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Incomplete Market Dynamics and Cross-Sectional Distributions

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Abstract

The size distributions of many economic variables seem to obey the *double power law*, that is, the power law holds in both the upper and the lower tails. I explain the emergence of the double power law—which has important economic, econometric, and social implications—using a tractable dynamic stochastic general equilibrium model with heterogeneous agents subject to aggregate and idiosyncratic investment risks. I establish theoretical properties such as existence, uniqueness, and constrained efficiency of equilibrium, and provide a numerical algorithm that is guaranteed to converge. The model is widely applicable: it allows for arbitrary homothetic CRRA recursive preferences, an arbitrary Markov process governing aggregate shocks, and an arbitrary number of technologies and assets with arbitrary portfolio constraints.

Keywords: applied general equilibrium analysis; emergence; incomplete markets; inequality; power law; robustness; wealth distribution.

JEL codes: D52, D53, F34, F36, G15.

1 Introduction

One of the most remarkable features of the size distributions of many economic variables is that they obey the *power law*: the fraction of units exceeding size x is proportional to $x^{-\alpha}$ when x is large, where $\alpha > 0$ is the power law exponent. The power law was discovered by Pareto (1896), who was studying the size distribution of income, and popularized by Gabaix (1999), who provided a simple explanation of Zipf's law (power law with exponent $\alpha = 1$) for cities.[1] Recently, a new phenomenon has been discovered: the *double power law*, which

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[1]See Gabaix (2009) for a review of the power law.

means that the power law holds not only in the upper tail but also in the *lower* tail: the fraction of units *below* size x is proportional to x^{β} for some exponent $\beta > 0$ when x is small. So far, the double power law has been reported in city size (Reed, 2002; Giesen et al., 2010), income (Reed, 2003; Toda, 2011, 2012), and consumption and its growth rate (Toda and Walsh, 2014a).

A question that often arises when talking about the power law is why we should care. Here I list four reasons: (i) Such an empirical regularity is interesting in its own right and should be explained. (ii) The behavior of a system with power law distributions will be strongly influenced by the behavior of the largest units.[2] (iii) If a variable obeys the power law, its exponent determines inequality.[3] However, before we do anything about inequality (with say policy), we should understand its determination. Having a positive theory of the tails should come before any normative analysis. Policy coming from the wrong model may be nonsense. (iv) Since power law variables have only finitely many moments, econometric techniques that assume the existence of moments (such as GMM) might be inapplicable.[4]

What is the origin of the double power law? By introducing birth and death in a mechanistic model with geometric Brownian motion, Reed (2001) showed that we can get the double Pareto distribution, whose tails satisfy the power law exactly. Benhabib et al. (2014) do the same with optimizing agents in a partial equilibrium model. But the real world is certainly more complicated than the i.i.d. world of Brownian motion. The question is, why is the double power law *robust*? This question was partly asked in the influential paper by Gabaix (1999). He argued that the power law (here in the upper tail) holds if individual units are hit by multiplicative shocks (Gibrat (1931)'s law of proportionate growth) and there is a small friction such as a reflecting barrier. But he formally proved the emergence of the power law under i.i.d. assumptions, leaving the robustness issue to subsequent research. In fact, he writes "[I]t does not matter if this mean rate is time varying [..., which] is a conjecture that we firmly believe to be true. [...] However, we could not find any argument in the mathematical literature" (p. 743, footnote 13).

This paper provides an answer to the robustness question in the context of general equilibrium with incomplete markets (GEI). The logic goes in two steps. First, I show that in a large class of dynamic general equilibrium models with incomplete markets where agents are hit by multiplicative aggregate and idiosyncratic shocks (AK models), the wealth of individual agents satisfies Gibrat's law. Second, I show that for a large class of stochastic processes, Gibrat's law and a constant probability of birth/death imply the double power law. Therefore to the extent that the AK world with incomplete markets and constant birth/death rate is a good assumption, we very naturally get the double power law. This explanation has virtues and vices. The vice is that the double power law emerges for a purely mechanistic reason (Gibrat + constant death) as in Reed (2001), so the underlying economics is qualitatively irrelevant. The virtue is robustness: since the double power law emerges whenever Gibrat's law

[2] For example, Gabaix (2011) shows that the idiosyncratic movements of the largest 100 firms in U.S. appear to explain a large part of the aggregate fluctuations.

[3] See Toda (2012) for the connection between the power law exponents and inequality measures such as the Gini coefficient.

[4] This point is examined by Kocherlakota (1997) and Toda and Walsh (2014b) in the context of the estimation of consumption-based capital asset pricing models.

holds and the age distribution is geometric, we do not need to fine-tune a model to get the power law. However, economics still quantitatively matters. Roughly, for the "symmetric" case I show that

$$\text{Power law exponent} = \frac{\sqrt{2 \times (\text{Probability of birth/death})}}{\text{Average idiosyncratic volatility}}. \tag{1.1}$$

Since the idiosyncratic volatility of wealth is endogenously determined in a model, it is economics that determines the *magnitude* of the power law exponent and hence the inequality across agents. My analysis shows that heavy-tailed inequality (both in the upper and lower tails) arises in a quite general economic setting, and I give a back-of-the-envelope formula for describing how unequal wealth and consumption should be, given fundamentals.

Following this two-step logic, this paper consists of two main parts. In the first part (Sections 2–4), I study the properties of a class of dynamic stochastic general equilibrium models with heterogeneous agents and incomplete markets from a theoretical perspective. This part is a contribution in itself and deserves some explanations. In the macroeconomics literature of dynamic general equilibrium with heterogeneous agents, models are usually solved numerically as in Krusell and Smith (1998). These models typically have only a few number of assets, say capital and risk-free asset. But in models with many assets, which necessarily arise in financial applications, since the portfolio enters only the value function (unlike consumption, which enters the period utility function as well), which is an unknown object, the numerical solution can become unstable. This calls for an alternative approach that I present. I prove the equilibrium existence, uniqueness, constrained efficiency, and provide a numerical algorithm that is guaranteed to converge. These models are highly tractable, allowing for semi closed-form solutions even for relatively complicated models—such as those featuring many assets with correlated and persistent payoff structure, complex trading constraints, and/or many states of the Markov chain. Therefore my contribution here is that I provide an accessible recipe for researchers interested in applied general equilibrium analysis, in addition to establishing theoretical properties.

Although tractable general equilibrium models have been applied in the literature in various contexts, they typically make special assumptions such as additive CRRA (constant relative risk aversion) or Epstein and Zin (1989) CRRA/CEIS (constant elasticity of intertemporal substitution) recursive preferences, a few technologies or assets with no portfolio constraints, i.i.d. shocks, and/or no aggregate shocks. Few papers rigorously prove properties such as equilibrium existence, uniqueness, and constrained efficiency. Since the objective of the paper is to show robustness, I present a general model with minimal assumptions that consolidates and extends many existing results within a unified framework. Again this approach has virtues and vices. The vice is that the model is abstract and that it has no particular applications in mind. The virtue is robustness and the wide range of applicability.

In the second part of the paper (Sections 5 and 6), I characterize the cross-sectional size distribution when individual units obey Gibrat's law and die with constant probability. I prove Gabaix's conjecture that the power law is robust: for a large class of stochastic processes, the cross-sectional distribution obeys the double power law. The power law exponents are governed by the average growth rate, the average idiosyncratic volatility, and the death probability. Although the focus of this paper is primarily theoretical, I also solve a general equilibrium model calibrated to the U.S. economy. The calibrated model is capable of matching many salient features of the cross-sectional consumption distribution. In addition, I verify that my approximation to the wealth distribution holds almost exactly when the model is taken to its continuous-time limit (or when calibrated to annual, quarterly, and monthly frequency). The model matches the U.S. equity premium, risk-free rate, and aggregate consumption growth, variance, and covariance with stocks with a relative risk aversion coefficient of 13, which is still high but much lower than typical representative agent models.

Obtaining a closed-form solution for the general equilibrium with heterogeneous agents and incomplete markets is of course not my invention. Papers that obtain closed-form solutions in the literature either use constant absolute risk aversion (CARA) preferences (Calvet, 2001; Angeletos and Calvet, 2005, 2006; Wang, 2007), additive CRRA preferences (Constantinides and Duffie, 1996; Saito, 1998; Krebs, 2003a,b, 2006), or CRRA/CEIS recursive preferences (Angeletos, 2007; Angeletos and Panousi, 2009, 2011; Panousi, 2010). Since I use general homothetic CRRA recursive preferences, my results complement those using CARA and subsume those using CRRA as special cases (at least in terms of preferences). All papers with (additive or recursive) CRRA preferences except Krebs (2006) in the literature assume i.i.d. shocks and do not prove properties of equilibria. My paper is technically closest to Krebs (2006), but I relax his assumptions on preferences and stochastic processes, weaken the sufficient condition for equilibrium existence, and provide a numerical algorithm.

My results on the cross-sectional distributions build on and extend those of Gabaix (1999) and Benhabib et al. (2011, 2014). Benhabib, Bisin, and Zhu derive the power law in an equilibrium model with optimizing agents, which is closest to my results. While they feature only idiosyncratic risk, my model allows for aggregate risk. This is an important distinction because we cannot study asset prices (except for risk-free assets) in models without aggregate risk. Section 5 contains further literature review.

2 Model

Time is infinite and is denoted by $t = 0, 1, \dots$. All random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$. In the economy there are a continuum of agents with mass 1 indexed by $i \in I = [0, 1]$.

2.1 Preferences

Agents have (identical) recursive preferences defined over (finite) consumption plans from time t onwards $\{c_{t+s}\}_{s=0}^{T-1}$ (where $t = 0, 1, \dots$ and $T = 1, 2, \dots$), constructed as follows. The one period utility at time t is $U_t^1 = u(c_t)$, where $u : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing. Given the T period recursive utility at time t, denoted by U_t^T, the $T+1$ period recursive utility is defined by

$$U_t^{T+1} = f(c_t, \mu_t(U_{t+1}^T)), \tag{2.1}$$

where $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is the aggregator, c_t is consumption, and $\mu_t(U_{t+1}^T)$ is the certainty equivalent of the distribution of time $t+1$ utility conditional on time

t information.

Throughout the paper I maintain the following assumptions regarding the aggregator and the certainty equivalent.

Assumption 1. *The terminal utility is consumption itself:* $u(c) = c$. *The aggregator* $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is upper semi-continuous, weakly increasing in both arguments, strictly quasi-concave, and homogeneous of degree 1,* i.e., $f(\lambda c, \lambda v) = \lambda f(c, v)$ *for any* $\lambda > 0$.

Assumption 2 (CRRA certainty equivalent)**.** *The certainty equivalent* μ_t *exhibits constant relative risk aversion (CRRA),* i.e.,

$$\mu_t(U) = \begin{cases} \mathrm{E}_t[U^{1-\gamma}]^{\frac{1}{1-\gamma}}, & (\gamma \neq 1) \\ \exp\left(\mathrm{E}_t[\log U]\right), & (\gamma = 1) \end{cases} \tag{2.2}$$

where $\gamma > 0$ *is the coefficient of relative risk aversion.*

Although it is not trivial that we should define the CRRA certainty equivalent for the case $\gamma = 1$ by using the exponential and logarithmic functions, Lemma A.1 in the Appendix shows that it is indeed a natural definition.

A typical homothetic aggregator is the Epstein and Zin (1989) constant elasticity of intertemporal substitution (CEIS) aggregator.

Example 1 (CEIS aggregator)**.** The aggregator f exhibits constant elasticity of intertemporal substitution (CEIS) if

$$f(c, v) = (c^{1-1/\varepsilon} + \beta v^{1-1/\varepsilon})^{\frac{1}{1-1/\varepsilon}},$$

where $\varepsilon > 0$ is the elasticity of intertemporal substitution and $\beta > 0$ is the discount factor.[5]

When $\varepsilon = 1/\gamma$, we get the standard additive CRRA preference.

Example 2 (Additive CRRA preference)**.** If $u(c) = c$ (one period utility is consumption itself), $\mu_t(U) = \mathrm{E}_t[U^{1-\gamma}]^{\frac{1}{1-\gamma}}$, and $f(c, v) = (c^{1-\gamma} + \beta v^{1-\gamma})^{\frac{1}{1-\gamma}}$, we obtain

$$U_t^T = \left(\mathrm{E}_t \sum_{s=0}^{T-1} \beta^s c_{t+s}^{1-\gamma}\right)^{\frac{1}{1-\gamma}},$$

which is ordinally equivalent to the additive CRRA utility $\mathrm{E}_t \sum_{s=0}^{T-1} \beta^s \frac{c_{t+s}^{1-\gamma}}{1-\gamma}$ with discount factor $\beta > 0$ and relative risk aversion $\gamma > 0$.

2.2 Technology and asset

Agent i is endowed with initial wealth (capital) $w_{i0} > 0$ in period 0 but nothing thereafter. However, each agent has access to stochastic, constant-returns-to-scale technologies (investment projects) indexed by $j \in J = \{1, \dots, J\}$, which

[5] Some authors define $f(c, v) = ((1-\beta)c^{1-1/\varepsilon} + \beta v^{1-1/\varepsilon})^{\frac{1}{1-1/\varepsilon}}$, where the discount factor is $0 < \beta < 1$. The two definitions are equivalent up to a multiplicative constant but the former is more flexible because it does not restrict the discount factor β to be less than 1 (Kocherlakota, 1990).

are subject to aggregate and idiosyncratic risks. When agent i invests one unit of good in technology j at the end of period t, he will receive $A^j_{i,t+1}$ units of good at the beginning of time $t+1$ (AK model).[6] Let $\mathbf{A}_{i,t+1} = (A^1_{i,t+1}, \dots, A^J_{i,t+1})$ be the vector of productivities of agent i. We can interpret technologies with idiosyncratic risk as human capital investment, farming in private land, private equity, etc.

There are also publicly traded assets in zero net supply (such as financial derivatives, Arrow securities, risk-free assets of various maturity) indexed by $k \in K$. The set of assets K need not be finite, but I assume that at any point in time the number of assets traded is finite. One share of asset $k \in K$ pays out dividend D^k_{t+1} at the beginning of time $t+1$ without default, independent of the identity of the asset holder. Let $\mathbf{D}_{t+1} = (D^k_{t+1})_{k\in K}$ be the collection of dividends.

The asset price P^k_t is to be determined in equilibrium and induces the asset return by $R^k_{t+1} = (P^k_{t+1} + D^k_{t+1})/P^k_t$, which is common across agents. Markets are incomplete in the sense that there is no insurance for the idiosyncratic component of investment returns, which can arise for a number of reasons but I take it as exogenous.[7]

2.3 Information and distributional assumptions

Agent i's information is represented by the filtration (increasing sequence of σ-algebras) $\{\mathcal{F}_{it}\}_{t=0}^{\infty}$. The public information is denoted by $\mathcal{F}_t = \bigcap_i \mathcal{F}_{it}$. Of course, productivity $\mathbf{A}_{it}$ is $\mathcal{F}_{it}$-measurable and dividend $\mathbf{D}_t$ is $\mathcal{F}_t$-measurable. I assume that agents are symmetric in the following sense.

Assumption 3. *Productivities $\{\mathbf{A}_{it}\}_{i\in I}$ are i.i.d. conditional on public information $\mathcal{F}_t$.*

Assumption 3 rules out individual-specific state dependence (*e.g.*, the nature of idiosyncratic risk changes depending on whether a worker is employed or unemployed). However, we can allow for such dependence in models with no aggregate risk because the distribution of wealth across agent types will converge to a fixed number.[8] Even with aggregate risk, with two types of agents the only additional state variable is the wealth share of one type (which lies in $[0,1]$).

I refer to the (common) conditional mean $A^j_t = \mathrm{E}\left[A^j_{it} \,\middle|\, \mathcal{F}_t\right]$ as the aggregate component of productivity of technology j. Letting $a^j_{it} = A^j_{it}/A^j_t$ be the purely idiosyncratic component, the productivity decomposes into the aggregate and idiosyncratic components as $A^j_{it} = a^j_{it} A^j_t$. Let $\mathbf{A}_t = (A^1_t, \dots, A^J_t)$ and $\mathbf{a}_{it} = (a^1_{it}, \dots, a^J_{it})$ be the vectors of aggregate and idiosyncratic shocks.

The second assumption concerns the individual's information and the public information.

[6] More generally, we can consider a constant-returns-to-scale (stochastic) production function $F : \mathbb{R}^J_+ \times \Omega \to \mathbb{R}_+$, but none of the subsequent analysis changes qualitatively.

[7] Angeletos and Panousi (2011) list moral hazard, adverse selection, costly state verification, inefficient legal and enforcement systems, or mere lack of sophistication as reasons for market incompleteness.

[8] For example, the model of Eisfeldt (2004) falls into the class of tractable general equilibrium models treated in this paper, even though she solves it numerically in a partial equilibrium setting (interest rate is exogenous).

