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VECTOR CURRENTS AND CURRENT ALGEBRA.

I. DUALITY AND ZERO-WIDTH MODELS*

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ABSTRACT

This is the first in a series of papers investigating the properties of vector currents consistent with the hadron bootstrap, assuming duality and zero resonance widths. First, on general grounds and independently of current algebra, we show that two current amplitudes must have fixed singularities in the angular momentum plane. Then we discuss some general consequences for current amplitudes of duality and the zero-width approximation. Throughout we treat amplitudes for one or two vector currents and an arbitrary number N of spinless hadrons.

I. INTRODUCTION

This is the first in a series of papers in which we initiate an investigation of amplitudes for currents consistent with the hadron bootstrap assuming duality and zero resonance widths.[1] In such an approach to currents, as first discussed by Dashen and Frautschi,[2,3] one assumes a bootstrap solution of the strong interaction problem and then investigates the consistency requirements imposed on the nonstrong (current) amplitudes by this solution along with analyticity and unitarity. Recent progress made in the hadron model based on infinitely rising Regge trajectories and zero resonance widths, originally proposed by Mandelstam,[4] provides a new basis for this investigation. In particular, the generalization of the four-body Veneziano[5] amplitude to N-body amplitudes[6] provides an important new starting point for the investigation of currents.

In this paper we discuss the general properties of amplitudes for one or two vector currents when duality and the zero-width approximation are assumed. These results provide the framework for our explicit investigation of currents in the N-point beta-function model[6] of the meson bootstrap in the following paper.[7] There we shall find consistency of the Gell-Mann current algebra[8] with that particular model of the hadron bootstrap in first approximation (single poles in form factors and factorization on leading trajectories). As yet, however, we have no definite answer to the question of whether current algebra is a consequence of, consistent with, or perhaps inconsistent with the hadron bootstrap.

In these papers we concentrate on the construction of amplitudes for conserved isoscalar and isovector vector currents consistent with current algebra; we discuss only occasionally the question of the uniqueness of currents with such quantum numbers and commutation relations. However, we believe that both the uniqueness and existence questions can eventually be fully answered, at least in the N-point beta-function model. Further we suggest the new possibility that the consistency problem for currents has a particularly simple solution (and current algebra is valid) only in dual zero-width models with linearly rising trajectories.

The chief dynamical constraint on current amplitudes is factorization, since in the zero-width approximation factorization is the chief remaining consequence of unitarity.[9] The power of the factorization constraints is seen clearly in the model of II. Factorization may or may not be enough to uniquely determine the current amplitudes; if it is not, current algebra may be required as an additional constraint. We also make the dynamical assumption that all energy variables have Regge behavior except when they are required on general grounds to have fixed-power behavior (see Sec. II.B).

Throughout we treat currents from an S-matrix point of view. We deal only with the covariant tensor amplitudes which are directly related to physical transition rates. We do not need to assume the existence of local current density operators.[10] Furthermore, asymptotic properties are most conveniently expressed in terms of the covariant amplitudes; for example, the angular momentum plane structure (moving poles, fixed poles, Kronecker deltas, etc.) can easily be deduced from their asymptotic behavior.

As an important technical convenience we discuss always amplitudes for an arbitrary number N of spinless hadrons. Such amplitudes give a convenient way of handling the important factorization constraints. It is also much easier to handle the kinematics of high-spin mesons by extracting the amplitudes for such particles from the residues of poles in many-particle amplitudes. A probable extension of our approach to fermions is to multiply our amplitudes by spinors for half-integral spin fermions. Finally, we note that, in the zero-width approximation, a solution of current algebra in terms of N-body amplitudes is equivalent to the saturation by single-particle states proposed by Dashen and Gell-Mann,[11] since the singularities in any variable are simple poles.

In Section II we discuss some general properties of current amplitudes independent of duality or the zero-width approximation. We discuss in some detail the consequences of the existence of a physical photon, since they imply important boundary conditions at the point $q^2 = 0$. The chief result of the section is a proof that the two-current amplitude must have fixed-power behavior (and hence fixed poles) independently of any consideration of current algebra.[12] In Section III we define our concept of duality and discuss the consequences of duality for current amplitudes. We shall find that the amplitudes must have a particular form as the momentum q_μ of a current goes to zero. The absence of exotic resonances and SU(2) internal symmetry imply that only isoscalar and isovector charges exist. The required properties of current amplitudes in zero-width models are listed in Section IV and their interrelationships are discussed.

We shall asume SU(2) symmetry for the hadron bootstrap; the extension to SU(3) or other symmetries is in most instances straightforward.

II. GENERAL PROPERTIES OF CURRENT AMPLITUDES

In this section we discuss some relevant properties of (A) single-current amplitudes and (B) two-current amplitudes that follow from Lorentz invariance and the usual analyticity and unitarity assumptions of S-matrix theory.

A. Single-Current Amplitudes

The description of the physical photon as the zero-mass limit (ZML) of a massive vector particle is very useful in constructing photon amplitudes with the correct kinematic properties.[13] The transformation law of the physical massless photon follows from that of the "massive photon" if the condition

$$m_\gamma / H_0(q) \to 0 \qquad \text{as} \qquad m_\gamma \to 0 \qquad (2.1)$$

holds, where m_γ is the photon mass, q its momentum, and H_0 the helicity zero amplitude (the hadron momenta have been suppressed). The condition (2.1) assures that the physical (helicity one) amplitudes transform independently of the unphysical (helicity zero) amplitude. The Wigner rotation of the massive photon then goes over into a pure z rotation of the proper amount.[14] The discontinuous change in little-group structure at $m_\gamma = 0$ often obscures the smoothness of amplitudes in m_γ.

If the condition (2.1) is satisfied, the helicity-one amplitudes will yield charge conservation and the low energy theorems, since these follow from their transformation law[13,15] (e.g., on-mass-shell gauge invariance). The undesired amplitude H_0 can merely be ignored. However, for physically reasonable off-mass-shell amplitudes, one would expect H_0 to vanish as $q^2 \to 0$, since a finite H_0 would correspond to a spin-zero photon, in contradiction with experiment.

The off-mass-shell amplitudes $H_\lambda(q)$ can be obtained from the electron scattering amplitude, A. This requires, of course, a complete knowledge of the electron form factor, the factorization at the $J = 1$ fixed photon singularity, and the weak coupling of the photon. The projection of $H_\lambda(q)$ from a diparticle (e.g. electron-electron) state gives a square-root kinematical singularity in H_0. In general the projection shows how the kinematic singularities associated with high spin can be derived from analytic amplitudes for many spinless particles. If A is to be analytic in q^2 and if H_0 is to be bounded, we must therefore have

$$H_0(q) = O[(q^2)^{1/2}] . \tag{2.2}$$

This is a nontrivial constraint on the off-shell amplitudes.

It is traditional and indeed <u>convenient</u> to introduce a covariant tensor (four-vector) amplitude for the photon through the expression

$$H_\lambda(q) = \epsilon_\mu(\lambda, q)\, T^\mu(q) . \tag{2.3}$$

The polarization vector $\epsilon^\mu(\lambda, q)$ is the standard one for massive particles; for $\vec{q} = 0$ it is $2^{-1/2}(0, -1, -i, 0)$, $2^{-1/2}(0, 1, -i, 0)$, and $(0, 0, 0, 1)$ for $\lambda = +1, -1,$ and 0 respectively.

The condition (2.2) for physical photons implies

$$q_\mu\, T^\mu(q) = O(q^2) . \tag{2.4}$$

This is the strongest conservation law demanded by the physical interpretation of the photon. For $q^2 \neq 0$ the tensor T^μ may be written as

$$T^\mu(q) = V^\mu(q) + \frac{q^\mu}{q^2} S(q) , \tag{2.5}$$

where the conserved (J = 1) part is

$$V^\mu = \mathcal{P}^{\mu\nu}(q)\, T_\nu , \tag{2.6}$$

with

$$\mathcal{P}^{\mu\nu}(q) = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} ,$$

and the scalar (J = 0) part is

$$S = q_\mu T^\mu .$$

In general, the two parts of (2.5) have compensating singularities at $q^2 = 0$ and the decomposition into nonsingular parts is impossible when $q^2 = 0$ (the axial current provides an example of such a phenomenon). The importance of (2.4) is that it removes this singularity in V^μ .

