LBL-12329
Preprint c.2

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

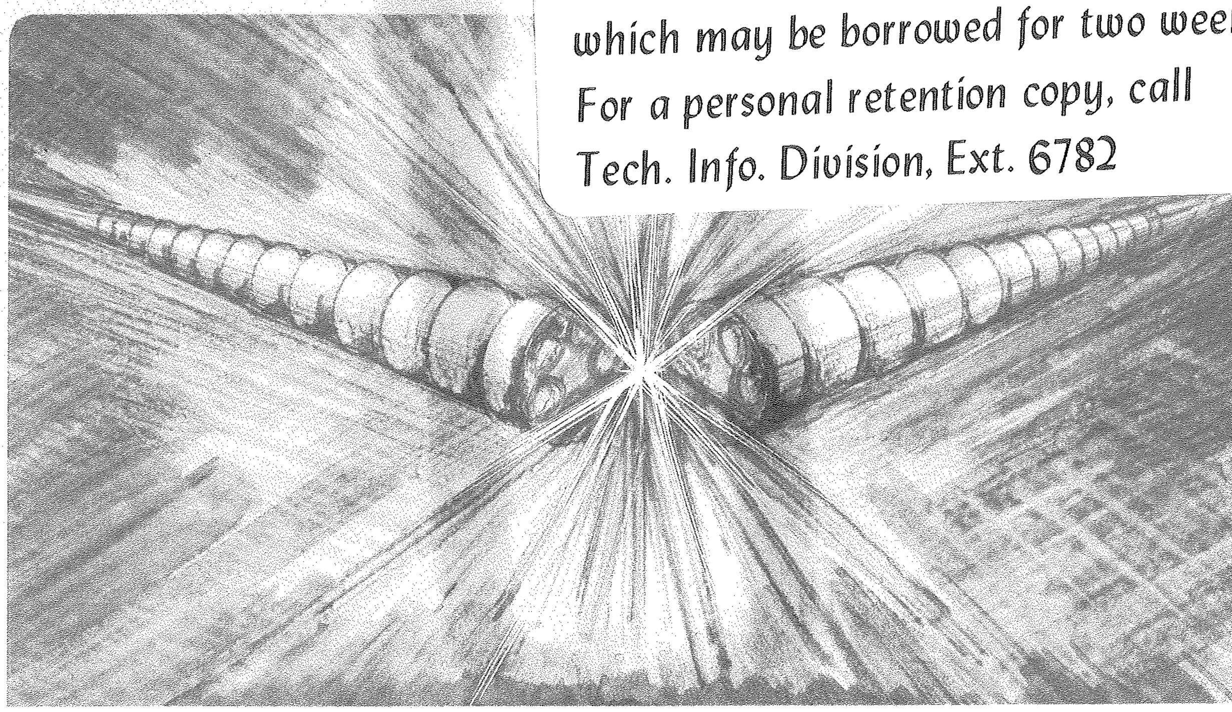
Accelerator & Fusion Research Division

Submitted to Physical Review A, Rapid Communications

DECAY OF STATISTICAL DEPENDENCE IN CHAOTIC ORBITS OF DETERMINISTIC MAPPINGS

Celso Grebogi and Allan N. Kaufman

July 1981



Prepared for the U.S. Department of Energy under Contract W-7405-ENG-48

DISCLAIMER

Submitted to Physical Review A, Rapid Communications

LBL-12329

Decay of Statistical Dependence in Chaotic Orbits of Deterministic Mappings*

Celso Grebogi and Allan N. Kaufman

Lawrence Berkeley Laboratory
University of California
Berkeley, CA. 94720

July, 1981

Abstract

A numerical study is made of the decay of statistical dependence in chaotic orbits of the iterated mappings of Chirikov and of Rannou. The decay appears to be exponential; the decay exponent is proportional to, but smaller than, the Liapunov exponent.

---

* This work was supported by the Director, Office of Energy Research, Division of Engineering, Mathematical and Geo-Sciences of the U.S. Department of Energy under Contract No. W-7405-ENG-48.

PACS 03.20.+i

It is now widely recognized that a deterministic equation of evolution, representing a dynamical system, may have chaotic solutions. A family of such solutions may be interpreted as the set of realizations of a random process.

Concepts from the theory of random processes may thus be carried over to the study of chaotic orbits of a deterministic system. In this paper we examine one such concept, the joint probability for two events separated by a time-interval τ.

A basic postulate in this theory is asymptotic statistical independence: that as $\tau \to \infty$, events become independent. It is widely believed that the decay of dependence with τ should be exponential. However, there is no theory for this, to our knowledge, nor predictions for the decay rate.

