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Three-dimensional acoustic scattering by vortical flows. II. Axisymmetric scattering by Hill's spherical vortex

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The general theory of Part I is applied to the the specific case of scattering of a wave incident along the axis of Hill's spherical vortex. The full asymptotic solution to the initial-value problem is calculated. Results agree with the general approach, showing that the conditions required for the latter to hold apply in the case of Hill's spherical vortex. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1401815]

I. INTRODUCTION

In Part I of this paper[1] we discussed the general problem of the scattering of acoustic plane waves by three-dimensional vortical structures with scales small compared to the wavelength of the incoming sound wave. Using an asymptotic expansion of the equations of motion, the leading-order scattered field can be calculated under quite general circumstances, and is identical to that predicted by what can be called the acoustic analogy approximation. In this approximation, a forced wave equation is solved, and the forcing can be computed without solving in detail for the interaction between the vortex and the acoustic wave. One important assumption must be made: the principal effect of the incident sound wave must simply be to cause the vortex to oscillate back and forth. Whether this is, in fact, the case, can only be determined for certain by solving the scattering problem in detail. In this paper, we solve in detail the problem of scattering of acoustic waves by Hill's spherical vortex (HSV), for the simple case in which the incident acoustic waves propagate along the axis of the vortex, so that the entire problem is axisymmetric.

The vorticity $\boldsymbol{\omega}$ takes the form

$$\boldsymbol{\omega} = \zeta \mathbf{e}_\phi, \tag{1}$$

where $\mathbf{e}_\phi$ is the unit vector in the azimuthal direction about the axis of symmetry. Within HSV, ζ is a linear function of the distance $r \sin\theta$ from the axis of symmetry. Outside HSV, the vorticity vanishes. By assuming, consistent with incompressible dynamics, that perturbations also possess this property, the problem reduces to solving for the location of the boundary of the vortex. This is because, for the incompressible axisymmetric Euler equations, $\zeta/(r \sin\theta)$ is a materially conserved quantity, i.e., it is advected by the flow. In a compressible fluid, the relevant materially conserved quantity is modified to $\zeta/(\rho r \sin\theta)$, where ρ is the density. A compressible version of HSV can be derived,[2] in which $\zeta/(\rho r \sin\theta)$ is uniform within the vortex, and vanishes without. Perturbations are then restricted to a class in which $\zeta/(\rho r \sin\theta)$ remains uniform.

Previous work on HSV has been concerned with an instability that grows in time, in a thin finger in the neighborhood of the rear stagnation point whose width decreases exponentially in time.[3,4] In these papers, prolate and oblate perturbations to HSV are considered, which result in the vortex either ejecting or entraining a thin finger of fluid at its rear stagnation point. We formulate our problem following Moffatt and Moore[3] (hereafter M and M), but in our problem the vortex is forced by the time-periodic flow of the incident acoustic wave.

We will employ the formalism of Part I. In the vortex, we will use coordinates centered on the vortex, while in the wave region we will use coordinates fixed in a frame at rest. In Part I we assumed that the vorticity was not expanded in powers of M, whereas here the vorticity will be expanded in powers of M. The analysis of Part I includes this, but effects due to the motion of vorticity, which in Part I are all represented together at $O(1)$, now arise at successive orders as the vorticity is expanded, as are its effects on the other flow variables.

We outline the physical situation briefly in Sec. II. Hill's spherical vortex is presented in Sec. III, together with the extension to weakly compressible flow that we use. The asymptotic solution to the scattering problem is calculated in Sec. IV, and we show that HSV meets the conditions required for the validity of the acoustic analogy approximation, as described in Part I. The acoustic scattering due to HSV has been computed previously using the acoustic analogy approximation.[5,6] We show that these results are recovered here. In Sec. V we summarize and conclude.

II. STATEMENT OF THE PROBLEM

We consider homentropic flow of an ideal gas, for which the governing equations are

[a)]Deceased.

$$\rho_a \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p_a, \tag{2}$$

$$\frac{\mathrm{D}\rho_a}{\mathrm{D}t} + \rho_a \nabla\cdot\mathbf{u} = 0, \tag{3}$$

$$\frac{p_a}{p_0} = \left(\frac{\rho_a}{\rho_0}\right)^{\gamma}, \tag{4}$$

where p_a and ρ_a denote absolute pressure and density. The physical situation consists of a vortex, in which the local Mach number is small, onto which sound waves with wavelength long compared to the size of the vortex are incident.

Taking the size of the vortex to be L and the local velocity to be U, the Mach number of the vortical flow is $M \equiv U/c_0$, where $c_0 \equiv (\gamma p_0/\rho_0)^{1/2}$. The Mach number is taken to be small and to leading order the vortex flow is incompressible. However, the vortex is modified by compressibility effects, which must be taken into account here.

The appropriate scalings for pressure and density are then

$$p_a = p_0(1+\gamma M^2 p), \quad \rho_a = \rho_0(1+M^2\rho); \tag{5}$$

these are the scalings appropriate for near-incompressible flow.[7]

Using the scaling (5), the nondimensional equations in the vortex region are

$$(1+M^2\rho)\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p, \tag{6}$$

$$M^2\left(\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u}\right) + \nabla\cdot\mathbf{u} = 0, \tag{7}$$

$$1+\gamma M^2 p = (1+M^2\rho)^{\gamma}. \tag{8}$$

In this paper, we consider Hill's spherical vortex, which translates at a constant velocity. Therefore, we take Eqs. (6)–(8) to apply in a frame of reference moving with the velocity $\mathbf{v}$ of the vortex centroid,[1,8] in which case the material derivative becomes

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{u}-\mathbf{v})\cdot\nabla. \tag{9}$$

We shall use the notation of Part I, in which the spatial coordinate in the vortical region is

$$\boldsymbol{\xi} = \mathbf{x} - \mathbf{x}_c, \tag{10}$$

where $\mathbf{x}_c$ denotes the ''center'' of the vortical region, and

$$\mathbf{v} = \frac{d\mathbf{x}_c}{dt}. \tag{11}$$

In Part I we discuss the conditions that must be satisfied by $\mathbf{v}$: in particular, it must be a slowly varying function of time. Because Hill's vortex translates at a constant velocity, this condition is automatically satisfied by taking $\mathbf{x}_c$ to be the center of Hill's vortex, for which $\mathbf{v}$ is constant.

Equations (6)–(8) are not appropriate far from the vortex. In the wave region, the appropriate spatial variable is $\mathbf{X} = M\mathbf{x}$ and the governing equations become

$$(1+M^2H)\left(\frac{\partial\mathbf{U}}{\partial t} + M^2\mathbf{U}\cdot\nabla\mathbf{U}\right) = -\nabla P, \tag{12}$$

$$\frac{\partial H}{\partial t} + \nabla\cdot\mathbf{U} + M^2\nabla\cdot(\mathbf{U}H) = 0, \tag{13}$$

$$1+\gamma M^2 P = (1+M^2H)^{\gamma}, \tag{14}$$

where the gradient operator acting on a wave-region quantity corresponds to differentiation with respect to $\mathbf{X}$. Nondimensional fields in the wave region are represented by capital letters (except for the density ρ, which is denoted there by H). In (12)–(14), the velocity field has been scaled by a factor of M. The leading-order dynamics in the wave region correspond to acoustic wave propagation with a mean flow. Note that, unlike the vortex region, no transformation to a moving frame is employed in the wave region.

In the wave region, because the acoustic waves are irrotational and the vortex is localized, the vorticity is small beyond all orders in M. This implies that, to all algebraic orders in M, the velocity field $\mathbf{U}$ may be written using a velocity potential Φ, defined by $\mathbf{U} = \nabla\Phi$.

The nondimensional amplitude $\delta \ll 1$ of the incoming acoustic wave is assumed to be sufficiently small that we may ignore nonlinear steepening effects. We are interested here in the linear scattering of waves by the vortex. This is reasonable in the case of experiments where the wave region is the region where waves are created and detected: farther away would be outside the experimental apparatus.

