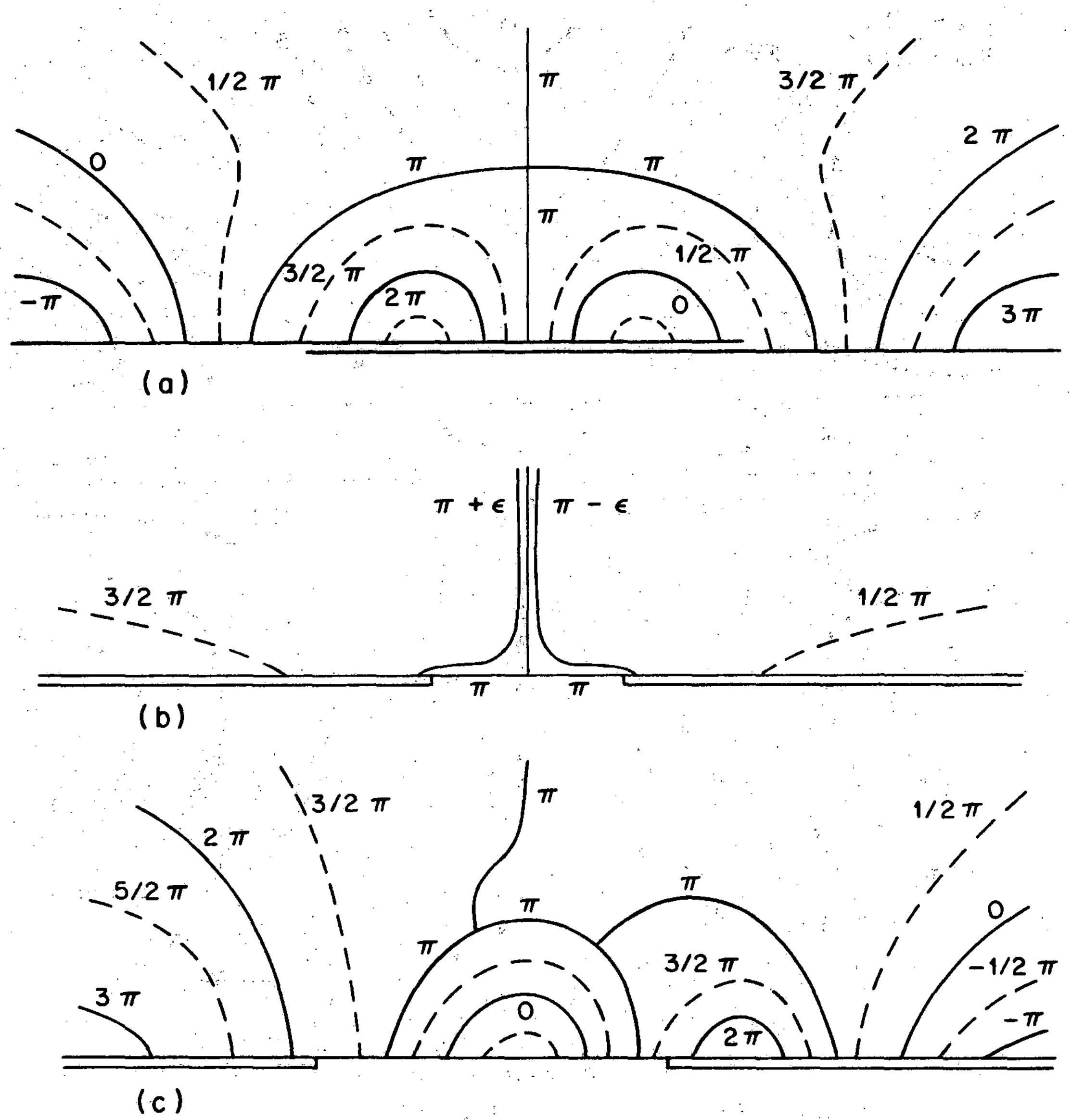


Fig. 3.3



XBL 6711-5648

Fig. 3.4

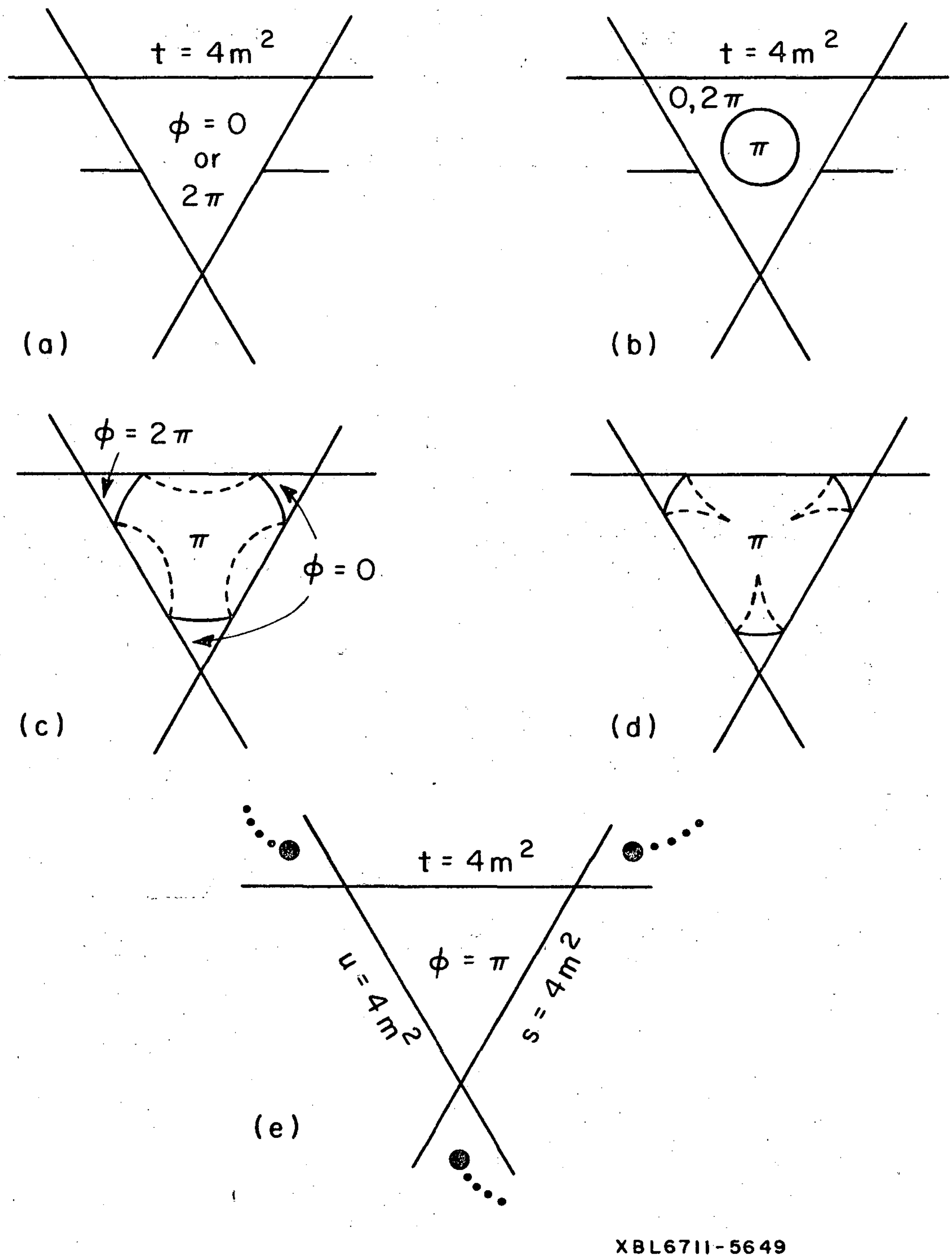

t = 4m²
φ = 0
or
2π
(a)
t = 4m²
0,2π
π
(b)
φ = 2π
π
φ = 0
(c)
π
(d)
t = 4m²
φ = π
u = 4m²
s = 4m²
(e)
XBL6711-5649