In this paper we present numerical evidence for exponential decay and values for the decay exponent ν, for two dynamical systems: the iterated area-preserving mappings of Chirikov[1] and of Rannou.[2] On heuristic grounds, it has been felt that there should be a relation between the decay exponent ν and the Liapunov characteristic exponent λ; but again there is as yet no theory. In the present paper, we report on numerical tests for the relation between these two exponents.

Let $x(t;x_0)$ represent an orbit in state space, for initial state x_0. For a mapping $T: x \mapsto x'$, the state evolves as $x(t+1;x_0) = Tx(t;x_0)$. Here we study the maps T_C and T_R, due to Chirikov and Rannou, respectively. These are area-preserving maps of a two-torus onto itself, and depend on a single parameter K. The coordinates of x are denoted by p,q (mod 2π); i.e., $0 \leq p,q < 2\pi$.

$$T_C: \quad p' = p + K \sin q$$
$$q' = q + p'$$

$$T_R: \quad p' = p + K\,(1 + \sin q - \cos q)$$
$$q' = q + p' + 1 - \cos p'$$

These systems are integrable for $K = 0$. As K increases, they become increasingly ergodic. For $K >> 1$, the ergodic region fills nearly the whole space. An initial state x_0 in this region has an unstable orbit, with Liapunov exponent $\lambda_C \approx \ell n\, K - \ell n\, 2$ and $\lambda_R \approx \ell n\, K - \frac{5}{8}\ \ell n\, 2$.

Although the state space (p,q) is continuous, it is convenient to partition this space into a finite set of A cells, labeled $j = 0,1,2,\ldots, A-1$. (Here we choose A=10). An orbit is then "coded" by a semi-infinite sequence: $\{j(0), j(1), j(2), \ldots\}$, where $j(t)$ is the label of the cell visited at time t. In our study,[3] we select the ten strips $0.1j \leq p/2\pi < 0.1(j+1)$, all q. The label j is then simply the first digit in the decimal representation of $p/2\pi$.

Thus the initial state ($p/2\pi = 0.30879$, $q/2\pi = 0.6703$) has the "coded orbit," under T_C with $K = 7.5$: (3,2,7,6,6,6,5,7,0,0,9,8...). The t'th digit thus identifies the cell visited at time t.

Now we identify the set of cells with an event space in probability theory. Then P_j, the probability of event j, is the relative frequency of label j in the coded orbit. Likewise $P_{ij}(\tau)$ is the relative frequency of the pair of cells i,j separated by τ iterations. Statistical dependence of events along a single orbit is thus represented by

$$Q_{ij}(\tau) = \frac{P_{ij}(\tau)}{P_i P_j} - 1 .$$

We observe that $Q_{ij}(\tau) \to 0$ as $\tau \to \infty$, and examine the τ-dependence.

In Fig. 1, we display Q_{ij} vs τ, for (i=9, j=0), K=7.5, initial state as above, and an orbit of length $t = 1.5 \times 10^6$. Since there are $A^2 = 100$ pairs (i,j), the expected error due to the finite orbit is $(t/100)^{-1/2}$; this is indicated by the error bar.

We note that Q_{ij} falls off significantly more rapidly for the Rannou map (---) than for the Chirikov map (——). This is found for all A^2 pairs (i,j). However, the oscillations dominate the fall-off for small τ, while the finite-sample error affects the large τ behavior. Hence, to determine the asymptotic ($\tau \to \infty$) behavior, we next average over all A^2 pairs, by forming the root-mean-square statistical dependence $Q(\tau)$:

$$Q^2(\tau) = A^{-2} \sum_{i,j=0}^{A-1} Q_{ij}^2(\tau).$$

This is exhibited in Fig. 2; the finite-sample expected error $(t/A^2)^{-1/2}$ has been subtracted from Q. The error bar represents the expected residual error $t^{-1/2}$.

The Rannou plot (---) appears to be an exponential fall-off, with decay exponent $\nu_R \approx 0.67$. For the Chirikov map, the decay is much slower, with a fit from the tail ($4 \leq \tau \leq 11$) given by $\nu_C \approx 0.27$. The respective ratios r of decay exponent ν to Liapunov exponent λ are $r_C \approx 0.2$, $r_R \approx 0.5$.

We have examined this ratio for a range of parameters: $5 < K < 15$. We have found that the ratio $r = \nu/\lambda$ is roughly independent of K for each map, but the two maps have significantly different ratios: $r_C = 0.25 \pm 0.05$, $r_R = 0.50 \pm 0.05$.

Rannou suggested[2] that the qualitative difference between her map and Chirikov's is due to the symmetry properties of the latter. This explanation seems reasonable to us.