The scalar part S of neutral vector currents is not measured in electron scattering. However, the scalar part of charged vector currents can be measured in the weak interactions ($e^+\nu$, $\mu^+\nu$, etc. probes). Hence the conserved vector current hypothesis (CVC) has a direct empirical consequence. Further we make the usual CVC assumption that the charged vector currents of weak interactions are part of the same isospin multiplet as the isovector part of the electromagnetic current. Because of their physical interest we shall for the most part study conserved vector currents and hence denote them by V^μ .

For single current amplitudes there is no rigorous necessity for non-Regge behavior. Dashen and Frautschi[2] have shown that pure Regge behavior is a consequence of the consistency conditions, Regge behavior for the hadron bootstrap, and unsubtracted dispersion relations in q^2. In Sec. IV we shall see that this conclusion follows particularly simply in the zero-width approximation.

B. Two-Current Amplitudes and Non-Regge Behavior

Two current tensor amplitudes, $M^{\mu\nu}(q_1,q_2)$ (covariant current correlation tensors), can be constructed from doubly nonstrong leptonic amplitudes in the same way as the single-current amplitudes. It is convenient to define the combinations

$$M_{(\pm)}{}^{\mu\nu}(q_1,q_2) \equiv \frac{1}{2}\left[M_{ab}{}^{\mu\nu}(q_1,q_2) \pm M_{ba}{}^{\mu\nu}(q_1,q_2)\right], \tag{2.7}$$

where a and b are the internal quantum numbers of the currents and are usually suppressed (as on the left-hand side). We also suppress the hadron momenta p_i, $i = 1,\cdots,N$. Due to Bose statistics, $M_+{}^{\mu\nu}$ and $M_-{}^{\mu\nu}$ are respectively symmetric and antisymmetric under the interchange $(q_1,\mu) \longleftrightarrow (q_2,\nu)$. We note that for isoscalar and isovector currents only the $I = 1$ combination of two isovector currents is antisymmetric,[16] and physical photons contribute only to symmetric amplitudes.

For physical photons, the arguments in (A) can be repeated to obtain the divergence conditions

$$q_{1\mu}M^{\mu\nu}(q_1,q_2) = O(q_1{}^2), \quad M^{\mu\nu}(q_1,q_2)q_{2\nu} = O(q_2{}^2) . \tag{2.8}$$

On the other hand, since there is no physical massless charged particle corresponding to the charged currents of the weak interactions we cannot directly obtain such divergence conditions for amplitudes involving these currents. Moreover, CVC does not imply (2.8) since $M^{\mu\nu}$ is only indirectly related to current operators (if they exist). In fact, we can easily show that $q_{1\mu} M_{(-)}^{\mu\nu}$ is nonvanishing as $q_{1\mu} \to 0$. In this limit only the soft poles due to the coupling of the current to an external line can contribute and one can easily show that as $q_{1\mu} \to 0$,

$$q_{1\mu} M_{(-)}^{\mu\nu}(q_1, q_2) \to V^{\nu}(q_1 + q_2) \tag{2.9}$$

but

$$q_{1\mu} M_{(+)}^{\mu\nu}(q_1, q_2) \to 0. \tag{2.10}$$

Roughly speaking, the nonvanishing of the divergence is due to the <u>lack</u> of an internal pole in the current channel at $t \equiv (q_1 + q_2)^2 = q_2^{\ 2}$ corresponding to the soft pole coupling to the second current. In lieu of such an unphysical q^2-dependent exchange pole, current algebra has an "exchanged" current. It corresponds to Kronecker delta and fixed pole singularities in the angular momentum plane rather than an ordinary physical particle pole.

The divergence conditions (2.9) and (2.10) are far less than that assumed by current density algebra, since they are restricted to the special point $q_{1\mu} = 0$. In fact (2.9) and (2.10) are equivalent to only the charge-current density algebra. Since in (2.9) all the "overlapping variables" $q_1 \cdot p_i$ are fixed at zero, it provides no evidence for fixed power behavior. However, we can extend the nonvanishing of the divergence to arbitrary $q_1 \cdot p_i$ for $q_1^2 = 0$ and $q_2^2 = t$. Before stating and proving this important theorem let us first discuss briefly the variables in $M^{\mu\nu}$. There are $3N-6$ independent hadronic variables $p_i \cdot p_j$ (t is included among these by energy momentum conservation, $q_1^\mu + q_2^\mu + \sum_{i=1}^{N} p_i^\mu = 0$). There are $N-1$ overlapping variables $q_1 \cdot p_i$, but for fixed $p_i \cdot p_j$ only two (denoted ν_1 and ν_2) are independent. The others are linearly related to them. There are thus the correct number $(3N-4)$ of on-mass-shell variables plus q_1^2 and q_2^2. A graphic way of visualizing the variables is to go to the two-current center of mass (called the t channel). The fixed hadronic momenta provide a coordinate system; t, q_1^2, and q_2^2 determine the length of the relative three-momentum and ν_1 and ν_2 determine its polar angles.

We now state the theorem:[12] If $M_{(-)}^{\mu\nu}$ is analytic on the physical sheet of ν_1 and ν_2,[17] except for singularities due to normal threshold cuts and bound state poles in the overlapping variables, and if CVC holds for V^μ, then $M_{(-)}^{\mu\nu}$ has fixed power behavior in the overlapping variables. It then directly follows that there are fixed poles or Kronecker delta singularities (or both) in the angular momentum plane of the two-current (t) channel at $J = 1$.

We first note that CVC implies that the discontinuity of $M^{\mu\nu}$ across the normal threshold cut in any overlapping variable has vanishing divergence. To prove this one uses unitarity to express the discontinuity as a (finite) sum over intermediate states in the given overlapping variable. Each term in the sum is the product of two single-current amplitudes and thus by CVC has vanishing divergence. Hence,

$$q_{1\mu}\left[\text{Disc}_{q_1\cdot p_i}\, M^{\mu\nu}(q_1,q_2)\right] = 0\,, \quad \text{all } i. \tag{2.11}$$

We now examine the divergence $q_{1\mu}\, M_{(-)}^{\mu\nu}$ at fixed $p_i \cdot p_j$, $q_1^{\,2}, q_2^{\,2}$, and t. Equation (2.11) and our assumption imply that $q_{1\mu}\, M_{(-)}^{\mu\nu}$ has no singularities in ν_1 and ν_2 and is therefore a polynomial in these variables. From (2.9) one sees that the constant coefficient is non-vanishing at $q_1^{\,2} = 0$ and $q_2^{\,2} = t$ and therefore $q_{1\mu}\, M_{(-)}^{\mu\nu}$ has a nonvanishing fixed power (constant) behavior in the overlapping variables. The amplitude $M_{(-)}^{\mu\nu}$ must then also have fixed power behavior.

We have thus proved that the two current channel cannot have only moving poles in the J plane; there are fixed poles and (or) Kronecker delta singularities at $J = 1$. This is the heart of the problem of finding consistent two current amplitudes.

We note that we have shown that $q_{1\mu}\, M_{(-)}^{\mu\nu}$ has a nonvanishing contribution at $q_1^{\,2} = 0$, $q_2^{\,2} = t$ which is a constant in ν_1 and ν_2. We have not obtained any information about higher-order terms in ν_1 and ν_2 which may in general be present. Also we do not learn anything about contributions proportional to $q_1^{\,\nu}$, since (2.9) gives information only at $q_{1\mu} = 0$. Thus without further information we do not have a detailed knowledge of the actual fixed power behavior.

However, if local current operators exist the divergence of $M^{\mu\nu}$ is determined for all $q_1^{\,2}$ and $q_2^{\,2}$ by the current commutation relations.[18] In particular the Gell-Mann current algebra gives[16]

$$q_{1\mu} M_{(-)}^{\mu\nu}(q_1,q_2) = V^{\nu}(q_1 + q_2) , \qquad (2.12)$$

$$q_{1\mu} M_{(+)}^{\mu\nu}(q_1,q_2) = 0 , \qquad (2.13)$$

which one sees is the simplest behavior consistent with the theorem and (2.9) and (2.10).[19] Similar expressions hold for the $q_{2\nu}$ divergence. Equation (2.12) clearly is a very nontrivial relation between the one- and two-current amplitudes.

