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A Global Torelli Theorem of Projective Manifolds

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Xiaojing Chen

2014

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2014

Abstract of the Dissertation

A Global Torelli Theorem of Projective Manifolds

by

Xiaojing Chen

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2014

Professor Kefeng Liu, Chair

This thesis has studied global Torelli problems for projective manifolds. In particular, we have focused on projective manifolds of Calabi–Yau type, which is a generalization of Calabi–Yau manifolds.

In collaboration with F. Guan and K. Liu, we have proven the injectivity of the period map on the Teichmüller space of polarized and marked Calabi–Yau type manifolds [10, 11]. Our approach has been to construct the holomorphic affine structure on the Teichmüller space and the Hodge metric completion of the Teichmüller space. As a corollary, we also prove that the Hodge metric completion space of the Teichmüller space for Calabi–Yau type manifolds is a domain of holomorphy and it admits a Kähler–Einstein metric.

More generally, we are interested in global Torelli problems for projective manifolds. We have been working on adopting our techniques to more general projective manifolds, such as Calabi–Yau manifolds, projective hypersurfaces, and projective hyperkähler manifolds.

As direct applications, we will prove some properties for the period map on the moduli space of Calabi–Yau type manifolds; we will also prove a general result about the period map to be biholomorphic from the Hodge metric completion space of the Teichmüller space of Calabi–Yau type manifolds to their period domains, and apply it to the cases of K3 surfaces and cubic fourfolds. Moreover, we will discuss some special cases when the period domain has the same dimension as the Teichmüller space and has a natural affine structure.

The dissertation of Xiaojing Chen is approved.

David Gieseker

Hongquan Xu

Kefeng Liu, Committee Chair

University of California, Los Angeles

2014

To my family.

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Vita

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Publications

X. Chen, F. Guan and K. Liu, Hodge metric completion of the Teichmüller space of Calabi–Yau manifolds. *(Summary, 10 pages, to appear in the Proceedings of the Six International Congress of Chinese Mathematicians).*

X. Chen, F. Guan and K. Liu, Affine structures on Teichmüller spaces and applications*: to appear in Methods and Applications of Analysis.*

X. Chen, F. Guan and K. Liu, Hodge metric completion of the Teichmüller space of Calabi–Yau manifolds. *Preprint: arXiv:1305.0231 (Submitted).*

X. Chen, F. Guan and K. Liu, Global Torelli theorem for projective manifolds of Calabi–Yau type. *Preprint arXiv:1205.4208.*

X. Chen, F. Guan and K. Liu, Applications of the affine structures on the Teichmüller spaces. *Preprint arXiv:1402.5570.*

CHAPTER 1

Introduction

In collaboration with Feng Guan and Kefeng Liu, we have proven a global Torelli theorem for Calabi–Yau type manifolds [10, 11]. A Calabi–Yau type manifolds is a projective manifold which is a generalization of Calabi–Yau manifolds. More precisely, a compact simply connected projective manifold M of complex dimension n is called a Calabi–Yau type manifold if it satisfies the following: there exists some $[n/2] < s \leq n$, such that $h^{s,n-s}(M) = 1$ and $h^{s',n-s'}(M) = 0$ for any $s' > s$; the contraction map $\lrcorner : H^{0,1}(M, T^{1,0}M) \to H^{s-1,n-s+1}_{pr}(M)$, $[\phi] \mapsto [\phi \lrcorner \Omega]$ is an isomorphism for any generator $[\Omega] \in H^{s,n-s}(M)$; $H^{s,n-s}(M) = H^{s,n-s}_{pr}(M)$. One can easily check that a Calabi–Yau manifold, which has a trivial canonical bundle and satisfies $H^i(M, \mathcal{O}_M) = 0$ for $0 < i < n$, is an example of Calabi–Yau type manifold. A polarized and marked Calabi–Yau type manifold is a triple $(M, L, \{\gamma_1, \cdots, \gamma_{b^n}\})$ consisting of a Calabi–Yau type manifold M, an ample line bundle L over M, and a basis $\{\gamma_1, \cdots, \gamma_{b^n}\}$ of the integral middle homology group modulo torsion $H_n(M, \mathbb{Z})/\mathrm{Tor}$.

In general, the Torelli problem asks whether an algebraic object is determined by a polarized abelian variety attached to it. In 1914, Torelli [59] originally asked whether two curves are isomorphic if they have the same periods. Weil reformulated the Torelli problem for Riemann surfaces with polarization in [68]; Andreotti proved Weil's version of the Torelli problem in [1]. In 1960s and 70s, Griffiths [21, 22, 23, 24] and Deligne [13] developed the general theory of variations of Hodge structures, which allowed for Torelli problems to be stated in terms of period domains and period maps.

Local Torelli problem has been well-studied ([8, 9, 19, 20, 21, 22]), and the local Torelli theorem has been proven for a large range of varieties. For example, the well-known theorem of Bogomolov–Tian–Todorov [56, 58] implies that the period map for Calabi–Yau manifolds is immersive. The local Torelli theorem holds for hypersurfaces in $\mathbb{P}^{n+1}$ ([21, 22]), and more generally complete intersections in $\mathbb{P}^{n+1}$ and weighted complete intersection ([60]), cyclic covers of $\mathbb{P}^{n+1}$ or of products of projective spaces ([32, 46]).

As far as global properties of period map, some global Torelli theorems have been proven for low dimensional projective manifolds besides Riemann surfaces. For 2-dimensional projective manifolds, global Torelli theorems for K3 surfaces was conjectured by Weil [69], and have been given by Shafarevich and Piatetski–Shapiro [47], Looijenga [38], and Burns–Rapoport [7]. Moreover, Todorov [57] and Siu [52] were able to show the surjectivity of the period mapping for K3 surfaces.

For higher dimensional cases, global Torelli theorem is confirmed for cubic fourfold in [67, 37] and is explicitly described in [36], and it induceds and isomorphism from the moduli space to an open subset of D/Γ. Also some generic Torelli theorems have been proven, but only for specific examples of projective manifolds. For example, Voisin [64] proved a generic Torelli theorem for quintic threefolds. She proved that the period map for quintic threefolds $U/PGL(4) \to D/\Gamma$ is of degree one, with $U \subseteq \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^4}(5))$ parametrizing smooth hypersurfaces and Γ, the group of automorphisms of $H^3(X_b, \mathbb{Z})$, preserving intersection form. Usui [61] proved a generic Torelli theorem for the quintic-mirror family. In particular, he proved that the period map for the quintic-mirror $\mathbb{P}^1 - \{0, 1, \infty\} \to D/\Gamma$ is of degree one. Donagi proved in [14] that the period map for smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$ is generically injective except for the cases when $n = 3, d = 3$; when d divides $n + 2$, and when $d = 4$ and $4|n$. Using analogous techniques in [14], Konno proved generic Torelli theorem for more hypersurfaces of more general homogenous spaces in [35]. Donagi and Tu also proved generic Torelli for weighted projective hypersufaces in [15]. There are also more results about generic Torelli theorems that one can find in [54] for certain types of hypersurfaces and prym varieties [17] and so on.

Our result asserts the global injectivity of the period map Φ from the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi–Yau type manifolds to period domain D of polarized

Hodge structures.

Theorem 1.0.1 *The period map $\Phi : \mathcal{T} \to D$ on the Teichmüller space of polarized and marked Calabi–Yau type manifolds is injective.*

In the theorem, $\mathcal{T}$ is the Teichmüller space, which is the deformation of the complex structure on the polarized and marked Calabi–Yau type manifold M, and D is the period domain of the Hodge structures of polarized and marked Calabi–Yau type manifolds of weight n. We will show that the Teichmüller space is precisely the universal cover of the smooth moduli space $\mathcal{Z}_m$ of polarized Calabi–Yau type manifolds with level m structure.

Our proof of this theorem consists of two main geometric constructions, namely the holomorphic affine structure on the Teichmüller space and the Hodge metric completion space $\mathcal{T}^H$ of the Teichmüller space. Then we use these constructions to prove the theorem as follows. We extend the affine structure on the Teichmüller space $\mathcal{T}$ to the Hodge metric completion space $\mathcal{T}^H$, which together with the completeness of $\mathcal{T}^H$ allows us to connect any two points with a straight line with respect to the affine structure on $\mathcal{T}^H$. We can thus reduce the question of global injectivity of the extended period map to the question of injectivity on any such straight line. This can then be proven using the local injectivity of the period map.

In the construction of the holomorphic affine structure on the Teichmüller space of polarized and marked Calabi–Yau type manifolds, we mainly rely on the local injectivity and horizontal property of the period map for Calabi–Yau type manifolds [22, 48] and structures of the period domain [25, 48]. We outline the construction of the holomorphic affine structure as follows. We fix a base point $p \in \mathcal{T}$ with its Hodge structure $\{H_p^{k,n-k}\}_{k=n-s}^{s}$ as the reference Hodge structure. With this fixed base point, we identify the unipotent subgroup N_+ with its orbit in $\check{D}$, where $\check{D}$ is the compact dual of D, and define $\check{\mathcal{T}} = \Phi^{-1}(N_+) \subseteq \mathcal{T}$. We first show that $\Phi : \check{\mathcal{T}} \to N_+ \cap D$ is a bounded map with respect to the Euclidean metric on N_+, and that $\mathcal{T} \backslash \check{\mathcal{T}}$ is an analytic subvariety. Here we use the structures of the period domain, and the horizontal property of the period map. Then by applying Riemann extension theorem, we conclude that $\Phi(\mathcal{T}) \subseteq N_+ \cap D$. Using this property, we then show Φ induces a global holomorphic map $\tau : \mathcal{T} \to \mathbb{C}^N$. This global map τ actually gives a local coordinate map around each point in $\mathcal{T}$ by mainly using the local injectivity of the period map. Thus $\tau : \mathcal{T} \to \mathbb{C}^N$ induces a global holomorphic affine structure on $\mathcal{T}$.

The most obvious approach of constructing the Hodge metric completion space of Teichmüller space is to directly take the metric completion of $\mathcal{T}$. Unfortunately, with this approach it becomes very difficult to prove essential properties of $\mathcal{T}^H$, such as smoothness and affineness. Instead we pass through the Hodge metric completion $\mathcal{Z}_m^H$ of $\mathcal{Z}_m$, and work with the universal cover $\mathcal{T}_m^H$ of $\mathcal{Z}_m^H$, where $\mathcal{Z}_m$ is the smooth moduli space of polarized Calabi–Yau type manifolds with level m structure. Let us outline this construction as follows. Denote the period map on the smooth moduli space $\mathcal{Z}_m$ by $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma$, where Γ denotes the global monodromy group which acts properly and discontinuously on D. Let $\pi_m : \mathcal{T} \to \mathcal{Z}_m$ denote the universal covering map. Then $\Phi : \mathcal{T} \to D$ is the lifting of $\Phi_{\mathcal{Z}_m} \circ \pi_m$. As we can show that both Φ and $\Phi_{\mathcal{Z}_m}$ are locally injective and we know that there is Hodge metric h on D, which is a complete homogeneous metric and is studied in [25], the pull-back of h on $\mathcal{Z}_m$ and $\mathcal{T}$ via $\Phi_{\mathcal{Z}_m}$ and Φ respectively are both well-defined Kähler metrics, and they are still called Hodge metrics. Let $\mathcal{Z}_m^H$ be the Hodge metric completion of the smooth moduli space $\mathcal{Z}_m$. Then we take the universal cover $\mathcal{T}_m^H$ of $\mathcal{Z}_m^H$, with the universal covering map $\pi_m^H : \mathcal{T}_m^H \to \mathcal{Z}_m^H$. We we will show that $\mathcal{Z}_m^H$ is a connected and complete smooth complex manifold, which implies that $\mathcal{T}_m^H$ is a connected and simply connected complete smooth complex manifold. We also obtain the following commutative diagram:

$$\begin{tikzcd} \mathcal{T} \arrow[r, "i_m"] \arrow[d, "\pi_m"] & \mathcal{T}_m^H \arrow[r, "\Phi_m^H"] \arrow[d, "\pi_m^H"] & D \arrow[d, "\pi_D"] \\ \mathcal{Z}_m \arrow[r, "i"] & \mathcal{Z}_m^H \arrow[r, "\Phi_{\mathcal{Z}_m}^H"] & D/\Gamma, \end{tikzcd} \tag{1.1}$$

where $\Phi_{\mathcal{Z}_m}^H$ is the natural extension map of the period map $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma$, i is the inclusion map, i_m is a lifting of $i \circ \pi_m$, and Φ_m^H is a lifting of $\Phi_{\mathcal{Z}_m}^H \circ \pi_m^H$. Let us denote $\mathcal{T}_m := i_m(\mathcal{T})$ and $\Phi_m := \Phi_m^H|_{\mathcal{T}_m}$. We then show the properties that there are induced holomorphic affine structure on the space $\mathcal{T}_m$ and extended holomorphic affine structure on

$\mathcal{T}^H_{_m}$ from the affine structure on $\mathcal{T}$. At this point, we carry out in detail the argument described at the outline of the proof, that is, we use the completeness and affineness of $\mathcal{T}^H_{_m}$ and the local injectivity of Φ^H_m to deduce the global injectivity of Φ^H_m. However, to complete the proof of the injectivity of $\Phi : \mathcal{T} \to D$, it remains to show that i_m is injective since $\Phi = \Phi^H_{_m} \circ i_m$. To achieve this, we first show that $\mathcal{T}^H_{_m}$ and $\mathcal{T}^H_{m'}$ are biholomorphic for any m, m', which is a corollary of the injectivity of $\Phi^H_{_m}$, and then use the markings of the Calabi–Yau type manifolds and the fact that $\mathcal{T}$ is simply connected to show i_m is an embedding.

The property that we obtained above, which states that $\mathcal{T}^H_{_m}$ and $\mathcal{T}^H_{m'}$ are biholomorphic for any m, m', allows us to define the complete complex manifold $\mathcal{T}^H$ with respect to the Hodge metric by $\mathcal{T}^H = \mathcal{T}^H_{_m}$, the holomorphic map $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}^H$ by $i_{\mathcal{T}} = i_m$, and the extended period map $\Phi^H : \mathcal{T}^H \to D$ by $\Phi^H = \Phi^H_{_m}$ for any m. By these definitions, we conclude that $\mathcal{T}^H$ is a complex affine manifold and that $\Phi^H : \mathcal{T}^H \to N_+ \cap D$ is a holomorphic injection.

As a direct corollary of the global Torelli theorem on the Teichmüller space of polarized and marked Calabi–Yau type manifolds, we prove another important result for the Hodge metric completion space of the Teichmüller space.

Theorem 1.0.2 *The completion space $\mathcal{T}^H$ is a bounded domain of holomorphy in $\mathbb{C}^N$; thus there exists a complete Kähler–Einstein metric on $\mathcal{T}^H$.*

To prove this theorem, we construct a plurisubharmonic exhaustion function on $\mathcal{T}^H$ by using Proposition 5.4.5 ([25]), the completeness of $\mathcal{T}^H$, and the injectivity of Φ^H. This shows that $\mathcal{T}^H$ is a bounded domain of holomorphy in $\mathbb{C}^N$. Then the existence of the Kähler-Einstein metric follows directly from a theorem in Mok–Yau in [42].

As an application of the global Torelli theorem on the Teichmüller space of polarized and marked Calabi–Yau type manifolds, we will prove several properties of the period map on the moduli space under given assumptions on the moduli space.

Proposition 1.0.3 *Let $\mathcal{M}$ be the moduli space of polarized Calabi–Yau type manifolds. If $\mathcal{M}$ is smooth and the global monodormy group Γ acts on D freely, then the period map*

$\Phi_{\mathcal{M}} : \mathcal{M} \to D/\Gamma$ *is a covering map from* $\mathcal{M}$ *to its image in* D/Γ*. As a consequence, if the period map* $\Phi_{\mathcal{M}}$ *is generically injective, then it is globally injective.*

Consider the homomorphism induced by the action of the subgroup MCG_L on the global monodromy group,

$$\sigma : \mathrm{MCG}_L \to \Gamma.$$

where MCG_L is a subgroup of the mapping class group containing the diffeomorphisms on the given Calabi–Yau type manifold that preserve the polarization L. With the above notation, we have the following result provided the kernel of σ is finite.

Corollary 1.0.4 *If the kernel of the map* $\sigma : MCG_L \to \Gamma$ *is finite, then there exists a finite cover* $\mathcal{M}'$ *such that the map* $\Phi' : \mathcal{M}' \to D/\Gamma$ *is injective.*

As another application, we consider the cases when the period domain is relatively small, especially for the case when the period domain has the same dimension as the Teichmüller space. We will first give a simple proof of the surjectivity of the period maps of K3 surfaces and cubic fourfolds.

Proposition 1.0.5 *Let* $\mathcal{T}$ *be the Teichmüller space of polarized and marked Calabi–Yau type manifolds. If* $\dim \mathcal{T}^H = \dim D$*, then the extended period map* $\Phi^H : \mathcal{T}^H \to D$ *is surjective. In particular, let* $\mathcal{T}^H$ *be the Hodge metric completion space of the Teichmüller space of polarized and marked K3 surfaces or cubic fourfolds. Then the extended period map* $\Phi^H : \mathcal{T}^H \to D$ *is surjective.*

We also consider the case when there is a modified period map such that the period domain can be realized as a ball of the same dimension as the Teichmüller space. To be more precise, suppose we can define a period map $\Phi_{\mathcal{M}} : \mathcal{M} \to B/\Gamma$ from the moduli space $\mathcal{M}$ of polarized projective manifolds to a ball quotient B/Γ such that $\dim_{\mathbb{C}} \mathcal{M} = \dim_{\mathbb{C}} B$ and that the period map $\Phi_{\mathcal{M}}$ is locally injective. And there exists lifting map $\Phi : \mathcal{T} \to B$ from Teichmüller space of polarized and marked projective manifolds to the period domain. Then with the above assumption, we have the following conclusion.

Proposition 1.0.6 *There exists an induced affine structure on the Teichmüller space from the affine structure on B, and that the period map $\Phi : \mathcal{T} \to B$ is injective.*

We remark that examples satisfy the above assumptions may be found in [2, 3, 4, 6, 39] for cubic surfaces and cubic threefolds.

There are also results about the existence of a global section on the Hodge bundles of Teichmüller spaces, as well as a global splitting property of the Hodge bundles by using the affine structure on the Teichmüller space. One may refer to [12] for more details.

This thesis is organized as follows. In Chapter 2, we review basic definitions and properties of Teichmüller space of polarized and marked projective manifolds. In particular, we focus on the study of the Teichmüller space of polarized and marked Calabi–Yau type manifolds. In Chapter 3, we will review some basic properties of polarized Hodge structures on projective manifolds. In Chapter 4, we study the properties of period map on the Teichmüller space of polarized and marked Calabi–Yau type manifolds, including the crucial boundedness property of the period map. In Chapter 5, we prove a global Torelli theorem of Calabi–Yau type manifolds. We mainly focus on the two geometric constructions: holomorphic affine structure and Hodge metric completion of the Teichmüller space. As a corollary, we also prove that the Hodge metric completion of the Teichmüller space is a domain of holomorphy and it admits a Kähler–Einstein metric. In Chapter 6, we give some applications of the result of the global Torelli theorem for Calabi–Yau type manifolds, and adopt the technique we use in the proof of global Torelli theorem to study periods of more general projective manifolds. In Appendix A, we provide the condition for smooth projective hypersurface of Calabi–Yau type; we also provide alternate proofs for some lemmas and proposition we have given throughout the thesis; and we also provide two basic topological lemmas we will use in the construction of Hodge metric completion space of the Teichmüller space.

CHAPTER 2

Teichmüller space of polarized and marked projective manifolds

In this chapter, we review basic definitions and properties about Teichmüller spaces of polarized and marked projective manifolds. In particular, we study the Teichmüller space of polarized and marked Calabi–Yau type manifolds. In §2.1, we briefly review the definitions of moduli space and Teichmüller space of projective manifolds. In §2.2, we review properties of Kuranishi families and versal families over the moduli spaces and Teichmüller space of projective manifolds. In §2.3, we restrict our interest to Calabi–Yau type manifolds. We first recall the definition of projective manifolds of Calabi–Yau type, which was originally introduced in [11] and a generalization of Fano manifolds of Calabi–Yau type in [26]. We then study the properties of Teichmüller space of polarized and marked Calabi–Yau type manifolds. In particular, we will show that Teichmüller space of polarized and marked Calabi–Yau type manifolds is a simply connected manifold, which is originally given in [11].

2.1 Moduli spaces and Teichmüller spaces

Let M be a complex projective compact manifold with $\dim_{\mathbb{C}} M = n$. Let L be an ample line bundle over M. We call the pair (M, L) a *polarized* projective manifold. Let $\{\gamma_1, \cdots, \gamma_{b^n}\}$ be a basis of the integral middle homology group modulo torsion $H_n(M,\mathbb{Z})/Tor$. We call the triple $(M, L, \{\gamma_1, \cdots, \gamma_{b^n}\})$ a *polarized and marked* projective manifold. For any integer $m \geq 3$, we call a basis $([\gamma_1], \cdots, [\gamma_{b^n}])$ of the quotient space $(H_n(M,\mathbb{Z})/\mathrm{Tor})/m(H_n(M,\mathbb{Z})/\mathrm{Tor})$ *a level m structure* on the polarized projective manifold.

The *moduli space* $\mathcal{M}$ of polarized complex structures on a given differential manifold X is a complex analytic space consisting of biholomorphically equivalent pairs (M, L), where M is a complex manifold diffeomorphic to X and L is an ample line bundle on M. Let us denote by $[M, L]$ the point in $\mathcal{M}$ corresponding to a pair (M, L), If there is a biholomorphic map f between M and M' with $f^*L' = L$, then $[M, L] = [M', L'] \in \mathcal{M}$.

Let $\mathcal{Z}_m$ be the moduli space of polarized projective manifolds with level m structure, which consists of biholomorphically equivalent triples $(M, L, ([\gamma_1], \cdots, [\gamma_{b^n}]))$, where M is a complex manifold diffeomorphic to the given differential manifold X, L is an ample line bundle on M, and $([\gamma_1], \cdots, [\gamma_{b^n}])$ is a level m structure on M.

Let $(M, L, \{\gamma_1, \cdots, \gamma_{b^n}\})$ be a polarized and marked projective manifold. We define the *Teichmüller space* $\mathcal{T}$ of polarized and marked projective manifold to be a complex analytic space consisting of biholomorphically equivalent triples of $(M, L, \{\gamma_1, \cdots, \gamma_{b^n}\})$. To be more precise, for two triples $(M, L, \{\gamma_1, \cdots, \gamma_{b^n}\})$ and $(M', L', \{\gamma_1', \cdots, \gamma_{b^n}'\})$, if there exists a biholomorphic map $f : M \to M'$ with

$$f^*L' = L, \quad \text{and} \quad f^*\gamma_i' = \gamma_i \ \text{ for } \ 1 \leq i \leq b^n,$$

then $[M, L, \{\gamma_1, \cdots, \gamma_{b^n})\}] = [M', L', \{\gamma_1', \cdots, \gamma_{b^n}')\}] \in \mathcal{T}$. By this definition, we know that the Teichmüller space $\mathcal{T}$ is a covering space of $\mathcal{Z}_m$, and we will denote the covering map by $\pi_m : \mathcal{T} \to \mathcal{Z}_m$.

2.2 Kuranishi family and versal family over the Teichmüller space

Let $\mathcal{U}, \mathcal{T}$ be a complex manifolds, and $\pi : \mathcal{U} \to \mathcal{T}$ a holomorphic map. Let $M_p =: \pi^{-1}(p)$ denote the fibre of π above the point $p \in \mathcal{T}$. We say that $\pi : \mathcal{U} \to \mathcal{T}$ is a *family of complex manifolds* if π is a proper holomorphic submersion.

A family of compact complex manifolds $\pi : \mathcal{U} \to \mathcal{T}$ is *complete* at a point $p \in \mathcal{T}$ if it satisfies the following condition:

- If given a complex analytic family $\iota : \mathcal{V} \to \mathcal{S}$ of compact complex manifolds with a point $s \in \mathcal{S}$ and a biholomorphic map $f_0 : V = \iota^{-1}(s) \to U = \pi^{-1}(p)$, then there exists a holomorphic map g from a neighbourhood $\mathcal{N} \subseteq \mathcal{S}$ of the point s to $\mathcal{T}$ and a holomorphic map $f : \iota^{-1}(\mathcal{N}) \to \mathcal{U}$ with $\iota^{-1}(\mathcal{N}) \subseteq \mathcal{V}$ such that they satisfy that $g(s) = p$ and $f|_{\iota^{-1}(s)} = f_0$, with the following commutative diagram

$$\begin{tikzcd} \iota^{-1}(\mathcal{N}) \arrow[r, "f"] \arrow[d, "\iota"] & \mathcal{U} \arrow[d, "\pi"] \\ \mathcal{N} \arrow[r, "g"] & \mathcal{T}. \end{tikzcd}$$

Moreover, $\pi : \mathcal{U} \to \mathcal{T}$ is said to be *versal* at the point p, if the following condition is satisfied,

- For all g satisfying the above condition, the tangent map $(dg)_s$ is uniquely determined.

If a family $\pi : \mathcal{U} \to \mathcal{T}$ is versal at every point in $\mathcal{T}$, then it is a *versal family* on $\mathcal{T}$. If a complex analytic family $\pi : \mathcal{U} \to \mathcal{T}$ of a compact complex manifold is complete at each point of $\mathcal{T}$ and versal at a point $p \in \mathcal{T}$, then the family $\pi : \mathcal{U} \to \mathcal{T}$ is called a Kuranishi family of the complex manifold $M_p = \pi^{-1}(p)$. If the family is complete at each point in a neighbourhood of $p \in \mathcal{T}$ and versal at p, then the family is called a *local Kuranishi family* at $p \in \mathcal{T}$. We also refer the reader to [53, page 8-10],[45, page 94] or [63, page 19] for more details about versal families and local Kuranishi families.

Let (M, L) be a polarized compact projective manifold. In this thesis, we assume that there exists $m_0 \in \mathbb{Z}$ with $m_0 \geq 3$ such that for any $m \geq m_0$ the moduli space $\mathcal{Z}_m$ is a connected quasi-projective smooth complex manifolds with a versal family $\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m$ of projective manifolds with level m structures, containing M as a fiber and polarized by an ample line bundle $\mathcal{L}_{\mathcal{Z}_m}$ on $\mathcal{X}_{\mathcal{Z}_m}$.

Remark 2.2.1 *One notices that this assumption can be satisfied for a large range of interesting manifolds, such as Calabi–Yau manifolds, projective hypersurfaces of Calabi–Yau type, and many complete intersections of Calabi–Yau type, which we will be our main objects of study.*

Recall that by the definition of the Teichmüller space of polarized and marked projective manifolds, the Teichmüller space is a covering space of $\mathcal{Z}_m$, and we denote the covering map by $\pi_m : \mathcal{T} \to \mathcal{Z}_m$. Thus we have the pull-back family $\pi : \mathcal{U} \to \mathcal{T}$ of $\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m$. With the above assumption on $\mathcal{Z}_m$, we have the following proposition about the Teichmüller space.

Proposition 2.2.2 *The Teichmüller space $\mathcal{T}$ of polarized and marked projective manifold is a smooth and connected complex manifold and the family*

$$\pi : \mathcal{U} \to \mathcal{T},$$

containing M as a fiber, is local Kuranishi at each point of $\mathcal{T}$.

Proof Recall that there is a natural covering map $\pi_m : \mathcal{T} \to \mathcal{Z}_m$ for any $m \geq m_0$. Thus the Teichmüller space $\mathcal{T}$ is a smooth and connected complex manifold as $\mathcal{Z}_m$ is a connected smooth complex manifold. Since the family $\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m$ is a versal family at each point of $\mathcal{Z}_m$ and that π_m is locally biholomorphic, the pull-back family via π_m is also versal at each point of $\mathcal{T}$, that is, $\pi : \mathcal{U} \to \mathcal{T}$ is local Kuranishi at each point of $\mathcal{T}$. ∎

2.3 Calabi–Yau manifolds and Calabi–Yau type manifolds and the Teichmüller space

Our main objects throughout this thesis are Calabi–Yau type manifolds. We first recall the definition of Calabi–Yau type manifolds, which is a generalization of Calabi–Yau manifolds.

Definition 2.3.1 *A compact simply connected projective manifold M of complex dimension n is called a Calabi–Yau type manifold, if it satisfies the following conditions:*

(i). *there exists some $[n/2] < s \leq n$, such that $h^{s,n-s}(M) = 1$ and $h^{s',n-s'}(M) = 0$ for any $s' > s$;*

(ii). *$H^{s,n-s}(M) = H^{s,n-s}_{pr}(M)$;*

(iii). *for any generator* $[\Omega] \in H^{s,n-s}(M)$*, the contraction map* $\lrcorner : H^{0,1}(M, T^{1,0}M) \to H^{s-1,n-s+1}_{pr}(M)$ *defined via* $[\phi] \mapsto [\phi \lrcorner \Omega]$ *is an isomorphism,*

where $H^{s,n-s}_{pr}(M)$ *and* $H^{s-1,n-s+1}_{pr}(M)$ *are the primitive cohomology group of corresponding type, which will be defined Chapter 3.*

Definition 2.3.2 *A compact projective complex manifold* M *of complex dimension* $n \geq 2$ *is called a Calabi–Yau manifold, if it has a trivial canonical bundle and satisfies* $H^i(M, \mathcal{O}_M) = 0$ *for any* $0 < i < n$.

Remark 2.3.3 *One notices that a Calabi–Yau manifold is in fact a special example of Calabi–Yau type manifold. Indeed, if* M *is a Calabi–Yau manifold of dimension* n *as defined above, then* M *is simple connected and* (i). $\dim H^{n,0}(M, \mathbb{C}) = 1$*;* (ii). *since* $H^2(M, \mathcal{O}_M) = 0$ *for a Calabi–Yau manifold, we have* $H^{n,0}_{pr}(M) = H^{n,0}(M)$ *and* $H^{n-1,1}_{pr}(M) = H^{n-1,1}(M)$*;* (iii) *by local Torelli theorem for a Calabi–Yau manifold, one has that any generator* $[\Omega] \in H^{n,0}(M)$*, the contraction map* $\lrcorner : H^{0,1}(M, T^{1,0}M) \to H^{n-1,1}(M)$, $[\phi] \mapsto [\phi \lrcorner \Omega]$ *is an isomorphism.*

Example 2.3.4 (Calabi–Yau) *It is well-known that* $K3$ *surfaces are Calabi–Yau manifolds of dimension* $n = 2$*. Non-singular quintic threefolds in* $\mathbb{P}^4$ *and the mirror family of quintic threefolds are Calabi–Yau manifolds of dimension* $n = 3$*, which have both been investigated in details. And for the example of higher dimensional Calabi–Yau manifolds, the zero set of non-singular homogeneous degree* d *polynomial in* $\mathbb{P}^{n+1}$ *such that* $d = n + 2$ *is a compact Calabi–Yau manifold of dimension* n.

Example 2.3.5 (Calabi–Yau type) *According to Proposition A.1.2 in Appendix A, one may conclude that hypersurfaces in* $\mathbb{P}^{n+1}$ *that are defined by the zero set of non-singular homogeneous degree* d *polynomial in* $\mathbb{P}^{n+1}$ *satisfying* $d | n + 2$ *is a Calabi–Yau type manifold of dimension* n*. Hence, Cubic fourfolds are obviously Calabi–Yau type manifolds. Moreover, one may also refer to [26] for many interesting examples of Calabi–Yau type manifolds of Fano type.*

Consider a polarized and marked Calabi–Yau type manifold $(M, L, \{\gamma_1, \cdots, \gamma_{b^n}\})$. Let $\mathcal{Z}_m$ be the moduli space of polarized Calabi–Yau type manifolds with level m structure as defined in the previous section. We assume that there exists $m_0 \in \mathbb{Z}$ such that for any $m \geq m_0$ the moduli space $\mathcal{Z}_m$ is a connected quasi-projective smooth complex manifolds with a versal family $\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m$ of projective manifolds of Calabi–Yau type with level m structures, containing M as a fiber and polarized by an ample line bundle $\mathcal{L}_{\mathcal{Z}_m}$ on $\mathcal{X}_{\mathcal{Z}_m}$.

Let $\mathcal{T}$ be the Teichmüller space of the polarized and marked Calabi–Yau type manifolds, then $\mathcal{T}$ is a covering space of $\mathcal{Z}_m$ with the covering map $\pi_m : \mathcal{T} \to \mathcal{Z}_m$. Thus there exists the pall-back family $\pi : \mathcal{U} \to \mathcal{T}$ via π_m. By Proposition 2.2.2, we conclude that the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi–Yau type manifold is a smooth and connected complex manifold and the family

$$\pi : \mathcal{U} \to \mathcal{T},$$

containing M as a fiber, is local Kuranishi at each point of $\mathcal{T}$.

