## Title
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**Journal**
The Astrophysical Journal, 486(2)

**ISSN**
0004-637X

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**Publication Date**
1997-09-10

**DOI**
10.1086/310838

Peer reviewed

THE ASTROPHYSICAL JOURNAL, 486:L111–L114, 1997 September 10


# GENERAL RELATIVISTIC EFFECTS IN THE NEUTRINO-DRIVEN WIND AND *r*-PROCESS NUCLEOSYNTHESIS

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*Received 1997 January 22; accepted 1997 July 1*

## ABSTRACT

We discuss general relativistic effects in the steady state neutrino-driven "wind" that may arise from nascent neutron stars. In particular, we generalize previous analytic estimates of the entropy per baryon $S$, the mass outflow rate $\dot{M}$, and the dynamical expansion timescale $\tau_{\rm dyn}$. We show that $S$ increases and $\tau_{\rm dyn}$ decreases with increasing values of the mass-to-radius ratio describing the supernova core. Both of these trends indicate that a more compact core will lead to a higher number of neutrons per seed nucleus. Such an enhancement in the neutron/seed ratio may be required for successful *r*-process nucleosynthesis in neutrino-heated supernova ejecta.

*Subject headings:* equation of state — nuclear reactions, nucleosynthesis, abundances — relativity — supernovae: general

The production site of the *r*-process elements (Burbidge et al. 1957; Cameron 1957) is a long-standing problem (see Mathews & Cowan 1990). One of the most promising candidate sites for *r*-process nucleosynthesis is the neutrino-heated ejecta from the post–core-bounce environment of a Type II or Type Ib supernova (Meyer et al. 1992; Woosley & Hoffman 1992; Takahashi, Witti, & Janka 1994; Woosley et al. 1994). These *r*-process calculations, though promising, cannot reproduce the solar system *r*-process abundance pattern without an artificial increase in the neutron-to-seed nucleus ratio over that predicted in hydrodynamical calculations and simple wind models (see, e.g., Hoffman, Woosley, & Qian 1996; Meyer, Brown, & Luo 1996).

Qian & Woosley (1996, hereafter QW) used a simple model of the neutrino-driven wind to obtain both analytic and numerical estimates of quantities upon which the nucleosynthesis abundance yield depends. These quantities include the electron fraction $Y_e$, the entropy per baryon $S$, the mass outflow rate $\dot{M}$, and the dynamic expansion timescale $\tau_{\rm dyn}$—all of which are important in setting the neutron/seed nucleus ratio prior to the epoch of rapid neutron capture.

In this Letter we generalize the analytic derivations in QW to include general relativistic effects. Such effects are of potential interest in light of "soft" nuclear matter equations of state—involving, for example, kaon condensation (Thorsson, Prakash, & Lattimer 1994)—which could lead to very compact supernova cores and even black holes as supernova remnants (Bethe & Brown 1995; Woosley & Timmes 1996). While QW reported two sample numerical calculations involving post-Newtonian corrections, they did not present corresponding analytic calculations. Numerical calculations reported in QW suggest that general relativistic effects go in the direction of making conditions more favorable for the *r*-process. We here allow for general relativistic effects in analytical calculations, including effects not included in the numerical calculations by QW, namely, the redshift of neutrino energies and bending of the neutrino trajectories.

In the analytic approximations performed by QW, the calculation of the quantities $S$, $\dot{M}$, and $\tau_{\rm dyn}$ essentially decouples from the calculation of $Y_e$. We will present analytic estimates for $S$, $\dot{M}$, and $\tau_{\rm dyn}$; general relativistic effects on $Y_e$ have been considered in another paper (Fuller & Qian 1996). Unless units are explicitly given, we take $\hbar = c = k = 1$.

