Title

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Abstract

This paper considers the problem of distribution estimation for the studentized sample mean in the context of Long Memory and Negative Memory time series dynamics, adopting the fixed-bandwidth approach now popular in the econometrics literature. The distribution theory complements the Short Memory results of Kiefer and Vogelsang (2005). In particular, our results highlight the dependence on the employed kernel, whether or not the taper is nonzero at the boundary, and most importantly whether or not the process has short memory. We also demonstrate that small-bandwidth approaches fail when long memory or negative memory is present since the limiting distribution is either a point mass at zero or degenerate. Extensive numerical work provides approximations to the quantiles of the asymptotic distribution for a range of tapers and memory parameters; these quantiles can be used in practice for the construction of confidence intervals and hypothesis tests for the mean of the time series.

Keywords. Confidence Intervals, Critical Values, Dependence, Gaussian, Kernel, Spectral Density, Tapers, Testing.

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1 Introduction

This paper considers the asymptotics of estimates of the variance of the sample mean constructed from a tapered sum of sample autocovariances, when the underlying data generating process (DGP) exhibits either short or long memory. As in Kiefer and Vogelsang (2002, 2005), we work out the so-called fixed b asymptotics, i.e., the case that bandwidth is a fixed proportion of sample size. In Kiefer, Vogelsang, and Bunzel (2000) results are obtained for the Bartlett kernel, which show that the limiting numerator and denominator are independent. However, this is not true in the case of long/intermediate memory, and more generally is not true when other kernels are used.

We study the situation that we have a sample $Y = \{Y_1, Y_2, \cdots, Y_n\}$ from a strictly stationary time series with mean $EY_t = \mu$, autocovariance $\gamma_h = Cov(Y_t, Y_{t+h})$, and integrable spectral density function $f(\lambda) = \sum_h \gamma_h e^{-ih\lambda}$. Memory strength can be parameterized through the partial sums of autocovariances:

$$\sum_{|h|<n} \gamma_h \sim CL(n)n^{\beta}, \tag{1}$$

where in general $A_n \sim B_n$ denotes $A_n/B_n \to 1$ as $n \to \infty$. In (1), C is a positive constant, and L is slowly varying at infinity (Embrechts, Klüppelberg, and Mikosch, 1997), with a limit that can be zero, one, or infinity. Then β and L parametrize memory as follows: $1 > \beta > 0$ or $\beta = 0$ and L tending to infinity correspond to long memory (LM) in which case $f(0) = \infty$; $\beta = 0$ and L tending to unity correspond to the usual short memory (SM) where $0 < f(0) < \infty$; finally, $-1 < \beta < 0$ or $\beta = 0$ and L tending to zero correspond to the less-studied case where $f(0) = 0$ which we will denoted by negative memory (NM). In this context, Brockwell and Davis (1991) used the terminology "intermediate memory", whereas others have used "anti-persistence" (Lieberman and Phillips, 2006) or "negative dependence" (Samorodnitsky and Taqqu, 1994) due to negative correlations; our choice of terminology follows these latter authors. These definitions encompass ARFIMA models (Hosking, 1981), FEXP models (Beran (1993, 1994)), and fractional Gaussian Noise models. Some authors prefer to parametrize memory in terms of the rate of explosion of f or $1/f$ at frequency zero, but it is more convenient for us to work in the time domain; see Palma (2007) for a recent overview.

We stipulate $\beta < 1$ to ensure stationarity, and $\beta > -1$ to ensure that Y_t is not over-differenced, i.e., equal to the first difference of another stationary process. The SM case was covered in Kiefer and Vogelsang (2005) who used the Bartlett kernel for smoothing; our results provide extensions to LM and NM DGPs with a variety of kernels. The chief problem of interest is to properly normalize the partial sums $S_n = \sum_{t=1}^{n} Y_t$, which have finite-sample variance V_n. In general V_n grows at a rate dependent on β (e.g., see Taqqu (1975)), which makes the problem of normalization more tricky. Supposing that $V_n^{-1/2}(S_n - n\mu)$ converges weakly to a nondegenerate distribution, it is of interest to develop an estimate of V_n that can be plugged in. We consider an estimator $V_{\Lambda,M}$ based on a tapered sum of sample autocovariances and bandwidth M, which grows at the same asymptotic rate as V_n.

Our main asymptotic result is contained in Section 2; it is the derivation of the limiting distribution of the studentized sample mean under different dependence structures, i.e., value of β in (1), and using any kernel from a general family of kernels. Since in practice β will be unknown, it must be estimated and plugged in. Aside from β, the limiting distribution is pivotal, facilitating the construction of confidence intervals and hypothesis tests. Interestingly—and conveniently—, the slowly varying function L in (1) does not affect the asymptotic distribution. Section 3 contains some additional theoretical results that are pertinent to understanding the impact of NM and LM. Section 4 investigates numerically the limiting distribution of the studentized sample mean, and presents tables of critical values that can be used by practitioners. Section 5 presents our conclusions, and technical proofs are deferred to the Appendix.

2 Asymptotic Results

As in Kiefer and Vogelsang (2005), let the bandwidth M be proportional to sample size n, i.e., $M = bn$ with $b \in (0,1]$. We first introduce the following notation: the sample autocovariance is $\widetilde{\gamma}_k = n^{-1}\sum_{t=1}^{n-k}(Y_{t+k}-\overline{Y})(Y_t-\overline{Y})$ for $0 \le k < n$, and $\overline{Y}$ the sample mean. Also let $\widehat{S}_i = \sum_{t=1}^{i}(Y_t-\overline{Y})$ (so that $\widehat{S}_n = 0$), and define the tapered sum of autocovariances by $V_{\Lambda,M} = \sum_h \Lambda(h/M)\widetilde{\gamma}_h$, where Λ is a taper.

We consider tapers $\Lambda(x)$ from the following general family:

$$\{\Lambda \text{ is an even function with support on [-1,1] such that } \Lambda(x) = 1 \text{ for } |x| \le c, \text{ for some } c \in [0,1). \text{ Furthermore, } \Lambda \text{ is continuous everywhere and twice continuously differentiable on } (c,1) \cup (-1,-c).\} \tag{2}$$

The above class of tapers includes the family of 'flat-top' kernels of Politis (2005) where $c > 0$, the Bartlett kernel (letting $c = 0$ and a linear decay of Λ), as well as other kernels considered in Kiefer and Vogelsang (2005).

A derivative of Λ from the left (with respect to x) is denoted $\dot{\Lambda}_-$, whereas from the right is $\dot{\Lambda}_+$; the second derivative is $\ddot{\Lambda}$. The greatest integer function is denoted by $[\cdot]$. With this notation, the following basic proposition is presented.