Assumption 4. *The distribution of productivity and dividend* $(\mathbf{A}_{i,t+1}, \mathbf{D}_{t+1})$ *conditional on* $\mathcal{F}_{it}$ *is the same as the distribution conditional on public information* $\mathcal{F}_t$.

Assumption 4 implies that idiosyncratic shocks $\{\mathbf{a}_{i,t+1}\}_{t=0}^{\infty}$ are independent over time, which might appear unrealistic. However, note that $a_{i,t+1}^j$'s are rate of returns and hence shocks are permanent in terms of the *level* of capital. Furthermore, since I place no distributional assumptions on the aggregate shock, the stochastic process governing productivities $\{\mathbf{A}_{i,t+1}\}_{t=0}^{\infty}$ is so far arbitrary.

Assuming multiplicative investment shocks as the only idiosyncratic shocks (thus ignoring additive labor income shocks) is of course an oversimplification, but there is some empirical evidence supporting this assumption. For instance, the time series of consumption inequality (measured by variance) for each cohort increases within a cohort almost linearly between the ages 20 and 80, as shown by Deaton and Paxson (1994) using U.S. household data from 1980 to 1990. Consumption inequality within a cohort for such a long period (60 years) is hard to explain by transitory idiosyncratic shocks, while it derives naturally from multiplicative (permanent) shocks.

2.4 Budget and portfolio constraints

I denote the portfolio share (relative position) in investments and asset holdings by a vector $(\theta, \phi) \in \mathbb{R}_+^J \times \mathbb{R}^K$, where $\sum_{j=1}^J \theta^j + \sum_{k\in K} \phi^k = 1$. $\phi^k > 0$ ($\phi^k < 0$) means a long (short) position in asset k. An agent's portfolio share is constrained to be in the set $\Pi_t \subset \mathbb{R}_+^J \times \mathbb{R}^K$ at time t, which can be interpreted as a constraint on leverage or other institutional constraints (limits on shortsales, restrictions on access to certain capital markets, etc.). The assumption that only finitely many assets are traded at any point in time is mathematically represented by $(\theta, \phi) \in \Pi_t$ implies $\phi^k = 0$ for all but finitely many $k \in K$. Letting $\pi_t = (\theta_t, \phi_t)$ and

$$R_{i,t+1}(\pi_t) = \sum_{j=1}^{J} A_{i,t+1}^j \theta_t^j + \sum_{k\in K} R_{t+1}^k \phi_t^k \tag{2.3}$$

be the gross return on portfolio of investments and assets, individual i faces the budget constraint

$$w_{i,t+1} = R_{i,t+1}(\pi_t)(w_{it} - c_{it}). \tag{2.4}$$

If shortsales are allowed, it may be the case that $R_{i,t+1}(\pi_t) \leq 0$ in some states, leaving the agent with negative wealth. I rule out this possibility by letting an agent with negative wealth bankrupt and get utility $-\infty$, so agents choose only portfolios that satisfy $R_{i,t+1}(\pi_t) > 0$ almost surely. By redefining the portfolio constraint if necessary, I assume that $R_{i,t+1}(\pi) > 0$ almost surely for any $\pi \in \Pi_t$.

3 Individual decision

As usual, the equilibrium is defined by individual optimality and market clearing. Thus I study the individual decision problem before proceeding to the general equilibrium. Throughout this section I suppress the individual subscript i. Also, since the portfolio return $R_{t+1}(\pi)$ is exogenous from an agent's point of view, I impose assumptions directly on the portfolio return $R_{t+1}(\pi)$ instead of the fundamentals (technology).

Formulating and solving for optimization problems in infinite horizon models with recursive (non-additive) preferences poses significant technical challenge compared to the additive case. In Section 3.1 I explain the technical difficulties and alleviate them by reformulating the problem in Section 3.2. Readers uninterested in technical subtleties may skip to the solution in Section 3.3, where I give an explicit formula for the Epstein-Zin CRRA/CEIS preferences in a quite general Markov setting.

3.1 Difficulties in infinite horizon recursive models

The standard way of formulating the optimization problem is as follows. First, one defines the utility function over feasible infinite horizon consumption plans by either showing that the recursion (2.1) admits a unique solution U_t^∞ regardless of the boundary condition at infinity ($T \to \infty$), or simply defines $U_t^\infty = \lim_{T\to\infty} U_t^T$, whenever the limit exists. Then one solves for the optimal consumption-portfolio applying dynamic programming techniques to the value function, which in our setting is defined by

$$V_t^*(w) = \sup \left\{ U_t^\infty \;\middle|\; w_t = w, (\forall \tau \geq t) \begin{matrix} 0 \leq c_\tau \leq w_\tau, \pi_\tau \in \Pi_\tau, \\ w_{\tau+1} = R_{\tau+1}(\pi_\tau)(w_\tau - c_\tau) \end{matrix} \right\}. \tag{3.1}$$

The most general treatment of this approach in the literature seems to be that of Ozaki and Streufert (1996), who employ the concept of biconvergence introduced by Streufert (1990, 1992). Roughly speaking, biconvergence means that the infinite horizon utility is insensitive to replacing the consumption plan in the far distant future by the most optimistic and pessimistic plans. Under the assumption of biconvergence, all standard dynamic programming techniques apply (Ozaki and Streufert, 1996, Theorems A, B, C): thus it is sufficient to show biconvergence. The trouble is that the sufficient conditions for biconvergence (see their assumptions N. 1–N. 12) are hard to verify, and the utility function fails to be biconvergent even in the special case of Epstein-Zin CRRA/CEIS recursive utility in Example 1 when the elasticity of intertemporal substitution is less than 1, which is the empirically relevant case.

Another approach is the fixed point techniques developed by Boyd (1990), Rincón-Zapatero and Rodríguez-Palmero (2003), and Marinacci and Montrucchio (2010), among others. In these papers the sufficient condition for existence and uniqueness of infinite horizon utility is often easier to verify than that of Ozaki and Streufert (1996). However, they all assume that the growth rate of consumption is bounded above, which is too stringent for stochastic dynamic programming because they *a priori* rule out distributions with unbounded support such as the lognormal or gamma distributions. A quick fix to this situation is to always use multinomial distributions that approximate unbounded distributions, but then the discount factor needs to be very small since these authors use the sup norm of the random variable, which makes the theory practically inapplicable. Ozaki and Streufert (1996) stress this difficulty and impose an upper bound on 'average growth' instead of 'maximum growth'.

3.2 Alternative approach

In order to alleviate these difficulties, I reformulate the dynamic optimization problem. Define the T period value function from time t on by

$$V_t^T(w) = \sup \left\{ U_t^T \;\middle|\; w_t = w, (\forall \tau) \begin{array}{l} 0 \le c_\tau \le w_\tau, \pi_\tau \in \Pi_\tau, \\ w_{\tau+1} = R_{\tau+1}(\pi_\tau)(w_\tau - c_\tau) \ge 0 \end{array} \right\}, \tag{3.2}$$

where the recursive utility U_t^T is defined by (2.1). I *define* the infinite horizon value function by

$$V_t^\infty(w) = \lim_{T \to \infty} V_t^T(w), \tag{3.3}$$

whenever the limit exists. We can interpret the infinite horizon value function as the maximum utility of a very long but finitely lived agent. Working with the infinite horizon value function V_t^∞ is technically much easier than with V_t^* since there is no need to define the infinite horizon *utility* (only the infinite horizon *value function* plays a role) and in principle V_t^∞ can be obtained by backward induction and taking the limit.

Since the T period value function is defined by backward induction, it trivially satisfies the Bellman equation

$$V_t^T(w) = \sup_{\substack{0 \le c \le w \\ \pi \in \Pi_t}} f\left(c, \mu_t\left(V_{t+1}^{T-1}(R_{t+1}(\pi)(w-c))\right)\right) \tag{3.4}$$

if $T \ge 2$. Thus letting $T \to \infty$ and assuming that the order of the limit and the certainty equivalent operator μ_t can be interchanged, we obtain

$$V_t^\infty(w) = \sup_{\substack{0 \le c \le w \\ \pi \in \Pi_t}} f\left(c, \mu_t\left(V_{t+1}^{\infty}(R_{t+1}(\pi)(w-c))\right)\right). \tag{3.5}$$

Now we can define two optimality concepts.

Definition 3.1 (Optimality)**.** Given initial wealth w_0, the consumption-portfolio plan $\{c_t, \pi_t\}_{t=0}^\infty$ is *recursively optimal* if (c_t, π_t) solves the right-hand side of (3.5) with $w = w_t$ for all t, where w_t is the wealth at time t implied by the budget constraint (2.4). $\{c_t, \pi_t\}_{t=0}^\infty$ is *optimal (in the usual sense)* if

$$\lim_{T \to \infty} U_0^T(c_0, \dots, c_{T-1}) = V_0^*(w_0).$$

From a computational point of view, defining optimality recursively or in the usual sense makes little difference. In either case we solve for the optimal plan by value function iteration, so as long as the Bellman equation admits a unique solution and the computational algorithm converges, we get the same answer. The economic interpretation of the two concepts are quite different, however. The usual approach regards an agent as inherently infinitely lived, defines the utility function over infinite horizon consumption plans, and optimizes. On the other hand, my alternative approach regards an agent as very long but finitely lived, optimizes over finite horizons, and takes the limit. Thus recursive optimality is an *infinite horizon approximation of the behavioral rule in finite horizon.* Clearly usual optimality and recursive optimality coincide in finite horizon. The point of introducing recursive optimality is because it is technically simpler (there is no need to define the infinite horizon utility, optimization in finite dimension is always easier than in infinite dimension, taking the limit often makes the model stationary, etc.) and has a natural interpretation since nobody or no dynasty lives forever.[9]

Mathematically, usual optimality and recursive optimality are distinct concepts and neither imply the other, although they are related. For instance, since $U_t^T \le V_t^T$ for any feasible consumption plan $\{c_\tau\}_{\tau=t}^{t+T-1}$, letting $T \to \infty$ and taking the supremum over $\{c_\tau\}$, we obtain $V_t^* \le V_t^\infty$. Therefore a sufficient condition that recursive optimality implies optimality is that

$$\lim_{T\to\infty} U_0^T(c_0, \dots, c_{T-1}) \ge V_0^\infty(w_0) \tag{3.6}$$

for a recursively optimal $\{c_t\}_{t=0}^\infty$, which is a sort of 'transversality condition'. In fact, for the additive case recursive optimality and the transversality condition imply optimality, as shown by the following proposition.

Proposition 3.2. *Suppose that the agent has the additive (expected) utility function* $\mathrm{E}_t \sum_{s=0}^\infty \beta^s u_{t+s}(c_{t+s})$ *(which is assumed to be well-defined for feasible consumption plans), where* u_t *is the felicity function at time* t*. Let* $V_t^T(w)$ *be the* T *period value function and assume that* $V_t^\infty(w) = \lim_{T\to\infty} V_t^T(w)$ *exists and* $\{c_t, \pi_t\}_{t=0}^\infty$ *is recursively optimal with associated wealth* $\{w_t\}_{t=0}^\infty$*. If the transversality condition* $\limsup_{t\to\infty} \mathrm{E}_0\, \beta^t V_t^\infty(w_t) \le 0$ *holds, then* $\{c_t, \pi_t\}_{t=0}^\infty$ *is optimal in the usual sense.*

Proof. See Appendix. □

Proposition 3.2 illustrates why the usual optimization in infinite horizon is difficult: it is the transversality condition that is hard to check. In the infinite horizon general equilibrium model of Duffie et al. (1994) (as well as its extension to recursive utility by Ma (1993)), the transversality condition easily obtains due to the boundedness of the state variable (they study endowment economies in a Markov setting) and discounting. For stochastic growth models (optimal consumption-portfolio problems), the 'transversality condition' with recursive preference (3.6) is hard to verify because there is no closed-form expressions *and* both the level and the growth rate of the state variable (wealth) are potentially unbounded.

3.3 Solution

In this subsection I provide a semi closed-form solution for the (recursively) optimal consumption-portfolio rule under homotheticity assumptions. For simplicity I assume that the coefficient of relative risk aversion γ is different from 1. The case with $\gamma = 1$ is analogous and it suffices to replace all expressions $x^{1-\gamma}$ and $y^{\frac{1}{1-\gamma}}$ below by $\log x$ and $\exp(y)$, respectively.

[9] In economics infinity is often introduced not for realism but for mathematical convenience (*e.g.*, making the model stationary). If that is the case, there seems to be little harm in reformulating the behavioral rule in infinite horizon by adopting recursive optimality to make the analysis simpler.

3.3.1 General case

First I characterize the recursively optimal consumption-portfolio rule in finite and infinite horizon without any distributional assumptions.

Proposition 3.3 (Characterization in finite horizon)**.** *Suppose that assumptions 1 and 2 hold except the quasi-concavity of the aggregator f. Then the T period value function is linear in wealth, $V_t^T(w) = b_t^T w$, where $b_t^1 = 1$ and*

$$b_t^T = \sup_{0 \le \tilde{c} \le 1} f\left(\tilde{c}, (1-\tilde{c}) \sup_{\pi \in \Pi_t} \mathrm{E}_t\left[(b_{t+1}^{T-1} R_{t+1}(\pi))^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right) \tag{3.7}$$

for $T \ge 2$. If the supremum in (3.7) is attained by $(\tilde{c}_t^T, \pi_t^T)$, then the optimal consumption-portfolio rule is $c = \tilde{c}_t^T w$ and $\pi = \pi_t^T$.

Proof. See Appendix. $\square$

Remark. Since $\mathrm{E}_t\left[(b_{t+1}^{T-1} R_{t+1}(\pi))^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$ is quasi-concave in π, if the portfolio constraint Π_t is convex, then the set of optimal portfolios is convex, and is a singleton if there are no redundant assets (*i.e.*, the individual asset returns are linearly independent). If the aggregator f is quasi-concave, then the set of optimal consumption rules is convex, and is a singleton if f is strictly quasi-concave.

Since recursive optimality is defined by taking the limit of the finite period value function, it is straightforward to characterize the recursively optimal consumption-portfolio rule in infinite horizon.

Proposition 3.4. *Let everything be as in Proposition 3.3. Suppose that $b_t := \lim_{T\to\infty} b_t^T$ exists almost surely and*

$$\mathrm{E}_t\left[\sup_T \left|b_{t+1}^{T-1}\right|^{1-\gamma} \sup_{\pi \in \Pi_t} R_{t+1}(\pi)^{1-\gamma}\right] < \infty.$$

Then the infinite horizon value function is $V_t^\infty(w) = b_t w$, and $\{b_t\}$ satisfy

$$b_t = \sup_{0 \le \tilde{c} \le 1} f\left(\tilde{c}, (1-\tilde{c}) \sup_{\pi \in \Pi_t} \mathrm{E}_t\left[(b_{t+1} R_{t+1}(\pi))^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right). \tag{3.8}$$

If the supremum in (3.8) is attained by $(\tilde{c}_t, \pi_t)$, then the optimal consumption-portfolio rule is $c = \tilde{c}_t w$ and $\pi = \pi_t$.

Proof. Immediate by letting $T \to \infty$ in (3.7) and invoking the Dominated Convergence Theorem. $\square$

3.3.2 Markov case

Propositions 3.3 and 3.4 show how to compute the value function and the optimal consumption-portfolio rule, *provided that they exist.* By imposing Markov assumptions and some regularity conditions (which can be verified by looking only at the aggregator f), I prove the existence and the essential uniqueness of the optimal consumption-portfolio rule and provide a computational algorithm that is guaranteed to converge.

The Markov assumptions are the following.

Assumption 5. *The state of the economy at time t is denoted by $s_t \in S$, where $\{s_t\}$ follows an exogenous stationary Markov process. The portfolio constraint Π_t and the distribution of the portfolio return $R_{t+1}(\pi)$ conditional on time t information depend only on s_t.*

The support S of the Markov process s_t need not be finite. Let $L_+^\infty(S)$ be the space of bounded positive functions defined on S and define $B : L_+^\infty(S) \to L_+^\infty(S)$ by

$$(Bx)(s) = \max_{0\le\tilde{c}\le 1} f\left(\tilde{c}, (1-\tilde{c}) \max_{\pi\in\Pi_s} \mathrm{E}\left[(x(s')R(\pi))^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}\right), \tag{3.9}$$

where $x \in L_+^\infty(S)$. Of course, in order for B to be well-defined, the maximum with respect to $\tilde{c}$ and π in (3.9) must be attained, but this follows from the upper semi-continuity of f and imposing mild restrictions on the portfolio constraint Π_s and the portfolio return $R(\pi)$.

The following theorem provides a sufficient condition for the existence of a solution and a computational algorithm.