The wind equations in Duncan, Shapiro, & Wasserman (1986) and QW can be generalized to allow for relativistic outflow velocities and general relativistic effects in a static Schwarzschild spacetime. A detailed derivation and discussion will appear elsewhere (Cardall & Fuller 1997, hereafter CF). We here simply present the general relativistic analogs of equations (24)–(26) of QW, which give the radial evolution of the velocity, "flow energy" per baryon $\epsilon_{\rm flow}$, and entropy per baryon:

$$\frac{v}{1 - v^2}\left(1 - \frac{v_s^2}{v^2}\right)\frac{dv}{dr} = \frac{1}{r}\left[\frac{2}{3}\frac{TS}{m_N} - \frac{(1 - TS/3m_N)}{(1 - 2GM/r)}\frac{GM}{r}\right] - \frac{q}{3vy(1 + TS/m_N)}, \tag{1}$$

$$\frac{q}{(1 + TS/m_N)} = vy\frac{d}{dr}\left[\ln\left(1 + \frac{TS}{m_N}\right) - \frac{1}{2}\ln(1 - v^2) + \frac{1}{2}\ln\left[\left(1 - \frac{2GM}{r}\right)\right]\right] \equiv vy\frac{d}{dr}\epsilon_{\rm flow}, \tag{2}$$

$$vy\frac{dS}{dr} = \frac{m_N q}{T}. \tag{3}$$

In these equations, $v$ is the outflow velocity as measured by an observer at rest in the Schwarzschild spacetime, $v_s \equiv (TS/3m_N)^{1/2}$ is the sound speed, $T$ is the temperature, $S$ is the entropy per baryon, $m_N$ is the baryon rest mass, $M$ is the mass of the proto–neutron star, $q$ is the heating rate per unit mass, and $y \equiv [(1 - 2GM/r)/(1 - v^2)]^{1/2}$. These equations assume radiation-dominated conditions, i.e., that the pressure and energy density are dominated by relativistic particles; a steady state outflow; and that the proto–neutron star is the dominant source of gravity. These equations could describe the environment above the proto–neutron star in the time frame of a few to about 20 s after core bounce. During this time, intense neutrino heating of the outer layer of the proto–neutron star drives mass loss from the surface. This epoch just follows shock liftoff and the earliest stage of deleptonization and rapid collapse of the proto–neutron star mantle. In the spirit of the present work on exploring the possible effects of general relativity, we assume a net one-dimensional outflow and ignore possible multidimensional hydrodynamic effects and late-time accretion.

The specific net heating rate $q$ includes a number of processes. Heating processes include $\nu$ absorption on nucleons, $\nu e$ scattering, and $\nu\bar{\nu}$ annihilation, both of which produce low-energy neutrinos that readily escape. The major cooling processes are $e^{\pm}$ capture on free nucleons and $e^{\pm}$ annihilation. Following QW, we will consider only the nucleonic processes as the dominant contributors to the net heating rate.

The heating rates depend on the neutrino distribution functions and geometric factors involving the maximum neutrino deviation angle from the radial direction at a given radius. General relativistic effects can be introduced through redshift factors and by suitably altering the geometric factors to account for curved neutrino trajectories (CF). For example, if the neutrino distribution functions can be parametrized as Fermi-Dirac functions with "temperature parameters" $T_\nu$ and "degeneracy parameters" $\eta_\nu$, then the rate for neutrino absorption on nucleons is roughly $q_{\nu N} \propto T_\nu^6 F_5(\eta_\nu)$, where $F_5(s) \equiv \int_0^\infty dx\, x^5[e^{(x-s)} + 1]^{-1}$. This heating rate would be subject to six redshift factors, one for each power of $T_\nu$.