III. DUALITY

In the Veneziano model[5] for four-body amplitudes, the full amplitude is decomposed into a sum of three terms $[B(-\alpha_s, -\alpha_t), B(-\alpha_t, -\alpha_u),$ and $B(-\alpha_u, -\alpha_s)]$, one for each permutation of the external particles (excluding cyclic and anticyclic permutations). A similar decomposition is used in the N-point beta function model[6] for N-body amplitudes. For each permutation of the external particles (excluding cyclic and anti-cyclic permutations) there is a separate term given by an N-point beta function. Each such function has poles at fixed (real) values of sub-energies of adjacent lines [e.g., $B(p_1, p_2, \cdots, p_N)$ with particle ordering $1, 2, \cdots, N$ has poles in $s_{ij} = (p_i + p_{i+1} + \cdots + p_j)^2$] and also has Regge behavior in these subenergies [e.g. $B \sim (s_{ij})^{\alpha}$ as $s_{ij} \to \infty$]. These functions are hence dual in the sense that poles are generated by divergences in sums over poles in overlapping variables.[20] In this section, we consider some general consequences for current amplitudes of a concept of duality based on this decomposition.

There is an interesting correspondence between the terms in this decomposition and the sets of planar (Cutkosky or Feynman) diagrams for various fixed permutations of the external lines. The set of planar diagrams for a fixed permutation has cuts (and bound state poles) in precisely those variables for which the corresponding N-point beta function has poles. In this sense each beta function approximates an infinite set of planar diagrams by an infinite set of tree diagrams. The decomposition of the set of all planar diagrams into its subsets for the various permutations of the external lines corresponds to the decomposition of the amplitude in terms of beta functions. We suggest that an appropriate name for this type of duality is planar duality since it assumes that each term in such a decomposition of the amplitude is self-dual.[21]

At present, the most successful Reggeized zero-width models for hadrons exhibit planar duality. Besides this pragmatic justification for studying its general consequences we can give some crude arguments why it might be a good approximation. Whereas Feynman diagrams provide no reliable estimate of the relative importance of planar and nonplanar diagrams in hadron amplitudes, the multiperipheral model enables one to make such an estimate, at least in a restricted kinematic region, and so far the indications are that the planar diagrams dominate.[22] Also Mandelstam[23] has constructed Veneziano-like amplitudes corresponding to nonplanar diagrams with more than four external lines and has concluded that these have a more degenerate and hence less desirable hadron spectrum.

In the remainder of this section we shall assume planar duality for current amplitudes and investigate the consequences of the "dual decomposition" into a sum of terms, one for each permutation of the external momenta, each of which has singularities only in subenergies of adjacent lines (and is itself dual). Most of the discussion deals with the poles that contribute at $q_\mu = 0$, since they have a distinguished role in current amplitudes. We call such soft pole terms external line insertions (ELI)--see Fig. 1. Finally, we note that in the zero-width approximation where the only singularities in q^2 are poles, duality for vector meson amplitudes implies duality for single-current amplitudes.

A. Single Current Amplitudes

For simplicity we first neglect isospin symmetry and consider the amplitude $V^{\mu}(q)$ for a single photon and N hadrons. This amplitude has the dual decomposition

$$V^{\mu}(q) = \sum_{i,\{P\}} C_{i,P} V^{\mu}_{i,P}(q) , \tag{3.1}$$

where $V^{\mu}_{i,P}(q)$ corresponds to the permutation P of the hadron momenta $(p_1, p_2, \cdots, p_N)$ and the photon momenta q to the left of p_i.

As we have seen in Sec. II.A, we must have $q_{\mu} V^{\mu}(q) = 0$. Each term in the dual decomposition (3.1) has right-hand singularities in a different set of variables, and thus there is no possibility of cancellation between them in the divergence. Since duality rules out terms without singularities in the full set of variables, we have the important condition

$$q_{\mu} V^{\mu}_{i,P}(q) = 0 . \tag{3.2}$$

From now on we consider only the hadron ordering $p_1, \cdots, p_N$ and drop the subscript P.

The term V_i^{μ} has only two soft photon pole terms (ELI); i.e., those corresponding to p_{i-1} and p_i at $(q + p_{i-1})^2 = m^2_{i-1}$ and $(q + p_i)^2 = m_i^2$. Since for $q_{\mu} \to 0$ these are the only possible contributions to (3.2), the residues of these poles must be equal and opposite,

$$V_i^{\mu}(q) \approx \begin{cases} \dfrac{2p_i^{\mu} + q^{\mu}}{(q + p_i)^2 - m_i^2} A \\ -\dfrac{2p_{i-1}^{\mu} + q^{\mu}}{(q + p_{i-1})^2 - m_{i-1}^2} A , \end{cases} \tag{3.3}$$

where A is the purely hadronic amplitude for $p_1, \cdots, p_N$. From now on the V_i^{μ} will always be understood to have their ELI poles normalized as in (3.3) (e.g., unit coupling of the current to the external lines at $q^2 = 0$). With this normalization we rewrite (3.1) as

$$V^{\mu}(q) = \sum_{i=1}^{N} Q_i V_i^{\mu}(q) . \tag{3.4}$$

We note the normalization condition (3.3) applies only to the soft poles, and hence in (3.4) any contribution not containing any such poles need not be proportional to Q_i.

From (3.3), (3.4), and the definition of the charge e_i of the ith hadron we easily obtain the condition

$$e_i = Q_i - Q_{i+1} . \tag{3.5}$$

From (3.5) charge conservation,

$$\sum_{i=1}^{N} e_i = 0 , \tag{3.6}$$

trivially follows as it must because V^{μ} is divergenceless (see Sec. II.A).

A natural diagrammatic representation of the photon amplitude is immediately suggested by (3.5)--see Fig. 2. The lines indicate a flow of charge and show clearly how (3.5) and (3.6) are satisfied. The diagram also shows that in V_i^{μ}, the photon couples to Q_i. One is thus led naturally to diagrams strikingly similar to the "duality diagrams" drawn by several authors.[24,25] Finally we note that the solution of (3.5) is defined only up to the translation $Q_i \to Q_i + C$. The constant C corresponds to an additional closed loop in Fig. 2 which does not connect to the external lines. Such a translation gives an additional contribution of $CV^{\mu(\Sigma)}$ to (3.4), where

$$V^{\mu(\Sigma)}(q) = \sum_{i=1}^{N} V_i^{\mu}(q) . \qquad (3.7)$$

The V_i^{μ} in (3.7) may be entirely different functions from those in (3.4), they are constrained only by the conditions (3.3).

We now assume SU(2) symmetry and discuss arbitrary conserved vector currents. First we describe a particularly convenient way of handling the isospin indices. We may represent the isospin representation and state of each external particle, p_i, in the hadronic amplitude A as a direct product of isospin one-half spinors--"quarks or antiquarks," i.e. lower indices $\alpha_i, \alpha_i', \cdots, \alpha_i^{(k)}$ and upper indices $\beta_i, \beta_i', \cdots, \beta_i^{(\ell)}$ (one may require some symmetry and tracelessness in these indices but we may ignore this inessential complication). The amplitude A may thus be labeled

$$A^{\beta_1\beta_1' \cdots \beta_N^{(\ell)}}_{\alpha_1\alpha_1' \cdots \alpha_N^{(k)}} . \qquad (3.8)$$

The number, M, of upper and lower indices can always be made equal by using the raising and lowering matrix $C_\alpha^{\ \beta}$. We note that in this notation the requirement of isospin invariance for infinitesimal transformations yields

$$\sum_{\bar{\alpha}_x}{}' (\tau_a)_{\alpha_x}^{\ \bar{\alpha}_x} A^{\beta_1 \cdots \beta_N (\ell)}_{\alpha_1 \cdots \bar{\alpha}_x \cdots \alpha_N (k)} - \sum_{\bar{\beta}_y} A^{\beta_1 \cdots \bar{\beta}_y \cdots \beta_N (\ell)}_{\alpha_1 \cdots \alpha_N (k)} (\tau_a)_{\bar{\beta}_y}^{\ \beta_y}$$

$$= 0\,, \qquad \text{for} \quad a = 1, 2, 3, \qquad (3.9)$$

where τ_a are the usual Pauli matrices.

