Chapter 1
An Approximation of the Invariant Measure for the Stochastic Navier-Stokes

Björn Birnir
Dept. of Math. and CNLS
UC Santa Barbara

1.1 Introduction

Kolmogorov's statistical theory of turbulence is based on the existence of the invariant measure of the Navier-Stokes flow. Recently the existence of the invariant measure was established in the three-dimensional case [2]. It was established for uni-directional flow in [1] and for rivers in [3]. Below we will discuss how one can try to go about approximating the invariant measure in three dimensions.

1.2 Fluid Flow and Ito Diffusion

A reasonable model for the motion of an Eulerian fluid particle is given by the stochastic ordinary differential equation (SODE)

$$dX_t = -u(X_t, t)dt + \sqrt{2\nu} dB_t$$

Here $u(x,t)$ is the fluid velocity and we expect the fluid particle to move upstream with velocity u and also to move randomly. This random motion is modeled by the second term, where B_t is Brownian motion and dB_t models the white noise affecting the motion of the particle. There is noise in any fluid flow and we expect fluid particles to diffuse under influence of the noise. The above equation is the equation of Ito's diffusion X_t with the generator

$$A = \nu\Delta - u(x,t) \cdot \nabla$$

The backward Kolmogorov equation, corresponding to X_t and A, is the dissipative Burger's equation, or the Navier-Stokes equation without pressure

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u$$

$$u(x,0) = f(x)$$

This initial value problem has the *implicit* solution

$$u(x,t) = E[f(X_t)]$$

where

$$X_t = X_0 - \int_0^t u(X_s,s)ds + \sqrt{2\nu}B_t$$

Notice that this is not the *explicit* solution of Burger's equation that is obtained by the Cole-Hopf transformation.

Anologously the Navier-Stokes equation for fully-developed turbulent flow, with periodic boundary conditions, can be written in the form

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u - \nabla p + \sum_{k \neq 0} h_k^{1/2} db_t^k e_k(x) \tag{1.1}$$

$$u(x,0) = f(x)$$

In laminar flow the driving term

$$f(x,t) = \sum_{k \neq 0} h_k^{1/2} db_t^k e_k(x)$$

is absent, but in fully developed turbulence the small ambient white noise is magnified into large turbulent noise which is modeled by f, see [1, 2]. The coefficients $h_k^{1/2} \in \mathbb{R}^3$ decay as $k \to \infty$, the $e_k(x)$ are Fourier components, that each come with their independent Brownian motion b_t^k, and we have imposed periodic boundary conditions in $x \in \mathbb{T}^3$. Thus the large turbulent noise is modeled by (independent) white noise in time in all directions, in function space, but the decay of the coefficients $h_k^{1/2}$ makes this noise colored in space. The color is characteristic for turbulent noise in three dimensions.

1.3 The Approximation

We can proceed further if we now project onto the space of divergence-free vectors eliminating the pressure gradient. Let P denote the projection operator, then we will model the difference between the projection of the inertial terms and the intertial terms themselves as

$$P[u \cdot \nabla u] - u \cdot \nabla u \approx \sum_{k \neq 0} g_k^{1/2} db_t^k e_k(x) \cdot \nabla u$$

This expression is of course not exact, but the modeling is motivated by numerical simulations where an analogous difference the ”eddy viscosity”, is shown to depend on the gradient ∇u.

Now the initial value problem (1.1) can be written in the form

$$\frac{\partial u}{\partial t} + w \cdot \nabla u = \nu \Delta u + \sum_{k \neq 0} h_k^{1/2} db_t^k e_k(x) \tag{1.2}$$

$$u(x,0) = f(x)$$

where

$$w(x,t) = u + \sum_{k \neq 0} g_k^{1/2} db_t^k e_k(x)$$

and we use the same notation for the divergence free $k \cdot h_k^{1/2} = 0$ vectors as for the original $h_k^{1/2}$. Then introducing the Ito diffusion

$$dX_t = -w(X_t,t)dt + \sqrt{2\nu} dB_t$$

we can write the solution of (1.2) of the form

$$u(x,t) = E[f(X_t)] + \sum_{k \neq 0} h_k^{1/2} \int_0^t E[e_k(X_{t-s})] db_s^k$$

Now by Girsanov’s theorem, see [4], we can rewrite u in the form

$$u(x,t) = E[f(B_t)M_t] + \sum_{k \neq 0} h_k^{1/2} \int_0^t E[e_k(B_{t-s})M_{t-s}] db_s^k \tag{1.3}$$

where

$$M_t = exp\{-\int_0^t w(B_s,s) \cdot dB_s - \frac{1}{2}\int_0^t |w(B_s,s)|^2 ds\}$$

This implies that (1.2) has the invariant measure

$$d\mu = \lim_{t \to \infty} M_t d \, [\mathcal{N}(e^{\nu \Delta t}, \sqrt{2\nu t}) * \mathcal{N}(B_t^{\infty}, Q_t)] \tag{1.4}$$

where B_t^{∞} is the evolution operator for the infininite-dimensional Brownian motion and the variance Q_{∞} is

$$Q_{\infty}^{-1} = \sum_{k \neq 0} \frac{h_k}{2\nu \lambda_k}$$

the coefficients being $h_k = |h_k^{1/2}|^2$. The statistical theory of (1.2) is determined by the invariant measure (1.4).

We can also write the approximate invariant measure in terms of densities

$$d\mu \approx \lim_{t\to\infty} e^{\{-\int_0^t w(x,s)\cdot dx - \frac{1}{2}\int_0^t |w(x,s)|^2 ds\}} \frac{e^{-\frac{|x|^2}{2\nu}}}{\sqrt{2\nu}} dx \prod_{k\neq 0} \frac{e^{-\frac{h_k \hat{u}_k^2}{2\nu\lambda_k}}}{\sqrt{2\nu\lambda_k/h_k}} d\hat{u}_k$$

where $\hat{u}_k$ are the Fourier coefficients of u and the approximation holds for large t.

In numerical simulation and fluid experiments the approximate velocity w will have similar statistical properties as the real velocity u. Thus w can be approximated by simulated or measured values of the fluid velocity u itself.

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