Proposition 1 *Let Λ be a kernel from family (2), and assume (1) with $|\beta| < 1$. Then*

$$nV_{\Lambda,M} = \sum_{i,j=1}^{n} \widehat{S}_i \widehat{S}_j \left(2\Lambda\left(\frac{i-j}{M}\right) - \Lambda\left(\frac{i-j+1}{M}\right) - \Lambda\left(\frac{i-j-1}{M}\right) \right)$$

$$= -\frac{2}{bn} \sum_{i=1}^{n-[cbn]} \widehat{S}_i \widehat{S}_{i+[cbn]} \left(\dot{\Lambda}_+(c) + \frac{1}{2bn}\ddot{\Lambda}(c) + O(n^{-2}) \right)$$

$$- \frac{1}{b^2 n^2} \sum_{[cbn]<|i-j|<[bn]} \widehat{S}_i \widehat{S}_j \left(\ddot{\Lambda}\left(\frac{|i-j|}{bn}\right) + O(n^{-1}) \right) + \frac{2}{bn} \sum_{i=1}^{n-[bn]} \widehat{S}_i \widehat{S}_{i+[bn]} \left(\dot{\Lambda}_-(1) + O(n^{-1}) \right).$$

Remark 1 In case the taper is continuously differentiable at c, $\dot{\Lambda}_+(c) = 0$ and the second derivative becomes dominant in the first term, which can then be recombined with the second term to yield

$$-\frac{1}{b^2 n^2} \sum_{[cbn]\leq|i-j|<[bn]} \widehat{S}_i \widehat{S}_j \left(\ddot{\Lambda}\left(\frac{|i-j|}{bn}\right) + O(n^{-1}) \right).$$

Likewise, if there is no kink at $|x| = 1$, then $\dot{\Lambda}_-(1) = 0$ and the third term vanishes completely.

Since we want the asymptotics of $nV_{\Lambda,M}$, we need functional limit theorems for the partial sums, since $\hat{S}_i = S_i - i/n\, S_n$. To that end we suppose that

$$V_n^{-1/2} \left(S_{[nr]} - [nr]\mu \right) \overset{\mathcal{L}}{\Longrightarrow} B(r) \tag{3}$$

in the sense that the corresponding probability measures on $\mathcal{D}[0,1]$ (the space of functions on $[0,1]$ that are right continuous with left limits, endowed with the Skorohod topology – Taniguchi and Kakizawa (2000)) converge weakly. Here $B(\cdot)$ is a Fractional Brownian Motion (FBM) process of parameter $(\beta+1)/2$ (Samorodnitsky and Taqqu, 1994).

Sufficient conditions for (3) include linearity and a moment condition (see Theorem 5.2.4 of Taniguchi and Kakizawa (2000)), as well as supposing that the process is an instantaneous functional of a long memory Gaussian (see Theorem 5.1 of Taqqu (1975)). In the interest of brevity we will henceforth assume that (3) holds, from which it follows that $\widehat{S}_{[rn]}/\sqrt{V_n}$ converges weakly to the process $\widetilde{B}(r) = B(r) - rB(1)$, which is a Fractional Brownian Bridge (FBB). Then we may conclude the following result:

Theorem 1 *Let Λ be a kernel from family (2), and suppose that $\{Y_t\}$ is a DGP such that (3) holds. Also assume that (1) holds with $|\beta| < 1$. Then*

$$\frac{S_n - n\mu}{\sqrt{nV_{\Lambda,M}}} \overset{\mathcal{L}}{\Longrightarrow} \frac{B(1)}{\sqrt{Q(b)}} \tag{4}$$

as $n \to \infty$, where $Q(b)$ is defined by

$$-\frac{2}{b}\dot{\Lambda}_+(c) \int_0^{1-cb} \widetilde{B}(r)\widetilde{B}(r+cb)\, dr - \frac{1}{b^2} \int_{cb<|r-s|<b} \widetilde{B}(r)\widetilde{B}(s)\ddot{\Lambda}\left(\frac{|r-s|}{b}\right) drds$$
$$+ \frac{2}{b}\dot{\Lambda}_-(1) \int_0^{1-b} \widetilde{B}(r)\widetilde{B}(r+b)\, dr.$$

Interestingly, the limit $Q(b)$ does not depend at all on the slowly varying function L that appears in (1), and thus L does not affect the asymptotic distribution of the studentized sample mean.

Remark 2 Note that $B(1)$ and $Q(b)$ in (4) will not be independent except in special cases. Such a special case is the set-up of Kiefer, Vogelsang and Bunzel (2000) who consider the case that $b = 1$ and $c = 0$, the kernel is the Bartlett, and $\beta = 0$. Then $Q(1) = 2\int_0^1 \widetilde{B}^2(r)\,dr$, and $B(1)$ is independent of $\widetilde{B}(r)$ because the covariance of $B(1)$ and $\widetilde{B}(r)$ is zero in that case. However, in general,

$$Cov\left(B(1), \widetilde{B}(r)\right) = -r + \left(1 + r^{\beta+1} - (1-r)^{\beta+1}\right)/2$$

which is nonzero unless of course $\beta = 0$. So, in general (say when $\beta \neq 0$ and/or the kernel is not the Bartlett) $B(1)$ and $Q(b)$ will be dependent. However, it is a simple matter to determine the limiting distribution of (4) numerically for any given value of β, and any choice of taper and bandwidth b.

Example 1 The trapezoidal taper is the benchmark flat-top taper whose use was proposed by Politis and Romano (1995); it is defined by

$$\Lambda^{T,c}(x) = \begin{cases} 1 & \text{if } |x| \leq c \\ \frac{|x|-1}{c-1} & \text{if } c < |x| \leq 1 \\ 0 & \text{else.} \end{cases}$$

Hence the second derivative for $|x| \in (c, 1]$ is zero, and

$$Q(b) = \frac{2}{b(c-1)}\left(\int_0^{1-b} \widetilde{B}(r)\widetilde{B}(r+b)\,dr - \int_0^{1-cb} \widetilde{B}(r)\widetilde{B}(r+cb)\,dr\right).$$

3 Theoretical Properties

We next discuss a few of the theoretical properties of $Q(b)$, which shed light on why the $\beta = 0$ case is so different from the LM and NM cases. Using the abbreviation

$$A(x) = 2\int_0^{1-x} \widetilde{B}(r)\widetilde{B}(r+x)\,dr,$$

we can re-express $Q(b)$ as

$$Q(b) = -\frac{1}{b}\int_c^1 \ddot{\Lambda}(x)A(bx)\,dx + \frac{1}{b}\left(\dot{\Lambda}_-(1)A(b) - \dot{\Lambda}_+(c)A(cb)\right). \tag{5}$$

The following proposition provides the first moment of $Q(b)$.

Proposition 2 *Under the conditions of Theorem 1,*

$$\mathbb{E}[A(x)] = \frac{2}{(\beta+2)(\beta+3)}\left((1-x)^{\beta+3} - x^{\beta+3} + 1\right) + \frac{1}{3}(x^3-1) + \frac{1}{\beta+2}\left(2x + \beta x^{\beta+2} - (\beta+2)x^{\beta+1}\right).$$

Denoting this function by $g(x)$*,*

$$\mathbb{E}[Q(b)] = -\frac{1}{b}\int_c^1 \ddot{\Lambda}(x) g(bx)\,dx + \frac{1}{b}\left(\dot{\Lambda}_-(1)g(b) - \dot{\Lambda}_+(c)g(cb)\right) = \int_c^1 \dot{\Lambda}(x)\dot{g}(bx)\,dx,$$

where $\dot{g}$ *is an integrable function for* $\beta > -1$ *and is given by*

$$\dot{g}(x) = -\frac{2}{\beta+2}\left((1-x)^{\beta+2} + x^{\beta+2}\right) + x^2 + \frac{1}{\beta+2}\left(2 + (\beta+2)x^{\beta}(\beta x - \beta - 1)\right).$$

Moreover,

$$\lim_{b\to 0}\mathbb{E}[Q(b)] = \begin{cases} 0 & \beta > 0 \\ 1-\Lambda(1) & \beta = 0 \\ \infty & \beta < 0 \end{cases}. \tag{6}$$

Example 1 Continuing the example of the trapezoidal taper, the mean is calculated to be

$$\mathbb{E}[Q(b)] = \frac{2}{(\beta+2)(\beta+3)(1-c)b}\left[(1-cb)^{\beta+3} - (1-b)^{\beta+3} - b^{\beta+3}(c^{\beta+3}-1)\right] - \frac{b^2}{3}(1+c+c^2)$$
$$+ \frac{1}{(\beta+2)(\beta+3)}\left[2(c-1) + b^{\beta+1}\beta(c^{\beta+2}-1) - b^{\beta}(\beta+2)(c^{\beta+1}-1)\right].$$

Hence the small bandwidth behavior is given by

$$\lim_{b\to 0}\mathbb{E}[Q(b)] = \lim_{b\to 0} b^{\beta}\,\frac{1-c^{\beta+1}}{1-c},$$

which equals ∞, one, or zero depending on whether the DGP is NM ($\beta < 0$), SM ($\beta = 0$), or LM ($\beta > 0$).

