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Virendra Singh

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Contents

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ABSTRACT

The pion-nucleon scattering amplitude and its analytic continuation $\pi+\pi \rightarrow N+\overline{N}$ are studied from the point of view of the analyticity in the complex angular momentum J plane. The quantum numbers that characterize the Regge trajectories are thereby settled. They are in the $\pi + N$ scattering channel, baryon number = 1, isospin = $\frac{1}{2}$ or $\frac{3}{2}$, J parity = even or odd, space parity = even or odd for Regge poles; and, for the $\pi+\pi \rightarrow N+\overline{N}$ channel, baryon number = 0, G-parity = +1, isospin = 0 or 1, J-parity = even or odd, space parity = even or odd. It turns out that in the $\pi+\pi \rightarrow N+\overline{N}$ channel, only the amplitudes having the same J parity and space parity, which is even for isospin 0 and odd for isospin 1, are nonvanishing. Experimentally observed particles and resonances are also discussed in terms of the Regge trajectories with definite quantum numbers, and certain experimental consequences pointed out. We also give the expressions for the differential cross section in forward and backward scattering cones for πN scattering. A discussion of the rigorous Froissart-type upper bounds on asymptotic behavior and their implications for undertermined single spectral functions is also given, which should be useful in any further work on the determination of the meromorphy domain in the J plane of the various partial-wave amplitudes.

I. INTRODUCTION

The prescription for simultaneously continuing two-body S-matrix elements into the complex plane, regarding them as a function of invariant energy and momentum-transfer variables, was given by Mandelstam.[1,2] Through analytic continuation, the two-body scattering amplitude for any process describes two more physical processes, which are related to the first one through the substitution law. Roughly speaking, these two other processes--i.e., the crossed channels in relation to the direct one--provide the ordinary and exchange potential for the direct process. This vague notion of two-body potential was made more precise for two-body relativistic S-matrix elements by Chew and Frautschi.[3] This, then, allows considerable insight by placing at our disposal all the previous experience with the potential concept. In particular, Regge has introduced for nonrelativistic potential scattering the notion of simultaneous analyticity in the complex energy and angular-momentum plane,[4,5] and this notion can be taken over to the relativistic S matrix. The implications of this notion, then, are many and important.[5-10] In this work we discuss the scattering amplitude for pion-nucleon scattering, $\pi+N \rightarrow \pi+N$, and its analytic continuation, describing $\pi+\pi \rightarrow N+\overline{N}$, from the point of view of analyticity in the angular-momentum plane.

II. GENERALIZED POTENTIAL

The class of potentials for which the analyticity in the complex angular-momentum J plane has been investigated is roughly that of the superposition of Yukawa potentials.[4] The existence of a two-body potential, which essentially belongs to this class, makes it plausible that the J-plane analyticity properties of the relativistic S-matrix elements will be similar to those enjoyed by potential scattering matrix elements, as they both have Mandelstam representation.[4, 11]

We now introduce this concept of potential for two-body relativistic S-matrix elements according to Chew and Frautschi.[3] It is convenient to do so through the Mandelstam diagram and Cutkosky graphs.[12]

Mandelstam Diagram

We shall use the usual invariant variables s, u, and t, defined by

$$\begin{aligned} s &= -(P_1+P_2)^2, \\ u &= -(P_1+P_4)^2, \\ t &= -(P_1+P_3)^2, \end{aligned} \tag{2.1}$$

where P_1, P_3 are four momenta of pions, and P_2, P_4 of nucleons, all ingoing (Fig. 1). They satisfy

$$s + u + t = 2m^2 + 2 \equiv \Sigma \tag{2.2}$$

(we take $\hbar = c$ = pion mass $\mu = 1$, and hence, when all three invariant variables are real, can be used as triangular coordinates; the height of the equilatreal triangle formed by straight lines $s = 0$, $u = 0$, $t = 0$ is equal to Σ. These variables shall also be used to refer to the channels for which they have the significance of the energy squared, in its center-of-mass system. Thus, s, u, and t channels refer respectively to pion-nucleon scattering, crossed pion-nucleon scattering, and two pions going into a nucleon-antinucleon pair.

The Mandelstam diagram (Fig. 2) exhibits--with s, t, and u (for their real values) as the triangular coordinates--the physical regions of the three channels, together with the regions in which the double spectral functions (dsf) are nonvanishing. The boundary curves

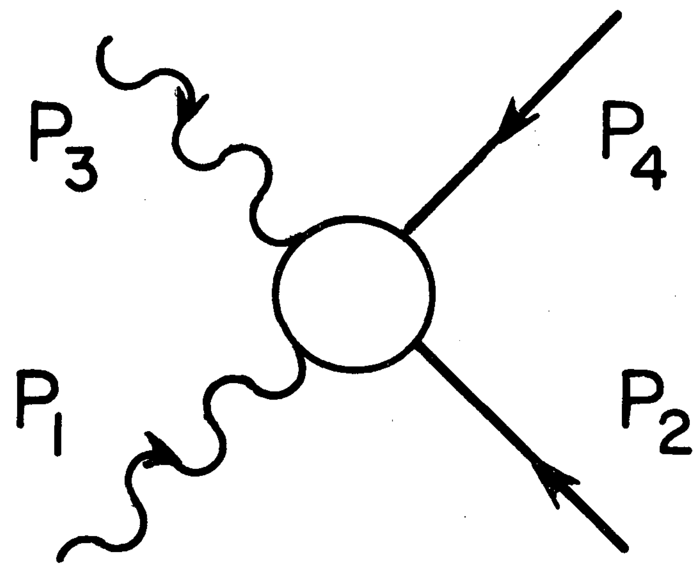

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Fig. 1. The four-line diagram for the π-N problem.

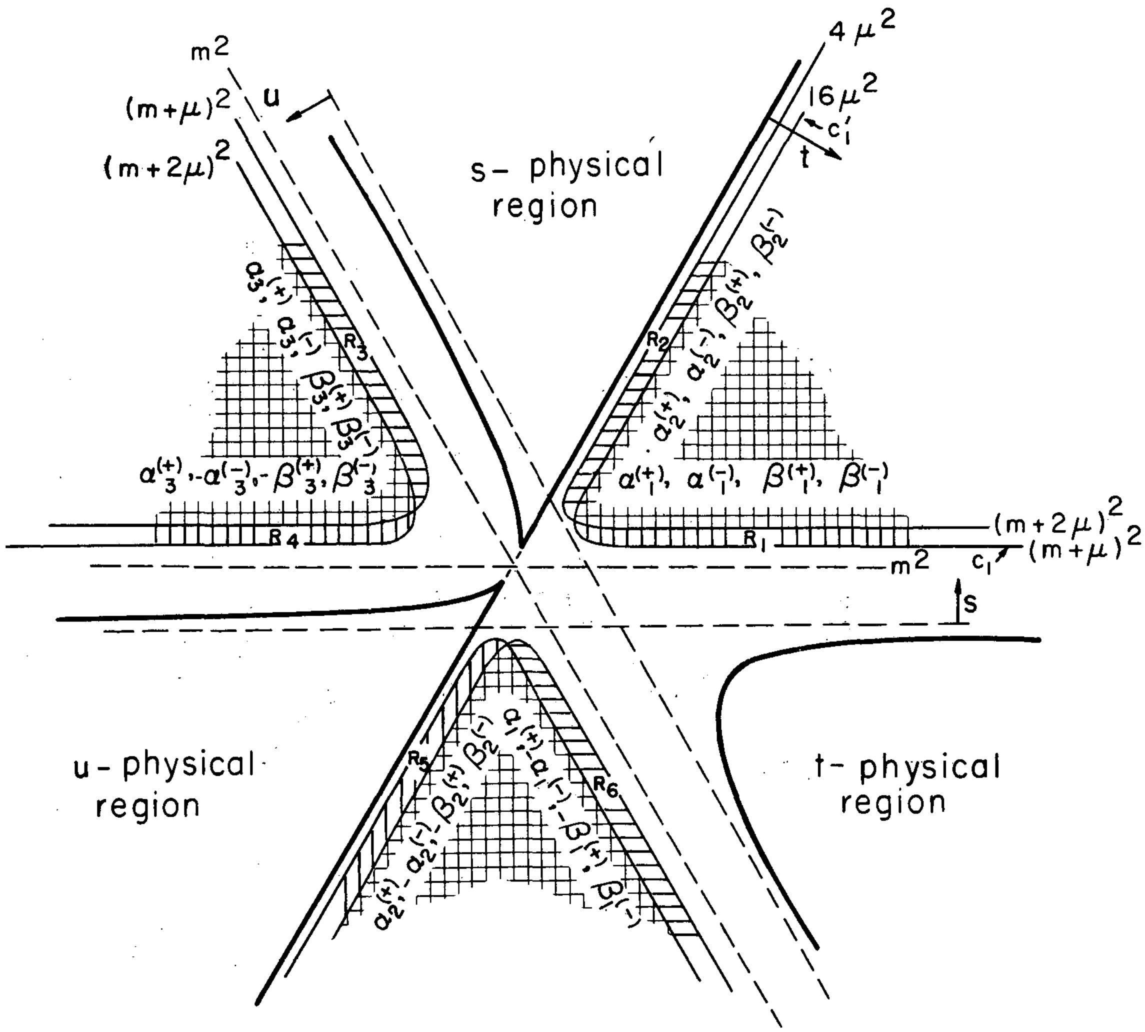

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Fig. 2. The Mandelstam diagram for the π-N problem.

for the physical regions of the three channels are given by

$$t = 0,$$
$$su = (m^2 - \mu^2)^2. \tag{2.3}$$

The boundary curves for the regions in which dsf are nonvanishing are given by Mandelstam.[13]

Cutkosky Graphs

Cutkosky has shown how, by use of generalized unitarity, the Mandelstam dsf's can be expressed as a sum of contributions of all possible four- (or more-) vertex graphs--Cutkosky graphs--where the four external particle lines (i.e., two nucleons and two pions) are attached to four separate vertices.[12] The contribution to a dsf $\rho(s_1, s_2)$, where s_1, s_2 are two invariant variables, of those Cutkosky graphs which have only the lowest-mass two-particle system going in the s_1(or s_2) channel while an arbitrary number of particles may be exchanged in the s_2(or s_1) channel, shall be called "elastic in s_1 (or s_2)" part of dsf $\rho(s_1, s_2)$. As an illustration, the Cutkosky graphs, giving rise to "elastic in s" parts of the dsf $A_{13}^{\pm}(s, t)$ and $B_{13}^{\pm}(s, t)$ are those four-vertex graphs which have only one nucleon and one pion going in the s direction--i.e., $s > (m + \mu)^2$--while any number of particles (at least four pions due to conservation of G parity, i.e., $t > 16\,\mu^2$) are exchanged in the t channel. These are shown in Fig. 3, (a) and (b). Their contribution is nonvanishing in the entire region bounded by the curve C_1C_1', while in the strip region R_1, these are the only Cutkosky graphs that contribute to dsf.