Some assumption must also be made about the dependence on $GM/R$ of the initial neutrino distribution functions at the proto–neutron star surface ($R$ = coordinate radius of the proto–neutron star in the Schwarzschild geometry). Since we are not using actual neutrino spectra calculated from proto–neutron star structure and neutrino transport calculations with various equations of state, we will make the simple assumption[1] that the product $L_{\bar{\nu}_e}\varepsilon^2_{\bar{\nu}_e}$ is a constant function of $GM/R$, where $L_{\bar{\nu}_e}$ is the $\bar{\nu}_e$ luminosity at $R$ and $\varepsilon^2_{\bar{\nu}_e} \equiv \langle E^3_{\bar{\nu}_e}\rangle/\langle E_{\bar{\nu}_e}\rangle$ at $R$. This assumption corresponds to the *Ansatz* that

$$T_\nu(R)^6 F_5(\eta_\nu) \propto (2GM/R)^2,\ R > 3GM, \tag{4}$$

$$T_\nu(R)^6 F_5(\eta_\nu) \propto (1 - 2GM/R)^{-1},\ R < 3GM. \tag{5}$$

Thus, according to this *Ansatz*, the average neutrino energies at the proto–neutron star surface increase somewhat with increasing $GM/R$. The dependence of $S$ on the precise nature of this *Ansatz* is weak, but we caution that the dependence of $\dot{M}$ and $\tau_{\rm dyn}$ on this assumption could be significant. The proportionalities in equations (4) and (5) are normalized by taking $L_{\bar{\nu}_e} = 10^{51}$ ergs s$^{-1}$ and $\sqrt{\varepsilon^2_{\bar{\nu}_e}} = 20$ MeV for a proto–neutron star with $M = 1.4\ M_\odot$ and $R = 10$ km.

Before proceeding to analytic estimates, we briefly discuss initial and boundary conditions. Treatment begins at the surface of the proto–neutron star, taken to be Schwarzschild coordinate radius $R$. As in QW, the initial entropy is taken to be $S_i = 4$ ($S > 4$ assures radiation-dominated conditions). The initial temperature $T_i$ is obtained by equating at $r = R$ the rates for heating from $\nu$ absorption on nucleons and neutrino cooling by $e^{\pm}$ capture on nucleons. Typical values are $T_i \sim$ 3–5 MeV, with the higher initial temperatures arising from larger values of $2GM/R$. Neutrino heating becomes negligible by $T \approx 0.5$ MeV, where $\epsilon_{\rm flow}$ and $S$ reach their final values, after which they remain constant. These final values will be denoted by a subscript $f$. We assume that a boundary is provided by the supernova shock at some large radius, where the temperature is $T_b$ and where the flow proceeds asymptotically to small velocity. We then have from equation (2)

$$\ln\left(1 + \frac{TS_f}{m_N}\right) - \frac{1}{2}\ln(1 - v^2) + \frac{1}{2}\ln\left(1 - \frac{2GM}{r}\right) \approx \epsilon_{\rm flow,f} \approx \ln\left(1 + \frac{T_b S_f}{m_N}\right) \tag{6}$$

between $T = 0.5$ MeV and $T_b$.

We now obtain an approximate expression for $S_f$. From equation (2),

$$\int_R^{r_f} \frac{q}{(1 + TS/m_N)}\frac{dr}{vy} = \epsilon_{{\rm flow},f} - \epsilon_{{\rm flow},i}. \tag{7}$$

Since the heating rate $q$ has a fairly strong peak (QW), we approximate the left-hand side of equation (7) to obtain

$$\frac{1}{(1 + T_{\rm eff}S_{\rm eff}/m_N)}\int_R^{r_f} q\frac{dr}{vy} \approx \ln\left(1 + \frac{T_b S_f}{m_N}\right) - \frac{1}{2}\ln\left(1 - \frac{2GM}{R}\right), \tag{8}$$

where $T_{\rm eff}$ and $S_{\rm eff}$ are the temperature and entropy at the maximum of $q$. From equation (3),

$$S_f \approx S_f - S_i = m_N \int_R^{r_f} \frac{q}{T}\frac{dr}{vy}. \tag{9}$$

Combining these last two equations, taking $\ln(1 + T_b S_f/m_N) \approx T_b S_f/m_N$, and using the definitions (QW) $S_{\rm eff} \approx S_f/2$ and $T_{\rm eff} \equiv [\int(vy)^{-1}q\,dr]/[\int(vyT)^{-1}q\,dr]$, we obtain

$$S_f \approx -\frac{m_N \ln(1 - 2GM/R)}{2T_{\rm eff}} \times \left[1 - \frac{T_b}{T_{\rm eff}} - \frac{1}{4}\ln\left(1 - \frac{2GM}{R}\right)\right]^{-1}. \tag{10}$$

