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DATA ASSIMILATION ALGORITHM FOR 3D BÉNARD CONVECTION IN POROUS MEDIA EMPLOYING ONLY TEMPERATURE MEASUREMENTS

ASEEL FARHAT, EVELYN LUNASIN, AND EDRISS S. TITI

Abstract. In this paper we propose a continuous data assimilation (downscaling) algorithm for the Bénard convection in porous medium using only coarse mesh measurements of the temperature field. In this algorithm, we incorporate the observables as a feedback (nudging) term in the evolution equation of the temperature. We show that under an appropriate choice of the nudging parameter and the size of the mesh, and under the assumption that the observed data is error free, the solution of the proposed algorithm approaches at an exponential rate asymptotically in time to the unique exact unknown reference solution of the original system, associated with the observed (finite dimensional projection of) temperature data. Moreover, in the case where the observational measurements are not error free, one can estimate the error between the solution of the algorithm and the exact reference solution of the system in terms of the error in the measurements.

MSC Subject Classifications: 35Q30, 93C20, 37C50, 76B75, 34D06.
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1. Introduction

Linking mathematical and computational models to a set of observed data is crucial to obtain a more systematic representation of the state of a dynamical system in many physical applications. A particular application that we have in mind is flows in porous media, subjected to heating from below and cooling from the top. Flows in porous media are connected to many important problems in geophysics (see, e.g., [27] and references therein) as well as in biological systems and biotechnology (see, e.g., [30]).

We consider $\Omega = [0, L] \times [0, l] \times [0, 1]$ to be a box in $\mathbb{R}^3$ of porous media saturated with a fluid. The side walls of the box are insulated, and the box is heated from below with constant temperature T_0 and cooled from above with constant temperature $T_1 < T_0$. After some change of variables and proper scaling, (see, e.g. [6, 27]), the governing non-dimensional equations of the convecting fluid through

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the porous medium are:

$$\gamma \frac{\partial u}{\partial t} + u + \nabla p = Ra\, \theta \hat{k}, \tag{1.1a}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta + (u\cdot\nabla)\theta - u\cdot\hat{k} = 0, \tag{1.1b}$$

$$\nabla \cdot u = 0, \tag{1.1c}$$

$$u(0;x,y,z) = u^0(x,y,z), \quad \theta(0;x,y,z) = \theta^0(x,y,z), \tag{1.1d}$$

subject to the boundary conditions:

$$\theta(t;x,y,0) = \theta(t;x,y,1) = 0, \tag{1.1e}$$

$$\frac{\partial\theta}{\partial x}(t;0,y,z) = \frac{\partial\theta}{\partial x}(t;L,y,z) = \frac{\partial\theta}{\partial y}(t;x,0,z) = \frac{\partial\theta}{\partial y}(t;x,l,z) = 0, \tag{1.1f}$$

and

$$u\cdot\hat{n} = 0, \quad \text{on } \partial\Omega. \tag{1.1g}$$

Here, $u(t;x,y,z) = (u_1(t;x,y,z), u_2(t;x,y,z), u_3(t;x,y,z))$ is the fluid velocity, $p = p(t;x,y,z)$ is the pressure. $\theta = \theta(t;x,y,z)$ is the scaled fluctuation of the temperature around the steady state background temperature profile $(T_1 - T_0)z + T_0$ and it is given by $\theta = T - (\frac{T_0}{T_0 - T_1} - z)$, where $T = T(t;x,y,z)$ is the temperature of the fluid inside the box Ω, T_0 is the temperature of the fluid at the bottom and T_1 is the temperature of the fluid at the top. The non-dimensional parameter γ is equal to $\frac{1}{Pr}$, where Pr is the Darcy-Prandtl number representing a measure of the ratio of the viscosity to the thermal diffusion coefficient, and the non-dimensional parameter. Ra is the Rayleigh-Darcy number which is a measure of the ratio of the driving force (coming from the imposed average temperature gradient), to the damping (the viscosity and thermal diffusion) in the system. The vector $\hat{k}$ is the unit vector in the z-direction and $\hat{n} = \hat{n}(x,y,z)$ is the normal vector to the boundary $\partial\Omega$ at the point (x,y,z). We remark that the temperature fluctuation $\theta(t;x,y,z)$ satisfies an advection-diffusion equation (1.1b). Also, the evolution of the fluid velocity $u(t;x,y,z)$ and pressure $p(t;x,y,,z)$ is described by Darcy's law (1.1a), this replaces the conservation of momentum equations in the Navier-Stokes equations.

The Bénard convection in porous medium problem (1.1) with $\gamma > 0$ (namely, the Darcy-Prandtl number Pr is finite), was studied by Fabrie in [7] and [8]. There, the existence and uniqueness of global weak and strong solutions were established. It was also shown that the temperature $\theta(t;x,y,z)$ satisfies the maximum principle and has an absorbing ball in $L^p(\Omega)$ for each finite p. Fabrie and Nicolaenko showed in [9] that for initial data $(u^0, \theta^0) \in L^2(\Omega) \times L^\infty(\Omega)$, the system has a finite-dimensional global attractor $\mathcal{A} \subset L^2(\Omega) \times L^\infty(\Omega)$ which attracts in the $L^2(\Omega) \times L^2(\Omega)$ metric. Moreover, they showed that the system has exponential attractors. The Gevrey regularity (spatial analyticity) of the solutions on the global attractor was later studied by Oliver and Titi in [23]. There, a rigorous lower bound estimate on the radius of analyticity was obtained.