Theorem 3.5. *Suppose that (i) Assumptions 1, 2, 5 hold, (ii) for all $s \in S$ the portfolio constraint Π_s is nonempty, compact, convex, and (iii) for all $s \in S$ the portfolio return $R(\pi)$ is positive almost surely*[10] *and $\mathrm{E}\left[\sup_{\pi\in\Pi_s} R(\pi)^{1-\gamma} \,\middle|\, s\right] < \infty$. Let $\rho_s := \sup_{\pi\in\Pi_s} \mathrm{E}\left[R(\pi)^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}$ and suppose further that there exists $0 < \epsilon < 1$ such that either*

$$(\forall s) \qquad f(\epsilon, \rho_s) < 1 \le \sup_{0\le c\le 1} f(c, \rho_s(1-c)), \ \textit{or} \tag{3.10a}$$

$$(\forall s) \qquad \sup_{0\le c\le 1} f(c, \rho_s(1-c)) \le 1 < \sup_{0\le c\le 1} f(c/\epsilon, \rho_s(1-c)) \tag{3.10b}$$

holds. Define $\left\{b^T\right\}_{T=1}^\infty \subset L_+^\infty(S)$ by $b^1 = 1$ and $b^T = Bb^{T-1}$ for $T \ge 2$. Then $\left\{b^T\right\}_{T=1}^\infty$ is well-defined, monotonically converges pointwise to some $b \in L_+^\infty(S)$, and the infinite horizon value function in state s is $V^\infty(w, s) = b(s)w$. Letting

$$\pi_s \in \arg\max_{\pi\in\Pi_s} \frac{1}{1-\gamma} \mathrm{E}\left[(b(s')R(\pi))^{1-\gamma} \,\middle|\, s\right], \tag{3.11a}$$

$$\tilde{c}_s = \arg\max_{0\le\tilde{c}\le 1} f\left(\tilde{c}, (1-\tilde{c}) \mathrm{E}\left[(b(s')R(\pi_s))^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}\right), \tag{3.11b}$$

the consumption-portfolio rule $(\tilde{c}_s w, \pi_s)$ is recursively optimal.

Proof. See Appendix. □

Since f is monotone, if (3.10) holds for some $0 < \epsilon < 1$, it also holds for any $0 < \epsilon' < \epsilon$. Hence letting $\epsilon' \to 0$ we obtain

$$(\forall s) \qquad f(0+, \rho_s) < 1 \le \sup_{0\le c\le 1} f(c, \rho_s(1-c)), \ \text{or} \tag{3.12a}$$

$$(\forall s) \qquad \sup_{0\le c\le 1} f(c, \rho_s(1-c)) \le 1 < f(\infty, \rho_s), \tag{3.12b}$$

[10] This assumption is not stringent since $R(\pi)$ is the *gross* return, so $R(\pi) \ge 0$ always.

which are easier than (3.10) to verify.[11] The following proposition shows that under mild conditions the converse is true. In particular, if S is a finite set (which is always true in numerical applications), it suffices to check (3.12).

Proposition 3.6. *Suppose that S is compact, ρ_s is continuous in s, and $f(c,v)$ is weakly increasing and continuous. Then* (3.12) *implies* (3.10).

Proof. See Appendix. □

Condition (3.10) (or (3.12)) cannot be weakened easily. In fact, in the i.i.d. case (S consists of a single point) (3.12) is also *necessary*. To see this, note that (3.12) is equivalent to $f(0+,\rho) < 1 < f(\infty,\rho)$, where I have suppressed s because it takes only a single value. If $f(0+,\rho) > 1$, then by the definition of the map B in (3.9) we obtain

$$\begin{aligned} Bx &= \max_{0\le c\le 1} f\left(c,(1-c)\max_{\pi\in\Pi}\mathrm{E}\left[(xR(\pi))^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right) \\ &\ge f(0+,x\rho) = xf(0+,\rho) > x \end{aligned}$$

whenever $x>0$, so B has no fixed point. If $f(\infty,\rho)\le 1$, then

$$\begin{aligned} Bx &= \max_{0\le c\le 1} f\left(c,(1-c)\max_{\pi\in\Pi}\mathrm{E}\left[(xR(\pi))^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right) \\ &= x\max_{0\le c\le 1} f(c/x,(1-c)\rho) < xf(\infty,\rho) \le x \end{aligned}$$

whenever $x>0$, so B has no fixed point. The only remaining case is $f(0,\rho)=1$ and $\arg\max_{0\le c\le 1} f(c,(1-c)\rho) = 0$. Then the agent postpones consumption indefinitely. (This is a case in which it is hard to justify recursive optimality.)

3.3.3 CRRA/CEIS preference

In most applications researchers use either the additive CRRA preference or Epstein-Zin CRRA/CEIS preference. Let us solve the optimal consumption-portfolio problem explicitly for these cases. Let $f(c,v) = (c^{1-1/\varepsilon}+\beta v^{1-1/\varepsilon})^{\frac{1}{1-1/\varepsilon}}$, where $\varepsilon>0$ is the elasticity of intertemporal substitution. The additive CRRA preference is a special case by setting $\varepsilon = 1/\gamma$.

Corollary 3.7. *Let $\rho_s = \sup_{\pi\in\Pi_s}\mathrm{E}\left[R(\pi)^{1-\gamma}\,\middle|\, s\right]^{\frac{1}{1-\gamma}}$ and suppose that*

$$0 < \beta\rho_s^{1-1/\varepsilon} < 1 \tag{3.13}$$

for all $s\in S$.[12] *Then the conclusion of Theorem 3.5 holds. Letting $\pi_s, \tilde{c}_s, b(s)$ be the optimal portfolio, optimal consumption rate, and coefficient of the value function in state s, the map B in* (3.9) *simplifies to*

$$(Bx)(s) = \left(1+\beta^{\varepsilon}\left(\max_{\pi\in\Pi_s}\mathrm{E}\left[(x(s')R(\pi))^{1-\gamma}\,\middle|\, s\right]^{\frac{1}{1-\gamma}}\right)^{\varepsilon-1}\right)^{\frac{1}{\varepsilon-1}} \tag{3.14}$$

[11] Here, $f(0+,\rho_s) = \lim_{c\to 0} f(c,\rho_s)$ and $f(\infty,\rho_s) = \lim_{c\to\infty} f(c,\rho_s)$.

[12] This condition is similar to the condition in Theorem 3.1 on (Epstein and Zin, 1989, p. 946), but more general. For instance, for $\varepsilon>1$ they assume that the growth rate is bounded above by $b>0$ and impose $\beta b^{1-1/\varepsilon}<1$, but in Theorem 3.5 the growth rate (asset returns) need not be bounded. In fact if $R(\pi)\le b$ almost surely for all portfolio π, then $\beta\rho_s^{1-1/\varepsilon} \le \beta b^{1-1/\varepsilon}$, so condition 3.13 is weaker.

and the optimal consumption rule (3.11b) *simplifies to* $\tilde{c}_s = b(s)^{1-\varepsilon}$.

Proof. See Appendix. □

Corollary 3.7 is very useful for those interested in applications with the CRRA/CEIS recursive utility.[13] (3.14) essentially prescribes the value function iteration, but since one iterates over the *coefficient* of the (linear) value function (which is finite-dimensional if the state space S is finite), the algorithm is much quicker and more stable to converge than iterating over the value function itself (which is an infinite-dimensional object). Furthermore, Theorem 3.5 *guarantees* convergence to the solution by starting at $b(s) = 1$.

Further specializing the preferences to the additive CRRA case, I can show that the recursively optimal rule is also optimal in the usual sense.

Proposition 3.8. *Let everything be as in Corollary 3.7 with* $\varepsilon = 1/\gamma$ *(additive CRRA preference). If* $\gamma < 1$*, strengthen* (3.13) *to* $0 < \beta\bar{\rho}^{1-\gamma} < 1$*, where* $\bar{\rho} = \sup_s \rho_s$*. Then the recursively optimal consumption-portfolio rule is also optimal in the usual sense.*

Proof. See Appendix. □

3.3.4 Comparison to the literature

Before concluding this section let us compare my results to those in the literature. That the optimal portfolio rule is independent of wealth and the optimal consumption rule is linear in wealth with additive CRRA preferences have been known for a long time (Samuelson, 1969; Hakansson, 1970, 1971), and it is not surprising that these properties carry over with homothetic CRRA recursive preferences. The novelty of Proposition 3.3 and Theorem 3.5, apart from their generality (arbitrary homothetic aggregator, arbitrary number of assets and portfolio constraints, Markovian returns, etc.), is the simplicity of the recursive formula (3.7) and (3.8). Since the consumption and portfolio decisions can be separated as in (3.11) (and the optimal consumption rule can often be solved by hand as in Corollary 3.7), my results are easy to implement.

Technically my results are much more general as well. Samuelson (1969) and Hakansson (1970) solve the infinite horizon optimization problem with additive CRRA utility and i.i.d. returns, and Angeletos (2007) solves with recursive CRRA/CEIS utility, i.i.d. returns, and two asseets. Note that they only derive necessary conditions for optimality since they do not discuss the transversality condition,[14] whereas I allow for recursive utility and Markovian

[13] Strictly speaking, Corollary 3.7 does not cover the empirically important isoelastic case ($\varepsilon = 1$). This case requires a separate treatment outlined below. First we define the aggregator by $f(c, v) = \exp((1-\beta)\log c + \beta \log v)$, where $0 < \beta < 1$ is the discount factor. (The coefficient $1-\beta$ on $\log c$ ensures that $f(c, v)$ is homothetic.) Working out the algebra, we can show that the optimal consumption rule is $\tilde{c}_s = 1 - \beta$ regardless of the state s and the mapping B in (3.14) becomes

$$(Bx)(s) = (1-\beta)^{1-\beta}\left(\beta \max_{\pi \in \Pi_s} \mathrm{E}\left[(x(s')R(\pi))^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}\right)^{\beta}.$$

Taking the logarithm, we can easily verify Blackwell's conditions and therefore the map $\log x \mapsto \log Bx$ is a contraction mapping. Thus $\left\{b^T\right\}_{T=1}^{\infty} \subset L_+^{\infty}(S)$ converges to the unique fixed point.

[14] Levhari and Srinivasan (1969) discuss the transversality condition, but their setting concerns a single asset with i.i.d. returns.

returns and prove the transversality condition for the additive case (Proposition 3.8). Hakansson (1971) solves the *finite* horizon optimization problem with additive CRRA utility, no distributional assumptions, time- and state-dependent preferences, probability of death, and bequest motive. While I abstract from state-dependent preferences, death, and bequest, they are straightforward to accommodate in my setting: it suffices to redefine the terminal utility and the aggregator. For time- and state-dependent preferences, for example, simply change the aggregator $f(c, v)$ to $f_t(c, v, s_t)$. Alvarez and Stokey (1998) prove the validity of dynamic programming for return functions that are homogeneous of degree $\theta < 1$ (which corresponds to $1-\gamma$ in my case), but they consider only the deterministic problem with additive utility and their proof depends on the value of θ. Although they have an arbitrary number of control variables, Theorem 3.5 can be easily extended to that case by reinterpreting that the dimension of c in the aggregator $f(c, v)$ is more than 1 but still assuming homogeneity $f(\lambda c, \lambda v) = \lambda f(c, v)$. Thus my results easily subsume those in the literature.

4 General equilibrium

Having solved the single agent problem, I proceed to the general equilibrium. First I define the equilibrium concept and show its essential uniqueness (asset prices and portfolio may be indeterminate, but the allocation of consumption is unique) and constrained efficiency without any distributional assumptions. Then under Markov assumptions I prove the existence of equilibrium and obtain a computational algorithm that is guaranteed to converge to the (essentially unique) equilibrium.

4.1 Definition of equilibrium

Let everything be as in Section 2. As usual the general equilibrium is defined by individual optimization and market clearing. The only nonstandard part in the definition is that I use recursive optimality (Definition 3.1) instead of optimality in the usual sense.

Definition 4.1 (Sequential GEI). $\left\{(c_{it}, w_{it}, \theta_{it}, \phi_{it})_{i\in I}, (P_t^k)_{k\in K}\right\}_{t=0}^{\infty}$ is a sequential general equilibrium with incomplete markets if

1. given the asset returns $R_{t+1}^k = (P_{t+1}^k + D_{t+1}^k)/P_t^k$ individual consumption c_{it} and portfolio $\pi_{it} = (\theta_{it}, \phi_{it})$ are optimal subject to the budget constraint (2.4) and the portfolio constraint $\pi_{it} \in \Pi_t$,
2. the markets for assets in zero net supply clear, *i.e.*, $\int_I \phi_{it}^k (w_{it} - c_{it}) \mathrm{d}i = 0$ for all $k \in K$, and
3. individual wealth w_{it} evolves according to the budget constraint (2.4).

4.2 Properties of equilibrium

In this subsection I prove the irrelevance of zero net supply assets for risk sharing and the essential uniqueness and constrained efficiency of equilibrium.

Proposition 4.2 (Irrelevance of zero net supply assets)**.** *Suppose that Assumptions 1–4 hold. If* $E = \left\{(c_{it}, w_{it}, \theta_{it}, \phi_{it})_{i\in I}, (P_t^k)_{k\in K}\right\}_{t=0}^{\infty}$ *is a sequential general equilibrium, there exists an equilibrium with the same consumption, wealth, and asset prices* $\bar{E} = \left\{(c_{it}, w_{it}, \bar{\theta}_t, \bar{\phi}_t)_{i\in I}, (P_t^k)_{k\in K}\right\}_{t=0}^{\infty}$ *with a common portfolio choice* $\bar{\pi}_t = (\bar{\theta}_t, \bar{\phi}_t)$ *across agents and no trade in assets:* $\bar{\phi}_t = 0$*.*

Proof. See Appendix. □

By Proposition 4.2, if there exists an equilibrium, there exists an equivalent symmetric (in the sense of common portfolio choice) equilibrium with no trade in zero net supply assets. Therefore the equilibrium allocation is the same as in autarky, which is not very interesting. The contraposition of Proposition 4.2 is that in order for risk sharing to take place, at least one asset must be in nonzero net supply.

Proposition 4.2 allows me to prove the essential uniqueness of equilibrium.

Proposition 4.3 (Essential uniqueness)**.** *Maintain Assumptions 1–4. If*

$$\begin{aligned} E &= \left\{(c_{it}, w_{it}, \theta_{it}, \phi_{it})_{i\in I}, (P_t^k)_{k\in K}\right\}_{t=0}^{\infty}, \\ \tilde{E} &= \left\{(\tilde{c}_{it}, \tilde{w}_{it}, \tilde{\theta}_{it}, \tilde{\phi}_{it})_{i\in I}, (\tilde{P}_t^k)_{k\in K}\right\}_{t=0}^{\infty} \end{aligned}$$

are sequential general equilibria, then $c_{it} = \tilde{c}_{it}$ *and* $w_{it} = \tilde{w}_{it}$ *almost surely.*

Proof. See Appendix. □

Proposition 4.3 does not say that the equilibrium is unique. If there are redundant assets the optimal portfolio is indeterminate. If the portfolio constraint is binding for some assets, the marginal utility condition (such as in Proposition 4.5 below) is an inequality, so there can be many asset prices that support the equilibrium. What Proposition 4.3 does say is that the consumption allocation is the same across all equilibria.

Since the sequential equilibrium is equivalent to autarky and there is no risk sharing across agents, it is clearly inefficient. Hence the next question is whether the equilibrium is constrained efficient or not. Diamond (1967) defined constrained efficiency by the impossibility of a Pareto improvement by intervening in the asset holdings alone. Here I define constrained efficiency by the impossibility of a Pareto improvement by shrinking the portfolio constraint Π_t. The consumption decision is left to the agents and asset prices may change, but individual budget constraints must be kept. This is because if we allow transfers of wealth from lucky agents (with high investment return) to unlucky agents, we can make everybody better off ex ante.

Proposition 4.4 (Constrained efficiency)**.** *Let* $\tilde{E}, E$ *be equilibria with portfolio constraints* $\tilde{\Pi}_t \subset \Pi_t$ *and corresponding infinite horizon value functions* $\tilde{V}_t^{\infty}, V_t^{\infty}$*. Then* $\tilde{V}_t^{\infty} \le V_t^{\infty}$*.*

Proof. See Appendix. □

In general equilibrium with incomplete markets (GEI) models, it is well-known that the competitive equilibrium is generically constrained inefficient (Geanakoplos and Polemarchakis, 1986). This is not the case in my model, but the efficiency property is not robust. In fact, Toda (2014) shows that the equilibrium is generically constrained inefficient if we consider general production functions instead of linear technologies but that the constrained efficient outcome can be achieved by linear taxes and subsidies.

Finally I derive an asset pricing formula.