The quantity $T_{\rm eff}$ is an average temperature, weighted by the heating rate. In QW it is approximated as the temperature at which the heating rate is a maximum, and they obtain $T_{\rm eff} = 6^{-1/6}T_i$. Neutrino redshift and the bending of neutrino trajectories partially offset each other, but the redshift is the dominant effect. The net result is that the heating occurs closer to the surface of the proto–neutron star, i.e., $T_{\rm eff}$ approaches $T_i$ for increasing values of $2GM/R$. Figure 1 shows the ratio $T_{\rm eff}/T_{\rm i}$ as a function of $2GM/R$. Figure 2 is a plot of $S$ as a function of $2GM/R$, obtained from equation (10) with $T_b = 0.1$ MeV. The circle shows the numerical result obtained in QW for their model 10B with post-Newtonian corrections. The agreement is excellent.

The method of estimating $\dot{M}$ closely follows that of QW. Integrating equation (2) above $r_{\rm eff}$, the radius where the heating is maximum, we have

$$\int_{r_{\rm eff}}^{r_f} \frac{q}{(1 + TS/m_N)}\frac{dr}{vy} = \epsilon_{{\rm flow},f} - \epsilon_{\rm flow,eff}, \tag{11}$$

[1] This assumption is motivated as follows. The luminosity is roughly equal to the gravitational binding energy released in the collapse ($\sim GM^2/R$) divided by the neutrino diffusion timescale. The diffusion timescale is approximately $R^2$ divided by the neutrino mean free path. The mean free path is essentially the inverse of nucleon density ($\propto M/R^3$) times cross section. Since the cross section is proportional to neutrino energy squared, the product $L\varepsilon^2$ scales like (binding energy)/($R^2 \times$ nucleon density) $\propto M$. In our calculations we assume $M = 1.4\ M_\odot$ and vary $GM/R$ by varying the proto–neutron star radius $R$.

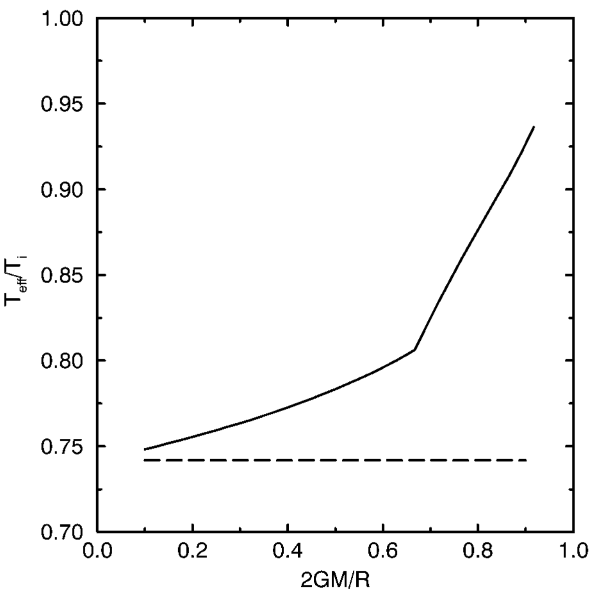


FIG. 1.—*Solid line*: The ratio of the effective temperature $T_{\rm eff}$, where the neutrino heating rate is a maximum, to the initial temperature $T_i$ at the proto–neutron star surface. Note the kink at $R = 3GM$; below this radius, there is a maximum angle of deviation from the radial direction beyond which massless particles cannot escape to infinity. *Dashed line*: The ratio $T_{\rm eff}/T_i$, ignoring neutrino redshift and trajectory bending effects.