It will be noticed that only those Cutkosky graphs contribute in strip regions R_1, $R_2, \ldots,$ R_6 which lead to "elastic" parts of the respective dsf's. There are no Cutkosky graphs that contribute only in the nonstrip regions and yet give rise to "elastic" parts. All the Cutkosky graphs that contribute to dsf's in strips R_1 (Fig. 3a, b), R_2 (Fig. 3c), and R_3 (Fig. 3d, e) are shown in Fig. 3. They respectively give rise to "elastic in s" part of $A_{13}^{\pm}(s, t)$ and $B_{13}^{\pm}(s, t)$, "elastic in t" part of $A_{13}^{\pm}(s, t)$ and $B_{13}^{\pm}(s, t)$, and "elastic in u" part of $A_{12}^{\pm}(s, u)$ and $B_{12}^{\pm}(s, u)$. The dsf's in the strips R_4, R_5, R_6 can be obtained by

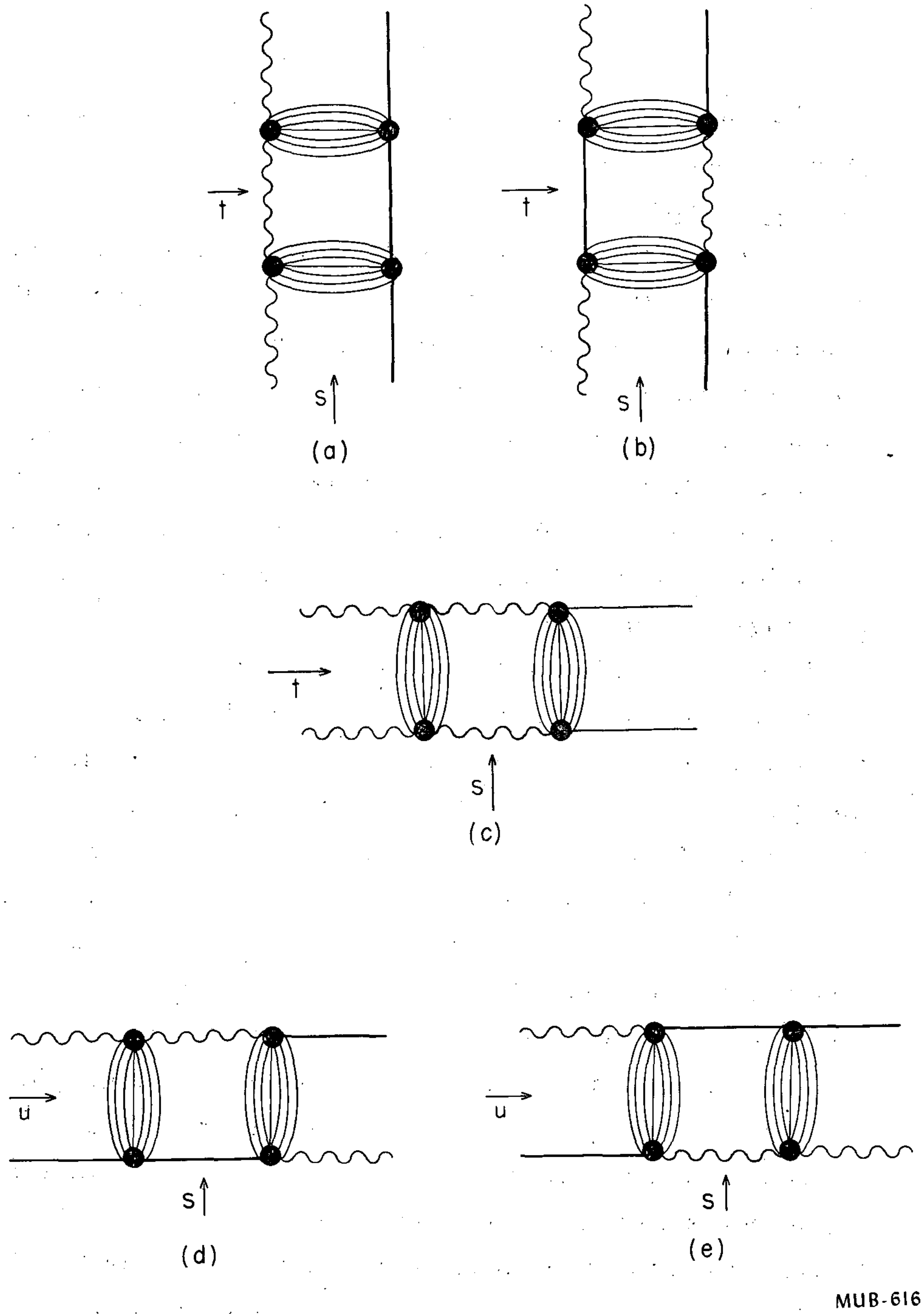

Fig. 3. The Cutkosky diagrams contributing to the dsf's in the strip regions R_1(a, b), R_2(c), and R_3(d, e).

using the crossing symmetry from those of R_1, R_2, R_3. The notation for the parts of dsf contributing in the strips R_1, R_2, R_3 shall be taken to be $\alpha_1^{\pm}(s,t)$, $\alpha_2^{\pm}(t,s)$, and $\alpha_3^{\pm}(u,s)$ for $A^{\pm}(s,u,t)$ amplitude, and $\beta_1^{\pm}(s,t)$, $\beta_2^{\pm}(t,s)$, and $\beta_3^{\pm}(u,s)$ for $B^{\pm}(s,u,t)$ amplitude, respectively. We will also refer to dsf $\alpha_1^{\pm}(s,t)$, $\beta_1^{\pm}(s,t)$ as strip R_1 dsf, and similarly for other strips.

Generalized Potential

We can defined and give expressions for the generalized potential for any channel. To be precise, let us consider π-N scattering, i.e., in the s channel. According to Chew and Frautschi,[3] the potential $V_{\pi N}^{d}(t,s)$ for direct scattering in s channel is given by the absorptive part in the t channel, of the scattering amplitude minus the contribution to it from the "elastic in s" part of the dsf's, i.e., from the strip R_1 dsf's, i.e.,

$$
\begin{aligned}
V_{\pi N}^{d\,(\pm)}(t,s) &= -\left[A_t^{\pm}(s,t) - \frac{1}{\pi}\int \frac{\alpha_1^{\pm}(s',t)ds'}{s'-s}\right] \\
&+ \gamma\cdot\left(\frac{P_1+P_3}{2}\right)\left[B_t^{(\pm)}(s,t) - \frac{1}{\pi}\int\frac{\beta_1^{\pm}(s',t)ds'}{s'-s}\right]. \qquad (2.4a)
\end{aligned}
$$

The potential $V_{\pi N}^{ex}(u,s)$ for exchange scattering in s channel is similarly given by the absorptive part in the u channel of the scattering amplitude minus the contribution to it from the "elastic in s" part of the dsf's, i.e., from the strip R_4 dsf's, i.e.,

$$
\begin{aligned}
V_{\pi N}^{ex\,(\pm)}(u,s) &= -\left[A_u^{\pm}(s,u) - \frac{1}{\pi}\int \frac{\alpha_3^{\pm}(s',u)ds'}{s'-s}\right] \\
&+ \gamma\cdot\left(\frac{P_1+P_3}{2}\right)\left[B_u^{(\pm)}(s,u) - \frac{1}{\pi}\int\frac{\beta_3^{\pm}(s',u)ds'}{s'-s}\right]. \qquad (2.4b)
\end{aligned}
$$

The potentials are obviously spin-, i-spin-, and velocity-dependent. Similarly for the other channels.

The expressions for the strip dsf's that we need for writing the expressions for the potential in the three channels can be derived by using generalized unitarity. The $\alpha_1^{\pm}$, $\beta_1^{\pm}$, $\alpha_3^{\pm}$, and $\beta_3^{\pm}$ can be obtained by using generalized unitarity in pion-nucleon scattering channels, keeping only the one-nucleon—one-pion intermediate state, and were calculated by S. Mandelstam.[9] The $\alpha_2^{\pm}$, $\beta_3^{\pm}$, on the other hand, are obtained by applying generalized unitarity in the $\pi+\pi \to N+\overline{N}$ channel with the two-pion intermediate state. This was done by this author and B. M. Udgaonkar.[14] The quantities are given by the following expressions.

$$\alpha_1^{(\pm)}(s,t) = \sum_{i=1}^{4} \frac{m}{8\pi^2 kW} \times$$

$$\times \Big[\int dt'\, dt''\, K_s(s;t,t',t'')\, \ell_i(s;t,t',t'')\, G_{i;tt}^{(\pm)}(s;t',t'')$$

$$+ \int du'\, du''\, K_s(s;t,\Sigma-s-u',\Sigma-s-u'')$$

$$\times\, \ell_i(s;t,\Sigma-s-u',\Sigma-s-u'')\, G_{i;uu}^{(\pm)}(s;u',u'')\Big], \tag{2.5}$$

$$\alpha_3^{\pm}(u,s) = \sum_{i=1}^{4} \frac{m}{8\pi^2 kW}\Big[\int dt'\, du''\, K_u(s,\Sigma-s-u,\ t'\ \Sigma-s-u'')$$

$$\times\, \ell_i(s;\Sigma-s-u,t',\Sigma-s-u'')\, \{G_{i,tu}^{(\pm)}(s;t',u'') + \text{H.C.}\}\Big]; \tag{2.6}$$

where

$$K_s(s;x_1,x_2,x_3) =$$

$$+\,[x_1^2+x_2^2+x_3^2-2(x_1x_2+x_1x_3+x_3x_1)-(x_1x_2x_3)/k^2]^{-\frac{1}{2}}\,\theta(x_1-x_{1+}), \tag{2.7}$$

and

$$K_u(s;x_1,x_2,x_3) =$$

$$-\,[x_1^2+x_2^2+x_3^3-2(x_1x_2+x_1x_3+x_3x_1)-(x_1x_2x_3)\,k^2]^{-\frac{1}{2}}\,\theta(x_{1-}-x_1), \tag{2.8}$$

with

$$x_{1\pm} = \{[x_2(1+x_3/4k^2)]^{\frac{1}{2}} \pm [x_3(t+x_2/4k^2)]^{\frac{1}{2}}\}^2; \tag{2.9}$$

and

$$W = \sqrt{s} = \sqrt{k^2+m^2} + \sqrt{k^2+1}$$

and $G_{i;\lambda\mu}(s;x,y)$ are bilinear combinations of absorptive parts defined by

$$G_{1;\lambda\mu}(s;x,y) = A_\lambda^{*(+)}(s,x)\,A_\mu^{(+)}(s,y) + 2A_\lambda^{*(-)}(s,x)\,A_\mu^{(-)}(s,y),$$

$$G_{2;\lambda\mu}^{(+)}(s;x,y) = A_\lambda^{*(+)}(s,x)B_\mu^{(+)}(s,y) + 2A_\lambda^{*(-)}(s,x)B_\mu^{(-)}(s,y)$$

$$= G_{3;\lambda\mu}^{*(+)}(s;y,x),$$

$$G_{4;\lambda\mu}^{(+)}(s;x,y) = B_\lambda^{*(+)}(s,x)\,B_\mu^{+}(s,y) + 2B_\lambda^{*(-)}(s,x)B_\mu^{(-)}(s,y),$$

$$G_{1;\lambda\mu}^{(-)}(s,x,y) = A_\lambda^{*(-)}(s,x)\,A_\mu^{(+)}(s,y) + A_\lambda^{*(+)}(s,x)\,A_\mu^{(-)}(s,y)$$

$$+ A_\lambda^{*(-)}(s,x)\,A_\mu^{(-)}(s,y),$$

$$G_{2;\lambda\mu}^{(-)}(s,x,y) = A_\lambda^{*(-)}(s,x)\,B_\mu^{(+)}(s,y) + A_\lambda^{*(+)}(s,x)\,B_\mu^{(-)}(s,y)$$

$$+ A_\lambda^{*(-)}(s,x)B_\mu^{(-)}(s,y) = G_{3;\lambda\mu}^{*(-)}(s;y,x),$$

$$G_{4;\lambda\mu}^{(-)}(s;x,y) = B_\lambda^{*(-)}(s,x)B_\mu^{(+)}(s,y) = B_\lambda^{*(+)}(s,x)B_\mu^{(-)}(s,y)$$

$$+ B_\lambda^{*(-)}(s,x)B_\mu^{(-)}(s,y); \qquad (2.10)$$

and the ℓ_i's are kinematical factors given by

$$\ell_1(s;t,t',t'') = 1 + \frac{(t'+t''-t)\,(s+1-m^2)}{4[(m^2-1)^2-su]},$$

$$\ell_2(s;t,t',t'') = \ell_3(s;t,t'',t') = \frac{(s-m^2-1)(t'-t''+t)}{4mt}$$

$$+ \frac{m(t-t'-t'')(s+1-m^2)}{4[(m^2-1)^2-su]},$$

and

$$\ell_4(s;t,t',t'') = \frac{(t-t'-t'')(s-m^2)(s+1-m^2)}{4[(m^2-1)^2-su]}. \qquad (2.11)$$