The solution is then written as an asymptotic expansion in M and δ that are independent small parameters (see Part I). In this situation, however, it is convenient to make use of the equation for mass-weighted vorticity, namely

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}}{\rho_a}\right) = \left(\frac{\boldsymbol{\omega}}{\rho_a}\right)\cdot\nabla\mathbf{u}, \tag{15}$$

which takes the following form in the axisymmetric case:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta}{\rho_a r\sin\theta}\right) = 0. \tag{16}$$

The conserved quantity is mass-weighted azimuthal vorticity. In common with the practice in geophysical fluid dynamics,[9] we shall refer to the materially conserved quantity $\zeta/(\rho_a r\sin\theta)$ as the ''potential vorticity.''

III. HILL'S SPHERICAL VORTEX

Hill's spherical vortex is a steadily translating exact solution of the Euler equations. It takes its simplest form when viewed in the frame moving with the vortex. In this frame there is a uniform flow $\mathbf{e}_z$ at infinity.

The Stokes streamfunction ψ_0' for Hill's spherical vortex in this frame is given by

$$\psi_0' = \begin{cases} -\frac{3}{4}(1-r^2)r^2\sin^2\theta, & r<1, \\ \frac{1}{2}(1-r^{-3})r^2\sin^2\theta, & r>1. \end{cases} \tag{17}$$

Here, $r = |\boldsymbol{\xi}|$ is the distance from the center of the vortex, and θ is the polar angle (or colatitude). This is an exact solution

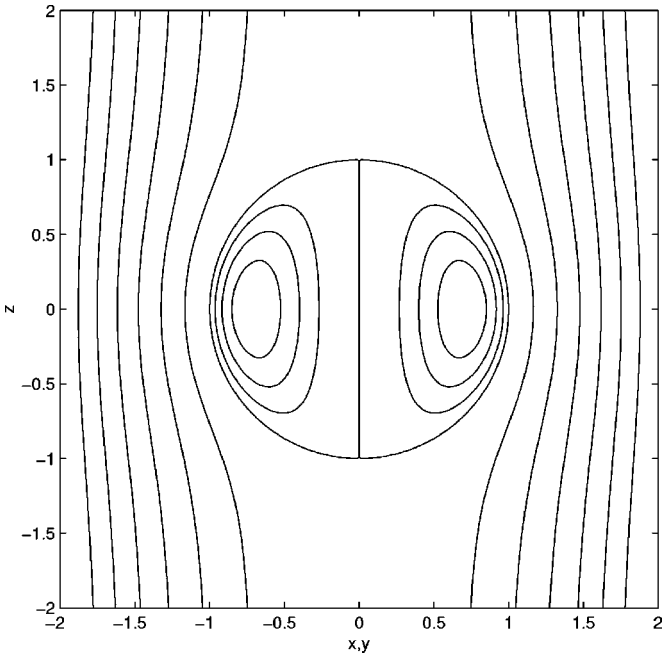

FIG. 1. Streamfunction of HSV in a frame moving with the vortex. The boundary of the vortex corresponds to $\psi_0=0$; the contour values inside are -0.2: 0.05: -0.05, the contour values outside are 0.25: 0.25: 1.5.

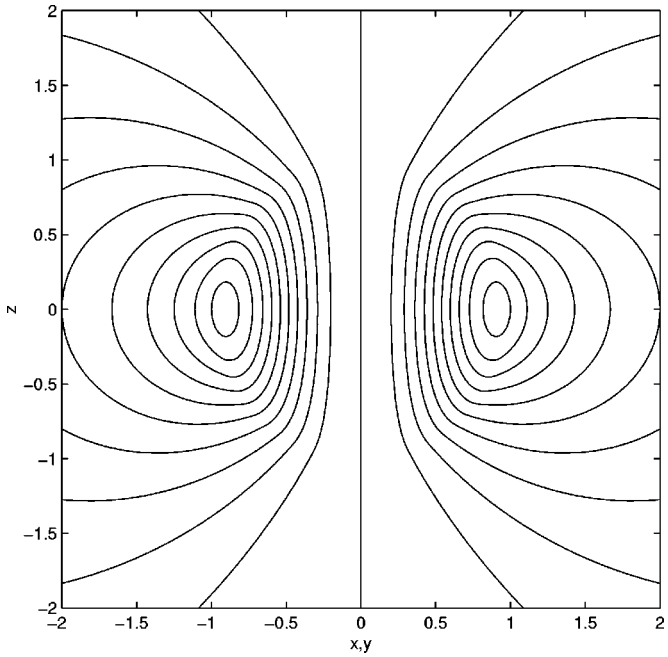

FIG. 2. Streamfunction of HSV in a frame at rest. The contour $\psi=0$ runs along the axis. The contour values are -0.5: 0.05: 0.

of the Euler equations [i.e., the limit of Eqs. (6)–(7) in which $M=0$]. The streamfunction ψ_0' defined by (17) is shown in Fig. 1.

In this paper, we are concerned with acoustic scattering from a vortex moving in a medium that is at rest at infinity. Therefore, we subtract the uniform flow at infinity from (17), which gives the streamfunction ψ_0 for HSV moving in a frame at rest:

$$\psi_0=\begin{cases} -\frac{1}{4}(5-3r^2)r^2\sin^2\theta, & r<1, \\ -\frac{1}{2}r^{-1}\sin^2\theta, & r>1. \end{cases} \tag{18}$$

Here, r is the distance from the center of the vortex, which translates at velocity $-\mathbf{e}_z$. The streamfunction ψ_0 defined by (18) is shown in Fig. 2.

The azimuthal component of vorticity corresponding to the flows (17) and (18) is given by

$$\zeta_0=-\frac{1}{r\sin\theta}D^2\psi_0=\begin{cases} -\frac{15}{2}r\sin\theta, & r<1, \\ 0, & r>1, \end{cases} \tag{19}$$

where

$$D^2=\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}. \tag{20}$$

The radial component of velocity is

$$u_{r0}=\frac{1}{r^2\sin\theta}\frac{\partial\psi_0}{\partial\theta}=\begin{cases} -\frac{1}{2}(5-3r^2)\cos\theta, & r<1, \\ -r^{-3}\cos\theta, & r>1, \end{cases} \tag{21}$$

and the meridional component is

$$u_{\theta 0}=-\frac{1}{r\sin\theta}\frac{\partial\psi_0}{\partial r}=\begin{cases} \frac{1}{2}(5-6r^2)\sin\theta, & r<1, \\ -\frac{1}{2}r^{-3}\sin\theta, & r>1. \end{cases} \tag{22}$$

This formulation of HSV satisfies (6)–(8) with $\mathbf{v}_0=-\mathbf{e}_z$, and $\mathbf{u}\to\mathbf{0}$ as $r\to\infty$, where $\mathbf{e}_z$ is the unit vector in the z direction along the axis of HSV.

To proceed, we must first obtain the leading-order pressure, p_0. To do this, a Bernoulli integral of (6) can be formed, and the result is

$$\frac{1}{2}\mathbf{u}_0^2-\mathbf{u}_0\cdot\mathbf{v}_0+p_0=\begin{cases} \frac{15}{2}\psi_0, & r<1, \\ 0, & r>1. \end{cases} \tag{23}$$

This Bernoulli integral exists in $r<1$ because of the special form of the vorticity inside HSV.

Hence, the pressure p_0 is given by

$$p_0=\begin{cases} -\frac{5}{8}-\frac{9}{8}r^2(1-3\cos^2\theta)+\frac{9}{8}r^4(1-2\cos^2\theta), \\ \quad r<1, \\ -\dfrac{1-3\cos^2\theta}{2r^3}-\dfrac{1+3\cos^2\theta}{8r^6}, \quad r>1. \end{cases} \tag{24}$$

To calculate the scattering in the wave region, it turns out that we will need the first correction to the HSV due to compressibility, which occurs at $O(M^2)$. This problem has been investigated previously,[2] and, as then, we need to extend the HSV to the compressible case, although we only need the small M version of the result. We will do this by specifying that the "potential vorticity" of the vortex be constant inside a region $r<h(\theta)$. For incompressible flow, $h(\theta)=1$ and the value of the constant is $-15/2$. Extending this condition gives

$$\zeta=\begin{cases} -\frac{15}{2}(1+M^2\rho)r\sin\theta, & r<h, \\ 0, & r>h. \end{cases} \tag{25}$$

The boundary between the inner and outer portions of the vortex becomes a function of the angular coordinate θ.