$$\frac{1}{(1 + T_{\rm eff}S_{\rm eff}/m_N)}\int_{r_{\rm eff}}^{r_f} q\,\frac{dr}{vy} \approx -\frac{1}{2}\left[\frac{1}{2}\ln\left(1 - \frac{2GM}{r_{\rm eff}}\right)\right], \quad (12)$$

where we have taken $T_b\,S_f/m_N \approx 0$ and $\epsilon_{\rm flow,eff} \approx T_{\rm eff}\,S_{\rm eff}/m_N + \frac{1}{2}\ln(1 - 2GM/r_{\rm eff}) \approx \frac{1}{2}[\frac{1}{2}\ln(1 - 2GM/r_{\rm eff})]$. On the left-hand side of equation (11), $q \approx q_{\nu N}$ has a factor of $r^{-2}$, so that $1/(r^2vy)$ can be replaced using baryon mass conservation in the Schwarzschild geometry ($\dot{M} = 4\pi r^2\rho_b vy$ = constant). The resulting integral involving $\rho_b$ can be estimated by similar

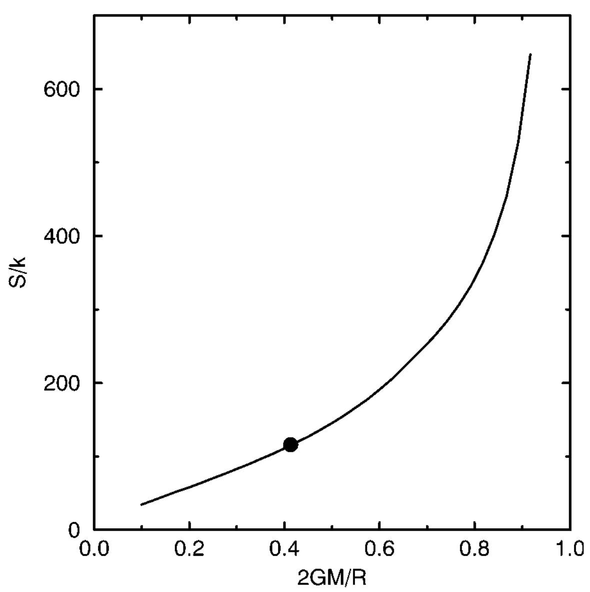


FIG. 2.—The final entropy per baryon in units of Boltzmann's constant, as a function of the Schwarzschild radius divided by the proto–neutron star radius. The circle is from the QW numerical calculation of model 10B with post-Newtonian corrections.

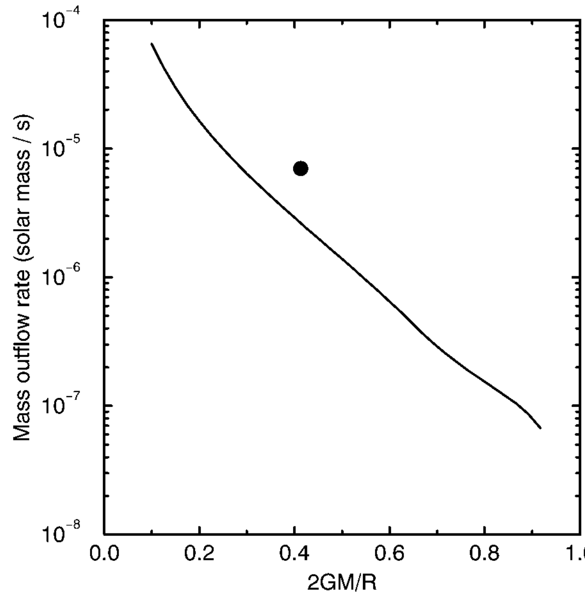


FIG. 3.—The mass outflow rate as a function of the Schwarzschild radius divided by the proto–neutron star radius. The circle is from the QW numerical calculation of model 10B with post-Newtonian corrections.

methods as in QW (see CF for details), and the equation thus derived from equation (11) can be solved for $\dot{M}$. A plot of $\dot{M}$ as a function of $2GM/R$ is given in Figure 3.