The corresponding expressions for the strip functions $\beta_1^{(\pm)}(s, t)$ and $\beta_3^{(\pm)}(u, s)$ are obtained from Eqs. (2.5) and (2.6) by replacing the kinematical factors ℓ_i therein by m_i, defined by

$$m_1(s;t,t',t'') = \frac{(s+m^2-1)(t-t'-t'')}{4m[(m^2-1)^2-su]},$$

$$m_2(s;t,t',t'') = m_3(s;t,t'',t') = \frac{t-t'+t''}{2t} - \frac{(t-t'-t'')(s+m^2-1)}{4[(m^2-1)^2-su]},$$

$$m_4(s;t,t',t'') = \frac{s-m^2-1}{2m} - \frac{(s-m^2)(s+m^2-1)(t-t'-t'')}{4m[(m^2-1)^2-su]}. \tag{2.12}$$

The expressions for $\alpha_2^{(\pm)}(t, s)$, $\beta_2^{(\pm)}(t, s)$ are given by

$$\alpha_2^{(\pm)}(t, s) = \frac{1}{\pi pq^2 W_t} \iint ds' ds'' K_t(t;s, s', s'') \{A_s^{*(\pm)}(t, s'')$$
$$-n_\alpha(t;s, s', s'') B_s^{*(\pm)}(t, s'')\} \mathcal{A}^{(\pm)}(t, s'') \tag{2.13}$$

and

$$\beta_2^{(\pm)}(t, s) = \frac{1}{\pi pq^2 W_t} \iint ds' ds'' K_t(t;s, s', s:) n_\beta(t;s, s', s'') \times$$
$$B_s^{(\pm)}(t, s'') \mathcal{A}^{(\pm)}(t, s'), \tag{2.14}$$

where

$$t = W_t^2 = 4(p^2+m^2) = 4(q^2+1),$$

and where the kinematical factors n_α, n_β, kernel function K_t, and $A_s^+(t, s)$ are quantities defined in Appendix I, where the derivation of these expressions is given.

In the first-order strip approximation of Chew and Frautschi,[15] the different absorptive parts occurring in the expressions for potentials can be expressed in terms of the strip $R_1, \cdots, R_6$ dsf and single spectral functions, which might be present. Higher-order strip approximation would consist in including more and more inner strips parallel to s_1 and s_2 axes in dsf $\rho(s_1, s_2)$. However, for the purpose of this work, we do not concern ourselves with any particular approximation to calculate the potential, as all we really use is the existence of the notion of potential for relativistic S-matrix elements.

III. REGGE POLES IN POTENTIAL SCATTERING

We would now like to discuss the analyticity in the J plane for potential scattering of two spin-zero particles. This would serve as a model for the more complicated case of πN scattering and $\pi+\pi \rightarrow N+\bar{N}$ which we wish to consider in the next two sections.

The usual partial-wave decomposition of the scattering amplitude is given by

$$A(E, z) = \sum_{J=0, 1, 2, \cdots} (2J+1)\, a_J(k)\, P_J(z), \tag{3.1}$$

where $a_J(E)$ is the Jth partial-wave amplitude. The $A(E, z)$ is known to be analytic inside the Lehman ellipse with focii at $z = \pm 1$ and semi-major axis $1 + \mu^2/2E$ for a superposition of Yukawa potentials with maximum range μ^{-1}.

In order to give an expression for $A(E, z)$ that can be used outside the Lehman ellipse, Regge[4] used an artifice due to Sommerfeld and Watson, and transformed the sum over integer J values into a contour integral along the path C in the complex J plane (Fig. 4):

$$A(E, z) = \frac{i}{2} \int_C \frac{a(J, k) P_J(-z) dJ \cdot (2J+1)}{\cos \pi(J + \frac{1}{2})}. \tag{3.2}$$

This transform--Watson transform--of the sum in (3.1) assumes a continuation, which we denote by $a(J, k)$, of the physical partial-wave amplitudes $a_J(k)$, which are defined only for $J = 0, 1, 2, \cdots$, into the complex plane. This continuation is obtained from the Schrödinger equation by solving it for complex J. The contour C is such that the only singularities of the integrand enclosed by C are poles of integrand arising from zeroes of the cosine function in the denominator. A little later we give a prescription for analytic continuation in J, which does not make any reference to the Schrödinger equation.

It was shown by Regge that $a(J, k)$ is a meromorphic function of J in the right half plane (Re $J > -\frac{1}{2}$). The poles occur only in the first quadrant for real E in the right half plane. The contour C in (3.1) can be distorted to C' along Re $J = -\frac{1}{2}$ and an infinite semicircle in the right half plane (Fig. 5) provided we include the contribution of poles enclosed--Regge poles.

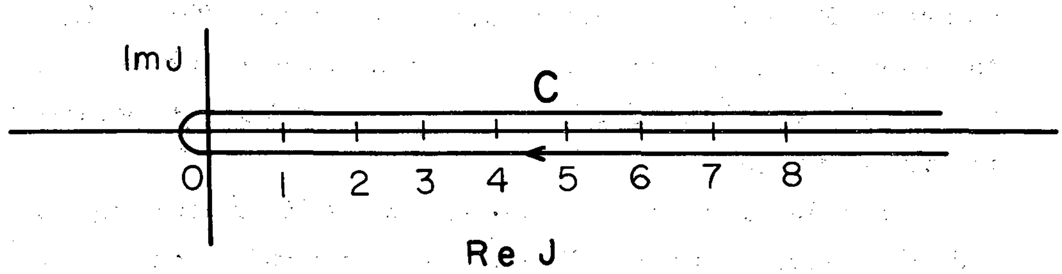

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Fig. 4. The contour C for the Sommerfeld-Watson transform (SWT) for the potential scattering.

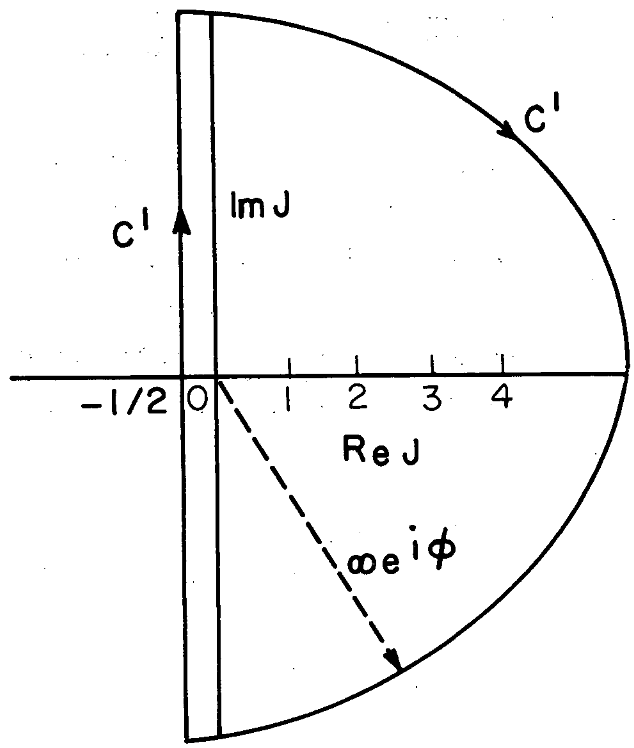

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Fig. 5. The displaced contour C' for SWT for potential scattering.

We now write

$$A(E, z) = \frac{1}{i} \int_{C'} \frac{(2J+1)a(J, k)P_J(-z)dJ}{\sin \pi J} + \sum_{\substack{i \\ \text{Regge poles}}} \frac{\beta_i(k)P_{\alpha i(E)}(-z)}{\sin \pi\alpha_i(E)} \tag{3.3}$$

$$= \frac{1}{i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{(2J+1)a(J,k)P_J(-z)dJ}{\sin \pi J} + \sum_{\text{Re } \alpha_i > -\frac{1}{2}} \frac{\beta_i(k)P_{\alpha i(E)}(-z)}{\sin \pi\alpha_i(E)} . \tag{3.4}$$

The integral along the semicircle is seen to vanish as a(J, k) goes to zero sufficiently fast on the semicircle. Later work[16, 17, 18] has shown that the amplitude is meromorphic in the entire J plane, except for an essential singularity at infinity.

We now come to the question of the analytic continuation in J in the Mandelstam framework. The total amplitude satisfies the Mandelstam representation,

$$A(E, t) = \frac{1}{\pi} \int_{\mu^2}^{\infty} \frac{V(t')dt'}{t'-t} + \frac{1}{\pi^2} \int_0^{\infty} \int_{\mu^2}^{\infty} \frac{\rho(E', t)dE', dt'}{(E'-E)(t'-t)} \tag{3.5}$$

$$\equiv \frac{1}{\pi} \int_{\mu^2}^{\infty} \frac{D_t(t', E)dt'}{t'-t} , \tag{3.6}$$

where $t = -2k^2(1-z)$ and D_t is the t- absorptive part of the amplitude.

Projecting out the Jth ($J = 0, 1, 2, \cdots$) partial wave, we get

$$a_J(k) = \frac{1}{2} \int_{-1}^{+1} A(E, z)P_J(z)dz \tag{3.7}$$

$$= \frac{1}{2\pi k^2} \int_{\mu^2}^{\infty} dt' \, D_t(t', E) \, Q_J(1 + t'/2k^2). \tag{3.8}$$

We now define, for complex J, the analytic continuation a(J, k) of $a_J(k)$ by

$$a(J, k) = \frac{1}{2\pi k^2} \int_{\mu^2}^{\infty} dt' \, D_t(t', E)Q_J(1 + t'/2k^2). \tag{3.9}$$

It can be shown that $a(J, k)$ as defined in terms of the Q_J projection are

(a) in agreement with $a_J(k)$ for physical J values ($J = 0, 1, 2, \cdots$), i.e., for a countable infinite set of physical J values;

(b) regular analytic to the right of some line Re $J = J_m$, where J_m depends on the asymptotic behavior in the t' of $D_t(t', E)$, and is thus the real part of the Regge pole on the farthest right in the J plane;

(c) vanishing sufficiently rapidly with J, owing to the Q_J factor, so as to allow the Watson transform with no contribution from the infinite semicircle in the right half plane;

(d) unitary; that is $\text{Im}\, a(J, E) = |a(J, E)|^2$ (Real J, E). (3.10)

Now it has been shown that the analytic continuation $a(J, E)$ in J of $a_J(E)$ away from positive integers is unique if the first three conditions, satisfied by our $a(J, E)$, in terms of Q_J projection, are satisfied.[16] Thus the Q_J projection is the meaningful Regge continuation in terms of the Mandelstam framework.