We carry out this calculation only in the vortical region. We expand the velocity field and boundary in powers of M^2:

$$\mathbf{u}=\mathbf{u}_0+M^2\mathbf{u}_2+\cdots, \quad h=1+M^2h_2+\cdots. \tag{26}$$

The equations that we need to solve are then

$$\zeta_2=\zeta_0\rho_0, \quad \mathbf{u}_0\cdot\nabla\rho_0+\nabla\cdot\mathbf{u}_2=0. \tag{27}$$

Note that from (4), $\rho_0=p_0$. We now decompose the velocity field $\mathbf{u}_2$ into a streamfunction and a velocity potential.

We hence solve for the velocity potential from

$$\nabla^2\phi_2=-(\mathbf{u}_0-\mathbf{v}_0)\cdot\nabla p_0, \tag{28}$$

to which a solution is

$$\phi_2=\begin{cases}(\frac{27}{40}r^3-\frac{297}{560}r^5+\frac{1}{8}r^7)P_1(\mu)+\frac{3}{40}r^5P_3(\mu), & r<1,\\ (-\frac{1}{5}r^{-5}+\frac{1}{24}r^{-8})P_1(\mu)+(-\frac{3}{10}r^{-2}-\frac{3}{10}r^{-5}\\ \qquad +\frac{3}{176}r^{-8})P_3(\mu), & r>1,\end{cases} \tag{29}$$

where $P_n(\mu)$ is the Legendre polynomial of order n with argument $\mu=\cos\theta$. This solution satisfies boundary conditions of regularity at $r=0$ and decay as $r\to\infty$, but does not satisfy continuity conditions at the boundary of the vortex. Continuity conditions must be applied to the velocity, but not necessarily to the rotational and divergent parts separately. Therefore, we will take (29) for ϕ_2, and apply continuity conditions when we determine the streamfunction ψ_2.

The streamfunction ψ_2 is determined by the potential vorticity equation, which takes the form

$$D^2\psi_2=\begin{cases}\frac{15}{2}\rho_0r^2\sin^2\theta, & r<1,\\ 0, \quad 1.\end{cases} \tag{30}$$

In total, ψ_{21} is expressed as the sum of two parts, $\psi_2^{(i)}$ and $\psi_2^{(h)}$. A solution to (30) in $r<1$ is

$$\psi_2^{(i)}=(-\tfrac{15}{32}r^4-\tfrac{27}{224}r^6+\tfrac{3}{32}r^8)\sin^2\theta\,P_1'(\mu) +(\tfrac{3}{16}r^6-\tfrac{9}{176}r^8)\sin^2\theta\,P_3'(\mu), \tag{31}$$

with $\psi_2^{(i)}=0$ in $r>1$. The form of $\psi_2^{(h)}$ is

$$\psi_2^{(h)}=\begin{cases}\sum_{n=1}^{\infty}A_nr^{n+1}\sin^2\theta\,P_n'(\mu), & r<1,\\ \sum_{n=1}^{\infty}B_nr^{-n}\sin^2\theta\,P_n'(\mu), & r>1.\end{cases} \tag{32}$$

The conditions at the boundary of the vortex boundary are continuity of the radial velocity u_r and the azimuthal velocity u_θ, and the kinematic condition that the boundary be a material surface. These may be rewritten as

$$[u_{r2}]=0, \quad [u_{\theta2}+h_2u_{\theta0}']=0, \qquad u_{r2}(1)+h_2u_{r0}'(1)=u_{\theta0}(1)\frac{\partial h_2}{\partial\theta}, \tag{33}$$

where [ ] represents the jump in that quantity at $r=1$, and primes represent derivatives with respect to r.

Each of these conditions is expressed as a sum of Legendre polynomials. Only the terms corresponding to $n=1,3$ remain in the continuity conditions for u_r and u_θ, and the third condition can be satisfied provided the boundary displacement takes the form $h_2=H_0P_0(\mu)+H_2P_2(\mu)$. The six resulting linear equations are

$$2B_1-2A_1=-\tfrac{148}{105}, \quad 12B_3-12A_3=\tfrac{21}{440}, \tag{34}$$

$$B_1+2A_1+\tfrac{15}{2}H_0-\tfrac{3}{2}H_2=\tfrac{1193}{840}, \qquad 3B_3+4A_3+\tfrac{3}{2}H_2=-\tfrac{1209}{880}, \tag{35}$$

$$2B_1+3H_0-\tfrac{3}{5}H_2=-\tfrac{2}{3}, \quad 12B_3+\tfrac{18}{5}H_2=-\tfrac{108}{55}. \tag{36}$$

The solution to this system is

$$A_1=-\tfrac{15}{112}, \quad A_3=-\tfrac{241}{880}, \quad B_1=-\tfrac{1409}{1680}, \quad B_3=-\tfrac{95}{352}, \quad H_0=\tfrac{137}{336}, \quad H_2=\tfrac{17}{48}, \tag{37}$$

with A_n and B_n equal to zero for $n\neq1,3$. The coefficients of the boundary displacement H_0 and H_2 agree with the results of Ref. 2. The nonzero value of H_0 ensures that the translation speed of the vortex remains unity to $O(M^2)$, and so

$$\mathbf{v}_2=\mathbf{0}. \tag{38}$$

IV. AXISYMMETRIC SCATTERING BY HSV

A. The $O(\delta)$ solution in the wave region

We take the incident acoustic wave to be of the form

$$P_{01}=F(t-\sigma Z), \tag{39}$$

which is a plane wave propagating along the Z axis. Here, $\sigma=\pm1$ determines the direction of propagation of the wave. The velocity and pressure are related by

$$\mathbf{u}_{01}=\sigma P_{01}\mathbf{e}_z. \tag{40}$$

Note that there is a distinction between $\sigma=1$ and $\sigma=-1$; in the former case, the vortex and the incident wave are propagating in opposite directions, while in the latter case they are propagating in the same direction. We shall assume that, for large values of its argument, $F(\tau)$ becomes a monochromatic function, $e^{-i\omega\tau}$. For smaller values of its argument, we assume that it turns on smoothly. This is important because, although the leading-order scattered field is monochromatic, the detailed dynamics within HSV can only be resolved by solving the initial-value problem.

In the neighborhood of the vortex, the pressure and velocity of the incident acoustic wave must be expanded, in order to provide asymptotic matching conditions for the flow in the vortex. The location of the vortex is $\mathbf{X}=\mathbf{X}_c=Z_c\mathbf{e}_z$, and

$$Z-Z_c=M\xi=Mr\cos\theta, \tag{41}$$

where Z_c is the location of the center of the vortex. In the case of Hill's vortex,

$$Z_c=-Mt. \tag{42}$$

The result is that the pressure and velocity in the wave region take the form

$$P_{01}=F(t-\sigma Z_c)-M\sigma\xi F'(t-\sigma Z_c)+\cdots, \tag{43}$$

$$\mathbf{U}_{01}=[\sigma F(t-\sigma Z_c)-M\xi F'(t-\sigma Z_c)]\mathbf{e}_z+\cdots, \tag{44}$$

in the limit $|\Xi|\to 0$.

B. The $O(\delta)$ solution in the vortical region

The solution at this order in the vortex region that satisfies the equations and the matching conditions is just a pressure oscillation,

$$\mathbf{u}_{01}=\mathbf{0}, \quad p_{01}=\rho_{01}=F(t-\sigma Z_c). \tag{45}$$

This is a general result, not specific to Hill's vortex, and it is discussed in Part I.