When $\gamma = 0$ (namely, the Darcy-Prandtl number Pr is infinite), system (1.1) was studied by Ly and Titi in [21]. All the constraints on the boundary are similar to the nonzero Darcy-Prandtl number case, while we note that in the $\gamma = 0$ case, there is no need to specify the initial data $u^0(x,y,z)$. One can recover $u^0(x,y,z)$ by solving equation (1.1a) given $\theta^0(x, y, z)$. This is unlike the non-zero Darcy-Prandtl case where one has to specify both initial data $u^0(x, y, z)$ and $\theta^0(x, y, z)$. Ly and Titi showed that the system for the $\gamma = 0$ case has global real analytic solutions and admits a real analytic global attractor. Moreover, they also showed that the standard Galerkin solution of the system, when $\gamma = 0$, converges exponentially fast, in the wave number, to the exact solution of the system. This in turn justifies the computational results reported by Graham, Steen and Titi in [16].

In [3], a continuous data assimilation (downscaling) algorithm was introduced based on an idea from control theory [2] (see also [20] for a computational study related to [2] and other relevant applications). This algorithm was designed to work for general dissipative dynamical systems. The algorithm, in general setting, is of the form

$$\frac{dv}{dt} = F(v) - \mu(I_h(v) - I_h(u)), \tag{1.2a}$$

$$v(0) = v^0, \tag{1.2b}$$

where $\mu > 0$ is a relaxation (nudging) parameter and v^0 is taken to be arbitrary initial data. $I_h(\cdot)$ represents an interpolant operator based on the observational measurements of a system at a coarse spatial resolution of size h, for $t \in [0, T]$. Notice that if system (1.2) is globally well-posed and $I_h(v)$ converge to $I_h(u)$ in time, then we recover the reference $u(t, x)$ from the approximate solution $v(t, x)$. The main task is to find estimates on $\mu > 0$ and $h > 0$ such that the approximate solution $v(t)$ is with increasing accuracy to the reference solution $u(t)$ as more continuous data in time is supplied.

In this paper we propose a continuous data assimilation algorithm for the 3D Bénard convection in porous medium employing coarse mesh measurements of only the temperature field. We incorporate the observables as a feedback term in the original evolution equation for the system. This algorithm can be implemented with a variety of finitely many observables: low Fourier modes, nodal values, finite volume averages, or finite elements. Our algorithm is given by the following system:

$$\gamma \frac{\partial v}{\partial t} + v + \nabla q = Ra\, \eta \hat{k}, \tag{1.3a}$$

$$\frac{\partial \eta}{\partial t} - \Delta \eta + (v \cdot \nabla)\eta - v \cdot \hat{k} = -\mu(I_h(\eta) - I_h(\theta)), \tag{1.3b}$$

$$\nabla \cdot v = 0, \tag{1.3c}$$

$$v(0; x, y, z) = v^0(x, y, z), \quad \eta(0; x, y, z) = \eta^0(x, y, z), \tag{1.3d}$$

subject to the boundary conditions:

$$\eta(t; x, y, 0) = \eta(t; x, y, 1) = 0, \tag{1.3e}$$

$$\frac{\partial \eta}{\partial x}(t; 0, y, z) = \frac{\partial \eta}{\partial x}(t; L, y, z) = \frac{\partial \eta}{\partial y}(t; x, 0, z) = \frac{\partial \eta}{\partial y}(t; x, l, z) = 0, \tag{1.3f}$$

and

$$v \cdot \hat{n} = 0, \quad \text{on } \partial\Omega. \tag{1.3g}$$

Here, q is a modified pressure, and we note that the initial conditions v_0 and η_0, may chosen arbitrarily, (for e.g., both initial conditions are set to zero). In this paper, we will only consider interpolant observables that are given by a linear interpolant operator $I_h : H^1(\Omega) \to L^2(\Omega)$ satisfying the approximation property

$$\|\varphi - I_h(\varphi)\|_{L^2(\Omega)} \le c_0 h \|\varphi\|_{H^1(\Omega)}, \tag{1.4}$$

for every $\varphi \in H^1(\Omega)$, where $c_0 > 0$ is a dimensionless constant. One example of an interpolant observable that satisfies (1.4) is the orthogonal projection onto the low Fourier modes with wave numbers less than $1/h$. A more physical example are the volume elements that were studied in [19]. The algorithm and our analysis in this paper will carry on to the second type of interpolants treated by Azouani, Olson and Titi in [3]. This second type is given by a linear operator $I_h : H^2(\Omega) \to L^2(\Omega)$, together with

$$\|\varphi - I_h(\varphi)\|_{L^2(\Omega)} \le c_1 h \|\varphi\|_{H^1(\Omega)} + c_2 h^2 \|\varphi\|_{H^2(\Omega)}, \tag{1.5}$$

for every $\varphi \in H^2(\Omega)$, where $c_1, c_2 > 0$ are dimensionless constants. An example of this type of interpolant observables that satisfies (1.5) is given by the measurements at a discrete set of nodal points in Ω. We are not treating the second type of interpolants in this paper only for the simplicity of presentation. For full details on the analysis for the second type of interpolants we refer to [10] and [11].