Proposition 4.5 (Asset pricing)**.** *Suppose that the portfolio constraint* $(\theta, \phi) \in \Pi_t$ *is not binding at the symmetric equilibrium portfolio* $(\theta_t, 0)$ *for asset* k*. Then the asset price* P_t^k *satisfies the recursive formula*

$$P_t^k = \frac{\mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta_t, 0)^{-\gamma}(P_{t+1}^k + D_{t+1}^k)]}{\mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta_t, 0)^{1-\gamma}]}. \tag{4.1}$$

In particular, the one period gross risk-free rate at time t *is given by*

$$R_t^f = \frac{\mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta_t, 0)^{1-\gamma}]}{\mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta_t, 0)^{-\gamma}]}. \tag{4.2}$$

Proof. See Appendix. □

4.3 Existence of Markov equilibrium

So far I have studied the properties of equilibrium, assuming its existence. In this subsection I prove the existence under the following Markov assumption.

Assumption 6. *The state of the economy at time* t *is denoted by* $s_t \in S$*, where* $\{s_t\}$ *follows an exogenous stationary Markov process. The portfolio constraint* Π_t *and the distribution of productivity* $\mathbf{A}_{i,t+1}$ *conditional on time* t *public information depend only on* s_t*.*

Theorem 4.6. *Suppose that Assumptions 1–4 hold. Let everything else be as in Theorem 3.5 except that Assumption 5 is replaced by 6,* Π_s *is replaced by* $\Theta_s = \{\theta \mid (\theta, 0) \in \Pi_s\}$*,* $\pi \in \Pi_s$ *is replaced by* $\theta \in \Theta_s$*, and* $R(\pi)$ *is replaced by* $R(\theta, 0)$*. Then there exists an essentially unique equilibrium, where the recursively optimal consumption-portfolio rule is given by* (3.11)*.*

Proof. By Propositions 4.2 and 4.3, if an equilibrium exists it is essentially unique and has the optimal consumption-portfolio rule derived in Theorem 3.5. Therefore it suffices to show that the optimal consumption-portfolio rule given by (3.11) together with asset prices in Proposition 4.5 constitute an equilibrium. By Theorem 3.5, the first-order conditions for the optimal portfolio problem (3.11a) holds for θ. By the construction of asset prices in Proposition 4.5, the first-order conditions hold for ϕ as well. Since the objective function in the optimal portfolio problem (3.11a) is concave in π, the first-order conditions are sufficient for the maximum. Therefore $(\tilde{c}_s, \pi_s)$ is individually optimal, where $\pi_s = (\theta_s, 0)$. Market clearing is trivial because there is no trade in assets. □

The construction of a symmetric (no trade) equilibrium in the proof of Theorem 4.6 is standard in the literature (Constantinides and Duffie, 1996; Krebs, 2006; Krueger and Lustig, 2010). Since the intertemporal marginal rates of substitution are independent of wealth, the marginal valuation of income in the next period conditional on the current aggregate state is the same for all agents. Thus, one can use the concavity of the utility function to find supporting prices (as in Proposition 4.5) so that in equilibrium the agents consume the same as in autarky and there is no trade in financial assets. But the essential uniqueness of equilibrium does not follow from this argument because the dividend of assets need not be Markovian.[15] The essential uniqueness derives from the no trade result in Proposition 4.2, where I impose no distributional assumptions. To the best of my knowledge, Theorem 4.6 is the only one in the literature that proves the uniqueness of equilibrium with a continuum of heterogeneous agents.

The implications of Theorem 4.6 are rather strong. By Propositions 4.2 and 4.3, the equilibrium is essentially unique, with no trade in zero net supply assets. The unique consumption rule and value function can be computed using the algorithm in Theorem 3.5, which is guaranteed to converge.

Several authors have studied tractable general equilibrium models with heterogeneous agents (Saito, 1998; Krebs, 2003a,b; Angeletos, 2007). Since these papers are oriented toward specific applications, they typically make strong assumptions (additive CRRA or recursive CRRA/CEIS preferences, i.i.d. returns, small number of assets, no portfolio constraints, etc.) and abstract from theoretical subtleties such as the transversality condition. Only Krebs (2006) develops the model with Markovian shocks, assets in zero net supply, and discusses the transversality condition.[16] Although my model is similar to Krebs (2006), there are a few technical differences. For instance, (i) in his model the aggregate state and productivities jointly follow a finite state Markov chain, whereas in my model the distribution of productivities need not be finitely supported, (ii) his model has only two inputs (physical and human capital) with no portfolio constraints, whereas I allow for an arbitrary number of technologies with arbitrary portfolio constraints, and (iii) his sufficient condition for existence (see his equation (9)) is stronger than mine (3.13). These differences may appear minor, but I believe at least the second point is crucial. If one wants to apply this type of tractable general equilibrium models to finance, there are typically many assets with complex portfolio constraints (which is the case with securitization (Toda, 2013) and international finance (Walsh, 2014)), so having a recipe readily available as in Theorem 4.6 seems helpful to the modeler.

5 Cross-sectional distributions

What do the cross-sectional wealth and consumption distributions look like in equilibrium? Since by Theorem 3.5 consumption is proportional to wealth, we only need to look at the wealth distribution. Before presenting the main result, I introduce some definitions and a limit theorem.

5.1 Definitions and a limit theorem

A nonnegative random variable X obeys the *power law* (in the upper tail) with exponent $\alpha > 0$ if

$$\lim_{x \to \infty} x^\alpha P(X > x) > 0$$

[15] I am grateful to an anonymous referee for pointing this out and spotting an error in an earlier proof.

[16] Krebs's proof of the transversality condition seems to be incorrect, however. In deriving his equation (A.9), he takes the maximum of the left-hand side with respect to S, but fails to take the maximum of the right-hand side at the same time which also depends on S.

exists (Pareto, 1896; Mandelbrot, 1960). It obeys the *power law in the lower tail* with exponent $\beta > 0$ if

$$\lim_{x \to 0} x^{-\beta} P(X < x) > 0$$

exists.[17] Toda (2012) coined the word the *double power law*, which means that the power law holds in both the upper and the lower tails. It is easy to show that if X obeys the double power law with exponents (α, β), then the ηth moment $\mathrm{E}[X^\eta]$ exists if and only if $-\beta < \eta < \alpha$. Since many econometric techniques rely on the existence of some moments, recognizing a power law is important (Kocherlakota, 1997; Toda and Walsh, 2014a,b).

The most common distribution that obeys the double power law is the *double Pareto* distribution (Reed, 2001, 2003), which has the density

$$f_{\mathrm{dP}}(y) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta} \frac{1}{M} \left(\frac{y}{M}\right)^{-\alpha-1}, & (y \geq M) \\ \frac{\alpha\beta}{\alpha+\beta} \frac{1}{M} \left(\frac{y}{M}\right)^{\beta-1}, & (0 \leq y < M) \end{cases} \tag{5.1}$$

where $M > 0$ is the mode and $\alpha, \beta > 0$ are shape parameters (power law exponents). The classical Pareto distribution (with minimum size M) is a special case of the double Pareto distribution by letting $\beta \to \infty$ in (5.1).

If Y is a double Pareto random variable, $X = \log Y$ is said to be *Laplace*.[18] Changing variables in (5.1) such that $x = \log y$ and $m = \log M$, the density of the Laplace distribution is given by

$$f_{\mathrm{L}}(x) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta} \mathrm{e}^{-\alpha|x-m|}, & (x \geq m) \\ \frac{\alpha\beta}{\alpha+\beta} \mathrm{e}^{-\beta|x-m|}, & (x < m) \end{cases} \tag{5.2}$$

where m is the mode and $\alpha, \beta > 0$ are scale parameters. X is said to be *asymmetric Laplace* if $\alpha \neq \beta$. Using (5.2), the characteristic function of X is

$$\begin{aligned} \phi_X(t) &= \int_{-\infty}^{m} \mathrm{e}^{itx} \frac{\alpha\beta}{\alpha+\beta} \mathrm{e}^{-\beta|x-m|} \mathrm{d}x + \int_{m}^{\infty} \mathrm{e}^{itx} \frac{\alpha\beta}{\alpha+\beta} \mathrm{e}^{-\alpha|x-m|} \mathrm{d}x \\ &= \frac{\mathrm{e}^{imt}}{1 - i(\frac{1}{\alpha} - \frac{1}{\beta})t + \frac{t^2}{\alpha\beta}}, \end{aligned} \tag{5.3}$$

from which we obtain the mean $m + \frac{1}{\alpha} - \frac{1}{\beta}$ and the variance $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$. It is often convenient to parameterize the Laplace distribution in terms of its characteristic function. Letting $a = \frac{1}{\alpha} - \frac{1}{\beta}$ be an asymmetry parameter and $\sigma = \sqrt{\frac{2}{\alpha\beta}}$ be a scale parameter in (5.3), we write $X \sim \mathcal{AL}(m, a, \sigma)$ if

$$\phi_X(t) = \frac{\mathrm{e}^{imt}}{1 - iat + \frac{\sigma^2 t^2}{2}}. \tag{5.4}$$

The mean, mode, and variance of $\mathcal{AL}(m, a, \sigma)$ is $m + a$, m, and $a^2 + \sigma^2$, respectively. Comparing (5.3) and (5.4), we obtain $1/\alpha - 1/\beta = a$ and $\alpha\beta = 2/\sigma^2$, so $-\alpha$ and β are the solutions to the quadratic equation

$$\frac{\sigma^2}{2}\zeta^2 - a\zeta - 1 = 0. \tag{5.5}$$

[17] With a slight abuse of notation, I use the letter β for both the discount factor and the power law exponent of the lower tail, but the meaning should be clear from the context.

[18] Hence the Laplace and the double Pareto distributions have the same relation as the normal and the lognormal distributions.

Kotz et al. (2001) is a comprehensive review of the Laplace distribution.

Perhaps the most important property of the Laplace distribution is that it is the only limit distribution of geometric sums. Theorem 5.1 below shows that it is a robust property that the limit of a geometric sum is a Laplace distribution.

Theorem 5.1. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of zero mean random variables such that the central limit theorem holds, $N^{-1/2}\sum_{n=1}^{N} X_n \xrightarrow{\mathrm{d}} \mathcal{N}(0,\sigma^2)$; $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $N^{-1}\sum_{n=1}^{N} a_n \to a$; and ν_p be a geometric random variable with mean $1/p$ independent from $\{X_n\}_{n=1}^{\infty}$. Then as $p \to 0$ we have*

$$p^{\frac{1}{2}} \sum_{n=1}^{\nu_p} (X_n + p^{\frac{1}{2}} a_n) \xrightarrow{\mathrm{d}} \mathcal{AL}(0, a, \sigma). \tag{5.6}$$

Proof. See Appendix. □

Theorem 5.1 has been known when $\{X_n\}_{n=1}^{\infty}$ is i.i.d. (Kotz et al., 2001, p. 152), but the generalization to the non i.i.d. case is nontrivial and appears to be new.

5.2 Characterization of cross-sectional distributions

5.2.1 Equation of motion

Combining the budget constraint (2.4) and the optimal consumption rule in Theorem 3.5, individual wealth evolves according to

$$S_{i,t+\Delta t} = G_{i,t+\Delta t} S_{it}, \tag{5.7}$$

where $S_{it} = w_{it}$ is individual wealth (the "size" of an individual unit), $\Delta t = 1$ is the time step, and

$$G_{i,t+\Delta t} = (1 - \tilde{c}_{s_t})\, R_{i,t+1}(\theta_{s_t}, 0)$$

is the (gross) growth rate of unit i. Since the growth rate $G_{i,t+\Delta t}$ is independent of unit size S_{it}, (5.7) represents the celebrated Gibrat's law of proportionate growth. Since consumption is proportional to wealth, (5.7) also holds for $S_{it} = c_{it}$ (by changing the definition of $G_{i,t+\Delta t}$). Therefore in order to study the cross-sectional distributions of wealth and consumption, it suffices to study the cross-sectional size distribution when units grow multiplicatively according to (5.7). Iterating (5.7), the log size of unit i is given by

$$\log S_{it} = \log S_{i0} + \sum_{n=1}^{N_{it}} \log G_{i,t+\Delta t - n\Delta t}, \tag{5.8}$$

where S_{i0} is the initial size and N_{it} is the number of time periods unit i has been alive up to time t.

If units are infinitely lived, then $N_{it} = t$ in (5.8). By conditioning on the history of the aggregate shock and applying the Lindeberg-Feller central limit theorem, letting $t \to \infty$ the cross-sectional size distribution (relative to initial wealth) becomes approximately lognormal, but does not admit a stationary distribution since the cross-sectional variance increases indefinitely.

5.2.2 Distributional assumptions

The easiest way to obtain a stationary distribution is to let units die with constant probability $0 < p < 1$ each period and replace a dead unit by a newborn,[19] which is used in Constantinides and Duffie (1996) and Angeletos (2007). Of course, assuming a constant probability of death is clearly inappropriate for human beings: in reality mortality of humans is higher before age 1 and after age 60, and nobody lives beyond age 120. Since I am concerned with the wealth of financially active people, however, it is more appropriate to interpret "death" as the arrival of major life events such as personal bankruptcy, retirement, divorce, death, etc., in which case the constant probability of "death" seems a reasonable first approximation. According to this interpretation, our agents do not literally live a finite life but are infinitely lived dynastic households that are subject to an exogenous risk of life events.

Furthermore, for simplicity assume that there are no life insurance or annuity markets and the initial size of newborn units is common, S_0. Although these assumptions may be plausible for human capital, which is not transferable across agents and disappear upon death, they may not be appropriate for physical capital. We may easily enrich the model by introducing life insurances, bequest, estate tax, and redistributions as in Benhabib et al. (2014), but the basic structure of the model (homotheticity and individual units obeying multiplicative processes that are i.i.d. conditional on aggregate shock) is preserved. None of the subsequent discussion qualitatively change by considering a richer model.

Let $\mathcal{F}_{it} = \sigma(\{G_{is}\}_{s \le t})$ be the σ-algebra generated by the individual stochastic process $\{G_{it}\}$, and $\mathcal{F}_t = \bigcap_i \mathcal{F}_{it}$ be the history of aggregate shocks. Assuming that $\{G_{it}\}$ is i.i.d. across units conditional on $\mathcal{F}_t$, $\log G_{it}$ can be decomposed into the aggregate component (conditional mean, $\mathrm{E}_t[\log G_{it}]$) and the purely idiosyncratic component ($\log G_{it} - \mathrm{E}_t[\log G_{it}]$). As will be shown below, the double Pareto distribution emerges in the continuous-time limit $\Delta t \to 0$. Therefore let

$$\log G_{it} = X_{at}\Delta t + X_{it}\sqrt{\Delta t} \tag{5.9}$$

be the decomposition of $\log G_{it}$ into the aggregate and idiosyncratic shocks, where $X_{at} = \mathrm{E}_t[\log G_{i,t+\Delta t}]/\Delta t$ is the aggregate shock per unit of time and $X_{it}\sqrt{\Delta t}$ is the idiosyncratic component, which by assumption has zero mean and is i.i.d. across i conditional on $\mathcal{F}_t$. The reason why X_{it} is scaled by $\sqrt{\Delta t}$ is because (as in Brownian motion) we want the conditional variance per unit of time (volatility)

$$\frac{1}{\Delta t}\mathrm{Var}[\log G_{it}|\mathcal{F}_t] = \frac{1}{\Delta t}\mathrm{E}_t[X_{it}^2 \Delta t] = \mathrm{E}_t[X_{it}^2]$$

to be finite in the limit $\Delta t \to 0$. One can think of X_{at} as the "drift". Let us further assume that the idiosyncratic component $\{X_{it}\}$ is independent over time and that the central limit theorem holds.

[19] The general equilibrium model of Section 4 requires only minor changes if we introduce death. For example, the certainty equivalent (2.2) should be redefined by $\mu_t(U) = \mathrm{E}_t[(1 - p)U^{1-\gamma}]^{\frac{1}{1-\gamma}}$ to account for mortality risk, which is equivalent to using the aggregator $g(c, v) = f(c, (1-p)^{\frac{1}{1-\gamma}} v)$ instead of f. Since g satisfies Assumption 1 if f does, the discussion in Sections 4 is virtually unchanged.

5.2.3 Emergence of double Pareto distribution

In the continuous-time limit, the death probability is $p = \delta\Delta t$, where $\delta > 0$ is the Poisson rate of death. The following theorem shows that for large t the cross-sectional size distribution is approximately double Pareto with mode being the initial size.