More generally, by (6) we see that the small b mean of $Q(b)$ is ∞, $1-\Lambda(1)$, or zero depending on whether the DGP is NM, SM, or LM. Since $Q(b)$ is in the denominator of the limiting distribution in (4), this implies a small-bandwidth limiting distribution of zero, normal (standard normal if $\Lambda(1) = 0$), or infinity (informally speaking) for the self-normalized statistic (4) in the cases NM, SM, or LM respectively.

Mean calculations for other tapers, such as Parzen, Bohman, Daniell, etc., are quite involved and are not included here. However, when $\beta = 0$ the results of Proposition 2 reduce to those of Kiefer and Vogelsang (2005), but extended to include flat-top tapers as well as tapers with $\Lambda(1) \neq 0$ (that are not included in their Definition 1). Letting $\mu_0 = \int_0^1 \Lambda(x)\,dx$ and $\mu_1 = \int_0^1 \Lambda(x)x\,dx$, we obtain for the $\beta = 0$ case that

$$\mathbb{E}[Q(b)] = 1 - \Lambda_-(1) + 2b\left(\Lambda_-(1) - \mu_0\right) + 2b^2\left(\mu_1 - \Lambda_-(1)/2\right).$$

We compute these expectations for several tapers whose definition is standard; see e.g. Bohman (1960), Priestley (1981) or Politis (2005).

$$\begin{aligned}
\text{Trapezoidal} &: 1 - b(1+c) + b^2(1+c+c^2)/3 \\
\text{Parzen} &: 1 - 3b/4 + 7b^2/40 \\
\text{Modified Quadratic Spectral} &: 1 - 3/\pi^2 + b\left(9/\pi^2 - 3\zeta\right) + 3b^2/\pi^2 \\
\text{Daniell} &: 1 - 2\zeta b + 4b^2/\pi^2 \\
\text{Tukey-Hanning} &: 1 - b + b^2\left(1/2 - 2/\pi^2\right) \\
\text{Bohman} &: 1 - 8b/\pi^2 + 2b^2/\pi^2,
\end{aligned}$$

where $\zeta = \int_0^1 \sin(\pi x)/(\pi x)\, dx \approx .589$. We wanted to include the Quadratic Spectral (QS) taper so that our results would be in conformity with Kiefer and Vogelsang (2005) – and also because it has some optimality properties among second order kernels (Priestley, 1981) – but the natural domain of this taper is $\mathbb{R}$. Therefore in restricting its support to $[-1, 1]$ we are greatly modifying its properties; the resulting restricted QS taper will be referred to as the Modified Quadratic Spectral (MQS), with formula given by $3(\sin(\pi x)/(\pi x) - \cos(\pi x))/(\pi x)^2$ for $x \in [-1, 1]$ and zero for $|x| > 1$. Then the MQS has a small-bandwidth bias, since $\mathbb{E}[Q(0)] = 1 - 3/\pi^2$; this is due to the fact that $\Lambda(\pm 1) = 3/\pi^2$. This bias causes an inflation to the variance of the limiting distribution, such that the limit is normal with variance 1.4367, rather than unity.

4 Numerical results

In this section we investigate the distribution $B(1)/\sqrt{Q(b)}$ of eq. (4) for various choices of β, b, and taper. Following Kiefer and Vogelsang (2005), we calculate upper quantiles of this distribution using the device or regressing on a convenient function of b for fixed taper, α-level, and β; these quantiles can then be used to construct confidence intervals or to find critical values for hypothesis tests regarding the mean. Since the distribution of $B(1)/\sqrt{Q(b)}$ is symmetric for all $|\beta| < 1$, it is sufficient to consider the upper quantiles. However, there are some differences in our presentation from Kiefer and Vogelsang (2005), which we discuss below; we also present some discussion on the simulation of $Q(b)$ since the details are non-trivial.

In Kiefer and Vogelsang (2005) each quantile is approximated by a cubic function of b but with the intercept (which corresponds to $b = 0$) set equal to the normal quantiles. As noted in the previous section, this is inappropriate for the MQS kernel, unless one first rescales the normal distribution by $\sqrt{1.4367}$. Moreover, since in the NM and LM cases we may expect the small-bandwidth case ($b = 0$) to correspond to zero and infinity respectively, it is nonsensical to fix the intercept when $\beta \neq 0$. For coherency of results, neither do we fix the intercept when $\beta = 0$.

Next, we consider an appropriate function to regress our quantiles on. When regressing on a cubic function, the intercept differed substantially from the normal quantiles in the $\beta = 0$ case for several of our tapers. This is due to increased variation in the quantile function for higher values of b; this heteroscedasticity is stabilized by taking the logarithm of the quantiles. When the log-quantile is regressed on a quintic polynomial, the resulting intercepts actually corresponds to the normal quantiles (when exponentiated). Therefore for all values of β, we regress the log-quantiles on a quintic function of b, namely

$$cv(b) = \exp\{a_0 + a_1 b + a_2 b^2 + a_3 b^3 + a_4 b^4 + a_5 b^5\}$$

and report the corresponding coefficients $a_0, a_1, \cdots, a_5$, as well as the R^2 between the log-quantile and the above $cv(b)$.

Note that the log of the normal quantiles at level .90, .95, .975, and .99 are given by .248, .498, .673, and .844; these can be compared with the $cv(b)$ corresponding to the coefficients given in the entries of our Tables 1–9 which correspond to different values of the memory parameter $\beta = -.8, -.6, -.4, -.2, 0, .2, .4, .6, .8$.

For each value of β, the entries of the appropriate table were obtained by simulating 50,000 sample-paths of a FBB of length 1,000, and computing $B(1)/\sqrt{Q(b)}$ for 50 choices of b (evenly spaced between .02 and 1.0) and one of eight tapers. The tapers considered are: Bartlett, Parzen, Daniell, Modified Quadratic Spectral, Tukey-Hanning, Bohman, Trapezoidal ($c = .25$), and Trapezoidal ($c = .5$).