Remark 2.3.6 *As a consequence, since the versal family $\pi : \mathcal{U} \to \mathcal{T}$ is local Kuranishi at any point $p \in \mathcal{T}$, the Kodaira-Spencer map*

$$\kappa : T_p^{1,0}\mathcal{T} \to H^{0,1}(M_p, T^{1,0}M_p) \quad \textit{for any} \quad p \in \mathcal{T}$$

is an isomorphism. Therefore, one may conclude that $\dim \mathcal{T} = \dim H^{0,1}(M_p, T^{1,0}(M_p)) = \dim H_{pr}^{s-1,n-s+1}(M_p) = N$. *Here we refer the reader to [33, 34] for more details about deformation of complex structures and the Kodaira-Spencer map.*

To close this chapter, we will show that the Teichmüller space of polarized and marked Calabi–Yau type manifolds is simply connected. To proceed, we will first show a lemma about extensions of families.

Assume that $\tilde{\mathcal{T}}$ is the universal cover of $\mathcal{T}$ with the covering map $\pi_{\tilde{\mathcal{T}}} : \tilde{\mathcal{T}} \to \mathcal{T}$. Then for each point $p = [M, L, \{\gamma_1, \cdots, \gamma_{b^n}\}] \in \mathcal{T}$, the preimage $\pi_{\tilde{\mathcal{T}}}^{-1}(p) = \{p_i |\ i \in I\} \subseteq \tilde{\mathcal{T}}$ satisfies that $|I| \geq 1$. Let $\tilde{\pi} : \tilde{\mathcal{U}} \to \tilde{\mathcal{T}}$ be the pull back family of $\pi : \mathcal{U} \to \mathcal{T}$ with the following commutative diagram,

$$\begin{tikzcd}
\tilde{\mathcal{U}} \arrow[r] \arrow[d, "\tilde{\pi}"] & \mathcal{U} \arrow[d, "\pi"] \\
\tilde{\mathcal{T}} \arrow[r, "\pi_{\tilde{\mathcal{T}}}"] & \mathcal{T}.
\end{tikzcd}$$

Then $\tilde{\pi} : \tilde{\mathcal{U}} \to \tilde{\mathcal{T}}$ is also a versal family of polarized and marked Calabi–Yau type manifolds.

On the other hand, for a polarized Calabi–Yau type manifold (M, L), let

$$\mathrm{Aut}(M, L) = \{\alpha : \ M \to M | \ \alpha \text{ is a biholomorphism, and } \alpha^* L = L\}$$

be the group of biholomorphic maps on M preserving the polarization L. We have a natural representation of $\mathrm{Aut}(M, L)$,

$$\sigma : \mathrm{Aut}(M, L) \to \mathrm{Aut}(H_n(M, \mathbb{Z})/\mathrm{Tor}), \qquad \alpha \mapsto \alpha^*.$$

As $H^n(M, \mathbb{C})$ is the dual space of $H_n(M, \mathbb{C})$, we may also view α^* as an automorphism of $H^n(M, \mathbb{C})$ via the duality.

Let us now prove the following lemma. We remark that we mainly adopt the analogous arguments of Lemma 2.6 in [55] in the first part of the proof of the following lemma.

Lemma 2.3.7 *For any $\alpha \in \ker \sigma$, there is an extension $\tilde{\alpha}$ on $\tilde{\mathcal{U}}$ leaving the base space $\tilde{\mathcal{T}}$ fixed and also the polarization fixed on each fiber of the family.*

Proof Let $p \in \tilde{\mathcal{T}}$ with $\tilde{\pi}^{-1}(p) = [M, L, \{\gamma_1, \cdots, \gamma_{b^n}\}]$. Since the family $\tilde{\pi} : \tilde{\mathcal{U}} \to \tilde{\mathcal{T}}$ is local Kuranishi at any point, there exists a neighborhood U_p of $p \in \tilde{\mathcal{T}}$ with holomorphic morphisms $\tilde{\alpha} : \tilde{\pi}^{-1}(U_p) \to \tilde{\mathcal{U}}$ and $f : U_p \to \tilde{\mathcal{T}}$ such that the following diagram is commutative,

$$\begin{tikzcd}
\tilde{\pi}^{-1}(U_p) \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tilde{\pi}"] & \tilde{\mathcal{U}} \arrow[d, "\tilde{\pi}"] \\
U_p \arrow[r, "f"] & \tilde{\mathcal{T}}.
\end{tikzcd}$$

To see $\tilde{\alpha}$ leaves the base space $\tilde{\mathcal{T}}$ fixed, it is sufficient to show that f is the identity map on U_p. Notice that $f(p) = p$ from the definition of f. Suppose towards a contradiction that f

were not the identity map on U_p. Then the tangent map $f_* : T_pU_p \to T_p\tilde{\mathcal{T}}$ is not identity either. By the analogous discussion in Remark 2.3.6, we know that the Kodaria-Spencer map $\kappa : T_pU_p = T_p^{1,0}\tilde{\mathcal{T}} \to H^{0,1}(M_p, T^{1,0}M_p)$ is an isomorphism. Moreover, since the contraction map $\lrcorner : H^{0,1}(M_p, T^{1,0}M) \to H^{s-1,n-s+1}_{pr}(M_p)$ is also an isomorphism, we have the following commutative diagram,

$$\begin{array}{ccc}
T_p^{1,0}\tilde{\mathcal{T}} & \xrightarrow{f_*^{-1}} & T_p^{1,0}U_p \\
\downarrow & & \downarrow \\
H^{0,1}(M, T^{1,0}M) & & H^{0,1}(M, T^{1,0}M) \\
\downarrow & & \downarrow \\
H^{s-1,n-s+1}_{pr}(M) & \xrightarrow{\alpha^*} & H^{s-1,n-s+1}_{pr}(M).
\end{array}$$

But $\alpha \in \ker(\sigma)$ implies the map $\alpha^* : H^{s-1,n-s+1}_{pr}(M) \to H^{s-1,n-s+1}_{pr}(M)$ is the identity map. This contradicts the assumption that f_* is not the identity map.

It is not hard to show that for each $q \in U_p$ the biholomorphic map $\tilde{\alpha}_q$ on the fiber M_q preserves the polarization L. Indeed, because $H^2(M, \mathbb{Z})$ is a discrete group, we have

$$c_1(\tilde{\alpha}_q^* L) = c_1(\tilde{\alpha}_p^* L) = c_1(L) \quad \text{for any} \quad q \in U_p,$$

where c_1 denotes the first Chern class. Since M is simply connected, holomorphic line bundles on M are uniquely determined by the first Chern class. Therefore we conclude that $\tilde{\alpha}_q^*(L) = L$ for any $q \in U_q$.

Let us define a sheaf $\Im$ on the base space $\tilde{\mathcal{T}}$ as follows: for any open set $U \subset \tilde{\mathcal{T}}$, we assign the group $\Im(U)$ to be all the biholomorphic maps $\alpha_{_U} : \tilde{\pi}^{-1}(U) \to \tilde{\pi}^{-1}(U)$ which leaves the open set U fixed and preserving the polarization on each fiber. In other words, for any $\alpha_{_U} \in \Im(U)$, we have the following commutative diagram,

$$\begin{array}{ccc}
\tilde{\pi}^{-1}(U) & \xrightarrow{\alpha_U} & \tilde{\pi}^{-1}(U) \\
\downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \tilde{\pi}} \\
U & \xrightarrow{id} & U;
\end{array}$$

and the restriction of $\alpha_{_U}$ on the fiber $M_q = \tilde{\pi}^{-1}(q)$ preserves the polarization L over M_q for any $q \in U$. If $V \subseteq U$ is open, then the restriction map of the sheaf is given by

$$res: \Im(U) \to \Im(V), \qquad \alpha_U \to \alpha_U|_{\tilde{\pi}^{-1}(V)}.$$

From the local extension result discussed above, we have that for any point $p \in \tilde{\mathcal{T}}$, there exists a neighborhood $U_p \subset \tilde{\mathcal{T}}$ such that any $\alpha \in \ker \sigma$ on M_p can be extended to the family $\tilde{\pi}^{-1}(U_p)$. This means the restriction map

$$res: \Im(U_p) \to \ker \sigma, \qquad \alpha_{_U} \mapsto \alpha_{_U}|_{M_p}$$

is an isomorphism. Therefore the sheaf $\Im$ is a locally constant sheaf. Using the fact that $\tilde{\mathcal{T}}$ is simply connected and Proposition 3.9 in [66], one concludes that $\Im$ is a constant sheaf. This means $\Im(\tilde{\mathcal{T}}) = \ker \sigma$. Therefore, for each point $p \in \tilde{\mathcal{T}}$ and $\alpha \in \ker \sigma$, there is a global section $\tilde{\alpha} \in \Im(\tilde{\mathcal{T}})$ such that $\tilde{\alpha}|_{M_p} = \alpha$. By the definition of the sheaf $\Im$, we have the commutative diagram,

$$\begin{tikzcd} \tilde{\mathcal{U}} \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tilde{\pi}"] & \tilde{\mathcal{U}} \arrow[d, "\tilde{\pi}"] \\ \tilde{\mathcal{T}} \arrow[r, "id"] & \tilde{\mathcal{T}}. \end{tikzcd}$$

And restricted to each fiber M_q the morphism $\tilde{\alpha}|_{M_q}$ preserves the polarization L. ∎

Corollary 2.3.8 *The action of ker(σ) on $\tilde{\mathcal{T}}$ is trivial.*

Proof For each element $\alpha \in \ker(\sigma)$, we have a global extension $\tilde{\alpha}$ acts on the family $\tilde{\mathcal{U}}$ with the commutative diagram

$$\begin{tikzcd} \tilde{\mathcal{U}} \arrow[r, "\tilde{\alpha}"] \arrow[d, "\pi"] & \tilde{\mathcal{U}} \arrow[d, "\pi"] \\ \tilde{\mathcal{T}} \arrow[r, "\bar{f}"] & \tilde{\mathcal{T}}, \end{tikzcd}$$

where α acts on $\tilde{\mathcal{T}}$ by the holomorphic map $\bar{f}: \tilde{\mathcal{T}} \to \tilde{\mathcal{T}}$. Then Lemma 2.3.7 implies that $\bar{f}$ is the identity map. Therefore the action of α on $\tilde{\mathcal{T}}$ is trivial. ∎

Theorem 2.3.9 *The Teichmüller space of polarized and marked Calabi–Yau type manifolds is simply connected.*

Proof Suppose towards a contradiction that $\mathcal{T}$ is not simply connected. Recall that $\tilde{\mathcal{T}}$ is the universal cover of $\mathcal{T}$ with a covering map $\pi_{\tilde{\mathcal{T}}} : \tilde{\mathcal{T}} \to \mathcal{T}$ with the pull-back family $\tilde{\pi} : \tilde{\mathcal{U}} \to \tilde{\mathcal{T}}$ and the following commutative diagram

$$\begin{tikzcd} \tilde{\mathcal{U}} \arrow[r] \arrow[d, "\tilde{\pi}"] & \mathcal{U} \arrow[d, "\pi"] \\ \tilde{\mathcal{T}} \arrow[r, "\pi_{\tilde{\mathcal{T}}}"] & \mathcal{T}. \end{tikzcd}$$

Then for each point $p = [M, L, \{\gamma_1, \cdots, \gamma_{b^n}\}] \in \mathcal{T}$, the preimage $\pi_{\tilde{\mathcal{T}}}^{-1}(p) = \{p_i |\ i \in I\} \subseteq \tilde{\mathcal{T}}$ satisfies that $|I| > 1$. For any $i \neq j \in I$, there exists a deck transformation $\alpha : \tilde{\mathcal{T}} \to \tilde{\mathcal{T}}$ of the covering map $\tilde{\pi} : \tilde{\mathcal{T}} \to \mathcal{T}$ such that $\alpha(p_i) = p_j \neq p_i$. Because $\pi_{\tilde{\mathcal{T}}}(p_i) = \pi_{\tilde{\mathcal{T}}}(p_j) = [M, L, \{\gamma_1, \cdots, \gamma_{b^n}\}]$, this α can be viewed as a biholomorphic map on M which preserves the polarization L and the marking $\{\gamma_1, \cdots, \gamma_{b^n}\}$. Therefore $\alpha \in \ker(\sigma)$. However, Corollary 2.3.8 shows that the action of $\ker(\sigma)$ on $\tilde{\mathcal{T}}$ is trivial. Thus $\alpha = \mathrm{Id} : \tilde{\mathcal{T}} \to \tilde{\mathcal{T}}$, which contradicts with the assumption that $\alpha(p_i) = p_j \neq p_i$. ∎

Remark 2.3.10 *As the same construction in Section 2 of [55], we can also realize the Teichmüller space $\mathcal{T}$ as a quotient space of the universal cover of the Hilbert scheme of Calabi–Yau manifolds by special linear group $SL(N+1, \mathbb{C})$. Here the dimension is given by $N + 1 = p(k)$ where p is the Hilbert polynomial of each fiber (M, L) and k satisfies that for any polarized algebraic variety $(\tilde{M}, \tilde{L})$ with Hilbert polynomial p, the line bundle $\tilde{L}^{\otimes k}$ is very ample. Under this construction, Teichmüller space $\mathcal{T}$ is automatically simply connected, and there is a natural covering map $\pi_m : \mathcal{T} \to \mathcal{Z}_m$.*

CHAPTER 3

Classifying space of polarized Hodge structures

In this chapter, we will review some basic properties of classifying space for polarized Hodge structures on projective manifolds. In § 3.1, we recall the definition of polarized Hodge structures. In § 3.2, we review the definition of classifying space of polarized Hodge structures, and some properties for further use. Most of the concepts and results in this chapter are standard in the literature. One may refer to [13, 25] for more details.

3.1 Polarized Hodge structures

Let M be a complex projective manifold of dimension n with background smooth manifold X. We identify the basis of $H_n(M,\mathbb{Z})/\mathrm{Tor}$ to a lattice Λ as in [55]. This gives us a canonical identification of the middle dimensional de Rham cohomology of M to that of the background manifold X, that is,

$$H^n(M,\mathbb{C}) \cong H^n(X,\mathbb{C}),$$

where the coefficient ring can also be $\mathbb{Q}$ or $\mathbb{R}$. As is well-known, this middle dimensional cohomology groups have decomposition according to the Hodge type, that is

$$H^n(M,\mathbb{C}) = \bigoplus_{k=0}^{n} H^{k,n-k}(M,\mathbb{C}),$$

where $H^{k,n-k}(M,\mathbb{C}) = \overline{H^{n-k,k}(M,\mathbb{C})}$ for all $0 \le k \le n$.

Let L be a holomorphic ample line bundle on M and consider the integral Kähler class $[\omega] = c_1(L) \in H^{1,1}(M,\mathbb{Z})$. Since $[\omega]$ is an integral cohomology class, it defines a map

$$L_{[\omega]}:\ H^n(M,\mathbb{Q}) \to H^{n+2}(M,\mathbb{Q}), \qquad A \mapsto [\omega] \wedge A.$$

We denote by $H^n_{pr}(M,\mathbb{Q}) = \ker(L_{[\omega]})$ the *primitive cohomology groups*, where the coefficient ring can also be $\mathbb{R}$ or $\mathbb{C}$. Let $H^{k,n-k}_{pr}(M,\mathbb{C}) = H^{k,n-k}(M) \cap H^n_{pr}(M,\mathbb{C})$ and denote the Hodge numbers $h^{k,n-k} = \dim_{\mathbb{C}} H^{k,n-k}_{pr}(M,\mathbb{C})$ for any $0 \le k \le n$. We then have the Hodge decomposition

$$H^n_{pr}(M,\mathbb{C}) = H^{n,0}_{pr}(M) \oplus \cdots \oplus H^{0,n}_{pr}(M). \tag{3.1}$$

The Poincaré bilinear form Q on $H^n_{pr}(X,\mathbb{Q})$ is defined by

$$Q(u,v) = (-1)^{\frac{n(n-1)}{2}} \int_X u \wedge v$$

for any d-closed n-forms u, v on X. Furthermore, Q is nondegenerate and can be extended to $H^n_{pr}(X,\mathbb{C})$ bilinearly. The bilinear form Q is symmetric if n is even, skew-symmetric if n is odd. Moreover, the Poincaré bilinear form Q also satisfies the Hodge-Riemann relations

$$Q\left(H^{k,n-k}_{pr}(M), H^{l,n-l}_{pr}(M)\right) = 0 \quad \text{unless} \quad k+l=n, \quad \text{and} \tag{3.2}$$

$$\left(\sqrt{-1}\right)^{2k-n} Q\left(v,\bar{v}\right) > 0 \quad \text{for} \quad v \in H^{k,n-k}_{pr}(M) \setminus \{0\}. \tag{3.3}$$

The *polarized Hodge structure of weight* n for the polarized projective manifold (M,L) is given by $(\{H^{k,n-k}_{pr}(M)\}_{k=0}^n, Q)$, where Q is the bilinear form given above. We may simply use $H^{k,n-k}$ to denote $H^{k,n-k}_{pr}(M)$ when there is no confusion.

To the Hodge structure of weight n, let $f^k = \sum_{i=k}^n h^{i,n-i}$, denote $f^0 = m$, and $F^k = F^k(M) = H^{n,0}_{pr}(M) \oplus \cdots \oplus H^{k,n-k}_{pr}(M)$. Then we get a decreasing filtration $H^n_{pr}(M,\mathbb{C}) = F^0 \supseteq \cdots \supseteq F^n$. The filtration has the properties that

$$\dim_{\mathbb{C}} F^k = f^k, \tag{3.4}$$

$$H^n_{pr}(X,\mathbb{C}) = F^k \oplus \overline{F^{n-k+1}}, \quad \text{and} \quad H^{k,n-k}_{pr}(M) = F^k \cap \overline{F^{n-k}}. \tag{3.5}$$

In terms of the Hodge filtration, the Hodge-Riemann relations (3.2) and (3.3) are

$$Q\left(F^k, F^{n-k+1}\right) = 0, \quad \text{and} \tag{3.6}$$

$$Q\left(Cv,\bar{v}\right) > 0 \quad \text{if} \quad v \neq 0, \tag{3.7}$$

where C is the Weil operator given by $Cv = \left(\sqrt{-1}\right)^{2k-n} v$ for $v \in H^{k,n-k}_{pr}(M)$.

3.2 Classifying spaces for polarized Hodge structures

3.2.1 Basic properties of classifying space

The *classifying space* D for polarized Hodge structures of the polarized projective manifold (M, L) with data (3.4) is the space given as follows

$$D = \left\{F^n \subseteq \cdots \subseteq F^0 = H^n_{pr}(X, \mathbb{C}) \mid (3.4), (3.6) \text{ and } (3.7) \text{ hold}\right\}.$$

The compact dual $\check{D}$ of D is

$$\check{D} = \left\{F^n \subseteq \cdots \subseteq F^0 = H^n_{pr}(X, \mathbb{C}) \mid (3.4) \text{ and } (3.6) \text{ hold}\right\}.$$

The classifying space $D \subseteq \check{D}$ is an open subset.

The orthogonal group of the bilinear form Q in the definition of Hodge structure is a linear algebraic group, defined over $\mathbb{Q}$. Let us simply denote $H_{\mathbb{C}} = H^n_{pr}(M, \mathbb{C})$ and $H_{\mathbb{R}} = H^n_{pr}(M, \mathbb{R})$. The group of the $\mathbb{C}$-rational points is

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}}) |\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\},$$

One can check that $G_{\mathbb{C}}$ acts on $\check{D}$ transitively by using elementary arguments in [21]. The group of real points in $G_{\mathbb{C}}$ is

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}}) |\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\},$$

One can check that $G_{\mathbb{R}}$ acts transitively on D by using elementary arguments in [21] as well.

We will use some quotients of $G_{\mathbb{C}}$ and $G_{\mathbb{R}}$ to exhibit $\check{D}$ and D respectively. To do so, let us fix a reference Hodge structure $\{H_0^{k,n-k}\}_{k=0}^n$, which corresponds to a point $o = \{F_0^n \subseteq F_0^{n-1} \subseteq \cdots \subseteq F_0^0\} \in D$, that is, $F_0^k = H_0^{n,0} \oplus \cdots H_0^{k,n-k}$ for any $0 \le k \le n$. We may refer this fix reference Hodge structure as the *base point* or *reference point*. A linear transformation $g \in G_{\mathbb{C}}$ preserves the base point if and only if $gF_0^k = F_0^k$ for any $0 \le k \le n$. Thus one obtains the identification

$$\check{D} \simeq G_{\mathbb{C}}/B \quad \text{with} \quad B = \{g \in G_{\mathbb{C}} |\ gF_0^k = F_0^k, \text{ for any } 0 \le k \le n\}.$$

As $G_{\mathbb{C}}$ is a complex Lie group and B is a closed complex Lie subgroup in $G_{\mathbb{C}}$, the quotient $G_{\mathbb{C}}/B$ has the structure of a complex manifolds. Thus the above isomorphism is actually a complex analytic isomorphism. Similarly, one obtains an analogous identification

$$D \simeq G_{\mathbb{R}}/V \hookrightarrow \check{D}, \text{ with } V = G_{\mathbb{R}} \cap B,$$

where the embedding corresponds to the inclusion $G_{\mathbb{R}}/V = G_{\mathbb{R}}/G_{\mathbb{R}} \cap B \subseteq G_{\mathbb{C}}/B$. Since for any $g \in V$, g preserves the individual subspaces $H_0^{k,n-k}$ and the Weil operator C of the reference Hodge structure $\{H_0^{k,n-k}\}_{k=0}^n$, the subspace V leaves invariant a positive definite Hermitian form. Moreover, as $V = G_{\mathbb{R}} \cap B$ is the intersection of closed subgroups of $Gl(H_{\mathbb{C}})$, V is also closed. Therefore, one may conclude that V is compact.

Let us now describe the Lie algebras of the above Lie groups. The Lie algebra $\mathfrak{g}$ of the complex Lie group $G_{\mathbb{C}}$ can be described as

$$\mathfrak{g} = \{X \in \mathrm{End}(H_{\mathbb{C}}) |\ Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_{\mathbb{C}}\}.$$

It is a simple complex Lie algebra, and it contains $\mathfrak{g}_0 = \{X \in \mathfrak{g} |\ XH_{\mathbb{R}} \subseteq H_{\mathbb{R}}\}$ as a real form, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. With the inclusion $G_{\mathbb{R}} \subseteq G_{\mathbb{C}}$, $\mathfrak{g}_0$ becomes Lie algebra of $G_{\mathbb{R}}$. One observes that the reference Hodge structure $\{H_0^{k,n-k}\}_{k=0}^n$ of $H_{\mathbb{C}}$ induces a Hodge structure of weight zero on $\mathrm{End}(H_{\mathbb{C}})$, namely,

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k,-k} \quad \text{with} \quad \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g} | XH_0^{r,n-r} \subseteq H_0^{r+k,n-r-k}\}$$

is a Hodge structure of weight zero. As a consequence, we have

$$\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, \quad i.e. \quad [\mathfrak{g}^{k,-k}, \mathfrak{g}^{r,-r}] \subseteq \mathfrak{g}^{k+r,-(k+r)}$$

is a morphism of Hodge structure.

Since the Lie algebra $\mathfrak{b}$ of B consists of those $X \in \mathfrak{g}$ that preserves the reference Hodge filtration $\{F_0^n \subseteq \cdots \subseteq F_0^0\}$, one thus has

$$\mathfrak{b} = \bigoplus_{k \geq 0} \mathfrak{g}^{k,-k}.$$

The Lie algebra $\mathfrak{v}_0$ of V is $\mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \bar{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}$. With the above isomorphisms, the holomorphic tangent space of $\check{D} \cong G_{\mathbb{C}}/B$ at the base point is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$. Under this isomorphism, the action of the isotopy group B on the tangent space corresponds to the adjoint action of B on $\mathfrak{g}/\mathfrak{b}$. Thus the holomorphic tangent bundle $T^{1,0}\check{D} \to \check{D}$ coincides with the vector bundle associated to the holomorphic principal bundle

$$B \to G_{\mathbb{C}} \to G_{\mathbb{C}}/B \cong \check{D}$$

by the adjoint representation of B on $\mathfrak{g}/\mathfrak{b}$.

Consider the reference Hodge structure $\{H_0^{k,n-k}\}_{k=0}^n$. Let us take

$$H^+ = \bigoplus_{k\ even} H_0^{k,n-k}, \quad H^- = \bigoplus_{k\ odd} H_0^{k,n-k}.$$

Then as can be directly checked ([48, § 8]), the following sub-Lie group

$$K = \{g \in G_{\mathbb{R}} | gH^+ = H^+\}$$

is a maximal compact subgroup of $G_{\mathbb{R}}$. Clearly, K contains $V \subseteq G_{\mathbb{R}}$ as a subgroup. The corresponding Lie algebra of K is

$$\mathfrak{k}_0 = \{X \subseteq \mathfrak{g}_0 | XH^+ \subseteq H^+\} = \mathfrak{g}_0 \cap \bigoplus_{k\ even} \mathfrak{g}^{k,-k}.$$

The adjoint action of K preserves the following

$$\mathfrak{p}_0 = \mathfrak{g}_0 \cap \bigoplus_{k\ odd} \mathfrak{g}^{k,-k},$$

thus we have $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. We denote the complexification of $\mathfrak{k}_0$ and $\mathfrak{p}_0$ by $\mathfrak{k}$ and $\mathfrak{p}$, respectively. We will study more about the maximal compact subgroup K in the next chapter.

Remark 3.2.1 *Since we will only need to use the component of $G_{\mathbb{R}}$ containing the identity element, we will denote by $G_{\mathbb{R}}$ again the component of the original real group containing the identity element, and K as well the component of the maximal subgroup which contains the identity. Furthermore, as we will only consider primitive cohomology classes, we may drop the subscribe "pr" from now on.*

3.2.2 Horizontal tangent bundle and a nilpotent sub-Lie algebra

As $\mathrm{Ad}(g)(\mathfrak{g}^{k,-k})$ is in $\bigoplus_{i\geq k}\mathfrak{g}^{i,-i}$ for each $g\in B$, the sub-Lie algebra $\mathfrak{b}\oplus\mathfrak{g}^{-1,1}/\mathfrak{b}\subseteq\mathfrak{g}/\mathfrak{b}$ defines an $\mathrm{Ad}(B)$-invariant subspace. By left translation via $G_{\mathbb{C}}$, $\mathfrak{b}\oplus\mathfrak{g}^{-1,1}/\mathfrak{b}$ gives rise to a $G_{\mathbb{C}}$-invariant holomorphic subbundle of the holomorphic tangent bundle at the base point. It will be denoted by $T^{1,0}_{o,h}\check{D}$, and will be referred to as the holomorphic horizontal tangent bundle at the base point. One can check that this construction does not depend on the choice of the base point. The horizontal tangent subbundle at the base point o, restricted to D, determines a subbundle $T^{1,0}_{o,h}D$ of the holomorphic tangent bundle $T^{1,0}_{o}D$ of D at the base point. The $G_{\mathbb{C}}$-invariace of $T^{1,0}_{o,h}\check{D}$ implies the $G_{\mathbb{R}}$-invariance of $T^{1,0}_{o,h}D$. As another interpretation of this holomorphic horizontal bundle at the base point, one has

$$T^{1,0}_{o,h}\check{D}\simeq T^{1,0}_{o}\check{D}\cap\bigoplus_{k=1}^{n}\mathrm{Hom}(F^k_0/F^{k+1}_0,F^{k-1}_0/F^k_0). \tag{3.8}$$

Let us consider the nilpotent Lie subalgebra $\mathfrak{n}_+:=\oplus_{k\geq 1}\mathfrak{g}^{-k,k}$. Then one gets the holomorphic isomorphism $\mathfrak{g}/\mathfrak{b}\cong\mathfrak{n}_+$. We take the unipotent group $N_+=\exp(\mathfrak{n}_+)$. Since D is an open set in $\check{D}$, we have the following relation:

$$T^{1,0}_{o,h}D=T^{1,0}_{o,h}\check{D}\cong\mathfrak{b}\oplus\mathfrak{g}^{-1,1}/\mathfrak{b}\hookrightarrow\mathfrak{g}/\mathfrak{b}\cong\mathfrak{n}_+. \tag{3.9}$$

3.2.3 Matrix representations of Lie groups and Lie algebras related to the classifying space

Let us introduce the notion of an adapted basis for the given Hodge decomposition or the Hodge filtration. Let $\{H^{k,n-k}_{pr}(M)\}_{k=0}^{n}$ be a given Hodge structure for the given polarized projective manifold (M,L) and $\{F^n\subseteq\cdots\subseteq F^0\}$ the corresponding Hodge filtration. Denote $f^k=\dim F^k$ for any $0\leq k\leq n$, we call a basis

$$\xi=\left\{\xi_0,\xi_1,\cdots,\xi_N,\cdots,\xi_{f^{k+1}},\cdots,\xi_{f^k-1},\cdots,\xi_{f^2},\cdots,\xi_{f^1-1},\xi_{f^0-1}\right\}$$

of $H^n_{pr}(M,\mathbb{C})$ an *adapted basis for the given Hodge decomposition*

$$H^n_{pr}(M,\mathbb{C})=H^{n,0}_{pr}(M)\oplus H^{n-1,1}_{pr}(M)\oplus\cdots\oplus H^{1,n-1}_{pr}(M)\oplus H^{0,n}_{pr}(M),$$

if it satisfies $H_p^{k,n-k} = \mathrm{Span}_{\mathbb{C}}\left\{\xi_{f^{k+1}}, \cdots, \xi_{f^k-1}\right\}$ with $\dim H_p^{k,n-k} = f^k - f^{k+1}$. We call a basis $\zeta = \{\zeta_0, \zeta_1, \cdots, \zeta_N, \cdots, \zeta_{f^{k+1}}, \cdots, \zeta_{f^k-1}, \cdots, \zeta_{f^2}, \cdots, \zeta_{f^1-0}, \zeta_{f^0-1}\}$ of $H^n_{pr}(M, \mathbb{C})$ an *adapted basis for the given filtration* $F^n \subseteq F^{n-1} \subseteq \cdots \subseteq F^0$, if it satisfies $F^k = \mathrm{Span}_{\mathbb{C}}\{\zeta_0, \cdots, \zeta_{f^k-1}\}$ with $\dim_{\mathbb{C}} F^k = f^k$.

Moreover, unless otherwise pointed out, the matrices in this paper are $m \times m$ matrices, where $m = f^0$. The blocks of the $m \times m$ matrix T is set as follows: for each $0 \leq \alpha, \beta \leq n$, the (α, β)-th block $T^{\alpha,\beta}$ is

$$T^{\alpha,\beta} = \left[T_{ij}(\tau)\right]_{f^{-\alpha+n+1} \leq i \leq f^{-\alpha+n}-1,\ f^{-\beta+n+1} \leq j \leq f^{-\beta+n}-1}, \tag{3.10}$$

where T_{ij} is the entries of the matrix T, and f^{n+1} is defined to be zero. In particular, $T = [T^{\alpha,\beta}]$ is called a *block lower triangular matrix* if $T^{\alpha,\beta} = 0$ whenever $\alpha < \beta$.

We remark that by fixing a reference Hodge structure $\{F_0^n \subseteq F_0^{n-1} \subseteq \cdots \subseteq F^0\} \in D$, we can identify the above quotient Lie groups or Lie algebras with their orbits in the corresponding quotient Lie algebras or Lie groups. For example, $\mathfrak{n}_+ \cong \mathfrak{g}/\mathfrak{b}$, $\mathfrak{g}^{-1,1} \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$, and $N_+ \cong N_+B/B \subseteq \check{D}$. We can also identify a point $\{F_0^n \subseteq F_0^{n-1} \subseteq \cdots \subseteq F_0^0\} \in D$ with its Hodge decomposition $\bigoplus_{k=0}^n H_{pr}^{k,n-k}(M_p)$, and thus with any fixed adapted basis of the corresponding Hodge decomposition for the base point, we have matrix representations of elements in the above Lie groups and Lie algebras. We list some matrix representations of elements as follows.