J Parity

The above analysis could be extended to include exchange potentials also. This leads to the notion of J parity, which we now proceed to explain. We have

$$A(E, t, u) = \frac{1}{\pi}\int \frac{V^d(t')dt'}{t'-t} + \frac{1}{\pi}\int \frac{V^{ex}(u')du'}{u'-u} + \frac{1}{\pi^2}\iint \frac{\rho_1(E', t')dE', dt'}{(E'-E)(t'-t)} \tag{3.11}$$

$$+\frac{1}{\pi^2}\iint \frac{\rho_2(E', u')dE'du'}{(E'-E)(u'-u)}$$

$$= \frac{1}{\pi}\int \frac{D_t(t', E)dt'}{t'-t} + \frac{1}{\pi}\int \frac{D_u(u', E)du'}{u'-u}, \tag{3.12}$$

where $u = -2k^2(1+z)$. (3.13)

Projecting out the Jth partial wave, we get, after some simplifications,

$$a_J(E) = \frac{1}{2\pi k^2} \int dx' \left[D_t(x', E) + (-)^J D_u(x', E)\right] Q_J\left(1 + \frac{x'}{2k^2}\right). \quad (3.14)$$

If we use this projection to define the continuation $a(J, E)$, we realize that it is no longer possible to make a Watson Transform because of the presence of the factor $(-)^J$, i. e., $e^{+i\pi J}$, which diverges badly as $J \to \infty\, e^{i\phi} (\frac{\pi}{2} > \phi > -\frac{\pi}{2})$. Therefore, what we do is to define two new amplitudes, $a^e(J, E)$ and $a^0(J, E)$, by the expressions

$$a^e(J, E) = \frac{1}{2\pi k^2} \int (D_t + D_u)\, Q_J\left(1 + \frac{x'}{2k^2}\right) dx', \quad (3.15)$$

$$a^0(J, E) = \frac{1}{2\pi k^2} \int (D_t - D_u)\, Q_J\left(1 + \frac{x'}{2k^2}\right) dx'. \quad (3.16)$$

These two analytic funtions of J satisfy the conditions (b) and (c) stated above, and the condition (a) reads, for them, as

(a') $a^e(J, E) = a_J(E)$, for $J = 0, 2, 4, \cdots$,

$$a^0(J, E) = a_J(E), \text{ for } J = 1, 3, 5, \cdots. \quad (3.17)$$

Thus we now have, instead of one, two continuations $a^e(J, E)$, $a^o(J, E)$. The superscript e or o stands for the label (even or odd) of the quantum number distinguishing the two continuations, and shall be designated as J parity. The even J-parity continuation is meaningful only for even J values, the odd one only for odd J values. The Sommerfeld-Watson artifice now takes the form

$$A(E, z) = \left[\frac{i}{2} \int_C \frac{a^e(J, k)[J + \frac{1}{2}][P_J(z) + P_J(-z)]\, dJ}{\cos \pi (J + \frac{1}{2})} + \frac{i}{2} \int_C \frac{a^o(J, k)[J + \frac{1}{2}][P_J(-z) - P_J(z)]\, dJ}{\cos \pi (J + \frac{1}{2})} . \right] \quad (3.18)$$

We can again distort the contour C and write an expression similar to (3.4), where the sum will now run over the poles of both $a^e(J, E)$ and $a^o(J, E)$.

It is to be noted that unitarity does not mix the even and the odd J-parity parts of the amplitude $A(k, z)$. In general, thus, the Regge poles for the even and odd J-parity parts of the amplitude are different. On the other hand, if unitarity coupled these two different J-parity parts of the amplitudes, then they would share the same singularities.

Physical Significance of the Regge Poles

We note that the large-momentum-transfer behavior of the amplitude is given by the Regge pole farthest to the right in the J plane. From Eq. (3.4) we have

$$A(E, t) \sim \frac{\beta_i(E) P_{\alpha(E)}\left(-1 - \frac{t}{2k^2}\right)}{\sin \pi\alpha(E)} \sim t^{\alpha(E)}. \qquad (3.19)$$

The interesting point about this for the relativistic S matrix is that the large unphysical momentum transfer in one channel is also the physical large energy for crossed channels. Thus the high-energy behavior in any one channel is related to the Regge poles in the crossed channels.

As the energy (real) varies the positions of Regge poles trace out a trajectory in the J plane. The system has bound states or resonances whenever the real part of any Regge pole (i) assumes a physical J value, and (ii) is an increasing function of the energy while passing through this J value. Thus the high-energy asymptotic behavior in any one channel is related to the bound states and resonances in the crossed channels.

IV. REGGE POLES IN THE PION-NUCLEON SCATTERING CHANNEL

In this section, we discuss analyticity in the J plane for the relativistic pion-nucleon scattering channel. Even though parity is conserved, the discussion is carried out for the general parity nonconserving case. The reason for doing this is that a certain confusion prevailed as to the question of J parity and ordinary parity (i.e., space parity). The source of the confusion is the circumstance that, in the scattering of two spin-zero particles (which we discussed in the preceding section), the separation of the amplitude in even and odd J-parity parts coincides with the separation into evan and odd ordinary parity parts. Also in this case angular momentum conservation implies parity conservation. Thus there is no way to resolve the confusion, unless one treats the scattering of two particles, one of which at least has nonzero spin. This is what we proceed to do now for the spin-zero–spin-half situation.

There are now four independent invariant amplitudes, instead of the usual two amplitudes A and B. The T matrix can be expressed as

$$T = -A + i\gamma\cdot QB + i\gamma_5\,\gamma\cdot QC - D\gamma_5, \qquad (4.1)$$

where A, B, C, D have only Mandelstam singularities. The differential cross section $d\sigma/d\Omega$ can be written as

$$\frac{d\sigma}{d\Omega} = \sum_{\text{spins}} \left| \left\langle \text{final} \,\middle|\, f \,\middle|\, \text{initial} \right\rangle \right|^2, \qquad (4.2)$$

where

$$f = f_1 + f_2\,\sigma\cdot\hat{k}_f\,\sigma\cdot\hat{k}_i + f_3\sigma\cdot(\hat{k}_f + \hat{k}_i) + f_4\sigma\cdot(\hat{k}_f - \hat{k}_i), \qquad (4.3)$$

$\hat{k}_f$, $\hat{k}_i$ = unit vectors in the direction of final and initial pion three-momentum, and

$$f_1 = \frac{(E+m)}{8\pi W}\,[A + (W-m)B],$$

$$f_2 = \frac{E-m}{8\pi W}\,[-A + (W+m)B],$$

$$f_3 = -\frac{kC}{8\pi},$$

$$f_4 = -\frac{kD}{8\pi W}. \qquad (4.4)$$

We might note that the time-reversal invariance implies that the D and consequently f_4 should be zero. We shall not need to assume the time-reversal invariance, however.

The partial-wave decomposition of the four f's is given by

$$f_1 = \sum_J \left(a^J_{J-\frac{1}{2}, J-\frac{1}{2}} P'_{J+\frac{1}{2}}, -a^J_{J+\frac{1}{2}, J+\frac{1}{2}} P'_{J-\frac{1}{2}} \right),$$

$$f_2 = \Sigma \left(a^J_{J+\frac{1}{2}, J+\frac{1}{2}} P'_{J+\frac{1}{2}} -a^J_{J-\frac{1}{2}, J-\frac{1}{2}} P'_{J-\frac{1}{2}} \right),$$

$$f_3 = \frac{1}{2} \Sigma \left(a^J_{J-\frac{1}{2}, J+\frac{1}{2}} + a^J_{J+\frac{1}{2}, J-\frac{1}{2}} \right)\left(P'_{J-\frac{1}{2}} - P'_{J+\frac{1}{2}} \right),$$

$$f_4 = \frac{1}{2} \Sigma \left(a^J_{J-\frac{1}{2}, J+\frac{1}{2}} - a^J_{J+\frac{1}{2}, J-\frac{1}{2}} \right)\left(P'_{J-\frac{1}{2}} + P'_{J+\frac{1}{2}} \right), \tag{4.5}$$

where $a^J_{L, L'}$ = the T-matrix element from the state $|J, L'\rangle$ to $|J, L\rangle$,

$|J, L\rangle$ = the state with total and orbital angular momentum equal to J and L respectively.

The summation over J runs over $J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. The argument of Lependre polynomials is

$$\cos\theta = 1 + \frac{t}{2k^2} = 1 + \frac{2m^2 + 2\mu^2 - s - u}{2k^2}. \tag{4.6}$$

The projection formulas for $a^J_{L, L'}$ are given by

$$a^J_{J+\frac{1}{2}, J+\frac{1}{2}} = \frac{1}{2} \int_{-1}^{+1} (f_1 P_{J+1/2} + f_2 P_{J-1/2}) dz,$$

$$a^J_{J-\frac{1}{2}, J-\frac{1}{2}} = \frac{1}{2} \int_{-1}^{+1} (f_1 P_{J-1/2} + f_2 P_{J+1/2}) dz,$$

$$a^J_{J-\frac{1}{2}, J+\frac{1}{2}} + a^J_{J+\frac{1}{2}, J-\frac{1}{2}} = \int_{-1}^{+1} f_3(P_{J-1/2} + P_{J+1/2})\,dz,$$

$$a^J_{J-\frac{1}{2}, J+\frac{1}{2},} - a^J_{J+\frac{1}{2}, J-\frac{1}{2}} = \int_{-1}^{+1} f_4(P_{J-1/2} - P_{J+1/2})dz. \quad (4.7)$$

The details of the derivation of these formulas can be found in Appendix II.

It is clear from (4.4), and from the fact that A, B, C, D have only Mandelstam singularities, that the f's satisfy fixed energy-dispersion relations of the forms

$$f_i = \frac{1}{\pi} \int_4^\infty \frac{D_{i,t}(t', s)dt'}{t'-t} + \frac{1}{\pi} \int_{(m+1)^2}^\infty \frac{D_{i,u}(u', s)du'}{u'-u} \quad (4.8)$$

$$= \frac{1}{\pi} \int_4^\infty \frac{D_{i,t}(x', s)dx'}{x'+2k^2(1-\cos\theta)} + \frac{1}{\pi} \int_{(m+1)^2-(m^2-1)^2/s}^\infty \frac{D_{i,u}(x'+\frac{(m^2-1)^2}{s}, s)dx'}{x'+2k^2(1+\cos\theta)}. \quad (4.9)$$

Using (4.7) and the above dispersion relations (4.9), we can carry out the projections for physical L, J, and we get

$$a^J_{J\pm\frac{1}{2}, J\pm\frac{1}{2}} = \frac{1}{4\pi k^2} \Big[\int \{D_{1,t}(x', s) + (-)^{J\pm\frac{1}{2}} D_{1,u}(x'+\frac{(m^2-1)^2}{s})\}$$

$$\times Q_{J\pm\frac{1}{2}}(1+\frac{x'}{2k^2}) + \int \{D_{2,t}(x', s) - (-)^{J\pm\frac{1}{2}} D_{2,u}(x'+\frac{(m^2-1)^2}{s}, s)\}$$

$$\times Q_{J\mp 1/2}(1+\frac{x'}{2k^2})\Big], \quad (4.10)$$

$$a^J_{J-\frac{1}{2}, J+\frac{1}{2}} \pm a^J_{J+\frac{1}{2}, J-\frac{1}{2}} = \frac{1}{2\pi k^2}\Big\{\int \Big[D_{\substack{3,t\\4,t}}(x's)$$

$$+ (-)^{J-1/2} D_{\substack{3,u\\4,u}}(x'+(\frac{(m^2-1)^2}{s}, s)\Big]\,\Big[Q_{J-1/2}(1+\frac{x'}{2k^2})\Big]$$

$$\pm \int [D_{\substack{3,t\\4,t}}(x', s) + (-)^{J+1/2} D_{\substack{3,u\\4,u}}(x' + \frac{(m^2-1)^2}{s}, s)] Q_{J+1/2}(1 + \frac{x'}{2k^2})\}. \tag{4.11}$$

One sees that these expressions have all the nice properties we demanded of them in Section 3 for the unique analytic continuation into the complex J plane, but for the factors $(-)^{J-1/2}$. Therefore, to define proper continuations, we split them into the even and odd J-parity parts, as before; i.e., we define

$$a^{J,e}_{J\pm 1/2, J\pm 1/2} = \frac{1}{4\pi k^2}\left\{\int [D_{1t} \mp D_{1,u}] Q_{J\pm 1/2} + \int (D_{2t} \pm D_{2,u}) Q_{J\mp 1/2}\right\}, \tag{4.12}$$

$$a^{J,o}_{J\pm 1/2, J\pm 1/2} = \frac{1}{4\pi k^2}\left\{\int [D_{1t} \pm D_{1,u}] Q_{J\pm 1/2} + \int [D_{2,t} \mp D_{2,u}] Q_{J\mp 1/2}\right\}, \tag{4.13}$$

$$a^{J,e}_{J-1/2, J+1/2} + (\pm) a^{J,e}_{J+1/2, J-1/2} = \frac{1}{2\pi k^2}\left\{\int [D_{\binom{3t}{4t}} + D_{\binom{3,u}{4,u}}] Q_{J-1/2} + (\pm)\int [D_{\binom{3,t}{4,t}} - D_{\binom{3,u}{4,u}}] Q_{J+1/2}\right\}, \tag{4.14}$$

$$a^{J,o}_{J-1/2, J+1/2} + (\pm) a^{J,o}_{J+1/2, J-1/2} = \frac{1}{2\pi k^2}\int [D_{\binom{3,t}{4,t}} - D_{\binom{3,u}{4,u}}] Q_{J-1/2} + (\pm)\int [D_{\binom{3,t}{4,t}} + D_{\binom{3,u}{4,u}}] Q_{J+1/2}. \tag{4.15}$$

We have replaced, in writing Eqs. (4.12) through (4.15), the factors $(-)^{J-1/2}$ in (4.10) and (4.11) by +1 for even J-parity and by (-1) for odd J-parity continuations. Thus even J-parity continuations are physically meaningful for $J = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \cdots$, and odd J-parity ones for $J = \frac{3}{2}, \frac{7}{2}, \cdots$. The eight amplitudes defined by Eqs. (4.12) through (4.15) are such that analytic continuation to the complex J plane is unique and is obtained by regarding J complex. We see that the notion of the J parity comes out naturally even in the non-parityconserving case, and therefore has nothing whatever to do with the parity conservation or symmetry properties, such as are present for $\pi\pi$ scattering. Rather, the notion of J parity is a direct consequence of the presence of exchange forces together with direct forces.