Fig. 4.1

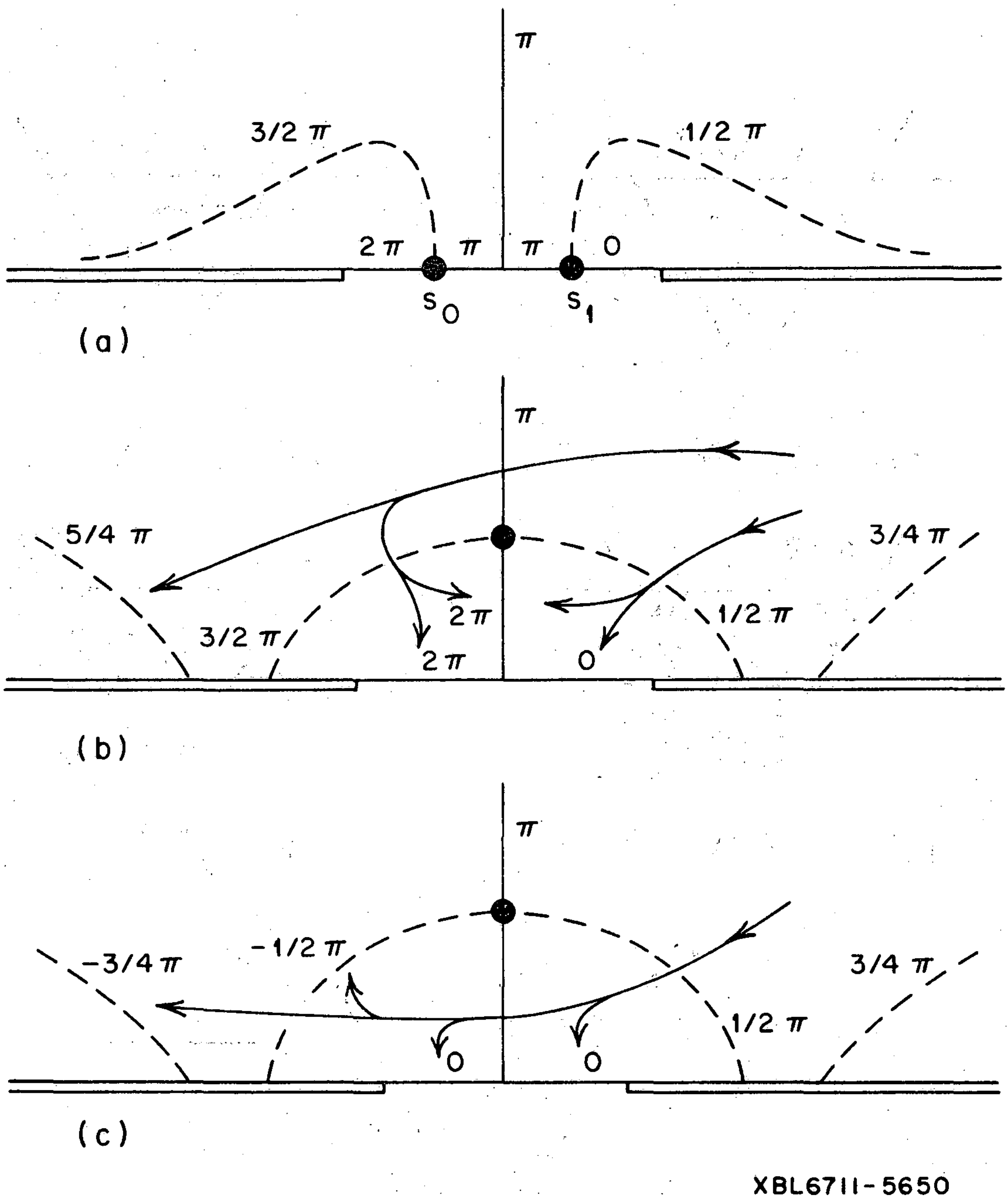

π
3/2 π
1/2 π
2π
π
π
0
s_0
s_1
(a)
π
5/4 π
3/4 π
2π
1/2 π
3/2 π
2π
0
(b)
π
-3/4 π
-1/2 π
3/4 π
1/2 π
0
0
(c)
XBL6711-5650