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}}) \mid gQg^T = Q\};$$
$$G_{\mathbb{R}} = \{g = [g^{i,j}] \in G_{\mathbb{C}} \mid \overline{g^{i,j}} = g^{n-i,n-j} \quad \text{for any} \quad 0 \leq i, j \leq n\};$$
$$B = \{g \in G_{\mathbb{C}} | g^{i,j} = 0 \quad \text{for any} \quad i > j\};$$
$$V = \{g \in G_{\mathbb{R}} | g^{i,j} = 0 \quad \text{for any} \quad i \neq j\};$$
$$g = [g^{i,j}] \in N_+ \Leftrightarrow g \in G_{\mathbb{C}} \quad \text{and} \quad \left\{ \begin{array}{c} g^{i,j} = 0 \quad \text{for any} \quad i < j, \\ g^{i,i} = Id \quad \text{for any} \quad 0 \leq i \leq n; \end{array} \right.$$

$$X = [X^{i,j}] \in \mathfrak{g}_0 \Leftrightarrow \begin{cases} XQ + QX^T = 0, & \\ \overline{X^{i,j}} = X^{n-i,n-j} & \text{for any } 0 \le i, j \le n; \end{cases}$$

$$X = [X^{\alpha,\beta}] \in \mathfrak{n}_+ \Leftrightarrow X^{\alpha,\beta} = 0 \text{ for any } \alpha \le \beta;$$

$$X = [X^{i,j}] \in \mathfrak{k}_0 \Leftrightarrow \begin{cases} XQ + QX^T = 0, & \\ \overline{X^{i,j}} = X^{n-i,n-j} & \text{for any} \quad 0 \le i, j \le n, \\ X^{i,j} = 0 & \text{when } i - j \text{ is odd}; \end{cases}$$

$$X = [X^{i,j}] \in \mathfrak{p}_0 \Leftrightarrow \begin{cases} XQ + QX^T = 0, & \\ \overline{X^{i,j}} = X^{n-i,n-j} & \text{for each } 0 \le i, j \le n, \\ X^{i,j} = 0 & \text{when } i - j \text{ is even}. \end{cases}$$

Remark 3.2.2 *We remark that there exists an embedding from $\mathfrak{n}_+$ into $\mathfrak{g}_0$ as derived as follows. For any element $X \in \mathfrak{n}_+$ with matrix representation $[X^{\alpha,\beta}]$, we construct a matrix $[Y^{\alpha,\beta}]$ such that*

$$Y^{\alpha,\beta} = \begin{cases} X^{\alpha,\beta}, & \text{if} \quad \alpha > \beta; \\ 0, & \text{if} \quad \alpha = \beta; \\ \overline{X^{n-\alpha,n-\beta}}, & \text{if} \quad \alpha < \beta. \end{cases}$$

Then the element represented by this matrix $[Y^{\alpha,\beta}]$ is an element in $\mathfrak{g}_0$. In this way, we can define an embedding from $\mathfrak{n}_+$ to $\mathfrak{g}_0$, and denote it by

$$\boldsymbol{i} : \mathfrak{n}_+ \to \mathfrak{g}_0, \quad [X^{\alpha,\beta}] \mapsto [Y^{\alpha,\beta}]. \tag{3.11}$$

In particular, the embedding $\boldsymbol{i}$ induces a map $\tilde{\boldsymbol{i}} : \mathfrak{n}_+ \to \mathfrak{g}_0/\mathfrak{v}_0$, and one can easily check that $\tilde{\boldsymbol{i}}$ is an isomorphism.

For the above description, one can easily get the following isomorphisms. Note that in Remark 3.2.1, we assume that $G_{\mathbb{R}}$ is the connected component in the original real group, which contains the identity element. One may also check [53, page 126] for details.

(1). when $n = 2m$, let us denote

$$h^{k,n-k} = \dim_{\mathbb{C}} H^{k,n-k}_{pr}, \quad h_+ = \sum_{j=0}^{m} h^{2j,n-2j}, \quad h_- = \sum_{k=0}^{m-1} h^{2j+1,n-2j+1}.$$

Then we have

$$G_{\mathbb{R}} \cong SO(h_+, h_-), \quad V \cong U(h^{0,n}) \times U(h^{1,n-1}) \times \cdots \times U(h^{m-1,m+1}) \times O(h^{m,m}).$$

In this case, when m is even, h_- is even; when m is odd, h_+ is even. Thus if $h_+ + h_-$ is even, then h_+ and h_- must both be even; otherwise, $h_+ + h_-$ is odd.

(2). when $n = 2m+1$, let us similarly denote

$$h^{k,n-k} = \dim_{\mathbb{C}} H_{pr}^{k,n-k}, \quad h = \sum_{k=0}^{n} h^{k,n-k}.$$

Then we have

$$G_{\mathbb{R}} \cong Sp(h, \mathbb{R}), \quad V \cong U(h^0) \times U(h^1) \times \cdots \times U(h^m).$$

To close this chapter, we make an observation about the homogenous space $G_{\mathbb{R}}/K$. First of all, it is well-known that $G_{\mathbb{R}}/K$ is a Remanian symmetric space. See [28, 41, 25] for more details about this. In general, $G_{\mathbb{R}}/K$ may not be Hermitian. However, the following theorem (see [28, Ch VIII] and [41, Ch III]) gives a condition for $G_{\mathbb{R}}/K$ being Hermitian.

Theorem 3.2.3 *The manifold $G_{\mathbb{R}}/K$ is noncompact irreducible Hermitian symmetric spaces exactly when $G_{\mathbb{R}}$ is a connected noncompact simple centerless Lie group with and K has non discrete center and is a maximal compact subgroup G. In particular, the center of K is isomorphic to the circle group S^1 and the complex structure in unique.*

Recall that we are considering the Hodge structure of weight n of a given polarized projective manifold (M, L) with $\dim_{\mathbb{C}} M = n$. Using Theorem 3.2.3, we can show

Proposition 3.2.4 *When n is odd , the manifold $G_{\mathbb{R}}/K$ is a noncompact irreducible Hermitian symmetric space.*

Proof In our case, first we know that $G_{\mathbb{R}}$ is a noncompact Lie group. We also know that when n is odd, $G_{\mathbb{R}} \simeq \mathrm{Sp}(h, \mathbb{R})$, where $h = \sum_{k=0}^{n} h^{k,n-k}$ is even as n is odd. Therefore $G_{\mathbb{R}}$ is a centerless simple Lie group. Thus to show $G_{\mathbb{R}}/K$ is Hermitian, it is enough to show that K has nontrivial center. This suffices to show the center of its Lie algebra $\mathfrak{k}_0$ is nontrivial. Recall that, we have described $\mathfrak{k}_0$ as follows,

$$X = [X^{i,j}] \in \mathfrak{k}_0 \Leftrightarrow \begin{cases} XQ + QX^T = 0 & \\ \overline{X^{i,j}} = X^{n-i,n-j} & \text{for each } 0 \leq i, j \leq n. \\ X^{i,j} = 0 & \text{when } i - j \text{ is odd.} \end{cases} \tag{3.12}$$

Let us define the following block diagonal matrices:

$$Z = \begin{cases} Z^{i,j} = 0 & \text{for any } i \neq j, \quad 0 \leq i, j \leq n; \\ Z^{i,j} = (-1)^i \sqrt{-1} I_{h^{i,n-i} \times h^{i,n-i}}, & \text{for any } 0 \leq i = j \leq n, \end{cases}$$

where $I_{h^{i,n-i} \times h^{i,n-i}}$ denotes the $h^{i,n-i} \times h^{i,n-i}$ identity matrix. First of all, it is easy to check that $Z \in \mathfrak{g}_0$. Moreover, let us denote the center of $\mathfrak{k}_0$ by $Z(\mathfrak{k}_0)$. Then we will show that $Z \in Z(\mathfrak{k}_0)$. Indeed, we know for any $X \in \mathfrak{k}_0$ that the (i,j)-th block of the matrix XZ and ZX respectively are the following

$$(XZ)^{i,j} = (-1)^j \sqrt{-1} X^{i,j}, \quad (ZX) = (-1)^i \sqrt{-1} X^{i,j}.$$

Since $X \in \mathfrak{k}_0$, we know that $X^{i,j} = 0$ if $i - j$ is odd. Therefore, when $i - j$ is odd $(-1)^j \sqrt{-1} X^{i,j} = (-1)^i \sqrt{-1} X^{i,j} = 0$; when $i - j$ is even, $(-1)^j \sqrt{-1} X^{i,j} = (-1)^i \sqrt{-1} X^{i,j}$. Therefore, for any $X \in \mathfrak{k}_0$, we have

$$[X, Z] = 0.$$

Thus $\{0\} \neq \mathrm{Span}_{\mathbb{R}}\{Z\} \subseteq Z(\mathfrak{k}_0)$. Therefore, we get that $\mathfrak{k}_0$ has nontrivial center when n is odd. Thus by theorem 3.2.3, we conclude that $G_{\mathbb{R}}/K$ is a Hermitian symmetric space when n is odd. ∎

Moreover, by Theorem 3.2.3, we know that if $G_{\mathbb{R}}/K$ is Hermitian, then the center of K is isomorphic to the circle group, thus $\dim_{\mathbb{R}} Z(\mathfrak{k}_0) = 1$. Hence

$$Z(\mathfrak{k}_0) = \mathrm{Span}_{\mathbb{R}}\{Z\}.$$

Remark 3.2.5 *When $n = 2$, if M is K3 surfaces, we have $N = h^{1,1} = 19$. Let us define $Z = [Z^{i,j}]$ such that*

$$Z^{i,j} = 0 \text{ for any } i \neq j, \quad 0 \leq i, j \leq 2; \quad Z^{0,0} = -Z^{2,2} = \sqrt{-1} \quad Z^{1,1} = 0_{19\times 19}$$

where $0_{19\times 19}$ is the 19×19 zero matrix. Then $Z \in \mathfrak{k}_0$, and using similar argument as above, we can get $Z \in Z(\mathfrak{k}_0)$. Similarly, if M is a cubic fourfold in $\mathbb{P}^5$, the analogous arguments will imply that the corresponding Lie algebra $\mathfrak{k}_0$ has nontrivial center. Therefore by Theorem 3.2.3, if M is a $K3$ surface or a cubic fourfold in $\mathbb{P}^5$, then $G_{\mathbb{R}}/K$ is a noncompact irreducible Hermitian symmetric space.

However, when n is even in more general cases, one can find examples when the center of $\mathfrak{k}_0$ is trivial. For example, when $n = 4$ with $h^{3,1} > 1$ and $h^{2,2} > 1$, one can show that the center of $\mathfrak{k}_0$ is trivial.

CHAPTER 4

Period map on the Teichmüller space for Calabi–Yau type manifolds and its boundedness property

In this chapter, we study the period map on the Teichmüller space of polarized and marked Calabi–Yau type manifolds. In § 4.1, we review the definition of period maps on moduli spaces and Teichmüller spaces. One may refer to [21] for more details. In § 4.2, we recall some local properties for the period maps. One may use [22] as a reference for this section. In § 4.3, we mainly prove a preliminary lemma for the boundedness property of the period map. This lemma was the result of [51, Lemma 3], which is also a generalization of [27, Lemma 7, Lemma 8]. In § 4.4, we briefly review the Hodge metric on the period domain, which is originally introduced in [25]; and we also introduce the induced Hodge metric on the moduli space and the Teichmüller space. In § 4.5, we prove that the period map is a bounded map. The results in this section are collaborating work with F. Guan and K. Liu. One may find the original work in [10, 11].

4.1 Period map on the Teichmüller space

Let $(M, L, \{\gamma_1, \cdots, \gamma_{b^n}\})$ be a polarized and marked Calabi–Yau type manifold where M is a Calabi–Yau type manifold of dimension n, L is the polarization on M and γ is a basis for $H_n(M,\mathbb{Z})/\mathrm{Tor}$. Let $\mathcal{Z}_m$ be the moduli space of polarized Calabi–Yau type manifold of level m structure (M, L, γ_m), where γ_m is a basis of $H_n(M,\mathbb{Z})/\mathrm{Tor}/(mH_n(M,\mathbb{Z})/\mathrm{Tor})$. Let $\mathcal{T}$ be the Teichmüller space of polarized and marked projective manifold $(M, L, \{\gamma_1, \cdots, \gamma_{b^n}\})$. Recall that the Teichmüller space is a covering space of $\mathcal{Z}_m$ and we denote the covering map by $\pi_m : \mathcal{T} \to \mathcal{Z}_m$.

Recall that we assume in this thesis that there exists $m_0 \in \mathbb{Z}$ such that for any $m \geq m_0$ the moduli space $\mathcal{Z}_m$ is a connected quasi-projective smooth complex manifolds with a versal family $\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m$ of projective manifolds with level m structures, containing M as a fiber and polarized by an ample line bundle $\mathcal{L}_{\mathcal{Z}_m}$ on $\mathcal{X}_{\mathcal{Z}_m}$. Thus we have the pull-back family $\pi : \mathcal{U} \to \mathcal{T}$ of $\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m$ via the covering map $\pi_m : \mathcal{T} \to \mathcal{Z}_m$. Moreover, we showed in Proposition 2.2.2 that $\mathcal{T}$ is a smooth and connected complex manifold and the family $\pi : \mathcal{U} \to \mathcal{T}$ is local Kuranishi at each point of $\mathcal{T}$, containing M as a fiber.

As showed in Theorem 2.3.9, the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi–Yau type manifold is a simply connected smooth complex manifold. Therefore, since we have fixed the basis of the middle homology group modulo torsions, we identify the basis of $H_n(M, \mathbb{Z})/\mathrm{Tor}$ to a lattice Λ as in [55]. This gives us a canonical identification of the middle dimensional cohomology of M to that of the background manifold X, that is, $H^n(M, \mathbb{C}) \simeq H^n(X, \mathbb{C})$. Therefore, we can use this to identify $H^n(M_p, \mathbb{C})$ for all fibers on $\mathcal{T}$. Thus we get a canonical trivial bundle $H^n(M_p, \mathbb{C}) \times \mathcal{T}$.

The period map from $\mathcal{T}$ to D is defined by assigning to each point $p \in \mathcal{T}$ the polarized Hodge structure on M_p which is defined in Chapter 3, that is

$$\Phi : \mathcal{T} \to D, \qquad p \mapsto \Phi(p) = \{F^n(M_p) \subset \cdots \subset F^0(M_p)\},$$

where $F^k(M_p) = H^{n,0}_{pr}(M_p, \mathbb{C}) \oplus H^{n-1,1}_{pr}(M_p, \mathbb{C}) \oplus \cdots \oplus H^{k,n-k}_{pr}(M_p, \mathbb{C})$ for any $0 \leq k \leq n$ with $H^{j,n-j}_{pr}(M_p, \mathbb{C})$ the $(j, n-j)$-th primitive cohomology of M_p. We denote $F^k(M_p)$ by F^k_p for simplicity. We call D the *period domain* for the above period map.

Let us now describe the period map on the moduli space $\mathcal{Z}_m$. Denote the first fundamental group of $\mathcal{Z}_m$ by $\pi_1(\mathcal{Z}_m)$. Then $\pi_1(\mathcal{Z}_m)$ acts on $H^n(M, \mathbb{Z})$ via the monodromy representation

$$\pi_1(\mathcal{T}) \to \mathrm{Aut}(H^n(M, \mathbb{Z})).$$

We denote the descrete group $\Gamma := \mathrm{Aut}(H^n(M, \mathbb{Z}))$. The descrete group Γ acts on $H^n(M, \mathbb{C})$

properly and discontinuously on D, thus one has the following commutative diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\Phi"] \arrow[d] & D \arrow[d] \\ \mathcal{Z}_m \arrow[r, "\Phi_m"] & D/\Gamma \end{tikzcd}$$

where D/Γ analytic and Φ_m is holomorphic.

The period map has several nice properties, and one may refer to [21, 22] and [65, Chapter 10] for details. Among them, one of the most important is the following Griffiths transversality: the period map Φ is a holomorphic map and its tangent map satisfies that

$$\Phi_*(v) \in \bigoplus_{k=1}^{n} \mathrm{Hom}\left(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k\right) \quad \text{for any} \quad p \in \mathcal{T} \ \text{ and } \ v \in T_p^{1,0}\mathcal{T}$$

with $F^{n+1} = 0$, or equivalently, $\Phi_*(v) \in \bigoplus_{k=0}^{n} \mathrm{Hom}(F_p^k, F_p^{k-1})$. In particular, one can also conclude that the period map is horizontal in the sense of [48].

4.2 Local property of period maps for Calabi–Yau type manifolds

Consider the period map $\Phi : \mathcal{T} \to D$ from the Teichmüller space of polarized and marked Calabi–Yau type manifolds to the period domain. The results of Griffiths in [21, 22] gave a cohomological interpretation for the tangent map of Φ as follows: the tangent map Φ_* is induced by the cup product (which is given by the contraction map)

$$H^{0,1}(M, T^{1,0}(M)) \otimes H_{pr}^{p,q}(M) \to H_{pr}^{p-1,q+1}(M) \quad \text{for some} \quad p+q=n,$$

in other words:

$$\Phi_* : H^{0,1}(M, T^{1,0}M) \to \bigoplus_{p+q=n} \mathrm{Hom}(H_{pr}^{p,q}(M), H_{pr}^{p-1,q+1}(M)).$$

For any $p \in \mathcal{T}$, let us denote the following projection map from

$$P_p^k : \bigoplus_{j=0}^{n} \mathrm{Hom}(H_{pr}^{j,n-j}(M_p), H_{pr}^{j-1,n-j+1}(M_p)) \to \mathrm{Hom}(H_{pr}^{k,n-k}(M_p), H_{pr}^{k-1,n-k+1}(M_p)).$$

Then the period map on the Teichmüller space of polarized and marked Calabi–Yau type manifolds has the following local property.

Proposition 4.2.1 *For any $p \in \mathcal{T}$ and any generator $[\Omega_p]$ of $H^{s,n-s}_{pr}(M_p)$, the map*

$$P^s_p \circ \Phi_* : T^{1,0}_p \mathcal{T} \cong H^{0,1}(M_p, T^{1,0}M_p) \to Hom(H^{s,n-s}_{pr}(M_p), H^{s-1,n-s+1}_{pr}(M_p)) \cong H^{s-1,n-s+1}_{pr}(M_p)$$

is an isomorphism, where Φ_* is the tangent map of Φ.

Proof The first isomorphism $T^{1,0}_p\mathcal{T} \cong H^{0,1}(M_p, T^{1,0}M_p)$ follows from that Kodaira-Spencer map is an isomorphism for Calabi–Yau type manifolds. The second isomorphism

$$\mathrm{Hom}(H^{s,n-s}_{pr}(M_p), H^{s-1,n-s+1}_{pr}(M_p)) \cong H^{s-1,n-s+1}_{pr}(M_p)$$

follows from the property that $\dim H^{s,n-s}_{pr}(M_p) = 1$ for Calabi–Yau type manifolds, where this isomorphism is determined by the choice of the generator $[\Omega_p]$. It is clear now that the map

$$P^s_p \circ \Phi_* : H^{0,1}(M_p, T^{1,0}M_p) \to H^{s-1,n-s+1}_{pr}(M_p)$$

is given by contraction $P^s_p \circ \Phi_*(v) = [\kappa(v) \lrcorner \Omega_p]$. This contraction map is an isomorphism by the definition of Calabi–Yau type manifolds. ∎

Remark 4.2.2 *Notice that the above proposition implies that the period map is injective in a small neighborhood around any point of $\mathcal{T}$. Moreover, as the period map $\Phi : \mathcal{T} \to D$ is a covering map of $\Phi_m : \mathcal{Z}_m \to D/\Gamma$, the period map Φ_m is also locally injective.*

4.3 A preliminary lemma

In this section, the Lie groups $G_{\mathbb{C}}, G_{\mathbb{R}}, N_+, B, V, K$ and the Lie algebras $\mathfrak{g}, \mathfrak{g}_0, \mathfrak{n}_+, \mathfrak{b}, \mathfrak{v}, \mathfrak{v}_0, \mathfrak{k}, \mathfrak{k}_0$ are the same as the ones in Chapter 3. We shall review and collect some facts about the structure of simple Lie algebra $\mathfrak{g}$ in our case. One may refer to [25, 48] for more details.

Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be the Weil operator, which is defined by

$$\theta(X) = (-1)^p X \quad \text{for } X \in \mathfrak{g}^{p,-p}.$$

Then θ is an involutive automorphism of $\mathfrak{g}$, and is defined over $\mathbb{R}$. The $(+1)$ and (-1) eigenspaces of θ are then $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Recall that we have

$$\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0, \quad \mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0.$$

The fact that θ is an involutive automorphism implies

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \quad [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}.$$

Let us consider $\mathfrak{g}_c = \mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0$. Then $\mathfrak{g}_c$ is a real form for $\mathfrak{g}$. The killing form $B(\cdot, \cdot)$ on $\mathfrak{g}$ is defined by

$$B(X, Y) = \mathrm{Trace}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) \quad \text{for } X, Y \in \mathfrak{g}. \tag{4.1}$$

A semisimple Lie algebra is *compact* if and only if the Killing form is negative definite; otherwise it is *non-compact*. It is not hard to check that $\mathfrak{g}_c$ is actually a compact real form of $\mathfrak{g}$, while $\mathfrak{g}_0$ is a non-compact real form. Recall that $G_{\mathbb{R}} \subseteq G_{\mathbb{C}}$ is the subgroup which correpsonds to the subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{g}$. Let us denote the connected subgroup $G_c \subseteq G_{\mathbb{C}}$ which corresponds to the subalgebra $\mathfrak{g}_c \subseteq \mathfrak{g}$. Let us denote the complex conjugation of $\mathfrak{g}$ with respect to the compact real form $\mathfrak{g}_c$ by τ_c, and the complex conjugation of $\mathfrak{g}$ with respect to the non-compact real form $\mathfrak{g}_0$ by τ_0.

We know that in our cases, $G_{\mathbb{C}}$ is a connected simple Lie group, B a parabolic subgroup in $G_{\mathbb{C}}$ with $\mathfrak{b}$ as its Lie algebra. The Lie algebra $\mathfrak{b}$ has a unique maximal nilpotent ideal $\mathfrak{n}_-$. It is not hard to see that

$$\mathfrak{g}_c \cap \mathfrak{n}_- = \mathfrak{n}_- \cap \tau_c(\mathfrak{n}_-) = 0.$$

By using Bruhat's lemma, one concludes $\mathfrak{g}$ is spanned by the parabolic subalgebras $\mathfrak{b}$ and $\tau_c(\mathfrak{b})$. Moreover $\mathfrak{v} = \mathfrak{b} \cap \tau_c(\mathfrak{b})$, $\mathfrak{b} = \mathfrak{v} \oplus \mathfrak{n}_-$. In particular, we also have

$$\mathfrak{n}_+ = \tau_c(\mathfrak{n}_-).$$

As remarked [25, §1] of Griffiths and Schmid, one gets that $\mathfrak{v}$ must have the same rank of $\mathfrak{g}$ as $\mathfrak{v}$ is the intersection of the two parabolic subalgebras $\mathfrak{b}$ and $\tau_c(\mathfrak{b})$. Moreover, $\mathfrak{g}_0$ and $\mathfrak{v}_0$

are also of equal rank, since they are real forms of $\mathfrak{g}$ and $\mathfrak{v}$ respectively. Therefore, we can choose a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0$ is also a Cartan subalgebra of $\mathfrak{v}_0$. Since $\mathfrak{v}_0 \subseteq \mathfrak{k}_0$, we also have $\mathfrak{h}_0 \subseteq \mathfrak{k}_0$. A Cartan subalgebra of a real Lie algebra is a maximal abelian subalgebra. Therefore $\mathfrak{h}_0$ is also a maximal abelian subalgebra of $\mathfrak{k}_0$, hence $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{k}_0$. Summarizing the above, we get

Proposition 4.3.1 *There exists a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{h}_0$ is also a Cartan subalgebra of $\mathfrak{k}_0$.*

One may find an alternate proof for the conclusion that $\mathfrak{k}$ and $\mathfrak{g}$ has the same rank in Appendix A.2 Proposition A.2.1.

In [50, §4], a simple Lie algebra $\mathfrak{g}_0$ is of *first category* if and only if it has a Cartan subaglebra which is contained in $\mathfrak{k}_0$. Thus by Proposition 4.3.1, the simple Lie algebra $\mathfrak{g}$ we are considering is of first category. The preliminary lemma (Lemma 4.3.6) we will prove in this section is the result of [51, Lemma 3] of simple Lie algebra of first category. Sugiura gave a sketch proof of this lemma in [51], and we will give a detailed proof following his original ideas.

Let us first review some general structure theory of simple Lie algebra. By Proposition 4.3.1, we can choose a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{h}_0$ is also a Cartan subalgebra of $\mathfrak{k}_0$. Let us denote the complexification of $\mathfrak{h}_0$ by $\mathfrak{h}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ with the property that $\mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{k}$.

Write $\mathfrak{h}_0^* = \mathrm{Hom}(\mathfrak{h}_0, \mathbb{R})$ and $\mathfrak{h}_{\mathbb{R}}^* = \sqrt{-1}\mathfrak{h}_0^*$. Then $\mathfrak{h}_{\mathbb{R}}^*$ can be identified with $\mathfrak{h}_{\mathbb{R}} := \sqrt{-1}\mathfrak{h}_0$ by duality using the restriction of the Killing form B (see (4.1)) to $\mathfrak{h}_{\mathbb{R}}$. For any $\rho \in \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$, one can define the following subspace of $\mathfrak{g}$:

$$\mathfrak{g}^\rho = \{x \in \mathfrak{g} | [h, x] = \rho(h)x \quad \text{for all } h \in \mathfrak{h}\}.$$

An element $\varphi \in \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$ is called a *root* of $\mathfrak{g}$ with respect to $\mathfrak{h}$ if $\mathfrak{g}^\varphi \neq \{0\}$. Let $\Delta \subseteq \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$ denote the space of nonzero $\mathfrak{h}$-roots. Then each root space

$$\mathfrak{g}^\varphi = \{x \in \mathfrak{g} | [h, x] = \varphi(h)x \text{ for all } h \in \mathfrak{h}\}$$

associated with $\varphi \in \Delta$ is one-dimensional over $\mathbb{C}$, i.e $\dim_{\mathbb{C}} g^{\varphi} = 1$, and we will denote the generator of g^{φ} by a root vector e_{φ} for any $\varphi \in \Delta$. The adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ determins a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\varphi \in \Delta} \mathfrak{g}^{\varphi}.$$

There exists a Weyl base $\{h_i, 1 \leq i \leq l;\ e_{\varphi},$ for any $\varphi \in \Delta\}$ with $l = \mathrm{rank}(\mathfrak{g})$ such that $\mathrm{Span}_{\mathbb{C}}\{h_1, \cdots, h_l\} = \mathfrak{h}$, $\mathrm{Span}_{\mathbb{C}}\{e_{\varphi}\} = g^{\varphi}$ for each $\varphi \in \Delta$.

Since the involution θ is a Lie algebra automorphism fixing $\mathfrak{k}$, we have $[h, \theta(e_{\varphi})] = \varphi(h)\theta(e_{\varphi})$ for any $h \in \mathfrak{h}$ and $\varphi \in \Delta$. Thus $\theta(e_{\varphi})$ is also a root vector belonging to the root φ. Therefore so e_{φ} must also be an eigenvector of θ. Thus there is a decomposition of the roots Δ into $\Delta = \Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}}$, where

$$\Delta_{\mathfrak{k}} = \{\varphi \in \Delta : \mathfrak{g}^{\varphi} \subseteq \mathfrak{k}\}, \qquad \Delta_{\mathfrak{p}} = \{\varphi \in \Delta : \mathfrak{g}^{\varphi} \subseteq \mathfrak{p}\}.$$

Moreover we have

$$\begin{aligned}
&\tau_c(h_i) = \tau_0(h_i) = -h_i \quad \text{for any } 1 \leq i \leq l; \\
&\tau_c(e_{\varphi}) = \tau_0(e_{\varphi}) = -e_{-\varphi} \quad \text{for any } \varphi \in \Delta_{\mathfrak{k}}; \\
&\tau_0(e_{\varphi}) = -\tau_c(e_{\varphi}) = e_{\varphi} \quad \text{for any } \varphi \in \Delta_{\mathfrak{p}}.
\end{aligned}$$

With respect to this Weyl base, we have

$$\begin{aligned}
\mathfrak{k}_0 &= \mathfrak{h}_0 + \sum_{\varphi \in \Delta_{\mathfrak{k}}} \mathbb{R}(e_{\varphi} - e_{-\varphi}) + \sum_{\varphi \in \Delta_{\mathfrak{k}}} \mathbb{R}\sqrt{-1}(e_{\varphi} + e_{-\varphi}); \\
\mathfrak{p}_0 &= \sum_{\varphi \in \Delta_{\mathfrak{p}}} \mathbb{R}(e_{\varphi} + e_{-\varphi}) + \sum_{\varphi \in \Delta_{\mathfrak{p}}} \mathbb{R}\sqrt{-1}(e_{\varphi} - e_{-\varphi}).
\end{aligned}$$

Recall that if we fix a reference Hodge structure $\{H_0^{k,n-k}\}_{k=0}^n$ of $H^n_{pr}(M, \mathbb{C})$, there is an induced Hodge structure of weight zero on $\mathrm{End}(H^n_{pr}(M, \mathbb{C}))$, namely,

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k,-k} \quad \text{with} \quad \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g} | X H_0^{r,n-r} \subseteq H_0^{r+k,n-r-k}\},$$

with the morphism $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, i.e. $[\mathfrak{g}^{k,-k}, \mathfrak{g}^{r,-r}] \subseteq \mathfrak{g}^{k+r,-(k+r)}$. More importantly, as the Cartan subalgebra $\mathfrak{h}$ we choose satisfies $\mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{g}^{0,0} \subseteq \mathfrak{k}$, each $\mathfrak{h}$-root space is of pure type in the sense of the following lemma.

Lemma 4.3.2 *Let Δ be the set of $\mathfrak{h}$-roots as above. Then for each root $\varphi \in \Delta$, there is an integer $-n \leq k \leq n$ such that $e_\varphi \in \mathfrak{g}^{k,-k}$.*

Proof Let $\varphi \in \Delta$ be a root, and e_φ be the generator of the root space $\mathfrak{g}^\varphi$. Then $e_\varphi = \sum_{j=-n}^{n} e^{-j,j}$, where $e^{-j,j} \in \mathfrak{g}^{-j,j}$. Since $\mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{g}^{0,0}$, the Lie bracket $[e^{-j,j}, h] \in \mathfrak{g}^{-j,j}$ for each j. Then the condition $[e_\varphi, h] = \varphi(h)e_\varphi$ implies that

$$\sum_{k=-n}^{n} [e^{-j,j}, h] = \sum_{j=-n}^{n} \varphi(h)e^{-j,j} \quad \text{for each } h \in \mathfrak{h}.$$

Since $h \in \mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{g}^{0,0}$ and $[\mathfrak{g}^{0,0}, \mathfrak{g}^{-j,j}] \subseteq \mathfrak{g}^{-j,j}$ and by comparing the type, we get

$$[e^{-j,j}, h] = \varphi(h)e^{-j,j} \quad \text{for each } h \in \mathfrak{h}.$$

Therefore $e^{-j,j} \in g^\varphi$ for each $-n \leq j \leq n$. As $\{e^{-j,j},\}_{j=-n}^{n}$ forms a linear independent set, but g^φ is one dimensional, thus there exists some k such that $e^{-k,k} \neq 0$ and for any $j \neq k$, $e^{-j,j} = 0$. This proves $e_\varphi \in \mathfrak{g}^{-k,k}$ for some k. ∎

Let us now introduce a lexicographic order (for example, see [50, pp. 416] and [70, pp. 41]) in the real vector space $\mathfrak{h}_{\mathbb{R}}$ as follows: we fix an ordered basis $e_1, \cdots, e_l$ for $\mathfrak{h}_{\mathbb{R}}$. Then for any $h = \sum_{i=1}^{l} \lambda_i e_i \in \mathfrak{h}_{\mathbb{R}}$, we call $h > 0$ if the first nonzero coefficient is positive, that is, if $\lambda_1 = \cdots = \lambda_k = 0, \lambda_{k+1} > 0$ for some $1 \leq k < l$. For any $h, h' \in \mathfrak{h}_{\mathbb{R}}$, we say $h > h'$ if $h - h' > 0$, $h < h'$ if $h - h' < 0$ and $h = h'$ if $h - h' = 0$.

We introduce a linear order in the dual space $\mathfrak{h}_{\mathbb{R}}^*$ by fixing an ordered basis of $\mathfrak{h}_{\mathbb{R}}$. Then there is a simple root system $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ with respect to this linear order such that $\{\alpha_1, \cdots, \alpha_l\}$ is a linear independent roots with $\alpha_1 < \cdots < \alpha_l$ and $l = \dim_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}^*$; and any positive root can be written as $m_1\alpha_1 + \cdots + m_l\alpha_l$ where $m_1, \cdots, m_l$ are all nonnegative integers. In this way, we can also define

$$\Delta^+ = \{\varphi > 0 : \varphi \in \Delta\}, \quad \Delta_{\mathfrak{p}}^+ = \Delta^+ \cap \Delta_{\mathfrak{p}}, \quad \Delta_{\mathfrak{k}}^+ = \Delta^+ \cap \Delta_{\mathfrak{k}};$$
$$\Delta^- = \{\varphi < 0 : \varphi \in \Delta\}, \quad \Delta_{\mathfrak{p}}^- = \Delta^- \cap \Delta_{\mathfrak{p}}, \quad \Delta_{\mathfrak{k}}^- = \Delta^- \cap \Delta_{\mathfrak{k}}.$$

To prepare the proof of Propsition 4.3.6, we will first show the following lemma. A sketched proof of this lemmas can be found [51] while we provide a detail proofs here.