We can now write

$$f = f^{(e)} + f^{(o)}, \tag{4.16}$$

where

$$f^{e,o} = f_1^{e,o} + f_2^{e,o}\, \sigma\cdot\hat{k}_f\, \sigma\cdot\hat{k}_i + f_3^{e,o}\, \sigma\cdot(\hat{k}_f+\hat{k}_i) + f_4^{e,o}\, \sigma\cdot(\hat{k}_f-\hat{k}_i), \tag{4.17}$$

where f_i^e and f_i^o are respectively the parts of partial-wave sums over $J = \frac{1}{2}, \frac{5}{2}, \cdots$ and $J = \frac{3}{2}, \frac{7}{2}, \cdots$, i.e.,

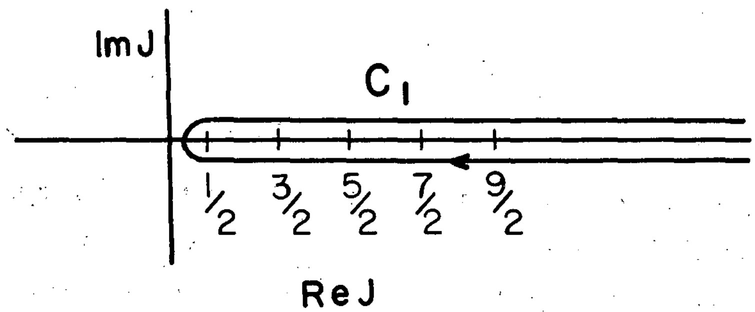
$$f_1^e = \sum_{J=1/2,5/2,\cdots} a^{J,e}_{J-1/2,J-1/2} P'_{J+1/2} - \sum_{J=1/2,5/2,\cdots} \times a^{J,e}_{J+1/2,J+1/2}\, P'_{J-1/2}, \tag{4.18}$$

and similar expressions for the other seven parts, $f_1^o, f_{2,3,4}^{e,o}$.
Equation (4.18) can alternatively be written as a sum over all physical J values,

$$f_1^e = \sum_{J=1/2,3/2,5/2} a^{J,e}_{J-1/2,J-1/2} \left(\frac{P'_{J+1/2}(\cos\theta) + P'_{J+1/2}(-\cos\theta)}{2}\right)$$
$$- \sum_{J=1/2,3/2,\cdots} a^{J,e}_{J+1/2,J+1/2} \left(\frac{P'_{J-1/2}(\cos\theta) - P'_{J-1/2}(-\cos\theta)}{2}\right). \tag{4.19}$$

We can convert the sum over physical J values in this and similar expressions into an integral in the same way as in Section 3 over the Contour C_1 (Fig. 6), and get

$$f_1 = \frac{i}{2} \int_{C_1} \frac{dJ}{\cos\pi J}\, a^{J,e}_{J-1/2,J-1/2}\, (W) \left(\frac{P'_{J+1/2}(z) + P'_{J+1/2}(-z)}{2}\right)$$



MU-27362

Fig. 6. The contour C_1 for SWT for π-N scattering.

$$+\frac{i}{2}\int_{C_1}\frac{dJ}{\cos\pi J}\;a^{J,e}_{J+1/2,J+1/2}(W)\left(\frac{P'_{J-1/2}(z)-P'_{J-1/2}(-z)}{2}\right)$$

$$+\frac{i}{4}\int_{C_1}\frac{dJ}{\cos\pi J}\left\{a^{J,0}_{J-1/2,J-1/2}\left(P'_{J-1/2}(z)-P'_{J+1/2}(-z)\right)\right.$$

$$\left.+\,a^{J,0}_{J+1/2,J+1/2}\left(P'_{J-1/2,}(z)+P'_{J-1/2}(z)\right)\right\}. \qquad (4.20)$$

We have drawn the contour C_1 so that it includes all the physical J values. To be accurate, the contour C_1 should extend only up to the smallest J value, say J_o, which is within the natural boundary of the eight function elements defined by Eqs. (4.12) through (4.15) for Re $J \geqslant$a constant, say $J_m > J_o$. The contribution of the partial waves Re $J_o > J$ should be explicitly added to the integrals. We shall later show that all the physical partial waves, with the possible exception of $J = 1/2$ waves, are completely determined in terms of the same double-spectal functions.

One could now displace the contour C_1, just as for spin-zero scattering, provided one included the contribution of the singularities—in particular, Regge poles—that one encounters in shifting in contour C. The J-plane singularitites of the pion-nucleon scattering amplitudes would be thus the J-plane singularities of the four even J-parity amplitudes $a^{J,e}_{J\pm 1/2,J\pm 1/2}$, $a^{J,e}_{J\pm 1/2,J\mp 1/2}$, and of the four odd J-parity amplitudes $a^{J,o}_{J\pm 1/2,J\pm 1/2}$, $a^{J,o}_{J\pm 1/2,J\mp 1/2}$.

Now in this case, with parity nonconservation, all the four amplitudes of the same J parity are coupled to one another by unitarity, and as a result share the same singularities. Thus the J-plane singularities are labeled only by the quantum numbers,

Baryon number = 1,

J parity = even or odd.

In particular, Regge trajectories—the trajectory of the real part of a Regge pole with energy taken real—are now divided into two families, one with even and the other with odd J parity.

For the real physical case of conserved parity, we have

$$a^{J,e}_{J\pm 1/2, J\mp 1/2} = 0,$$

$$a^{J,0}_{J\pm 1/2, J\mp 1/2} = 0. \tag{4.21}$$

Further, unitarity no longer couples the even and odd space-parity parts. That is, we now have

$$f = f^e + f^0,$$

$$f^{e,0} = f^{e,0}_{(P=+1)} + f^{e,0}_{(P=-1)},$$

where

$$f^e_{1,(P=+1)} = \frac{1}{2} \sum_J a^{J,e}_{J-1/2, J-1/2}\left(P'_{J+1/2}(z) + P'_{J+1/2}(-z)\right),$$

$$f^e_{1,\ (P=-1)} = \frac{1}{2} \sum_J a^{J,e}_{J+1/2, J+1/2}\left(P'_{J-1/2}(z) - P'_{J-1/2}(+z)\right),$$

etc. (P = space parity). (4.22)

Unitarity never mixes the four amplitudes $f^{e,0}_{(P=\pm 1)}$; for example, $f^e_{1,(P=+1)}$ is coupled only to $f^e_{i,\ P=+1}$ (i = 1, 2, 3, 4). This means that the J-plane singularities of the π-N scattering amplitude are now the J-plane singularities of the four amplitudes $a^{J;\,e,0}_{J\pm 1/2, J\pm 1/2}$. Thus the singularities—Regge trajectories in particular—can be labeled by

Baryon number = 1,
J parity = even or odd,
Parity = +1 .

So far we have not considered isospin. The inclusion of isospin gives one more quantum number;

Total isospin = I = 1/2, 3/2.

Thus there is a total of eight families of Regge trajectories. In general we expect to see roughly one trajectory from each family—the top one.

These eight families of Regge trajectories, can be specified by assigning an ordered triplet (isospin, ~~space~~ parity, J parity) to each family. The elements of the set of pion-nucleon scattering states, characterized by these triplets, are given by

(1/2 or 3/2, -1, +1): $S\frac{1}{2}$, $D\frac{5}{2}$, $\cdots$, with I = 1/2 or 3/2,

(1/2 or 3/2, -1, -1): $D\frac{3}{2}$, $G\frac{7}{2}$, $\cdots$, with I = 1/2 or 3/2,

(1/2 or 3/2, +1, -1): $P\frac{3}{2}$, $F\frac{7}{2}$, $\cdots$, with I = 1/2 or 3/2,

(1/2 or 3/2, +1, +1): $P\frac{1}{2}$, $F\frac{5}{2}$, $\cdots$, with I = 1/2 or 3/2.

Now, experimentally, a number of particle and resonance states have been observed having baryon number one.[19] For isospin one-half, one has to start with nucleon N.

$$N_{1/2}\text{: } J = 1/2,\ P = +1,\ \text{mass} = 939\ \text{MeV}.$$

Then there are two resonances observed in pion-nucleon scattering:

$$N_{1/2}^{*}\text{: } J = 3/2, P = -1\ (\text{i.e., } D\tfrac{3}{2}\ \pi\text{-N state}),$$
$$\text{mass} = 1510\ \text{MeV};$$

$$N_{1/2}^{**}\text{: } J = 5/2, P = +1\ (\text{i.e., } F\tfrac{5}{2}\ \pi\text{-N state}),$$
$$\text{mass} = 1680\ \text{MeV}.$$

If we regard these particles and resonances as Regge poles, they can be interpreted as follows:

(a) N and $N_{1/2}^{**}$ may be regarded as the first two members of the Regge family (1/2, +1, +1). It must be observed that without J parity, it would not have been possible to explain the absence of a $P\frac{3}{2}$, $I = \frac{1}{2}$ π-N resonance in the Regge picture. We further have to have both these objects, N, $N_{1/2}^{**}$ lying on the same Regge trajectory, since otherwise one will expect another particle with nucleon quantum numbers and mass depending on where Regge trajectory N^{**} crosses J = 1/2. Then from observed mass, we can get an idea of average slope of the Regge trajectory in this region of energy. This turns out to be $d\alpha/ds \approx (1 \text{ BeV})^{-2}$.

(b) $N^*_{1/2}$ has to be regarded as the first member of the family (1/2, -1, -1). Whether one observes a second member of this family depends on whether this Regge trajectory ever crosses $J = 7/2$. On the basis of the above estimate of the slope of Regge trajectories, we might expect this to happen at around 4.5 $(\text{BeV})^2$, if one were allowed such an extrapolation. More likely, however, is that this is the only observable member of this family.

It would further be noticed that no members of the families (1/2, -1, +1) and (1/2, +1, -1) have been experimentally observed. Obviously, no Regge trajectory belonging to these families reaches the lowest physical J value available to the family, i.e., $J = 1/2$, $3/2$ respectively. The absence of the observable members in these families can be expressed alternatively: Given isospin (i.e., one-half) and space parity, the Regge trajectory corresponding to only one of the two possible J-parity values shows up. This could be understood probably as follows: Looking at Formulas (4.12) through (4.15), one sees that the direct force has the same, while the exchange force has the opposite sign in the two states with opposite J parity. Now if the direct and exchange forces have roughly the same magnitude, the total forces are strong in only one of the two J-parity states and weak in the other.