Theorem 5.2. *Let everything be as above. Let Ω^* be the set of realizations $\omega \in \Omega$ such that the "drift" and "volatility" are Riemann integrable and their time averages have limits:*

$$\mu_S(\omega) := \lim_{t\to\infty} \frac{1}{t}\int_0^t X_{as}(\omega)\mathrm{d}s,$$
$$\sigma_S^2(\omega) := \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{E}[X_{is}^2|\mathcal{F}_s](\omega)\mathrm{d}s.$$

Then for $\omega \in \Omega^*$ the cross-sectional distribution of $\{S_{it}\}_{i\in I}$ converges in distribution to the double Pareto distribution with mode S_0 and power law exponents α, β as $t \to \infty$ and $\Delta t \to 0$, where $-\alpha$ and β are solutions to the quadratic equation

$$\frac{\sigma_S^2(\omega)}{2}\zeta^2 - \mu_S(\omega)\zeta - \delta = 0. \tag{5.10}$$

Proof. Letting $p = \delta\Delta t$, the number of time periods unit i has been alive is $N_{it} = \min\{\nu_p, t/\Delta t\}$, where ν_p is distributed as a geometric random variable with mean $1/p$. By (5.8) and (5.9), the log size of unit i at time t is given by

$$\log S_{it} = \log S_0 + p^{\frac{1}{2}} \sum_{n=1}^{N_{it}} \left(X_{i,t+\Delta t-n\Delta t}/\sqrt{\delta} + p^{\frac{1}{2}} X_{a,t+\Delta t-n\Delta t}/\delta \right). \tag{5.11}$$

Identify $X_{i,t+\Delta t-n\Delta t}/\sqrt{\delta}$ and $X_{a,t+\Delta t-n\Delta t}/\delta$ as X_n and a_n in Theorem 5.1. Since by assumption $s \mapsto X_{as}$ is Riemann integrable, letting $N = t/\Delta t$ and $\Delta t \to 0$, we have

$$\frac{1}{N}\sum_{n=1}^{N} X_{a,t-n\Delta t} = \frac{\Delta t}{t}\sum_{n=1}^{N} X_{a,t-n\Delta t} \to \frac{1}{t}\int_0^t X_{as}(\omega)\mathrm{d}s.$$

Letting $t \to \infty$ in (5.11), the asymmetry parameter in Theorem 5.1 is

$$a = a(\omega) = \lim_{t\to\infty} \frac{1}{\delta t}\int_0^t X_{as}(\omega)\mathrm{d}s = \frac{\mu_S(\omega)}{\delta}. \tag{5.12}$$

Since by assumption the central limit theorem holds for X_{it}, conditioning on $\mathcal{F}_t$, letting $\Delta t \to 0$ and $t \to \infty$ in (5.11), by a similar argument the scale parameter $\sigma = \sigma(\omega)$ satisfies

$$\sigma(\omega)^2 = \lim_{t\to\infty} \frac{1}{\delta t}\int_0^t \mathrm{E}[X_{is}^2|\mathcal{F}_s](\omega)\mathrm{d}s = \frac{\sigma_S(\omega)^2}{\delta}. \tag{5.13}$$

Hence by Theorem 5.1, we obtain

$$\log S_{it} \xrightarrow{\mathrm{d}} \mathcal{AL}(m, a(\omega), \sigma(\omega)).$$

Substituting (5.12) and (5.13) into (5.5), the cross-sectional size distribution is approximately double Pareto with mode S_0 and power law exponents α, β, where $-\alpha$ and β are solutions to (5.10). $\square$

By (5.12) and (5.13), the asymmetry parameter a and the scale parameter σ can be interpreted as the time average of the "drift" and the "volatility" multiplied by the average age of units $1/\delta$. Of course, these quantities are history dependent, *i.e.*, they depend on the realization of the aggregate shocks $\{X_{at}\}$ and $\left\{\mathrm{E}_t[X_{it}^2]\right\}$. However, if these processes are stationary, by the ergodic theorem $a(\omega)$ and $\sigma(\omega)^2$ will be constant almost surely. This argument might explain why empirically the power law exponents α, β have been historically very stable.[20]

Why is it necessary to take the simultaneous limit $t \to \infty$ and $\Delta t \to 0$ in Theorem 5.2? The necessity of $t \to \infty$ is intuitive: unless we let the system run for a very long time, the system will not settle down to the stationary distribution. It is clearest to see this in continuous-time with no aggregate shock as in the following proposition.

Proposition 5.3. *Suppose that the units obey the Brownian motion* $\mathrm{d}\log S_{it} = \mu \mathrm{d}t + \sigma \mathrm{d}B_{it}$ *in log size, die at Poisson rate* $\delta > 0$*, and are reborn at size 1 (*$\log S_0 = 0$*). Then the density function of the cross-sectional log size distribution at time* t *is*

$$f(x,t) = \frac{1}{\sigma\sqrt{2\pi t}} \mathrm{e}^{-\frac{(x-\mu t)^2}{2\sigma^2 t} - \delta t} + \frac{\delta \mathrm{e}^{\frac{\mu x}{\sigma^2}}}{\kappa\sigma} \left[\mathrm{e}^{-\frac{\kappa|x|}{\sigma}} \Phi\left(-\frac{|x|}{\sigma\sqrt{t}} + \kappa\sqrt{t}\right) - \mathrm{e}^{\frac{\kappa|x|}{\sigma}} \Phi\left(-\frac{|x|}{\sigma\sqrt{t}} - \kappa\sqrt{t}\right) \right], \quad (5.14)$$

where $\kappa = \sqrt{2\delta + \frac{\mu^2}{\sigma^2}}$ *and* $\Phi(\cdot)$ *is the cumulative distribution function of the standard normal distribution.*

Proof. See Appendix. □

Since the asymptotic expansion of $\Phi(x)$ is $\Phi(x) \sim -\frac{2\sqrt{2}\mathrm{e}^{-x^2/2}}{\sqrt{\pi}x}$ as $x \to -\infty$, all terms in (5.14) are dampened by the factor $\mathrm{e}^{-\frac{x^2}{2\sigma^2 t}}$ as $|x| \to \infty$. Hence for finite t the tails of the density (5.14) are always thinner than exponential, which implies that the size distribution will not obey the (double) power law. On the other hand, letting $t \to \infty$ in (5.14), the density converges to

$$f(x,\infty) = \frac{\delta}{\kappa\sigma} \mathrm{e}^{-\frac{\kappa|x|}{\sigma} + \frac{\mu x}{\sigma^2}},$$

which is the Laplace density (5.2) with $\alpha, \beta = \frac{\kappa\sigma \mp \mu}{\sigma^2}$.[21] Therefore the cross-sectional size distribution converges to the double Pareto distribution with exponents α, β as $t \to \infty$.

To understand the necessity of $\Delta t \to 0$, go back to Theorem 5.1 and consider the special case that $\{X_n\}$ is i.i.d. with zero mean, variance σ^2, characteristic function $\phi_X(s) = \mathrm{E}[\mathrm{e}^{isX}]$, and $a_n = 0$ for all n. Conditioning on ν_p, the characteristic function of $Y_p := p^{\frac{1}{2}} \sum_{n=1}^{\nu_p} X_n$ is

$$\phi_{Y_p}(s) = \sum_{n=1}^{\infty} p(1-p)^{n-1} [\phi(p^{\frac{1}{2}} s)]^n = \frac{p\phi_X(p^{\frac{1}{2}} s)}{1-(1-p)\phi_X(p^{\frac{1}{2}} s)}, \quad (5.15)$$

[20] According to sources cited by Gabaix (2009), the exponent of the upper tail α is typically in the range $[1.5, 2.5]$ for income and wealth and very close to 1 for cities (Zipf's law).

[21] We can easily see that $-\alpha, \beta$ are solutions to the quadratic equation $\frac{\sigma^2}{2}\zeta^2 - \mu\zeta - \delta = 0$, in agreement with (5.10).

which is not necessarily the characteristic function of the Laplace distribution (5.5).[22] But since $\phi_X(s) = 1 - \frac{\sigma^2}{2}s^2 + o(s^2)$ as $s \to 0$, it follows that

$$(5.15) = \frac{p(1 - p\sigma^2 s^2/2 + o(p))}{1 - (1-p)(1 - p\sigma^2 s^2/2 + o(p))} = \frac{1}{1 + \frac{\sigma^2 s^2}{2}} + o(p),$$

which converges to the characteristic function of $\mathcal{AL}(0, 0, \sigma)$ as $p \to 0$.

Intuitively, the reason why we have to take the continuous-time limit $\Delta t \to 0$ in Theorem 5.2 is because with finite time step Δt the expected number of idiosyncratic shocks (the average number of time steps an agent lives) is $1/\delta\Delta t < \infty$, so just as the central limit theorem holds (the Gaussian distribution arises regardless of the underlying stochastic process) in the limit that infinitely many terms are added, we cannot expect the cross-sectional size distribution to be independent of those of idiosyncratic shocks, even if the stationary distribution may be characterized as in (5.15).

5.2.4 Robustness of double power law

So far I have assumed that agents are ex ante identical, *i.e.*, they have the same recursive preference and initial wealth. However, the double power law holds under weaker assumptions.

First, instead of assuming a common initial size $S_{i0} = S_0$, assume that S_{i0} is random (i.i.d. across agents). Since the double power law implies that the cross-sectional log size distribution has exponential tails (which follows from the same argument as the relation between the double Pareto distribution (5.1) and the Laplace distribution (5.2)), the double power law still holds as long as the distribution of the initial log size has tails thinner than exponential.

Second, instead of assuming a constant initial size S_0, suppose that initial log size of an agent born at time t is the cross-sectional average of log size at time t. (Think about inheriting financial and human capital wealth when born.) Then (5.11) becomes

$$\log S_{it} = \log S_0 + \Delta t \sum_{n=1}^{t/\Delta t} X_{a,t+\Delta t - n\Delta t} + p^{\frac{1}{2}} \sum_{n=1}^{N_{it}} X_{i,t+\Delta t - n\Delta t}/\sqrt{\delta}. \tag{5.16}$$

We can still apply Theorem 5.1 to the third term in (5.16) and see that the log size distribution is approximately (symmetric) Laplace. Applying (5.10) with $\mu_S(\omega) = 0$, the power law exponent is

$$\alpha = \beta = \frac{\sqrt{2\delta}}{\sigma_S(\omega)}, \tag{5.17}$$

which is the back-of-the-envelope formula (1.1). The first and second terms of (5.16) simply determine the common mode.

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[22] Y_p is (symmetric) Laplace if and only if X is. To see this, note that

$$\frac{p\phi_X(p^{\frac{1}{2}}s)}{1 - (1-p)\phi_X(p^{\frac{1}{2}}s)} = \frac{1}{1 + \frac{\sigma^2 s^2}{2}} \iff \phi_X(t) = \frac{1}{1 + \frac{\sigma^2 t^2}{2}}$$

by setting $t = p^{\frac{1}{2}}s$.

Finally, suppose that there are finite types of agents denoted by $h \in H = \{1, \dots, H\}$ with heterogeneous aggregator, risk aversion, and death probability.[23] By Theorem 5.2, the double power law holds for each agent type, with corresponding power law exponents $(\alpha_h, \beta_h)_{h \in H}$. Letting $\alpha = \min_h \alpha_h$ and $\beta = \min_h \beta_h$, the double power law with exponents α, β holds in the entire economy because the tail of the entire population is governed by the fattest tail among all subpopulations.

5.3 Comparison to the literature

To the best of my knowledge, Reed (2001) was the first to recognize that combining multiplicative growth and geometric age distribution (via constant birth/death probability or population growth) yields the double Pareto distribution. However, his stochastic process is very special, namely the geometric Brownian motion with no aggregate shock. As shown in Theorems 5.1 and 5.2, the double Pareto property is robust in the sense that it depends only on multiplicative growth and the geometric age distribution and not on the details of the stochastic process governing growth. Gabaix (1999) conjectures that the power law should hold even if the multiplicative process is time-varying, but notes in footnote 13 that no mathematical results are known. Since in my model the stochastic processes are almost arbitrary, even Theorems 5.1 and 5.2 alone are significant achievement.

The models of Reed (2001) is mechanistic in the sense that there is no optimizing behavior by agents. Benhabib et al. (2014) derive the double Pareto distribution in an economy with optimizing agents subject to idiosyncratic investment risk, which is similar to my results. Although they have features absent in my model (bequest, fiscal policies, etc.), their setup is very special—additive CRRA preferences, geometric Brownian motion, two technologies (one technology is risk-free with an exogenous interest rate, so it is a partial equilibrium model), no portfolio constraints, etc. Thus it is not clear whether their result is robust, but Theorem 5.2 shows the double Pareto property is indeed robust. Most importantly, since their model has no aggregate risk, unlike my model it cannot be applied to asset pricing.

A few more papers characterize the wealth distribution in an economy with optimizing agents.[24] Wang (2007) solves for the optimal consumption-portfolio rule of an agent with CARA utility subject to a fairly general income process, characterizes the general equilibrium in a Bewley (1986) economy (only riskless lending and borrowing is allowed), and analytically relates the moments of the wealth distribution to those of the income distribution. Since his model features labor income shock but abstracts from investment and does not refer to the power law, my results are highly complementary.

Panousi (2010) develops a model similar to Angeletos (2007) in continuous-time with transitions between occupations (entrepreneurs and workers) and

[23] In this case, since different types typically choose different portfolios, the wealth distribution across types becomes a state variable to describe the equilibrium. Thus solving for the equilibrium is more challenging.

[24] There are also substantial works in firm size dynamics, for example Luttmer (2007) and Rossi-Hansberg and Wright (2007). These papers focus on the upper tail and derive Zipf's law (power law in the upper tail with exponent 1), but since they impose a minimum size, the stationary distribution does not obey the power law in the lower tail.

studies the general equilibrium effect of capital taxation. She derives a second-order ordinary differential equation that the stationary distribution satisfies, but neither explicitly solve it nor refer to the power law.

Benhabib et al. (2011) study an overlapping generations model with inheritance. Since between generations the wealth is subject to idiosyncratic investment risk and estate tax, and each generation receives lump-sum transfers (*e.g.*, present value of labor income), the wealth of dynasties obeys the so-called Kesten (1973) process, $X_{t+1} = M_{t+1}X_t + Q_{t+1}$.[25] Using the general properties of the Kesten process, they show that the wealth distribution obeys the power law in the upper tail and characterize the tail exponent α (but not the entire distribution). To get an idea of what the stationary distribution looks like in these models, consider the continuous-time analog of the Kesten process

$$\mathrm{d}X_t = gX_t\mathrm{d}t + vX_t\mathrm{d}B_t + q\mathrm{d}t, \tag{5.18}$$

where B_t is Brownian motion, g is expected growth rate, v is volatility, and $q > 0$ is the drift. The Fokker-Planck equation (forward Kolmogorov equation) of the diffusion (5.18) that the density f satisfies is

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}[(gx+q)f] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[v^2x^2f].$$

Setting $\partial f/\partial t = 0$ and solving the ordinary differential equation,[26] we can show that the stationary distribution is inverse gamma (the distribution of the reciprocal of a gamma variable) with density

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{-\alpha-1}\mathrm{e}^{-\frac{\beta}{x}},$$

where $\alpha = 1 - \frac{2g}{v^2}$ and $\beta = \frac{2q}{v^2}$. Although the inverse gamma distribution obeys the power law in the upper tail with exponent α (and Zipf's law when $g \ll v^2$), the lower tail is rapidly decreasing due to the factor $\mathrm{e}^{-\frac{\beta}{x}}$. Hence my results are again complementary.

In summary, only Benhabib et al. (2014) obtains the double Pareto distribution with optimizing agents, and none of the papers that analytically characterize the cross-sectional distributions in the literature show the robustness of the double power law or allow for aggregate shocks. Note that the introduction of aggregate shocks is indispensable for applications in asset pricing or business cycles but poses a significant challenge for studying cross-sectional distributions because the stochastic process governing idiosyncratic shocks often becomes nonstationary (time-dependent) conditional on the history of aggregate shocks, even if aggregate and idiosyncratic shocks are jointly stationary.

Finally I briefly discuss the empirical literature. I focus on the *double power law* since there is a vast literature documenting the power law in the *upper tail* of cross-sectional distributions. The double power law has been found in city size (Reed, 2002; Giesen et al., 2010), income (Reed, 2003; Toda, 2011, 2012), and consumption and its growth rate (Toda and Walsh, 2014a). The power law

[25] Nirei and Souma (2007) also apply the Kesten process, but their model is mechanistic (no agent optimization).

[26] The ODE can be solved by separation of variables and imposing the boundary condition $f(\infty) = 0$ and the normalization $\int_0^\infty f(x)\mathrm{d}x = 1$.

exponents vary across variables (Table 1), but are generally stable over time. Some stylized facts are $1 \approx \alpha < \beta$ for city size (Zipf's law), $2 \approx \alpha > \beta$ for income, and $\alpha \approx \beta \approx 4$ for consumption and consumption growth.

Table 1. Power law exponents in empirical studies.

Source	Variable	α	β
Reed (2002)	city size	1–2	1–5
Giesen et al. (2010)	city size	1–1.5	2–7
Toda (2012)	income	2–2.5	1–1.5
Toda and Walsh (2014a)	consumption	3–5	3–5

6 Quantitative analysis

In this section I analytically solve a continuous-time version of the general equilibrium model in Section 4 calibrated to the U.S. economy, and compare it to the numerical solutions to the discrete-time versions with various time steps.