Fig. 4.2

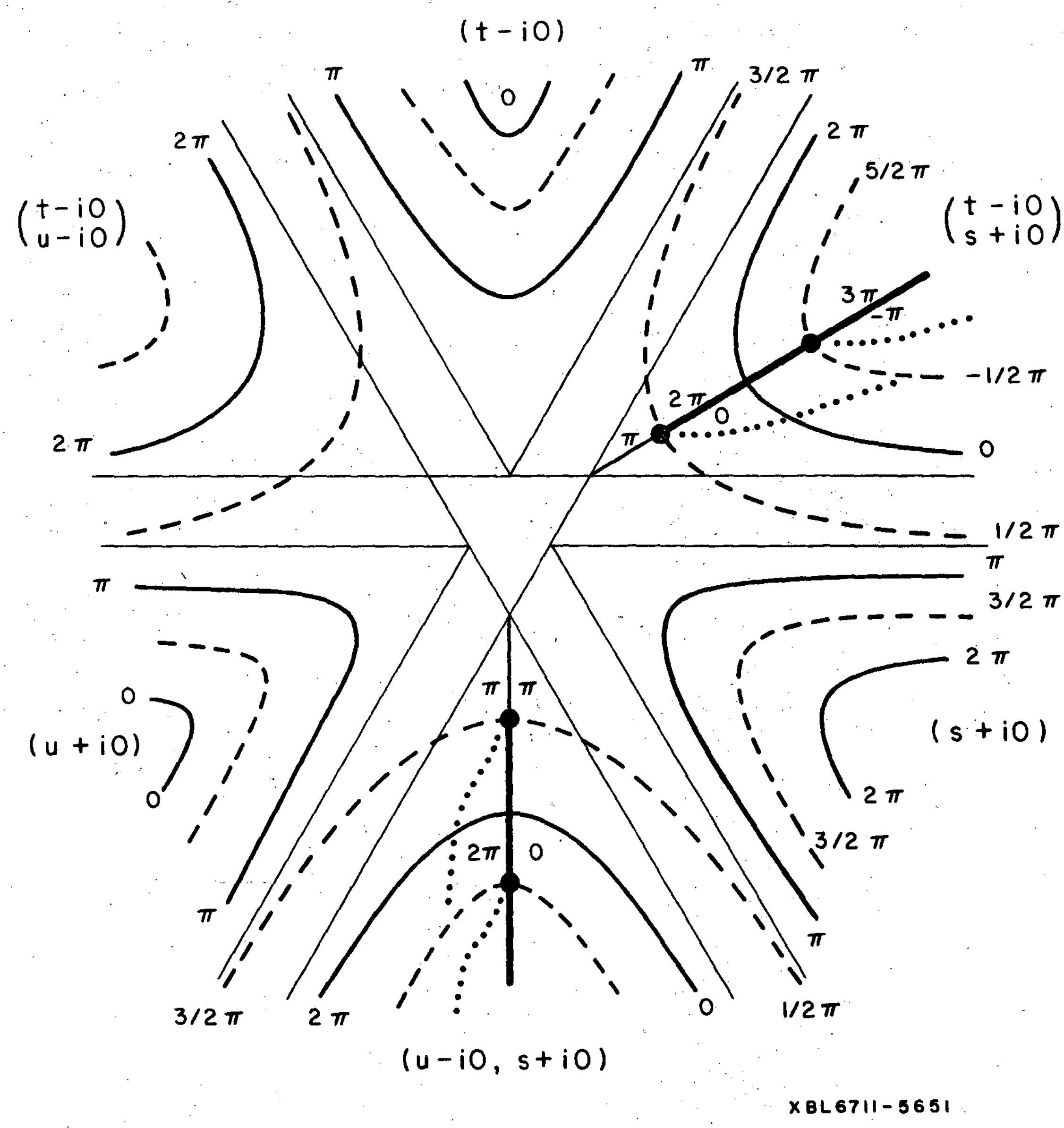


Fig. 4.3

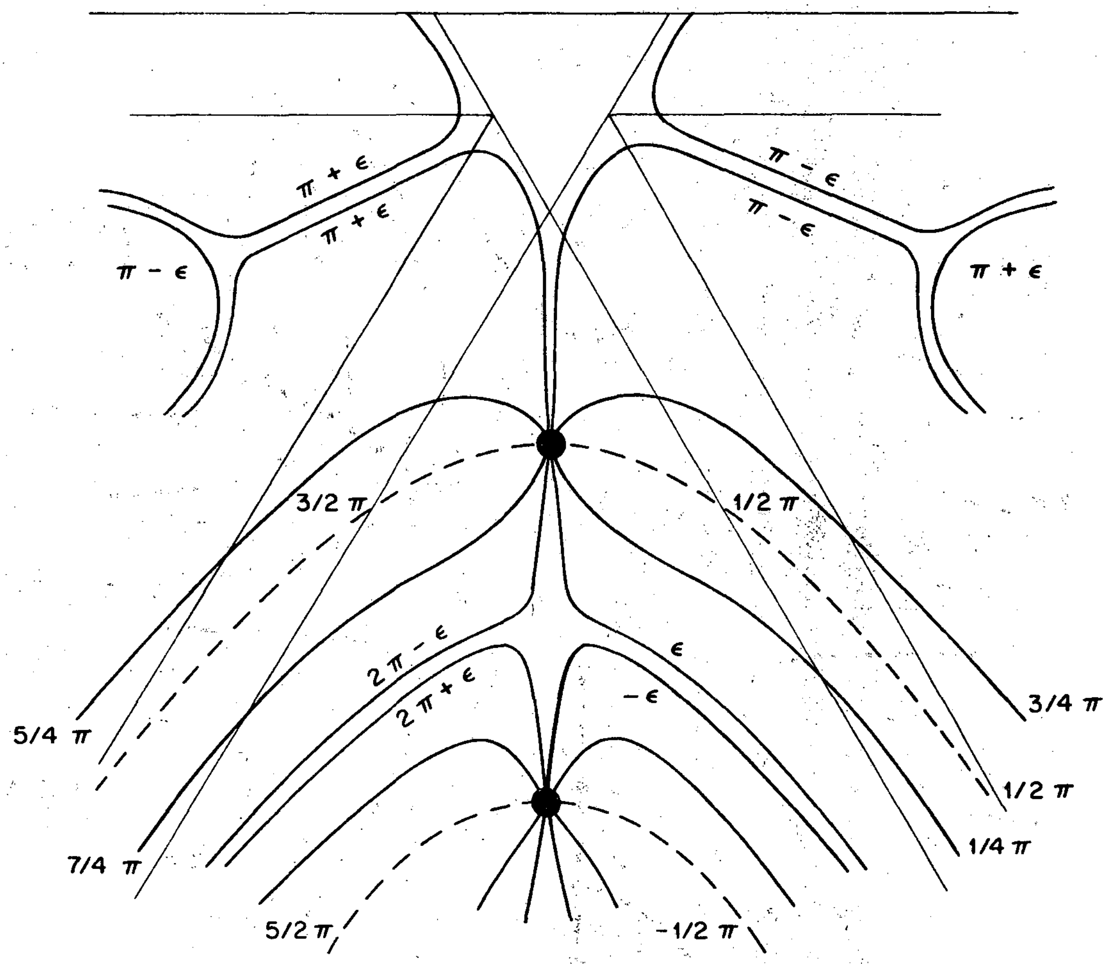

π + ε
π + ε
π − ε
π − ε
π − ε
π + ε
3/2 π
1/2 π
2π − ε
2π + ε
ε
−ε
5/4 π
7/4 π
3/4 π
1/2 π
1/4 π
5/2 π
−1/2 π
XBL6711-5652