Now coming to isospin 3/2, first there is the well-known 3, 3 resonance in the π-N scattering,

$$N^*_{3/2}:\ J = 3/2,\ P = +1 \text{ (i.e., } P\tfrac{3}{2}\ \pi\text{-N state), mass} = 1238 \text{ MeV}.$$

One has also observed a bump at mass $\approx$ 1900 MeV, whose quantum numbers are not known. Now using the above estimate of slope $\approx (1 \text{ BeV})^{-2}$, one expects in just this neighborhood of energy, the second member of the Regge family (3/2, +1, -1) to which $N^*_{3/2}$ belongs. If this bump is really the second member of this family, then one would predict that it would occur in the $F\frac{7}{2}$ state. Besides, there is also a definite shoulder on the low-energy side of this 1900-MeV bump, which might be the first member of the family (3/2, -1, -1), i.e., $D\frac{3}{2}$, $T = \frac{3}{2}$, as suggested by Moyer and Carruthers and Bethe.

We show the tentative Regge trajectories, which are physically mainfest in Fig. 7.

The Range of the Exchange Potential

One has so far never been sure what quantity should properly be called the range of the exchange potential in case of the scattering of two unequal-mass particles, such as π-N scattering. The above discussion–in addition to the question of the J parity–clarifies this situation.

It will be seen from Expressions (4.10) and (4.11) that the absorptive parts in the t and u channels having the same value of the integration variable x' are superimposed on each other. Now,

$$x' = t \quad \text{for t-absorptive part,}$$

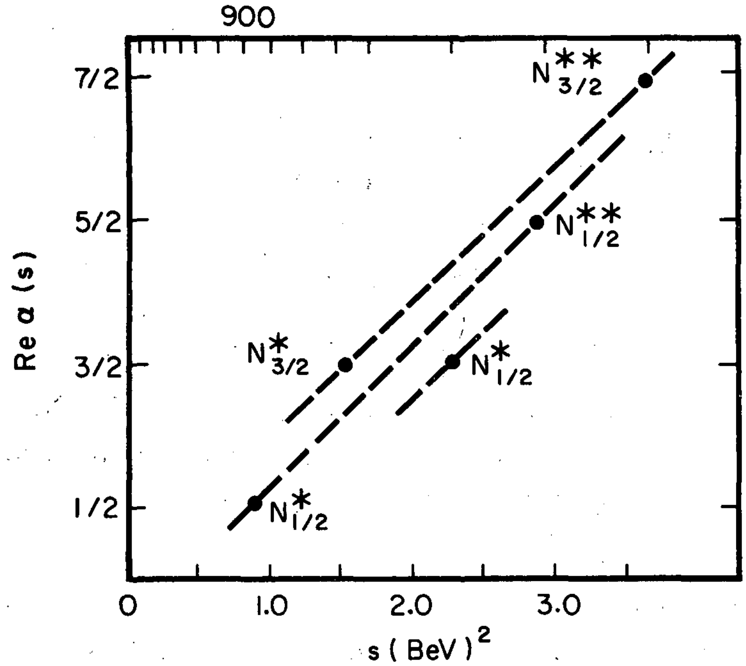
and

$$x' = u - \frac{(m^2 - 1)^2}{s} \quad \text{for u-absorptive part.}$$

Hence the range of the exchange force arising from the exchange of mass $\sqrt{u}$ is

$$\left(u - \frac{(m^2 - 1)^2}{s}\right)^{-1/2}$$

in the same sense as $(t)^{-1/2}$ is the range of the direct force arising from an exchange of mass $\sqrt{t}$. The range of the exchange force is thus–unlike the direct force–energy-dependent, and becomes smaller as energy becomes larger. In particular, the exchange of a single nucleon gives rise at low energies to a force of range $\approx (2M)^{-1/2}$ and approaches the naively expected M^{-1} only at very high energy.



MU-27363

Fig. 7. The Regge trajectories in I = 1/2, 3/2 for π-N scattering.

V. REGGE POLES IN THE $\pi + \pi \leftrightarrow N + \bar{N}$ CHANNEL

We now come to a discussion of the J-plane analyticity in the $\pi + \pi \rightarrow N + \bar{N}$ channel. The partial-wave decomposition in this channel is given by[20]

$$A^{\pm} = -\frac{8\pi i}{p^2}\left(\frac{p}{q}\right)^{1/2} \sum_J (J+1/2)\left\{\frac{m \cos\theta_3}{[J(J+1)]^{1/2}} S^{(\pm)}_{-J} \quad P'_J(\cos\theta_3) - \frac{t^{1/2}}{2} P_J(\cos\theta_3)\, S^{\pm}_{+J}\right\}, \tag{5.1}$$

$$B^{\pm} = -\frac{8\pi i}{pq}\left(\frac{p}{q}\right)^{1/2} \sum_J \frac{(J+1/2)}{[J(J+1)]^{1/2}} S^{\pm}_{-J} \quad P'_J(\cos\theta_3), \tag{5.2}$$

where

$t = 4(q^2 + 1) = 4(p^2 + m^2)$,

$\cos\theta_3 = (s + p^2 + q^2)/2pq$,

$S^{\pm}_{\pm, J}$ = S-matrix elements for π-π $\rightarrow N + \bar{N}$. The subscript + and - refer to nucleon and antinucleon having the same or opposite helicity. The superscripts + and - refer respectively to total i spin and 0 or 1.

The sum over J runs over $J = 0, 2, 4, \cdots$ for A^+, B^+, i.e., $T = 0$; and over $J = 1, 3, 5, \cdots$ for A^-, B^-, i.e., $T = 1$.

In what follows, we do not consider the analytic continuation of $S^{\pm}_{\pm J}$ into complex J plane, but rather

$$(\underline{S})^{\pm}_{+J} = S^{\pm}_{+J}, \tag{5.3}$$

$$(\underline{S})^{\pm}_{+J} = \frac{J + 1/2}{[J(J+1)]^{1/2}} \quad S^{\pm}_{-J}, \tag{5.4}$$

as these are the quantities we always encounter. This gets rid of the fixed branch points in $J = 0, -1$. We have, for physical J values,

$$(\underline{S})^{\pm}_{+J} = \frac{i}{4\pi}\left(\frac{q}{pt}\right)^{1/2}\left(-p^2 A^{\pm}_J + \frac{m\,p\,q}{2J+1}\left((J+1)\, B^{\pm}_{J+1} + J\, B^{\pm}_{J-1}\right)\right), \tag{5.5}$$

$$(\underline{S})^{\pm}_{-J} = \frac{i}{8\pi}\left(\frac{q}{p}\right)^{1/2} \frac{pq}{2}\left(B^{\pm}_{J-1} - B^{\pm}_{J+1}\right). \tag{5.6}$$

Using these expressions to project out these partial waves, we obtain, after certain simplifications,

$$(\underline{S})^{\pm}_{+J} = [1 \pm (-)^J] \left(\frac{-i}{8\pi^2}\right)\left(\frac{p}{qt}\right)^{1/2} \left[\int_{(m+1)^2}^{\infty} ds' \left\{ A_s^{\pm}(s', t) - \frac{(s' + p^2 + q^2)m}{2p^2} B_s^{\pm}(s', t) \right\} Q_J \left(\frac{s' + p^2 + q^2}{2\, p\, q}\right) \right], \tag{5.7}$$

$$(\underline{S})^{\pm}_{-J} = [1 \pm (-)^J] \frac{i}{32\,\pi^2} \left(\frac{q}{p}\right)^{1/2} \left[\int_{(m+1)^2}^{\infty} ds' \, B_s(s', t) \, Q_{J-1}\left(\frac{s' + p^2 + q^2}{2\, p\, q}\right) - Q_{J+1}\left(\frac{s' + p^2 + q^2}{2\, p\, q}\right) \right]. \tag{5.8}$$

We have also used crossing symmetry (Bose statistics for the pions) in writing these expressions. Looking at the expressions (5.7) and (5.8), we again see that apart from the factors $[1 \pm (-)^J]$, the quantities $(\underline{S})^{\pm}_{+J}$ and $(\underline{S})^{\pm}_{-J}$ define the suitable unique analytic continuations, in the sense of Section 3. In order to get rid of these factors, we—as in the last two sections—define the new amplitudes for even and odd J parity. The even-J-parity ones we obtain by replacing $(-)^J$ by $+1$; the odd-parity ones by replacing $(-)^J$ by (-1) in Eqs. (5.7) and (5.8). We thus get

$$(\underline{S})^{+e}_{+J},\ (\underline{S})^{+e}_{-J},\ (\underline{S})^{-0}_{+J},\ (\underline{S})^{-0}_{-J} \neq 0, \tag{5.9}$$

$$(\underline{S})^{+0}_{+J} = (\underline{S})^{+0}_{-J} = (\underline{S})^{-e}_{+J} = (\underline{S})^{-e}_{-J} = 0. \tag{5.10}$$

Thus only the even-J-parity continuations are nonvanishing for isospin zero, and only odd-J-parity ones for isospin one. This is a particular instance in which a symmetry property--in this case crossing symmetry--tells us that only one J parity is physical.

The partial-wave sums in (5.1) and (5.2) can again be expressed as contour integrals in the J plane over the contour C (same as in Fig. 4).

The contour C, strictly speaking as in the last section, has to include only those integer J values which can be reached from J continuation from the function elements defined above by (5.7) and (5.8). We show later that with the possible exception of J = 0, 1, all other physical partial waves are determined by the same double-spectral functions. The main interest in expressing the partial-wave sum as a contour integral is, of course, in displacing the contour so as to be parallel to the imaginary J axis. This can be done provided the contribution of the singularities in the J plane, which we cross in doing so, is included. Also, we should be careful not to cross the natural boundary (if any) of the analytic continuation. In particular, we have to include the contribution of the Regge poles in these channels.

The J-plane singularities of $\pi+\pi \rightarrow N + \overline{N}$ amplitudes are the singularities of $(\underline{S})^{+e}_{\pm J}$ for isospin zero and of $(\underline{S})^{-e}_{\pm J}$ for isospin one. Since both $(\underline{S})^{+e}_{+J}$ and $(\underline{S})^{+e}_{-J}$ are connected by unitarity to a number of channels that are the same for both (for example, pion-pion scattering amplitude for isospin zero), they share the same Regge poles with pion-pion amplitude. Similarly the $(\underline{S})^{-0}_{+J}$ and $(\underline{S})^{-0}_{-J}$ both have the same Regge poles as $\alpha^{(1)}(J)$, the isospin-one pion-pion scattering amplitude for angular momentum J.

Thus there would be two families of Regge Poles which can be labeled by quantum numbers,

Baryon numbers	= 0,
G parity	= +1,
Isospin = I	= 0, 1,
J parity	= even for isospin 0 and odd for isospin 1,
space parity	= even for isospin 0 and odd for isospin 1,

and thus differ in having different isospin, together with uniquely associated J parity and space parity.

Experimentally the only observed resonance for B = 0, G = 1, I = 1 is the ρ meson,

$$\rho: \; J = 1, \; P = -1, \; \text{mass} \approx 750 \text{ MeV}.$$

For isospin zero, I = 0, B = 0, G = 1, only the interaction in J = 0, I = 0 $\pi\pi$ s wave observed by Abashian et al. comes anywhere near being a resonance. This occurs very near the elastic threshold, i.e., mass $\approx$ 280 MeV. If we regard these as manifestations of the Regge poles, then these would be first members of the two Regge trajectories. One might observe the second member of the ρ-Regge trajectory with $I = 1$, $J = 3$ at about 1600 MeV, if the trajectory had not already turned down. For ABC trajectory, however, one does not expect to see the second member, as the force leading to ABC phenomena does not seem to be strong enough. This trajectory, it is believed, just reaches to $\mathrm{Re}\, J = 0$ before turning down.

As we shall see later, the constancy of high-energy cross sections implies in the Regge picture also the existence of another trajectory with $I = 0$, which passes through $J = 1$ at zero mass. There is some inconclusive experimental evidence for the first physical manifestation (i.e., $I = 0$, $J = 2$ $\pi\pi$ resonance) of this trajectory, to be called the Pomeranchuk trajectory.

We represent the tentative Regge trajectories in this channel in Fig. 8.