Before summarizing the results, we comment on the method of simulation. The best approach is to discretize (5) starting with a discretization of $A(x)$. For N the chosen mesh size, let A_j for $j = 0, \cdots, N$ be defined by

$$A_j = \frac{2}{N} \sum_{k=1}^{N-[bj]} W_k W_{k+[bj]},$$

where $[\cdot]$ is the greatest integer function, and W_k is a discretization of FBB. Let $X_{j+1} = B(j + 1/N) - B(j/N)$ be an increment of FBM; then computation shows that this time series (for fixed N) is stationary with autocovariance function

$$\gamma(h) = \frac{1}{2}\left(\frac{h+1}{N}\right)^{\beta+1} - \left(\frac{h}{N}\right)^{\beta+1} + \frac{1}{2}\left|\frac{h-1}{N}\right|^{\beta+1}; \qquad (7)$$

cf. Hall, Jing, and Lahiri (1998). Hence the cumulation of the X_j time series (with initial condition $B(0) = 0$) will be a discrete sampling from FBM, and the FBB is then obtained by $W_k = B(k/N) - \frac{k}{N}B(1)$, for $k = 1, \cdots, N$. Hence the first step is to generate a Gaussian time series with autocovariance function (7)[1], then cumulate to get the FBM, and finally obtain the

[1]Simply find the corresponding $N \times N$ Toeplitz covariance matrix, compute the square root (or Cholesky factor), and right multiply by a standard normal Gaussian vector.

FBB. As a result, A_j is an approximation to $A(bj/N)$. We plug these A_j into a discretization of (5) utilizing the trapezoidal rule:

$$\widehat{Q}(b) = -\frac{1}{bN}\left(\frac{1}{2}\ddot{\Lambda}(0)A_0 + \sum_{j=1}^{N-1}\ddot{\Lambda}(j/N)A_j + \frac{1}{2}\ddot{\Lambda}(1)A_N\right) + \frac{1}{b}\left(\dot{\Lambda}_-(1)A_N - \dot{\Lambda}_+(c)A_{[cN]}\right).$$

Surprisingly, using the trapezoidal rule reduced substantial bias that arose in using the left-hand and right-hand integral discretization techniques; this bias was identified using the exact expectation for $\beta = 0$ from Section 3. In our simulations we used $N = 1000$, and had 50,000 repetitions for each choice of b, β, and taper. Each of these $50 \times 9 \times 8 = 360$ simulations required one to two hours of computing time on a 3.20 GHz processor (with 3 GB RAM), though the Trapezoidal kernels went much faster. The code was written in R.

The results in the tables demonstrate the sensitivity of these critical values to β; in particular, the cost of falsely assuming $\beta = 0$ is acute for small bandwidth ($b \approx 0$). When $\beta > 0$, there is a change in the shape of the quantile function as b increases; initially the quantiles start out high, then they drop down a bit, and then rise steadily. This is reflected in the large positive value of the a_2 coefficient, and is consistent with the results of Proposition 2. But when $\beta < 0$ there is a downward facing parabolic shape, reflected in the negative values of a_2. The variation of results between tapers is not as great (excepting the biased MQS taper).

5 Conclusion

The paper at hand investigates the distribution of the studentized sample mean in the context of NM and LM time series dynamics, adopting the fixed-bandwidth approach now popular in the econometrics literature. We derive the limiting distribution in Theorem 1, thus generalizing the results of Kiefer and Vogelsang (2005) not only to different dependence structures but also employing kernels other than the Bartlett. Our results highlight the influence of the kernel – e.g., whether or not the taper is nonzero at the boundary of its support – and the influence of the DGP's type of memory. Notably, the cost of using the SM quantiles when NM or LM is present increases with $|\beta|$. A main finding from our calculations – see (6) – is that small-bandwidth approaches are doomed to failure when NM or LM is present, since the limiting distribution of the usual studentized sample mean is either a point mass at zero or degenerate; this provides further support for the fixed-bandwidth approach to hypothesis testing and confidence intervals for the mean.

If the practitioner suspects that the time series is NM or LM, it is important to get an accurate estimate of β, say $\widehat{\beta}$, so that the correct critical values can be used. There is a large literature on the estimation of the memory parameter β; available methods are either parametric (e.g., Giraitis and Taqqu (1999)), semiparametric (Giraitis and Surgailis (1990) and Hurvich (2002)), or even nonparametric (McElroy and Politis (2007)). Once estimator $\widehat{\beta}$ is obtained, it will–of course–not be exactly equal to one of the values $-.8, -.6, -.4, -.2, 0, .2, .4, .6, .8$ considered in our Tables. Interpolation between the two closest β values (or regression using more than two values) can then be used to get the desired quantile corresponding to the estimated β; in this way the tables of this paper can be used for practical data analysis.

Appendix

Proof of Proposition 1. For shorthand let $W_t = Y_t - \overline{Y}$. Then using summation by parts as in Kiefer and Vogelsang (2002, 2005),

$$\begin{aligned} nV_{\Lambda,M} &= \sum_{|h|<n} \Lambda(h/M) \sum_{t=1}^{n-|h|} W_t W_{t+|h|} \\ &= \sum_{i,j=1}^{n} W_i W_j \Lambda\left(\frac{|i-j|}{bn}\right) \\ &= \sum_{i=1}^{n} W_i \left[\sum_{j=1}^{n-1} \left(\Lambda\left(\frac{i-j}{bn}\right) - \Lambda\left(\frac{i-j-1}{bn}\right)\right) \widehat{S}_j \right] \\ &= \sum_{i,j=1}^{n} \widehat{S}_i \widehat{S}_j \left(2\Lambda\left(\frac{i-j}{bn}\right) - \Lambda\left(\frac{i-j+1}{bn}\right) - \Lambda\left(\frac{i-j-1}{bn}\right)\right). \end{aligned}$$

Consider $2\Lambda\left(\frac{h}{bn}\right) - \Lambda\left(\frac{h+1}{bn}\right) - \Lambda\left(\frac{h-1}{bn}\right)$. If $[cbn] < h < [bn]$, then the approximation $-b^{-2}n^{-2}\ddot{\Lambda}\left(\frac{h}{bn}\right)$ holds. If $h = [cbn]$, we obtain $2\Lambda(c) - \Lambda(c+1/bn) - 1 + o(1) = -\dot{\Lambda}_+(c)/bn - \ddot{\Lambda}(c)/(2b^2n^2) + O(n^{-3})$. Finally, if $h = [bn]$ we obtain $-\Lambda(1-1/bn) + o(1) = \dot{\Lambda}_-(1)/bn + O(n^{-2})$. This completes the proof of the Proposition. □

Proof of Theorem 1. This follows at once from Proposition 1 and (3), noting that $V_n^{-1/2}(S_n - n\mu) \stackrel{\mathcal{L}}{\Longrightarrow} B(1)$ jointly with $V_n^{-1/2}\widehat{S}_i$ tending to $\widetilde{B}(i/n)$. The second order terms in $nV_{\Lambda,M}$ in Proposition 1 drop out, and the summations become integrals. In the case that $\dot{\Lambda}_+(c) = 0$, we can apply Remark 1 and extend the integral to $|r-s| = cb$. Since this set has measure zero, it has no impact on the final limit $Q(b)$. □

Proof of Proposition 2. The expectation of $A(x)$ hinges on the mean of $\widetilde{B}(r)\widetilde{B}(r+x)$, which by the definition of FBM is

$$\frac{1}{2}\left(r^{\beta+1} + (1-r)(r+x)^{\beta+1} - x^{\beta+1} + r\left((1-r-x)^{\beta+1} - 1\right) - (r+x)(1 + r^{\beta+1} - (1-r)^{\beta+1} - 2r)\right).$$