6.1 Model

There are two technologies indexed by $j = 1, 2$. Capital invested in technology 1 (stock market) evolves according to the geometric Brownian motion

$$\mathrm{d}K_t/K_t = \mu_1 \mathrm{d}t + \sigma_1 \mathrm{d}B_{1t}.$$

Capital invested in technology 2 (private equity, human capital, etc.) is subject to aggregate and idiosyncratic risks and evolves according to

$$\mathrm{d}K_t/K_t = \mu_2 \mathrm{d}t + \sigma_2 \mathrm{d}B_{2t} + \sigma_i \mathrm{d}B_{it}.$$

Here B_{1t}, B_{2t} are standard Brownian motions satisfying $\mathrm{d}B_1 \mathrm{d}B_2 = \rho \mathrm{d}t$, where $-1 \le \rho \le 1$ is the correlation coefficient, and B_{it} is a standard Brownian motion for agent i, which is i.i.d. across agents and independent from aggregate shocks B_1, B_2. Each agent maximizes the continuous-time version of the CRRA/CEIS recursive utility (see Duffie and Epstein (1992) for details) with discount rate β (I assume the death rate δ is already included), relative risk aversion (RRA) coefficient γ, and elasticity of intertemporal substitution (EIS) ε subject to the budget constraint

$$\begin{aligned} \mathrm{d}w_t = (1-\theta_1-\theta_2) r w_t \mathrm{d}t + \theta_1(\mu_1 \mathrm{d}t + \sigma_1 \mathrm{d}B_{1t}) w_t \\ + \theta_2(\mu_2 \mathrm{d}t + \sigma_2 \mathrm{d}B_{2t} + \sigma_i \mathrm{d}B_{it}) w_t - c_t \mathrm{d}t, \end{aligned}$$

where r is the (equilibrium) risk-free rate and $\theta = (\theta_1, \theta_2)$ is the portfolio of technologies. This problem is a standard Merton (1971) type optimal consumption-portfolio problem except that I use recursive utility, which has been solved by Svensson (1989). Letting

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 + \sigma_i^2 \end{bmatrix},$$

the optimal portfolio is $\theta = \frac{1}{\gamma}V^{-1}(\mu - r\mathbf{1})$ and the optimal consumption rule is $c_t = mw_t$, where the marginal propensity to consume out of wealth is

$$m = \beta\varepsilon + (1-\varepsilon)\left(r + \frac{1}{2\gamma}(\mu - r\mathbf{1})'V^{-1}(\mu - r\mathbf{1})\right).$$

The market clearing condition for the risk-free asset is $1 - \theta_1 - \theta_2 = 0$, which pins down the interest rate $r = \frac{\mathbf{1}'V^{-1}\mu - \gamma}{\mathbf{1}'V^{-1}\mathbf{1}}$. If agents inherit the average cross-sectional wealth, by (5.17) the power law exponent is $\alpha = \frac{\sqrt{2\delta}}{\theta_2\sigma_i}$.

The discrete-time model with time step Δt has discount factor $\mathrm{e}^{-\beta\Delta t}$, death probability $1 - \mathrm{e}^{-\delta\Delta t}$, and productivities with joint distribution

$$\begin{bmatrix}\log A_1 \\ \log A_2 \\ \log a_i\end{bmatrix} = \mathcal{N}\left(\begin{bmatrix}\mu_1 - \sigma_1^2/2 \\ \mu_2 - \sigma_2^2/2 \\ -\sigma_i^2/2\end{bmatrix}\Delta t, \begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2 & 0 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_i^2\end{bmatrix}\Delta t\right).$$

By Theorem 4.6 the equilibrium portfolio solves for

$$k =: \max_{\theta_1,\theta_2} \mathrm{E}[(A_1\theta_1 + a_iA_2\theta_2)^{1-\gamma}]^{\frac{1}{1-\gamma}}$$

subject to $\theta_1 + \theta_2 = 1$, $\theta_1, \theta_2 \geq 0$. By Corollary 3.7 with $\sigma = 1/\varepsilon$ and $S = \{1\}$, the optimal consumption rule is then $\tilde{c} = 1 - \mathrm{e}^{-\beta\varepsilon\Delta t}k^{\varepsilon-1}$.

6.2 Calibration

The continuous-time model has 10 parameters, discount rate β, RRA coefficient γ, EIS ε, death rate δ, expected returns μ_1, μ_2, volatilities $\sigma_1, \sigma_2, \sigma_i$, and correlation coefficient ρ. There are 8 targets, the expected stock market return μ_1, volatility σ_1, interest rate r, average aggregate log consumption growth $\mathrm{E}[\Delta \log C]$, variance $\mathrm{Var}[\Delta \log C]$, covariance with stock market return $\mathrm{Cov}[\Delta \log C, \log R]$, power law exponent of cross-sectional consumption distribution α, and idiosyncratic volatility of log consumption. Table 2 shows these numbers from data[27] as well as the expressions using model parameters.

Since there are more parameters (10) than targets (8), we can choose two more parameters freely. Vissing-Jørgensen (2002) finds that EIS is about 0.8–1 for non-stock holders, so I set $\varepsilon = 1$. Lustig and Van Nieuwerburgh (2008) report that the portfolio share of human wealth is 0.77–0.80, so I set $\theta_2 = 0.8$. Table 3 shows the implied values of model parameters that match the targets in Table 2 exactly.[28]

The values of $\beta, \mu_2, \sigma_2, \sigma_i$ seem reasonable. Interestingly, $\mu_2 \approx 10\%$ is a typical value for returns on human capital found in the labor economics literature. The RRA coefficient of $\gamma = 13$ is high, but much lower than the implied value from a representative agent model in order to explain asset returns (Hansen and Singleton, 1983). It may appear counter-intuitive that the

[27] I use Robert Shiller's spreadsheet for the U.S. 1889–2009 annual data of real total stock returns (S&P500), real interest rate, and aggregate consumption, available at `http://www.econ.yale.edu/~shiller/data.htm`. The power law exponent is taken from Toda and Walsh (2014a). The idiosyncratic volatility of log consumption is taken from Deaton and Paxson (1994).

[28] I obtained these values by numerically solving 10 nonlinear equations in 10 unknowns by Mathematica 7.0.

Table 2. Targets for calibration and their values.

Target	Expression	Value
Stock return $\mathrm{E}[\log R]$	μ_1	0.0588
Stock volatility	σ_1	0.1789
Interest rate	r	0.0179
$\mathrm{E}[\Delta \log C]$	$\mu'\theta - \frac{1}{2}\theta' V\theta - m$	0.0200
$\mathrm{Var}[\Delta \log C]$	$\theta'\Sigma\theta$	$\frac{2}{3}(0.0352)^2$
$\mathrm{Cov}[\Delta \log C, \log R]$	$\theta_1\sigma_1^2 + \theta_2\rho\sigma_1\sigma_2$	$\frac{2}{3}0.0035$
Power law exponent	$\frac{\sqrt{2\delta}}{\theta_2\sigma_i}$	4
$\mathrm{Var}_t\, \mathrm{E}[\log(c_{i,t+1}/c_{it})]$	$(\theta_2\sigma_i)^2$	$\frac{2}{3}0.0069$

Σ is the variance-covariance matrix of aggregate shock (V above with $\sigma_i = 0$). The factor $\frac{2}{3}$ comes from the Grossman et al. (1987) adjustment for time-aggregated data.

Table 3. Implied values of model parameters

Description	Parameter	Value
Discount rate	β	0.0657092
RRA coefficient	γ	12.997
EIS	ε	1
Death rate	δ	0.0368
Expected returns	μ_1	0.0588
	μ_2	0.0958277
Volatilities	σ_1	0.1789
	σ_2	0.0363895
	σ_i	0.0847791
Correlation	ρ	-0.624832

correlation between physical and human capital is negative ($\rho = -0.62$), but Lustig and Van Nieuwerburgh (2008) challenge the conventional wisdom of positive correlation. My ρ corresponds to $Corr(DR^y_\infty, DR^a_\infty) = -0.63$ in their Table 4 (p. 2118), which is virtually identical to my number. The death rate of $\delta = 0.0368$ (which implies an average lifespan of $1/\delta = 27.2$ years) is too high if we interpret death literally. However, "death" in the model should be interpreted as major life events that affect households, such as children leaving home, death, divorce, personal bankruptcy, retirement, etc. When interpreted this way, an average of 27.2 years without life events seems reasonable. Also note that since the power law exponent is $\alpha = \sqrt{2\delta}/(\theta_2\sigma_i)$ and the values of α and $\theta_2\sigma_i$ are known from data, the only free parameter in matching the power law exponent is δ: I do not match α by fine-tuning many parameters. Thus the current heterogeneous agent model goes a long way toward explaining asset returns, aggregate consumption dynamics, and cross-sectional distributions.[29]

[29] Since in my model consumption is proportional to wealth, the wealth and consumption distributions have the same shape, which is counter-factual since wealth is more skewed than consumption in actual data. The quantitative macro literature such as Guvenen (2006) resolves this issue by introducing two types of agents (asset holders and non-holders), which can be done in my setting. Another solution is to use HARA instead of CRRA preferences.

6.3 Simulation

Next I simulate the model economies with various time steps in order to evaluate whether the continuous-time approximation in Theorem 5.2 is good with reasonable parameters. The length of one time step ranges from 10 years to 1 month. All parameters are taken from Table 3. For each time step Δt, I compute the death probability $p = 1 - \mathrm{e}^{-\delta \Delta t}$ and run the economy for $T = \lceil 5/p \rceil$ periods (5 times the expected number of time steps in the lifespan) with 10,000 agents starting from initial wealth 1. Newborn agents inherit the average cross-sectional wealth. I compute the continuously compounded annual interest rate r_f using (4.2), the optimal portfolio share of stocks θ_1 using (3.11a), the power law exponent α using the continuous-time approximation (5.10), and the maximum likelihood estimate of the power law exponent by fitting the double Pareto distribution to the last period's consumption distribution. I simulate each economy for 500 times and compute the sample mean and standard deviation of the estimated power law exponent.

Table 4 shows the simulation results. The discrete-time model is virtually identical to the continuous-time model at monthly or quarterly frequency. With coarse approximation (time steps of 5 or 10 years), the continuous-time approximation of the power law exponent is biased downwards, while the fitted power law exponent is biased upwards compared to the continuous-time limit.

Table 4. Model parameters when changing the length of the time step Δt. $\mathrm{E}[\nu_p]$: expected number of time steps in the lifespan ($= 1/p$), r_f: risk-free rate (continuously compounded annual rate), θ_1: portfolio share of stocks, α: theoretical power law exponent computed by (5.10), $\hat{\alpha}_{\mathrm{ML}}$: maximum likelihood estimate of power law exponent (standard deviation in parenthesis).

Frequency	Δt	$\mathrm{E}[\nu_p]$	r_f	θ_1	α	$\hat{\alpha}_{\mathrm{ML}}$
10 years	10	3.2480	1.6387%	0.1897	3.4070	4.57 (0.22)
5 years	5	5.9501	1.6485%	0.1923	3.6639	4.19 (0.12)
1 year	1	27.677	1.7482%	0.1978	3.9234	3.98 (0.047)
3 months	1/4	109.20	1.7860%	0.2014	3.9900	3.98 (0.042)
1 month	1/12	326.59	1.7883%	0.2001	3.9952	3.97 (0.042)
continuous	0	∞	1.7900%	0.2000	4.0000	-

Figure 1 shows the time series of the power law exponent estimated by maximum likelihood from one simulation at the quarterly frequency ($\Delta t = 1/4$). We can see that the cross-sectional distribution settles down after around $3/p$ periods (three times the average life span, or 330 periods). The power law exponent fluctuates around the true value ($\alpha = 4$) due to the sampling error from the finiteness of the number of agents and periods.

7 Concluding remarks

In the presidential address delivered at the sixty-seventh Annual Meeting of the American Economic Association, Kuznets (1955) asked the following questions.

> Does inequality in the distribution of income increase or decrease in the course of a country's economic growth? What factors determine the secular level and trends of income inequalities?

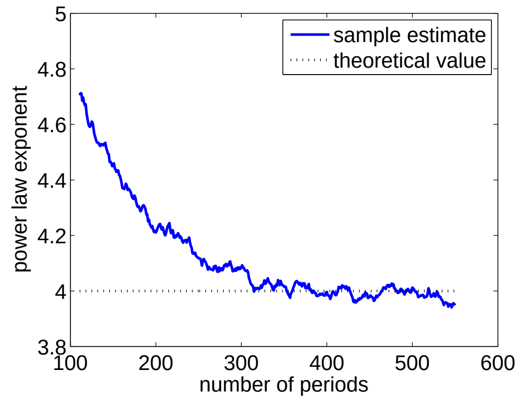

Figure 1. Time series of power law exponent estimated by maximum likelihood from one simulation at quarterly frequency.

My back-of-the-envelope formula (1.1) answers the second question: it is the (broadly interpreted) birth/death rate and the idiosyncratic volatility that determines inequality. With regard to the first question, the answer depends on the model. Economic growth may or may not be associated with idiosyncratic volatility, and one can only answer the first question within a concrete model.

My results on cross-sectional distributions show that the double power law is a robust property. Thus I provide a positive theory to understand inequality, which might help applied economists given the growing interest in inequality in recent years (Piketty and Saez, 2003). The consequence of the power law on econometric estimation is sometimes severe due to nonexistence of high order moments. This topic is beyond the scope of this paper, but I refer the interested readers to the companion papers Toda and Walsh (2014a,b).

In this paper I also proved the properties of tractable general equilibrium models with heterogeneous agents and incomplete markets, and provided a numerical algorithm. My results provide an accessible recipe for building models oriented toward applications, especially those with a complex financial structure where standard numerical techniques are difficult to apply. Two recent particular applications of this type of model are securitization (Toda, 2013) and sovereign debt and default (Walsh, 2014). Other interesting but unexplored topics are monetary policy and social security.

A Proofs

Lemma A.1. *Let X be an almost surely positive random variable and suppose that $\mathrm{E}[X^r]$ is finite for $0 < |r| < \epsilon$ and $\mathrm{E}[\log X]$ is finite. Then*

$$\lim_{r \to 0} \mathrm{E}[X^r]^{\frac{1}{r}} = \exp(\mathrm{E}[\log X]).$$

Proof. It suffices to prove when X is a discrete random variable (simple function) since the Lebesgue integral of a measurable function is defined by the limit of the integrals of approximating simple functions. Suppose that X takes values

$x_1, \dots, x_N > 0$ with probability $p_1, \dots, p_N$, and let

$$f(r) = \log \mathrm{E}[X^r] = \log \left(\sum_{n=1}^{N} p_n x_n^r \right).$$

Since $f(0) = \log \left(\sum_{n=1}^{N} p_n \right) = 0$, it follows that

$$\begin{aligned}
\lim_{r \to 0} \frac{1}{r} \log \mathrm{E}[X^r] &= \lim_{r \to 0} \frac{f(r) - f(0)}{r} = f'(0) \\
&= \left. \frac{\sum_{n=1}^{N} p_n x_n^r \log x_n}{\sum_{n=1}^{N} p_n x_n^r} \right|_{r=0} = \sum_{n=1}^{N} p_n \log x_n = \mathrm{E}[\log X].
\end{aligned}$$

Therefore $\lim_{r \to 0} \mathrm{E}[X^r]^{\frac{1}{r}} = \exp\left(\mathrm{E}[\log X]\right)$. □

Proof of Proposition 3.2. By the definition of recursive optimality, for any T we have

$$V_0^\infty(w) = \mathrm{E}_0 \sum_{t=0}^{T-1} \beta^t u_t(c_t) + \mathrm{E}_0\, \beta^T V_T^\infty(w_T).$$

Letting $T \to \infty$, we obtain

$$\begin{aligned}
V_0^*(w) &\geq \liminf_{T \to \infty} \mathrm{E}_0 \sum_{s=0}^{T-1} \beta^t u_t(c_t) = V_0^\infty(w) - \limsup_{t \to \infty} \mathrm{E}_0\, \beta^t V_t^\infty(w_t) \\
&\geq V_0^\infty(w) \geq V_0^*(w).
\end{aligned}$$

Therefore $\mathrm{E}_0 \sum_{s=0}^{\infty} \beta^t u_t(c_t) = V_0^*(w)$, so $\{c_t\}$ attains the maximum. □

Proof of Proposition 3.3. For $T = 1$, since the terminal utility is consumption itself, we obtain $V_t^1(w) = \max_{0 \leq c \leq w} c = w$, so $b_t^1 = 1$.