The dynamic expansion timescale $\tau_{\rm dyn}$ is defined to be

$$\tau_{\rm dyn} \equiv \frac{1}{(vy)_f}\left|\frac{1}{T}\frac{dT}{dr}\right|_f^{-1}, \quad (13)$$

where the subscript $f$ means that the quantity is evaluated at $T$ = 0.5 MeV. The dynamic expansion timescale is closely related to the proper time a fluid element experiences in going from $T_1$ = 0.5 MeV to, say, $T_2$ = 0.2 MeV:

$$\int_{T_1}^{T_2} d\tau_{\rm proper} = \int_{T_1}^{T_2}\frac{dr}{vy} = \int_{T_1}^{T_2}\frac{T}{vy}\frac{dr}{dT}\frac{dT}{T}$$
$$\approx \frac{1}{(vy)_f}\left|\frac{1}{T}\frac{dT}{dr}\right|_f^{-1}\ln\left(\frac{0.5}{0.2}\right) \approx \tau_{\rm dyn}. \quad (14)$$

We estimate the temperature scale height by approximating the boundary condition in equation (6). Taking $T_bS_f/m_N \approx 0$ and $v^2 \ll TS_f/m_N$, $2GM/r \ll 1$ for $T < 0.5$ MeV, equation (6) gives $TS_f/m_N \approx GM/r$. Then (recalling that $S_f$ is now constant out at this radius),

$$\left|\frac{1}{T}\frac{dT}{dr}\right|_f^{-1} = r_f = \frac{GMm_N}{S_fT_f}. \quad (15)$$

The quantity $(vy)_f$ can be obtained from the previously computed quantities $\dot{M}$ and $S_f$. A plot of $\tau_{\rm dyn}$ as a function of $2GM/R$ is given in Figure 4.

Neutron-to-seed ratios on the order of 100 or greater are necessary for a successful $r$-process, and this can be achieved under a variety of combinations of $S$, $\tau_{\rm dyn}$, and $Y_e$ (Hoffman et al. 1996; Meyer et al. 1996; Meyer & Brown 1997). We have here confirmed that general relativistic effects increase $S$ and reduce $\tau_{\rm dyn}$, as seen in selected numerical calculations in QW. Both of these trends lead to a higher neutron-to-seed ratio. On

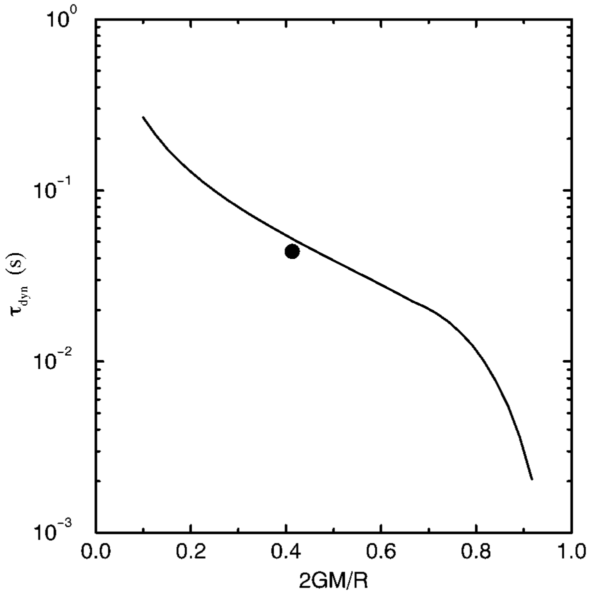


FIG. 4.—The dynamic expansion timescale as a function of the Schwarzschild radius divided by the proto–neutron star radius. The circle is from the QW numerical calculation of model 10B with post-Newtonian corrections.