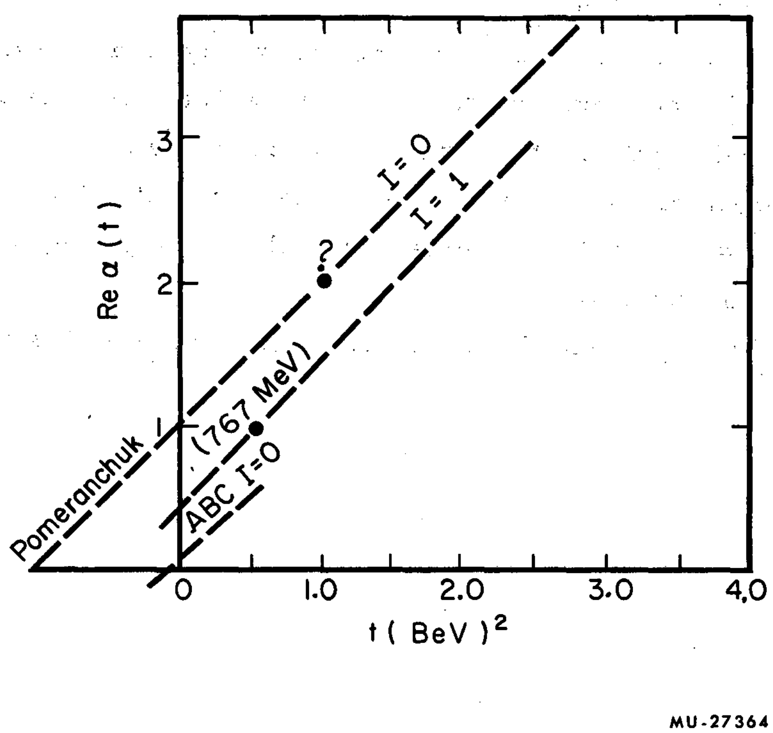


Fig. 8. The Regge trajectories in I = 0, 1 for the $\pi+\pi \rightarrow N+\overline{N}$ channel.

VI. HIGH-ENERGY PION-PROTON SCATTERING IN THE FORWARD AND BACKWARD CONES

The present thinking about strong interactions, from the point of view of analyticity in angular momenta, is inclined to regard all the baryons and mesons, as well as resonances, as Regge poles.[7] One believes that the high-energy behavior in any one channel is dominated by the Regge poles in the crossed channels. Further, there does not seem to be any hesitation to apply this notion to either nucleon or pion, which customarily have been treated as elementary–i.e., on a different footing from, say, the 3, 3 resonance in the π-N system. In view of what we have been able to establish in the foregoing sections about the J-plane analyticity for pion-nucleon amplitudes, such a notion obviously has not been shown to follow from assumed analyticity in linear momenta, i.e., the Mandelstam representation. It is, however, experimentally possible to test the implications of the above Regge pole hypothesis. To that purpose, we present in this section the expressions for high-energy pion-nucleon scattering.

<u>High-Energy Forward Elastic Scattering</u>

This would be dominated by the Regge poles in the crossed channel $\pi+\pi \longleftrightarrow N+\bar{N}$, i.e., t channel. We have, for differential and total cross sections,

$$\frac{d\sigma}{d\Omega} = |f_1+f_2|^2 + \frac{t}{k^2} \text{Re } f_1^* f_2, \tag{6.1}$$

$$\sigma^{\text{total}} = \frac{4\pi W}{m\sqrt{\omega^2-1}} \text{I m} (f_1+f_2)_{t=0}, \tag{6.2}$$

$$\omega = (s - m^2 - 1)/2m, \tag{6.3}$$

where one has to substitute proper isospin combinations for f_1, f_2.

Thus, for example,

$$f_i = f_i^{(+)} + f_i^{(-)} \text{ for } \pi^- p \rightarrow \pi^- p,$$

$$= f_i^{(+)} - f_i^{(-)} \text{ for } \pi^+ p \rightarrow \pi^+ p, \tag{6.4}$$

$$= -\sqrt{2}\, f_i^{(-)} \text{ for } \pi^- p \rightarrow \pi^0 n.$$

Re-expressing Eqs. (6.1) and (6.2) in terms of amplitudes,

$$A' = A + \frac{\omega + t/4m}{1 - t/4m^2} B, \tag{6.5}$$

$$B' = B,$$

we obtain

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{4\pi W}\right)^2 \Big[(1 - t/4m^2)\,|A'|^2 + \frac{Wt}{4m^2}(A'^*B - B^*A') +$$

$$+ \frac{t}{4m^2}\left(s - \frac{(m+\omega)^2}{1 - t/4m^2}\right)|B|^2\Big], \tag{6.6}$$

$$\sigma^{\text{total}} = \frac{1}{\sqrt{\omega^2 - 1}}\, \operatorname{Im} A'(s, t = 0). \tag{6.7}$$

Now we have, from Eqs. (5.1) and (5.2),

$$A'^{\pm} = \frac{2\pi i}{p^2}\left(\frac{tp}{q}\right)^{1/2} \sum_J (2J+1)(\underline{S})^{\pm}_{+J}\, P_J(\cos\theta_3), \tag{6.8}$$

$$B^{\pm} = -\frac{8\pi i}{pq}\left(\frac{p}{q}\right)^{1/2} \sum_J (\underline{S})^{\pm}_{-J}\, P'_J(\cos\theta_3). \tag{6.9}$$

On the hypothesis that large s (i.e., $\cos\theta_3$) behavior is dominated by the Regge poles in the t channel, we shall have, as in Section 3,

$$A'^{\pm} \xrightarrow[s \rightarrow \infty]{} b_+^{\pm}(t)\left(\frac{pq}{m}\right)^{\alpha^{\pm}(t)} \left[\frac{P_{\alpha^{\pm}(t)}\left(-\frac{s+p^2+q^2}{2pq}\right) \pm P_{\alpha^{\pm}(t)}\left(\frac{s+p^2+q^2}{2pq}\right)}{\sin \pi\alpha^{\pm}(t)}\right], \tag{6.10}$$

$$B^{\pm} \xrightarrow[s \to \infty]{} b_-^{(\pm)}(t) \left(\frac{pq}{m}\right)^{a^{\pm}(t)} \left[\frac{P'_{a^{\pm}(t)}\left(\frac{s+p^2+q^2}{2pq}\right) \pm P'_{a^{\pm}(t)}\left(-\frac{s+p^2+q^2}{2pq}\right)}{\sin \pi a^{\pm}(t)} \right], \tag{6.11}$$

where

$a^+(t), a^-(t)$ = Regge poles with maximum real parts for isospin 0 and 1 channels respectively,

$$b_+^{\pm}(t) = \mathop{\mathrm{Lt}}_{J \to a^{\pm}(t)} \left[\frac{2\pi}{p^2} \left(\frac{pt}{q}\right)^{1/2} (2J+1) \left(\underline{S}\right)_{+J}^{+} \left(\frac{m}{pq}\right)^{J} (J - a^{\pm}(t)) \right], \tag{6.12}$$

$$b_-^{\pm}(t) = \mathop{\mathrm{Lt}}_{J \to a^{\pm}(t)} \left[-\frac{8\pi}{pq} \left(\frac{p}{q}\right)^{1/2} \left(\underline{S}\right)_{-J}^{\pm} \left(\frac{m}{pq}\right)^{J-1} (J - a^{\pm}(t)) \right]. \tag{6.13}$$

It should be mentioned that, in writing these equations, we have used the results on the J parity and sharing of Regge poles by different amplitudes that were established in Section V.

The expressions (6.10) and (6.11) could be further simplified to

$$A'^{\pm} \xrightarrow[s \to \infty]{} b_+^{\pm}(t) \left(\frac{s}{2m}\right)^{a^{\pm}(t)} \left[\frac{1 \pm e^{-i\pi a^{\pm}(t)}}{\sin \pi a^{\pm}(t)} \right], \tag{6.14}$$

$$B^{\pm} \xrightarrow[s \to \infty]{} a^{\pm}(t)\, b_-^{\pm}(t) \left(\frac{s}{2m}\right)^{a^{\pm}(t)-1} \left[\frac{1 \pm e^{-i\pi a^{\pm}(t)}}{\sin \pi a^{\pm}(t)} \right]. \tag{6.15}$$

Using the behaviors (6.10) in our expression for total cross sections, we get

$$\sigma^{\mathrm{total}}(\pi^- p) + \sigma^{\mathrm{total}}(\pi^+ p) \xrightarrow[s \to \infty]{} b_+^{+}(0) \left(\frac{s}{2m}\right)^{a^+(0)-1}, \tag{6.16}$$

$$\sigma^{\mathrm{total}}(\pi^- p) - \sigma^{\mathrm{total}}(\pi^+ p) \xrightarrow[s \to \infty]{} b_+^{-}(0) \left(\frac{s}{2m}\right)^{a^-(0)-1}. \tag{6.17}$$

Now, if constancy and equality of the $\pi^+ p$ and $\pi^- p$ cross sections are to be achieved in this picture, then we must have

$$a^+(0) = 1,$$
$$a^-(0) < 1. \tag{6.18}$$

Thus there must be a trajectory with zero baryon number, even G parity, even J parity, even space parity, and zero isospin--in short, the quantum numbers of the vacuum, which must pass through 1 at $t = 0$. This is the Pomeranchuk trajectory to which we alluded in the preceding section. There cannot be any trajectory that passes through a point $J > 1$, $t = 0$; otherwise we would have a cross section increasing like a power of energy, which is certainly not allowed by Mandelstam representation, as we shall show later in Section VII. Also, for isospin one, we expect the ρ-Regge trajectory to be the same as $\alpha^-(t)$. Now Rt $\alpha^-(t) = 1$ at $t \approx 29\, m_\pi^2$; hence at $t \approx 0$, we would automatically have $\alpha^-(0) < 1$. A preliminary analysis of experimental cross sections using (6.16) and (6.17) gives $\alpha^-(0) \approx 0.3$, and within experimental accuracy the observed cross sections can be reproduced by the formulas (6.16) and (6.17).[10]

Thus at high energies the π^+-p and π^--p scattering are both dominated by the Pomeranchuk Regge pole, and we have

$$\frac{d\sigma}{dt}(\pi^\pm p \to \pi^\pm p) \underset{s\to\infty}{\longrightarrow} \frac{1}{16\pi}\left(\frac{s}{2m}\right)^{2(\alpha^+(t)-1)}$$

$$\left\{ |b_+^+(t)|^2 - \frac{t}{4m^2}\left(|b_+^+(t)|^2 + |\alpha^+(t) b_-^+(t)|^2 \right)\right\} \times \left| \frac{1 + e^{-i\pi\alpha^+(t)}}{\sin \pi\, \alpha^+(t)} \right|^2 . \tag{6.19}$$

For charge exchange, $\pi^- + p \to \pi^0 + n$, however, the Pomeranchuk Regge pole cannot contribute, as one has to exchange charge in the crossed channel, and the Pomeranchuk trajectory has zero isospin. Charge exchange is a pure $I = 1$ process when looked at from t channel. Thus the process is dominated by the ρ-Regge pole, and we have

$$\frac{d\sigma}{dt}(\pi^- p \to \pi^0 n) \underset{s\to\infty}{\longrightarrow} \frac{1}{8\pi}\left(\frac{s}{2m}\right)^{2(\alpha^-(t)-1)}$$

$$\left\{ |b_+^-(t)|^2 - \frac{t}{4m^2}\left(|b_+^-(t)|^2 + |\alpha^-(t)\, b_-^-(t)|^2 \right)\right\} \times \left| \frac{1 - e^{-i\pi\alpha^-(t)}}{\sin \pi\, \alpha^-(t)} \right|^2 . \tag{6.20}$$

Using these two expressions, (6.19) and (6.20), it should be experimentally possible to determine the Pomeranchuk and ρ trajectories for negative values of t, if such a description applies. A significant feature of the Regge pole hypothesis is the logarithmic shrinkage of the width of the diffraction peak with increasing energy.

High-Energy Backward Scattering

This would be controlled by Regge poles in the u channel; i.e., the crossed pion-nucleon channel. One knows from the analysis in Section IV the general features of J-plane analyticity for the pion-nucleon scattering.