Lemma 4.3.3 *There exists a linear order of $\mathfrak{h}_{\mathbb{R}}^*$ such that the sum of any three positive non-compact roots is not a root.*

Proof Recall that we introduced a linear order in the dual space $\mathfrak{h}_{\mathbb{R}}^*$ by fixing an ordered basis of $\mathfrak{h}_{\mathbb{R}}$. Thus there is a simple root system $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ with respect to this linear order such that $\{\alpha_1, \cdots, \alpha_l\}$ is a linear independent roots with $\alpha_1 < \cdots < \alpha_l$ and $l = \dim_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}^*$; and any positive root can be written as $m_1\alpha_1 + \cdots + m_l\alpha_l$ where $m_1, \cdots, m_l$ are all nonnegative integers. Let us denote the maximal root with respect to this linear order by

$$\beta = n_1\alpha_1 + \cdots + n_l\alpha_l. \tag{*}$$

Then for any positive root $m_1\alpha_1 + \cdots + m_l\alpha_l$, we have

$$m_i \leq n_i, \quad \text{for all} \quad i = 1, \cdots, l.$$

By [44, §2, Theorem 1], the maximal compact subalgebra $\mathfrak{k}_0$ of $\mathfrak{g}_0$ is one of the following kinds:

1). $\mathfrak{k}_0 = \mathfrak{k}_0' \oplus Z(\mathfrak{k}_0)$ with $\dim_{\mathbb{R}} Z(\mathfrak{k}_0) = 1$ and $\operatorname{rank}(\mathfrak{k}_0') = l - 1$, where $\mathfrak{k}_0'$ is a compact semi-simple Lie subalgebra of $\mathfrak{k}_0$ and $Z(\mathfrak{k}_0)$ is the center of $\mathfrak{k}_0$. Moreover, the simple roots system of $\mathfrak{k}_0'$ is [1]isomorphic to $\Pi - \{\alpha_j\}$ for some j. In particular, $n_j = 1$ in (*).

2). $\mathfrak{k}_0$ is a compact semi-simple Lie algebra of rank l, and the simple root system is isomorphic to $\{-\beta, \alpha_1, \cdots, \alpha_l\} - \{\alpha_j\}$ for some j. In particular, $n_j = 2$ in (*).

Therefore by [44, §2, Lemma 1], by changing to a different linear order of $\mathfrak{h}_{\mathbb{R}}^*$, we may assume that the above simple root system $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ satisfies that $\Pi - \{\alpha_j\}$ consists of all compact roots. Using the above notation, let us prove that $\{\alpha_j\}$ must be non-compact in both case 1) and 2):

[1]Here two systems of roots are isomorphic means that there is a bijection between them that preserves the scalar products between the corresponding elements, up to a multiplication of a constant.

1). In this case, as $\mathfrak{k}_0 = \mathfrak{k}_0' \oplus Z(\mathfrak{k}_0)$ with $\dim_{\mathbb{R}}(Z(\mathfrak{k}_0)) = 1$, one can easily conclude that $\dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}\Delta_{\mathfrak{k}}) \leq l-1$. Thus since $\dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}\Pi) = \dim_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}^*) = l$, and $\Pi - \{\alpha_j\}$ is a linear independent compact roots, we conclude that α_j must be a non-compact root.

2). In this case, we may assume that there exists another linear order of $\mathfrak{h}_{\mathbb{R}}^*$ such that for the semi-simple Lie algebra $\mathfrak{k}_0$, its root system $\Delta_{\mathfrak{k}}$ has the simple root system $\{-\beta, \alpha_1, \cdots, \alpha_{j-1}, \alpha_{j+1}, \cdots, \alpha_l\}$. Recall that in this case, we also have $n_j = 2$ in (*), that is,

$$\beta = n_1\alpha_1 + \cdots n_{j-1}\alpha_{j-1} + 2\alpha_j + n_{j+1}\alpha_{j+1} + \cdots + n_l\alpha_l.$$

Now suppose towards a contradiction that α_j was a compact root. Then

$$\alpha_j = -\frac{1}{2}\beta + \frac{1}{2}(n_1\alpha_1 + \cdots + n_{j-1}\alpha_{j-1} + n_{j+1}\alpha_{j+1} + \cdots n_l\alpha_l). \tag{4.2}$$

We know that $\{-\beta, \alpha_1, \cdots, \alpha_{j-1}, \alpha_{j+1}, \cdots, \alpha_l\}$ forms a basis for the $\mathfrak{h}_{\mathbb{R}}^*$, thus (4.2) is a unique linear combination. Moreover, under the given linear order, either α_j or α_{-j} is positive. However, neither α_j nor $-\alpha_j$ can be written as non-negative integer linear combination of the simple roots system $\{-\beta, \alpha_1, \cdots, \alpha_{j-1}, \alpha_{j+1}, \cdots, \alpha_l\}$, contradiction. Thus α_j is non-compact.

To conclude, there exists a linear order of $\mathfrak{h}_{\mathbb{R}}^*$ such that the simple root system $\{\alpha_1, \cdots, \alpha_l\}$ with respect to this linear order satisfying the following: i). there is exactly one non-compact root, which we will denoted by α_j among this simple root system; ii). the maximal root with respect to this order can be written as

$$\beta = n_1\alpha_1 + \cdots + n_l\alpha_l \quad \text{satisfying} \quad n_i \geq 0 \quad \text{and} \quad 1 \leq n_j \leq 2.$$

Moreover we claim that with these properties of the simple root system, if $\gamma = \sum_{i=1}^{l} m_i\alpha_i$ is a non-compact positive roots, then $m_j = 1$, this claim will be proved in Lemma 4.3.4. Followed by the claim, if we assume γ_1, γ_2, and γ_3 be any 3 positive non-compact roots with

$$\gamma_1 = a_1\alpha_1 + \cdots + a_l\alpha_l; \quad \gamma_2 = b_1\alpha_1 + \cdots + b_l\alpha_l; \quad \gamma_3 = c_1\alpha_1 + \cdots + c_l\alpha_l$$

where a_i, b_i, c_i are all non-negative integers for $1 \leq i \leq l$, then $a_j = b_j = c_j = 1$. Thus

$$\gamma_1 + \gamma_2 + \gamma_3 = (a_1 + b_1 + c_1)\alpha_1 + \cdots + (a_l + b_l + c_l)\alpha_l,$$

where $a_i + b_i + c_i$ are all non-negative integers for all $1 \leq i \leq l$, and $a_j + b_j + c_j = 3$. Thus if $\gamma_1+\gamma_2+\gamma_3$ were a root, then $a_j+b_j+c_j \leq n_j = 2$, which is a contradiction. Thus $\gamma_1+\gamma_2+\gamma_3$ can never be a root. $\blacksquare$

To complete the proof of Lemma 4.3.3, we still need to show the claim, which is given in the following lemma.

Lemma 4.3.4 *With the above new linear order of $\mathfrak{h}_{\mathbb{R}}^*$, if $\gamma = \sum_{i=1}^{l} m_i\alpha_i$ is a non-compact positive roots, then $m_j = 1$.*

Proof For any positive root

$$\gamma = k_1\alpha_1 + \cdots + k_{j-1}\alpha_{j-1} + k_j\alpha_j + k_{j+1}\alpha_{j+1} + \cdots + k_l\alpha_l,$$

we have k_i is a non-negative integer for all $1 \leq i \leq l$ and $k_j \leq 2$. To prove the lemma, it suffices to show that if $k_j = 0$ or 2, then γ is a compact root.

Denote $k = \sum_{i=1}^{l} k_i$. Then there exists a sequence of roots

$$\alpha_{i_1}, \quad \alpha_{i_1} + \alpha_{i_2}, \quad \cdots, \quad \alpha_{i_1} + \cdots + \alpha_{i_{k-1}}, \quad \alpha_{i_1} + \cdots + \alpha_{i_k} = \gamma, \tag{4.3}$$

where $\alpha_{i_s} \in \Pi$ for all $1 \leq s \leq k$. For any $1 \leq s \leq k$, denote $\gamma_s := \alpha_{i_1} + \cdots + \alpha_{i_s}$. Then the root space of γ_s is a one dimensional space, and we denote it by $g^{\gamma_s} = \mathrm{Span}_{\mathbb{C}}\{e_s\}$, where e_s is the generator of the root space. Let us denote the root space of the root α_{i_s} by $g^{\alpha_{i_s}} = \mathrm{Span}_{\mathbb{C}}\{e_{\alpha_{i_s}}\}$. Since $\gamma_{s-1} + \alpha_{i_s} = \gamma_s$ is a root, $[e_{\gamma_{s-1}}, e_{\alpha_{i_s}}] \neq 0$. Second,

$$\begin{aligned}
[h, [e_{\gamma_{s-1}}, e_{\alpha_{i_s}}]] &= [[h, e_{\gamma_{s-1}}], e_{\alpha_{i_s}}] + [[e_{\alpha_{i_s}}, h], e_{\gamma_{s-1}}] \\
&= \gamma_{s-1}(h)[e_{\gamma_{s-1}}, e_{\alpha_{i_s}}] + \alpha_{i_s}(h)[e_{\gamma_{s-1}}, e_{\alpha_{i_s}}], \\
&= \gamma_s(h)[e_{\gamma_{s-1}}, e_{\alpha_{i_s}}] \quad \text{for any } h \in \mathfrak{h}.
\end{aligned}$$

Thus $[e_{s-1}, e_{\alpha_{i_s}}]$ is a nonzero root vector of the root γ_s. Thus we may assume $e_s = [e_{s-1}, e_{\alpha_{i_s}}]$.

Case 1: when $k_j = 0$, considering the sequence (4.3), we have $\alpha_{i_s} \in \Pi - \{\alpha_j\}$ for all $1 \le s \le k$. First we note that each vector in the above sequence (4.3) is a root. We will show that γ is compact by induction on s. When $s = 1$, the root α_{i_1} is compact. Suppose now $\gamma_{s-1} = \alpha_{i_1} + \cdots + \alpha_{i_{s-1}}$ is a compact root, we need to show that $\gamma_s = \alpha_{i_1} + \cdots + \alpha_{i_{s-1}} + \alpha_{i_s} = \gamma_{s-1} + \alpha_{i_s}$ is also compact. Indeed, since γ_{s-1} is compact, its root vector $e_{s-1} \in \mathfrak{k}$. Since $e_{\alpha_{i_s}} \in \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{k}] \in \mathfrak{k}$, we have $e_s = [e_{s-1}, e_{\alpha_{i_s}}] \in \mathfrak{k}$, where e_s is the root vector of γ_s. Therefore, γ_s is also a compact root.

Case 2: when $k_j = 2$, we have that in (4.3), there exists $1 \le s_1 < s_2 \le k$, such that $\alpha_{i_{s_1}} = \alpha_{i_{s_2}} = \alpha_j$, and $\{\alpha_{i_1}, \cdots, \alpha_{i_s}\} - \{\alpha_{i_{s_1}}, \alpha_{i_{s_2}}\} \in \Pi - \{\alpha_j\}$. By realizing that $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, we can use similar arguments as in Case 1 to determine if γ_s is a compact or a non-compact root. In particular, we can conclude that $\{\gamma_1, \cdots, \gamma_{s_1-1}\}$ are all compact roots, $\{\gamma_{s_1}, \cdots, \gamma_{s_2-1}\}$ are all non-compact roots, and $\{\gamma_{s_2}, \cdots, \gamma_k = \gamma\}$ are all compact roots. ∎

Definition 4.3.5 *Two different roots $\varphi, \psi \in \Delta$ are said to be strongly orthogonal if and only if $\varphi \pm \psi \notin \Delta \cup \{0\}$, which is denoted by $\varphi \perp\!\!\!\perp \psi$.*

Now we are ready to prove the following main lemma of this section.

Lemma 4.3.6 *There exists a set of strongly orthogonal roots $\{\gamma_1, \cdots, \gamma_r\} \subseteq \Delta_{\mathfrak{p}}^+$ such that the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is*

$$\mathfrak{a} = \sum_{i=1}^{r} \mathbb{C}(e_{\gamma_i} + e_{-\gamma_i}).$$

In particular, the maximal abelian subspace in $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$ is $\mathfrak{a}_0 = \sum_{i=1}^{r} \mathbb{R}(e_{\gamma_i} + e_{-\gamma_i})$.

Proof Let φ_1 be the minimum in $\Delta_{\mathfrak{p}}^+$, and φ_2 be the minimal element in $\{\varphi \in \Delta_{\mathfrak{p}}^+ : \varphi \perp\!\!\!\perp \varphi_1\}$, then we obtain inductively an maximal ordered set of roots $\Lambda = \{\varphi_1, \cdots, \varphi_r\} \subseteq \Delta_{\mathfrak{p}}^+$, such that for each $1 \le k \le r$

$$\varphi_k = \min\{\phi \in \Delta_{\mathfrak{p}}^+ : \varphi \perp\!\!\!\perp \varphi_j \text{ for } 1 \le j \le k-1\}.$$

Because $\varphi_i \perp\!\!\!\perp \varphi_j$ for any $1 \le i < j \le r$, we have $[e_{\pm\varphi_i}, e_{\pm\varphi_j}] = 0$. Therefore $\mathfrak{a} = \sum_{i=1}^r \mathbb{C}\left(e_{\varphi_i} + e_{-\varphi_i}\right)$ is an abelian subspace of $\mathfrak{p}$. In particular, $\mathfrak{a}_0 = \sum_{i=1}^r \mathbb{R}\left(e_{\varphi_i} + e_{-\varphi_i}\right)$ is an abelian subspace of $\mathfrak{p}_0$. Also because a root can not be strongly orthogonal to itself, the ordered set Λ contains distinct roots. Thus $\dim_{\mathbb{C}} \mathfrak{a} = \dim_{\mathbb{R}} \mathfrak{a}_0 = r$.

Now we prove that $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$. Suppose towards a contradiction that there was a nonzero vector $X \in \mathfrak{p}_0$ as follows

$$X = \sum_{\alpha \in \Delta_{\mathfrak{p}}^+ \setminus \Lambda} \lambda_\alpha \left(e_\alpha + e_{-\alpha}\right) + \sum_{\alpha \in \Delta_{\mathfrak{p}}^+ \setminus \Lambda} \mu_\alpha \sqrt{-1}\left(e_\alpha - e_{-\alpha}\right), \quad \text{where } \lambda_\alpha, \mu_\alpha \in \mathbb{R},$$

such that $[X, e_{\varphi_i} + e_{-\varphi_i}] = 0$ for each $1 \le i \le r$. We denote $c_\alpha = \lambda_\alpha + \sqrt{-1}\mu_\alpha$. Because $X \ne 0$, there exists $\psi \in \Delta_{\mathfrak{p}}^+ \setminus \Lambda$ with $c_\psi \ne 0$. Also ψ is not strongly orthogonal to φ_i for some $1 \le i \le r$. Thus we may first define k_ψ for each ψ with $c_\psi \ne 0$ as the following:

$$k_\psi = \min_{1 \le i \le r} \{i : \psi \text{ is not strongly orthogonal to } \varphi_i\}.$$

Then we know that $1 \le k_\psi \le r$ for each ψ with $c_\psi \ne 0$. Then we define k to be the following,

$$k = \min_{\psi \in \Delta_{\mathfrak{p}}^+ \setminus \Lambda \text{ with } c_\psi \ne 0} \{k_\psi\}. \tag{4.4}$$

Here, we are taking the minimum over a finite set in the (4.4) and $1 \le k \le r$. Moreover, we get the following non-empty set,

$$S_k = \{\psi \in \Delta_{\mathfrak{p}}^+ \setminus \Lambda : c_\psi \ne 0 \text{ and } k_\psi = k\} \ne \emptyset. \tag{4.5}$$

Recall the notation $N_{\beta,\gamma}$ for any $\beta, \gamma \in \Delta$ is defined as as follows: if $\beta + \gamma \in \Delta \cup \{0\}$, $N_{\beta,\gamma}$ is defined such that $[e_\beta, e_\gamma] = N_{\beta,\gamma} e_{\beta+\gamma}$; if $\beta + \gamma \notin \Delta \cup \{0\}$ then one defines $N_{\beta,\gamma} = 0$. Now let us take k as defined in (4.4) and consider the Lie bracket

$$\begin{aligned} 0 &= [X, e_{\varphi_k} + e_{-\varphi_k}] && (4.6) \\ &= \sum_{\psi \in \Delta_{\mathfrak{p}}^+ \setminus \Lambda} \Big(c_\psi (N_{\psi,\varphi_k} e_{\psi+\varphi_k} + N_{\psi,-\varphi_k} e_{\psi-\varphi_k}) + \bar{c}_\psi (N_{-\psi,\varphi_k} e_{-\psi+\varphi_k} + N_{-\psi,-\varphi_k} e_{-\psi-\varphi_k}) \Big). && (4.7) \end{aligned}$$

We notice that in the above equality, the only term that belongs to $\mathfrak{g}^{\psi+\varphi_k}$ is the term $c_\psi N_{\psi,\varphi_k} e_{\psi+\varphi_k}$, since if there were $\delta_1, \delta_2, \delta_3 \in \Delta_{\mathfrak{p}}^+$ such that $\psi + \varphi_k = \delta_1 - \varphi_k = -\delta_2 + \varphi_k =$

$-\delta_3 - \varphi_k$, then we would get

$$\psi + \varphi_k + \varphi_k = \delta_1 \in \Delta_{\mathfrak{p}}^+, \quad \psi + \varphi_k + \delta_2 = \varphi_k \in \Delta_{\mathfrak{p}}^+, \quad \psi + \varphi_k + \delta_3 = -\varphi_k \in \Delta_{\mathfrak{p}}^-,$$

which are contradictions to Lemma 4.3.3 which states that the sum of any three positive noncompact roots is not a root. Therefore $c_\psi N_{\psi,\varphi_k} e_{\psi+\varphi_k} = 0$ for all $\psi \in \Delta_{\mathfrak{p}}^+ \backslash \Lambda$ in (4.7). Similarly, we also have $c_{-\psi} N_{-\psi,-\varphi_k} e_{-\psi-\varphi_k} = 0$ for all $\psi \in \Delta_{\mathfrak{p}}^+ \backslash \Lambda$ in (4.7). Then we have the simplified expression

$$0 = [X, e_{\varphi_k} + e_{-\varphi_k}] = \sum_{\psi \in \Delta_{\mathfrak{p}}^+ \backslash \Lambda} \left(c_\psi N_{\psi,-\varphi_k} e_{\psi-\varphi_k} + \bar{c}_\psi N_{-\psi,\varphi_k} e_{-\psi+\varphi_k} \right). \tag{4.8}$$

Now let us take $\psi_0 \in S_k \neq \emptyset$. Then $c_{\psi_0} \neq 0$. By the definition of k, we have ψ_0 is not strongly orthogonal to φ_k while $\psi_0 + \varphi_k \notin \Delta \cup \{0\}$. Thus we have $\psi_0 - \varphi_k \in \Delta \cup \{0\}$. Therefore $c_{\psi_0} N_{\psi_0,-\varphi_k} e_{\psi_0-\varphi_k} \neq 0$. Since $0 = [X, e_{\varphi_k} + e_{-\varphi_k}]$, there must exist one element $\psi_0' \neq \psi_0 \in \Delta_{\mathfrak{p}}^+ \setminus \Lambda$ such that $\varphi_k - \psi_0 = \psi_0' - \varphi_k$ and $c_{\psi_0'} \neq 0$. This implies $2\varphi_k = \psi_0 + \psi_0'$, and consequently one of ψ_0 and ψ_0' is smaller then φ_k. Then we have the following two cases:

(i). if $\psi_0 < \varphi_k$, then we find $\psi_0 < \varphi_k$ with $\psi_0 \perp\!\!\!\perp \varphi_i$ for all $1 \leq i \leq k-1$, and this contradicts to the definition of φ_k as the following

$$\varphi_k = \min\{\phi \in \Delta_{\mathfrak{p}}^+ : \varphi \perp\!\!\!\perp \varphi_j \text{ for } 1 \leq j \leq k-1\}.$$

(ii). if $\psi_0' < \varphi_k$, since we have $c_{\psi_0'} \neq 0$, we have

$$k_{\psi_0'} = \min_{1 \leq i \leq r} \{i : \psi_0' \text{ is not strongly orthogonal to } \varphi_i\}.$$

Then by the definition of k in (4.4), we have $k_{\psi_0'} \geq k$. Therefore we found $\psi_0' < \varphi_k$ such that $\psi_0' < \varphi_i$ for any $1 \leq i \leq k-1 < k_{\psi_0'}$, and this contradicts with the definition of φ_k.

Therefore in both cases, we found contradictions. Thus we conclude that $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$. Therefore, $\mathfrak{a}$ is a maximal abelican subspace of $\mathfrak{p}$. ∎

Remark 4.3.7 *We remark that this proposition is a generalization of the same property for $G_{\mathbb{R}}$ when $G_{\mathbb{R}}/K$ is a Hermitian symmetric space, which is originally proved in [27, Lemma*

7, Lemma 8]. We will give the original proof in Appendix A Lemma A.2.2 for Lemma 4.3.6, which is provided in [51] and analogous to the proof of [27, Lemma 7, Lemma 8].

For further use, we also state a proposition about the maximal abelian subspaces of $\mathfrak{p}_0$ according to [28, Ch V],

Proposition 4.3.8 *Let $\mathfrak{a}'_0$ be an arbitrary maximal abelian subspaces of $\mathfrak{p}_0$, then there exists an element $k \in K$ such that $k \cdot \mathfrak{a}_0 = \mathfrak{a}'_0$. Moreover, we have*

$$\mathfrak{p}_0 = \bigcup_{k \in K} \mathrm{Ad}(k) \cdot \mathfrak{a}_0,$$

where Ad *denotes the adjoint action of K on $\mathfrak{a}_0$.*

4.4 Hodge metric

In [25], Griffiths and Schmid defined a Hermitian metric on D, which is the so-called Hodge metric on the period domain D. Here we will briefly describe this Hermitian metric. As $D \cong G_{\mathbb{R}}/V$, the tangent space of D at eV is isomorphic to $\mathfrak{g}_0/\mathfrak{v}_0$, where e is the identity element in $G_{\mathbb{R}}$. Moreover, there is an natural isomorphism between $\mathfrak{n}_+ = \tau_c(\mathfrak{n}_-) = \tau_0(\mathfrak{n}_-)$ and $\mathfrak{g}_0/\mathfrak{v}_0$ via the quotient map induced by the embedding $\mathfrak{i}_1 : \mathfrak{n}_+ \to \mathfrak{g}_0$, which is given in (3.11). And

$$(X, Y) = -B(X, \tau_c(Y)), \quad X, Y \in \mathfrak{n}_+$$

defines an adjoint V-invariant inner product on $\mathfrak{n}_+$. By translation, the inner product gives rise to a $G_{\mathbb{R}}$-invariant Hermitian metric on D. Because of its homogeneity, this metric turns D into a complete Hermitian manifold.

By Proposition 4.2.1 for Calabi–Yau type manifolds, we know that $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma, \Phi : \mathcal{T} \to D$ are both locally injective. Thus it follows from [25] that the pull-backs of h by $\Phi_{\mathcal{Z}_m}$ and Φ on $\mathcal{Z}_m$ and $\mathcal{T}$ respectively are both well-defined Kähler metrics. By abuse of notation, we still call these pull-back metrics on $\mathcal{Z}_m$ and $\mathcal{T}$ the *Hodge metrics*.

4.5 Boundedness of the period map on the Teichmüller space

Now let us fix a base point $p \in \mathcal{T}$ with $\Phi(p) \in D$. Then according to the discussion in §3.2, N_+ can be viewed as a subset in $\check{D}$ by identifying it with its orbit in $\check{D}$ with base point $\Phi(p)$. Thus $N_+ \cap D \subseteq D$. We define

$$\check{\mathcal{T}} = \Phi^{-1}(N_+ \cap D).$$

At the base point $\Phi(p) = o \in N_+ \cap D$, the holomorphic tangent space $T_o^{1,0}N_+ = T_o^{1,0}D \simeq \mathfrak{n}_+ \simeq N_+$, then the Hodge metric on $T_o^{1,0}D$ induces an Euclidean metric on N_+. In the proof of the following lemma, we require all the root vectors to be unit vectors with respect to this Euclidean metric.

Recall the definition of $T_{o,h}^{1,0}\check{D}$, as another interpretation of this holomorphic horizontal bundle at the base point, one has

$$T_{o,h}^{1,0}\check{D} \simeq T_o^{1,0}\check{D} \cap \bigoplus_{k=1}^{n} \mathrm{Hom}(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k). \tag{4.9}$$

In [48], a holomorphic mapping $\Psi : M \to \check{D}$ of a complex manifold M into $\check{D}$ is called *horizontal* if at each point o of M, the induced map between the holomorphic tangent spaces takes values in the horizontal holomorphic tangent space $T_{o,h}^{0,1}\check{D}$. It is easy to see that the period map $\Phi : \mathcal{T} \to D$ is horizontal since $\Phi_*(T_p^{1,0}\mathcal{T}) \subseteq T_{o,h}^{1,0}D$ for any $p \in \mathcal{T}$. Since D is an open set in $\check{D}$, we have the following relation:

$$T_{o,h}^{1,0}D = T_{o,h}^{1,0}\check{D} \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \hookrightarrow \mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+. \tag{4.10}$$

In particular, we may also identify $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$ with $\mathfrak{g}^{-1,1}$ by fixing the based point. Thus $\Phi_*(T_p^{1,0}\mathcal{T}) \subseteq T_{o,h}^{1,0}D \subseteq \mathfrak{g}^{-1,1}$.

Since the period map is a horizontal map, and the geometry of horizontal slices of the period domain D is similar to Hermitian symmetric space as discussed in detail in [25], the proof the following theorem is basically an analogue of the proof of the Harish-Chandra embedding theorem for Hermitian symmetric spaces, see for example [27, 41].

Proposition 4.5.1 *The restriction of the period map $\Phi : \check{\mathcal{T}} \to N_+$ is bounded in N_+ with respect to the Euclidean metric on N_+.*

Proof We need to show that there exists $0 \leq C < \infty$ such that for any $q \in \check{\mathcal{T}}$, $d_E(\Phi(p), \Phi(q)) \leq C$, where d_E is the Euclidean distance on N_+.

For any $q \in \check{\mathcal{T}}$, as the period map is horizontal, there exists a vector $X^+ \in \mathfrak{g}^{-1,1} \subseteq \mathfrak{n}_+$ and a real number T_0 such that $\beta(t) = \exp(tX^+)$ defines a geodesic $\beta : [0, T_0] \to N_+ \subseteq G_{\mathbb{C}}$ from $\beta(0) = I$ to $\beta(T_0)$ with $\pi_1(\beta(T_0)) = \Phi(q)$, where $\pi_1 : N_+ \to N_+B/B \subseteq \check{D}$ is the projection map with the fixed base point $\Phi(p) = o \in D$. In this proof, we will not distinguish $N_+ \subseteq G_{\mathbb{C}}$ from its orbit $N_+B/B \subseteq \check{D}$ with the fixed base point $\Phi(p) = o$.

Consider $X^- = \tau_0(X^+) \in \mathfrak{g}^{1,-1}$, then $X = X^+ + X^- \in (\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}) \cap \mathfrak{g}_0$. For any $q \in \check{\mathcal{T}}$, there exists T_1 such that $\gamma = \exp(tX) : [0, T_1] \to G_{\mathbb{R}}$ defines a geodesic from $\gamma(0) = I$ to $\gamma(T_1) \in G_{\mathbb{R}}$ such that $\pi_2(\gamma(T_1)) = \Phi(q) \in D$, where $\pi_2 : G_{\mathbb{R}} \to G_{\mathbb{R}}/V \simeq D$ denotes the projection map with the fixed base point $\Phi(p) = o$.

Let $\Lambda = \{\varphi_1, \cdots, \varphi_r\} \subseteq \Delta^+_{\mathfrak{p}}$ be a set of strongly orthogonal roots given in Proposition 4.3.6, such that the maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ is

$$\mathfrak{a} = \sum_{i=1}^{r} \mathbb{C}(e_{\varphi_i} + e_{-\varphi_i}).$$

We denote $x_{\varphi_i} = e_{\varphi_i} + e_{-\varphi_i}$ and $y_{\varphi_i} = \sqrt{-1}(e_{\varphi_i} - e_{-\varphi_i})$ for any $\varphi_i \in \Lambda$. Then

$$\mathfrak{a}_0 = \mathbb{R}x_{\varphi_1} \oplus \cdots \oplus \mathbb{R}x_{\varphi_r}, \quad \text{and} \quad \mathfrak{a}_c = \mathbb{R}y_{\varphi_1} \oplus \cdots \oplus \mathbb{R}y_{\varphi_r},$$

are maximal abelian spaces in $\mathfrak{p}_0$ and $\sqrt{-1}\mathfrak{p}_0$ respectively.

Since $X \in \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1} \subseteq \mathfrak{p}_0$, by Proposition 4.3.8, there exists $k \in K$ such that $X \in Ad(k) \cdot \mathfrak{a}_0$. As the adjoint action of K on $\mathfrak{p}_0$ is unitary action and we are considering the length in this proof, we may simply assume that $X \in \mathfrak{a}_0$ up to a unitary transformation. With this assumption, there exists $\lambda_i \in \mathbb{R}$ for $1 \leq i \leq r$ such that

$$X = \lambda_1 x_{\varphi_1} + \lambda_2 x_{\varphi_2} + \cdots + \lambda_r x_{\varphi_r}$$

Since $\mathfrak{a}_0$ is commutative, we have

$$\exp(tX) = \exp(\sum_{i=0}^{r} \lambda_i x_{\varphi_i}) = \prod_{i=1}^{r} \exp(t\lambda_i x_{\varphi_i}).$$

Now for each $\varphi_i \in \Lambda$, we have $\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\} \simeq \mathfrak{sl}_2(\mathbb{C})$ with

$$h_{\varphi_i} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_{\varphi_i} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{-\varphi_i} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};$$

and $\mathrm{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\} \simeq \mathfrak{sl}_2(\mathbb{R})$ with

$$\sqrt{-1}h_{\varphi_i} \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad x_{\varphi_i} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad y_{-\varphi_i} \mapsto \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Since $\Lambda = \{\varphi_1, \cdots, \varphi_r\}$ is a set of strongly orthogonal roots, we have that

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}}(\Lambda) &= \mathrm{Span}_{\mathbb{C}}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\}_{i=1}^{r} \simeq (\mathfrak{sl}_2(\mathbb{C}))^r, \\ \text{and} \quad \mathfrak{g}_{\mathbb{R}}(\Lambda) &= \mathrm{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{-\varphi_i}, \sqrt{-1}h_{\varphi_i}\}_{i=1}^{r} \simeq (\mathfrak{sl}_2(\mathbb{R}))^r. \end{aligned}$$

In fact, we know that for any $\varphi, \psi \in \Lambda$ with $\varphi \neq \psi$, since φ is strongly orthogonal to ψ, we have $[e_{\pm\varphi}, e_{\pm\psi}] = 0$; since $\mathfrak{h}$ is abelian, we have$[h_{\phi}, h_{\psi}] = 0$; and $[h_{\varphi}, e_{\pm\psi}] = \sqrt{-1}[[e_{\varphi}, e_{-\varphi}], e_{\pm\psi}] = \sqrt{-1}[e_{-\varphi}, [e_{\varphi}, e_{\pm\psi}]] = 0$.

Then we denote $G_{\mathbb{C}}(\Lambda) = \exp(\mathfrak{g}_{\mathbb{C}}(\Lambda)) \simeq (SL_2(\mathbb{C}))^r$ and $G_{\mathbb{R}}(\Lambda) = \exp(\mathfrak{g}_{\mathbb{R}}(\Lambda)) = (SL_2(\mathbb{R}))^r$, which are subgroups of $G_{\mathbb{C}}$ and $G_{\mathbb{R}}$ respectively. With the fixed reference point $o = \Phi(p)$, we denote $D(\Lambda) = G_{\mathbb{R}}(\Lambda)(o)$ and $S(\Lambda) = G_{\mathbb{C}}(\Lambda)(o)$ to be the corresponding orbits of these two subgroups, respectively. Then we have the following isomorphisms,

$$D(\Lambda) = G_{\mathbb{R}}(\Lambda) \cdot B/B \simeq G_{\mathbb{R}}(\Lambda)/G_{\mathbb{R}}(\Lambda) \cap V, \tag{4.11}$$

$$S(\Lambda) \cap (N_+B/B) = (G_{\mathbb{C}}(\Lambda) \cap N_+) \cdot B/B \simeq (G_{\mathbb{C}}(\Lambda) \cap N_+)/(G_{\mathbb{C}}(\Lambda) \cap N_+ \cap B). \tag{4.12}$$

With the above notations, we claim the following:

(i). $D(\Lambda) \subseteq S(\Lambda) \cap (N_+B/B) \subseteq \check{D}$;

(ii). $D(\Lambda)$ is bounded inside $S(\Lambda) \cap (N_+B/B)$;

(iii). $S(\Lambda) \cap (N_+B/B)$ is totally geodesic in N_+B/B.