the other hand, Fuller & Qian (1996) have shown that general relativistic effects tend to increase $Y_e$ because of the differential redshift of the $\nu_e$ and $\bar{\nu}_e$ emitted from the supernova core. This differential redshift decreases the difference in the average energies of the $\nu_e$ and $\bar{\nu}_e$ populations, driving $Y_e$ larger and thus closer to 0.5. This is probably unfavorable for a high neutron-to-seed ratio, since the general trend is that a higher $Y_e$ value requires a higher $S$ value to obtain a given neutron-to-seed ratio. The magnitude of the differential redshift effect on $Y_e$ is uncertain because of its dependence on unknown details of the nuclear equation of state.

On balance, however, general relativistic effects may increase the neutron-to-seed ratio. For one thing, an increased $Y_e$ value due to differential neutrino redshifts is not always bad: examination of Figure 10 and Table 5 of Hoffman et al. (1996) shows that the entropy requirements to obtain a given neutron-to-seed ratio actually become *less* severe as $Y_e$ gets very close to 0.5. More important than this, however, is the fact that salutary effects on $S$ and $\tau_{dyn}$ may be able to compensate for deleterious differential redshift effects on $Y_e$. For example, consider the case of $2GM/R = 0.72$, for which our analytic estimates give $\tau_{dyn} \simeq 0.019$ s and $S \simeq 265$. In Table 5 of Hoffman et al. (1996), for "expansion time" 0.025 s (which corresponds to $\tau_{dyn} \simeq 0.020$ s), an entropy of only 255 is needed to obtain a neutron-to-seed ratio $\gtrsim 100$, even for the "worst case" value of $Y_e$.

It should be pointed out that the case we just considered, $2GM/R = 0.72$, slightly violates the causality limit $R > 1.52 \times 2GM$ imposed on any equation of state allowing a cold stable neutron star. However, we only consider this particular case for comparison with Hoffman et al. (1997). Unfortunately, their Table 5 does not contain values of "expansion time" between 0.025 s and 0.05 s. For values of "expansion time" in this range for which $2GM/R$ does not violate the causality limit, relativistic effects on $S$ may allow for a larger $Y_e$. Also, even beyond the causality limit, our calculations may give an idea of what happens near a metastable supernova core that collapses to a black hole at relatively late times. Of course, the timescale of collapse may be fast enough to violate grossly our assumption of steady state conditions and static geometry. Reliable exploration of this case would require detailed numerical modeling and good knowledge of the equation of state of nuclear matter.

There are other questions regarding the viability of a highly relativistic supernova core as a suitable environment for *r*-process nucleosynthesis. While a small dynamic expansion timescale is conducive to a large neutron-to-seed ratio, it may not be compatible with the attainment of steady flow equilibrium in the *r*-process (see, e.g., McLaughlin & Fuller 1997). If weak steady flow is desired, some mechanism would be needed to reduce the material expansion rate at later times; perhaps the shock used in the boundary condition could serve this purpose. Also, we have seen that a more compact core leads to a smaller mass outflow rate. One of the attractions of neutrino-heated supernova ejecta as an *r*-process site is that the total mass loss, together with the estimated Galactic supernova rate, roughly fits the observed amount of Galactic *r*-process material (Meyer et al. 1992). One would not want $\dot{M}$ to become so small that this agreement is ruined. However, even for the extreme case of $2GM/R = 0.72$ considered above, our estimated $\dot{M} \simeq 2.6 \times 10^{-7}\,M_\odot\,s^{-1}$ may still be large enough to be viable.

We wish to thank Y.-Z. Qian and J. R. Wilson for helpful discussions. This work was supported by grants NSF PHY95-03384 and NASA NAG5-3062 at UCSD.

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