Since $\delta_\alpha^{\ \beta}$ is the only invariant tensor in SU(2), A can always be expanded in a sum of terms, each consisting of a product of M δ's and an isospin invariant amplitude.[24] Each term has the natural diagrammatic representation shown in Fig. 3(a). Each term in the dual decomposition of A has a similar expansion. For these it is convenient to draw the lines around the periphery of the diagram by introducing extra δ's, using the trivial identity

$$\delta_\alpha^{\ \beta} = \sum_x \delta_\alpha^{\ x} \ \delta_x^{\ \beta} \ ;$$

compare Figs. 3(a) and 3(b).

The isospin factors for current amplitudes can be treated in the same way. Consider first for simplicity an isovector current with spinor indices α and β. There are now M + 1 δ's in each internal symmetry factor and the isospin one current is obtained by using the projection operator $\sum_{\alpha,\beta} (\tau_a)^\alpha_{\ \beta}$. Note that (3.9) assures that

$$q_\mu V^{\mu}{}^{\beta\beta_1\cdots\beta_N(\ell)}_{\alpha\alpha_1\cdots\alpha_N(k)} = 0$$

for $q_\mu \to 0$ since the ELI's are given by

$$V^{\mu}_a{}^{\beta_1\cdots\beta_N(\ell)}_{\alpha_1\cdots\alpha_N(k)} \approx \frac{2\,p_i^{\mu} + q^{\mu}}{(q+p_i)^2 - m_i^2}\left\{\sum_{\bar{\alpha}_i(m)} (\tau_a)_{\alpha_i(m)}{}^{\bar{\alpha}_i(m)} A^{\beta_1\cdots\beta_N(k)}_{\alpha_1\cdots\bar{\alpha}_i(m)\cdots\alpha_N(k)} - \sum_{\bar{\beta}_i(n)} A^{\beta_1\cdots\bar{\beta}_i(n)\cdots\beta_N(k)}_{\alpha_1\cdots\alpha_N(k)} (\tau_a)_{\bar{\beta}_i(n)}{}^{\beta_i(n)}\right\}. \tag{3.10}$$

We now consider a particular isospin invariant amplitude. For definiteness suppose it corresponds to $\delta^{\bullet}_{\bullet} \cdots \delta_{\alpha}^{\beta_j}\, \delta_{\alpha_k}^{\beta} \cdots \delta^{\bullet}_{\bullet}$, where only the δ's involving the currents have been explicitly shown and α_k and β_j are any indices for p_k and p_j respectively. This isospin invariant amplitude has a dual decomposition of the form (3.1) and (3.2) holds for each term. Thus by (3.3) each term must have equal and opposite contributions from its two ELI as $q_\mu \to 0$. This condition and the requirement that the full ELI residue be given by (3.10) can easily be shown to imply that this term has the form

$$\delta^{\bullet}_{\bullet} \cdots (\tau_a)_{\alpha_k}{}^{\beta_j} \cdots \delta^{\bullet}_{\bullet}\left[\sum_{i=j}^{k-1} V_i^{\mu}(q) + C\,V^{\mu(\Sigma)}(q)\right], \tag{3.11}$$

where $V^{\mu(\Sigma)}$ is of the form (3.7) and has no ELI. In terms of diagrams, this means that in V_i^{μ} the current couples (by τ_a) to each "quark" line passing between p_{i-1} and p_i, as shown in Fig. 4. As seen in (3.11), for each quark line in the diagram there is the possibility of an arbitrary contribution from a closed loop which does not change the ELI. These arguments may be repeated for isoscalar currents with the replacement $(\tau_a)_{\alpha}^{\ \beta} \to \delta_{\alpha}^{\ \beta}$.

The above results [e.g., Eqs. (3.4) and (3.11)] apply rigorously only for the ELI at $q^2 = 0$. However, when the dual nature of the amplitudes is fully taken into account we expect similar results for the full amplitude, since the soft poles are closely related to the full amplitude (for example, they lie on Regge trajectories). In II we shall find that results like (3.4) and (3.11) do indeed hold for the full amplitude.

Exotic currents with isospin greater than one may be introduced straightforwardly by adding further indices α', β', etc. However, if we assume that there are no exotic resonances in the hadron spectrum, exotic currents are excluded. The absence of exotic resonances forbids the presence of more than one quark line between each adjacent pair of external momenta[25,27] and requires all C's to be zero. This applies equally well to amplitudes with currents, and thus only isoscalar and isovector currents can be formed. If the zero-width approximation is made this result is completely trivial, since only vector mesons with these quantum numbers exist.

The prescription[25,27] for eliminating exotic resonances is easily extended to SU(n), and we see that it leads satisfyingly to the existence of only n^2 conserved currents transforming according to the n^2-1 dimensional adjoint representation of SU(n) plus the trivial representation. These are just the currents whose charges generate precisely the symmetry SU(n), and no smaller or larger one.

B. Two Current Amplitudes

With the restriction of the hadron momenta to the permutation $p_1, \cdots, p_N$, the dual decomposition of $M^{\mu\nu}(q_1,q_2)$ is given by

$$M^{\mu\nu}(q_1,q_2) = \sum_{i\neq j} C_{ij}\, M^{\mu\nu}_{ij}(q_1,q_2) + \sum_i C_{ii}\, M^{\mu\nu}_{ii}(q_1,q_2) + \sum_i C'_{ii}\, M'^{\mu\nu}_{ii}(q_1,q_2)\,, \tag{3.12}$$

where $M^{\mu\nu}_{ij}(q_1,q_2)$ corresponds to the permutation $p_1, \cdots, p_{i-1}, q_1, p_i, \cdots, p_{j-1}, q_2, p_j, \cdots, p_N$ (or similarly for $i < j$) and the adjacent-current terms for the two different orderings have been explicitly exhibited. (M_{ii} is the term with q_1 to the left of q_2, and M'_{ii} is the term for q_1 to the right of q_2.)

There is an important new feature which has to do with the adjacent isovector currents and the ELI poles for q_1. First we note that the $M^{\mu\nu}_{ij}$ for $i \neq j$ have two ELI's, as in A . They may thus be taken to be individually divergenceless; in fact, if the divergence has only fixed poles in the two current (t) channel where they are necessary by the theorem of Sec. II.B (and as is the case in current algebra), they must be divergenceless. However, the adjacent current terms have only <u>one</u> ELI each; both $M^{\mu\nu}_{ii}(q_1, q_2)$ and $M'^{\mu\nu}_{ii}(q_1, q_2)$ are needed to supply the usual two ELI on p_{i-1} and p_i. This implies, following the method of (A) and suppressing constants C, the condition

$$q_{1\mu}\, M^{\mu\nu}_{ii;ab}\,{}^{\beta_1\cdots\beta_N}_{\alpha_1\cdots\alpha_N}{}^{(\ell)}(k)(q_1, q_2) \longrightarrow \sum_{\bar{\alpha}_x} (\tau_a)_{\alpha_x}^{\bar{\alpha}_x}\, V^{\nu}_{i;b}\,{}^{\beta_1\cdots\beta_N}_{\alpha_1\cdots\bar{\alpha}_x\cdots\alpha_N}{}^{(\ell)}(k)(q_1 + q_2)$$

$$- \sum_{\bar{\beta}_y} V^{\nu}_{i;b}\,{}^{\beta_1\cdots\bar{\beta}_y\cdots\beta_N}_{\alpha_1\cdots\alpha_N}{}^{(\ell)}(k)\,(q_1 + q_2)(\tau_a)_{\bar{\beta}_y}^{\beta_y} \tag{3.13}$$

as $q_{1\mu} \to 0$, where the sums are over all quark lines between p_{i-1} and p_i. Since $M^{\mu\nu}_{ii}$ satisfies the assumptions of the theorem of Sec. II.B, we may conclude from (3.13) that it has fixed power behavior at least for $q_1^2 = 0$ and $q_2^2 = t$ (i.e., $q_1 \cdot q_2 = 0$). This result imposes an important boundary condition on the $M^{\mu\nu}_{ii}$; it is more restrictive than the general theorem [see (2.9) and (2.10)], as it requires fixed poles in these terms for <u>both</u> symmetric and antisymmetric amplitudes. Note, however, that it does not require fixed poles in the full symmetric amplitude, since the contributions of the two adjacent diagrams cancel.