To prove the claim, notice that since $X \in \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}$. By Lemma 4.3.2, we know that for each pair of roots $\{e_{\varphi_i}, e_{-\varphi_i}\}$, either $e_{\varphi_i} \in \mathfrak{g}^{-1,1} \subseteq \mathfrak{n}_+$ and $e_{-\varphi_i} \in \mathfrak{g}^{1,-1}$, or $e_{\varphi_i} \in \mathfrak{g}^{1,-1}$ and $e_{-\varphi_i} \in \mathfrak{g}^{-1,1} \subseteq \mathfrak{n}_+$. For notation simplicity, for each pair of root vectors $\{e_{\varphi_i}, e_{-\varphi_i}\}$, we may assume the one in $\mathfrak{g}^{-1,1} \subseteq \mathfrak{n}_+$ to be e_{φ_i} and denote the one in $\mathfrak{g}^{1,-1}$ by $e_{-\varphi_i}$. In this way, one can check that $\{\varphi_1, \cdots, \varphi_r\}$ may not be a set in $\Delta^+_{\mathfrak{p}}$, but it is a set of strongly orthogonal roots in $\Delta_{\mathfrak{p}}$. Therefore, we have the following description of the above groups,

$$
\begin{aligned}
&G_{\mathbb{R}}(\Lambda) = \exp(\mathfrak{g}_{\mathbb{R}}(\Lambda)) = \exp(\mathrm{Span}_{\mathbb{R}}\{x_{\varphi_1}, y_{\varphi_1}, \sqrt{-1}h_{\varphi_1}, \cdots, x_{\varphi_r}, y_{\varphi_r}, \sqrt{-1}h_{\varphi_r}\})\\
&G_{\mathbb{R}}(\Lambda) \cap V = \exp(\mathfrak{g}_{\mathbb{R}}(\Lambda) \cap \mathfrak{v}_0) = \exp(\mathrm{Span}_{\mathbb{R}}\{\sqrt{-1}h_{\varphi_1}, \cdot, \sqrt{-1}h_{\varphi_r}\})\\
&G_{\mathbb{C}}(\Lambda) \cap N_+ = \exp(\mathfrak{g}_{\mathbb{C}}(\Lambda) \cap \mathfrak{n}_+) = \exp(\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_1}, e_{\varphi_2}, \cdots, e_{\varphi_r}\});\\
&G_{\mathbb{C}}(\Lambda) \cap B = \exp(\mathfrak{g}_{\mathbb{C}}(\Lambda) \cap \mathfrak{b}) = \exp(\mathrm{Span}_{\mathbb{C}}(h_{\varphi_1}, e_{-\varphi_1}, \cdots, h_{\varphi_r}, e_{-\varphi_r}\}).
\end{aligned}
$$

Thus by the isomorphisms in (4.11) and (4.12), we have

$$
\begin{aligned}
D(\Lambda) &\simeq \prod_{i=1}^{r} \exp(\mathrm{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\})/\exp(\mathrm{Span}_{\mathbb{R}}\{\sqrt{-1}h_{\varphi_i}\},\\
S(\Lambda) \cap (N_+B/B) &\simeq \prod_{i=1}^{r} \exp(\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_i}\}).
\end{aligned}
$$

Let us denote $G_{\mathbb{C}}(\varphi_i) = \exp(\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\}) \simeq SL_2(\mathbb{C})$, $S(\varphi_i) = G_{\mathbb{C}}(\varphi_i)(o)$, and $G_{\mathbb{R}}(\varphi_i) = \exp(\mathrm{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\}) \simeq SL_2(\mathbb{R})$, $D(\varphi_i) = G_{\mathbb{R}}(\varphi_i)(o)$. On one hand, each point in $S(\varphi_i) \cap (N_+B/B)$ can be represented by

$$\exp(ze_{\varphi_i}) = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \quad \text{for some } z \in \mathbb{C}.$$

Thus $S(\varphi_i) \cap (N_+B/B) \simeq \mathbb{C}$. One the other hand, denote $z = a + bi$ for some $a, b \in \mathbb{R}$, then

$$\exp(ax_{\varphi_i} + by_{\varphi_i}) = \begin{bmatrix} \cosh|z| & \frac{\bar{z}}{|z|}\sinh|z| \\ \frac{z}{|z|}\sinh|z| & \cosh|z| \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \frac{z}{|z|}\tanh|z| & 1 \end{bmatrix} \begin{bmatrix} \cosh|z| & 0 \\ 0 & (\cosh|z|)^{-1} \end{bmatrix} \begin{bmatrix} 1 & \frac{\bar{z}}{|z|}\tanh|z| \\ 0 & 1 \end{bmatrix}$$

$$= \exp\left[(\frac{z}{|z|}\tanh|z|)e_{\varphi_i}\right] \exp\left[(-\log\cosh|z|)h_{\varphi_i}\right] \exp\left[(\frac{\bar{z}}{|z|}\tanh|z|)e_{-\varphi_i}\right].$$

So elements in $D(\varphi_i)$ can be represented by $\exp[(z/|z|)(\tanh|z|)e_{\varphi_i}]$, i.e. the lower triangular matrix

$$\begin{bmatrix} 1 & 0 \\ \frac{z}{|z|}\tanh|z| & 1 \end{bmatrix},$$

Therefore $D(\varphi_i) \subseteq S(\varphi_i) \cap N_+B/B \cong \mathbb{C}$. Moreover, as $\frac{z}{|z|}\tanh|z|$ is a point in the unit disc $\mathbb{D}$ of the complex plane, $D(\varphi_i)$ is isomorphic to unit disc $\mathbb{D}$ in the complex plane $S(\varphi_i) \cap (N_+B/B) \cong \mathbb{C}$. Therefore $D(\Lambda) \subseteq S(\Lambda) \cap N_+B/B$ and

$$D(\Lambda) \simeq \mathbb{D}^r \quad \text{and} \quad S(\Lambda) \cap N_+ \simeq \mathbb{C}^r.$$

This proves both (i) and (ii) in the claim. To complete the proof of the claim, we only need to show that $S(\Lambda) \cap (N_+B/B)$ is totally geodesic in N_+B/B. In fact, the tangent space of N_+ at the base point is $\mathfrak{n}_+$ and the tangent space of $S(\Lambda) \cap N_+B/B$ at the base point is $\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_1}, e_{\varphi_2}, \cdots, e_{\varphi_r}\}$. Since $\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_1}, e_{\varphi_2}, \cdots, e_{\varphi_r}\}$ is a sub-Lie algebra of $\mathfrak{n}_+$, the corresponding orbit $S(\Lambda) \cap N_+B/B$ is totally geodesic in N_+B/B. Here the basis $\{e_{\varphi_1}, e_{\varphi_2}, \cdots, e_{\varphi_r}\}$ is an orthonormal basis with respect to the pull-back Euclidean metric. This proves the claim.

As a consequence, we get that for any $q \in \check{\mathcal{T}}$, $\Phi(q) \in D(\Lambda)$. This implies

$$d_E(\Phi(p), \Phi(q)) \leq \sqrt{r}$$

where d_E is the Euclidean distance on $S(\Lambda) \cap N_+B/B$. Since $S(\Lambda) \cap N_+B/B$ is totally geodesic in N_+, it follows that $d_E(\Phi(p), \Phi(q))$ is also bounded in N_+B/B, with d_E the Euclidean distance on N_+B/B. ∎

Although not needed in the proof of the above theorem, we can also show that the above inclusion of $D(\varphi_i)$ in D is totally geodesic in D with respect to the Hodge metric. In fact, the tangent space of $D(\varphi_i)$ at the base point is $\mathrm{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}\}$ which satisfies

$$[x_{\varphi_i}, [x_{\varphi_i}, y_{\varphi_i}]] = 4y_{\varphi_i},$$
$$[y_{\varphi_i}, [y_{\varphi_i}, x_{\varphi_i}]] = 4x_{\varphi_i}.$$

So the tangent space of $D(\varphi_i)$ forms a Lie triple system, and consequently $D(\varphi_i)$ gives a totally geodesic in D. The fact that the exponential map of a Lie triple system gives a totally geodesic in D is from (cf. [28] Ch 4 §7), and we note that this result still holds true for locally homogeneous spaces instead of only for symmetric spaces. And the pull-back of the Hodge metric on $D(\varphi_i)$ is $G(\varphi_i)$ invariant metric, therefore must be the Poincare metric on the unit disc. In fact, more generally, we have

Lemma 4.5.2 *If $\tilde{G}$ is a subgroup of $G_{\mathbb{R}}$, then the orbit $\tilde{D} = \tilde{G}(o)$ is totally geodesic in D, and the induced metric on $\tilde{D}$ is $\tilde{G}$ invariant.*

Proof Firstly, $\tilde{D} \simeq \tilde{G}/(\tilde{G} \cap V)$ is a quotient space. The induced metric of the Hodge metric from D is $G_{\mathbb{R}}$-invariant, and therefore $\tilde{G}$-invariant. Now let $\gamma: [0,1] \to \tilde{D}$ be any geodesic, then there is a local one parameter subgroup $S: [0,1] \to \tilde{G}$ such that, $\gamma(t) = S(t)\cdot\gamma(0)$. On the other hand, because $\tilde{G}$ is a subgroup of $G_{\mathbb{R}}$, we have that $S(t)$ is also a one parameter subgroup of $G_{\mathbb{R}}$, therefore the curve $\gamma(t) = S(t)\cdot\gamma(0)$ also gives a geodesic in D. Since geodesics on $\tilde{D}$ are also geodesics on D, we have proved $\tilde{D}$ is totally geodesic in D.

In order to prove Theorem 4.5.5, we first show that $\mathcal{T}\backslash\check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}\backslash\check{\mathcal{T}}) \geq 1$.

Recall that for a Calabi–Yau type manifold M, we have that there exists $s \geq [n/2]+1$ such that $H^{s',n-s'}(M,\mathbb{C}) = 0$ for any $s' > s$, and $\dim_{\mathbb{C}} H^{s,n-s}_{pr}(M,\mathbb{C}) = 1$. Let us also fix an adapted basis $(\eta_0, \cdots, \eta_{m-1})$ for the Hodge decomposition of the base point $\Phi(p) \in D$. The we can identify elements is N_+ with nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrices.

Lemma 4.5.3 *Let $p \in \mathcal{T}$ be the base point with $\Phi(p) = \{F_p^s \subseteq F_p^{s-1} \subseteq \cdots \subseteq F_p^{n-s}\}$. Let $q \in \mathcal{T}$ be any point with $\Phi(q) = \{F_q^s \subseteq F_q^{s-1} \subseteq \cdots \subseteq F_q^{n-s}\}$, then $\Phi(q) \in N_+$ if and only if F_q^k is isomorphic to F_p^k for all $n-s \leq k \leq s$.*

Proof For any $q \in \mathcal{T}$, we choose an arbitrary adapted basis $\{\zeta_0, \cdots, \zeta_{m-1}\}$ for the given Hodge filtration $\{F_q^s \subseteq F_q^{s-1} \subseteq \cdots \subseteq F_q^{n-s}\}$. Recall that $\{\eta_0, \cdots, \eta_{m-1}\}$ is the adapted basis for the Hodge filtration $\{F_p^s \subseteq F_p^{s-1} \subseteq \cdots \subseteq F_p^{n-s}\}$ for the base point p. Let $[A^{i,j}(q)]_{0\leq i,j\leq 2s-n}$ be the transition matrix between the basis $\{\eta_0, \cdots, \eta_{m-1}\}$ and $\{\zeta_0, \cdots, \zeta_{m-1}\}$ for the same vector space $H^n(M, \mathbb{C})$, where $A^{i,j}(q)$ are the corresponding blocks. Recall that elements in N_+ and B have matrix representations with the fixed adapted basis at the base point: elements in N_+ can be realized as nonsingular block lower triangular matrices with identity blocks in the diagonal; elements in B can be realized as nonsingular block upper triangular matrices. Therefore $\Phi(q) \in N_+ = N_+B/B \subseteq \check{D}$ if and only if its matrix representation $[A^{i,j}(q)]_{0\leq i,j\leq 2s-n}$ can be decomposed as $L(q) \cdot U(q)$, where $L(q)$ is a nonsingular block lower triangular matrix with identities in the diagonal blocks, and $U(q)$ is a nonsingular block upper triangular matrix. By basic linear algebra, we know that $[A^{i,j}(q)]_{0\leq i,j\leq 2s-n}$ has such decomposition if and only if $\det[A^{i,j}(q)]_{0\leq i,j\leq k} \neq 0$ for any $0 \leq k \leq 2s-n$. In particular, we know that $[A(q)^{i,j}]_{0\leq i,j\leq k}$ is the transition map between the bases of F_p^{s-k} and F_q^{s-k}. Therefore, $\det([A(q)^{i,j}]_{0\leq i,j\leq k}) \neq 0$ if and only if F_p^{s-k} is isomorphic to F_q^{s-k}. ∎

Lemma 4.5.4 *The subset $\check{\mathcal{T}}$ is an open dense submanifold in $\mathcal{T}$, and $\mathcal{T}\backslash\check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}\backslash\check{\mathcal{T}}) \geq 1$.*

Proof From Lemma 4.5.3, one can see that $\check{D} \setminus N_+ \subseteq \check{D}$ is defined as a analytic subvariety by equations

$$\det[A^{i,j}(q)]_{0\leq i,j\leq k} = 0 \quad \text{for each } 0 \leq k \leq 2s-n.$$

Therefore N_+ is dense in $\check{D}$, and that $\check{D} \setminus N_+$ is an analytic subvariety, which is close in $\check{D}$ and $\mathrm{codim}_{\mathbb{C}}(\check{D}\backslash N_+) \geq 1$. We consider the period map $\Phi : \mathcal{T} \to \check{D}$ as a holomorphic map to $\check{D}$, then $\mathcal{T} \setminus \check{\mathcal{T}} = \Phi^{-1}(\check{D} \setminus N_+)$ is the pre-image of the holomorphic map Φ. So $\mathcal{T} \setminus \check{\mathcal{T}}$

is also an analytic subvariety and a close set in $\mathcal{T}$. Because $\mathcal{T}$ is smooth and connected, if $\dim(\mathcal{T} \setminus \check{\mathcal{T}}) = \dim \mathcal{T}$, then $\mathcal{T} \setminus \check{\mathcal{T}} = \mathcal{T}$ and $\check{\mathcal{T}} = \emptyset$. But this contradicts to the fact that the reference point $p \in \check{\mathcal{T}}$. Therefore we have $\dim(\mathcal{T} \setminus \check{\mathcal{T}}) < \dim \mathcal{T}$, and as a consequent $\operatorname{codim}_{\mathbb{C}}(\mathcal{T} \backslash \check{\mathcal{T}}) \geq 1$. ∎

We give another proof for this lemma in a more direct manner in Appendix A Lemma A.2.3.

Theorem 4.5.5 *The image of $\Phi : \mathcal{T} \to D$ lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on N_+.*

Proof According to Lemma 4.5.4, $\mathcal{T} \backslash \check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ and the complex codimension of $\mathcal{T} \backslash \check{\mathcal{T}}$ is at least one; by Proposition 4.5.1, the holomorphic map $\Phi : \check{\mathcal{T}} \to N_+ \cap D$ is bounded in N_+ with respect to the Euclidean metric. Thus by the Riemann extension theorem, there exists a holomorphic map $\Phi' : \mathcal{T} \to N_+ \cap D$ such that $\Phi'|_{\check{\mathcal{T}}} = \Phi|_{\check{\mathcal{T}}}$. Since as holomorphic maps, Φ' and Φ agree on the open subset $\check{\mathcal{T}}$, they must be the same on the entire $\mathcal{T}$. Therefore, the image of Φ is in $N_+ \cap D$, and the image is bounded with respect to the Euclidean metric on N_+. As a consequence, we also get $\mathcal{T} = \check{\mathcal{T}} = \Phi^{-1}(N_+)$. ∎

CHAPTER 5

A global Torelli theorem for Calabi–Yau type manifolds

In this chapter, we will prove a global Torelli theorem on the Teichmüller space of polarized and marked Calabi–Yau type manifolds, which states that the period map $\Phi : \mathcal{T} \to D$ is injective. The results in this chapter are collaborating work with F. Guan and K. Liu. The original work can be found in [10, 11].

We take a geometric approach to the problem, which has traditionally been studied via more algebraic techniques. Our proof consists of two main geometric constructions, the holomorphic affine structure on the Teichmüller space and the Hodge metric completion space of the Teichmüller space. Then we use these constructions to prove the theorem as follows. We extend the affine structure to the Hodge metric completion space, which together with the completeness allows us to connect any two points with a straight line. We can thus reduce the question of global injectivity of the extended period map to the question of injectivity on any such straight line. This can then be proven using the local injectivity of the period map.

Following the above idea, in § 5.1, we construct the affine structure on the Teichmüller space. In § 5.2, we construct the Hodge metric completion space and the extended period map on the Hodge metric completion space. In § 5.3, we prove the global Torelli theorem on the Teichmüller space of polarized and marked Calabi–Yau type manifolds. In § 5.4, we prove that the Hodge metric completion of the Teichmüller space is a domain of holomorphy using the global Torelli theorem.

5.1 Affine structure on the Teichmüller space

We first review the definition of complex affine manifolds. One may refer to [40, page 215] for more details.

Definition 5.1.1 *Let M be a complex manifold of complex dimension n. If there is a coordinate cover $\{(U_i,\, \phi_i);\, i \in I\}$ of M such that $\phi_{ik} = \phi_i \circ \phi_k^{-1}$ is a holomorphic affine transformation on $\mathbb{C}^n$ whenever $U_i \cap U_k$ is not empty, then $\{(U_i,\, \phi_i);\, i \in I\}$ is called a complex affine coordinate cover on M and it defines a holomorphic affine structure on M.*

Let us still fix an adapted basis $(\eta_0, \cdots, \eta_{m-1})$ for the Hodge decomposition of the base point $\Phi(p) \in D$. Recall that we can identify elements in N_+ with nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix, and element in B with nonsingular block upper triangular matrices. Therefore $N_+ \cap B = \{Id\}$.

By Theorem 4.5.5, we know that $\mathcal{T} = \Phi^{-1}(N_+)$. Therefore, each $\Phi(q)$ can be represented by a block lower triangular matrices whose diagonal blocks are all identity submatrices for any $q \in \mathcal{T}$. In particular, such representation is unique. In fact, for any $\Phi(q) \in N_+$, let us assume there are two lower triangular matrices A and A', whose diagonal matrix are all identity submatrices, and both represent elements $\Phi(q) \in N_+$. Then as here we view $N_+ \cong N_+B/B \subseteq \check{D}$, there exists a matrix C representing an element in B such that $A = A'C$. Since $(A')^{-1}A = C \in N_+ \cap B = \{Id\}$, the matrix C mush be the identity matrix. Thus $A = A'$. We denote by $\phi(q)$, which is nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrices and represents the point $\Phi(q) \in N_+$ under the fixed adapted basis for the Hodge decomposition of the base point. Let us denote $\phi(q) = [\phi_{ij}(q)]_{0 \leq i,j \leq m-1} = [\phi^{i,j}(q)]_{0 \leq i,j \leq 2n-s}$, where $\phi_{ij}(q)$ is the (i,j)-th entry of $\phi(q)$, and $\phi^{i,j}(q)$ is the (i,j)-th block of $\phi(q)$. Thus we can define a holomorphic map

$$\tau : \mathcal{T} \to \mathbb{C}^N \cong H_p^{s,n-s}, \quad \tau(q) = (\tau_1(q), \tau_2(q), \cdots, \tau_N(q)) = (\phi_{10}(q), \phi_{20}(q), \cdots, \phi_{N0}(q)).$$

Notice that according to our convention of block matrix in (3.10), $\tau(q)$ is actually the transpose of the $(1,0)$-block of $\phi(q)$. The following Proposition 5.1.3, which originally given in

[10, 11], will show that τ gives desired affine structure on $\mathcal{T}$.

Remark 5.1.2 *If we define the following projection map with respect to the base point and the its pre-fixed adapted basis to the Hodge decomposition,*

$$P : N_+ \cap D \to H_p^{s,n-s} \cong \mathbb{C}^N, \quad \psi \mapsto (\eta_1, \cdots, \eta_N)\psi^{(1,0)} = \psi_{10}\eta_1 + \cdots + \psi_{N0}\eta_N, \tag{5.1}$$

where $\psi^{(1,0)}$ is the $(1,0)$-block of the unipotent matrix F, according to our convention in (3.10)*, then $\tau = P \circ \Phi : \mathcal{T} \to \mathbb{C}^N$.*

Proposition 5.1.3 *The holomorphic map $\tau = (\tau_1, \cdots, \tau_N) : \mathcal{T} \to \mathbb{C}^N$ defines a coordinate chart around each point $q \in \mathcal{T}$.*

Proof We have that the generator $\{\eta_0\} \subseteq H_p^{s,n-s}$, the generators $\{\eta_1, \cdots, \eta_N\} \subseteq H_p^{s-1,n-s+1}$, and the generators $\{\eta_{N+1}, \cdots, \eta_{m-1}\} \in \oplus_{k \geq n-s+2} H_p^{n-k,k}$.

Recall that $\phi(q) = [\phi_{ij}(q)]_{0 \leq i,j \leq m-1} = [\phi^{i,j}(q)]_{0 \leq i,j \leq 2s-n}$ is a block lower triangular matrix with the diagonal blocks all equal to identity submatrices, where $\phi_{ij}(q)$ is the (i,j)-th entry of $\phi(q)$, and $\phi^{i,j}(q)$ is the (i,j)-th block of $\phi(q)$. On one hand, the submatrix $[\phi_{i0}(q)]_{0 \leq i \leq m-1}$ allows one to define the following map:

$$\Omega : \mathcal{T} \to F_q^s, \quad \Omega(q) = (\eta_0, \cdots, \eta_{m-1})(\phi_{00}(q), \phi_{10}(q), \cdots, \phi_{N0}(q), \cdots)^T.$$

As $\phi_{00}(q) = \phi^{0,0}(q) = 1$, we have $\Omega(q) = \eta_0 + \tau_1(q)\eta_1 + \tau_2(q)\eta_2 + \cdots \tau_N(q)\eta_N + g_0(q) \in F_q^s$, where $g_0(q) \in \bigoplus_{k \geq n-s+2} H_p^{n-k,k}$. On the other hand, the submatrix $[\phi_{ij}(q)]_{0 \leq i \leq m-1, 1 \leq j \leq N}$ allows up to define the following map:

$$(\theta_1, \cdots, \theta_N) : \mathcal{T} \to F_q^{s-1}, \quad (\theta_1(q), \cdots, \theta_N(q)) \mapsto (\eta_1, \cdots, \eta_{m-1})[\phi_{ij}(q)]_{0 \leq i \leq m-1, 1 \leq j \leq N}.$$

Again, as $[\phi_{ij}(q)]_{1 \leq i,j \leq N} = \phi^{1,1}(q) = I_{N \times N}$, where $I_{N \times N}$ is the $N \times N$ identity matrix, we can also write,

$$\theta_1(q) = \eta_1 + g_1(q), \quad \cdots\cdots, \quad \theta_N(q) = \eta_N + g_N(q) \in F_q^{s-1}, \tag{5.2}$$

where $g_k(q) \in \bigoplus_{k \geq n-s+2} H_p^{n-k,k}$.

As $\{\Omega(q), \theta_1(q), \cdots, \theta_N(q)\}$ are linear independent, it forms a basis for F_q^{s-1} for each $q \in \mathcal{T}$. By Proposition 4.2.1, we also know that for any holomorphic coordinate $\{\sigma_1, \cdots, \sigma_N\}$ around q, $\{\Omega(q), \frac{\partial\Omega(q)}{\partial\sigma_1}, \cdots, \frac{\partial\Omega(q)}{\partial\sigma_N}\}$ forms a basis of F_q^{s-1}. Thus as both $\{\Omega(q), \theta_1(q), \cdots, \theta_N(q)\}$ and $\{\Omega(q), \frac{\partial\Omega(q)}{\partial\sigma_1}, \cdots, \frac{\partial\Omega(q)}{\partial\sigma_N}\}$ are bases for F_q^{s-1}, there exists $\{X_1, \cdots, X_N\}$ with

$$X_k = \sum_{i=1}^{N} a_{ik} \frac{\partial}{\partial\sigma_i} \quad \text{for each} \quad 1 \leq k \leq N$$

such that

$$\begin{aligned}
&\theta_k = X_k(\Omega(q)) + \lambda_k\Omega(q) = \eta_k + g_k(q) \\
&= X_k(\tau_1(q))\eta_1 + \cdots + X_k(\tau_N(q))\eta_N + X_k(g_0(q)) + \lambda_k(\eta_0 + \tau_1(q)\eta_1 + \cdots \tau_N(q)\eta_N + g_0(q))
\end{aligned}$$

for all $1 \leq k \leq N$, as $\Omega(q) = \eta_0 + \tau_1(q)\eta_1 + \tau_2(q)\eta_2 + \cdots \tau_N(q)\eta_N + g_0(q)$. By comparing the coefficient of η_0 for the expressions of θ_k, we get $\lambda_k = 0$. Thus we have

$$X_k(\Omega(q)) = \theta_k(q) = \eta_k + g_k(q) \quad \text{for each } 1 \leq k \leq N. \tag{5.3}$$

Since $\{\theta_1(q), \cdots, \theta_N(q)\}$ is a linearly independent subset of F_q^{s-1}, we know that $\{X_1, \cdots, X_N\}$ is also linearly independent in $T_q^{1,0}(\mathcal{T})$. Therefore $\{X_1, \cdots, X_N\}$ forms a basis for $T_q^{1,0}(\mathcal{T})$. Without loss of generality, we may assume $X_k = \frac{\partial}{\partial\sigma_k}$ for each $1 \leq k \leq N$. Thus by (5.3), we have

$$\frac{\partial\Omega(q)}{\partial\sigma_k} = \theta_k = \eta_k + g_k(q) \quad \text{for any} \quad 1 \leq k \leq N.$$

Since we also have

$$\frac{\partial\Omega(q)}{\partial\sigma_k} = \frac{\partial}{\partial\sigma_k}(\eta_0 + \tau_1(q)\eta_1 + \tau_2(q)\eta_2 + \cdots \tau_N(q)\eta_N + g_0(q)),$$

we get $\left[\frac{\partial\tau_i(q)}{\partial\sigma_j}\right]_{1\leq i,j\leq N} = \mathrm{I}_N$. This shows that $\tau_* : T_q^{1,0}(\mathcal{T}) \to T_{\tau(q)}(\mathbb{C}^N)$ is an isomorphism for each $q \in \mathcal{T}$, as $\{\frac{\partial}{\partial\sigma_1}, \cdots, \frac{\partial}{\partial\sigma_N}\}$ is a basis for $T_q^{1,0}(\mathcal{T})$. ∎

By Proposition 5.1.3, the global holomorphic map $\tau : \mathcal{T} \to \mathbb{C}^N$ defines local coordinate map around each point $q \in \mathcal{T}$. In particular, the map τ itself gives a global holomorphic coordinate for $\mathcal{T}$. Thus the transition maps are all identity maps. Therefore,

Theorem 5.1.4 *The global holomorphic coordinate map $\tau : \mathcal{T} \to \mathbb{C}^N$ defines a holomorphic affine structure on $\mathcal{T}$.*

Remark 5.1.5 *This affine structure on $\mathcal{T}$ depends on the choice of the base point p. Affine structures on $\mathcal{T}$ defined in this ways by fixing different base point may not be compatible with each other.*

5.2 Hodge metric completion spaces

Recall that we assume that there exists $m_0 \in \mathbb{Z}$ such that for any $m \geq m_0$, the moduli space $\mathcal{Z}_m$ is a connected quasi-projective smooth complex manifolds with a versal family $\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m$ of projective manifolds with level m structures, containing M as a fiber and polarized by an ample line bundle $\mathcal{L}_{\mathcal{Z}_m}$ on $\mathcal{X}_{\mathcal{Z}_m}$. We then proved that the Teichmüller space $\mathcal{T}$ is the universal cover of $\mathcal{Z}_m$ for any $m \geq m_0$ with the universal covering map $\pi_m : \mathcal{T} \to \mathcal{Z}_m$. Then by the work of Viehweg in [63], we can find a smooth projective compactification $\bar{\mathcal{Z}}_m$ such that $\mathcal{Z}_m$ is open in $\bar{\mathcal{Z}}_m$ and the complement $\bar{\mathcal{Z}}_m \backslash \mathcal{Z}_m$ is a divisor of normal crossing. Therefore, $\mathcal{Z}_m$ is dense and open in $\bar{\mathcal{Z}}_m$ with the complex codimension of the complement $\bar{\mathcal{Z}}_m \backslash \mathcal{Z}_m$ at least one. Moreover, as $\bar{\mathcal{Z}}_m$ a compact space, it is a complete space. Let us now take $\mathcal{Z}^H_{_m}$ to be completion space of $\mathcal{Z}_m$ with respect to the Hodge metric on $\mathcal{Z}_m$. Then $\mathcal{Z}^H_{_m}$ is the smallest complete space with respect to the Hodge metric that contains $\mathcal{Z}_m$. In particular, $\mathcal{Z}^H_{_m} \subseteq \bar{\mathcal{Z}}_m$. Therefore, the complex codimension of the complement $\mathcal{Z}^H_{_m} \backslash \mathcal{Z}_m$ is at least one. Moreover, we have the following properties of the completion space $\mathcal{Z}^H_{_m}$, which is originally proved in [10, 11].

Lemma 5.2.1 *The Hodge metric completion $\mathcal{Z}^H_{_m}$ is a dense and open smooth submanifold in $\bar{\mathcal{Z}}_m$, and the complex codimenison of $\mathcal{Z}^H_{_m} \backslash \mathcal{Z}_m$ is at least one.*

Proof It suffices to show that $\mathcal{Z}^H_{_m}$ is an open subset in $\bar{\mathcal{Z}}_m$. Since $\bar{\mathcal{Z}}_{_m} \backslash \mathcal{Z}_m$ is a normal crossing divisor in $\bar{\mathcal{Z}}_{_m}$, any point $q \in \bar{\mathcal{Z}}_{_m} \backslash \mathcal{Z}_m$ has a neighborhood $U_q \subset \bar{\mathcal{Z}}_{_m}$ such that $U_q \cap \mathcal{Z}_m$ is biholomorphic to $(\Delta^*)^k \times \Delta^{N-k}$. Given a fixed reference point $p \in \mathcal{Z}_m$, and any point $p' \in U_q \cap \mathcal{Z}_m$, there is a piecewise smooth curve $\gamma_{p,p'}: [0,1] \to \mathcal{Z}_m$ connecting p and p' since $\mathcal{Z}_m$ is a path connected manifold. There is also a piecewise smooth curve $\gamma_{p',q}$ in $U_q \cap \mathcal{Z}_m$ connecting p' and q. Therefore, there is a piecewise smooth curve $\gamma_{p,q}: [0,1] \to \bar{\mathcal{Z}}_m$ connecting p and q with $\gamma_{p,q}((0,1)) \subset \mathcal{Z}_m$. Thus for any point $q \in \bar{\mathcal{Z}}_m$, we can define the Hodge distance between q and p by

$$d_H(p,q) = \inf_{\gamma_{p,q}} L_H(\gamma_{p,q}),$$

where $L_H(\gamma_{p,q})$ denotes the length of the curve $\gamma_{p,q}$ with respect to the Hodge metric on $\mathcal{Z}_m$, and the infimum is taken over all piecewise smooth curves $\gamma_{p,q}: [0,1] \to \bar{\mathcal{Z}}_m$ connecting p and q with $\gamma_{p,q}((0,1)) \subset \mathcal{Z}_m$.

Then $\mathcal{Z}_m^H$ is the set of all the points q in $\bar{\mathcal{Z}}_m$ with $d_H(p,q) < \infty$, and the complement set $\bar{\mathcal{Z}}_m \backslash \mathcal{Z}_m^H$ contains all the points q in $\bar{\mathcal{Z}}_m$ with $d_H(p,q) = \infty$. Now let $\{q_k\}_{k=1}^{\infty} \subset \bar{\mathcal{Z}}_m \backslash \mathcal{Z}_m^H$ be a convergent sequence and assume that the limit of this sequence is q_∞. Then the Hodge distance between the reference point p and the limiting point q_∞ is clearly $d(p, q_\infty) = \infty$ as well, since $d(p, q_k) = \infty$ for all k. Thus $q_\infty \in \bar{\mathcal{Z}}_m \backslash \mathcal{Z}_m^H$. Therefore, the set of the points of Hodge infinite distance form p is a close set in $\bar{\mathcal{Z}}_m$. We conclude that $\mathcal{Z}_m^H$ is an open submanifold of $\bar{\mathcal{Z}}_m$. ∎

We give an another proof of the above in Appendix A Lemma A.2.4, which is also originally given in [10].