C. The $O(M\delta)$ solution in the vortical region

The velocity at this order is irrotational, since no vorticity can be introduced into the system by the acoustic waves. The streamfunction inside the vortex thus takes the form

$$\psi_{11}^{-}=\sum_{n=1}^{\infty} A_n^{(1)} r^{n+1}\sin^2\theta P_n'(\mu), \tag{46}$$

while the streamfunction outside the vortex takes the form

$$\psi_{11}^{+}=\sum_{n=1}^{\infty} (B_n^{(1)} r^{-n}+C_n^{(1)} r^{n+1})\sin^2\theta P_n'(\mu). \tag{47}$$

We impose the continuity of velocity at the vortex boundary, and also the condition that the boundary of the vortex is a material boundary, to obtain evolution equations for the coefficients $A_n^{(1)}$. The result is

$$(2n+1)\left(\frac{dA_n^{(1)}}{dt}-\frac{dC_n^{(1)}}{dt}\right)$$

$$=3(n-1)\left(\frac{n(n-3)}{2n-1}A_{n-1}^{(1)}-\frac{(n+1)(n+2)}{2n+3}A_{n+1}^{(1)}\right)$$

$$-\tfrac{3}{2}[n(n-1)C_{n-1}^{(1)}-(n+1)(n+2)C_{n+1}^{(1)}]. \tag{48}$$

The procedure is identical to that of M and M, except that here an additional set of coefficients, C_n, is retained. These coefficients are determined by matching to the incident sound wave. At $O(M\delta)$, these matching conditions give

$$C_1^{(1)}=\tfrac{1}{2}\sigma F(t-\sigma Z_c) \quad C_n=0, \quad \text{for all } n>1. \tag{49}$$

Now, when all the $C_n^{(1)}$ vanish except for $C_1^{(1)}$, the system of difference equations (48) has a special solution in which

$$A_1^{(1)}=C_1^{(1)}; \quad \frac{dA_2^{(1)}}{dt}=-C_1^{(1)}, \quad A_n^{(1)}=0, \quad \text{for all } n>2. \tag{50}$$

On solving the equation for $A_2^{(1)}$, we recall that, although Z_c is a function of time, its time-derivative is taken to be $O(M)$, and $A_2^{(1)}$ satisfies

$$\left.\frac{dA_2^{(1)}}{dt}\right|_{O(1)}=-\frac{\sigma}{2}F(t-\sigma Z_c), \tag{51}$$

where the notation for the time-derivative is as in Part I. The solution for $A_2^{(1)}$ is

$$A_2^{(1)}=-\frac{\sigma}{2}\int_{-\infty}^{t} F(\tau-\sigma Z_c)\,d\tau, \tag{52}$$

in which Z_c is taken to be a constant.

Continuity of ψ at the boundary of the vortex gives

$$B_2^{(1)}=A_2^{(1)}, \quad B_n^{(1)}=0, \quad \text{for all } n\neq 2. \tag{53}$$

Note that if the function $F(t-\sigma z)$ tends to a time-harmonic function of its argument for large times, $A_1^{(1)}$ will become time harmonic in the limit $t\to\infty$, and $A_2^{(1)}$ will become time harmonic, albeit possibly offset by a constant value. This constant value may be neglected if the acoustic source is turned on over a time that is long compared with one period of the wave, so that

$$F(t)=\Lambda(\epsilon t)e^{-i\omega t}, \tag{54}$$

in which $\Lambda(t)\to 0$ as $t\to-\infty$ and $\Lambda(t)\to 1$ as $t\to\infty$, and $\epsilon/|\omega|\ll 1$. Henceforth we shall assume that $F(t)$ takes the form given by (54). Then, in the limit of large time, $F(t-\sigma z)\to e^{-i\omega(t-\sigma z)}$, and the coefficients $A_1^{(1)}$, $A_2^{(1)}$, and $B_2^{(1)}$ approach their time-harmonic values:

$$A_1^{(1)}\to\frac{k}{2\omega}e^{ikZ_c-i\omega t}, \tag{55}$$

$$A_2^{(1)}\to-\frac{i}{2k}e^{ikZ_c-i\omega t}, \tag{56}$$

$$B_2^{(1)}\to-\frac{i}{2k}e^{ikZ_c-i\omega t}, \tag{57}$$

where

$$k=\sigma\omega \tag{58}$$

is the wave number of the incident acoustic wave. (Recall that $\sigma=\pm 1$.)

The pressure p_{11} can be obtained by expressing (2) in the form

$$\left.\frac{\partial\mathbf{u}_{11}}{\partial t}\right|_{O(1)}+\zeta_0\mathbf{e}_\phi\times\mathbf{u}_{11}+\nabla[(\mathbf{u}_0-\mathbf{v}_0)\cdot\mathbf{u}_{11}]=-\nabla p_{11}, \tag{59}$$

at $O(M\delta)$, where here we have used the fact that the vorticity vanishes at $O(M\delta)$. Hence there exists a velocity potential ϕ_{11}, such that $\mathbf{u}_{11}=\nabla\phi_{11}$. Consequently, (59) may be integrated to give

$$\left.\frac{\partial\phi_{11}}{\partial t}\right|_{O(1)}+(\mathbf{u}_0-\mathbf{v}_0)\cdot\mathbf{u}_{11}+p_{11}=\begin{cases}\frac{15}{2}\psi_{11}, & r<1,\\ 0, & r>1.\end{cases} \tag{60}$$

When ϕ_{11} is obtained from ψ_{11}, and the result substituted into (60), the expression for p_{11} in the limit $t\to\infty$ is found to be

$$p_{11}=\begin{cases}\left(ikr-\dfrac{9i}{2k}r+\dfrac{27i}{10k}r^3\right)P_1(\mu)e^{ikZ_c-i\omega t}+\dfrac{9i}{5k}r^3P_3(\mu)e^{ikZ_c-i\omega t}, & r<1,\\ \left(ikr-\dfrac{9i}{5k}r^{-7}\right)P_1(\mu)e^{ikZ_c-i\omega t}+\left(\dfrac{3i}{k}r^{-4}-\dfrac{6i}{5k}r^{-7}\right)P_3(\mu)e^{ikZ_c-i\omega t}, & r>1.\end{cases} \tag{61}$$

Thus the incoming wave induces an $O(M\delta)$ response in the vortex region whose pressure is time harmonic and decays as r^{-4} in the farfield. This matches to a pressure at $O(M^5\delta)$ in the wave region. The velocity also decays as r^{-4} at this order, and this implies a velocity potential that decays as r^{-3}. This matches to terms that are $O(M^4\delta)$ in the wave region. This is at first sight surprising, since one expects pressure and velocity potential to exist at the same order for a radiating acoustic wave. The apparent contradiction here is resolved by the fact that the flow in the wave region is forced at $O(M^4\delta)$, and so the anticipated relation between velocity and pressure does not hold, except many wavelengths away from the vortex.

D. The $O(M^2\delta)$ solution in the vortical region

At $O(M^2\delta)$, the effects of compressibility enter in two ways. First, the continuity equation (7) is

$$\left.\frac{\partial\rho_{01}}{\partial t}\right|_{O(1)}+\nabla\cdot\mathbf{u}_{21}=0, \tag{62}$$

and so we see that at $O(M^2\delta)$ the flow is compressible. Also, constancy of the potential vorticity implies that

$$\zeta_{21}=\zeta_0\rho_{01}, \tag{63}$$

and so we see that the vorticity compensates for changes in density, in such a way that the potential vorticity takes the value $\frac{15}{2}$.

We decompose the velocity field into streamfunction and potential, so that

$$\mathbf{u}_{21}=\nabla\phi_{21}-\frac{1}{r\sin\theta}\mathbf{e}_\phi\times\nabla\psi_{21}. \tag{64}$$

Then

$$\nabla^2\phi_{21}=-F'(t-\sigma Z_c). \tag{65}$$

We may pick the solution to the Poisson equation (65) to be

$$\phi_{21}=-\tfrac{1}{2}\xi^2F'(t-\sigma Z_c); \tag{66}$$

the unspecified harmonic functions will all be incorporated into the streamfunction at this order. For large r, $\nabla\phi_{21}\sim-\xi F'(t-\sigma Z)\mathbf{e}_z$, which exactly matches the second term in (44). Therefore the matching condition on ψ_{21} is that it decay in the farfield.