The isospin analysis proceeds very similarly to A, and we shall not give the details. We remark only that there are three free constants for each quark line, one associated with its coupling to each current individually and one associated with its coupling to both currents. For example, for physical photons we have $C_{ij} = (Q_i + C_1)(Q_j + C_2) + C$. For consistency the constants C_1 and C_2 should be those corresponding to $V^{\mu}(q_1)$ and $V^{\nu}(q_2)$ respectively.

Finally, for future reference we state the divergence conditions (3.13) for isovector currents assuming no exotic resonances (see Fig. 5). In this case, in order to satisfy Bose statistics, $M_{ii}^{\mu\nu}(q_1, q_2)$ and $M'^{\mu\nu}_{ii}(q_1, q_2)$ must be related by the interchange $(q_1, \mu) \longleftrightarrow (q_2, \nu)$,

$$M'^{\mu\nu}_{ii}(q_1, q_2) = M_{ii}^{\nu\mu}(q_2, q_1) \quad . \tag{3.14}$$

Hence with the unit coupling of the currents to the external lines at $q^2 = 0$ [see Eq. (3.3)], we obtain

$$q_{1\mu} M_{ii}^{\mu\nu}(q_1, q_2) = V_i^{\ \nu}(q_1 + q_2), \tag{3.15a}$$

$$M_{ii}^{\nu\mu}(q_2, q_1) q_{1\mu} = - V_i^{\ \nu}(q_1 + q_2) \tag{3.15b}$$

for $q_1^{\ 2} = 0$ and $q_2^{\ 2} = t$; possible additional terms on the right-hand sides which vanish as $q_{1\mu} \to 0$ have been suppressed. The $q_{2\nu}$ divergence conditions are now equivalent to (3.15a) and (3.15b) under the interchange $(q_1, \mu) \longleftrightarrow (q_2, \nu)$. We find this "signature" decomposition of the adjacent current terms very useful, particulary in the exchange-degenerate (i.e., no exotic resonances) model of II. There one needs only construct a single function $M_{ii}^{\mu\nu}(q_1, q_2)$ and extract the symmetric and anti-symmetric part to obtain both $M^{\mu\nu}_{(\pm)}$.

IV. ZERO-WIDTH MODELS

In the zero-width approximation all singularities are represented by simple poles. In other words the amplitude is approximated by a sum of tree diagrams. In this section we list and discuss the properties of (A) single-current amplitudes and (B) two-current amplitudes in Reggeized zero-width models.

A. Single-Current Amplitudes

We require the following properties:

(i) Divergence Condition:

$$q_\mu V^\mu(q) = 0, \qquad \text{i.e., CVC hypothesis.}$$

(ii) Generalized Vector-Meson Dominance (GVMD):

The only singularities in q^2 are simple poles and their residues completely determine V^μ (no subtractions in q^2 dispersion relations). The residue of the pole at $q^2 = m_{V_n}{}^2$ is a product of the vector meson (V_n) scattering amplitude and a current-vector meson coupling constant (f_{V_n}).

(iii) Regge Asymptotics:

V^μ has Regge behavior in all subenergies $s_{ij\cdots k} \equiv (p_i + p_j + \cdots + p_k)^2$.

(iv) Particle Spectrum:

The only singularities in $s_{ij\cdots k}$ are simple poles with polynomial residues in overlapping variables. Each pole is located at a fixed positive and real value of some invariant $(s_{ij\cdots k} = m^2)$.

(v) Factorization:

At any pole in $s_{ij\cdots k}$ the residue of V^μ factorizes into a current amplitude with fewer hadrons and a purely hadronic amplitude.

Not all the above properties are independent. The Regge behavior, particle spectrum, and factorization of the vector meson amplitudes and the no-subtraction assumption (ii) directly imply the properties (iii) - (v).[28] Hence the self-consistency conditions[2] can be rather easily satisfied.

The couplings f_{V_n} of the V_n to the vector current are arbitrary in the solution to the single-current problem. This is the analog of the freedom discussed by Dashen and Frautschi[2] in specifying the q^2 dependence of their self-consistent currents. In their language, if we assume an infinite family of V_n, there will be an infinite number of undetermined constants in the solution to the Omnes equation. One can hope to find constraints on the f_{V_n} only by studying amplitudes involving more than one current.

Finally, we note that in constructing a GVMD amplitude one cannot just write

$$V^\mu(q) = \sum_n \frac{m_{V_n}^{2} f_{V_n}^{-1}}{q^2 - m_{V_n}^{2}} \left(g^\mu{}_\nu - \frac{q^\mu q_\nu}{q^2} \right) A^\nu{}_{V_n} , \qquad (4.1)$$

where $A^\nu{}_{V_n}$ is the purely hadronic on-shell amplitude for V_n. If (4.1) is viewed as a dispersion relation in q^2 for a fixed independent set of the $s_{ij\cdots k}$, it is clear that for $N > 2$ some singularities in q^2 due to singularities in the dependent $s_{ij\cdots k}$ have been omitted,[29] thus causing a violation of (iv). One may, however, collect together all terms from the dispersion integral corresponding to a given V_n and regard the resulting q^2—dependent object as an off-shell continuation of $A^\nu{}_{V_n}$. Also in order to satisfy (i), the constraint

$$\sum_n f_{V_n}^{-1} q_\nu A^\nu_{V_n} = O(q^2)$$

must be imposed in order that V^μ not have a spurious singularity at $q^2 = 0$. This requires that $A^\nu_{V_n}$ depend upon q^2. As we shall see in II, the N-point beta function model provides a natural way of handling these complications.

B. Two Current Amplitudes

We require the following properties:

(i) Divergence Conditions:

(a) Charge-Current Density Algebra;

$$q_{1\mu} M^{\mu\nu}_{(+)}(q_1, q_2) \to 0 ,$$

$$q_{1\mu} M^{\mu\nu}_{(-)}(q_1, q_2) \to V^\nu(q_1 + q_2)$$

for $q_{1\mu} \to 0$.

(b) Photon Correspondence:

$$q_{1\mu} M^{\mu\nu}_{\gamma\gamma}(q_1, q_2) = O(q_1^2) ,$$

and similarly for q_2.

(ii) Generalized Vector Meson Dominance:

The only singularities in q_1^2 and q_2^2 are simple poles and the residues at $q_1^2 = m_{V_n}^2$ (or $q_2^2 = m_{V_n}^2$) are single-current amplitudes for the production of V_n.

(iii) Regge Asymptotics:

$M^{\mu\nu}$ has Regge behavior in all $s_{ij\cdots k}$ except possibly those invariants $q_1 \cdot p_i$ that overlap the two current channel.

(iv) Particle Spectrum:

The only singularities in $s_{ij\cdots k}$ are simple poles with polynomial residues in overlapping variables. Each pole is located at a fixed positive and real value of some invariant.

(v) Factorization:

The amplitude factorizes as indicated in Fig. 6;

(α) "Hadronic Factorization," at poles in $s_{ij\ldots k}$ <u>not</u> overlapping t,

(β) "Current Factorization," at poles in $s_{ij\ldots k}$ overlapping t.

Comparison of this list of properties with the list A shows that there are essentially two new features: (ia) nonvanishing divergences and (vb) current factorization. These lead to nontrivial connections between two-current and one-current amplitudes and probably give the crucial dynamical constraints on the q^2 dependence of form factors.