Integrating and consolidating yields the stated expression for $g(x)$, and the derivative follows at once. The expressions for $Q(b)$ follow from (5) and integration by parts. As for (6), if $\beta > 0$ we can use the Dominated Convergence Theorem (DCT) to obtain the limit $\int_c^1 \dot{\Lambda}(x)\dot{g}(0)\,dx$, which is zero since $\dot{g}(0) = 0$. If $\beta = 0$, we easily find $\dot{g}(x) = -(1-x)^2$, and the result follows again by the DCT. But if $\beta < 0$, observe that all terms in $\dot{g}$ are bounded except $-(\beta+1)x^\beta$. Therefore this term in $\dot{g}(bx)$ yields $-(\beta+1)b^\beta$ times integrable functions, and hence the mean tends to infinity at rate b^β (the minus sign is accounted for by the remaining factor $\int_c^1 \dot{\Lambda}(x)x^\beta\,dx$). $\square$

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	a_0	a_1	a_2	a_3	a_4	a_5	R^2
Bartlett							
.90	-1.410	9.695	-34.317	63.846	-56.797	19.224	.9967
.95	-1.159	9.540	-33.465	61.782	-54.575	18.368	.9967
.975	-.989	9.551	-33.931	63.595	-56.951	19.394	.9962
.99	-.821	9.559	-34.378	65.216	-58.973	20.242	.9959
Parzen							
.90	-1.088	9.058	-31.482	58.387	-51.795	17.500	.9974
.95	-.836	8.988	-31.152	58.112	-51.685	17.473	.9977
.975	-.660	8.888	-30.478	57.181	-51.128	17.358	.9977
.99	-.482	8.776	-29.408	54.918	-48.646	16.293	.9977
MQS							
.90	-.677	8.967	-30.305	57.247	-51.862	17.976	.9980
.95	-.429	9.080	-30.216	58.663	-54.848	19.513	.9982
.975	-.245	8.963	-27.897	54.380	-51.698	18.600	.9985
.99	-.066	8.690	-21.987	40.008	-37.625	13.511	.9984
Daniell							
.90	-.886	9.137	-31.976	62.030	-57.255	20.026	.9980
.95	-.639	9.258	-32.272	64.311	-60.716	21.521	.9985
.975	-.460	9.236	-31.493	64.633	-62.641	22.520	.9987
.99	-.292	9.309	-28.922	59.095	-57.544	20.579	.9989
TH							
.90	-.965	9.154	-31.898	60.325	-54.196	18.450	.9981
.95	-.711	8.981	-30.533	57.708	-51.621	17.423	.9982
.975	-.540	9.037	-30.177	57.385	-51.260	17.140	.9986
.99	-.365	8.904	-27.993	52.779	-46.490	15.213	.9985
Bohman							
.90	-1.061	9.140	-32.077	60.142	-53.690	18.173	.9973
.95	-.813	9.137	-32.048	60.617	-54.452	18.516	.9975
.975	-.631	8.904	-30.504	57.451	-51.337	17.342	.9976
.99	-.449	8.667	-28.566	53.245	-47.195	15.836	.9981
Trap, $c = 1/4$							
.90	-.834	9.584	-32.394	60.973	-55.547	19.461	.9977
.95	-.582	9.698	-31.018	58.153	-53.808	19.238	.9980
.975	-.411	9.890	-28.377	50.514	-46.053	16.487	.9984
.99	-.277	11.191	-27.825	42.499	-34.758	11.667	.9984
Trap, $c = 1/2$							
.90	-.673	9.486	-32.366	56.970	-46.597	14.423	.9963
.95	-.393	9.760	-31.125	51.790	-40.265	11.905	.9954
.975	-.196	10.965	-34.642	56.959	-44.042	13.001	.9946
.99	.025	13.887	-48.143	85.364	-71.414	22.810	.9932

Table 1: Fixed-b asymptotic critical value function coefficients for $cv(b) = \exp\{a_0 + a_1 b + a_2 b^2 + a_3 b^3 + a_4 b^4 + a_5 b^5\}$. The memory parameter is $\beta = -.8$. The R^2 indicates the fit of the polynomial through the log simulated asymptotic critical values.

	a_0	a_1	a_2	a_3	a_4	a_5	R^2
Bartlett							
.90	-1.012	7.763	-27.159	51.464	-46.627	16.011	.9978
.95	-.753	7.591	-25.846	48.185	-43.152	14.688	.9977
.975	-.575	7.547	-25.412	47.203	-42.182	14.326	.9978
.99	-.413	7.785	-26.355	49.255	-44.355	15.195	.9980
Parzen							
.90	-.884	7.153	-23.675	44.017	-39.242	13.285	.9982
.95	-.635	7.250	-23.778	44.531	-39.935	13.587	.9985
.975	-.453	7.175	-22.618	41.672	-36.891	12.421	.9985
.99	-.287	7.370	-22.841	42.420	-37.866	12.833	.9985
MQS							
.90	-.545	7.522	-23.534	44.649	-40.780	14.150	.9992
.95	-.296	7.765	-23.605	46.104	-43.555	15.511	.9990
.975	-.115	7.804	-21.481	41.398	-39.392	14.091	.9992
.99	.058	7.882	-17.119	30.182	-28.503	10.285	.9992
Daniell							
.90	-.749	7.718	-25.650	50.491	-47.222	16.607	.9990
.95	-.497	7.864	-25.312	50.594	-47.965	16.951	.9991
.975	-.318	7.908	-24.092	49.067	-47.536	16.987	.9991
.99	-.129	7.719	-20.079	41.198	-41.320	15.086	.9993
TH							
.90	-.800	7.478	-24.359	45.544	-40.432	13.550	.9986
.95	-.537	7.360	-22.948	42.980	-38.284	12.847	.9987
.975	-.357	7.390	-22.061	41.235	-36.498	12.094	.9985
.99	-.194	7.656	-21.750	40.837	-36.068	11.796	.9985
Bohman							
.90	-.859	7.113	-23.061	41.989	-36.565	12.094	.9981
.95	-.611	7.255	-23.462	43.321	-38.150	12.722	.9984
.975	-.433	7.242	-22.837	42.356	-37.586	12.636	.9986
.99	-.267	7.595	-24.227	46.306	-42.110	14.404	.9989
Trap, $c = 1/4$							
.90	-.703	7.580	-21.985	39.914	-36.015	12.562	.9986
.95	-.462	7.918	-20.946	37.280	-34.251	12.281	.9987
.975	-.289	8.111	-17.620	27.583	-24.416	8.836	.9989
.99	-.138	8.800	-13.560	12.129	-6.481	1.916	.9992
Trap, $c = 1/2$							
.90	-.632	8.738	-27.311	46.583	-37.689	11.616	.9983
.95	-.401	10.023	-30.596	50.310	-39.324	11.730	.9982
.975	-.257	11.979	-37.015	60.362	-46.841	13.896	.9984
.99	-.142	15.826	-52.471	88.995	-71.582	21.961	.9975

Table 2: Fixed-b asymptotic critical value function coefficients for $cv(b) = \exp\{a_0 + a_1 b + a_2 b^2 + a_3 b^3 + a_4 b^4 + a_5 b^5\}$. The memory parameter is $\beta = -.6$. The R^2 indicates the fit of the polynomial through the log simulated asymptotic critical values.