Inside the vortex, the streamfunction ψ_{21}^- satisfies

$$D^2\psi_{21}^-=\tfrac{15}{2}r^2\sin^2\theta F(t-\sigma Z_c). \tag{67}$$

The solution to this equation is

$$\psi_{21}^-=\tfrac{3}{4}r^4\sin^2\theta F(t-\sigma Z_c)+\sum_{n=1}^{\infty}A_n^{(2)}r^{n+1}\sin^2\theta P_n'(\mu). \tag{68}$$

The solution outside the vortex is

$$\psi_{21}^+=\sum_{n=1}^{\infty}B_n^{(2)}r^{-n}\sin^2\theta P_n'(\mu). \tag{69}$$

The matching conditions that must be enforced across the vortex boundary are the continuity of ψ, $\partial\psi/\partial r$, and the kinematic condition. Note that no terms of form $C_nr^{n+1}\sin^2\theta P_n(\mu)$ are required in (69), because the velocity $\nabla\phi_{21}$ matches to the incident wave at this order, where ϕ_{21} is given by (66).

The continuity of both components of velocity at the boundary of the vortex, plus the kinematic condition, are expressed at this order as

$$\begin{gathered}[\psi_{21}]=0,\quad [h_{21}\psi_0''+\psi_{21}']=0,\\ \left.\frac{\partial h_{21}}{\partial t}\right|_{O(1)}+\left.\frac{\partial h_{11}}{\partial t}\right|_{O(M)}+v_0(1)\frac{\partial h_{21}}{\partial\theta}=h_{21}u_0'(1)+u_{21}(1).\end{gathered} \tag{70}$$

Following the derivation of (48) for the evolution of the coefficients $A_n^{(1)}$, we have

$$\begin{aligned}&(2n+1)\left.\frac{dA_n^{(2)}}{dt}\right|_{O(1)}+5\delta_{2n}\left.\frac{dA_2^{(1)}}{dt}\right|_{O(M)}\\ &\quad=3(n-1)\left(\frac{n(n-3)}{2n-1}A_{n-1}-\frac{(n+1)(n+2)}{2n+3}A_{n+1}\right)\\ &\qquad+\delta_{3n}F'(t-\sigma Z_c)-\tfrac{9}{4}\delta_{1n}F'(t-\sigma Z_c).\end{aligned} \tag{71}$$

To obtain the scattered field at $O(M^4\delta)$, it appears that we need only determine the value of $A_1^{(2)}$, since it is only the flow corresponding to $n=1$ that can match to flow in the wave region at $O(M^4\delta)$. Taking $n=1$ in (71), we see that $A_1^{(2)}$ satisfies

$$3\left.\frac{dA_1^{(2)}}{dt}\right|_{O(1)}=-\frac{9}{4}F'(t-\sigma Z_c). \tag{72}$$

The causal solution, which satisfies $A_1^{(2)}\to0$ as $t\to-\infty$, is

$$A_1^{(2)}=-\tfrac{3}{4}F(t-\sigma Z_c). \tag{73}$$

The corresponding expression for $B_1^{(2)}$ is

$$B_1^{(2)}=0. \tag{74}$$

Unlike the solution at $O(M\delta)$, however, Eq. (71) has forcing at $n=1$, $n=2$, and $n=3$. At this point, we note that the system (71) can be solved for $n\geqslant3$ independently of the

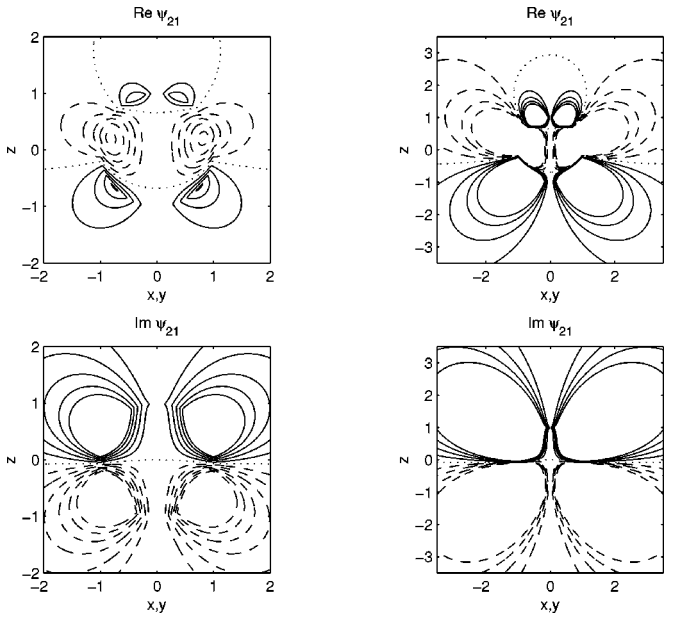

FIG. 3. The $O(M^2\delta)$ streamfunction ψ_{21} in the case $\omega=1$. The upper two panels show the real part of ψ_{21}, and the lower two panels show the imaginary part. The contour interval in the left two panels is 0.05; solid contours correspond to positive values, and dashed contours correspond to negative values. The dotted contour is the zero contour. The contour interval in the upper right panels is 0.005, and in the lower right panel it is 0.01.

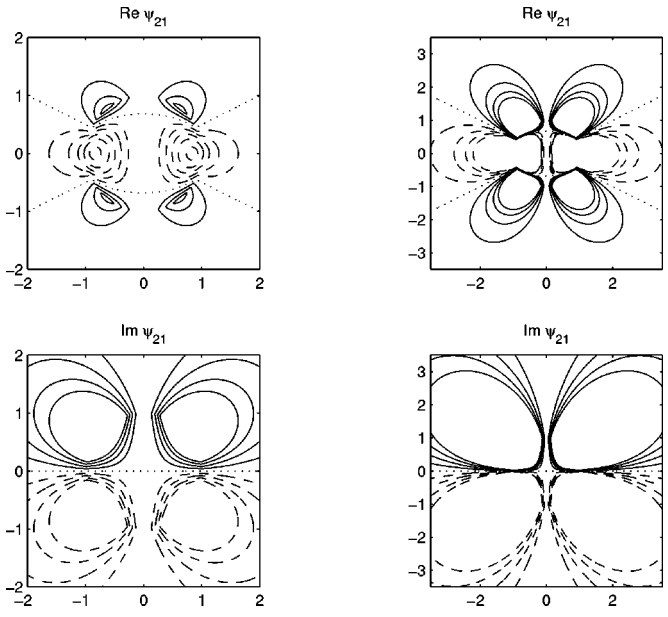

FIG. 4. The $O(M^2\delta)$ streamfunction ψ_{21} in the case $\omega=10$. The upper two panels show the real part of ψ_{21}, and the lower two panels show the imaginary part. The contour interval in the top left panel is 0.05, and in the top right panel it is 0.005; solid contours correspond to positive values, and dashed contours correspond to negative values. The dotted contour is the zero contour. The contour intervals in the lower panels is 0.005 (left) and 0.001 (right), implying that the real part of the solution is approximately ten times larger than the imaginary part.

values of $A_1^{(2)}$ and $A_2^{(2)}$. Therefore, if the $A_n^{(2)}$ can be obtained for $n\geqslant 3$, the value of $A_2^{(2)}$ can then be evaluated *a posteriori*.

Thus, although forcing at modes $n=1$ and $n=2$ in (71) admits a solution in which $A_n^{(2)}=0$ for all $n>2$, the forcing at $n=3$ in (71) implies that, in general, the $A_n^{(2)}$ will be nonzero for all n. It is now important to determine the character of these solutions in the limit $t\to\infty$, for all n. If any of these coefficients grows without bound as $t\to\infty$ then the asymptotic expansion for the flow in the vortex may become disordered, and the results derived in Part I may no longer apply.

We start by investigating time-periodic solutions to (71), with time dependence $e^{-i\omega t}$. There are two branches of solution, and in the limit $n\to\infty$ they take the form

$$A_n^{(2)}=n^{\nu_1}; \quad A_n^{(2)}=(-1)^n n^{\nu_2}. \tag{75}$$

It can readily be shown that

$$\nu_1=\tfrac{2}{3}i\omega-2; \quad \nu_2=-\tfrac{2}{3}i\omega-2. \tag{76}$$

[The expressions given for ν_1 and ν_2 in M and M, Eq. (3.35), are incorrect.]