In Sec. II.B we have shown that the condition (ia), which is a consequence of kinematics and internal symmetries, can be extended to all $q_{1\mu}$ with $q_1^2 = 0$ and $q_2^2 = t$, to within terms that vanish as $q_{1\mu} \to 0$. We should like to give a rigorous example of our proof of this statement in the zero-width approximation. This is possible since the factorization property (vb) is a consequence of the unitarity assumed in Sec. II.B. One simply notes that at a pole in $q_1 \cdot p_i$, $M^{\mu\nu}$ factorizes into a product of single current amplitudes. The contribution of this pole to the divergence is required to be zero by CVC. In the zero-width model these poles are the only possible singularities in the $q_1 \cdot p_i$ plane, and hence there must be pure polynomial behavior in this variable. Hence (ia), (vb) and CVC rigorously imply fixed power behavior for $M^{\mu\nu}_{(-)}$. Thus (ia) holds for $q_1^{\,2} = 0$ and $q_2^{\,2} = t$ to within terms vanishing at $q_{1\mu} = 0$ (e.g., proportional to $q_1^{\,\nu}$, $q_1 \cdot p_i$, etc.).

The above discussion indicates the importance of the factorization constraints and especially current factorization (vb). This is further illustrated in II where we find that (vb) is the most difficult condition to satisfy. If the hadron bootstrap in fact uniquely determines the divergences (i.e., the current algebra), we expect the crucial constraint is (vb).

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FOOTNOTES AND REFERENCES

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2. R. F. Dashen and S. C. Frautschi, Phys. Rev. 143, 1171 (1966).

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9. This is also the case, of course, in the purely hadronic problem. See, for example, S. Mandelstam, A Relativistic Quark Model Based on the Veneziano Model. I. Meson Trajectories, Berkeley preprint, 1969; K. Bardakci and M. B. Halpern, A Possible Born Term for the Hadron Bootstrap, Berkeley preprint, 1969.

10. If such operators exist, the matrix elements of their commutators can be determined from the divergences of certain covariant amplitudes; see Sec. II.B.

11. See, for example, R. F. Dashen and M. Gell-Mann, Phys. Rev. Letters 17, 340 (1966).

12. For $N = 2$ an equivalent result has been proven by J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. 157, 1448 (1967), Sec. III.

13. F. Arbab and R. C. Brower, Phys. Rev. 178, 2470 (1969) and Phys. Rev. (to be published). In these papers the ZML is explicitly shown to imply the conservation of charge and the low energy theorems when combined with analyticity and crossing for four-body helicity amplitudes.

14. R. C. Brower, Lawrence Radiation Laboratory report UCRL-19220, June 1969. (unpublished). An explicit demonstration of these statements is given here. Also crossing, analyticity, and Regge asymptotics are studied in the ZML to show how the zero-width model for hadrons can be "naturally" extended to photon amplitudes.

15. S. Weinberg, Phys. Rev. 135, B1049 (1964) first proved that charge conservation follows from Lorentz invariance.

16. In the following, when $M^{\mu\nu}_{(-)}$ refers to such amplitudes, we shall use this symbol to refer to $M^{\mu\nu}_{(-)c}$, where

$$M^{\mu\nu}_{(-)ab} = i\,\epsilon_{abc}\, M^{\mu\nu}_{(-)c} .$$

17. For fixed $p_i \cdot p_j$ in some interval, judiciously chosen to remove anomalous thresholds from the physical sheet, if possible.

18. S. L. Adler and R. F. Dashen, Current Algebras (W. B. Benjamin, New York, 1968), Cp. 3.

19. See H. G. Doesch and D. Gordon, Nuovo Cimento 57A, 82 (1968), for a discussion of the J-plane structure implied by these divergences in the case N = 2 and arbitrary spin hadrons.

20. D. Sivers and J. Yellin, Lawrence Radiation Laboratory preprint UCRL-18784, 1969, give an interesting discussion of this property for the π-π scattering amplitude.

21. Since we consider only this type of duality here, we shall usually refer to it simply as duality. Other applications of planar duality have been discussed by G. Frye and L. Susskind, Yeshiva University preprints, 1969.

22. D. Amati, A. Stanghellini, and S. Fubini, Nuovo Cimento 26, 896 (1962). In this paper, the pion production amplitude is represented by a product of $\pi\pi$ elastic amplitudes and pion propagators. Since the $\pi\pi$ amplitudes are well approximated by planar diagrams (e.g. the Veneziano representation), the entire production amplitude is planar. Moreover, if "crossed rungs" are negligible, the unitarity sum does not generate nonplanar terms. E. L. Berger (private communication) has numerically estimated a nonplanar term in the unitarity equation resulting from crossing the $\pi^-\pi^-$ pair in the final state of $\pi^- p \to \pi^+\pi^-\pi^- p$ to be less than 10%. Although the discovery of a large nonplanar term in some amplitude can not be precluded, present evidence indicates that they are generally small.

23. S. Mandelstam, General Trajectories in the Relativistic Quark Model, Berkeley preprint, 1969.

24. S. Mandelstam, see Ref. 9. This reference makes use of the delta-function formalism which we have found particularly useful in the following.

25. H. Harari, Phys. Rev. Letters 22, 562 (1969); J. Rosner, ibid. 22, 689 (1969).

26. For SU(n), $n > 2$, this analysis is insufficient due to the existence of the antisymmetric n index ϵ tensor in addition to δ. For strange currents in SU(2) one would have to allow quark lines to terminate at the current.

27. Chan Hung-Mo and J. Paton, CERN preprint TH994, 1969.

28. We assume that the sum over V_n and the asymptotic limit can be interchanged.

29. We thank S. Mandelstam for emphasizing this point to us. As an example, for $N = 3$, if s and t are held fixed, then a singularity at $u = u_0$ gives a contribution to the dispersion integral at $q^2 = u_0 + s + t - m_1^2 - m_2^2 - m_3^2$.

FIGURE CAPTIONS

Fig. 1. An external line insertion (ELI) for the particle X.

Fig. 2. Duality diagram for $V_i^{\mu}(q)$.

Fig. 3. (a) Diagram for the hadronic isospin factor

$$\delta_{\alpha_1}^{\beta_3} \delta_{\alpha_1'}^{\beta_6} \delta_{\alpha_2}^{\beta_6'} \delta_{\alpha_3}^{\beta_4} \delta_{\alpha_6}^{\beta_3'} .$$

Each line represents a δ.

(b) Modified diagram. Each cusp represents a sum, e.g.

$$\sum_{x_1} \delta_{\alpha_2}^{x_1} \delta_{x_1}^{\beta_6'} .$$

Fig. 4. Diagram for isovector current.

Fig. 5. Duality diagrams for (a) $M_{ij}^{\mu\nu}(q_1, q_2)$ and (b) $M_{ii}^{\mu\nu}(q_1, q_2)$.

Fig. 6. (a) Hadronic Factorization; (b) Current Factorization.