	a_0	a_1	a_2	a_3	a_4	a_5	R^2
Bartlett							
.90	-.548	4.971	-15.265	28.021	-24.666	8.258	.9982
.95	-.303	5.119	-15.335	28.184	-24.810	8.285	.9981
.975	-.134	5.357	-15.893	29.589	-26.420	8.961	.9984
.99	.030	5.703	-17.068	32.980	-30.438	10.584	.9984
Parzen							
.90	-.884	7.153	-23.675	44.017	-39.242	13.285	.9988
.95	-.635	7.250	-23.778	44.531	-39.935	13.587	.9990
.975	-.453	7.175	-22.618	41.672	-36.891	12.421	.9991
.99	-.287	7.370	-22.841	42.420	-37.866	12.833	.9991
MQS							
.90	-.251	5.222	-13.989	26.530	-24.577	8.645	.9993
.95	-.003	5.550	-14.401	28.960	-28.566	10.523	.9994
.975	.171	5.690	-12.799	25.863	-26.536	10.083	.9992
.99	.345	5.696	-8.054	13.768	-14.639	5.831	.9991
Daniell							
.90	-.454	5.414	-15.575	30.209	-28.085	9.789	.9992
.95	-.211	5.785	-16.648	34.425	-33.912	12.329	.9994
.975	-.036	6.066	-16.851	36.392	-37.125	13.716	.9995
.99	.143	6.120	-14.225	31.851	-34.370	13.121	.9991
TH							
.90	-.498	5.245	-15.184	27.980	-24.708	8.271	.9994
.95	-.237	5.188	-13.539	23.827	-20.015	6.328	.9993
.975	-.048	5.171	-12.380	21.587	-17.777	5.394	.9993
.99	.122	5.376	-12.083	22.611	-20.255	6.672	.9991
Bohman							
.90	-.539	5.127	-15.803	29.174	-25.769	8.643	.9991
.95	-.295	5.273	-15.534	28.105	-24.288	7.945	.9991
.975	-.112	5.303	-14.869	26.455	-22.462	7.186	.9989
.99	.056	5.557	-15.377	28.275	-24.893	8.218	.9987
Trap, $c = 1/4$							
.90	-.445	5.591	-14.035	25.320	-22.995	8.031	.9991
.95	-.191	5.663	-11.543	19.585	-18.153	6.573	.9991
.975	-.000	5.379	-5.462	3.761	-2.357	1.006	.9992
.99	.154	5.722	.794	-16.005	18.875	-6.737	.9990
Trap, $c = 1/2$							
.90	-.413	6.810	-18.394	29.252	-22.409	6.584	.9990
.95	-.179	7.759	-18.945	25.890	-16.438	3.774	.9992
.975	-.043	9.522	-23.601	31.248	-18.877	4.001	.9992
.99	.058	12.974	-35.633	51.040	-34.468	8.747	.9988

Table 3: Fixed-b asymptotic critical value function coefficients for $cv(b) = \exp\{a_0 + a_1 b + a_2 b^2 + a_3 b^3 + a_4 b^4 + a_5 b^5\}$. The memory parameter is $\beta = -.4$. The R^2 indicates the fit of the polynomial through the log simulated asymptotic critical values.

	a_0	a_1	a_2	a_3	a_4	a_5	R^2
Bartlett							
.90	-.165	3.086	-8.381	15.691	-14.494	5.076	.9986
.95	.081	3.258	-8.480	15.584	-14.397	5.076	.9988
.975	.250	3.491	-8.662	15.075	-13.387	4.580	.9990
.99	.413	3.724	-8.620	14.000	-11.913	3.973	.9990
Parzen							
.90	-.163	2.860	-7.703	14.503	-13.032	4.422	.9994
.95	.091	2.911	-7.215	13.556	-12.297	4.225	.9993
.975	.259	3.229	-8.037	15.247	-13.879	4.761	.9993
.99	.416	3.788	-10.190	20.591	-19.695	7.012	.9990
MQS							
.90	.083	3.188	-6.488	13.145	-12.965	4.737	.9996
.95	.337	3.290	-5.215	11.095	-11.872	4.547	.9996
.975	.523	3.166	-2.064	4.408	-6.124	2.673	.9992
.99	.688	3.438	.735	-2.004	-1.028	1.266	.9993
Daniell							
.90	-.105	3.170	-7.261	15.668	-16.037	5.999	.9995
.95	.144	3.378	-7.095	16.661	-18.260	7.074	.9997
.975	.316	3.704	-7.338	18.579	-21.425	8.466	.9996
.99	.502	3.657	-4.141	12.201	-16.082	6.637	.9993
TH							
.90	-.133	2.951	-6.407	11.770	-10.420	3.451	.9993
.95	.112	3.251	-6.895	13.147	-11.902	3.961	.9994
.975	.291	3.421	-6.629	12.989	-11.997	3.995	.9993
.99	.475	3.311	-3.716	6.061	-4.497	.920	.9991
Bohman							
.90	-.539	5.127	-15.803	29.174	-25.769	8.643	.9991
.95	-.295	5.273	-15.534	28.105	-24.288	7.945	.9991
.975	-.112	5.303	-14.869	26.455	-22.462	7.186	.9989
.99	.056	5.557	-15.377	28.275	-24.893	8.218	.9987
Trap, $c = 1/4$							
.90	-.145	2.692	-6.294	11.131	-9.625	3.195	.9991
.95	.105	2.868	-6.480	11.652	-10.155	3.347	.9993
.975	.286	2.902	-5.529	9.128	-7.418	2.281	.9994
.99	.449	3.204	-5.746	9.418	-7.755	2.430	.9991
Trap, $c = 1/2$							
.90	-.089	3.928	-5.118	2.540	2.157	-1.851	.9993
.95	.152	4.594	-4.067	-3.820	10.383	-5.238	.9993
.975	.294	5.949	-6.122	-4.369	13.688	-7.053	.9991
.99	.378	9.279	-16.717	11.008	3.580	-4.670	.9984

Table 4: Fixed-b asymptotic critical value function coefficients for $cv(b) = \exp\{a_0 + a_1 b + a_2 b^2 + a_3 b^3 + a_4 b^4 + a_5 b^5\}$. The memory parameter is $\beta = -.2$. The R^2 indicates the fit of the polynomial through the log simulated asymptotic critical values.