Suppose the claim holds up to $T - 1$. Then

$$\begin{aligned}
V_t^T(w) &= \sup_{\substack{0 \leq c \leq w \\ \pi \in \Pi_t}} f\left(c, \mathrm{E}_t \left[(b_{t+1}^{T-1} R_{t+1}(\pi)(w - c))^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \right) \\
&= \sup_{0 \leq c \leq w} f\left(c, (w - c) \sup_{\pi \in \Pi_t} \mathrm{E}_t \left[(b_{t+1}^{T-1} R_{t+1}(\pi))^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \right) \\
&= w \sup_{0 \leq \tilde{c} \leq 1} f\left(\tilde{c}, (1 - \tilde{c}) \sup_{\pi \in \Pi_t} \mathrm{E}_t \left[(b_{t+1}^{T-1} R_{t+1}(\pi))^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \right),
\end{aligned}$$

where I used the principle of optimality in the first equality, monotonicity and upper semi-continuity of f in the second, and homogeneity of f in the third. □

Proof of Theorem 3.5. Suppose that the sequence $\left\{ b^T(s) \right\}_{T=1}^{\infty}$ is monotone and there exist $0 < m < M$ such that $b^T(s) \in [m, M]$ for all s, T. Then $\left\{ b^T \right\}_{T=1}^{\infty}$ monotonically converges pointwise to some $b \in L_+^\infty(S)$. By Proposition 3.4, $0 < m \leq b^T(s) \leq M$, $\mathrm{E}\left[\sup_{\pi \in \Pi_s} R(\pi)^{1-\gamma} \,\middle|\, s \right] < \infty$, it follows that (3.8) holds. Since $R(\pi)$ is continuous in π, by the Dominated Convergence Theorem $\mathrm{E}\left[(b(s') R(\pi))^{1-\gamma} \,\middle|\, s \right]$ is continuous in π. Since the portfolio constraint Π_s is nonempty and compact, the supremum in (3.8) (which is equivalent to (3.11)) is attained and there exists a recursively optimal portfolio rule. The existence and uniqueness of the optimal consumption rule (3.11b) follows by the strict quasi-concavity and upper semi-continuity of the aggregator f.

To complete the proof, it remains to show that $\left\{b^T\right\}_{T=1}^\infty$ is monotone and there exist $0 < m < M$ such that $b(s)^T \in [m, M]$ for all s, T. By the monotonicity of f, the map B is a monotone map. Therefore $\left\{b^T\right\}_{T=1}^\infty$ is monotone if and only if either $b^2 \geq b^1$ or $b^2 \leq b^1$. Let us show either of these inequalities and the existence of bounds m, M.

Case 1: (3.10a) **holds.** Since $b^1(s) = 1$ for all s, by the definition of B, ρ_s, and the right inequality of (3.10a) we obtain

$$\begin{aligned} b^2(s) &= \max_{0\leq c\leq 1} f\left(c, (1-c)\max_{\pi\in\Pi_s} \mathrm{E}\left[R(\pi)^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}\right) \\ &= \max_{0\leq c\leq 1} f(c, (1-c)\rho_s) \geq 1 = b^1(s). \end{aligned}$$

Therefore $\left\{b^T\right\}$ is monotone increasing. Let $0 < \epsilon < 1$ be as in (3.10a) and $M = 1/\epsilon > 1$. Clearly $b^1(s) = 1 < M$ for all s. If $b^T(s) \leq M$ for all s, then by the left inequality of (3.10a) and the monotonicity and homotheticity of f, we obtain

$$\begin{aligned} b^{T+1}(s) &= \max_{0\leq c\leq 1} f\left(c, (1-c)\max_{\pi\in\Pi_s} \mathrm{E}\left[\left(b^T(s')R(\pi)\right)^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}\right) \\ &\leq \max_{0\leq c\leq 1} f\left(c, (1-c)\max_{\pi\in\Pi_s} \mathrm{E}\left[M^{1-\gamma}R(\pi)^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}\right) \\ &= \max_{0\leq c\leq 1} f(c, M\rho_s(1-c)) = M \max_{0\leq c\leq 1} f(c/M, \rho_s(1-c)) \\ &\leq M f(\epsilon, \rho_s) \leq M. \end{aligned}$$

Therefore $1 \leq b^T(s) \leq b^{T+1}(s) \leq M$ for all s, T.

Case 2: (3.10b) **holds.** By a similar argument we can show $b^{T+1} \leq b^T$ for all T. To show that $\left\{b^T\right\}_{T=1}^\infty$ is bounded away from zero, let $0 < \epsilon < 1$ be as in (3.10b). Clearly $b^1(s) = 1 \geq \epsilon$ for all s. If $b^T(s) \geq \epsilon$ for all s, then by the right inequality of (3.10b) and the monotonicity and homotheticity of f, we obtain

$$\begin{aligned} b^{T+1}(s) &= \max_{0\leq c\leq 1} f\left(c, (1-c)\max_{\pi\in\Pi_s} \mathrm{E}\left[\left(b^T(s')R(\pi)\right)^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}\right) \\ &\geq \max_{0\leq c\leq 1} f\left(c, (1-c)\max_{\pi\in\Pi_s} \mathrm{E}\left[\epsilon^{1-\gamma}R(\pi)^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}\right) \\ &= \max_{0\leq c\leq 1} f(c, \epsilon\rho_s(1-c)) = \epsilon \max_{0\leq c\leq 1} f(c/\epsilon, \rho_s(1-c)) \geq \epsilon. \end{aligned}$$

Therefore $\epsilon \leq b^{T+1}(s) \leq b^T(s) \leq 1$ for all s, T. □

Proof of Proposition 3.6. Suppose that (3.10a) is false. Then for every n

$$S_n := \{s \in S \mid f(1/n, \rho_s) \geq 1\}$$

is nonempty and compact. By the monotonicity of f, we have $S_{n+1} \subset S_n$. Therefore we can take $s \in \bigcap_n S_n$. Letting $n \to \infty$ in $f(1/n, \rho_s) \geq 1$, we get $f(0+, \rho_s) \geq 1$, which contradicts (3.12a).

Now suppose that (3.10b) is false. Then for every n

$$S_n := \left\{ s \in S \,\middle|\, \max_{0 \le c \le 1} f(nc, \rho_s(1-c)) \le 1 \right\}$$

is nonempty and compact. By the monotonicity of f, we have $S_{n+1} \subset S_n$. Therefore we can take $s \in \bigcap_n S_n$. Letting $n \to \infty$ in $f(nc, \rho_s(1-c)) \le 1$, we get

$$f(\infty, \rho_s) = \max_{0 \le c \le 1} f(\infty, \rho_s(1-c)) \le 1,$$

which contradicts (3.12b). □

Proof of Corollary 3.7. Letting $c \to 0, \infty$ in $f(c,v) = (c^{1-1/\varepsilon} + \beta v^{1-1/\varepsilon})^{\frac{1}{1-1/\varepsilon}}$, we obtain

$$f(0+, \rho_s) = \begin{cases} 0, & (\varepsilon < 1) \\ \beta^{\frac{1}{1-1/\varepsilon}} \rho_s, & (\varepsilon > 1) \end{cases} \quad \text{and} \quad f(\infty, \rho_s) = \begin{cases} \beta^{\frac{1}{1-1/\varepsilon}} \rho_s, & (\varepsilon < 1) \\ \infty. & (\varepsilon > 1) \end{cases}$$

Therefore independent of $\varepsilon \gtrless 1$, we have

$$f(0+, \rho_s) < 1 < f(\infty, \rho_s) \iff \beta \rho_s^{1-1/\varepsilon} < 1.$$

Furthermore, by simple calculus for any $k > 0$ the maximum of $f(c, k(1-c))$ is attained at $c = (1 + \beta^\varepsilon k^{\varepsilon-1})^{-1}$ and the maximum is $(1 + \beta^\varepsilon k^{\varepsilon-1})^{\frac{1}{\varepsilon-1}}$, so substituting $k = \rho_s$, (3.12a) holds if $\varepsilon > 1$ and (3.12b) holds if $\varepsilon < 1$. Therefore the conclusion of Theorem 3.5 holds. (3.14) follows by setting

$$k = \max_{\pi \in \Pi_s} \mathrm{E}\left[(x(s')R(\pi))^{1-\gamma} \,\middle|\, s \right]^{\frac{1}{1-\gamma}}.$$

Similarly, substituting $k = \max_{\pi \in \Pi_s} \mathrm{E}\left[(b(s')R(\pi))^{1-\gamma} \,\middle|\, s \right]^{\frac{1}{1-\gamma}}$ and noting that b is a fixed point of the map B, (3.11b) implies the optimal consumption rule $\tilde{c}_s = b(s)^{1-\varepsilon}$. □

Proof of Proposition 3.8. If we write the utility function in the additive form $\mathrm{E}_0 \sum_{t=0}^\infty \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$, the infinite horizon value function becomes

$$W^\infty(w,s) := \frac{1}{1-\gamma} (V^\infty(w,s))^{1-\gamma} = b(s)^{1-\gamma} \frac{w^{1-\gamma}}{1-\gamma}.$$

By Proposition 3.2 it suffices to show the transversality condition

$$\limsup_{t \to \infty} \mathrm{E}_0 \, \beta^t W^\infty(w_t, s_t) \le 0.$$

If $\gamma > 1$, this is trivial because $W^\infty(w,s) \le 0$ always. Assume $\gamma < 1$.[30] Since consumption is nonnegative, by the budget constraint $w' = R(\pi)(w-c)$ wealth grows at most at the rate $R(\pi)$ each period. Hence $w_t \le w_0 \prod_{\tau=0}^{t-1} R_{\tau+1}(\pi_{s_\tau})$. Taking the $(1-\gamma)$th power, taking the expectation, and using (3.13) with $\varepsilon = 1/\gamma$, we obtain

$$0 \le \mathrm{E}_0[\beta^t w_t^{1-\gamma}] \le \left(\beta \bar{\rho}^{1-\gamma} \right)^t \to 0,$$

so the transversality condition holds. □

[30]The case $\gamma = 1$ requires a separate argument, but I omit since it is straightforward.

Proof of Proposition 4.2. Let $V_{it}^T(w) = b_{it}^T w$ be the T period value function, taking asset prices as given. By Proposition 3.3, we have $b_{it}^1 = 1$ and

$$b_{it}^T = \sup_{0\le\tilde{c}\le 1} f\left(\tilde{c}, (1-\tilde{c}) \sup_{\pi\in\Pi_t} \mathrm{E}_{it}\left[(b_{i,t+1}^{T-1} R_{i,t+1}(\pi))^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right). \tag{A.1}$$

Suppose that b_{it}^{T-1} is common across agents, b_t^{T-1}. Then by Assumption 3 and 4, we can replace E_{it} in (A.1) by E_t, so it follows that b_{it}^T is common across agents. Hence by induction $V_{it}^T(w) = b_t^T w$ for all T, and letting $T \to \infty$ the infinite horizon value function $V_{it}^\infty(w) = b_t w$ is common across agents. By Proposition 3.4, b_t satisfies (3.8), which determines the optimal consumption and portfolio. Let

$$\begin{bmatrix}\bar{\theta}_t \\ \bar{\phi}_t\end{bmatrix} = \int_I \begin{bmatrix}\theta_{it} \\ \phi_{it}\end{bmatrix}(w_{it} - c_{it})\mathrm{d}i \Big/ \int_I (w_{it} - c_{it})\mathrm{d}i$$

be the value weighted average portfolio. Since ϕ_{it} clears the asset markets, it follows that $\bar{\phi}_t = 0$, which trivially clears the asset markets. Since $\mathrm{E}_t[R_{t+1}(\pi)^{1-\gamma}]^{\frac{1}{1-\gamma}}$ is quasi-concave in π, if π_{it} solves (3.8) for all i, so does $\bar{\pi}_t$. Therefore $\bar{E}$ is an equilibrium. $\square$

Proof of Proposition 4.3. By Proposition 4.2 we may assume that $\phi_{it} = \tilde{\phi}_{it} = 0$, so there is no trade in assets. Then $\theta_{it}, \tilde{\theta}_{it}$ are optimal in the smaller portfolio constraint $\Theta_t := \{\theta \mid (\theta, 0) \in \Pi_t\}$. Since the portfolio return $R_{t+1}(\theta, 0)$ with no asset holdings does not depend on asset prices, (c_{it}, θ_{it}) and $(\tilde{c}_{it}, \tilde{\theta}_{it})$ solve the same optimization problem. Since $R \mapsto \mathrm{E}[R^{1-\gamma}]^{\frac{1}{1-\gamma}}$ is strictly quasi-concave in R, it follows that $R_{i,t+1}(\theta_{it}, 0) = R_{i,t+1}(\tilde{\theta}_{it}, 0)$ almost surely. Since by Assumption 1 the aggregator f is strictly quasi-concave, it follows that $c_{it} = \tilde{c}_{it}$ almost surely. $w_{it} = \tilde{w}_{it}$ follows by the budget constraint. $\square$

Proof of Proposition 4.4. By Proposition 3.4 it suffices to show $\tilde{b}_t^T \le b_t^T$ for all t, T. For $T = 1$ this is trivial since $\tilde{b}_t^T = b_t^T = 1$. Suppose that the claim holds up to $T-1$. By Proposition 4.2 without loss of generality we may assume that $\tilde{E}, E$ are autarky (no trade in zero net supply assets). Hence by (3.8) and the monotonicity of f we obtain

$$\begin{aligned}\tilde{b}_t^T &= \sup_{0\le\tilde{c}\le 1} f\left(\tilde{c}, (1-\tilde{c}) \sup_{(\theta,0)\in\tilde{\Pi}_t} \mathrm{E}_t\left[(\tilde{b}_{t+1}^{T-1} R_{i,t+1}(\theta, 0))^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right) \\ &\le \sup_{0\le\tilde{c}\le 1} f\left(\tilde{c}, (1-\tilde{c}) \sup_{(\theta,0)\in\Pi_t} \mathrm{E}_t\left[(b_{t+1}^{T-1} R_{i,t+1}(\theta, 0))^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right) = b_t^T. \quad \square\end{aligned}$$

Proof of Proposition 4.5. Let $R_{t+1}^k = (P_{t+1}^k + D_{t+1}^k)/P_t^k$ be the return of asset k. Consider the return $(1-\alpha)R_{t+1}(\theta_t, 0) + \alpha R_{t+1}^k$, which can be attained by investing the fraction of wealth $1-\alpha$ in the symmetric equilibrium portfolio and α in asset k. Since by assumption the portfolio constraint is not binding, the consumer can choose a small positive or negative α, and of course $\alpha = 0$ is optimal. Hence for small enough $\epsilon > 0$ by (3.11a) we obtain

$$0 = \arg\max_{\alpha\in[-\epsilon,\epsilon]} \frac{1}{1-\gamma} \mathrm{E}_t[b_{t+1}^{1-\gamma}[(1-\alpha)R_{t+1}(\theta_t, 0) + \alpha R_{t+1}^k]^{1-\gamma}].$$

The first-order condition with respect to α at $\alpha = 0$ is

$$\mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta_t, 0)^{1-\gamma}(R_{t+1}^k - R_{t+1}(\theta_t, 0))] = 0.$$

Using $R_{t+1}^k = (P_{t+1}^k + D_{t+1}^k)/P_t^k$ and rearranging terms, we obtain (4.1).

By setting $D_t = 1$ and zero thereafter (hence $P_{t+1} = 0$) in (4.1), we obtain the price of the one period risk-free bond as the reciprocal of (4.2). □

Proof of Theorem 5.1. The idea of the proof is to condition on the geometric random variable ν_p, take the limit as $p \to 0$, and then obtain the unconditional distribution. Since ν_p is a geometric random variable that is independent from everything else, without loss of generality we can construct it as follows.