We defer to a longer publication[4] discussions of sample error and round-off error, the question of time reversal, the dependence on the partition, studies of three-point probabilities and the Markov property, and the relation of statistical dependence to correlation functions.

Acknowledgements

We have benefited from discussions with H. Abarbanel and from the mathematical advice of O. Lanford.

This work was supported by the Director, Office of Energy Research, Division of Engineering, Mathematical and Geo-Sciences of the U.S. Department of Energy under Contract No. W-7405-ENG-48.

Figure Captions

1. Statistical dependence Q_{ij} vs. τ for the Chirikov (——) and Rannou (---) maps. $K = 7.5$ and $t = 1.5 \times 10^6$ for both mappings.

2. Root-mean-square statistical dependence Q vs. τ for the Chirikov (——) and Rannou (---) maps. $K = 7.5$ for both mappings, but $t = 1.2 \times 10^6$ for Chirikov and $t = 0.45 \times 10^6$ for Rannou. The expected numerical error $(t/A^2)^{-1/2}$ has been subtracted.

References

1. B. V. Chirikov, Phys. Reports 52, 263 (1979).

2. F. Rannou, Astron. and Astrophys. 31, 289 (1974).

3. A. N. Kaufman, H. D. I. Abarbanel and C. Grebogi, Proc. Int'l. Conf. on Plasma Physics (Nagoya, Japan), Vol. II (1980).