Our current work was inspired to sidetrack a common problem in data assimilation where the dimension of the observation vector is less than the dimension of the model's state vector. Analyzing the validity and success of a data assimilation algorithm when some state variable observations are not available as an input on the numerical forcast model is a crucial task. In particular, our current work is motivated by [10] and [11]. In [10], a continuous data assimilation scheme for the two-dimensional incompressible Bénard convection problem was introduced. The data assimilation algorithm in [10] construct the approximate solutions for the velocity u and temperature fluctuations θ using only the observational data, $I_h(u)$, of the velocity field and without any measurements for the temperature (or density) fluctuations. Inspired by the recent algorithms proposed in [3, 10], in [11] we introduced an abridged dynamic continuous data assimilation for the 2D Navier-Stokes equations (NSE). The proposed algorithm in [3] for the 2D NSE requires measurements for the two components of the velocity vector field. On the other hand, in [11], we establish convergence results for an improved algorithm where the observational data needed to be measured and inserted into the model equation is reduced or subsampled. Our algorithm there requires observational measurements of only one component of the velocity vector field. It is worth mentioning that our work in [11], can be extended for the corresponding convergence analysis for the 2D Bénard problem, where the approximate solutions constructed using observations in only one component of the two-dimensional velocity field and without any measurements on the temperature, converge in time to the reference solution of the 2D Bénard system ([13]). This will be a progression of a recent result in [10] where convergence results were established, given that observations are known at discrete points on *two* components of the velocity field and without any measurements of the temperature. On a similar note, ideally, one would like to design an algorithm based on temperature measurements only.

Our main contribution in this current study is to give a rigorous justification that a data assimilation algorithm based on temperature measurements alone can be designed for the 3D Bénard convection in porous medium. We provide explicit estimates on the relaxation (nudging) parameter μ and the spatial resolution h of the observational measurements, in terms of physical parameters, that are needed in order for the proposed downscaling algorithm to recover the reference solution under the assumption that the supplied data are error free. In the case where the observational measurements are not error free, one can estimate the error between the solution of the algorithm and the exact reference solution of the system in terms of the error in the measurements. While the typical scenario in data assimilation is to choose μ depending on h, in our convergence analysis we choose our parameters μ and h to depend on physical parameters. More explicitly, we choose μ to depend on the bounds of the solution on the global attractor of the system and then choose h to depend on μ. The philosophy here is that in order to prove the convergence theorems, we need to have a complete resolution of the flow, so h has to depend on the physical parameters, such as the Rayleigh number (Ra) in this case.

It is worth noting that the continuous data assimilation in the context of the incompressible 2D NSE studied in [3] under the assumption that the data is noise free. The case when the observational data contains stochastic noise is treated in [4]. The authors in [4] established resolution conditions which guarantee that the limit supremum, as the time tends to infinity, of the expected value of the L^2-norm of the difference between the approximating solution and the actual reference solution, (i.e. the error), is bounded by an estimate involving the variance of the noise in the measurements and the spatial resolution of the collected data, h. The analysis of uncertainty quantification for the Bénard convection problem in porous medium, in the case where the measurements and observations contain noise, is a subject of future work by a subset of us.

It is also worthwhile to mention that the authors in [22] analyzed an algorithm for continuous data assimilation for 3D Brinkman-Forchheimer-extended Darcy (3D BFeD) model of porous medium, a model equation when the velocity is too large for Darcy's law to be valid. A similar algorithm for stochastically noisy data is at hand combining ideas from the work in [11] and [4]. Furthermore, the proposed data assimilation algorithm can also be applied to several three-dimensional subgrid scale turbulence models. In [1], it was shown that approximate solutions constructed using observations on all three components of the unfiltered velocity field converge in time to the reference solution of the 3D NS-α model. An abridged data assimilation algorithm for the 3D Leray-α model is analyzed in [12].

Numerical simulations in [14] and [15] (see also [17]) have shown that, in the absence of measurements errors, the continuous data assimilation algorithm for the 2D Navier-Stokes equations performs much better than analytical estimates in [3] would suggest. This was also noted in a different context in [24] and [25]. It is likely that the data assimilation algorithm studied in this paper will also perform much better than suggested by the analytical results, i.e. under more relaxed conditions than those assumed in the rigorous estimates. This is a subject of future work.

The outline of the paper is as follows. In section 2 we give some preliminaries. In section 3.1, we present the convergence analysis of our proposed data assimilation algorithm in the case of zero Darcy-Prandtl case and the nonzero Darcy-Prandtl case in section 3.2.

2. Preliminaries

For the sake of completeness, this section presents some preliminary material and notation commonly used in the mathematical study of hydrodynamics models, in particular in the study of the Navier-Stokes equations (NSE) and the Euler equations. For more detailed discussion on these topics, we refer the reader to, e.g., [5], [26], [28] and [29].