Fig. 4.4

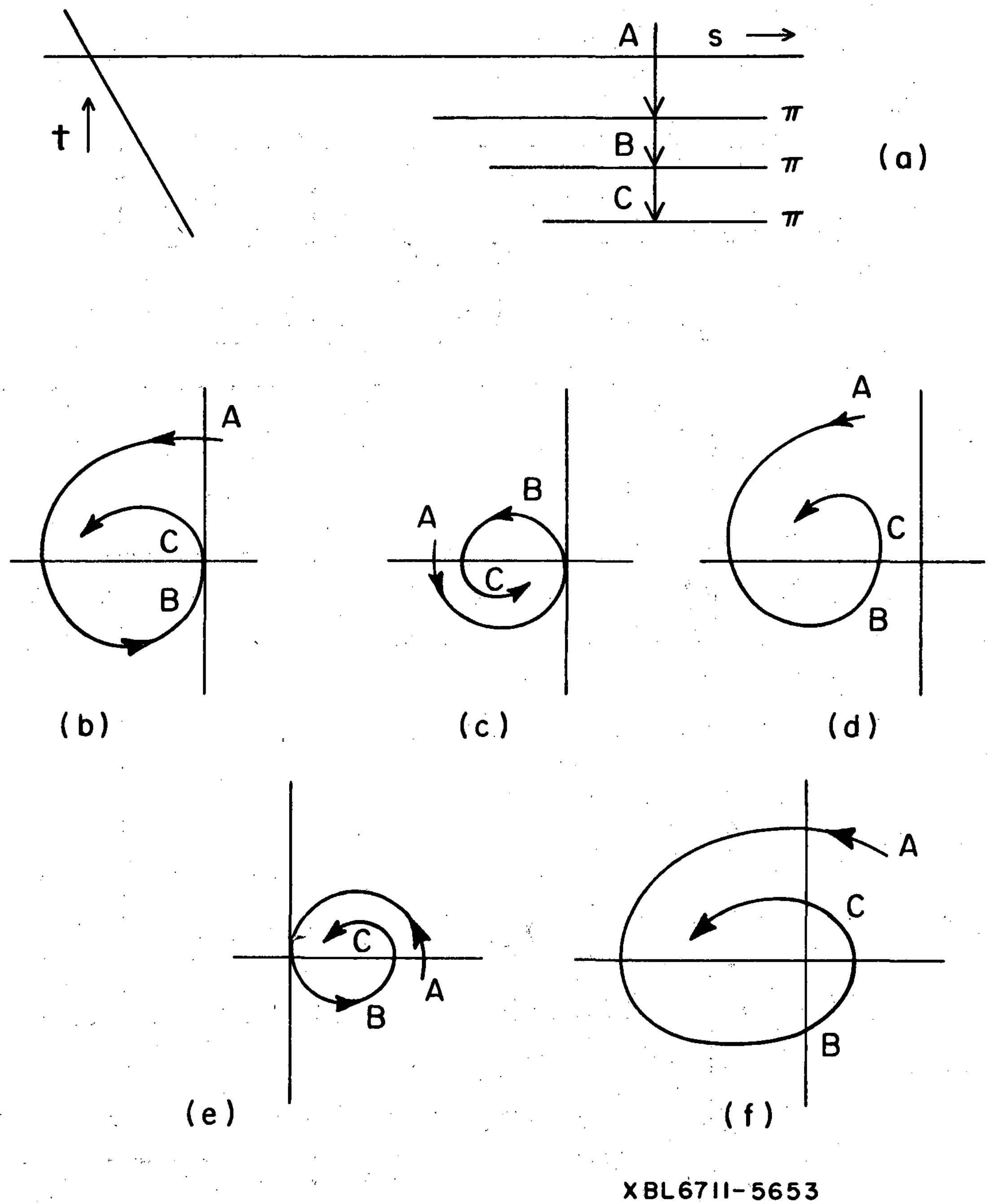
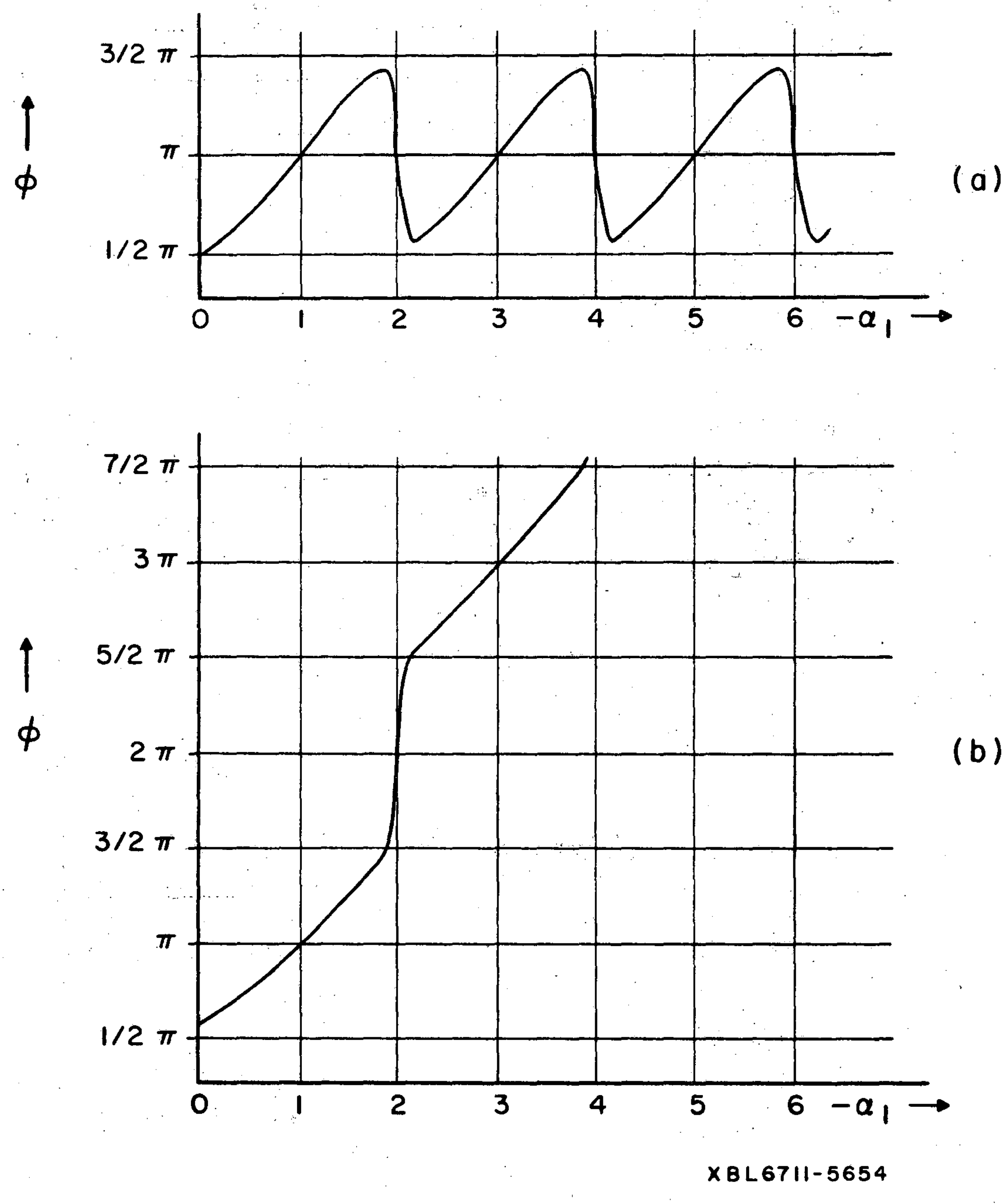


Fig. 5.1



3/2 π
π
1/2 π
φ
(a)
0
1
2
3
4
5
6
−α₁
7/2 π
3 π
5/2 π
2 π
3/2 π
π
1/2 π
φ
(b)
0
1
2
3
4
5
6
−α₁
XBL6711-5654