We recall some basic properties about metric completion space we are using in this paper. We know that the metric completion space of a connected space is still connected. Therefore, $\mathcal{Z}_m^H$ is connected.

Suppose (X, d) is a metric space with the metric d. Then the metric completion space of (X, d) is unique in the following sense: if $\bar{X}_1$ and $\bar{X}_2$ are complete metric spaces that both contain X as a dense set, then there exists an isometry $f: \bar{X}_1 \to \bar{X}_2$ such that $f|_X$ is the identity map on X. Moreover, as mentioned above that the metric completion space $\bar{X}$ of X is the smallest complete metric space containing X in the sense that any other complete space that contains X as a subspace must also contains $\bar{X}$ as a subspace.

Moreover, suppose $\bar{X}$ is the metric completion space of the metric space (X, d). If there is a continuous map $f : X \to Y$ which is a local isometry with Y a complete space, then there exists a continuous extension $\bar{f} : \bar{X} \to Y$ such that $\bar{f}|_X = f$.

In the rest of the thesis, unless otherwise pointed out, when we mention a complete space, the completeness is always with respect to the Hodge metric.

Let $\mathcal{T}^H_m$ be the universal cover of $\mathcal{Z}^H_m$ with the universal covering map $\pi^H_m : \mathcal{T}^H_m \to \mathcal{Z}^H_m$. Thus $\mathcal{T}^H_m$ is a connected and simply connected complete smooth complex manifold with respect to the Hodge metric. The complete manifold $\mathcal{T}^H_m$ is called the *Hodge metric completion space with level m structure*. Since $\mathcal{Z}^H_m$ is the Hodge metric completion of $\mathcal{Z}_m$, we have the continuous extension map $\Phi^H_{\mathcal{Z}_m} : \mathcal{Z}^H_m \to D/\Gamma$. In particular, we obtain the following commutative diagram:

$$\begin{tikzcd} \mathcal{T} \arrow[r, "i_m"] \arrow[d, "\pi_m"] & \mathcal{T}^H_m \arrow[r, "\Phi^H_m"] \arrow[d, "\pi^H_m"] & D \arrow[d, "\pi_D"] \\ \mathcal{Z}_m \arrow[r, "i"] & \mathcal{Z}^H_m \arrow[r, "\Phi^H_{\mathcal{Z}_m}"] & D/\Gamma, \end{tikzcd} \tag{5.4}$$

with i the inclusion map, i_m a lifting map of $i \circ \pi_m$, π_D the covering map and Φ^H_m a lifting map of $\Phi^H_{\mathcal{Z}_m} \circ \pi^H_m$. By Lemma A.3.1 in Appendix A, which is originally proved in [10, Lemma A.1], there exists a suitable choice of i_m and Φ^H_m such that $\Phi^H_m \circ i_m = \Phi$. We will fix such choice of i_m and Φ^H_m in the rest of the thesis. Moreover, we have the following result which is also originally proved in Proposition 4.3 in [10].

Proposition 5.2.2 *The image $\mathcal{T}_m := i_m(\mathcal{T})$ equals to the preimage $(\pi^H_m)^{-1}(\mathcal{Z}_m)$.*

Proof Because of the commutativity of diagram (5.6), we have that $\pi^H_m(i_m(\mathcal{T})) = i(\pi_m(\mathcal{T})) = \mathcal{Z}_m$. Therefore, $\mathcal{T}_m = i_m(\mathcal{T}) \subseteq (\pi^H_m)^{-1}(\mathcal{Z}_m)$. For the other direction, we need to show that for any point $q \in (\pi^H_m)^{-1}(\mathcal{Z}_m) \subseteq \mathcal{T}^H_m$, one has $q \in i_m(\mathcal{T}) = \mathcal{T}_m$.

Let $p = \pi^H_m(q) \in i(\mathcal{Z}_m)$, Let $x_1 \in \pi_m^{-1}(i^{-1}(p)) \subseteq \mathcal{T}$ be an arbitrary point, then $\pi^H_m(i_m(x_1)) = i(\pi_m(x_1)) = p$ and $i_m(x_1) \in (\pi^H_m)^{-1}(p) \subseteq \mathcal{T}^H_m$.

As $\mathcal{T}^H_m$ is a connected smooth complex manifold, $\mathcal{T}^H_m$ is path connected. Therefore, for $i_m(x_1)$ and $q \in \mathcal{T}^H_m$, there exists a curve $\gamma : [0,1] \to \mathcal{T}^H_m$ with $\gamma(0) = i_m(x_1)$ and $\gamma(1) = q$. Then the composition $\pi^H_m \circ \gamma$ gives a loop on $\mathcal{Z}^H_m$ with $\pi^H_m \circ \gamma(0) = \pi^H_m \circ \gamma(1) = p$. Lemma A.3.2 implies that there is a loop Γ on $\mathcal{Z}_m$ with $\Gamma(0) = \Gamma(1) = i^{-1}(p)$ such that

$$[i \circ \Gamma] = [\pi^H_m \circ \gamma] \in \pi_1(\mathcal{Z}^H_m),$$

where $\pi_1(\mathcal{Z}^H_m)$ denotes the fundamental group of $\mathcal{Z}^H_m$. Because $\mathcal{T}$ is universal cover of $\mathcal{Z}_m$, there is a unique lifting map $\tilde{\Gamma} : [0,1] \to \mathcal{T}$ with $\tilde{\Gamma}(0) = x_1$ and $\pi_m \circ \tilde{\Gamma} = \Gamma$. Again since $\pi^H_m \circ i_m = i \circ \pi_m$, we have

$$\pi^H_m \circ i_m \circ \tilde{\Gamma} = i \circ \pi_m \circ \tilde{\Gamma} = i \circ \Gamma : [0,1] \to \mathcal{Z}_m.$$

Therefore $[\pi^H_m \circ i_m \circ \tilde{\Gamma}] = [i \circ \Gamma] \in \pi_1(\mathcal{Z}_m)$, and the two curves $i_m \circ \tilde{\Gamma}$ and γ have the same starting points $i_m \circ \tilde{\Gamma}(0) = \gamma(0) = i_m(x_1)$. Then the homotopy lifting property of the covering map π^H_m implies that $i_m \circ \tilde{\Gamma}(1) = \gamma(1) = q$. Therefore, $q \in i_m(\mathcal{T})$, as needed. ∎

Since $\mathcal{Z}_m$ is an open submanifold in $\mathcal{Z}^H_m$ and $\pi^H_m : \mathcal{T}^H_m \to \mathcal{Z}^H_m$ is a holomorphic covering map, the preimage $\mathcal{T}_m$ is then a connected open submanifold in $\mathcal{T}^H_m$. In particular, since the complex codimension of $\mathcal{Z}^H_m \backslash \mathcal{Z}_m$ in $\mathcal{Z}^H_m$ is at least one, the complex codimension of $\mathcal{T}^H_m \backslash \mathcal{T}_m$ is at least one in $\mathcal{T}^H_m$ as well. On the other hand, it is easy to check that Φ_m is a holomorphic map from $\mathcal{T}_m$ to $N_+ \cap D$, where N_+ is identified with its unipotent orbit in $\check{D}$ by fixing the base point $\Phi(p) \in D$ with $p \in \mathcal{T}$. In fact, according to the commutativity of the diagram (5.6), we have $i_m : \mathcal{T} \to \mathcal{T}_m$ is the lifting of $i \circ \pi_m$, i_m is locally invertible. Since $\pi^H_m|_{\mathcal{T}_m} : \mathcal{T}_m \to \mathcal{Z}_m$ is holomorphic covering map, i_m is also holomorphic. As $\Phi = \Phi_m \circ i_m$ as $i_m(\mathcal{T}) = \mathcal{T}_m$ with Φ and i_m both holomorphic and i_m locally invertible, Φ_m is holomorphic. In particular, since $\Phi_m(\mathcal{T}_m) = \Phi_m(i_m(\mathcal{T})) = \Phi(\mathcal{T}) \subseteq N_+ \cap D$, the map Φ_m is a holomorphic map from $\mathcal{T}_m$ to $N_+ \cap D$. Therefore, as Φ_m is locally bounded, we can apply the Riemann extension theorem to $\Phi_m : \mathcal{T}_m \to N_+ \cap D$ to first conclude that there exists a holomorphic map $\Phi'_m : \mathcal{T}^H_m \to N_+ \cap D$ such that $\Phi'_m|_{\mathcal{T}_m} = \Phi_m$. Then since Φ'_m and Φ^H_m are both continuous extension of Φ_m and they agree on the open dense subset $\mathcal{T}_m$, we conclude that Φ^H_m and Φ'_m

must agree on the whole set of $\mathcal{T}_m^H$. Moreover, since $\Phi = i_m \cdot \Phi_m$ is bounded, so is Φ_m. Therefore Φ_m^H is also bounded. Thus we get the following proposition.

Proposition 5.2.3 *The map Φ_m^H is a bounded holomorphic map from $\mathcal{T}_m^H$ to $N_+ \cap D$ with respect to the Hodge metric on N_+.*

In the rest of this section, we will prove affineness results about the space $\mathcal{T}_m$ and $\mathcal{T}_m^H$. We remark that the affine structures on $\mathcal{T}_m$ and $\mathcal{T}_m^H$ are induced from the affine structure on $\mathcal{T}$.

Let P be the projection map given by (5.1) with the same fixed base point $\Phi(p) \in D$ and $p \in \mathcal{T}$ and the fixed adapted basis $(\eta_0, \cdots, \eta_{m-1})$ for the Hodge decomposition of $\Phi(p)$. Then based on the above proposition, we can define the following holomorphic map

$$\tau_m^H = P \circ \Phi_m^H : \mathcal{T}_m^H \to \mathbb{C}^N. \tag{5.5}$$

Moreover, we also have $\tau = P \circ \Phi = P \circ \Phi_m^H \circ i_m = \tau_m^H \circ i_m$. Let us denote the restriction map $\tau_m = \tau_m^H|_{\mathcal{T}_m} : \mathcal{T}_m \to \mathbb{C}^N$ in the following context. Then τ_m^H is the continuous extension of τ_m and $\tau = \tau_m \circ i_m$. By the definition of τ_m^H, we can easily conclude the following lemma, which is originally in [10, 11].

Lemma 5.2.4 *The restriction map $\tau_m : \mathcal{T}_m \to \mathbb{C}^N \cong H_p^{s,n-s}$ is a local embedding. In particular, τ_m defines a holomorphic affine structure on $\mathcal{T}_m$.*

Proof We know that $i : \mathcal{Z}_m \to \mathcal{Z}_m^H$ is the natural inclusion map, π_m, π_m^H are both universal covering map. Since $i \circ \pi_m = \pi_m^H \circ i_m$, we have that i_m is locally biholomorphic. On the other hand, we showed in Proposition 5.1.3 that τ is also a local embedding. Consider an open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of $\mathcal{T}_m$. We may assume that for each $U_\alpha \subseteq \mathcal{T}_m$, i_m is biholomorphic on U_α and thus the inverse $(i_m)^{-1}$ is also an embedding on U_α. We may also assume that τ is an embedding on $(i_m)^{-1}(U_\alpha)$. In particular, the relation $\tau = \tau_m \circ i_m$ implies that $\tau_m|_{U_\alpha} = \tau \circ (i_m)^{-1}|_{U_\alpha}$ is also an embedding on U_α. This shows that τ_m is a local embedding on $\mathcal{T}_m$. In particular, $\tau_m : \mathcal{T}_m \to \mathbb{C}^N$ defines a local coordinate map around each point of $\mathcal{T}_m$. Thus τ_m gives a holomorphic coordinate cover on $\mathcal{T}_m$ whose transition maps are all identity maps. To conclude, we get that τ_m defines a holomorphic affine structure on $\mathcal{T}_m$. ∎

As a corollary of the above lemma, we conclude the following property of the map $\tau_{_m}^H$. One may refer to [10, Lemma 4.7] for the original proof.

Corollary 5.2.5 *The map $\tau_{_m}^H : \mathcal{T}_{_m}^H \to \mathbb{C}^N \cong H_p^{s,n-s}$ is a local embedding.*

Proof The proof uses mainly the affineness of $\tau_m : \mathcal{T}_m \to \mathbb{C}^N \cong H_p^{s-1,n-s+1}$. By Proposition 5.2.2, we know that $\mathcal{T}_m$ is dense and open in $\mathcal{T}_{_m}^H$. Thus for any point $q \in \mathcal{T}_{_m}^H$, there exists $\{q_k\}_{k=1}^\infty \subseteq \mathcal{T}_m$ such that $\lim_{k\to\infty} q_k = q$. As $\tau_{_m}^H(q) \in H_p^{s-1,n-s+1}$, we can take a neighborhood $W \subseteq H_p^{s-1,n-s+1}$ of $\tau_{_m}^H(q)$ with $W \subseteq \tau_{_m}^H(\mathcal{T}_{_m}^H)$.

Consider the projection map $P : N_+ \to \mathbb{C}^N$ with $P(F) = F^{(1,0)}$ the $(1,0)$ block of the matrix F, and the decomposition of the holomorphic tangent bundle

$$T^{1,0}N_+ = \bigoplus_{n-s\le l\le k\le s} \mathrm{Hom}(F^k/F^{k+1}, F^l/F^{l+1}).$$

In particular, the subtangent bundle $\mathrm{Hom}(F^s, F^{s-1}/F^s)$ over N_+ is isomorphic to the pull-back bundle $P^*(T^{1,0}\mathbb{C}^N)$ of the holomorphic tangent bundle of $\mathbb{C}^N$ through the projection P. On the other hand, the holomorphic tangent bundle of $\mathcal{T}_m$ is also isomorphic to the holomorphic bundle $\mathrm{Hom}(F^s, F^{s-1}/F^s)$, where F^s and F^{s-1} are pull-back bundles on $\mathcal{T}_m$ via Φ_m from $N_+ \cap D$. Since the holomorphic map $\tau_m = P \circ \Phi_m$ is a composition of P and Φ_m, the pull-back bundle of $T^{1,0}W$ through τ_m is also isomorphic to the tangent bundle of $\mathcal{T}_m$.

Now with the fixed adapted basis $\{\eta_1, \cdots, \eta_N\}$, one has a standard coordinate $(z_1, \cdots, z_N)$ on $\mathbb{C}^N \cong H_p^{s-1,n-s+1}$ such that each point in $\mathbb{C}^N \cong H_p^{s-1,n-s+1}$ is of the form $z_1\eta_1 + \cdots + z_N\eta_N$. Let us choose one special trivialization of

$$T^{1,0}W \cong \mathrm{Hom}(F_p^s, F_p^{s-1}/F_p^{s-1}) \times W$$

by the standard global holomorphic frame $(\Lambda_1, \cdots, \Lambda_N) = (\partial/\partial z_1, \cdots, \partial/\partial z_N)$ on $T^{1,0}W$. Under this trivialization, we can identify $T_o^{1,0}W$ with $\mathrm{Hom}(F_p^s, F_p^{s-1}/F_p^s)$ for any $o \in W$. Then $(\Lambda_1, \cdots, \Lambda_N)$ are parallel sections with respect to the trivial affine connection on $T^{1,0}W$. Let $U_q \subseteq (\tau_{_m}^H)^{-1}(W)$ be a neighborhood of q and let $U = U_q \cap \mathcal{T}_m$. Then the pull back sections

$(\tau_{_m}^H)^*(\Lambda_1, \cdots, \Lambda_N) : U_q \to T^{1,0}U_q$ are tangent vectors of U_q, we denote them by $(\mu_1^H, \cdots, \mu_N^H)$ for convenience.

According to the proof of Lemma 5.2.4, we know that the restriction map τ_m is a local embedding. Therefore the tangent map $(\tau_m)_* : T^{1,0}_{q'}U \to T^{1,0}_o W$ is an isomorphism, for any $q' \in U$ and $o = \tau_m(q')$. Moreover, since τ_m is a holomorphic affine map, the holomorphic sections $(\mu_1, \cdots, \mu_N) := (\mu_1^H, \cdots, \mu_N^H)|_U$ form a holomorphic parallel frame for $T^{1,0}U$. Under the parallel frames $(\mu_1, \cdots, \mu_N)$ and $(\Lambda_1, \cdots, \Lambda_N)$, there exists a nonsingular matrix function $A(q') = (a_{ij}(q'))_{1\le i\le N, 1\le j\le N}$, such that the tangent map $(\tau_m)_*$ is given by

$$(\tau_m)_*(\mu_1, \cdots, \mu_N)(q') = (\Lambda_1(o), \cdots, \Lambda_N(o))A(q'), \quad \text{with } q' \in U \text{ and } o = \tau_m(q') \in D.$$

Moreover, since $(\Lambda_1, \cdots, \Lambda_N)$ and $(\mu_1, \cdots, \mu_N)$ are parallel frames for $T^{1,0}W$ and $T^{1,0}U$ respectively and τ_m is a holomorphic affine map, the matrix $A(q') = A$ is actually a constant nonsingular matrix for all $q' \in U$. In particular, for each $q_k \in U$, we have $((\tau_m)_*\mu_1, \cdots, (\tau_m)_*\mu_N)(q_k) = (\Lambda_1(o_k), \cdots, \Lambda_N(o_k))A$, where $o_k = \tau_m(q_k)$. Because the tangent map $(\tau_{_m}^H)_* : T^{1,0}U_q \to T^{1,0}W$ is a continuous map, we have that

$$\begin{aligned}(\tau_{_m}^H)_*(\mu_1^H(q), \cdots, \mu_N^H(q)) &= \lim_{k\to\infty}(\tau_m)_*(\mu_1(q_k), \cdots, \mu_N(q_k)) = \lim_{k\to\infty}(\Lambda_1(o_k), \cdots, \Lambda_N(o_k))A \\ &= (\Lambda_1(\bar o), \cdots, \Lambda_N(\bar o))A, \quad \text{where} \quad o_k = \tau_m(q_k) \ \text{ and } \ \bar o = \tau_{_m}^H(q).\end{aligned}$$

As $(\Lambda_1(\bar o), \cdots, \Lambda_N(\bar o))$ forms a basis for $T^{1,0}_{\bar o}W = \mathrm{Hom}(F_p^s, F_p^{s-1}/F_p^s)$ and A is nonsingular, we can conclude that $(\tau_{_m}^H)_*$ is an isomorphism from $T^{1,0}_q U_q$ to $T^{1,0}_{\bar o}W$. This shows that $\tau_{_m}^H : \mathcal{T}_{_m}^H \to \mathbb{C}^N \cong H_p^{s-1,n-s+1}$ is a local embedding. ∎

Theorem 5.2.6 *The holomorphic map $\tau_{_m}^H : \mathcal{T}_{_m}^H \to \mathbb{C}^N$ is a local embedding, and it defines a holomorphic affine structure on $\mathcal{T}_{_m}^H$.*

Proof Since $\tau_{_m}^H : \mathcal{T}_{_m}^H \to \mathbb{C}^N$ is a local embedding and $\dim \mathcal{T}_{_m}^H = N$, thus the same arguments as the proof of Lemma 5.2.4 can be adopted to conclude $\tau_{_m}^H$ defines a global holomorphic affine structure on $\mathcal{T}_{_m}^H$. ∎

5.3 Global Torelli Theorem on the Teichmüller space

In this section, we prove that the global Torelli theorem on the Teichmüller space of polarized and marked Calabi–Yau type manifolds, which states that the period map $\Phi : \mathcal{T} \to D$ is injectitive. The main idea is that by realizing the following commutative diagram,

$$\begin{tikzcd} \mathcal{T} \arrow[r, "i_m"] \arrow[rr, bend left, "\Phi"] & \mathcal{T}^H_m \arrow[r, "\Phi^H_m"] & D \end{tikzcd},$$

we prove that the map Φ^H_m and i_m are both injective to conclude that the period map Φ is also injective.

In the proof of injectivity of Φ^H_m, the affine structure on the Hodge metric completion space together with the completeness allows us to connect any two points with a straight line. Then to prove the global injectivity of the extended period map, it suffices to prove the injectivity of the period map on any such straight line. And finally the local injectivity of the period map implies its injectivity on any such straight line. In the proof of the injectivity of i_m, the properties that Φ^H_m is injective, that there are markings on the polarized Calabi–Yau type manifolds, and that the Teichmüller space is simply connected come in to play substantially

Proposition 5.3.1 *The holomorphic map $\Phi^H_m : \mathcal{T}^H_m \to D$ is injective.*

Proof Since $\tau^H_m = P \circ \Phi^H_m : \mathcal{T}^H_m \to \mathbb{C}^N$, where P is a projection map, to show Φ^H_m is injective, it is enough to prove τ^H_m is injective, which is given by the following Lemma 5.3.2. ∎

Lemma 5.3.2 *The holomorphic map $\tau^H_m : \mathcal{T}^H_m \to \mathbb{C}^N$ is injective.*

Proof Note that as $\mathcal{T}^H_m$ is a complex affine manifold, we have the notion of straight lines in it with respect to the affine structure. We first claim that for any two points in $\mathcal{T}^H_m$, there is a straight line in $\mathcal{T}^H_m$ connecting them. Let us take an arbitrary point $p \in \mathcal{T}^H_m$, and $S \subseteq \mathcal{T}$

be the set so that for any $q \in S$, there exists a straight line in $\mathcal{T}_{_m}^H$ that connects p and q. Then to show the claim, we need to show $S = \mathcal{T}_{_m}^H$, and this is enough to show that S is an open and closed set in $\mathcal{T}_{_m}^H$.

We first show that S is a closed set. Let $\{q_i\}_{i=1}^{\infty} \subseteq S$ be a Cauchy sequence with respect to the Hodge metric. Then for each i we have the straight line l_i connecting p and q_i such that $l_i(0) = p$, $l_i(T_i) = q_i$ for some $T_i \geq 0$ and $v_i := \frac{\partial}{\partial t} l_i(0)$ a unit vector with respect to the Euclidean metric on n_+. We can view these straight lines $l_i : [0, T_i] \to \mathcal{T}_{_m}^H$ as the solutions of the affine geodesic equations $l_i''(t) = 0$ with initial conditions $v_i := \frac{\partial}{\partial t} l_i(0)$ and $l_i(0) = p$ in particular $T_i = d_E(p, q_i)$ is the Euclidean distance between p and q_i. It is well-known that solutions of these geodesic equations analytically depend on their initial data.

Proposition 5.2.3 showed that $\Phi_{_m}^H : \mathcal{T}_{_m}^H \to N_+ \cap D$ is a bounded map, which implies that the image of $\Phi_{_m}^H$ is bounded with respect to the Euclidean metric on N_+. Because a linear projection will map a bounded set to a bounded set, we have that the image of $\tau_{_m}^H = P \circ \Phi_{_m}^H$ is also bounded in $\mathbb{C}^N$ with respect to the Euclidean metric on $\mathbb{C}^N$. Passing to a subsequence, we may therefore assume that $\{T_i\}$ and $\{v_i\}$ converge, with $\lim_{i\to\infty} T_i = T_\infty$ and $\lim_{i\to\infty} v_i = v_\infty$, respectively. Let $l_\infty(t)$ be the local solution of the affine geodesic equation with initial conditions $\frac{\partial}{\partial t} l_\infty(0) = v_\infty$ and $l_\infty(0) = p$. We claim that the solution $l_\infty(t)$ exists for $t \in [0, T_\infty]$. Consider the set

$$E_\infty := \{a : \, l_\infty(t) \quad \text{exists for} \quad t \in [0, a)\}.$$

If E_∞ is unbounded above, then the conclusion is obvious. Otherwise, let $a_\infty = \sup E_\infty$, and then we need to show $a_\infty > T_\infty$. Suppose towards a contradiction that $a_\infty \leq T_\infty$. One defines the sequence $\{a_k\}_{k=1}^{\infty}$ so that $a_k / T_k = a_\infty / T_\infty$. We have $a_k \leq T_k$ and $\lim_{k\to\infty} a_k = a_\infty$. Then the continuous dependence of solutions of the geodesic equation on initial data implies that the sequence $\{l_k(a_k)\}_{k=1}^{\infty}$ is a Cauchy sequence. Note that $\mathcal{T}_{_m}^H$ is a completion space, thus the sequence $\{l_k(a_k)\}_{k=1}^{\infty}$ converges to some $q' \in \mathcal{T}_{_m}^H$. Define $l_\infty(a_\infty) := q'$. Then the solution $l_\infty(t)$ exists for $t \in [0, a_\infty]$. On the other hand, q' is an inner point of $\mathcal{T}_{_m}^H$ since $\mathcal{T}_{_m}^H$ is a smooth manifold. Thus the affine geodesic equation has a local solution at

q' that extends the geodesic l_∞. That is, there exists $\epsilon > 0$ such that the solution $l_\infty(t)$ exists for $t \in [0, a_\infty + \epsilon)$. This contradicts the fact that a_∞ is an upper bound of E_∞. We have therefore proven that $l_\infty(t)$ exists for $t \in [0, T_\infty]$. Then again since the continuous dependence of solutions of the affine geodesic equations on the initial data, we conclude that $l_\infty(T_\infty) = \lim_{k\to\infty} l_k(T_k) = \lim_{k\to\infty} q_k = q_\infty$. This means the limit point $q_\infty \in S$, and hence S is a closed set.

We now show that S is an open set. For any point $q \in S$, there exists a straight line l connecting p and q. Then for each point $x \in l$ there exists an open neighborhood $U_x \subset \mathcal{T}^H_{_m}$ with diameter $2r_x$. Therefore the collection $\{U_x\}_{x\in l}$ forms an open cover of l. Since l is a compact set, there is a finite subcover $\{U_{x_i}\}_{i=1}^K$ of l. Let $r = \min\{r_{x_i} :\ 1 \le i \le K\}$, then the straight line l is covered by a cylinder C_r of radius r in $\mathcal{T}^H_{_m}$. As C_r is a convex set, each point in C_r can be connected to p by a straight line. Therefore we have found an open neighborhood $C_r \subseteq S$ of $q \in S$. This implies that S is an open set.

Therefore, we have proved the claim that any two points in $\mathcal{T}^H_{_m}$ can by connected by a straight line.

Let $p, q \in \mathcal{T}^H_{_m}$ be two different points. Suppose towards a contradiction that $\tau^H_{_m}(p) = \tau^H_{_m}(q) \in \mathbb{C}^N$. On one hand, we have showed there is a straight line $l \subseteq \mathcal{T}^H_{_m}$ connecting p and q. Since $\tau^H_{_m} : \mathcal{T}^H_{_m} \to \mathbb{C}^N$ is affine, we have that $\tau^H_{_m}|_l$ is a linear map. Thus the restriction of $\tau^H_{_m}$ to the straight line l is a constant map. On the other hand, Corollary 5.2.5 shows that $\tau^H_{_m} : \mathcal{T}^H_{_m} \to \mathbb{C}^N$ is locally injective. Therefore, let us take U_p to be a neighborhood of p in $\mathcal{T}^H_{_m}$ such that $\tau^H_{_m} : U_p \to \mathbb{C}^N$ is injective. Then the intersection of U_p and l contains infinitely many points, but the restriction of $\tau^H_{_m}$ to $U_p \cap l$ is a constant map. Therefore, we get the contraction that $\tau^H_{_m}$ is both injective and constant on $U_p \cap l$. Thus $\tau^H_{_m}(p) \neq \tau^H_{_m}(q)$ for any different $p, q \in \mathcal{T}^H_{_m}$. ∎

Corollary 5.3.3 *For any two integers $m, m' \ge m_0$, let $\mathcal{Z}_m$ and $\mathcal{Z}_{m'}$ be smooth moduli space of Calabi–Yau type manifolds with level structure and $\mathcal{T}^H_{_m}$ and $\mathcal{T}^H_{m'}$ be the universal cover spaces of $\mathcal{Z}^H_{_m}$ and $\mathcal{Z}^H_{m'}$ respectively. Then the complete complex manifolds $\mathcal{T}^H_{_m}$ and $\mathcal{T}^H_{m'}$ are* *biholomorphic to each other.*

Proof First, Proposition 5.3.1 shows that Φ_m and $\Phi_{m'}$ are embeddings for any $m, m' \geq 3$. Therefore, we have the isomorphisms $\mathcal{T}_m \cong \Phi_m(\mathcal{T}_m)$ and $\mathcal{T}_{m'} \cong \Phi_{m'}(\mathcal{T}_{m'})$. Secondly, since $\Phi = \Phi_m \circ i_m = \Phi_{m'} \circ i_{m'}$, and Φ is independent of m and m', we have $\Phi_m(\mathcal{T}_m) = \Phi_{m'}(\mathcal{T}_{m'}) = \Phi(\mathcal{T})$. Therefore $\mathcal{T}_m \cong \mathcal{T}_{m'} \cong \Phi(\mathcal{T})$ biholomorphically. Finally, since $\mathcal{T}^H_m$ is the completion space of $\mathcal{T}_m$, $\mathcal{T}^H_{m'}$ is the completion space of $\mathcal{T}_{m'}$, and that the metric completions space is unique up to biholomorphism, we conclude that $\mathcal{T}^H_m$ is biholomorphic to $\mathcal{T}^H_{m'}$. ∎

Proposition 5.3.4 *The map $i_m : \mathcal{T} \to \mathcal{T}^H_m$ is an embedding.*

Proof Recall that we have the following commutative diagram,

$$\begin{tikzcd} \mathcal{T} \arrow[r, "i_m"] \arrow[d, "\pi_m"] & \mathcal{T}^H_m \arrow[r, "\Phi^H_m"] \arrow[d, "\pi^H_m"] & D \arrow[d, "\pi_D"] \\ \mathcal{Z}_m \arrow[r, "i"] & \mathcal{Z}^H_m \arrow[r, "\Phi^H_{\mathcal{Z}_m}"] & D/\Gamma. \end{tikzcd} \tag{5.6}$$

Since $\mathcal{T}_m = i_m(\mathcal{T}) = (\pi^H_m)^{-1}(\mathcal{Z}_m)$ by Proposition 5.2.2, the map $\pi^H_m : \mathcal{T}_m \to \mathcal{Z}_m$ is a covering map. Thus the fundamental groups satisfies $\pi_1(\mathcal{T}_m) \subseteq \pi_1(\mathcal{Z}_m)$ for any $m \geq m_0$. Therefore, the universal property of the universal covering map $\pi_m : \mathcal{T} \to \mathcal{Z}_m$ and that $i \circ \pi_m = \pi^H_m|_{\mathcal{T}_m} \circ i_m$ implies that $i_m : \mathcal{T} \to \mathcal{T}_m$ is a covering map for any $m \geq m_0$.

We now claim that $\pi_1(\mathcal{T}_m)$ is a trivial group. To prove the claim, let $\{m_1, m_2, \cdots, m_k, \cdots\}$ be a sequence of positive integers satisfying $m_k | m_{k+1}$ and $m_k < m_{k+1}$ for each $k \geq 1$. Because each point in $\mathcal{Z}_{m_{k+1}}$ is a polarized Calabi–Yau type manifold with a basis $\gamma_{m_{k+1}}$ for the space $(H_n(M,\mathbb{Z})/\mathrm{Tor})/m_{k+1}(H_n(M,\mathbb{Z})/\mathrm{Tor})$ and $m_k | m_{k+1}$, then the basis $\gamma_{m_{k+1}}$ induces a basis for the space $(H_n(M,\mathbb{Z})/\mathrm{Tor})/m_k(H_n(M,\mathbb{Z})/\mathrm{Tor})$. Therefore we get a well-defined map $\mathcal{Z}_{m_{k+1}} \to \mathcal{Z}_{m_k}$ by assigning to a polarized Calabi–Yau type manifold with the basis $\gamma_{m_{k+1}}$ the same polarized Calabi–Yau type manifold with the basis $(\gamma_{m_{k+1}} (\mathrm{mod}\ m_k)) \in (H_n(M,\mathbb{Z})/\mathrm{Tor})/m_k(H_n(M,\mathbb{Z})/\mathrm{Tor})$. Moreover, using the versal properties of both the families $\mathcal{X}_{m_{k+1}} \to \mathcal{Z}_{m_{k+1}}$ and $\mathcal{X}_{m_k} \to \mathcal{Z}_{m_k}$, we can conclude that the map $\mathcal{Z}_{m_{k+1}} \to \mathcal{Z}_{m_k}$ is locally biholomorphic. Therefore we get a natural covering $\mathcal{Z}_{m_{k+1}} \to \mathcal{Z}_{m_k}$. Thus the fundamental group $\pi_1(\mathcal{Z}_{m_{k+1}})$ is a subgroup of $\pi_1(\mathcal{Z}_{m_k})$ for reach k. Hence, the inverse system of fundamental groups

$$\pi_1(\mathcal{Z}_{m_1}) \longleftarrow \pi_1(\mathcal{Z}_{m_2}) \longleftarrow \cdots\cdots \longleftarrow \pi_1(\mathcal{Z}_{m_k}) \longleftarrow \cdots$$

has an inverse limit, and this limit is the fundamental group of $\mathcal{T}$.