Let $L^p(\Omega) := L^p : \Omega \to \mathbb{R}$ and $H^k(\Omega) := H^k : \Omega \to \mathbb{R}$ be denote the usual L^p Lebesgue space and H^k-Sobolev space, respectively, for $1 \le p \le \infty$ and $k \in \mathbb{R}$. We define the spaces:

$$\mathcal{V} := \{u \in (C^\infty(\Omega))^3 : u \cdot \hat{n} = 0 \text{ on } \partial\Omega \text{ and } \nabla \cdot u = 0 \text{ in } \Omega\},$$
$$\tilde{\mathcal{V}} := \{\theta \in C^\infty(\Omega) : \theta \text{ satisfies the boundary conditions (1.1e) and (1.1f)}\},$$
$$\mathbf{H} := \text{the closure of } \mathcal{V} \text{ in the } (L^2(\Omega))^3 \text{ norm},$$
$$\mathbf{V} := \text{the closure of } \mathcal{V} \text{ in the } (H^1(\Omega))^3 \text{ norm},$$
$$H := \text{the closure of } \tilde{\mathcal{V}} \text{ in the } L^2(\Omega) \text{ norm},$$
$$V := \text{the closure of } \tilde{\mathcal{V}} \text{ in the } H^1(\Omega) \text{ norm}.$$

We define the inner products on $\mathbf{H}$ and H by

$$(u, w) = \sum_{i=1}^3 \int_\Omega u_i w_i \, dxdydz, \quad \text{and} \quad (\theta, \eta) = \int_\Omega \theta\eta \, dxdydz,$$

respectively. We also define the inner product on $\mathbf{V}$ and V by

$$((u, w)) = \sum_{i,j=1}^3 \int_\Omega \partial_j u_i \partial_j w_i \, dxdydz, \quad \text{and} \quad ((\theta, \eta)) = \sum_{j=1}^3 \int_\Omega \partial_j \theta \partial_j \eta \, dxdydz,$$

respectively. Note that, thanks to the boundary conditions (1.1e) and (1.1f), $\|\cdot\|_V = ((\cdot,\cdot))^{1/2} = \|\nabla\cdot\|_{L^2(\Omega)}$ is a norm on V due to the Poincaré inequality (2.2), below.

We define the Helmholtz-Leray projector P_σ as the orthogonal projection from $(L^2(\Omega))^3$ onto $\mathbf{H}$ and define $A = -\Delta$ subject to the boundary condition (1.1g) and (1.1f) with the domain

$$\mathcal{D}(A) = \{\theta \in H^2(\Omega) : A\theta \in H \text{ and } \theta \text{ satisfies (1.1e) and (1.1f)}\}.$$

Using the Lax-Milgram Theorem and the elliptic regularity in the box Ω, the linear operator A is self-adjoint and positive definite with compact inverse $A^{-1} : H \to \mathcal{D}(A)$. Thus, there exists a complete orthonormal set of eigenfunctions w_i in H such that $Aw_i = \lambda_i w_i$ where $0 < \lambda_i \le \lambda_{i+1}$ for $i \in \mathbb{N}$.

We define the bilinear from $\mathcal{B}(\cdot,\cdot) : \mathbf{V} \times \mathcal{D}(A) \to H$, such that

$$\mathcal{B}(u, \theta) := (u \cdot \nabla)\theta,$$

for every $u \in \mathbf{V}$ and $\theta \in \mathcal{D}(A)$. Using the boundary conditions (1.1e) and (1.1f), one can easily check that

$$(\mathcal{B}(u, \theta), \theta) = 0, \tag{2.1}$$

for every $u \in \mathbf{V}$ and $\theta \in \mathcal{D}(A)$.

We recall the Poincaré inequality:

$$\|\varphi\|^2_{L^2(\Omega)} \le \lambda_1^{-1} \|A^{1/2}\varphi\|^2_{L^2(\Omega)}, \quad \text{for all } \varphi \in V, \tag{2.2}$$

where λ_1 is the smallest eigenvalue of the operator A.

Let Y be a Banach space. We denote by $L^p([0,T];Y)$ the space of (Bochner) measurable functions $t \mapsto w(t)$, where $w(t) \in Y$, for a.e. $t \in [0,T]$, such that the integral $\int_0^T \|w(t)\|_Y^p \, dt$ is finite.

Furthermore, inequality (1.4) implies that

$$\|\theta - I_h(\theta)\|_{L^2(\Omega)} \leq c_0 h \left\| A^{1/2}\theta \right\|_{L^2(\Omega)}, \tag{2.3}$$

for every $\theta \in V$.

We recall the following existence and uniqueness results for the 3D Bénard convection problem in porous medium (1.1). Hereafter, c will denote a universal dimensionless positive constant that may change from line to line.

Proposition 2.1. *[21] Given $\theta \in V$, there exists a unique solution $u \in \mathbf{V}$ of the problem (1.1a) and (1.1c), with $\gamma = 0$, subject to the boundary condition (1.1g). Moreover, u satisfies*

$$\|u\|_{\mathbf{V}} \leq cRa \|\theta\|_V . \tag{2.4}$$

Theorem 2.2. *[21] Let $\theta^0 \in V$, then system (1.1), with $\gamma = 0$, has a unique global strong solution (u, θ) that satisfies:*

$$u \in C([0,T];\mathbf{V}) \cap L^2([0,T];(H^2(\Omega))^3), \quad \text{and} \quad \frac{du}{dt} \in L^2([0,T];\mathbf{H}),$$

$$\theta \in C([0,T];V) \cap L^2([0,T];\mathcal{D}(A)), \quad \text{and} \quad \frac{d\theta}{dt} \in L^2([0,T];H),$$

for any $T > 0$, and depends continuously on the initial data θ^0. Moreover, the solution satisfies the following bound:

$$\limsup_{t\to\infty} \|\theta(t)\|_{L^\infty(\Omega)} \leq 1, \tag{2.5}$$

and the system admits a global compact finite-dimensional global attractor $\mathcal{A} \subset \mathbf{V} \times V$.