Fig. 5.2

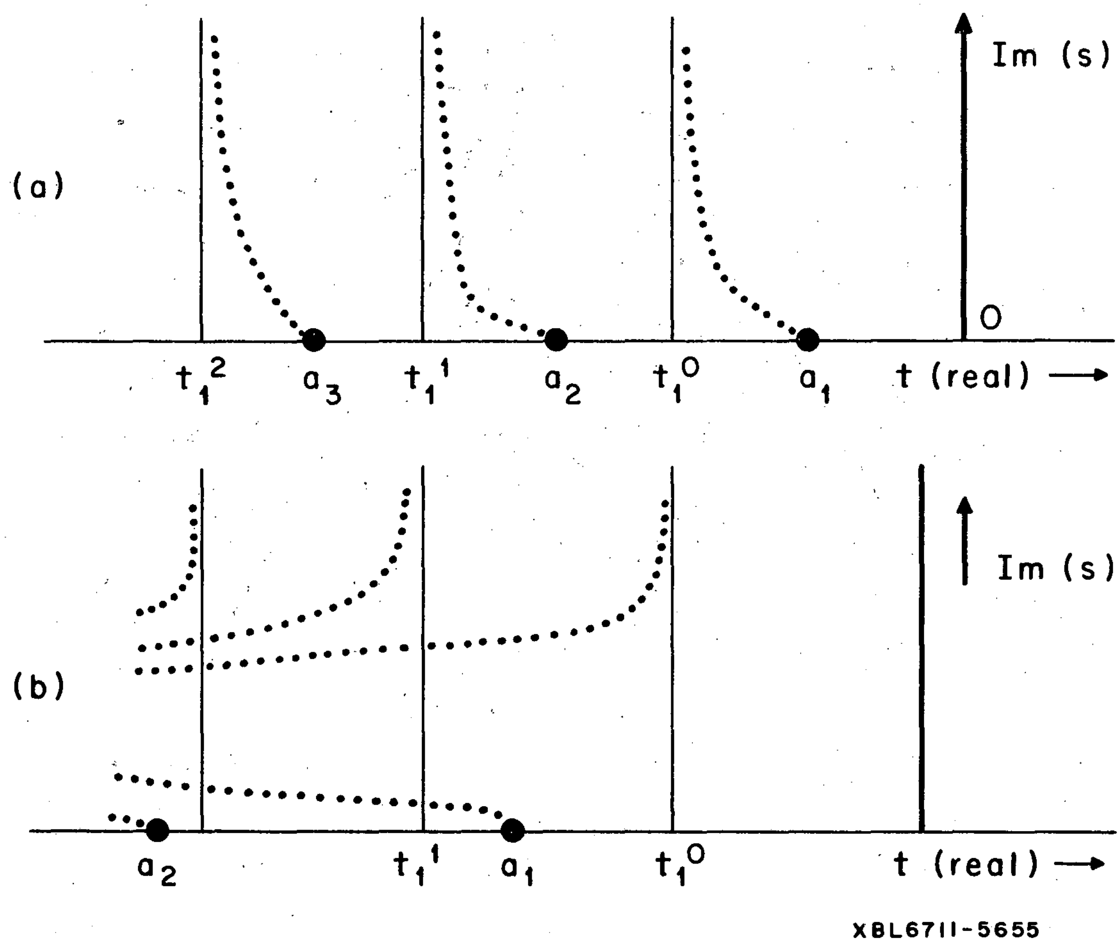

(a)
Im (s)
O
t_1^2
a_3
t_1^1
a_2
t_1^0
a_1
t (real) →
(b)
Im (s)
a_2
t_1^1
a_1
t_1^0
t (real) →
XBL6711-5655

Fig. 5.3

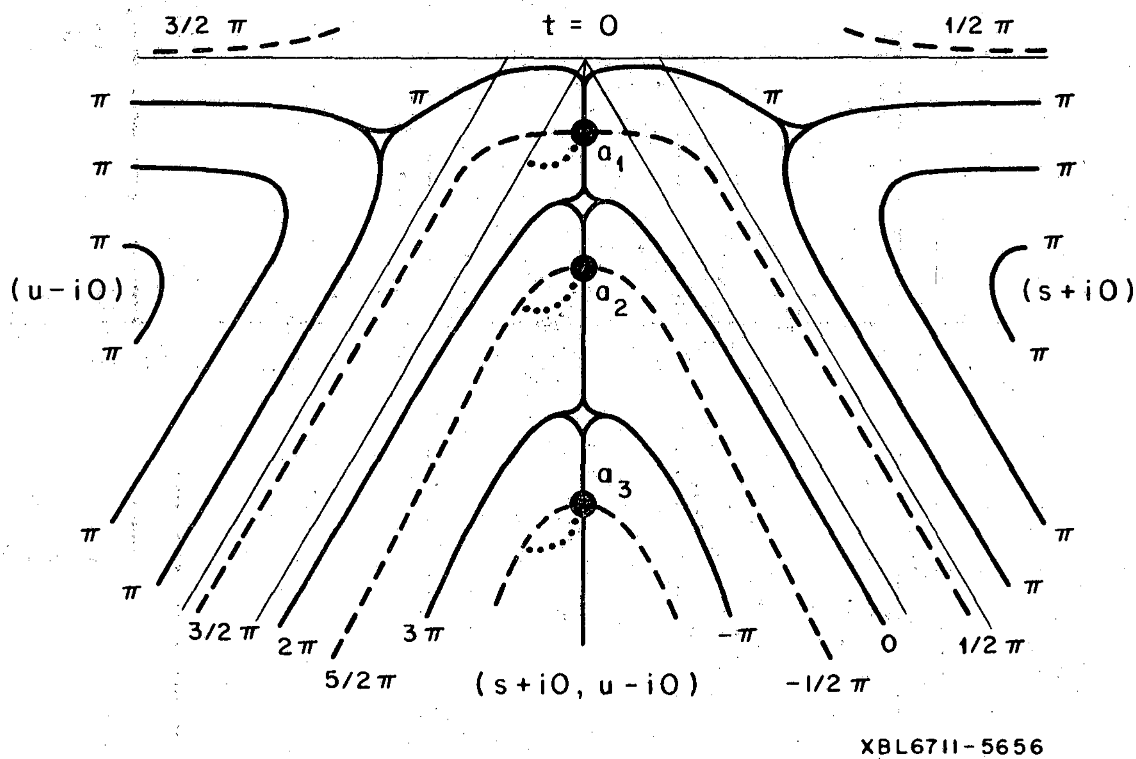

3/2 π
t = 0
1/2 π
π
π
π
π
π
π
π
π
π
π
(u - i0)
(s + i0)
a1
a2
a3
π
π
π
π
3/2 π
2π
3π
-π
0
1/2 π
5/2 π
(s + i0, u - i0)
-1/2 π
XBL6711-5656