On the other hand, recall that in the proof of Lemma 5.3.3, we also showed that $\mathcal{T}_m$ is biholomorphic to any $\mathcal{T}_{m'}$ for any $m' \geq m_0$. Thus $\pi_1(\mathcal{T}_m) = \pi_1(\mathcal{T}_{m'})$ for any $m' \geq m_0$. Since $\pi_1(\mathcal{T}_m) = \pi_1(\mathcal{T}_{m_k}) \subseteq \pi_1(\mathcal{Z}_{m_k})$ for each k, the fundamental group of $\mathcal{T}_m$ is also a subgroup of inverse limit $\pi_1(\mathcal{T})$. However, simply-connectedness of $\mathcal{T}$ implies that $\pi_1(\mathcal{T})$ is a trivial group. Therefore $\pi_1(\mathcal{T}_m)$ is also a trivial group. Thus the covering map $i_m : \mathcal{T} \to \mathcal{T}_m$ is a one-to-one covering and therefore $i_m : \mathcal{T} \to \mathcal{T}^H_m$ is an embedding. ∎

There is an alternate proof of the above proposition given in Appendix A Proposition A.2.5. One may find the original proof in [10].

Theorem 5.3.5 (Global Torelli) *The period map $\Phi : \mathcal{T} \to D$ is injective.*

Proof Since the period map $\Phi = \Phi^H_m \circ i_m$ for any $m \geq m_0$, and we have showed in Proposition 5.3.1 that Φ^H_m is injective, and in Propostion 5.3.4 that i_m is also injective, we can conclude that Φ is injective as well. ∎

5.4 Hodge metric completion space of the Teichmüller space and Domain of holomorphy

Corollary 5.3.3 shows that up to biholomorphisms, $\mathcal{T}^H_m$ is independent of the choice of the level m structure. Therefore, it allows us to give the following definition.

Definition 5.4.1 *We define the completion space $\mathcal{T}^H = \mathcal{T}^H_m$, the holomorphic map $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}^H$ by $i_{\mathcal{T}} = i_m$, and the holomorphic map $\Phi^H : \mathcal{T}^H \to D$ by $\Phi^H = \Phi^H_m$ for any $m \geq m_0$.*

Notice that with these new notations, we obtain the following commutative diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "i_{\mathcal{T}}"] \arrow[d, "\pi_m"] & \mathcal{T}^H \arrow[r, "\Phi^H"] \arrow[d, "\pi_m^H"] & D \arrow[d, "\pi_D"] \\ \mathcal{Z}_m \arrow[r, "i"] & \mathcal{Z}_m^H \arrow[r, "\Phi^H_{\mathcal{Z}_m}"] & D/\Gamma, \end{tikzcd} \tag{5.7}$$

which satisfies $\Phi = \Phi^H \circ i_{\mathcal{T}}$. Moreover, by the above definitions, we can also combine Theorem 5.2.6, Proposition 5.3.1 and Proposition 5.3.4 to conclude the following corollary.

Corollary 5.4.2 *The complete manifold $\mathcal{T}^H$ is a smooth holomorphic affine manifold, which can be embedded into a bounded domain in $\mathbb{C}^N$; and it is the completion space of $\mathcal{T}$ with respect to the Hodge metric. Moreover, the holomorphic map $\Phi^H : \mathcal{T}^H \to N_+ \cap D$ is a holomoprhic injection.*

The main theorem we will prove in this section is the following.

Theorem 5.4.3 *The completion space $\mathcal{T}^H$ is a bounded domain of holomorphy in $\mathbb{C}^N$; thus there exists a complete Kähler–Einstein metric on $\mathcal{T}^H$.*

We recall that a $\mathcal{C}^2$ function $\rho : \Omega \to \mathbb{R}$ on a domain $\Omega \subseteq \mathbb{C}^n$ is *plurisubharmornic* if and only if its Levi form is positive definite at each point in Ω. Given a domain $\Omega \subseteq \mathbb{C}^n$, a function $f : \Omega \to \mathbb{R}$ is called an *exhaustion function* if for any $c \in \mathbb{R}$, the set $\{z \in \Omega \,|\, f(z) < c\}$ is relatively compact in Ω. The following well-known theorem Proposition 5.4.4 provides a definition for domains of holomorphy. For example, one may refer to [29] for details. The other theorem we will use is from [25, Section 8], which gives us basic ingredients to construct a plurisubharmoic exhaustion function on $\mathcal{T}^H$.

Proposition 5.4.4 *An open set $\Omega \in \mathbb{C}^n$ is a domain of holomorphy if and only if there exists a continuous plurisubharmonic function $f : \Omega \to \mathbb{R}$ such that f is also an exhaustion function.*

Proposition 5.4.5 *On every manifold D, which is dual to a Kähler C-space, there exists an exhaustion function $f : D \to \mathbb{R}$, whose Levi form, restricted to $T^{1,0}_h(D)$, is positive definite at every point of D.*

We remark that in this proposition, in order to show f is an exhaustion function on D, Griffiths and Schmid showed that the set $f^{-1}(-\infty, c]$ is compact in D for any $c \in \mathbb{R}$.

Lemma 5.4.6 *The extended period map $\Phi^H : \mathcal{T}^H \to D$ still satisfies the Griffiths transversality.*

Proof Let us consider the tangent bundles $T^{1,0}\mathcal{T}^H$ and $T^{1,0}D$ as two differential manifolds, and the tangent map $(\Phi^H)_* : T^{1,0}\mathcal{T}^H \to T^{1,0}D$ as a continuous map. We only need to show that the image of $(\Phi^H)_*$ is contained in the horizontal tangent bundle $T_h^{1,0}D$.

The horizontal subbundle $T_h^{1,0}D$ is a close set in $T^{1,0}D$, so the preimage of $T_h^{1,0}D$ under $(\Phi^H)_*$ is a close set in $T^{1,0}\mathcal{T}^H$. On the other hand, because the period map Φ satisfies the Griffiths transversality, the image of Φ_* is in the horizontal subbundle $T_h^{1,0}D$. This means that the preimage of $T_h^{1,0}D$ under $(\Phi^H)_*$ contains both $T^{1,0}\mathcal{T}$ and the closure of $T^{1,0}\mathcal{T}$, which is $T^{1,0}\mathcal{T}^H$. This finishes the proof. ∎

Proof of Theorem 5.4.3 By Corollary 5.4.2, we can see that $\mathcal{T}^H$ is a bounded domain in $\mathbb{C}^N$. Therefore, once we show $\mathcal{T}^H$ is domain of holomorphy, the existence of Kähler-Einstein metric on it follows directly from the well-known theorem by Mok–Yau in [42].

In order to show that $\mathcal{T}^H$ is a domain of holomorphy in $\mathbb{C}^N$, it is enough to construct a plurisubharmonic exhaustion function on $\mathcal{T}^H$.

Let f be the exhaustion function on D constructed in Proposition 5.4.5, whose Levi form, when restricted to the horizontal tangent bundle $T_h^{1,0}D$ of D, is positive definite at each point of D. We claim that the composition function $f \circ \Phi^H$ is the demanded plurisubharmonic exhaustion function on $\mathcal{T}^H$.

By the Griffiths transversality of Φ^H, the composition function $f \circ \Phi^H : \mathcal{T}^H \to \mathbb{R}$ is a plurisubharmonic function on $\mathcal{T}^H$. Thus it suffices to show that the function $f \circ \Phi^H$ is an exhaustion function on $\mathcal{T}^H$, which is enough to show that for any constant $c \in \mathbb{R}$, $(f \circ \Phi^H)^{-1}(-\infty, c] = (\Phi^H)^{-1}\left(f^{-1}(-\infty, c]\right)$ is a compact set in $\mathcal{T}^H$. Indeed, it has already been shown in [25] that the set $f^{-1}(-\infty, c]$ is a compact set in D. Now for any sequence $\{p_k\}_{k=1}^{\infty} \subseteq$

$(f \circ \Phi^H)^{-1}(-\infty, c]$, we have $\{\Phi^H(p_k)\}_{k=1}^\infty \subseteq f^{-1}(-\infty, c]$. Since $f^{-1}(-\infty, c]$ is compact in D, the sequence $\{\Phi^H(p_k)\}_{k=1}^\infty$ has a convergent subsequence. We denote this convergent subsequence by $\{\Phi^H(p_{k_n})\}_{n=1}^\infty \subseteq \{\Phi^H(p_k)\}_{k=1}^\infty$ with $k_n < k_{n+1}$, and denote $\lim_{k\to\infty} \Phi^H(p_k) = o_\infty \in D$. On the other hand, since the map Φ^H is injective and the Hodge metric on $\mathcal{T}^H$ is induced from the Hodge metric on D, the extended period map Φ^H is actually a global isometry onto its image. Therefore the sequence $\{p_{k_n}\}_{n=1}^\infty$ is also a Cauchy sequence that converges to $(\Phi^H)^{-1}(o_\infty)$ with respect to the Hodge metric in $(f \circ \Phi^H)^{-1}(-\infty, c] \subseteq \mathcal{T}^H$. In this way, we have proved that any sequence in $(f \circ \Phi^H)^{-1}(-\infty, c]$ has a convergent subsequence. Therefore $(f \circ \Phi^H)^{-1}(-\infty, c]$ is compact in $\mathcal{T}^H$, as was needed to show. ∎

CHAPTER 6

Applications

In this chapter, we give some applications of the result of global Torelli theorem for Calabi–Yau type manifolds we proved in Chapter 5, and adopt the technique we use in the proof of global Torelli theorem to study periods of more general projective manifolds. The results in this chapter are collaborating work with F. Guan and K. Liu. The original work can be found in [11, 12]. We remark that there are also applications about the results of the existence of a global section on the Hodge bundles of Teichmüller spaces, as well as a global splitting property of the Hodge bundles by using the affine structure on the Teichmüller space in [12].

In § 6.1, we use the injectivity of the period map on the Teichmüller space for Calabi–Yau type manifolds to prove that the period map on the moduli space is a covering map onto its image if given smoothness condition on the moduli space. We also conclude a corollary of an injectivity result for the period map on a finite cover of moduli space under certain assumptions. In § 6.2, we first prove a simple result about when the extended period map is biholomorphic from the Hodge metric completion space of Teichmüller space to the period domain. Then we apply it to give a simple proof of the surjectivity for the period maps of K3 surfaces, cubic fourfolds. Moreover, we consider modified period maps when the period domain can be realized as a ball. Then we adopt our technique to conclude an injectivity result for such types of period maps.

6.1 Period map on the moduli space of polarized Calabi–Yau type manifolds

In this section, we make two assumptions: the moduli space $\mathcal{M}$ of polarized Calabi–Yau type manifolds is smooth and that the global monodromy group acts on the period domain freely. Then we use the global Torelli theorem on Teichmüller space of polarized and marked Calabi–Yau type manifolds to show that the period map on the moduli space of polarized Calabi–Yau type manifolds is a covering map onto its image. As a consequence, we derive that the generic Torelli theorem on the moduli space of polarized Calabi–Yau type manifolds implies the global Torelli theorem on the moduli space.

Let Γ denote the global monodromy group which acts properly and discontinuously on D. Then for the smooth moduli space $\mathcal{M}$, we consider the period map $\Phi_{\mathcal{M}} : \mathcal{M} \to D/\Gamma$. Thus for the two period maps Φ and $\Phi_{\mathcal{M}}$, we have the following commutative diagram,

$$\begin{tikzcd}
\mathcal{T} \arrow[r, hook, "\Phi"] \arrow[d, "\pi_{\mathcal{T}}"] \arrow[dr, "\tilde{\Phi}"] & D \arrow[d, "\pi_D"] \\
\mathcal{M} \arrow[r, "\Phi_{\mathcal{M}}"] & D/\Gamma,
\end{tikzcd}$$

where $\pi_{\mathcal{T}} : \mathcal{T} \to \mathcal{M}$ is a covering map and $\tilde{\Phi} = \pi_D \circ \Phi$. The image of the period map $\Phi_{\mathcal{M}}$ is an analytic subvariety of D/Γ. We refer the reader to page 156 of [23] for details of the analyticity of the image of the period mapping.

Global Torelli problem on the moduli space $\mathcal{M}$ asks when $\Phi_{\mathcal{M}}$ is injective, and generic Torelli problem asks when there exists an open dense subset $U \subseteq \mathcal{M}$ such that $\Phi_{\mathcal{M}}|_U$ is injective. In both cases, we need to understand the global Torelli property of the period map $\Phi_{\mathcal{M}}$ on the moduli space.

Furthermore, we assume Γ acts on D freely, thus the quotient space D/Γ is a smooth analytic variety. Therefore the quotient map $\pi_D : D \to D/\Gamma$ is a covering map. Then we will show the following theorem.

Proposition 6.1.1 *Let $\mathcal{M}$ be the moduli space of polarized Calabi–Yau type manifolds. If*

$\mathcal{M}$ is smooth and the global monodormy group Γ acts on D freely, then the period map $\Phi_{\mathcal{M}} : \mathcal{M} \to D/\Gamma$ is a covering map from $\mathcal{M}$ to its image in D/Γ. As a consequence, if the period map $\Phi_{\mathcal{M}}$ is generically injective, then it is globally injective.

Proof of Theorem 6.1.1 First, we show that $\Phi_{\mathcal{M}}$ is a covering map from $\mathcal{M}$ to its image in D/Γ. This follows from the following lemma,

Lemma 6.1.2 *Let $\tilde{\Phi} : \mathcal{T} \to D \to D/\Gamma$ be the composition of Φ and the covering map $\pi_D : D \to D/\Gamma$. If $p_1, p_2 \in \mathcal{T}$ are distinct points such that $\tilde{\Phi}(p_1) = \tilde{\Phi}(p_2)$, then there exist V_1 and V_2, which are neighborhoods of p_1 and p_2 respectively, such that $\tilde{\Phi}(V_1) = \tilde{\Phi}(V_2)$, $V_1 \cap V_2 = \emptyset$, and the map*

$$\tilde{\Phi} : V_i \to \tilde{\Phi}(V_i)$$

is biholomorphic for each $i = 1, 2$.

Proof of Lemma 6.1.2 For any $p \in \mathcal{T}$, we can identify a point $\Phi(p) = \{F_p^s \subset F_p^{s-1} \subset \cdots \subset F_p^{n-s}\} \in D$ with its Hodge decomposition $\Phi(p) = \{H_p^{k,n-k}\}_{k=n-s}^s$. Therefore, let $\Phi(p_1) = \{H_{p_1}^{k,n-k}\}_{k=n-s}^s$ and $\Phi(p_2) = \{H_{p_2}^{k,n-k}\}_{k=n-s}^s$ be the corresponding Hodge decompositions and thus a fixed adapted basis of the Hodge decomposition. Then the condition $\tilde{\Phi}(p_1) = \tilde{\Phi}(p_2)$ implies that there exists some $\alpha \in \Gamma \subseteq \mathrm{Aut}(H^n(M,\mathbb{Z}))$ such that $\alpha \cdot \Phi(p_1) = \Phi(p_2)$, where "$\cdot$" means α acts on a fixed adapted basis of the Hodge decomposition $\{H_{p_1}^{k,n-k}\}_{k=n-s}^s$.

We fix an adapted basis $\{\eta_0^{(1)}, \eta_1^{(1)}, \cdots, \eta_N^{(1)}, \cdots, \eta_{m-1}^{(1)}\}$ for the Hodge decomposition of $\Phi(p_1)$. Let $(\eta_0^{(2)}, \eta_1^{(2)}, \cdots, \eta_N^{(2)}, \cdots, \eta_{m-1}^{(2)}) = (\alpha \cdot \eta_0^{(1)}, \alpha \cdot \eta_1^{(1)}, \cdots, \alpha \cdot \eta_N^{(1)}, \cdots, \alpha \cdot \eta_{m-1}^{(1)})$. Then $(\zeta_0, \zeta_1, \cdots, \zeta_N, \cdots, \zeta_{m-1})$ forms an adapted basis for the Hodge decomposition of $\Phi(p_2)$.

Let $(U_{p_i}, \{\tau_1^{(i)}, \cdots, \tau_N^{(i)}\})$ be the local Kuranishi coordinate chart associated to the basis $\{\eta_0^{(i)}, \eta_1^{(i)}, \cdots, \eta_N^{(i)}\}$ of $H_{p_i}^{s,n-s} \oplus H_{p_i}^{s-1,n-s+1}$ for $i = 1, 2$ respectively, which are defined in Section 5.1. Then Proposition 5.1.3 shows that we have the following embeddings,

$$\rho_1 : U_{p_1} \to \mathbb{C}^N \simeq H_{p_1}^{s-1,n-s+1}, \quad \rho_2 : U_{p_2} \to \mathbb{C}^N \simeq H_{p_2}^{s-1,n-s+1}.$$

with $\rho_i(p^{(i)}) = \sum_{j=1}^N \tau_j^{(i)}(p^{(i)})\eta_j^{(i)}$ for any $p^{(i)} \in U_{p_i}$ and $i = 1, 2$.

Since $\dim_{\mathbb{C}} U_{p_i} = \dim_{\mathbb{C}} H_{p_i}^{s-1,n-s+1} = \dim_{\mathbb{C}} \mathcal{T}$ and that ρ_i is an embedding, the image $\rho_i(U_{p_i})$ is open in $H_{p_i}^{s-1,n-s+1}$ for each $i = 1, 2$. This implies that $\alpha \cdot \rho_1(U_{p_1})$ and $(\alpha \cdot \rho_1(U_{p_1})) \cap \rho_2(U_{p_2})$ are also open in $H_{p_2}^{s-1,n-s+1}$. Together with the fact that $\alpha \cdot \rho_1(p_1) = \rho_2(p_2) \in (\alpha \cdot \rho_1(U_{p_1})) \cap \rho_2(U_{p_2}) \neq \emptyset$, we get that there exists a neighborhood W of $\rho_2(p_2)$ in $H_{p_2}^{s-1,n-s+1}$, such that

$$W \subseteq (\alpha \cdot \rho_1(U_{p_1})) \cap \rho_2(U_{p_2}).$$

Let $V_1 = \rho_1^{-1}(\alpha^{-1} \cdot W) \subseteq U_{p_1}$ and $V_2 = \rho_2^{-1}(W) \subseteq U_{p_2}$, then the restriction maps

$$\rho_1|_{V_1} : V_1 \to W, \qquad \rho_2|_{V_2} : V_2 \to W \tag{6.1}$$

are biholomorphic maps since the ρ_1 and ρ_2 are embeddings. Therefore, we get the following composition maps,

$$\alpha \cdot \Phi_1|_{V_1} : V_1 \simeq W \hookrightarrow \mathbb{C}^N \cong H_{p_2}^{s-1,n-s+1}, \qquad \Phi_2|_{V_2} : V_2 \simeq W \hookrightarrow \mathbb{C}^N \cong H_{p_2}^{s-1,n-s+1}.$$

Notice that the composition maps from W to $H_{p_2}^{s-1,n-s+1}$ in the above two maps are the same. Together with the isomorphisms between $V_1 \simeq W$ and $V_2 \simeq W$ as defined in (6.1), we now can conclude that $\alpha \cdot \Phi(V_1) = \Phi(V_2)$, which implies $\tilde{\Phi}(V_1) = \tilde{\Phi}(V_2) = \pi_D(W)$. By shrinking W properly, we can make V_1 and V_2 disjoint, and also the map

$$\tilde{\Phi} : V_i \xrightarrow{\Phi} W \xrightarrow{\pi_D} \tilde{\Phi}(V_i)$$

is biholomorphic for each $i = 1, 2$. ∎

Proof of Theorem 6.1.1(continued). Notice that in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{T} \arrow[r, hook, "\Phi"] \arrow[d, "\pi_{\mathcal{T}}"] \arrow[dr, "\tilde{\Phi}"] & D \arrow[d, "\pi_D"] \\
\mathcal{M} \arrow[r, "\Phi_{\mathcal{M}}"] & D/\Gamma,
\end{tikzcd}$$

since $\mathcal{T}$ and $\mathcal{M}$ are both smooth, the map $\pi_{\mathcal{T}}$ is a covering map. This implies that $\pi_{\mathcal{T}}$ is locally biholomorphic.

For any Hodge structure $\{H^{k,n-k}\}_{k=n-s}^{s} \in D/\Gamma$, if the preimage $\Phi_{\mathcal{M}}^{-1}(\{H^{k,n-k}\}_{k=n-s}^{s}) = \{p'_i|\ i \in I\}$ is not empty, then take $\{p_j|\ j \in J\} = \pi_{\mathcal{T}}^{-1}(\{p'_i|\ i \in I\})$. By Lemma 6.1.2, for each $j \in J$, we get $V_j \subseteq \mathcal{T}$ which is a neighbourhood around p_j such that

$$\tilde{\Phi} : V_j \to \tilde{\Phi}(V_j)$$

is biholomorphic, $\cup_j V_j$ is a disjoint union, and all the $\tilde{\Phi}(V_j)$ are the same for any $j \in J$. Now take $\{V'_k \subseteq \mathcal{M}|\ k \in K\} = \{\pi_{\mathcal{T}}(V_j)|\ j \in J\}$. By shrinking the set W properly, we may assume that $\cup_k V'_k$ is a disjoint union and that the images $\Phi_{\mathcal{M}}(V'_k) = \pi_D(W)$ are still all the same for any k. Moreover, since $\pi_{\mathcal{T}}$ is covering map, the map

$$\Phi_{\mathcal{M}} : V'_k \to \Phi_{\mathcal{M}}(V'_k)$$

is still biholomorphic for each $k \in K$. This shows that the holomorphic map $\Phi_{\mathcal{M}} : \mathcal{M} \to \Phi_{\mathcal{M}}(\mathcal{M})$ is a covering map. In particular, if the map $\Phi_{\mathcal{M}} : \mathcal{M} \to \Phi_{\mathcal{M}}(\mathcal{M})$ is generically injective, then $\Phi_{\mathcal{M}} : \mathcal{M} \to \Phi_{\mathcal{M}}(\mathcal{M})$ is a degree one covering map, which must be globally injective. ∎

Note that in many cases it is possible to find a subgroup Γ_0 of Γ, which is of finite index in Γ, such that its action on D is free and D/Γ_0 is smooth. In such cases we can consider the lift $\Phi_{\mathcal{M}_0} : \mathcal{M}_0 \to D/\Gamma_0$ of the period map $\Phi_{\mathcal{M}}$, with $\mathcal{M}_0$ a finite cover of $\mathcal{M}$. Then our argument can be applied to prove that $\Phi_{\mathcal{M}_0}$ is actually a covering map onto its image for polarized Calabi–Yau type manifolds with smooth moduli spaces.

In this subsection, we require that the moduli space $\mathcal{M}$ of Calabi–Yau type manifolds are smooth, and there are many nontrivial examples satisfying this requirement. As in the special cases of Calabi–Yau manifolds, one may see from Popp [45], Viehweg [63] and Szendroi [55] that the moduli spaces $\mathcal{Z}_m$ of Calabi–Yau manifolds are smooth.

Let (M, L) be a polarized Calabi–Yau type manifold of dimension n. Let *Diff* be the group of oriented diffeomorphisms and *Diff*$_0$ the group of isotopies, which is a connected component of the group Diff. Then we call the quotient Diff/Diff$_0$ the *mapping class group*, and denote it by *MCG*. Let us denote the following subset

$$\mathrm{MCG}_L = \{\phi \in MCG | \phi^* L = L\}.$$

Then the moduli space of polarized Calabi–Yau type manifolds is

$$\mathcal{M} = \mathcal{T}/\mathrm{MCG}_L.$$

Consider the homomorphism induced by the action of the diffeomorphism group on the global monodromy group

$$\sigma : \mathrm{MCG}_L \to \Gamma.$$

With the above notation, we have the following result provided the kernel of σ is finite.

Corollary 6.1.3 *If the kernel of the map $\sigma : MCG_L \to \Gamma$ is finite, then there exists a finite cover $\mathcal{M}'$ such that the map $\Phi' : \mathcal{M}' \to D/\Gamma$ is injective.*

Proof Let us denote the kernel of σ by $\mathrm{Ker}(\sigma)$. Then since it is finite, there exists $\mathrm{MCG}'_L := \mathrm{MCG}_L/\mathrm{Ker}(\sigma)$ such that of $\mathcal{T}/\mathrm{MCG}'_L \to \mathcal{T}/\mathrm{MCG}_L$ is a finite cover. Let us denote $\mathcal{M}' := \mathcal{T}/\mathrm{MCG}'_L$, and thus we have a lifting map $\Phi' : \mathcal{M}' \to D/\Gamma$ as $\mathrm{MCG}'_L = \mathrm{MCG}_L/\mathrm{Ker}(\sigma)$. Moreover, since the period map $\Phi : \mathcal{T} \to D$ is injective and that $\mathrm{MCG}'_L = \mathrm{MCG}_L/\mathrm{Ker}(\sigma)$, we conclude that $\Phi' :, \mathcal{M}' \to D/\Gamma$ is also injective. ∎

Remark 6.1.4 *One can show that the kernel of this map for Calabi–Yau manifolds is finite. Thus the above corollary is true for Calabi–Yau manifolds.*

6.2 Surjectivity of the period map on the Teichmüller space

In this section we use our results on the Hodge completion space $\mathcal{T}^H$ to give a simple proof of the surjectivity of the period maps of K3 surfaces and cubic fourfolds. First we have the following general result,

Theorem 6.2.1 *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi–Yau type manifolds. If $\dim \mathcal{T}^H = \dim D$, then the extended period map $\Phi^H : \mathcal{T}^H \to D$ is surjective.*

Proof Since $\dim \mathcal{T}^H = \dim D$, the property that $\Phi^H : \mathcal{T}^H \to D$ is an local isomorphism shows that the image of $\mathcal{T}^H$ under the extended period map Φ^H is open in D. On the other hand, the completeness of $\mathcal{T}^H$ with respect to Hodge metric implies that the image of $\mathcal{T}^H$ under Φ^H is close in D. As $\mathcal{T}^H$ is not empty and that D is connected, we can conclude that $\Phi^H(\mathcal{T}^H) = D$. ∎

It is well known that for K3 surfaces, which are two dimensional Calabi–Yau manifolds, we have $\dim \mathcal{T}^H = \dim \mathcal{T} = \dim D = 19$; for cubic fourfolds, they are Calabi–Yau type manifolds, and $\dim \mathcal{T}^H = \dim \mathcal{T} = \dim D = 20$. Thus applying the above theorem, we can easily conclude the following corollary.

Corollary 6.2.2 *Let $\mathcal{T}^H$ be the Hodge metric completion space of the Teichmüller space for K3 surfaces or cubic fourfolds. Then the extended period map $\Phi^H : \mathcal{T}^H \to D$ is surjective.*

The period domain is generally too big to be dominant for the period map. In fact, if we consider all the projective hypersurfaces of Calabi–Yau type manifolds of dimension $2l$ of weight $2l$-Hodge structure of K3-type, that is, the hodge numbers satisfy $h^{k,2n-k} = 0$ for any $k \geq l-2$, and $h^{l-1,l+1} = 1$, K3 surfaces and cubic fourfolds are the only ones satisfying the condition that the dimension of the Teichmüller space and the period domain are the same. It would be interesting to find such examples for complete intersections or weighted projective hypersurfaces.

Remark 6.2.3 *Let $\mathcal{T}$ be the Teichmüller space of marked and polarized projective hyperkähler manifolds, $H^2_{pr}(M, \mathbb{C})$ the degree 2 primitive cohomology group, and D the period domain of weight two Hodge structures on $H^2_{pr}(M, \mathbb{C})$. Then our method can be applied without change to prove that the period map from $\mathcal{T}$ to D is also injective. Furthermore, let $\mathcal{T}^H$ be the Hodge completion of $\mathcal{T}$ with respect to the Hodge metric induced from the homogeneous metric on*

D, then the extended period map from $\mathcal{T}^H$ to D is also surjective. This follows from the same argument of above theorem. See [30] and [62] for different injectivity and surjectivity results for more general hyperkähler manifolds.

As mentioned above, the period domain is generally too big for the period map defined in Chapter 3 to be dominant. Recall that the construction of the affine structure on the Teichüller space is based on the property that the period map is a bounded map. Then we can construct the global affine coordinate by embedding the Techmüller space to a much smaller Eulidean space of the period domain. However, if the period domain is already known to be bounded affine space, has the same dimension as the Teichmüller space or moduli space and that the period map is locally injective, then there is an induced affine structure on the Teimüller space or moduli space from the bounded domain. With this induced affine structure and together with the above construction of Hodge metric completion of the Teichmüller space, we may conclude the injectivity of the period on the Teichüller space.

To summarize, we set up the above discussions as follows. Suppose we can define a period map $\Phi_{\mathcal{M}} : \mathcal{M} \to B/\Gamma$ from the moduli space $\mathcal{M}$ of polarized projective manifolds to a ball quotient B/Γ with $\dim_{\mathbb{C}} \mathcal{M} = \dim_{\mathbb{C}} B$, and there exists lifting map $\Phi : \mathcal{T} \to B$ from Teichmüller space of polarized and marked projective manifolds to the period domain. Here, we assume that the Teichmüller space is a smooth and connected and simply connected complex manifold. We also assume that the period map above is local injective. Then with the above assumption, we can conclude the following,

Lemma 6.2.4 *There is an induced affine structure on the Teichmüller space from B.*

Proof Consider an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of the $\mathcal{T}$. Since $\Phi : \mathcal{T} \to B \subseteq \mathbb{C}^N$ is locally injective and $\dim_{\mathbb{C}} \mathcal{T} = \dim_{\mathbb{C}} B = N$, we may assume that for each $U_\alpha \subseteq \mathcal{T}$, Φ is biholomorphic. Let us define

$$\tau|_{U_\alpha} := \Phi|_{U_\alpha} : U_\alpha \to \mathbb{C}^N.$$

Then $\tau|_{U_\alpha}$ defines a local coordinate on U_α. As τ is globally defined to be the global period map, τ gives a global holomorphic coordinate cover on $\mathcal{T}$ whose transition maps are all identity maps. In conclude, we get that τ defines a holomorphic affine structure on $\mathcal{T}$. ∎

Based on this construction of affine structure on the Teichmüller space, we can use the same construction of the metric completion of Teichmüller space $\mathcal{T}^H$ in §5.2 to conclude the following.

Lemma 6.2.5 *There exists a completion space $\mathcal{T}^H$ of the Teichmüller space $\mathcal{T}$, which is holomorphic affine space with the affine structure induced from $\mathcal{T}$.*

Finally, with the completion space $\mathcal{T}^H$ and the induced affine structure on $\mathcal{T}^H$, we can prove the following.

Proposition 6.2.6 *The period map $\Phi : \mathcal{T} \to B$ is injective.*

Remark 6.2.7 *We remark that there are many cases where one can construct the modified period map such that the moduli space can be realized as a ball quotient. One may refer to examples in [2, 3, 4, 6, 39] for modified period map for cubic surfaces and cubic threefolds.*

APPENDIX A

Appendix A

A.1 Hypersurfaces in projective space

In this section, we assume that X is a hypersurface in $\mathbb{P}^{n+1}$ which is defined to be the zero set of a nondegenerate homogeneous polynomial f of degree d. In another word, $X \subset \mathbb{P}^{n+1}$ is a dimension n smooth projective hypersurface, which is defined by

$$X = \{z \in \mathbb{P}^{n+1} : f(z) = 0 \quad \deg f = d \quad \text{and} \quad f \quad \text{is nondegenerate}\}.$$

Let $\mathcal{M}(n, d)$ denote the moduli space of stable hypersurfaces of degree d in $\mathbb{P}^{n+1}$. Let $\mathbb{C}[x_0, \cdots, x_{n+1}]_d$ be the space of all homogeneous polynomials of degree d in variable $x_0, \cdots, x_{n+1}$ and let K be the open subset of $\mathbb{P}(\mathbb{C}[x_0, \cdots, x_{n+1}]_d)$ consisting of all points are stable for the action of $SL(n+2, \mathbb{C})$. Then the moduli space $\mathcal{M}(n, d)$ is the quotient of K by $SL(n+2, \mathbb{C})$ (see [43]). It is easy to see that the dimension of $SL(n+2, \mathbb{C})$ is $(n+2)^2-1$, and the dimension of $\mathbb{P}(\mathbb{C}[x_0, \cdots, x_{n+1}]_d)$ is $\begin{pmatrix} n+1+d \\ n+1 \end{pmatrix} - 1$. Therefore, we have the following proposition.