Theorem 2.3. *[9] If $u^0 \in \mathbf{H}$ and $\theta^0 \in L^\infty(\Omega)$, then for any $T > 0$, system (1.1), with $\gamma > 0$, has a unique weak solution (u, θ) that satisfies*

$$u \in L^\infty([0,T];\mathbf{H}), \quad \text{and} \quad \theta \in L^\infty([0,T];L^\infty(\Omega)) \cap L^2([0,T];V).$$

Moreover, for every finite p,

$$\limsup_{t\to\infty} \|\theta(t)\|_{L^p(\Omega)} \leq |\Omega|^{1/p} . \tag{2.6}$$

More precisely, $\theta(t;x,y,z)$ can be decomposed such that

$$\theta(t;x,y,z) = \theta_1(t;x,y,z) + \theta_2(t;x,y,z), \tag{2.7}$$

where $-1 \leq \theta_1(t;x,y,z) \leq 1$, and that there exist two positive constants α_1 and α_2 depending on p and Ω such that

$$\|\theta_2(t)\|_{L^p(\Omega)} \leq \alpha_1 e^{-\alpha_2 t}.$$

If the initial data θ^0 satisfies $m \leq \theta^0(x,y,z) \leq M$, a.e. in Ω, for some $m \geq -1$ and $M \leq 1$, we have

$$m \leq \theta(t;x,y,z) \leq M, \quad \text{a.e. in } \Omega, \tag{2.8}$$

for all $t \geq 0$.

Finally, if $u^0 \in \mathbf{V}$ *and* $\theta^0 \in L^\infty(\Omega)$*, then there exists a unique strong solution* (u, θ) *that satisfies*

$$u \in C([0,T];\mathbf{V}), \quad \textit{and} \quad \frac{du}{dt} \in L^2([0,T];\mathbf{H}),$$
$$\theta \in L^\infty([0,T];L^\infty(\Omega)) \cap L^2([0,T];\mathcal{D}(A)), \quad \textit{and} \quad \frac{d\theta}{dt} \in L^2([0,T];H),$$

and the system admits a global compact finite-dimensional exponential global attractor $\mathcal{A} \subset \mathbf{V} \times V$*.*

3. Convergence Results

In this section, we will establish the global existence, uniqueness and stability of solutions of system (1.3), when the observable data satisfy (1.4).

Theorem 3.1 (Existence and Uniqueness)**.** *Suppose* I_h *satisfy* (1.4) *with* $\mu > 0$ *and* $h > 0$ *are chosen such that* $\mu c_0^2 h^2 \le 1$*, where* c_0 *is the constant in* (1.4)*.*

(1) For $\gamma = 0$*, if the initial data* $\eta^0 \in V$*, then for any* $T > 0$*, the continuous data assimilation system* (1.3)*, subject to the boundary conditions* (1.3g)–(1.3f)*, possess a unique global strong solution* (v, η) *that satisfies*

$$v \in C([0,T];\mathbf{V}) \cap L^2([0,T];(H^2(\Omega))^3), \quad \textit{and} \quad \frac{dv}{dt} \in L^2([0,T];\mathbf{H}),$$
$$\eta \in C([0,T];V) \cap L^2([0,T];\mathcal{D}(A)), \quad \textit{and} \quad \frac{d\eta}{dt} \in L^2([0,T];H).$$

(2) For $\gamma > 0$*, if the initial data* $v^0 \in \mathbf{V}$ *and* $\eta^0 \in V$*, then for any* $T > 0$*, the continuous data assimilation system* (1.3)*, subject to the boundary conditions* (1.3g)–(1.3f)*, possess a unique global strong solution* (v, η) *that satisfies*

$$v \in C([0,T];\mathbf{V}), \quad \textit{and} \quad \frac{dv}{dt} \in L^2([0,T];\mathbf{H})$$
$$\eta \in C([0,T];V) \cap L^2([0,T];\mathcal{D}(A)), \quad \textit{and} \quad \frac{d\eta}{dt} \in L^2([0,T];H).$$

Proof. Since we assume that (u, θ) is a reference solution of system (1.1), then it is enough to show the existence and uniqueness of the difference $(w, \xi) = (u - v, \theta - \eta)$. In the proofs of Theorem 3.3 and Theorem 3.4 below, we will drive formal *a-priori* bounds on the difference (w, ξ), under the condition $\mu c_0^2 h \le 1$. These *a-priori* estimates, together with the global existence of the solution (u, θ), form the key elements for showing the global existence of the solution (v, η) of system (1.3). The convergence of the approximate solution (v, η) to the exact reference solution (u, θ) will be established later under extra conditions on the nudging parameter μ stated in (3.1) and (3.9), when $\gamma = 0$ and $\gamma > 0$, respectively. Uniqueness can then be obtained using similar energy estimates. □

Remark 3.2. The estimates we provide in this section are formal, but can be justified by the Galerkin approximation procedure and then passing to the limit while using the relevant compactness theorems. We will omit the rigorous details of this standard procedure (see, e.g., [5, 7, 9, 21, 26, 29]) and provide only the formal *a-priori* estimates.

3.1. **The infinite Darcy-Prandtl number case ($\gamma = 0$).**

Theorem 3.3. *Let I_h satisfy the approximation property (1.4) and (u, θ) be a strong solution in the global attractor of (1.1) with $\gamma = 0$ and (v, η) be a strong solution of (1.3) with $\gamma = 0$. Let $\mu > 0$ be large enough such that*

$$\mu + \frac{\lambda_1}{2} \geq 2cRa^2 + 4Ra. \tag{3.1}$$

If h is chosen small enough such that $\mu c_0^2 h^2 \leq 1$ then, $\|u(t) - v(t)\|_{L^2(\Omega)}^2 \to 0$, and $\|\theta(t) - \eta(t)\|_{L^2(\Omega)}^2 \to 0$, at an exponential rate, as $t \to \infty$.