Fig. 5.4

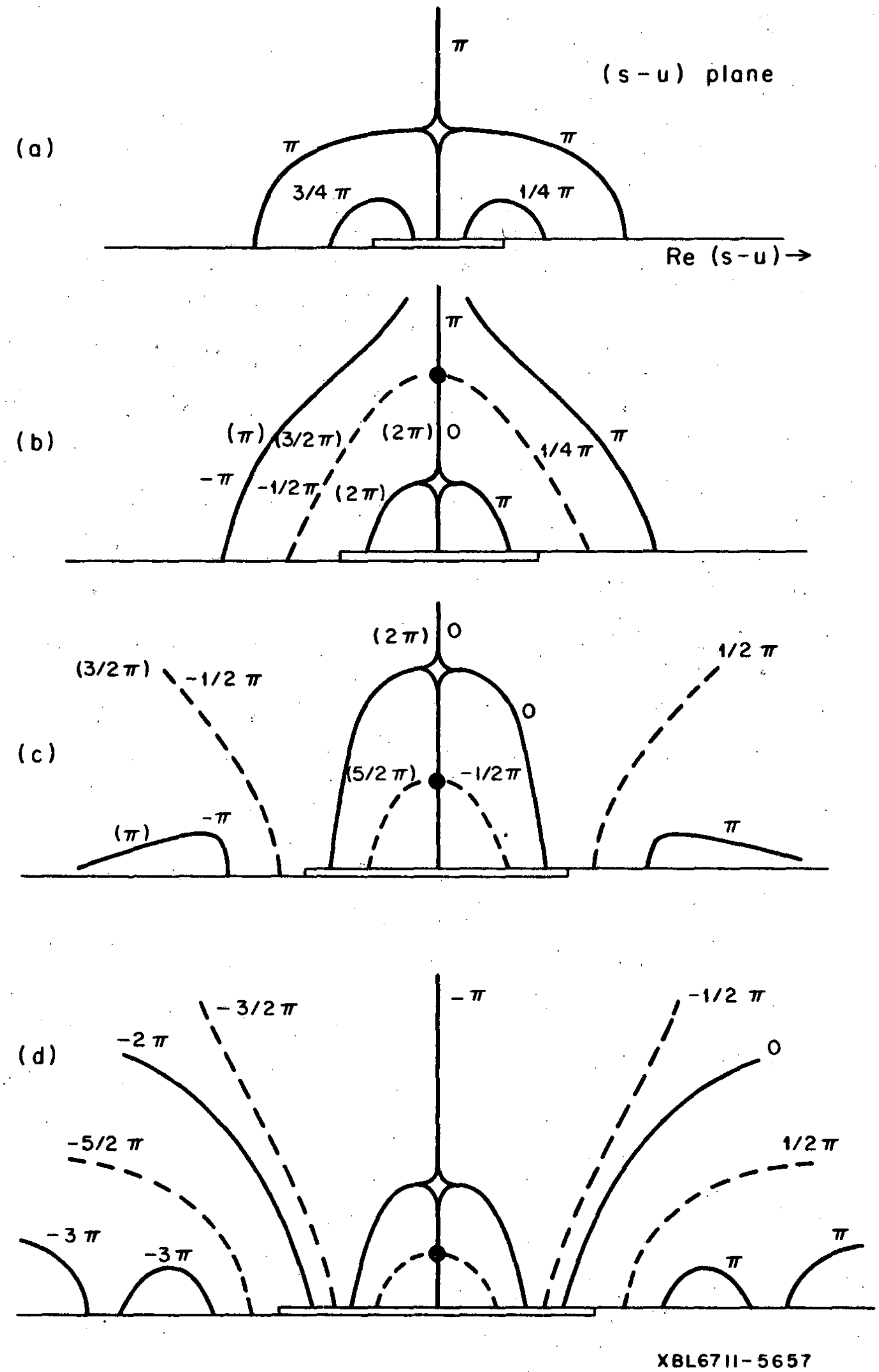

(s-u) plane
(a)
π
π
π
3/4 π
1/4 π
Re (s-u)→
(b)
π
(π)
(3/2π)
(2π)
0
1/4 π
π
-π
-1/2π
(2π)
π
(c)
(2π)
0
(3/2π)
-1/2 π
1/2 π
0
(5/2 π)
-1/2π
(π)
-π
π
(d)
-π
-3/2 π
-1/2 π
-2 π
0
-5/2 π
1/2 π
-3 π
-3 π
π
π
XBL6711-5657

Fig. 5.5

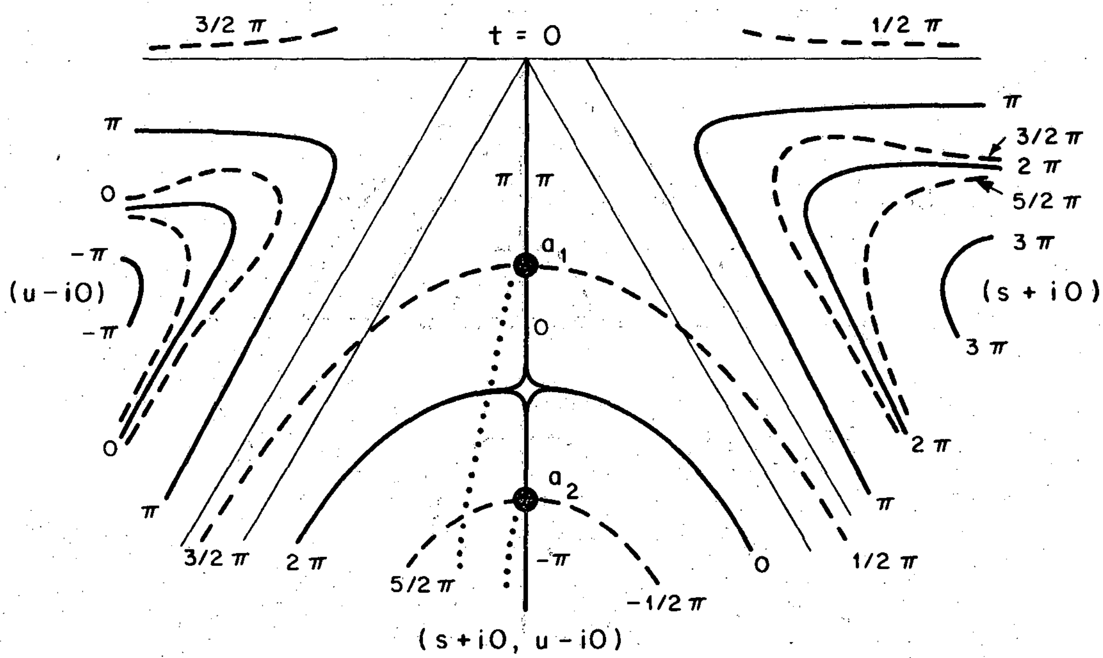

3/2 π
t = 0
1/2 π
π
π
3/2 π
2 π
5/2 π
0
π π
a1
−π
(u − i0)
−π
3 π
(s + i0)
3 π
0
0
2 π
π
π
a2
3/2 π
2 π
5/2 π
−π
−1/2 π
0
1/2 π
(s + i0, u − i0)
XBL6711-5658