For $w > 0$ and $0 < p < 1$, define $N_p(w) = \lceil -w/\log(1-p) \rceil$, the integer obtained by rounding up $-w/\log(1-p) > 0$. If W is standard exponential, then $\nu_p = N_p(W)$ (in distribution). In fact,

$$\begin{aligned}
&\Pr[N_p(W) = n] = \Pr[n - 1 < -W/\log(1-p) \le n] \\
&= \int_{-(n-1)\log(1-p)}^{-n\log(1-p)} \mathrm{e}^{-w} \mathrm{d}w = (1-p)^{n-1} - (1-p)^n = (1-p)^{n-1} p.
\end{aligned}$$

Since $-\log(1-p) \approx p$ for small p, it follows that $N_p(w) \to \infty$ and $pN_p(w) \to w$ as $p \to 0$. Conditioning on $W = w$, since by assumption the central limit theorem holds, as $p \to 0$ we obtain

$$p^{\frac{1}{2}} \sum_{n=1}^{N_p(w)} (X_n + p^{\frac{1}{2}} a_n) = \frac{\sqrt{pN_p(w)}}{\sqrt{N_p(w)}} \sum_{n=1}^{N_p(w)} X_n + \frac{pN_p(w)}{N_p(w)} \sum_{n=1}^{N_p(w)} a_n \xrightarrow{\mathrm{d}} \sqrt{w}\sigma Z + wa,$$

where Z is a standard normal variable. Therefore

$$p^{\frac{1}{2}} \sum_{n=1}^{\nu_p} (X_n + p^{\frac{1}{2}} a_n) = p^{\frac{1}{2}} \sum_{n=1}^{N_p(W)} (X_n + p^{\frac{1}{2}} a_n) \xrightarrow{\mathrm{d}} \sqrt{W}\sigma Z + aW,$$

where W is standard exponential that is independent of Z.[31] The claim follows since its characteristic function is

$$\begin{aligned}
&\mathrm{E}\left[\exp(it(\sqrt{W}\sigma Z + aW))\right] = \mathrm{E}\left[\mathrm{E}\left[\exp(it(\sqrt{W}\sigma Z + aW)) \,\middle|\, W\right]\right] \\
&= \int_0^\infty \mathrm{e}^{iatw - \frac{1}{2}\sigma^2 t^2 w} \mathrm{e}^{-w} \mathrm{d}w = \frac{1}{1 - iat + \frac{\sigma^2 t^2}{2}},
\end{aligned}$$

which is the characteristic function of $\mathcal{AL}(0, a, \sigma)$ in (5.4). □

Proof of Proposition 5.3. Since the log size follows the Brownian motion, the cross-sectional log size distribution of units with age s is Gaussian with mean μs and variance $\sigma^2 s$. Since units die at a Poisson rate δ, the age distribution of units is truncated exponential with parameter δ. In particular, the mass of units

[31] In general, if $X_n | Y \xrightarrow{\mathrm{d}} X | Y$, we have $X_n \xrightarrow{\mathrm{d}} X$. To see this, for any bounded measurable f, we have $\mathrm{E}[f(X_n)|Y] \to \mathrm{E}[f(X)|Y]$ almost surely. Since f is bounded, by the law of iterated expectations and the dominated convergence theorem we have $\mathrm{E}[f(X_n)] = \mathrm{E}[\mathrm{E}[f(X_n)|Y]] \to \mathrm{E}[\mathrm{E}[f(X)|Y]] = \mathrm{E}[f(X)]$ almost surely, so $X_n \xrightarrow{\mathrm{d}} X$. I thank Brendan Beare for this remark.

that have never died up to time t is $\mathrm{e}^{-\delta t}$. Therefore the entire cross-sectional log size distribution at time t is the Gaussian mixture

$$f(x,t) = \mathrm{e}^{-\delta t}\frac{1}{\sigma\sqrt{2\pi t}}\mathrm{e}^{-\frac{(x-\mu t)^2}{2\sigma^2 t}} + \int_0^t \delta \mathrm{e}^{-\delta s}\frac{1}{\sigma\sqrt{2\pi s}}\mathrm{e}^{-\frac{(x-\mu s)^2}{2\sigma^2 s}}\mathrm{d}s.$$

To derive (5.14), use the integration formula

$$\int \frac{1}{\sqrt{t}}\mathrm{e}^{-a^2 t-\frac{b^2}{t}}\mathrm{d}t = \frac{\sqrt{\pi}}{2a}\left[\mathrm{e}^{-2ab}\operatorname{erfc}\left(\frac{b}{\sqrt{t}} - a\sqrt{t}\right) - \mathrm{e}^{2ab}\operatorname{erfc}\left(\frac{b}{\sqrt{t}} + a\sqrt{t}\right)\right],$$

where $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-t^2}\mathrm{d}t$ is the complementary error function, set $a = \sqrt{2\delta + \frac{\mu^2}{\sigma^2}}$ and $b = \frac{|x|}{\sqrt{2}\sigma}$, and use the fact that $\operatorname{erfc}(x) = 2\Phi(-\sqrt{2}x)$. $\square$

References

Fernando Alvarez and Nancy L. Stokey. Dynamic programming with homogeneous functions. *Journal of Economic Theory*, 82(1):167–189, September 1998. doi:10.1006/jeth.1998.2431.

George-Marios Angeletos. Uninsured idiosyncratic investment risk and aggregate saving. *Review of Economic Dynamics*, 10(1):1–30, January 2007. doi:10.1016/j.red.2006.11.001.

George-Marios Angeletos and Laurent-Emmanuel Calvet. Incomplete-market dynamics in a neoclassical production economy. *Journal of Mathematical Economics*, 41(4-5):407–438, August 2005. doi:10.1016/j.jmateco.2004.09.005.

George-Marios Angeletos and Laurent-Emmanuel Calvet. Idiosyncratic production risk, growth and the business cycle. *Journal of Monetary Economics*, 53: 1095–1115, 2006. doi:10.1016/j.jmoneco.2005.05.016.

George-Marios Angeletos and Vasia Panousi. Revisiting the supply side effects of government spending. *Journal of Monetary Economics*, 56(2):137–153, March 2009. doi:10.1016/j.jmoneco.2008.12.010.

George-Marios Angeletos and Vasia Panousi. Financial integration, entrepreneurial risk and global dynamics. *Journal of Economic Theory*, 146 (3):863–896, May 2011. doi:10.1016/j.jet.2011.02.001.

Jess Benhabib, Alberto Bisin, and Shenghao Zhu. The distribution of wealth and fiscal policy in economies with finitely lived agents. *Econometrica*, 79(1): 123–157, January 2011. doi:10.3982/ECTA8416.

Jess Benhabib, Alberto Bisin, and Shenghao Zhu. The distribution of wealth in the Blanchard-Yaari model. *Macroeconomic Dynamics*, 2014. doi:10.1017/S1365100514000066.

Truman F. Bewley. Stationary monetary equilibrium with a continuum of independently fluctuating consumers. In Werner Hildenbrand and Andreu Mas-Collel, editors, *Contributions to Mathematical Economics in Honor of Gérard Debreu*, chapter 5, pages 79–102. North-Holland, Amsterdam, 1986.

John H. Boyd, III. Recursive utility and the Ramsey problem. *Journal of Economic Theory*, 50(2):326–345, April 1990. doi:10.1016/0022-0531(90)90006-6.

Laurent-Emmanuel Calvet. Incomplete markets and volatility. *Journal of Economic Theory*, 98(2):295–338, June 2001. doi:10.1006/jeth.2000.2720.

George M. Constantinides and Darrell Duffie. Asset pricing with heterogeneous consumers. *Journal of Political Economy*, 104(2):219–240, April 1996.

Angus Deaton and Christina Paxson. Intertemporal choice and inequality. *Journal of Political Economy*, 102(3):437–467, June 1994.

Peter A. Diamond. The role of a stock market in a general equilibrium model with technological uncertainty. *American Economic Review*, 57(4):759–776, September 1967.

Darrell Duffie and Larry G. Epstein. Stochastic differential utility. *Econometrica*, 60(2):353–394, March 1992. doi:10.2307/2951600.

Darrell Duffie, John Geanakoplos, Andreu Mas-Collel, and Andrew McLennan. Stationary Markov equilibria. *Econometrica*, 62(4):745–781, July 1994. doi:10.2307/2951731.

Andrea L. Eisfeldt. Endogenous liquidity in asset markets. *Journal of Finance*, 59(1):1–30, February 2004. doi:10.1111/j.1540-6261.2004.00625.x.

Larry G. Epstein and Stanley E. Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns:a theoretical framework. *Econometrica*, 57(4):937–969, July 1989. doi:10.2307/1913778.

Xavier Gabaix. Zipf's law for cities: An explanation. *Quarterly Journal of Economics*, 114(3):739–767, 1999. doi:10.1162/003355399556133.

Xavier Gabaix. Power laws in economics and finance. *Annual Review of Economics*, 1:255–293, 2009. doi:10.1146/annurev.economics.050708.142940.

Xavier Gabaix. The granular origins of aggregate fluctuations. *Econometrica*, 79(3):733–772, May 2011. doi:10.3982/ECTA8769.

John Geanakoplos and Heraklis M. Polemarchakis. Existence, regularity, and constrained suboptimality of competitive allocations when markets are incomplete. In Walter P. Heller, Ross M. Starr, and David A. Starrett, editors, *Uncertainty, Information, and Communication*, volume 3 of *Essays in Honor of Kenneth Arrow*, chapter 3, pages 65–95. Cambridge University Press, Cambridge, 1986.

Robert Gibrat. *Les Inégalités économiques*. Librairie du Recueil Sirey, Paris, 1931.

Kristian Giesen, Arndt Zimmermann, and Jens Suedekum. The size distribution across all cities—double Pareto lognormal strikes. *Journal of Urban Economics*, 68(2):129–137, September 2010. doi:10.1016/j.jue.2010.03.007.

Sanford J. Grossman, Angelo Melino, and Robert J. Shiller. Estimating the continuous-time consumption-based asset-pricing model. *Journal of Business and Economic Statistics*, 5(3):315–327, 1987. doi:10.1080/07350015.1987.10509594.

Fatih Guvenen. Reconciling conflicting evidence on the elasticity of intertemporal substitution: A macroeconomic perspective. *Journal of Monetary Economics*, 53(7):1451–1472, October 2006. doi:10.1016/j.jmoneco.2005.06.001.

Nils H. Hakansson. Optimal investment and consumption strategies under risk for a class of utility functions. *Econometrica*, 38(5):587–607, September 1970. doi:10.2307/1912196.

Nils H. Hakansson. Optimal entrepreneurial decisions in a completely stochastic environment. *Management Science*, 17(7):427–449, March 1971. doi:10.1287/mnsc.17.7.427.

Lars Peter Hansen and Kenneth J. Singleton. Stochastic consumption, risk aversion, and the temporal behavior of asset returns. *Journal of Political Economy*, 91(2):249–265, April 1983.

Harry Kesten. Random difference equations and renewal theory for products of random matrices. *Acta Mathematica*, 131(1):207–248, 1973. doi:10.1007/BF02392040.

Narayana R. Kocherlakota. On the 'discount' factor in growth economies. *Journal of Monetary Economics*, 25(1):43–47, January 1990. doi:10.1016/0304-3932(90)90043-4.

Narayana R. Kocherlakota. Testing the consumption CAPM with heavy-tailed pricing errors. *Macroeconomic Dynamics*, 1(3):551–567, September 1997. doi:10.1017/S136510059700401X.

Samuel Kotz, Tomasz J. Kozubowski, and Krzysztof Podgórski. *The Laplace Distribution and Generalizations*. Birkhäuser, Boston, 2001.

Tom Krebs. Human capital risk and economic growth. *Quarterly Journal of Economics*, 118(2):709–744, 2003a. doi:10.1162/003355303321675491.

Tom Krebs. Growth and welfare effects of business cycles in economies with idiosyncratic human capital risk. *Review of Economic Dynamics*, 6(4):846–868, October 2003b. doi:10.1016/S1094-2025(03)00030-9.

Tom Krebs. Recursive equilibrium in endogenous growth models with incomplete markets. *Economic Theory*, 29(3):505–523, 2006. doi:10.1016/S0165-1889(03)00062-9.

Dirk Krueger and Hanno Lustig. When is market incompleteness irrelevant for the price of aggregate risk (and when is it not)? *Journal of Economic Theory*, 145(1):1–41, January 2010. doi:10.1016/j.jet.2009.10.005.

Per Krusell and Anthony A. Smith, Jr. Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896, October 1998. doi:10.1086/250034.

Simon Kuznets. Economic growth and income inequality. *American Economic Review*, 45(1):1–28, March 1955.

David Levhari and T. N. Srinivasan. Optimal savings under uncertainty. *Review of Economic Studies*, 36(2):153–163, April 1969.

Hanno Lustig and Stijn Van Nieuwerburgh. The returns on human capital: Good news on Wall street is bad news on main street. *Review of Financial Studies*, 21(5):2097–2137, 2008. doi:10.1093/rfs/hhl035.

Erzo G. J. Luttmer. Selection, growth, and the size distribution of firms. *Quarterly Journal of Economics*, 122(3):1103–1144, 2007. doi:10.1162/qjec.122.3.1103.

Chenghu Ma. Market equilibrium with heterogeneous recursive-utility-maximizing agents. *Economic Theory*, 3(2):243–266, 1993. doi:10.1007/BF01212916.

Benoît Mandelbrot. The Pareto-Lévy law and the distribution of income. *International Economic Review*, 1(2):79–106, May 1960. doi:10.2307/2525289.

Massimo Marinacci and Luigi Montrucchio. Unique solutions for stochastic recursive utilities. *Journal of Economic Theory*, 145(5):1776–1804, September 2010. doi:10.1016/j.jet.2010.02.005.

Robert C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4):373–413, December 1971. doi:10.1016/0022-0531(71)90038-X.

Makoto Nirei and Wataru Souma. A two factor model of income distribution dynamics. *Review of Income and Wealth*, 53(3):440–459, September 2007. doi:10.1111/j.1475-4991.2007.00242.x.

Hiroyuki Ozaki and Peter A. Streufert. Dynamic programming for non-additive stochastic objectives. *Journal of Mathematical Economics*, 25(4):391–442, 1996. doi:10.1016/0304-4068(95)00737-7.

Vasia Panousi. Capital taxation with entrepreneurial risk. Finance and Economics Discussion Series 2010-56, Board of Governors of the Federal Reserve System (U.S.), 2010. URL `http://EconPapers.repec.org/RePEc:fip:fedgfe:2010-56`.

Vilfredo Pareto. *Cours d'économie politique*. F. Rouge, Lausanne, 1896.

Thomas Piketty and Emmanuel Saez. Income inequality in the United States, 1913–1998. *Quarterly Journal of Economics*, 118(1):1–41, 2003. doi:10.1162/00335530360535135.

William J. Reed. The Pareto, Zipf and other power laws. *Economics Letters*, 74(1):15–19, December 2001. doi:10.1016/S0165-1765(01)00524-9.

William J. Reed. On the rank-size distribution for human settlements. *Journal of Regional Science*, 42(1):1–17, February 2002. doi:10.1111/1467-9787.00247.

William J. Reed. The Pareto law of incomes—an explanation and an extension. *Physica A*, 319(1):469–486, March 2003. doi:10.1016/S0378-4371(02)01507-8.

Juan Pablo Rincón-Zapatero and Carlos Rodríguez-Palmero. Existence and uniqueness of solutions to the Bellman equation in the unbounded case. *Econometrica*, 71(5):1519–1555, September 2003. doi:10.1111/1468-0262.00457.

Esteban Rossi-Hansberg and Mark L. J. Wright. Urban structure and growth. *Review of Economic Studies*, 74(2):597–624, 2007. doi:10.1111/j.1467-937X.2007.00432.x.

Makoto Saito. A simple model of incomplete insurance: The case of permanent shocks. *Journal of Economic Dynamics and Control*, 22(5):763–777, May 1998. doi:10.1016/S0165-1889(97)00077-8.

Paul A. Samuelson. Lifetime portfolio selection by dynamic stochastic programming. *Review of Economics and Statistics*, 51(3):239–246, August 1969. doi:10.2307/1926559.

Peter A. Streufert. Stationary recursive utility and dynamic programming under the assumption of biconvergence. *Review of Economic Studies*, 57(1):79–97, January 1990. doi:10.2307/2297544.

Peter A. Streufert. An abstract topological approach to dynamic programming. *Journal of Mathematical Economics*, 21(1):59–88, 1992. doi:10.1016/0304-4068(92)90022-Y.

Lars E. O. Svensson. Portfolio choice with non-expected utility in continuous time. *Economics Letters*, 30(4):313–317, October 1989. doi:10.1016/0165-1765(89)90084-0.

Alexis Akira Toda. Income dynamics with a stationary double Pareto distribution. *Physical Review E*, 83(4):046122, 2011. doi:10.1103/PhysRevE.83.046122.

Alexis Akira Toda. The double power law in income distribution: Explanations and evidence. *Journal of Economic Behavior and Organization*, 84(1):364–381, September 2012. doi:10.1016/j.jebo.2012.04.012.

Alexis Akira Toda. Securitization and leverage in general equilibrium. 2013. URL `http://ssrn.com/abstract=2336582`.

Alexis Akira Toda. Asset prices and efficiency in a Krebs economy. *Review of Economic Dynamics*, 2014. doi:10.1016/j.red.2014.11.003.

Alexis Akira Toda and Kieran Walsh. The double power law in consumption and implications for testing Euler equations. *Journal of Political Economy*, 2014a. URL `http://ssrn.com/abstract=2319454`. Forthcoming.

Alexis Akira Toda and Kieran Walsh. Fat tails and Euler equations: A Monte Carlo study. 2014b. URL `http://ssrn.com/abstract=2495151`.

Annette Vissing-Jørgensen. Limited asset market participation and the elasticity of intertemporal substitution. *Journal of Political Economy*, 110(4):825–853, August 2002. doi:10.1086/340782.

Kieran Walsh. Portfolio choice and partial default in emerging markets: A quantitative analysis. 2014.

Neng Wang. An equilibrium model of wealth distribution. *Journal of Monetary Economics*, 54(7):1882–1904, October 2007. doi:10.1016/j.jmoneco.2006.11.005.