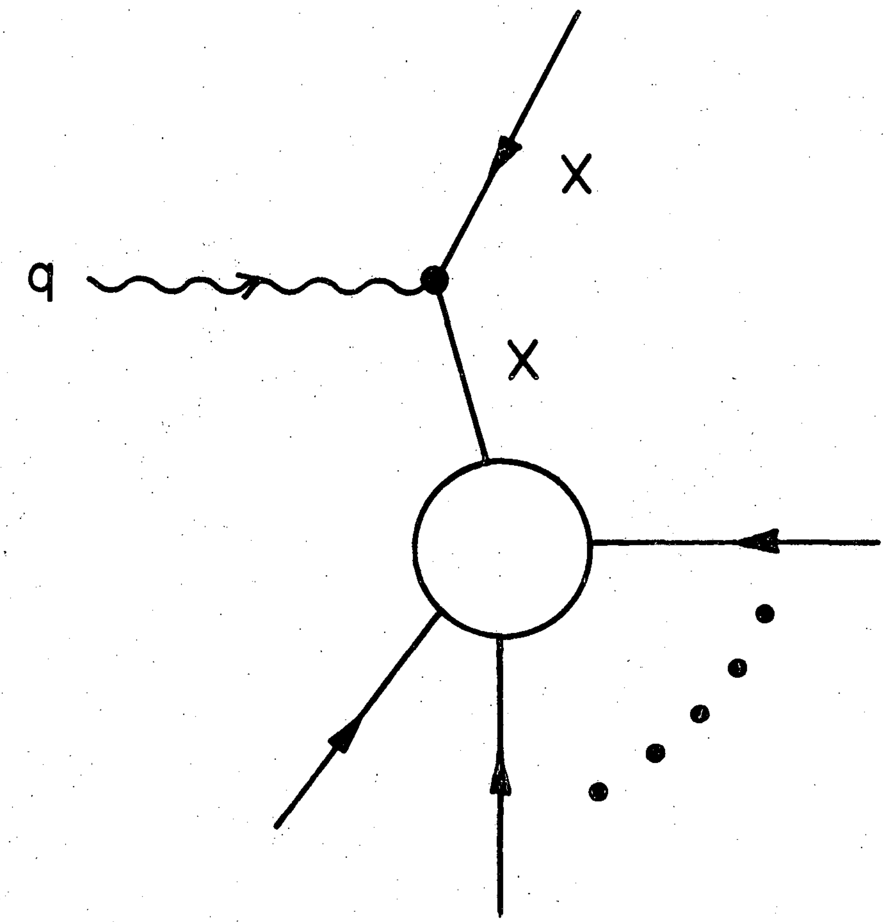

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Fig. 1

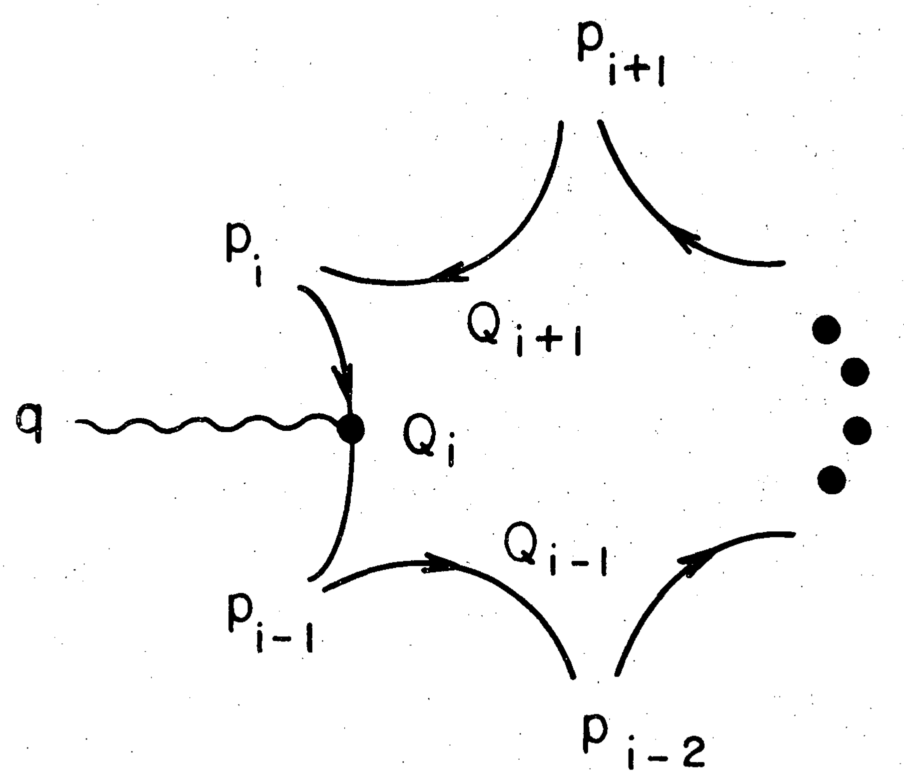

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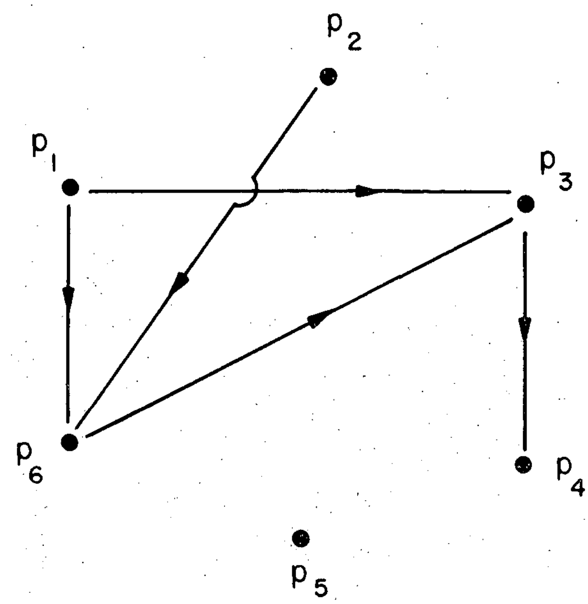
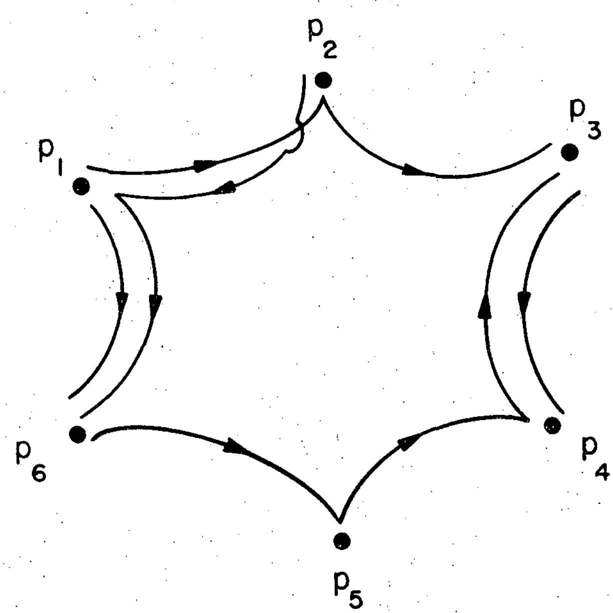
Fig. 2



(a)