Proof. Define $w := u - v$ and $\xi := \theta - \eta$. Then, in functional settings, w and ξ will satisfy the equations:

$$w = RaP_\sigma(\xi\hat{k}), \tag{3.2a}$$

$$\frac{\partial \xi}{\partial t} + A\xi + \mathcal{B}(v, \xi) + \mathcal{B}(w, \theta) - w \cdot \hat{k} = -\mu I_h(\xi). \tag{3.2b}$$

Taking the H-inner product of (3.2b) with ξ, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\xi\|_{L^2(\Omega)}^2 + \left\|A^{1/2}\xi\right\|_{L^2(\Omega)}^2 &+ (\mathcal{B}(v, \xi), \xi) + (\mathcal{B}(w, \theta), \xi) \\ &= (w \cdot \hat{k}, \xi) - \mu(I_h(\xi), \xi).\end{aligned}$$

Thanks to (2.1), we have

$$(\mathcal{B}(v, \xi), \xi) = 0. \tag{3.3}$$

We also notice from (3.2a) that

$$w = RaP_\sigma(\xi\hat{k}), \qquad \text{in } L^2([0, T]; \mathbf{H}),$$

for any $T > 0$. This implies that

$$(w \cdot \hat{k}, \xi) = Ra(P_\sigma(\xi\hat{k}), \xi) \leq Ra\left\|P_\sigma(\xi\hat{k})\right\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)} \leq Ra\|\xi\|_{L^2(\Omega)}^2. \tag{3.4}$$

Using and Hölder's inequality, we get

$$\begin{aligned}|(\mathcal{B}(w, \theta), \xi)| &= |(\mathcal{B}(w, \xi), \theta)| \\ &\leq \|w\|_{L^2(\Omega)}\|\theta\|_{L^\infty(\Omega)}\left\|A^{1/2}\xi\right\|_{L^2(\Omega)} \\ &\leq \frac{1}{8}\left\|A^{1/2}\xi\right\|_{L^2(\Omega)}^2 + c\|\theta\|_{L^\infty(\Omega)}^2\|w\|_{L^2(\Omega)}^2.\end{aligned}$$

Thanks to the equation (3.2a) we have

$$|(\mathcal{B}(w, \theta), \xi)| \leq \frac{1}{8}\left\|A^{1/2}\xi\right\|_{L^2(\Omega)}^2 + cRa^2\|\theta\|_{L^\infty(\Omega)}^2\|\xi\|_{L^2(\Omega)}^2. \tag{3.5}$$

The approximation inequality (2.3) and Young's inequality imply

$$\begin{aligned}-\mu(I_h(\xi), \xi) &= -\mu(I_h(\xi) - \xi, \xi) - \mu\|\xi\|_{L^2(\Omega)}^2 \\ &\leq \mu\|I_h(\xi) - \xi\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)} - \mu\|\xi\|_{L^2(\Omega)}^2 \\ &\leq \frac{\mu c_0^2 h^2}{2}\left\|A^{1/2}\xi\right\|_{L^2(\Omega)}^2 - \frac{\mu}{2}\|\xi\|_{L^2(\Omega)}^2.\end{aligned} \tag{3.6}$$

Using assumption $\mu c_0^2 h^2 \leq 1$ and estimates (3.3)–(3.6), we conclude that

$$\frac{d}{dt}\left\|\xi\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|A^{1/2}\xi\right\|_{L^2(\Omega)}^2 \leq \left(cRa^2\left\|\theta\right\|_{L^\infty(\Omega)}^2 + 2Ra - \mu\right)\left\|\xi\right\|_{L^2(\Omega)}^2.$$

The Poincare inequality (2.2) yields

$$\frac{d}{dt}\left\|\xi\right\|_{L^2(\Omega)}^2 + \frac{\lambda_1}{2}\left\|\xi\right\|_{L^2(\Omega)}^2 \leq \left(cRa^2\left\|\theta\right\|_{L^\infty(\Omega)}^2 + 2Ra - \mu\right)\left\|\xi\right\|_{L^2(\Omega)}^2.$$

We define

$$\alpha(t) := \mu + \frac{\lambda_1}{2} - 2Ra - cRa^2\left\|\theta\right\|_{L^\infty(\Omega)}^2.$$

Then,

$$\frac{d}{dt}\left\|\xi\right\|_{L^2(\Omega)}^2 + \alpha(t)\left\|\xi\right\|_{L^2(\Omega)}^2 \leq 0. \tag{3.7}$$

The uniform bound (2.5) implies that: for any fixed $\varepsilon > 0$, there exists a time $t_0(\varepsilon) > 0$ such that

$$\left\|\theta(t)\right\|_{L^\infty(\Omega)} \leq 1 + \varepsilon,$$

for all $t \geq t_0$. Taking $\varepsilon = 1$, then there exists a time $t_0 > 0$ such that $\left\|\theta(t)\right\|_{L^\infty(\Omega)} \leq 2$, for all $t \geq t_0$. Then, we have

$$\alpha(t) \geq \mu + \frac{\lambda_1}{2} - 2Ra - cRa^2,$$

and

$$\alpha(t) \leq \mu + \frac{\lambda_1}{2} + 2Ra + cRa^2 < \infty,$$

for all $t \geq t_0$. Now, the assumption (3.1) implies that

$$\alpha(t) \geq 2Ra + cRa^2 > 0, \tag{3.8}$$

for all $t \geq t_0$.