	a_0	a_1	a_2	a_3	a_4	a_5	R^2
Bartlett							
.90	.255	.779	1.109	-2.802	2.468	-.801	.9991
.95	.514	.833	1.611	-4.016	3.441	-1.053	.9994
.975	.679	1.195	.644	-2.276	1.703	-.383	.9992
.99	.841	1.692	-.962	.852	-1.392	.772	.9988
Parzen							
.90	.250	.825	-.614	2.014	-2.408	.962	.9992
.95	.496	1.015	-.771	2.405	-2.977	1.258	.9993
.975	.677	1.115	-.654	2.496	-3.302	1.405	.9993
.99	.847	1.311	-.317	1.458	-2.149	.942	.9992
MQS							
.90	.438	1.238	.441	1.139	-2.917	1.498	.9997
.95	.695	1.363	1.203	1.082	-4.550	2.541	.9997
.975	.876	1.537	2.116	.460	-5.544	3.298	.9996
.99	1.067	1.218	8.389	-14.561	8.976	-1.856	.9994
Daniell							
.90	.262	1.137	.239	1.891	-3.834	1.871	.9994
.95	.509	1.433	.041	3.174	-5.702	2.572	.9996
.975	.684	1.851	-1.483	9.764	-15.082	6.709	.9994
.99	.855	2.069	.037	8.223	-15.706	7.448	.9992
TH							
.90	.250	1.032	.063	.591	-.853	.270	.9995
.95	.492	1.372	-.599	2.455	-2.996	1.092	.9994
.975	.671	1.439	.579	-.452	.291	-.322	.9992
.99	.849	1.585	1.549	-1.738	.926	-.470	.9990
Bohman							
.90	.256	.607	1.287	-2.477	2.052	-.632	.9993
.95	.500	.921	.277	.236	-1.014	.569	.9994
.975	.671	1.215	-.291	1.393	-2.117	.934	.9994
.99	.841	1.466	-.287	1.335	-2.240	1.043	.9990
Trap, $c = 1/4$							
.90	.262	.785	4.191	-7.571	5.633	-1.549	.9996
.95	.521	.752	6.507	-11.099	6.913	-1.347	.9997
.975	.709	.598	11.065	-21.668	16.119	-4.177	.9995
.99	.916	-.310	23.179	-52.791	47.640	-15.519	.9991
Trap, $c = 1/2$							
.90	.244	1.686	3.454	-11.828	13.346	-5.171	.9994
.95	.483	2.158	5.704	-20.594	23.550	-9.124	.9992
.975	.629	3.244	5.448	-25.184	30.560	-12.131	.9986
.99	.738	5.472	1.321	-23.692	33.088	-13.879	.9974

Table 5: Fixed-b asymptotic critical value function coefficients for $cv(b) = \exp\{a_0 + a_1 b + a_2 b^2 + a_3 b^3 + a_4 b^4 + a_5 b^5\}$. The memory parameter is $\beta = 0$. The R^2 indicates the fit of the polynomial through the log simulated asymptotic critical values.

	a_0	a_1	a_2	a_3	a_4	a_5	R^2
Bartlett							
.90	.678	-1.228	8.798	-17.402	15.833	-5.465	.9982
.95	.924	-.980	8.748	-17.899	16.428	-5.681	.9983
.975	1.102	-.785	8.857	-18.730	17.348	-6.011	.9980
.99	1.263	-.377	8.496	-19.142	18.051	-6.247	.9976
Parzen							
.90	.675	-1.183	6.788	-11.619	9.471	-2.943	.9973
.95	.935	-1.250	8.194	-14.956	12.879	-4.222	.9984
.975	1.103	-.943	7.500	-13.557	11.640	-3.840	.9977
.99	1.278	-.764	7.907	-14.653	12.713	-4.220	.9985
MQS							
.90	.835	-.987	9.276	-16.213	13.433	-4.367	.9993
.95	1.093	-.912	10.392	-17.119	12.665	-3.649	.9993
.975	1.271	-.685	10.946	-16.575	10.091	-2.155	.9991
.99	1.460	-.988	16.481	-28.237	19.684	-5.002	.9993
Daniell							
.90	.658	-1.124	9.621	-17.263	14.728	-4.917	.9992
.95	.902	-.650	8.070	-11.860	7.578	-1.845	.9994
.975	1.067	-.070	5.907	-4.639	-1.606	1.945	.9993
.99	1.235	.359	5.943	-3.232	-4.180	2.957	.9991
TH							
.90	.654	-1.013	7.605	-13.186	11.180	-3.735	.9991
.95	.900	-.742	7.409	-12.634	10.684	-3.654	.9992
.975	1.087	-.887	10.176	-19.760	18.529	-6.779	.9995
.99	1.247	-.258	8.206	-13.893	11.531	-3.967	.9992
Bohman							
.90	.677	-1.228	7.315	-12.822	10.920	-3.610	.9980
.95	.930	-1.158	8.247	-15.692	14.442	-5.127	.9984
.975	1.110	-1.100	9.074	-17.817	16.763	-6.062	.9986
.99	1.282	-.887	9.685	-19.742	18.804	-6.798	.9989
Trap, $c = 1/4$							
.90	.656	-1.274	11.419	-20.659	17.395	-5.636	.9995
.95	.909	-1.121	12.425	-20.792	15.108	-4.116	.9994
.975	1.100	-1.334	16.315	-27.904	19.471	-4.818	.9993
.99	1.293	-1.859	25.179	-49.569	40.033	-11.772	.9991
Trap, $c = 1/2$							
.90	.620	-.760	13.406	-30.443	29.705	-10.622	.9990
.95	.867	-.531	17.119	-42.306	42.653	-15.443	.9992
.975	1.026	.039	19.961	-53.867	56.460	-20.869	.9986
.99	1.153	1.659	19.677	-61.712	68.686	-26.239	.9967

Table 6: Fixed-b asymptotic critical value function coefficients for $cv(b) = \exp\{a_0 + a_1 b + a_2 b^2 + a_3 b^3 + a_4 b^4 + a_5 b^5\}$. The memory parameter is $\beta = .2$. The R^2 indicates the fit of the polynomial through the log simulated asymptotic critical values.

	a_0	a_1	a_2	a_3	a_4	a_5	R^2
Bartlett							
.90	1.104	-3.019	15.728	-30.162	26.842	-9.049	.9954
.95	1.351	-2.627	14.613	-27.628	23.920	-7.842	.9965
.975	1.531	-2.409	14.789	-28.729	25.142	-8.281	.9963
.99	1.709	-2.101	14.812	-30.042	27.025	-9.095	.9965
Parzen							
.90	1.136	-3.363	15.645	-29.784	27.309	-9.562	.9943
.95	1.392	-3.289	16.238	-31.119	28.526	-9.973	.9961
.975	1.564	-2.934	14.954	-27.573	24.385	-8.291	.9974
.99	1.741	-2.791	15.530	-28.655	24.889	-8.226	.9980
MQS							
.90	1.256	-3.089	17.470	-31.882	27.623	-9.217	.9990
.95	1.496	-2.476	15.224	-24.382	17.734	-4.916	.9987
.975	1.682	-2.231	15.118	-21.643	12.673	-2.493	.9989
.99	1.863	-1.991	16.537	-22.569	10.943	-1.172	.9987
Daniell							
.90	1.062	-2.741	14.916	-25.881	21.416	-6.881	.9986
.95	1.306	-2.252	13.259	-20.056	13.581	-3.446	.9986
.975	1.474	-1.781	11.895	-14.905	6.753	-.656	.9989
.99	1.644	-1.197	10.263	-7.591	-3.776	3.918	.9987
TH							
.90	1.092	-3.055	15.551	-28.336	24.703	-8.265	.9986
.95	1.341	-2.795	15.587	-28.752	25.601	-8.838	.9989
.975	1.523	-2.589	15.643	-28.386	24.954	-8.590	.9991
.99	1.678	-1.821	12.830	-20.526	15.971	-5.055	.9987
Bohman							
.90	1.113	-2.924	13.161	-23.281	20.076	-6.706	.9969
.95	1.366	-2.843	13.789	-24.493	20.927	-6.892	.9980
.975	1.544	-2.680	14.144	-25.600	22.243	-7.471	.9985
.99	1.718	-2.499	14.851	-27.475	24.114	-8.161	.9985
Trap, $c = 1/4$							
.90	1.057	-3.070	17.831	-32.526	28.366	-9.580	.9991
.95	1.324	-3.142	19.916	-35.013	28.510	-8.993	.9990
.975	1.524	-3.295	22.595	-38.112	28.001	-7.668	.9991
.99	1.713	-3.461	27.486	-46.762	32.295	-7.720	.9990
Trap, $c = 1/2$							
.90	1.018	-2.753	20.574	-42.682	39.933	-13.986	.9980
.95	1.280	-2.831	25.667	-56.902	54.651	-19.299	.9981
.975	1.454	-2.577	29.791	-69.788	68.183	-24.089	.9975
.99	1.591	-1.448	32.541	-84.222	86.443	-31.450	.9961