4. C. Grebogi and A. N. Kaufman, Lawrence Berkeley Laboratory report LBL-12605 (1981) (to be published).

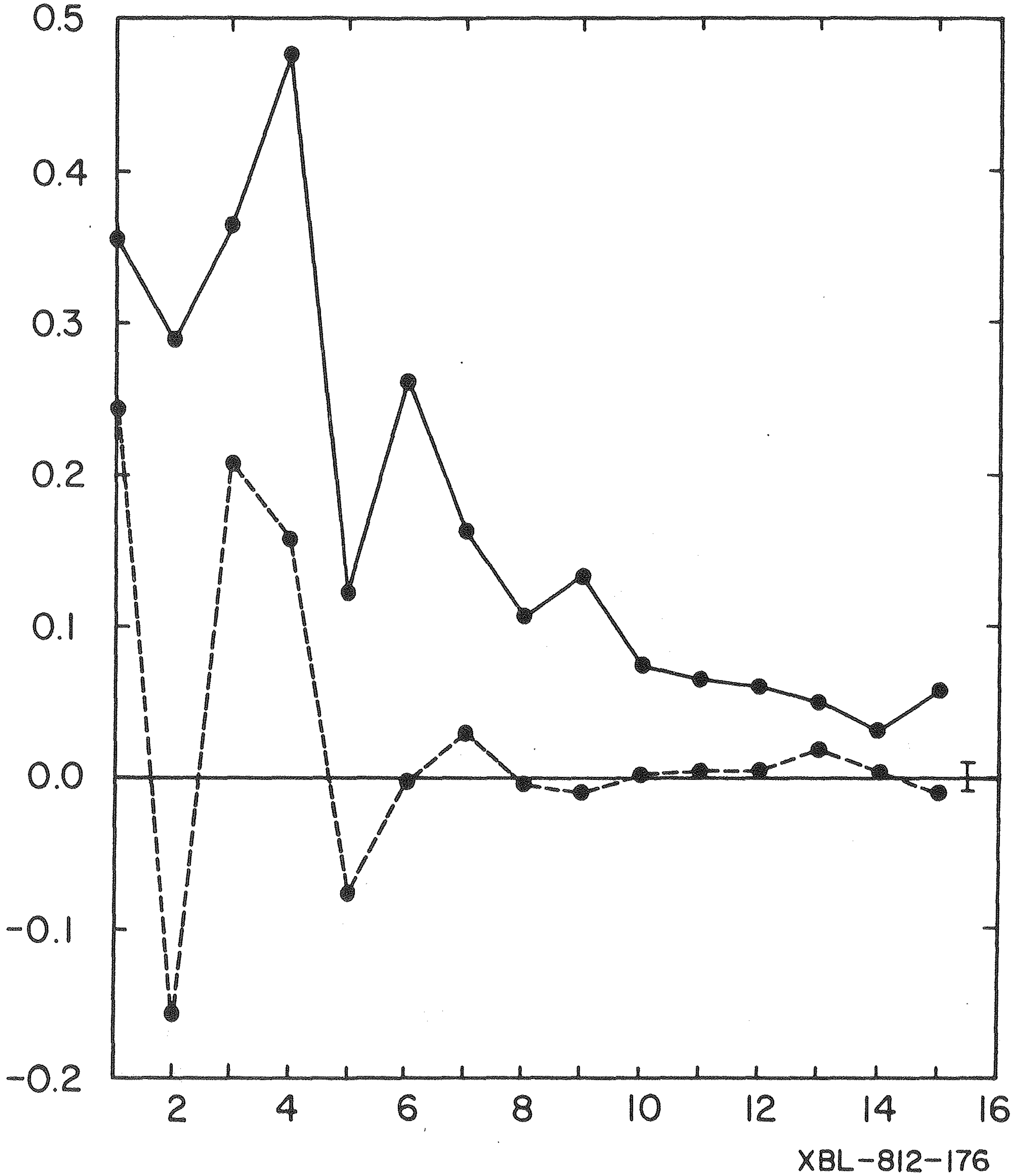


FIGURE 1

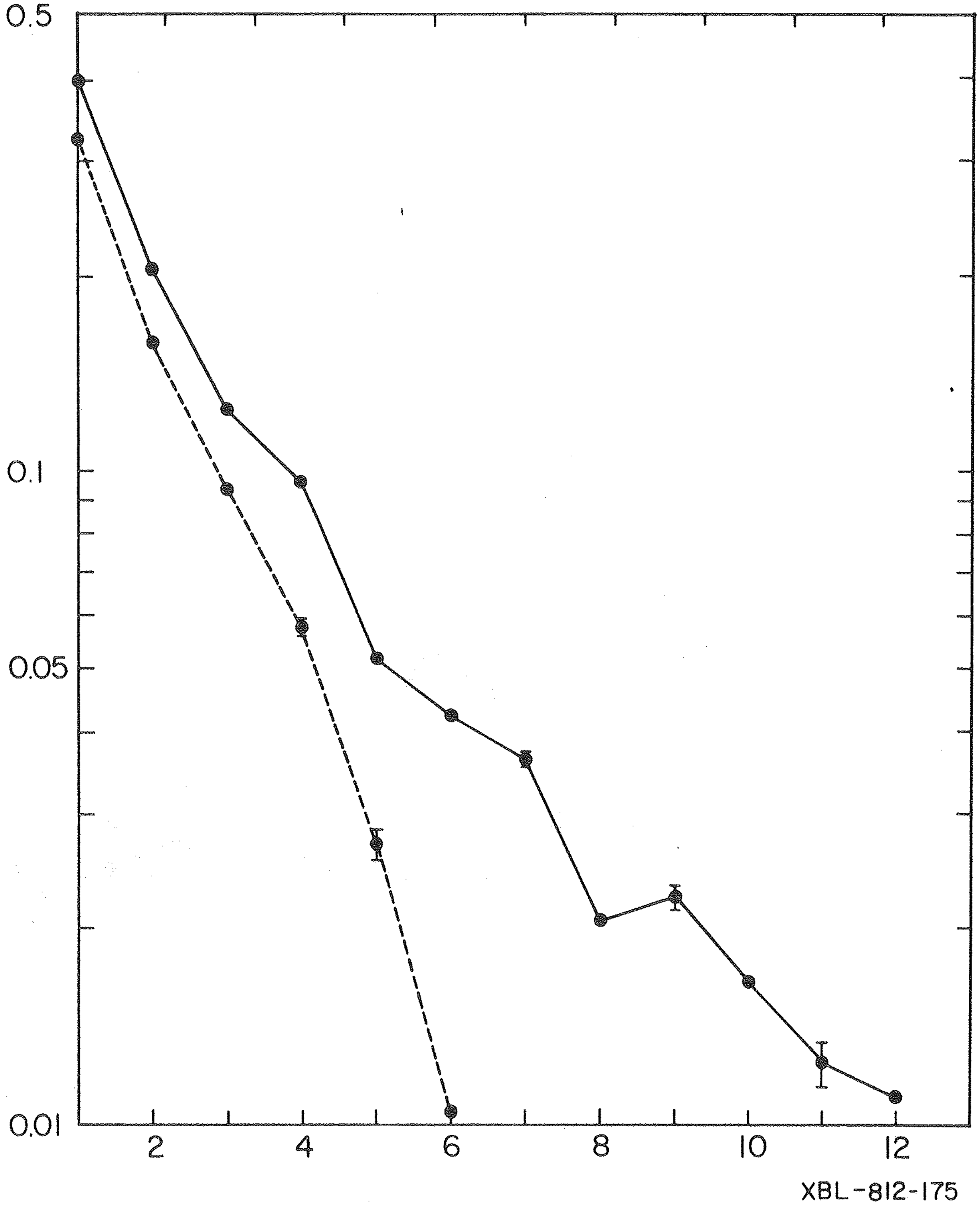


FIGURE 2