By Gronwall's Lemma, it follows that

$$\left\|\theta(t) - \eta(t)\right\|_{L^2(\Omega)}^2 = \left\|\xi(t)\right\|_{L^2(\Omega)}^2 \to 0,$$

at an exponential rate, as $t \to \infty$. Finally, the equation (3.2a) yields that

$$\left\|w(t)\right\|_{L^2(\Omega)}^2 \leq Ra\left\|\xi(t)\right\|_{L^2(\Omega)}^2,$$

thus,

$$\left\|u(t) - v(t)\right\|_{L^2(\Omega)}^2 = \left\|w\right\|_{L^2(\Omega)}^2 \to 0,$$

at an exponential rate, as $t \to \infty$.

□

3.2. **The finite Darcy-Prandtl number case ($\gamma > 0$).**

Theorem 3.4. *Fix $\gamma > 0$, and (u, θ) be a strong solution in the global attractor of (1.1) with and (v, η) be a strong solution of (1.3) corresponding to $\gamma > 0$, respectively. Suppose that I_h satisfies the approximation property (1.4). If $\mu > 0$ is large enough such that*

$$2\mu + \lambda_1 \geq 2\frac{cRa^4}{\gamma} + 2c\gamma(1 + \lambda_1^{-1})^2. \tag{3.9}$$

and h is small enough such that $\mu c_0^2 h^2 \leq 1$ then, $\|u(t) - v(t)\|^4_{L^2(\Omega)} + \|\theta(t) - \eta(t)\|^4_{L^2(\Omega)} \to 0$, at an exponential rate, as $t \to \infty$.

Proof. Define $w := u - v$ and $\xi := \theta - \eta$. Then, in functional settings, w and ξ will satisfy the equations:

$$\gamma\frac{dw}{dt} + w = RaP_\sigma(\xi\hat{k}), \tag{3.10a}$$

$$\frac{d\xi}{dt} + A\xi + \mathcal{B}(v, \xi) + \mathcal{B}(w, \theta) - w \cdot \hat{k} = -\mu I_h(\xi). \tag{3.10b}$$

Taking the **H**-inner product of (3.10a) with w and the H-inner product of(3.10b) with ξ, respectively, we obtain

$$\begin{aligned}
\frac{\gamma}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)} &= Ra(\xi, w \cdot \hat{k}),\\
\frac{1}{2}\frac{d}{dt}\|\xi\|^2_{L^2(\Omega)} + \left\|A^{1/2}\xi\right\|^2_{L^2(\Omega)} &+ (\mathcal{B}(v, \xi), \xi) + (\mathcal{B}(w, \theta), \xi)\\
&= (w \cdot \hat{k}, \xi) - \mu(I_h(\xi), \xi).
\end{aligned}$$

Thanks to (2.1), we have

$$(\mathcal{B}(v, \xi), \xi) = 0. \tag{3.11}$$

Also, estimate (3.6) shows that

$$-\mu(I_h(\xi), \xi) \leq \frac{\mu c_0^2 h^2}{2}\left\|A^{1/2}\xi\right\|^2_{L^2(\Omega)} - \frac{\mu}{2}\|\xi\|^2_{L^2(\Omega)}. \tag{3.12}$$

By Hölder's inequality, we have

$$\begin{aligned}
|(\mathcal{B}(w, \theta), \xi)| &= |(\mathcal{B}(w, \xi), \theta)|\\
&\leq \|w\|_{L^2(\Omega)}\|\theta\|_{L^\infty(\Omega)}\left\|A^{1/2}\xi\right\|_{L^2(\Omega)}\\
&\leq \frac{1}{8}\left\|A^{1/2}\xi\right\|^2_{L^2(\Omega)} + c\|\theta\|^2_{L^\infty(\Omega)}\|w\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.13}$$

The above estimates (3.11)–(3.13), the Cauchy-Schwarz inequality, and the Poincaré inequality (2.2) yield

$$\begin{aligned}
\frac{\gamma}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)} &= Ra(\xi, w \cdot \hat{k})\\
&\leq \frac{1}{2}\|w\|^2_{L^2(\Omega)} + \frac{Ra^2}{2}\|\xi\|^2_{L^2(\Omega)}\\
\frac{1}{2}\frac{d}{dt}\|\xi\|^2_{L^2(\Omega)} + \left\|A^{1/2}\xi\right\|^2_{L^2(\Omega)} + \frac{\mu}{2}\|\xi\|^2_{L^2(\Omega)} &\leq \frac{\mu c_0^2 h^2}{2}\left\|A^{1/2}\xi\right\|^2_{L^2(\Omega)} + \frac{1}{4}\left\|A^{1/2}\xi\right\|^2_{L^2(\Omega)}\\
&\quad + c\|\theta\|^2_{L^\infty(\Omega)}\|w\|^2_{L^2(\Omega)} + c\lambda_1^{-1}\|w\|^2_{L^2(\Omega)}.
\end{aligned}$$