Now, in both of these solutions, $|A_n^{(2)}|$ decays as n^{-2} for large n. It turns out that the sum for ψ_{21} converges, whichever is taken, but the expression for the velocities does not converge, whichever is taken. If a causal solution is sought, by setting $\omega\to\omega+i\epsilon$ and taking the limit $\epsilon\to 0^+$, then the expression for the velocity converges if the first case, $A_n^{(2)}=n^{\nu_1}$, is taken. Therefore, this represents the causal solution.

The solution for $\omega=1$ is shown in Fig. 3. Note that the farfield form of ψ_{21}, visible in the right-hand panels, clearly takes the form of a quadrupole, with angular dependence proportional to $\sin^2\theta P_2'(\cos\theta)\propto\sin^2\theta\cos\theta$. Note also that, for $\omega=1$, the real and imaginary parts are approximately equal in magnitude.

The solution for $\omega=10$ is shown in Fig. 4. Note that the magnitude of the real part of ψ_{21} is approximately ten times larger than the imaginary part. Note also that, for $r\to\infty$, the imaginary part has the angular dependence $\sin^2\theta\cos\theta$, consistent with matching to a quadrupole farfield, whereas the real part has the angular dependence $\sin^2\theta P_3'(\cos\theta)\propto\sin^2\theta(5\cos^2\theta-1)$.

The solution for $\omega=0.1$ is shown in Fig. 5. Note here that the magnitude of the imaginary part is approximately ten times greater than the real part. In this case, however, both real and imaginary parts have the quadrupolar form $\sin^2\theta\cos\theta$ as $r\to\infty$.

Now, in the time-harmonic limit, $\phi_{21}=\frac{1}{2}i\omega\xi^2$, and so the results presented in Figs. 4 and 5 show that the velocity induced in the vortex is in phase with the imposed velocity of the wave if ω is small, and $\pi/2$ out of phase if ω is large.

Although ψ_{21} is bounded in these periodic solutions, the radial velocity is not bounded in the neighborhood of the rear stagnation point. The solutions presented in Figs. 3–5 do not apply at any finite time, and although ψ_{21} approaches these solutions pointwise as $t\to\infty$, its gradient does not approach any finite limit as $t\to\infty$.

To investigate the solution in the limit $t\to\infty$, we solve an initial-value problem in which the incident wave takes the form given by (44), with F given by (54). The system (71) for $n\geqslant 3$ can now be solved to good approximation, following M and M, by first defining

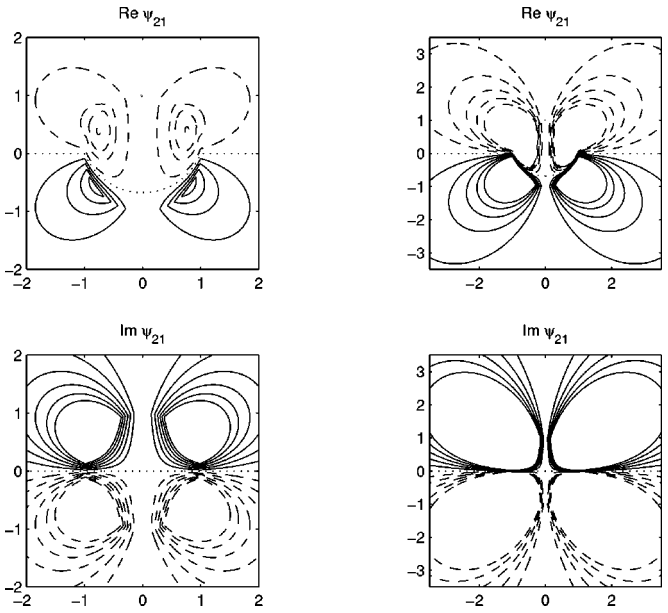

FIG. 5. The $O(M^2\delta)$ streamfunction ψ_{21} in the case $\omega=0.1$. The upper two panels show the real part of ψ_{21}, and the lower two panels show the imaginary part. The contour interval in the top left panel is 0.05, and in the top right panel it is 0.01; solid contours correspond to positive values, and dashed contours correspond to negative values. The dotted contour is the zero contour. The contour intervals in the lower panels is 0.5 (left) and 0.1 (right), implying that the imaginary part of the solution is approximately ten times larger than the real part.

$$\alpha_n=\frac{n(n+2)(n+3)}{2n+5}A^{(2)}_{n+2}, \quad \text{for } n\geqslant 0. \tag{77}$$

For convenience, we define a new time scale $t'=\frac{3}{4}t$. Then, dropping the prime on the new time scale, the α_n satisfy the difference equation

$$\frac{d\alpha_n}{dt}=n[1-1/(2n+5)^2](\alpha_{n-1}-\alpha_{n+1}) + \tfrac{12}{49}\,\delta_{n1}F'(\tfrac{4}{3}t-\sigma Z_c), \quad \alpha_0=0, \tag{78}$$

with $\alpha_n=0$ at $t=0$.

Following M and M, we approximate $(1-1/(2n+5)^2)$ by 1. As discussed by M and M, and also by Pozrikidis,[4] this makes an error of no more than 2%, when $n=1$, and the approximation improves as $n\to\infty$, which is our interest here. We therefore replace (78) by

$$\frac{d\alpha_n}{dt}-n(\alpha_{n-1}-\alpha_{n+1})=\frac{16}{49}\delta_{n1}F'\left(\frac{4}{3}t-\sigma Z_c\right). \tag{79}$$

Now, a solution of (79), with no right-hand side, is (see M and M)

$$\alpha_n=n\tanh^{n-1}t\,\mathrm{sech}^2 t. \tag{80}$$

At $t=0$, this solution satisfies $\alpha_1=1$ and $\alpha_n=0$ for all $n>1$. By using this solution as a Green's function for (79), we may express the solution to (79) as

$$\alpha_n(t)=\frac{16n}{49}\int_0^t \tanh^{n-1}(t-\tau)\,\mathrm{sech}^2(t-\tau) \times F'(\tfrac{4}{3}\tau-\sigma Z_c)\,d\tau. \tag{81}$$

The integrand is more conveniently rewritten in terms of a new variable, x, defined such that $\tanh(t-\tau)=e^{-x}$, and so

$$\alpha_n(t)=\frac{16n}{49}\int_{\ln(\coth t)}^{\infty} F'\left(\frac{4}{3}(t-\tanh^{-1}e^{-x})-\sigma Z_c\right) \times e^{-nx}dx. \tag{82}$$

Our primary aim in this analysis is to consider the behavior of $\alpha_n(t)$ in the limit $t\to\infty$, and so to establish whether the solution remains well behaved over large times. To analyze the limit $t\to\infty$ in (82), we first replace $\ln(\coth t)$ by $2e^{-2t}$. It follows immediately that if $2ne^{-2t}=N_1\gg 1$, say, then $\alpha_n(t)=O(e^{-N_1})$. Thus, for any given t, there exists an $N\gg e^{2t}$ such that for all $n>N$ the coefficients decay exponentially with n.

On the other hand, if t is large, and n is large but held fixed, then the dominant contribution to (82) comes from the region $x=O(n^{-1})$. Let $u=nx$. Then we may approximate $\tanh^{-1}e^{-u/n}$ by $\tanh^{-1}(1-u/n)\approx\frac{1}{2}(\ln n+\ln 2-\ln u)$. Then

$$\alpha_n(t)\sim\frac{16}{49}\int_{2ne^{-2t}}^{\infty}F'\left(\frac{4}{3}[t-\tfrac{1}{2}(\ln 2+\ln n-\ln u)]-\sigma Z_c\right) \times e^{-u}du. \tag{83}$$

Now, if $t\gg\frac{1}{2}\ln n$, then the argument of F' in (83) is large when $u=O(1)$. The exponential e^{-u} implies that contribution from $u\gg 1$ is then negligible, and so $F'(\varphi)$ may be approximated by $-i\omega e^{-i\omega\varphi}$ in the evaluation of (83). Hence, for $t\gg\frac{1}{2}\ln n$,

$$\alpha_n(t)\sim-\frac{16i\omega}{49}e^{ikZ_c-4i\omega t/3}(2n)^{2i\omega/3}\int_0^{\infty}u^{-2i\omega/3}e^{-u}du = -\frac{32\omega^2}{147}e^{ikZ_c-4i\omega t/3}(2n)^{2i\omega/3}\Gamma\left(-\frac{2i\omega}{3}\right). \tag{84}$$

This is the solution with behavior $A_n=n^{2i\omega/3-2}$ for large n, which we argued previously should correspond to the causal solution. The analysis here shows how this solution develops in time, starting with small n and propagating to large n, with the coefficients for sufficiently large n decaying exponentially with n for any time t. Thus, for any finite time, this solution remains bounded.