Proposition A.1.1 *Let $\mathcal{M}$ be the moduli space of projective hypersurface X, with* $\dim_{\mathbb{C}} X = n$, *deg(X)=d. Then*

$$\dim(\mathcal{M}) = \begin{pmatrix} n+1+d \\ n+1 \end{pmatrix} - (n+2)^2.$$

We are interested in the case infinitesimal Torelli theorem holds. According to [16, Theorem 3.1], the infinitesimal Torelli theorem holds for X except when $n = 2, d = 3$.

Moreover, we may consider the case when $d < n+2$ with $(d,n) \neq (3,2)$, and we denote $n+2 = \alpha d + \beta$ with $\beta < d$. We will also use the notations in [14] as follows,

$$V = H^0(\mathbb{P}^{n+1}, \mathcal{O}(1)), \dim V = n+2$$

$$S = S^*V: \text{ the homogeneous function ring of } \mathbb{P}^{n+1}.$$

$$J \subset S: \text{ the homogeneous ideal generated by the partial of } f.$$

$$R = S/J: \text{ the Jacobian ring of } f.$$

$$S^t, J^t, R^t: \text{ homogeneous pieces of degree } t \text{ in } S, J, R \text{ respectively.}$$

We will use the following basic results from [9] to carry out our computations.

Theorem A.1.2 *There are natural isomorphisms, depending holomorphically on f:*

$$\lambda_a : R^{t_a} \to F^a/F^{a+1} \simeq H^{a,n-a}_{pr}(X) \cong H^{n-a,a}_{pr}(X),$$

where $t_a = (n-a+1)d - (n+2)$, $d = \deg(X)$, $n = \dim(X)$.

Based on the above notations and results, we can compute the dimensions of the first two non-zero n-th primitive cohomology of X and we may conclude the following proposition:

Proposition A.1.3 *When $d < n+2$ and $d|n+2$, X is a Calabi–Yau type manifold. In particular, when $d = n+2$, X is a Calabi–Yau manifold.*

Proof Using Theorem A, we may can compute the dimensions of the first two non-zero n-th primitive cohomology of X as in the following cases.

Case 1: $\beta = 0$, i.e. $n+2 = \alpha d$:

Case 1.1: $\alpha = 1$, i.e. $n+2 = d$:

(i) when $n-a = 0$: $t_a = 0$, then $h^{n,0} = \dim R^0 = 1$;

(ii) when $n-a = 1$: $t_a = d$, then $h^{n-1,1} = \dim R^d$;

(iii) when $n-a = k > 1$: $t_a = kd$, then $h^{n-k,k} = \dim R^{kd}$.

Case 1.2: $\alpha > 1$:

(i) when $n-a=k<\alpha-1$: $t_a<0$, then $h^{n-k,k}=0$;

(ii) when $n-a=\alpha-1$: $t_a=0$, then $h^{n-\alpha+1,\alpha-1}=\dim R^0=1$;

(iii) when $n-a=\alpha$: $t_a=d$, then $h^{n-\alpha,\alpha}=\dim R^d$;

(iv) when $n-a=k>\alpha$: $t_a=(k-\alpha)d$ and $h^{n-k,k}=\dim R^{t_a}$.

Case 2: $\beta \neq 0$, i.e. $n+2=\alpha d+\beta$ with $0<\beta<d$. Then

(i) when $n-a=k<\alpha$, $t_a<-1<0$, then $h^{n-k,k}=0$.

(ii) when $n-a=\alpha$, $t_a=d-\beta$, $h^{n-\alpha,\alpha}=\dim R^{d-\beta}=n+2$.

(iii) when $n-a=\alpha+1$, $t_a=2d-1$, $h^{n-\alpha-1,\alpha+1}=\dim R^{2d-\beta}$.

(iv) when $n-a=k>\alpha+1$, $t_a=(k-\alpha+1)d-1$ and $h^{n-k,k}=\dim R^{t_a}$.

One can easily conclude from the computation in Case 1: when $d<n+2$ and $d|n+2$, X is a Calabi–Yau type manifold. In particular, when $d=n+2$, X is a Calabi–Yau manifold. ∎

Corollary A.1.4 (Corollary 2.4 in [14]) $\dim R^{t_a}$ *only depends on* a *and on* $d = deg(f)$.

By this corollary, we can choose $f=x_1^d+x_2^d+\cdots+x_{n+2}^d$ to compute the dimension of $\dim R^{t_a}$. Therefore $J=<x_1^{d-1},\cdots,x_{n+2}^{d-1}>$.

Let us denote $n+2=\alpha d+\beta$ with $\beta<d$. Then we have the following lemma about the dimension of R^t.

Proposition A.1.5 *The dimension of R^t for $t\leq 2d-2$ is listed as follows:*

(1). $\dim R^t=0$ *for any* $t<0$, *and* $\dim R^0=1$;

(2). $\dim R^{d-\beta}=\begin{pmatrix} n+1+d-\beta \\ n+1 \end{pmatrix}$ *for any* $1<\beta<d$;

(3). $\dim R^{d-1}=\begin{pmatrix} n+d \\ n+1 \end{pmatrix}-(n+2)$;

(4). $\dim R^d = \begin{pmatrix} n+1+d \\ n+1 \end{pmatrix} - (n+2)^2$;

(5). $\dim R^{2d-\beta} = \begin{pmatrix} n+1+2d-\beta \\ n+1 \end{pmatrix} - (n+2)\begin{pmatrix} n+1+d+1-\beta \\ n+1 \end{pmatrix}$ *for any* $2 < \beta < d$;

(6). $\dim R^{2d-2} = \begin{pmatrix} n+1+2d-2 \\ n+1 \end{pmatrix} - (n+2)\begin{pmatrix} n+d \\ n+1 \end{pmatrix} + \begin{pmatrix} n+2 \\ 2 \end{pmatrix}$.

(7). $\dim R^{2d-1} = \begin{pmatrix} n+1+2d-1 \\ n+1 \end{pmatrix} - (n+2)\begin{pmatrix} n+1+d \\ n+1 \end{pmatrix} + 2\begin{pmatrix} n+2 \\ 2 \end{pmatrix} + 3\begin{pmatrix} n+2 \\ 3 \end{pmatrix}$.

Proof (1) is obvious.

For the rest, we notice that $R^k = S^k/J^k$ with $\dim S^k = \begin{pmatrix} n+1+k \\ n+1 \end{pmatrix}$. Thus, we need to check the dimension of J^k.

For (2), with $1 < \beta < d$, we have $\dim J^{d-\beta} = 0$.

For (3), $J^{d-1} = n+2$.

For (4), since a basic of J^d contains all monomials that have the form of $x_i x_j^{d-1}$ for any $0 \le i, j \le n+1$. Thus $\dim J^d = (n+2)^2$.

For (5), it is easy to see that $J^{2d-\beta} \cong S^{d+1-\beta} \otimes J^{d-1}$ for $2 < \beta < d$. Thus

$$\dim J^{2d-\beta} = \dim S^{d+1-\beta} \times \dim J^{d-1} = (n+2)\begin{pmatrix} n+1+d+1-\beta \\ n+1 \end{pmatrix}.$$

For (6), a basis of $J^{2(d-1)}$ contains all the the monomials of the form $x^I = x_1^{i_1} x_2^{i_2} \cdots x_{n+2}^{i_{n+2}}$ with at least one of the $i_k \ge d-1$ and $i_1 + \cdots + i_{n+2} = 2d-2$. Thus

$$\dim J^{2(d-1)} = (n+2)\begin{pmatrix} n+d \\ n+1 \end{pmatrix} - \begin{pmatrix} n+2 \\ 2 \end{pmatrix}.$$

For (7), a basis of J^{2d-1} contains all the monomials of the form $x^I = x_1^{i_1}x_2^{i_2}\cdots x_{n+2}^{i_{n+2}}$ with at least one of the $i_k \geq d-1$ and $i_1 + \cdots i_{n+2} = 2d-1$. Thus

$$\dim J^{2d-1} = (n+2)\begin{pmatrix} n+1+d \\ n+1 \end{pmatrix} - 2\begin{pmatrix} n+2 \\ 2 \end{pmatrix} - 3\begin{pmatrix} n+2 \\ 3 \end{pmatrix}.$$

Remark A.1.6 *Using Proposition A.1.3 and Proposition A.1.5, and the isomorphism of D we discussed in §3.2.3, we may compute the dimension of D for projective hypersurfaces. Moreover, comparing the dimension of the moduli spaces of projective hypersurfaces, we can find the examples of projective hypersurfaces when the dimensions of D and the moduli space are the same. In fact, by simple computation, we found that the only K3-type (see § 6.2)of Calabi–Yau type of projective surfaces are the K3-surfaces and cubic fourfolds.*

A.2 Alternate proofs for some lemmas and propositions

The following proof is an alternate proof for Propostion 4.3.1.

Proposition A.2.1 *There exists a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{h}_0$ is also a Cartan subalgebra of $\mathfrak{k}_0$.*

Proof It suffices to show that $\mathfrak{k}_0$ and $\mathfrak{g}_0$ have equal rank, one realizes that that $\mathfrak{g}_0$ in our case is one of the following real simple Lie algebras: (1). $\mathfrak{sp}(2l, \mathbb{R})$, or (2). $\mathfrak{so}(p, q)$ with $p+q$ odd, or $\mathfrak{so}(p,q)$ with p and q both even. Let us write $s(G_{\mathbb{R}}/K) = \mathrm{rank}(G_{\mathbb{R}}/K) - \mathrm{rank}(G_{\mathbb{R}}) + \mathrm{rank}(K)$. Then according to [28, Ch. X, Table V] and [5, Table 7], (1). when $\mathfrak{g}_0 = \mathfrak{sp}(2l, \mathbb{R})$, $s(G_{\mathbb{R}}/K) = \mathrm{rank}(G_{\mathbb{R}}/K) = l$; (2). when $\mathfrak{g}_0 = \mathfrak{so}(p,q)$, if $p+q$ is odd, then $s(G_{\mathbb{R}}/K) = \mathrm{rank}(G_{\mathbb{R}}/K) = \min\{p,q\}$; if $p+q$ is even with p,q are both even, then $s(G_{\mathbb{R}}/K) = \mathrm{rank}(G_{\mathbb{R}}/K) = \min\{p,q\}$. Thus in both cases, $\mathrm{rank}(G_{\mathbb{R}}) = \mathrm{rank}(K)$. ∎

The following provides an alternate proof of Lemma 4.3.6, which is originally given in [51].

Lemma A.2.2 *There exists a set of strongly orthogonal roots* $\{\gamma_1, \cdots, \gamma_r\} \subseteq \Delta_{\mathfrak{p}}^+$ *such that the maximal abelian subspace* $\mathfrak{a}$ *of* $\mathfrak{p}$ *is*

$$\mathfrak{a} = \sum_{i=1}^{r} \mathbb{C}(e_{\gamma_i} + e_{-\gamma_i}).$$

In particular, the maximal abelian subspace in $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$ *is* $\mathfrak{a}_0 = \sum_{i=1}^{r} \mathbb{R}(e_{\gamma_i} + e_{-\gamma_i})$.

Proof For any subset Q in $\Delta_{\mathfrak{p}}^+$, let us denote

$$\mathfrak{p}(Q) = \sum_{\gamma \in Q} (\mathfrak{g}^{\gamma} + \mathfrak{g}^{-\gamma})$$

Let us also denote the generator of $\mathfrak{g}^{\gamma}$ by e_{γ}. If β is the lowest root in Q, we define

$$Q(\beta) = \{\gamma \in Q | \gamma \neq \beta, \gamma \pm \beta \in \Delta\}.$$

We claim that the centralizer $\mathfrak{z}$ of $e_{\beta} + e_{-\beta}$ in $\mathfrak{p}(Q)$ is equal to

$$\mathbb{C}(e_{\beta} + e_{-\beta}) + \mathfrak{p}(Q(\beta)).$$

To prove the claim, let X be an element in $\mathfrak{p}(Q)$ and $Q' = Q - \{\beta\}$. Then X can be written as

$$X = c_{\beta} e_{\beta} + c_{-\beta} e_{-\beta} + \sum_{\gamma \in Q'} (c_{\gamma} e_{\gamma} + c_{-\gamma} e_{-\gamma})$$

If $X \in \mathfrak{z}$, then $[X, e_{\beta} + e_{-\beta}] = 0$, that is,

$$(c_{\beta} - c_{-\beta})[e_{\beta}, e_{-\beta}] + \sum_{\gamma \in Q'} \big(c_{\gamma}[e_{\gamma}, e_{\beta}] + c_{\gamma}[e_{\gamma}, e_{-\beta}] + c_{-\gamma}[e_{-\gamma}, e_{\beta}] + c_{-\gamma}[e_{-\gamma}, e_{-\beta}]\big) = 0 \quad \text{(A.1)}$$

In (A.1), $(c_{\beta} - c_{-\beta})[e_{\beta}, e_{-\beta}]$ is the only term that is in $\mathfrak{h}$, thus $(c_{\beta} - c_{-\beta}) = 0$, that is

$$c_{\beta} = c_{-\beta}. \quad \text{(A.2)}$$

Moreover, for each $\gamma \in Q'$, $c_{\gamma}[e_{\gamma}, e_{\beta}]$ is the only term that belongs $\mathfrak{g}^{\gamma+\beta}$. In fact, if there were $\delta_1, \delta_2, \delta_3 \in Q'$ such that $\beta + \gamma = \delta_1 - \gamma = -\delta_2 + \gamma = -\delta_3 - \gamma$, then we would get

$$\beta + \gamma + \gamma = \delta_1 \in \Delta_{\mathfrak{p}}^+, \quad \beta + \gamma + \delta_2 = \gamma \in \Delta_{\mathfrak{p}}^+, \quad \beta + \gamma + \delta_3 = -\gamma \in \Delta_{\mathfrak{p}}^-,$$

which are contradictions to Lemma 4.3.3. Therefore

$$c_{\gamma}[e_{\gamma}, e_{\beta}] = 0 \quad \text{for any} \quad \gamma \in Q. \tag{A.3}$$

Similarly, we can also conclude that

$$c_{-\gamma}[e_{-\gamma}, e_{-\beta}] = 0 \quad \text{for any} \quad \gamma \in Q'. \tag{A.4}$$

Therefore (A.1) becomes

$$\sum_{\gamma \in Q'} \left(c_{\gamma}[e_{\gamma}, e_{-\beta}] + c_{-\gamma}[e_{-\gamma}, e_{\beta}] \right) = 0$$

We will show that

$$c_{\gamma}[e_{\gamma}, e_{-\beta}] = 0, \quad \text{and} \quad c_{-\gamma}[e_{-\gamma}, e_{\beta}] = 0 \tag{A.5}$$

In fact, let us suppose towards a contradiction that $c_{\gamma}[e_{\gamma}, e_{-\beta}] \neq 0$ for some $\gamma \in Q'$, then there exists $\delta \in Q'$ such that $\gamma - \beta = -\delta + \beta$. Therefore, $\beta = \delta + \gamma - \beta > \gamma \in Q$ since β is the lowest root in Q. However, it is a contradiction that $\beta > \gamma$ since β is the lowest root in Q.

From (A.2), (A.3), (A.4), and (A.5), we know that if either $\gamma - \beta \in \Delta$ or $\gamma + \beta \in \Delta$ for some $\gamma \in Q'$, then $c_\gamma = c_{-\gamma} = 0$. Therefore, we showed that the cneterlizer $\mathfrak{z}$ of $e_{\beta} + e_{-\beta}$ in $\mathfrak{p}(Q)$ is equal to $\mathbb{C}(e_{\beta} + e_{-\beta}) + \mathfrak{p}(Q(\beta))$.

Now we are ready to finish the prove of Proposition 4.3.6. Let $Q_1 = \Delta_{\mathfrak{p}}^{+}$ and let γ_1 be the lowest positive root in Q_1. Let $\mathfrak{p}_2$ denote the centralizer of $\mathfrak{g}^{\gamma_1} + \mathfrak{g}^{-\gamma_1}$ in $\mathfrak{p}_1 = \mathfrak{p}(Q_1)$. Let $Q_2 = Q_1(\gamma_1)$, then $\mathfrak{p}_2 = \mathfrak{p}(Q_2)$. Denoting by γ_2 the lowest positive root in Q_2. Let $\mathfrak{p}_3$ denote the centralizer of $\mathfrak{g}^{\gamma_2} + \mathfrak{g}^{-\gamma_2}$ in $\mathfrak{p}_2$. Let $Q_3 = Q_2(\gamma_2)$, then $\mathfrak{p}_2 = \mathfrak{p}(Q_3)$. Recursively, if we have defined γ_k and Q_k, then take $Q_{k+1} = Q_k(\gamma_k)$, then $\mathfrak{p}_{k+1} = \mathfrak{p}(Q_{k+1})$, which is exactly the centralizer of $\mathfrak{g}^{\gamma_k} + \mathfrak{g}^{-\gamma_k}$ in $\mathfrak{p}_k$. Then we denote the lowest root in Q_{k+1} by γ_{k+1}. In this way, we can define a sequence of spaces $\mathfrak{p} = \mathfrak{p}_1 \supseteq \mathfrak{p}_2 \supseteq \cdots \supseteq \mathfrak{p}_{s+1} = \{0\}$, each of which has the form $\mathfrak{p}_i = \mathfrak{p}(Q_i)$, where $(Q_i \subseteq \Delta_{\mathfrak{p}}^{+})$; and the set of roots $\gamma_1, \cdots, \gamma_s$. First, it is obvious that $\{\gamma_1, \cdots, \gamma_s\}$ is a set of strongly orthogonal sets. Let us denote

$$\mathfrak{a} = \sum_{i=1}^{s} \mathbb{C}(e_{\gamma_i} + e_{-\gamma_i}).$$

Then by the construction it is easy to see that $\mathfrak{a}$ is abelian. Moreover, suppose $X \in \mathfrak{p}$ commutes with each element in $\mathfrak{a}$. We claim that $X \in \mathfrak{a}$. Followed from the claim, we can conclude that $\mathfrak{a}$ is the maximal subspace in $\mathfrak{p}$. To prove the claim, suppose this were false. Then there exists an integer r $(1 \leq r \leq s)$ such that $X \in \mathfrak{p}_r + \mathfrak{a}$ but $X \notin \mathfrak{p}_{r+1} + \mathfrak{a}$. Let $X = Y + Z$ with $X \in \mathfrak{p}_r$ and $Z \in \mathfrak{a}$. Since X and Z commutes with $e_{\gamma_r} + e_{-\gamma_r}$, the same is true of Y. Thus the property that the centralizer $\mathfrak{z}$ of $e_\beta + e_{-\beta}$ in $\mathfrak{p}(Q)$ is equal to $\mathbb{C}(e_\beta + e_{-\beta}) + \mathfrak{p}(Q(\beta))$ implies that

$$Y = c(e_{\gamma_r} + e_{-\gamma_r}) + Y_1$$

where $Y_1 \in \mathfrak{p}_{r+1}$ and $c \in \mathbb{C}$. Now we have

$$X = Y + Z = Y_1 + c(e_{\gamma_r} + e_{-\gamma_r}) + Z.$$

Since $Z \in \mathfrak{a}$, we have $c(e_{\gamma_r} + e_{-\gamma_r}) + Z \in \mathfrak{a}$. Since $Y_1 \in \mathfrak{p}_{r+1}$, we conclude $X \in \mathfrak{p}_{r+1} + \mathfrak{a}$, a contradiction to the definition r. Thus we proved the claim. ∎

The following gives another proof for Lemma 4.5.4.

Lemma A.2.3 *The subset $\check{\mathcal{T}}$ is an open dense submanifold in $\mathcal{T}$, and $\mathcal{T} \backslash \check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $codim_{\mathbb{C}}(\mathcal{T} \backslash \check{\mathcal{T}}) \geq 1$.*

Proof We can also prove this lemma in a more direct manner. Let $p \in \mathcal{T}$ be the base point. For any $q \in \mathcal{T}$, let us still use $[A(q)^{i,j}]_{0 \leq i,j \leq 2s-n}$ as the transition matrix between the adapted bases $\{\eta_0, \cdots, \eta_{m-1}\}$ and $\{\zeta_0, \cdots, \zeta_{m-1}\}$ to the Hodge filtration at p and q respectively. In particular, we know that $[A(q)^{i,j}]_{0 \leq i,j \leq k}$ is the transition map between the bases of F_p^{s-k} and F_q^{s-k}. Therefore, $\det([A(q)^{i,j}]_{0 \leq i,j \leq k}) \neq 0$ if and only if F_q^{s-k} is isomorphic to F_p^{s-k}. We recall the inclusion

$$D \subseteq \check{D} \subseteq \check{\mathcal{F}} \subseteq Gr\left(f^s, H^n(X, \mathbb{C})\right) \times \cdots \times Gr\left(f^{n-s}, H^n(X, \mathbb{C})\right)$$

where $\check{\mathcal{F}} = \left\{F^s \subseteq \cdots \subseteq F^{n-s} \subseteq H^n(X, \mathbb{C}) \mid \dim_{\mathbb{C}} F^k = f^k\right\}$ and $Gr(f^k, H^n(X, \mathbb{C})$ is the complex vector subspaces of dimension f^k of $H^n(X, \mathbb{C})$. For each $n - s \leq k \leq s$ the points in $Gr\left(f^k, H^n(X,\mathbb{C})\right)$ whose corresponding vector spaces are not isomorphic to the reference vector space F_p^k form a divisor $Y_k \subseteq Gr\left(f^k, H^n(X,\mathbb{C})\right)$. Now we consider the divisor $Y \subseteq \prod_{k=n-s}^{s} Gr\left(f^k, H^n(X,\mathbb{C})\right)$ given by

$$Y = \sum_{k=n-s}^{s} \left(\prod_{j<k} Gr\left(f^j, H^n(X,\mathbb{C})\right) \times Y_k \times \prod_{j>k} Gr\left(f^j, H^n(X,\mathbb{C})\right) \right).$$

Then $Y \cap D$ is also a divisor in D. Since by Lemma 4.5.3, we know that $\Phi(q) \in N_+$ if and only if F_q^k is isomorphic to F_p^k for all $n-s \leq k \leq s$, so we have $\mathcal{T} \setminus \check{\mathcal{T}} = \Phi^{-1}(Y \cap D)$. Finally, by local Torelli theorem for Calabi–Yau manifolds, we know that $\Phi : \mathcal{T} \to D$ is a local embedding. Therefore, the complex codimension of $(\mathcal{T} \setminus \check{\mathcal{T}})$ in $\mathcal{T}$ is at least the complex codimension of the divisor $Y \cap D$ in D. ∎

The following is another proof of Lemma 5.2.1.

Lemma A.2.4 *The Hodge metric completion $\mathcal{Z}_m^H$ is a dense and open smooth submanifold in $\bar{\mathcal{Z}}_m$, and the complex codimenison of $\mathcal{Z}_m^H \backslash \mathcal{Z}_m$ is at least one.*

Proof Since $\bar{\mathcal{Z}}_m$ is a smooth manifold and $\mathcal{Z}_m$ is dense in $\bar{\mathcal{Z}}_m$, we only need to show that $\mathcal{Z}_m^H$ is open in $\bar{\mathcal{Z}}_m$. We can use a compactification $\overline{D/\Gamma}$ of D/Γ and a continuous extension of the period map $\Phi_{\mathcal{Z}_m} \to D/\Gamma$ to

$$\bar{\Phi}_{\mathcal{Z}_m} : \bar{\mathcal{Z}}_m \to \overline{D/\Gamma}.$$

Then since D/Γ is complete with respect to the Hodge metric, the Hodge metric completion space $\mathcal{Z}_m^H = \bar{\Phi}_{\mathcal{Z}_m}^{-1}(D/\Gamma)$. Notice that D/Γ is open and dense in the compacification in $\overline{D/\Gamma}$, $\mathcal{Z}_m^H$ is therefore an open subset of $\bar{\mathcal{Z}}_m$.

To define the compactification space $\overline{D/\Gamma}$, there are different approaches from literature. One natural compactification is given by Kao-Usuai following [49] as given in [31] and discussed in page 2 of [18], where one attaches to D a set $B(\Gamma)$ of equivalence classes of limiting mixed Hodge structures, then define $\overline{D/\Gamma} = (D \cup B(\Gamma))/\Gamma$ and extend the period map $\Phi_{\mathcal{Z}_m}$ continuously to $\bar{\Phi}_{\mathcal{Z}_m}$. An alternative and more recent method is to use the reduced limit period mapping as defined in page 29 and 30 in [18] to enlarge the period domain D. Different approaches can give us different compactification spaces $\overline{D/\Gamma}$. It is clear that both compactifications can be used for our above proof of the openness of $\mathcal{Z}_m^H = \bar{\Phi}_{\mathcal{Z}_m}^{-1}(D/\Gamma)$. ∎

The following gives another approach to Proposition 5.3.4.

Proposition A.2.5 *The map $i_m : \mathcal{T} \to \mathcal{T}_m^H$ is an embedding.*

Proof We will prove is a proof by contradiction. Suppose towards a contradiction that there were two points $p \neq q \in \mathcal{T}$ such that $i_m(p) = i_{\mathcal{T}}(q) \in \mathcal{T}_m^H$.

On one hand, since each point in $\mathcal{T}$ represents a polarized and marked Calabi–Yau manifold, p and q are actually triples (M_p, L_p, γ_p) and (M_q, L_q, γ_q) respectively, where γ_p and γ_q are two bases of $H_n(M, \mathbb{Z})/\text{Tor}$. On the one hand, each point in $\mathcal{Z}_m$ represents a triple (M, L, γ_m) with γ_m a basis of $(H_n(M, \mathbb{Z})/\text{Tor})/m(H_n(M, \mathbb{Z})\text{Tor})$. By the assumption that $i_m(p) = i_m(q)$ and the relation that $i \circ \pi_m = \pi_m^H \circ i_m$, we have $i \circ \pi_m(p) = i \circ \pi_m(q) \in \mathcal{Z}_m$. In particular, the image of (M_p, L_p, γ_p) and (M_q, L_q, γ_q) under π_m are the same in $\mathcal{Z}_m$. This implies that there exists a biholomorphic map $f_{pq} : M_p \to M_q$ such that $f_{pq}^*(L_q) = L_p$ and $f_{pq}^*(\gamma_q) = \gamma_p \cdot A$, where A is an integer matrix satisfying

$$A = (A_{ij}) \equiv \text{Id} \pmod{m} \qquad \text{for any} \quad m \geq m_0. \tag{A.6}$$

Let $|A_{ij}|$ be the absolute value of the ij-th entry of the matrix (A_{ij}). Since (A.6) holds for any $m \geq m_0$, we can choose an integer m_0 greater than any $|A_{ij}|$ such that

$$A = (A_{ij}) \equiv \text{Id} \pmod{m_0}.$$

Since each $A_{ij} < m_0$ and $A = (A_{ij}) \equiv \text{Id} \pmod{m_0}$, we have $A = \text{Id}$. Therefore, we found a biholomorphic map $f_{pq} : M_p \to M_q$ such that $f_{pq}^*(L_q) = L_p$ and $f_{pq}^*(\gamma_q) = \gamma_p$. This implies that the two triples (M_p, L_p, γ_p) and (M_q, L_q, γ_q) are equivalent to each other. Therefore, p and q in $\mathcal{T}$ are actually the same point. This contradicts to our assumption that $p \neq q$. ∎

A.3 Two topological lemmas

Lemma A.3.1 *There exists a suitable choice of* i_m *and* Φ^H_m *such that* $\Phi^H_m \circ i_m = \Phi$.

Proof Fix a reference point $p \in \mathcal{T}$. The relations $i \circ \pi_m = \pi^H_m \circ i_m$ and $\Phi^H_{\mathcal{Z}_m} \circ \pi^H_m = \pi_D \circ \Phi^H_m$ imply that $\pi_D \circ \Phi^H_m \circ i_m = \Phi^H_{\mathcal{Z}_m} \circ i \circ \pi_m = \Phi_{\mathcal{Z}_m} \circ \pi_m$. Therefore $\Phi^H_m \circ i_m$ is a lifting map of $\Phi_{\mathcal{Z}_m}$. On the other hand $\Phi : \mathcal{T} \to D$ is also a lifting of $\Phi_{\mathcal{Z}_m}$. In order to make $\Phi^H_m \circ i_m = \Phi$, one only needs to choose the suitable i_m and Φ^H_m such that these two maps agree on the reference point, i.e. $\Phi^H_m \circ i_m(p) = \Phi(p)$.

For an arbitrary choice of i_m, we have $i_m(p) \in \mathcal{T}^H_m$ and $\pi^H_m(i_m(p)) = i(\pi_m(p))$. Considering the point $i_m(p)$ as a reference point in $\mathcal{T}^H_m$, we can choose $\Phi^H_m(i_m(p))$ to be any point from $\pi_D^{-1}(\Phi^H_{\mathcal{Z}_m}(i(\pi_m(p)))) = \pi_D^{-1}(\Phi_{\mathcal{Z}_m}(\pi_m(p)))$. Moreover the relation $\pi_D(\Phi(p)) = \Phi_{\mathcal{Z}_m}(\pi_m(p))$ implies that $\Phi(p) \in \pi_D^{-1}(\Phi_{\mathcal{Z}_m}(\pi_m(p)))$. Therefore we can choose Φ^H_m such that $\Phi^H_m(i_m(p)) = \Phi(p)$. With this choice, we have $\Phi^H_m \circ i_m = \Phi$. ∎

Lemma A.3.2 *Let* $\pi_1(\mathcal{Z}_m)$ *and* $\pi_1(\mathcal{Z}^H_m)$ *be the fundamental groups of* $\mathcal{Z}_m$ *and* $\mathcal{Z}^H_m$ *respectively, and suppose the group morphism*

$$i_* : \pi_1(\mathcal{Z}_m) \to \pi_1(\mathcal{Z}^H_m)$$

is induced by the inclusion $i : \mathcal{Z}_m \to \mathcal{Z}^H_m$. *Then* i_* *is surjective.*

Proof First notice that $\mathcal{Z}_m$ and $\mathcal{Z}^H_m$ are both smooth manifolds, and $\mathcal{Z}_m \subseteq \mathcal{Z}^H_m$ is open. Thus for each point $q \in \mathcal{Z}^H_m \setminus \mathcal{Z}_m$ there is a disc $D_q \subseteq \mathcal{Z}^H_m$ with $q \in D_q$. Then the union of these discs

$$\bigcup_{q \in \mathcal{Z}^H_m \setminus \mathcal{Z}_m} D_q$$

forms a manifold with open cover $\{D_q : \ q \in \cup_q D_q\}$. Because both $\mathcal{Z}_m$ and $\mathcal{Z}^H_m$ are second countable spaces, there is a countable subcover $\{D_i\}_{i=1}^{\infty}$ such that $\mathcal{Z}^H_m = \mathcal{Z}_m \cup \bigcup_{i=1}^{\infty} D_i$, where the D_i are open discs in $\mathcal{Z}^H_m$ for each i. Therefore, we have $\pi_1(D_i) = 0$ for all $i \geq 1$. Letting

$\mathcal{Z}_{m,k} = \mathcal{Z}_m \cup \bigcup_{i=1}^{k} D_i$, we get

$$\pi_1(\mathcal{Z}_{m,k}) * \pi_1(D_{k+1}) = \pi_1(\mathcal{Z}_{m,k}) = \pi_1(\mathcal{Z}_{m,k-1} \cup D_k), \quad \text{for any } k.$$

We know that $\mathrm{codim}_{\mathbb{C}}(\mathcal{Z}_m^H \backslash \mathcal{Z}_m) \geq 1$. Therefore since $D_{k+1} \backslash \mathcal{Z}_{m,k} \subseteq D_{k+1} \backslash \mathcal{Z}_m$, we have $\mathrm{codim}_{\mathbb{C}}[D_{k+1} \backslash (D_{k+1} \backslash \mathcal{Z}_{m,k})] \geq 1$ for any k. As a consequence we can conclude that $D_{k+1} \cap \mathcal{Z}_{m,k}$ is path-connected. Hence we can apply the Van Kampen Theorem on $X_k = D_{k+1} \cup \mathcal{Z}_{m,k}$ to conclude that for every k, the following group homomorphism is surjective:

$$\pi_1(\mathcal{Z}_{m,k}) = \pi_1(\mathcal{Z}_{m,k}) * \pi_1(D_{k+1}) \twoheadrightarrow \pi_1(\mathcal{Z}_{m,k} \cup D_{k+1}) = \pi_1(\mathcal{Z}_{m,k+1}).$$

Thus we get the directed system:

$$\pi_1(\mathcal{Z}_m) \twoheadrightarrow \pi_1(\mathcal{Z}_{m,1}) \twoheadrightarrow \cdots\cdots \twoheadrightarrow \pi_1(\mathcal{Z}_{m,k}) \twoheadrightarrow \cdots\cdots$$

By taking the direct limit of this directed system, we get the surjectivity of the group homomorphism $\pi_1(\mathcal{Z}_m) \to \pi_1(\mathcal{Z}_m^H)$. ∎

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