Fig. 5.6

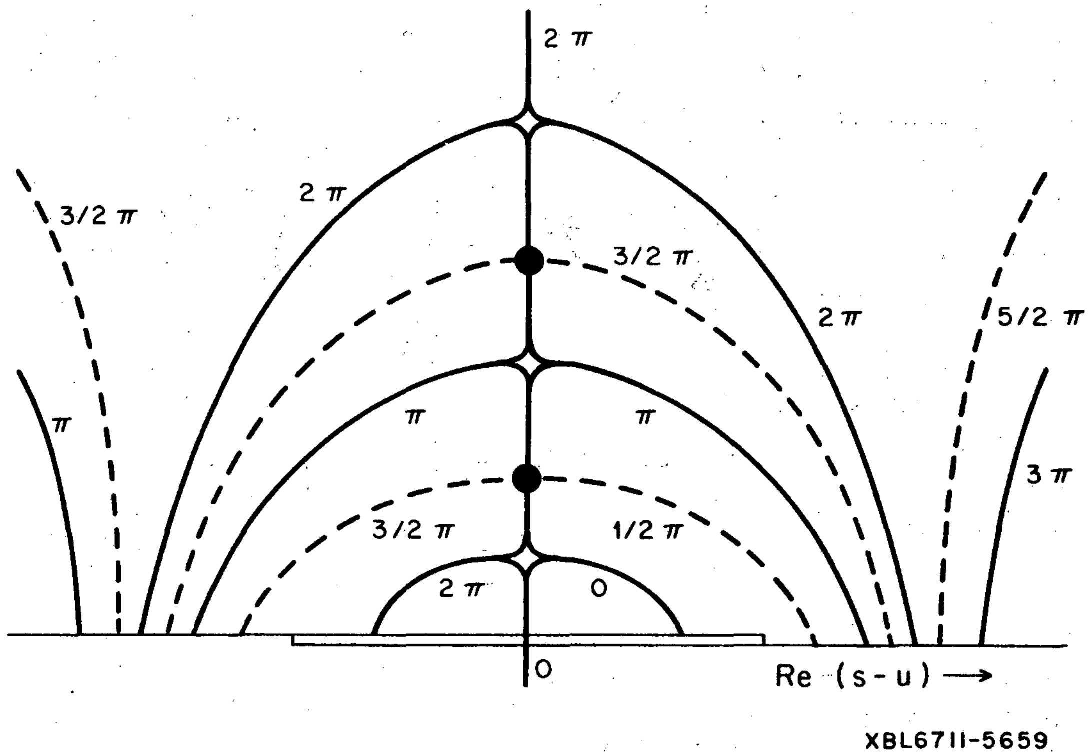


Fig. 5.7

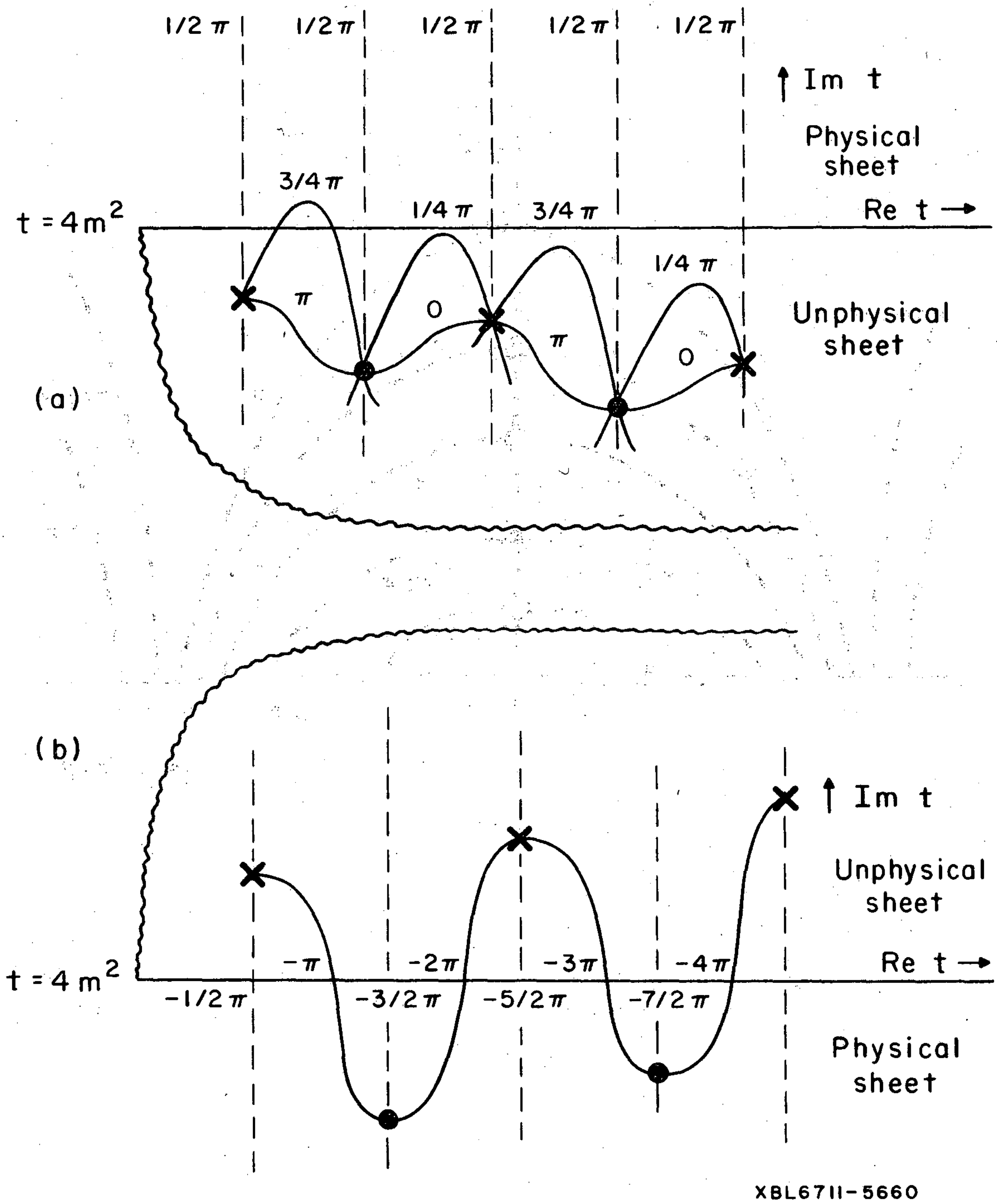

1/2 π
1/2 π
1/2 π
1/2 π
1/2 π
↑ Im t
Physical sheet
3/4 π
1/4 π
3/4 π
Re t →
t = 4m²
1/4 π
π
0
π
0
Unphysical sheet
(a)
(b)
↑ Im t
Unphysical sheet
−π
−2π
−3π
−4π
Re t →
t = 4 m²
−1/2 π
−3/2 π
−5/2 π
−7/2 π
Physical sheet
XBL6711-5660