(b) XBL696-3101

Fig. 3

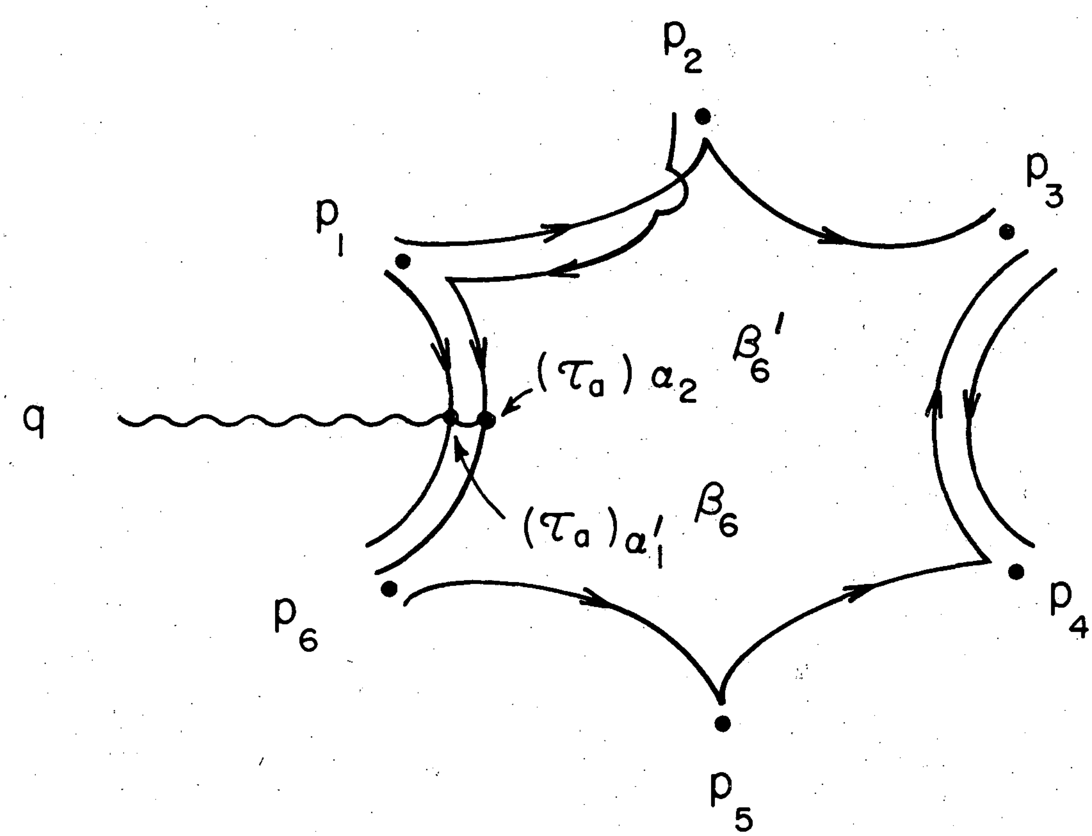


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Fig. 4

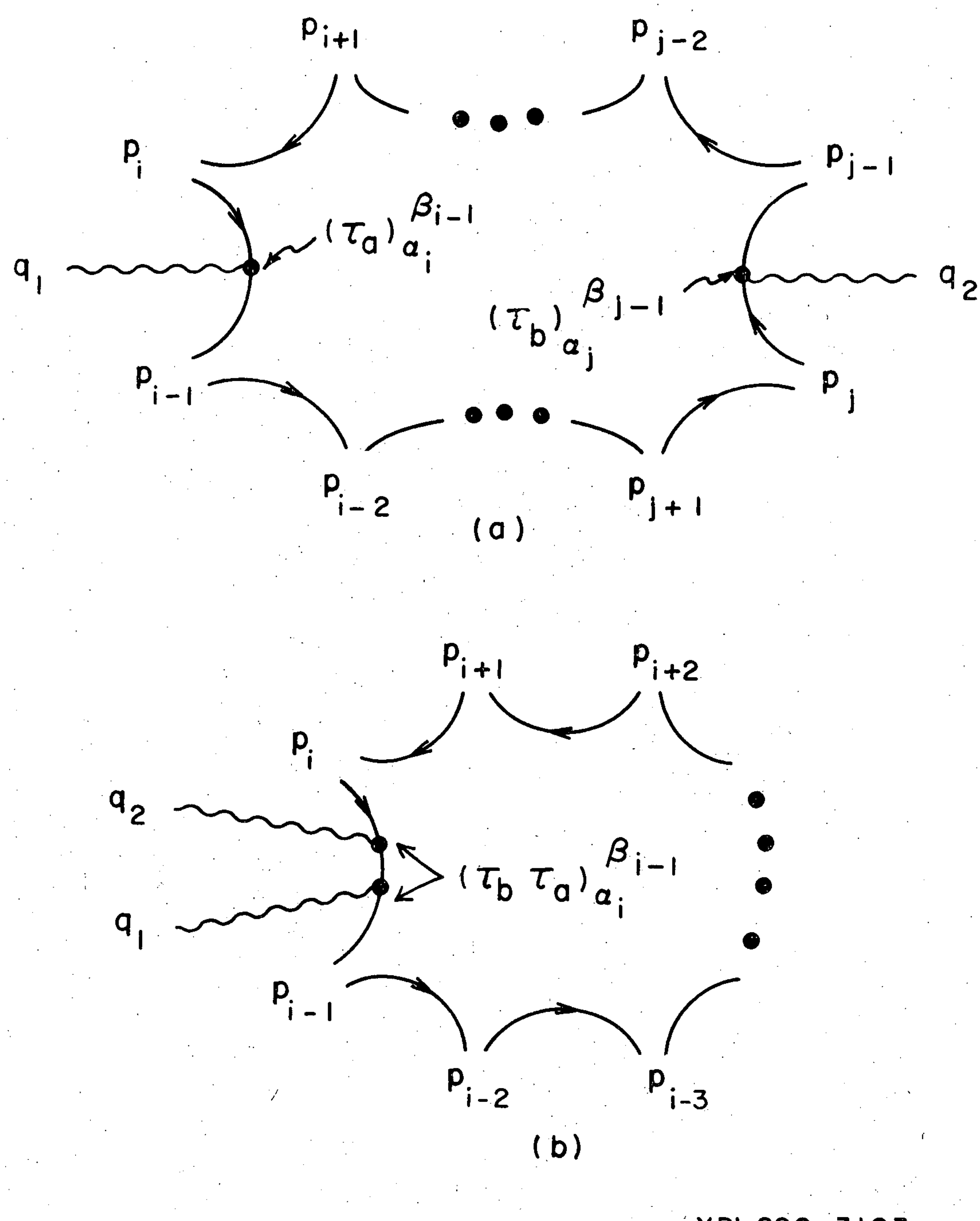

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Fig. 5

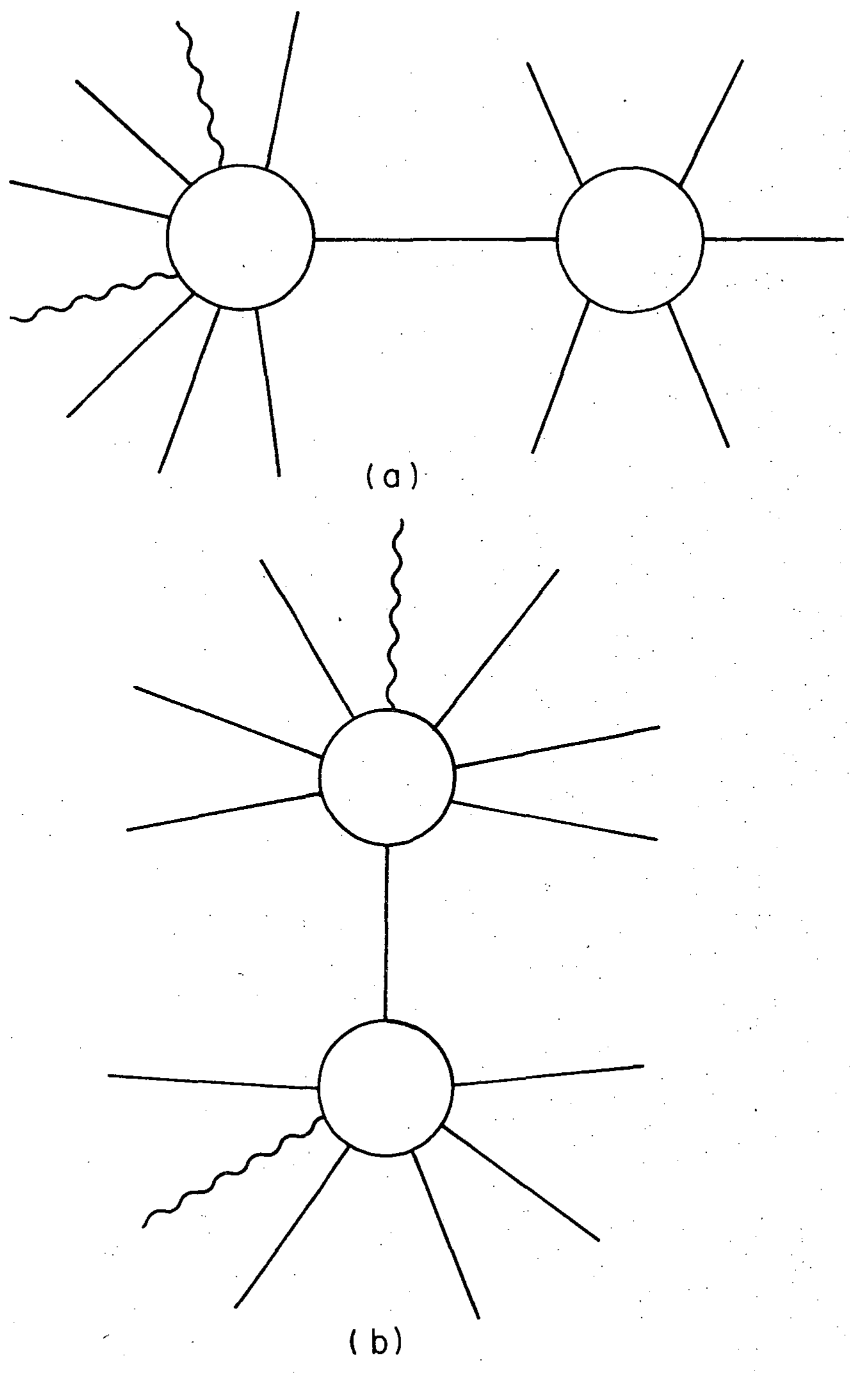

Fig. 6

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