Finally, A_2 is obtained by solving

$$\left.\frac{dA_2^{(2)}}{dt}\right|_{O(1)}=-\left.\frac{dA_2^{(1)}}{dt}\right|_{O(M)}-\frac{2}{5}A_1-\frac{3}{5}\alpha_1. \tag{85}$$

In the limit $t\to\infty$, the right-hand side of this equation is harmonic in time. Thus, in the limit $t\to\infty$, $A_2^{(2)}$ is also harmonic in time, possibly with a constant offset, corresponding to a fixed displacement of the vortex. This displacement is negligible provided the time over which the acoustic wave is turned on is large compared with one period of the wave.

Corresponding expressions for the coefficients $B_n^{(2)}$ can be obtained from continuity of normal velocity at the bound-

ary of the vortex. The result is that all of these coefficients remain bounded for all time. Consequently, the solution at $O(M^2\delta)$ remains bounded for all finite time, and (74) shows that it contains no component that could match to a scattered wave field at $O(M^4\delta)$. Finally, we must determine any terms in the vortex at $O(M^3\delta)$ that can match to a monopole wave field at $O(M^4\delta)$.

E. The $O(M^3\delta)$ solution in the vortical region

At this order, the equations are

$$\left.\frac{\partial\rho_{11}}{\partial t}\right|_{O(1)}+\left.\frac{\partial\rho_{01}}{\partial t}\right|_{O(M)}-\mathbf{v}_0\cdot\nabla\rho_{11}+\nabla\cdot(\mathbf{u}_0\rho_{11}+\mathbf{u}_{11}\rho_0)+\nabla\cdot\mathbf{u}_{31}=0,\quad \zeta_{31}=\zeta_0\rho_{11}. \tag{86}$$

In order to determine the scattered wave field, at $O(M^4\delta)$, we need to determine the streamfunction and velocity potential that together correspond to a monopole source in the limit $r\to\infty$. At this order, we shall be concerned only with establishing the time-harmonic response; the analysis of the previous section shows that the time-harmonic response is the the long-time limit of the initial value problem for finite spherical harmonic mode numbers.

Now, using (61), one can readily see that the streamfunction ψ_{31}^- inside the vortex satisfies

$$D^2\psi_{31}^-=\rho_{11}\zeta_0=\frac{15}{2}r^2\sin^2\theta\left[\left(ikr-\frac{9i}{2k}r+\frac{171i}{70k}r^3\right)\times P_2'(\mu)+\frac{9i}{35k}r^3P_4'(\mu)\right]e^{ikZ_c-i\omega t}. \tag{87}$$

It follows that, in $r<1$, ψ_{31}^- takes the form

$$\psi_{31}^-=\sum_{n=1}^{\infty}A_n^{(3)}(r)e^{ikZ_c-i\omega t}\sin^2\theta P_n'(\mu), \tag{88}$$

while in $r>1$, the streamfunction ψ_{31}^+ takes the form

$$\psi_{31}^+=\sum_{n=1}^{\infty}B_n^{(3)}e^{ikZ_c-i\omega t}r^{-n}\sin^2\theta P_n'(\mu). \tag{89}$$

Hence, ψ_{31}^+ corresponds to a velocity that is $O(r^{-3})$ in the limit $r\to\infty$, or, equivalently, a velocity potential r^{-2}. Thus, ψ_{31}^+ cannot contribute to matching to flow in the outer region at $O(M^4\delta)$. In fact, as we shall see below, matching conditions can be determined to flow in the wave region at $O(M^4\delta)$ without evaluating ψ_{31}.

Now, the velocity potential satisfies the equation

$$\nabla^2\phi_{31}=-\left.\frac{\partial\rho_{11}}{\partial t}\right|_{O(1)}-\left.\frac{\partial\rho_{01}}{\partial t}\right|_{O(M)}-(\mathbf{u}_0-\mathbf{v}_0)\cdot\nabla\rho_{11}-\mathbf{u}_{11}\cdot\nabla\rho_0-\rho_0\nabla\cdot\mathbf{u}_{11}. \tag{90}$$

In $r<1$, the velocity potential ϕ_{31}^- takes the form

$$\phi_{31}^-=\sum_{n=0}^{\infty}E_n(r)e^{ikZ_c-i\omega t}P_n(\mu), \tag{91}$$

and in $r>1$ the velocity potential ϕ_{31}^+ takes the form

$$\phi_{31}^+=\sum_{n=0}^{\infty}F_n(r)e^{ikZ_c-i\omega t}P_n(\mu). \tag{92}$$

For our purposes, it is necessary only to determine $F_0(r)$; this determines the strength of the monopole term, of form r^{-1}, that matches to flow in the wave region at $O(M^4\delta)$. We will see that this term can be evaluated without evaluating the other terms in ϕ_{31}, and without evaluating ψ_{31}. Henceforth we shall write

$$\overline{\phi}_{31}(r)=e^{ikZ_c-i\omega t}\begin{cases}E_0(r), & r<1,\\ F_0(r), & r>1.\end{cases} \tag{93}$$

Substituting the expressions already obtained onto the right-hand side of (90), and collecting the θ dependence into a sum over Legendre polynomials, we find that, in the large-t time-harmonic limit, $\overline{\phi}_{31}$ satisfies

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\overline{\phi}_{31}\right)=e^{ikZ_c-i\omega t}\times\begin{cases}\frac{5}{2}ik-\frac{27i}{4k}-\left(\frac{5}{2}ik-\frac{99i}{4k}\right)r^2-\frac{63i}{4k}r^4, & r<1,\\ -\frac{6i}{k}r^{-8}+\frac{6i}{k}r^{-11}, & r>1.\end{cases} \tag{94}$$

The solution is

$$\overline{\phi}_{31}=e^{ikZ_c-i\omega t}\times\begin{cases}\left(\frac{5}{12}ik-\frac{9i}{8k}\right)r^2+\left(-\frac{1}{8}ik+\frac{99i}{80k}\right)r^4-\frac{3i}{8k}r^6, & r<1,\\ -\frac{i}{5k}r^{-6}+\frac{i}{12k}r^{-9}+Cr^{-1}, & r>1,\end{cases} \tag{95}$$

where C is a constant that must be determined by matching the radial velocity at the boundary of the vortex. Expanding this condition to $O(M^3\delta)$ gives

$$h_2h_{11}[u_{r0}'']+h_{11}[u_{r2}']+h_2[u_{r11}']+[u_{r31}]=0, \tag{96}$$

where, as before [ ] denotes the jump in a quantity at $r=1$, and primes denote differentiation with respect to r.

The expressions for u_0, u_2, u_{11}, h_2, and h_{11} are substituted into (96), and expressed as a sum over Legendre polynomials. The result is that the coefficient of $P_0(\mu)$ obtained from the first three terms in (96) vanishes, and hence

$$\left[\frac{d\overline{\phi}_{31}}{dr}\right]=0. \tag{97}$$

It follows that C in (95) is given by

$$C=-\tfrac{1}{3}ike^{ikZ_c-i\omega t}. \tag{98}$$

Continuity conditions for the other modes in the expansion will yield a system of equations analogous to (71), except that forcing will be present for a larger (but finite) num-

ber of modes. An analysis equivalent to that performed at $O(M^2\delta)$ will also apply at $O(M^3\delta)$, with the number of significant modes increasing exponentially with time, but no secular growth in the lowest-order modes.