Fig. 6.1

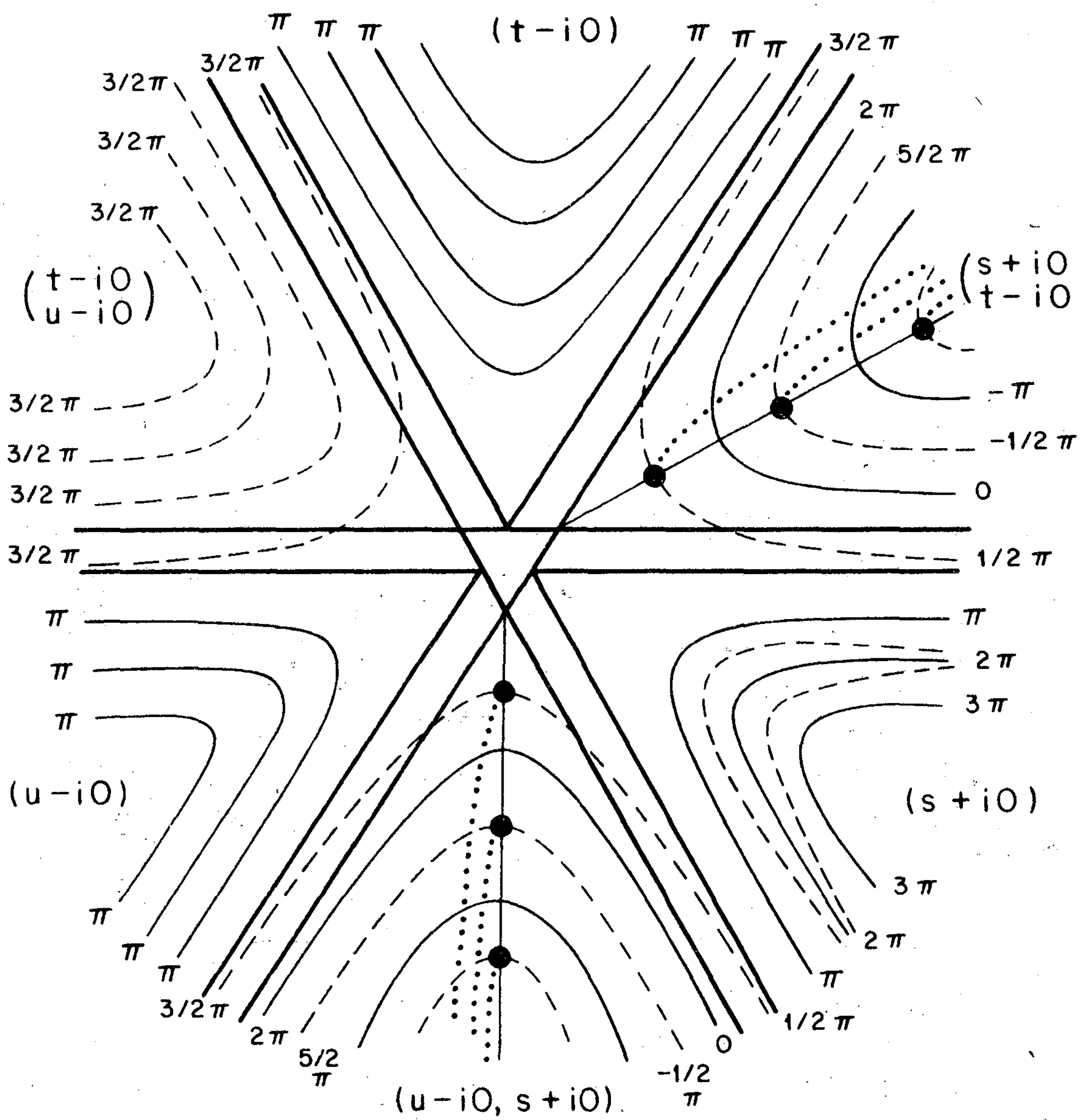

(t − i0)
(t − i0 / u − i0)
(s + i0 / t − i0)
(u − i0)
(s + i0)
(u − i0, s + i0)
π
3/2 π
2 π
5/2 π
3 π
−π
−1/2 π
0
1/2 π
XBL6711-5570

Fig. 7.1

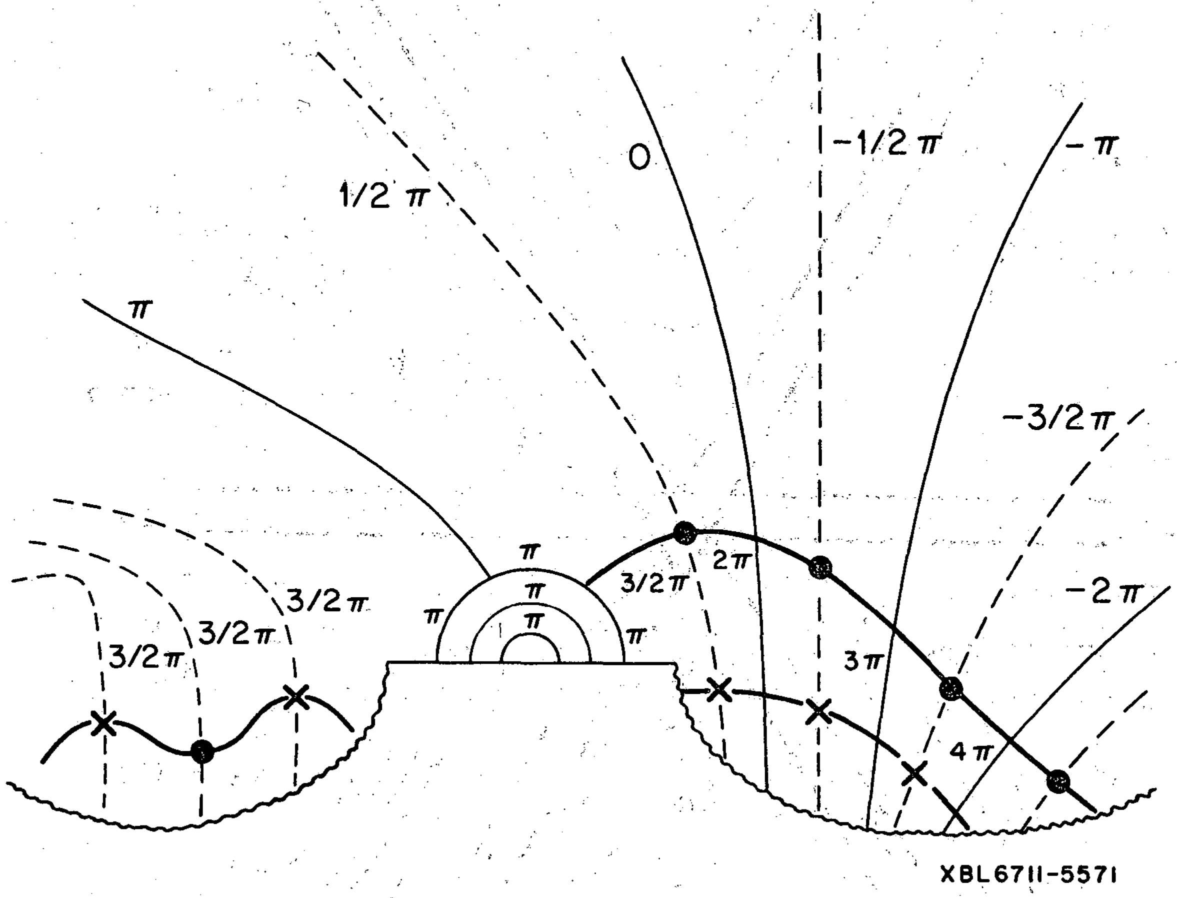


Fig. 7.2