Table 7: Fixed-b asymptotic critical value function coefficients for $cv(b) = \exp\{a_0 + a_1 b + a_2 b^2 + a_3 b^3 + a_4 b^4 + a_5 b^5\}$. The memory parameter is $\beta = .4$. The R^2 indicates the fit of the polynomial through the log simulated asymptotic critical values.

	a_0	a_1	a_2	a_3	a_4	a_5	R^2
Bartlett							
.90	1.581	-4.639	21.813	-41.475	36.907	-12.449	.9908
.95	1.842	-4.533	22.555	-43.441	38.659	-12.996	.9940
.975	2.020	-4.279	22.699	-44.664	40.180	-13.602	.9935
.99	2.203	-4.122	23.940	-49.053	45.372	-15.725	.9933
Parzen							
.90	1.644	-5.144	21.626	-39.643	34.880	-11.724	.9898
.95	1.895	-4.967	21.913	-40.801	36.433	-12.435	.9935
.975	2.077	-4.852	22.435	-42.248	38.014	-13.062	.9960
.99	2.255	-4.709	23.233	-44.332	40.117	-13.823	.9963
MQS							
.90	1.708	-4.643	22.859	-41.553	36.144	-12.111	.9977
.95	1.966	-4.419	23.174	-40.957	34.251	-11.106	.9983
.975	2.147	-4.114	22.973	-38.266	29.377	-8.782	.9990
.99	2.353	-4.372	26.798	-43.682	31.147	-8.421	.9987
Daniell							
.90	1.536	-4.717	22.867	-41.670	36.201	-12.087	.9978
.95	1.783	-4.131	20.435	-34.012	26.739	-8.190	.9980
.975	1.953	-3.603	18.659	-27.611	18.188	-4.548	.9985
.99	2.130	-2.984	16.299	-17.793	4.532	1.353	.9983
TH							
.90	1.558	-4.705	21.826	-40.980	37.089	-12.868	.9959
.95	1.814	-4.413	21.256	-38.959	34.540	-11.845	.9973
.975	2.000	-4.331	22.640	-42.946	39.376	-13.940	.9981
.99	2.175	-3.968	22.813	-43.576	40.497	-14.673	.9979
Bohman							
.90	1.630	-5.310	23.627	-45.094	40.975	-14.138	.9914
.95	1.876	-4.954	22.863	-43.646	39.735	-13.777	.9956
.975	2.042	-4.396	20.579	-37.843	33.359	-11.307	.9964
.99	2.210	-3.993	20.186	-37.623	33.474	-11.422	.9968
Trap, $c = 1/4$							
.90	1.517	-4.739	23.822	-43.753	38.407	-12.924	.9985
.95	1.773	-4.530	24.069	-41.863	34.278	-10.861	.9985
.975	1.967	-4.646	26.758	-45.659	35.436	-10.562	.9987
.99	2.160	-5.018	33.266	-59.748	47.282	-14.159	.9985
Trap, $c = 1/2$							
.90	1.461	-4.441	25.863	-50.195	44.941	-15.243	.9978
.95	1.728	-4.675	31.834	-66.064	60.887	-20.851	.9981
.975	1.911	-4.647	37.297	-82.432	78.629	-27.508	.9978
.99	2.086	-4.392	44.662	-106.439	105.577	-37.756	.9965

Table 8: Fixed-b asymptotic critical value function coefficients for $cv(b) = \exp\{a_0 + a_1 b + a_2 b^2 + a_3 b^3 + a_4 b^4 + a_5 b^5\}$. The memory parameter is $\beta = .6$. The R^2 indicates the fit of the polynomial through the log simulated asymptotic critical values.

	a_0	a_1	a_2	a_3	a_4	a_5	R^2
Bartlett							
.90	2.196	-6.339	28.370	-54.069	48.616	-16.617	.9877
.95	2.452	-6.094	28.774	-55.811	50.527	-17.327	.9899
.975	2.630	-5.656	27.583	-53.102	47.269	-15.916	.9911
.99	2.805	-5.249	27.490	-53.850	47.896	-15.985	.9925
Parzen							
.90	2.250	-6.557	27.165	-50.915	45.660	-15.554	.9817
.95	2.511	-6.289	26.375	-48.600	42.854	-14.379	.9871
.975	2.687	-5.961	25.733	-47.273	41.528	-13.902	.9901
.99	2.875	-5.785	26.058	-48.010	42.185	-14.135	.9931
MQS							
.90	2.295	-6.125	28.550	-53.038	47.183	-16.090	.9970
.95	2.556	-5.853	28.468	-51.183	43.742	-14.440	.9976
.975	2.730	-5.281	26.155	-42.562	32.023	-9.344	.9983
.99	2.923	-5.081	27.386	-42.506	28.889	-7.383	.9978
Daniell							
.90	2.125	-6.016	26.961	-48.723	42.362	-14.186	.9944
.95	2.376	-5.507	25.200	-42.918	34.972	-11.101	.9959
.975	2.543	-4.915	23.144	-36.322	26.759	-7.776	.9969
.99	2.715	-4.177	20.261	-25.867	12.992	-1.991	.9972
TH							
.90	2.176	-6.434	28.150	-52.233	46.427	-15.759	.9937
.95	2.427	-6.032	27.289	-50.164	44.346	-15.066	.9957
.975	2.606	-5.717	27.165	-49.951	44.163	-15.056	.9968
.99	2.792	-5.549	28.861	-55.266	51.102	-18.263	.9972
Bohman							
.90	2.232	-6.458	26.538	-48.330	42.122	-14.000	.9832
.95	2.483	-6.131	25.956	-47.290	41.298	-13.792	.9893
.975	2.660	-5.842	25.649	-46.988	41.296	-13.904	.9912
.99	2.848	-5.830	27.495	-51.674	46.060	-15.649	.9943
Trap, $c = 1/4$							
.90	2.100	-6.029	27.619	-49.835	43.346	-14.471	.9975
.95	2.363	-5.865	28.190	-49.156	40.910	-13.157	.9982
.975	2.549	-5.620	28.272	-46.086	34.587	-9.967	.9984
.99	2.743	-5.630	31.008	-48.080	31.387	-7.170	.9983
Trap, $c = 1/2$							
.90	2.034	-5.835	29.821	-55.220	47.814	-15.820	.9969
.95	2.302	-5.938	34.591	-67.807	60.418	-20.294	.9976
.975	2.509	-6.417	42.426	-88.224	80.751	-27.361	.9974
.99	2.691	-6.549	52.279	-118.432	114.355	-40.187	.9970

Table 9: Fixed-b asymptotic critical value function coefficients for $cv(b) = \exp\{a_0 + a_1 b + a_2 b^2 + a_3 b^3 + a_4 b^4 + a_5 b^5\}$. The memory parameter is $\beta = .8$. The R^2 indicates the fit of the polynomial through the log simulated asymptotic critical values.