Using the assumption $\mu c_0^2 h^2 \leq 1$ imply

$$\frac{d}{dt} \|w\|^2_{L^2(\Omega)} + \frac{1}{\gamma} \|w\|^2_{L^2(\Omega)} \leq \frac{Ra^2}{\gamma} \|\xi\|^2_{L^2(\Omega)}, \tag{3.14a}$$

$$\frac{d}{dt} \|\xi\|^2_{L^2(\Omega)} + \left(\mu + \frac{\lambda_1}{2}\right) \|\xi\|^2_{L^2(\Omega)} \leq c\left(\lambda_1^{-1} + \|\theta\|^2_{L^\infty(\Omega)}\right) \|w\|^2_{L^2(\Omega)}. \tag{3.14b}$$

Multiplying (3.14a) with $\|w\|^2_{L^2(\Omega)}$ and (3.14b) with $\|\xi\|^2_{L^2(\Omega)}$, respectively, we have

$$\frac{1}{2}\frac{d}{dt} \|w\|^4_{L^2(\Omega)} + \frac{1}{\gamma} \|w\|^4_{L^2(\Omega)} \leq \frac{Ra^2}{\gamma} \|w\|^2_{L^2(\Omega)} \|\xi\|^2_{L^2(\Omega)},$$

$$\frac{1}{2}\frac{d}{dt} \|\xi\|^4_{L^2(\Omega)} + \left(\mu + \frac{\lambda_1}{2}\right) \|\xi\|^4_{L^2(\Omega)} \leq c\left(\lambda_1^{-1} + \|\theta\|^2_{L^\infty(\Omega)}\right) \|w\|^2_{L^2(\Omega)} \|\xi\|^2_{L^2(\Omega)}.$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|w\|^4_{L^2(\Omega)} + \|\xi\|^4_{L^2(\Omega)}\right) \\
&\qquad \leq -\frac{1}{\gamma} \|w\|^4_{L^2(\Omega)} - \left(\mu + \frac{\lambda_1}{2}\right) \|\xi\|^4_{L^2(\Omega)} + \frac{Ra^2}{\gamma} \|w\|^2_{L^2(\Omega)} \|\xi\|^2_{L^2(\Omega)} \\
&\qquad\quad + c\left(\lambda_1^{-1} + \|\theta\|^2_{L^\infty(\Omega)}\right) \|w\|^2_{L^2(\Omega)} \|\xi\|^2_{L^2(\Omega)} \\
&\qquad \leq -\frac{1}{\gamma} \|w\|^4_{L^2(\Omega)} - \left(\mu + \frac{\lambda_1}{2}\right) \|\xi\|^4_{L^2(\Omega)} + \frac{1}{4\gamma} \|w\|^4_{L^2(\Omega)} + \frac{cRa^4}{\gamma} \|\xi\|^4_{L^2(\Omega)} \\
&\qquad\quad + \frac{1}{4\gamma} \|w\|^4_{L^2(\Omega)} + c\gamma\left(\lambda_1^{-1} + \|\theta\|^2_{L^\infty(\Omega)}\right)^2 \|\xi\|^4_{L^2(\Omega)} \\
&\qquad = -\frac{1}{2\gamma} \|w\|^4_{L^2(\Omega)} - \left(\mu + \frac{\lambda_1}{2} - \frac{cRa^4}{\gamma} - c\gamma\left(\lambda_1^{-1} + \|\theta\|^2_{L^\infty(\Omega)}\right)^2\right) \|\xi\|^4_{L^2(\Omega)}.
\end{aligned}$$

The above inequality can be rewritten as

$$\frac{d}{dt}\left(\|w\|^4_{L^2(\Omega)} + \|\xi\|^4_{L^2(\Omega)}\right) + \alpha(t)\left(\|w\|^4_{L^2(\Omega)} + \|\xi\|^4_{L^2(\Omega)}\right) \leq 0, \tag{3.15}$$

where

$$\alpha(t) = \min\left\{\frac{1}{\gamma},\, 2\mu + \lambda_1 - \frac{cRa^4}{\gamma} - c\gamma\left(\lambda_1^{-1} + \|\theta(t)\|^2_{L^\infty(\Omega)}\right)^2\right\}.$$

Recall that, by Theorem (2.3), the solution $\theta(t; x, y, z)$ of the 3D Bénard problem in porous medium satisfies the Maximum Principle and

$$\limsup_{t\to\infty} \|\theta(t)\|_{L^\infty(\Omega)} \leq 1.$$

Then, by a similar argument as in the proof of the previous theorem, we conclude that there exists a time $t_0 > 0$ such that

$$\alpha(t) \geq \min\left\{\frac{1}{\gamma},\, 2\mu + \lambda_1 - \frac{cRa^4}{\gamma} - c\gamma(1 + \lambda_1^{-1})^2\right\},$$

and

$$\alpha(t) \leq \min\left\{\frac{1}{\gamma},\, 2\mu + \lambda_1 + \frac{cRa^4}{\gamma} + c\gamma(1 + \lambda_1^{-1})^2\right\} < \infty,$$

for all $t \geq t_0$. Now, assumption (3.9) implies that

$$\alpha(t) \geq \min\left\{\frac{1}{\gamma}, \frac{cRa^4}{\gamma} + c\gamma(1+\lambda_1^{-1})^2\right\} > 0, \tag{3.16}$$

for all $t \geq t_0$.

By Gronwall's Lemma , it follows that

$$\|u(t) - v(t)\|_{L^2(\Omega)}^4 + \|\theta(t) - \eta(t)\|_{L^2(\Omega)}^4 \to 0,$$

at an exponential rate, as $t \to \infty$. □

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