F. The $O(M^4\delta)$ solution in the wave region

The equations satisfied in the wave region at this order are

$$\frac{\partial \mathbf{U}_{41}}{\partial t}+\mathbf{U}_2\cdot\nabla\mathbf{U}_{01}+\mathbf{U}_{01}\cdot\nabla\mathbf{U}_2=-\nabla P_{41}, \tag{99}$$

$$\frac{\partial H_{41}}{\partial t}+\nabla\cdot(\mathbf{U}_2 H_{01})+\nabla\cdot\mathbf{U}_{41}=0. \tag{100}$$

Since the vorticity vanishes in the wave region, we write $\mathbf{U}_{41}=\nabla\Phi_{41}$. The momentum equation is then integrated to give the unsteady irrotational Bernoulli equation. When this is used with the continuity equation, and previously-computed expressions for $\mathbf{U}_{01}$, $\mathbf{U}_2$ and H_{01} are substituted, the equation for Φ_{41} is

$$(\nabla^2+k^2)\Phi_{41}=2ikP_2(\cos\theta)R^{-3}e^{ikZ-i\omega t}, \tag{101}$$

where here $R=|\mathbf{X}-\mathbf{X}_c|$. It can readily be verified that a solution to (101) is

$$\Phi_{41}=-\frac{P_1(\mu)}{2R^2}e^{ikZ-i\omega t}+\Phi_{41}^H, \tag{102}$$

where the Φ_{41}^H is a solution of the homogeneous wave equation, which is determined by matching to the inner solution, as follows.

First, we define the monopole solution to Φ_{41}^H by

$$G_0\equiv\frac{1}{R}e^{i(k^2/\omega)R+ikZ_c-i\omega t}; \tag{103}$$

this solution has outgoing wave behavior. Note that, due to the time-dependence of Z_c, this is not a solution of the wave equation to all orders, but it is a solution to leading order, which is sufficient here.

Corresponding expressions for the dipole G_1 and quadrupole G_2 are

$$G_1\equiv\frac{\partial}{\partial Z}G_0=\left(\frac{ik^2}{\omega R}-\frac{1}{R^2}\right)e^{i(k^2/\omega)R+ikZ_c-i\omega t}P_1(\mu), \tag{104}$$

$$G_2\equiv\frac{1}{2}\left(\frac{\partial^2}{\partial Z^2}G_0+\frac{1}{3}k^2G_0\right)=\left(-\frac{1}{3}\frac{k^2}{R}-\frac{ik^2}{\omega R^2}+\frac{1}{R^3}\right)e^{i(k^2/\omega)R+ikZ_c-i\omega t}P_2(\mu). \tag{105}$$

In preparation for matching to these solutions in the wave region, the velocity corresponding to the streamfunction ψ_{11}^+ can be expressed as the gradient of a potential. The result is that, when contributions from the velocity field at orders $M\delta$, $M^2\delta$, and $M^3\delta$ are combined, the matching condition on Φ_{41} is

$$\Phi_{41}=\left(\frac{i}{kR^3}P_2(\mu)-\frac{ik}{3R}P_0(\mu)\right)e^{-i\omega t}+O(R^{-2}),\quad \text{as } R\to 0. \tag{106}$$

This determines Φ_{41}^H to be

$$\Phi_{41}^H=-\frac{ik}{6}G_0-\frac{1}{2}G_1+\frac{i}{k}G_2. \tag{107}$$

The dominant behavior in the farfield is given by

$$\Phi_{41}=R^{-1}\left(-\frac{1}{6}ik-\frac{1}{2}i\frac{k^2}{\omega}P_1(\mu)-\frac{1}{3}ikP_2(\mu)\right)\times e^{i(k^2/\omega R)+ikZ_c-i\omega t}+O(R^{-2})$$
$$=-\frac{ik}{2R}\cos\theta\left(\cos\theta+\frac{k}{\omega}\right)e^{i(k^2/\omega R)+ikZ_c-i\omega t}+O(R^{-2}). \tag{108}$$

Now recall that $k/\omega=\pm 1$. It follows that

$$\Phi_{41}=-R^{-1}ik\cos\vartheta\cos^2\tfrac{1}{2}\vartheta\, e^{i(k^2/\omega)R+ikZ_c-i\omega t}+O(R^{-2}), \tag{109}$$

as $R\to\infty$, where

$$\vartheta=\begin{cases}\theta, & \omega/k=1,\\ \theta-\pi, & \omega/k=-1.\end{cases} \tag{110}$$

In terms of the pressure, the scattered field far from the vortex is given by

$$P_{41}=R^{-1}\omega k\cos\vartheta\cos^2\tfrac{1}{2}\vartheta\, e^{i(k^2/\omega)R+ikZ_c-i\omega t}+O(R^{-2}). \tag{111}$$

It is easily shown that this expression agrees with Eq. (134) of Part I of this paper, which states that the scattered sound field is given by

$$P_{41}=-\frac{\omega^2}{4\pi R}\cos\vartheta\left(\frac{\mathbf{I}\cdot\boldsymbol{\Xi}}{R}+I_1\right)e^{i(kR+kZ_c-\omega t)}+O(R^{-2}), \tag{112}$$

where

$$\mathbf{I}=\frac{1}{2}\int\mathbf{x}\times\boldsymbol{\omega}(\mathbf{x})d^3\mathbf{x} \tag{113}$$

is the vortex impulse, and the acoustic wave here is assumed to propagate in the positive z direction. For HSV, we find that the impulse $\mathbf{I}$ is given by

$$\mathbf{I}=-2\pi\mathbf{e}_z. \tag{114}$$

There are two cases to consider: $\omega=k$ and $\omega=-k$.

If $\omega=k$, Eq. (112) can be applied directly to the present situation, giving

$$P_{41}=\frac{1}{2}\omega^2\cos\vartheta(1+\cos\vartheta)\frac{e^{i(kR+kZ_c-\omega t)}}{R}+O(R^{-2})$$
$$=\omega^2\cos\vartheta\cos^2\frac{1}{2}\vartheta\frac{e^{i(kR+kZ_c-\omega t)}}{R}+O(R^{-2}). \tag{115}$$

Equation (112) is derived on the assumption that $\omega=k$. In order to apply this equation to the case $\omega=-k$, we keep $\omega=k$ in (134), but reverse the direction of propagation of the vortex, which corresponds to reversing the direction of $\mathbf{I}$. Hence, in this second case the sign of P_{41} is reversed, and both cases may be represented together as

$$P_{41}=R^{-1}\omega k\cos\vartheta\cos^2\tfrac{1}{2}\vartheta\, e^{i[(k^2/\omega)R+kZ_c-\omega t]}+O(R^{-2}). \tag{116}$$

Hence the scattered sound field (116) predicted by the general theory of Part I agrees with the result of the explicit calculation for Hill's spherical vortex (111).

V. CONCLUSIONS

In Part I of this paper, we showed that the sound field scattered by a vortex in response to an incident plane acoustic wave could be evaluated without solving the problem in detail, provided it was assumed that $|\mathbf{J}-\mathbf{J}^{\mathrm{U}}|=o(M\delta)$ throughout the time period of interest. Here we see that this apparently remains true for all time in the case of HSV, because the coefficients of the low-order moments of fields at $O(M^2\delta)$ remain bounded, even over times $O(M^{-1})$. Hence, we appear to have demonstrated that this assumption is not vacuous, and is in fact satisfied by the most familiar example of a translating vortex.

However, throughout this pair of papers, we have assumed that δ is "sufficiently small that terms quadratic in δ can always be neglected." Now, although the coefficients of the low-order moments of the solution at $O(M^2\delta)$ remain bounded, the number of coefficients of significant magnitude grows as $n\sim e^{2t}$ so that, especially in the neighborhood of the rear stagnation point, $\mathbf{u}_{21}$ is expected to grow as e^{2t}. This suggests that δ may have to be exponentially small in M in order that nonlinear effects may be neglected over times $O(M^{-1})$. The precise dependence of δ on M can only be determined by solving the weakly nonlinear problem, which has not been attempted here. Over times $O(1)$, however, the analysis is valid under the much weaker, and more reasonable, assumption that δ is small compared with M itself.

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