We have

$$\frac{d\sigma}{d\Omega}(\pi^{\pm}p \to \pi^{\pm}p) = \left|f_1^{(+)} \mp f_1^{(-)} - (f_2^{(+)} \mp f_2^{(-)})\right|^2 - \frac{1}{k^2}\left(u - \frac{(m^2-1)^2}{s}\right) \times$$

$$\mathrm{Re}\,(f_1^{(+)} \mp f_1^{(-)})^* (f_2^{(+)} \mp f_2^{(-)}). \tag{6.21}$$

when crossing symmetry is used, this is

$$\frac{d\sigma}{d\Omega}(\pi^{\pm}p \to \pi^{\pm}p) = \left|f_1^{(+)c} \pm f_1^{(-)c} - (f_2^{(+)c} \pm f_2^{(-)c})\right|^2$$

$$- \frac{(u - (m^2-1)^2/s)}{k^2}\,\mathrm{Re}(f_1^{(+)c} \pm f_1^{(-)c})^* (f_2^{(+)c} \pm f_2^{(-)c}), \tag{6.22}$$

since we have

$$f_{1,2}^{(+)c}(u,s,t) \pm f_{1,2}^{(-)c}(u,s,t) = f_{1,2}^{(+)}(s,u,t) \mp f_{1,2}^{(-)}(s,u,t). \tag{6.23}$$

Now, on the basis of formulas given in Section IV and the tentative Regge trajectories corresponding to N, $N^*_{1/2}$, $N^*_{3/2}$, which have the largest real parts, we can write, for large-momentum-transfer s-behavior,

$$f_1^{(1/2)}(u,s,t) = f_1^{(+)c} + 2f_1^{(-)c}$$

$$\xrightarrow[s \to \infty]{} \beta_N(u)\cos^{-1}\pi\alpha_N(u)\left[P'_{\alpha_N(u)-1/2}(z_c) - P'_{\alpha_N(u)-1/2}(-z_c)\right]$$

$$+ \beta_{N^*_{1/2}}(u)\sec\pi\alpha_{N^*_{1/2}}(u)\left[P'_{\alpha_{N^*_{1/2}}(u)-1/2}(z_c) + P'_{\alpha_{N^*_{1/2}}(u)-1/2}(-z_c)\right], \tag{6.24}$$

$$f_1^{(3/2)}(u, s, t) = f_1^{(+)c} - f_1^{(-)c} \xrightarrow[s \to \infty]{} \left[P'_{\alpha_{33}+\frac{1}{2}}(z_c) - P'_{\alpha_{33}+\frac{1}{2}}(-z_c) \right]$$

$$\times \beta_{33}(u) \operatorname{Sec} \pi \alpha_{33}(u), \tag{6.25}$$

and similar expressions for $f_2^{(1/2)}$, $f_2^{(3/2)}$. Here

$$z_c = -(s - m^2 - 1 + 2 E_u (W_u - E_u) / 2q_u^2,$$

q_u, E_u = three momenta of the pion and the energy of the nucleon (c.m.) in the u channel,

$W_u^2 = u.$

Thus we would have

$$\frac{d\sigma}{d\Omega}(\pi^+ p \to \pi^+ p) \approx \beta(u)\, s^{2(\alpha_>(u) - 1/2)}, \tag{6.26}$$

where $\alpha_>(u)$ is that Regge pole out of $\alpha_N(u)$, $\alpha_{N^*_{1/2}}(u)$, $\alpha_{33}(u)$ which has the largest real part. The power of s, $2(\alpha_>(u) - 1/2)$, comes from f_2 for the N, $N^*_{1/2}$ trajectory and from f_1 for the 3,3 trajectory. Looking at the tentative trajectories shown in Section IV, one expects

$$\alpha_>(u) \approx 0 \quad \text{at } u = 0, \tag{6.27}$$

i.e., we have roughly

$$\frac{d\sigma}{d\Omega}(\pi^+ p \to \pi^+ p) \sim \frac{\beta(u)}{s}. \tag{6.28}$$

On the other hand, if the nucleon were elementary--i.e., had definite spin J = 1/2-- we would get

$$\frac{d\sigma}{d\Omega}(\pi^+ p \to \pi^+ p) \sim \text{const.} \tag{6.29}$$

Thus the experiment can distinguish in principle the two possibilities for the nucleon. Similarly

$$\frac{d\sigma}{d\Omega}(\pi^- p \to \pi^- p) \sim \beta'(u)\, s^{2(\alpha_{33}(u) - 1/2)}. \tag{6.30}$$

VII. FROISSART-TYPE UPPER BOUNDS ON HIGH-ENERGY BEHAVIOR OF THE SCATTERING AMPLITUDES IN THE PHYSICAL REGIONS

We now come to the other part of the work, i.e., the determination of the asymptotic behavior of the scattering amplitudes. This is preliminary to any further discussion about the possible number of subtractions in the Mandelstam representation, the domain of meromorphy in the J plane of the partial-wave amplitudes, and so on.

We obtain in this section strict upper bounds on the high-energy behavior of our amplitudes. It is not implied that these bounds actually obtain in nature.

A. Channel $\pi + N \to \pi + N$

First we obtain a strict upper bound on the partial-wave amplitudes for large angular momenta, which is stronger than the unitarity bound.[21, 22] Unitarity requires

$$a^J_{J\mp\frac{1}{2},\,J\mp\frac{1}{2}} = f_{\ell\pm} = \frac{1}{k}\, e^{i\delta_{\ell\pm}} \sin\delta_{\ell\pm}, \tag{7.1}$$

where $\delta_{\ell\pm}$ does have a negative imaginary part above the inelastic threshold and is real in the elastic region. This gives the unitarity bound

$$|f_{\ell\pm}| < \frac{1}{k}. \tag{7.2}$$

To obtain a stronger bound, we use analyticity of the scattering amplitude in the Lehman Ellipse, given by Mandelstam representation, in the $\cos\theta = z$ plane. We have

$$f_{\ell\pm} = a^J_{J\mp\frac{1}{2},\,J\mp\frac{1}{2}} = \frac{1}{2}\int_{-1}^{+1} dz\left[f_1(s,z)P_{J\mp\frac{1}{2}}(z) + f_2(s,z)P_{J\pm\frac{1}{2}}(z)\right]. \tag{7.3}$$

Also, from Cauchy's theorem,

$$f_{1,2}(s,z) = \frac{1}{2\pi i}\oint_E \frac{f_{1,2}(s,z')dz'}{z'-z} \tag{7.4}$$

where E is any ellipse within the Lehman ellipse (major axis = 2a).

Using (7.4) in (7.3), we get

$$f_{\ell\pm} = \frac{1}{2\pi i} \oint_E dz' \left[f_1(s,z')\, Q_{J\mp\frac{1}{2}}(z') + f_2(s,z')\, Q_{J\pm\frac{1}{2}}(z') \right]. \tag{7.5}$$

Therefore,

$$|f_{\ell\pm}| \leq \left[\oint_E dz \right] \left\{ \|f_1(s,z)\| \cdot \|Q_{J\mp\frac{1}{2}}(z)\| + \|f_2(s,z')\| \|Q_{J\pm\frac{1}{2}}(z)\| \right\}, \tag{7.6}$$

where the $\|\cdots\|$ denotes the maximum value attained on the ellipse E.

We now have the bounds

$$\|f_{1,2}(s,z)\| < |R_{1,2}(W)|, \tag{7.7}$$

$$\|Q_n(z)\| < \left(\frac{\pi}{n}\right)^{1/2} \left(1 - \frac{1}{d^2}\right)^{1/2} \frac{1}{d^{n+1}}, \tag{7.8}$$

$$\oint_E dz < 2d, \tag{7.9}$$

where $d = a + (a^2-1)^{1/2}$ and $R_{1,2}(W)$ are some polynomials in W.

Using these upper bounds (7.7) and (7.8), we get

$$|f_{\ell\pm}| \leq (\pi)^{1/2} \left(1 - \frac{1}{d^2}\right)^{1/2} \left[\frac{|R_1(W)|}{(J\mp\frac{1}{2})^{1/2}} \frac{1}{d^{J\mp\frac{1}{2}}} + \frac{|R_2(W)|}{(J\pm\frac{1}{2})^{1/2} d^{J\pm\frac{1}{2}}} \right]$$

$$\leq \frac{|R(W)|}{k\, d^{J-1/2}}, \tag{7.10}$$

where R(W) is another suitable polynomial. This bound is weaker than the unitarity bound for

$$\left(J - \frac{1}{2}\right) < \frac{\ell n\, R(W)}{\ell n\, d} = J_m - \frac{1}{2}. \tag{7.11}$$

As the semimajor axis a is given by

$$a = 1 + \frac{4\mu^2}{2k^2}, \tag{7.12}$$

the $J_m - \frac{1}{2}$ for large W is given by

$$J_m - 1/2 \approx \text{const } k\, \ell n\, W \approx \text{const } s^{1/2}\, \ell n s. \tag{7.13}$$

The bounds can now be used to find the high-energy behavior for the various amplitudes. We have

$$A^{\pm} = 4\pi \left[\frac{W+m}{E+m} f_1^{(\pm)} - \frac{W-m}{E-m} f_2^{(\pm)} \right]_{s \to \infty} \longrightarrow 8\pi \left(f_1^{(\pm)} - f_2^{(\pm)}\right), \tag{7.14}$$

$$B^{\pm} = 4\pi \left[\frac{1}{E+m} f_1^{(\pm)} + \frac{1}{E-m} f_2^{(\pm)} \right] \underset{s \to \infty}{\longrightarrow} \frac{8\pi}{s^{1/2}} \left(f_1^{(\pm)} + f_2^{(\pm)}\right). \tag{7.15}$$

Using (4.5), one then readily sees

$$A^{(\pm)} \underset{s \to \infty}{\longrightarrow} 8\pi \sum_{\ell} \left(f_{\ell+}^{\pm} - f_{\ell+1-}^{\pm}\right) \left(P'_{\ell+1} + P'_{\ell}\right), \tag{7.16}$$

$$B^{\pm} \underset{s \to \infty}{\longrightarrow} \frac{8\pi}{s^{1/2}} \sum_{\ell} \left(f_{\ell+}^{(\pm)} - f_{\ell+1,-}^{(\pm)}\right)\left(P'_{\ell+1} - P'_{\ell}\right). \tag{7.17}$$

We also have

$$\left|P'_{\ell}(z)\right| < \frac{1}{2}\ell\,(\ell+1), \quad \text{for } -1 < z < 1, \tag{7.18}$$

$$\left|P'_{\ell}(z)\right| = \frac{1}{2}\ell\,(\ell+1), \quad \text{for } z = \pm 1; \tag{7.19}$$

$$\underset{\ell \to \infty}{\mathrm{Lt}}\; P'_{\ell} = -\sqrt{\frac{2\ell}{\pi}}\; \frac{\cos\left((\ell+1/2)\,\theta\right)}{(\sin\theta)^{3/2}}, \quad \text{for } \epsilon < \theta < \pi - \epsilon. \tag{7.20}$$

Using (7.16) through (7.19), we obtain

$$A^{\pm}(s, \theta = 0 \underset{s \to \infty}{\longrightarrow} 8\pi \sum_{J} \left(a^{J(\pm)}_{J-\frac{1}{2}, J-\frac{1}{2}} - a^{J(\pm)}_{J+\frac{1}{2}, J+\frac{1}{2}} \right) \left(J + \frac{1}{2}\right)^2$$

$$\leqslant \frac{16\pi}{k} \sum_{J=1/2}^{J=J_m} \left(J+\frac{1}{2}\right)^2 + \frac{16\pi}{k} \sum_{J=J_m+1}^{\infty} \left(J+\frac{1}{2}\right)^2 \frac{|R(W)|}{d^{J-1/2}}$$

$$\approx s\,(\ell n\, s)^3. \tag{7.21}$$

Similarly

$$B^{\pm}(s, \theta = 0) \leqslant (\ell n\, s)^2 \tag{7.22}$$

These behaviors allow the total cross sections to increase as $\ln^3 s$. Using (7.16) through (7.20), we similarly obtain

$$A^{\pm}(s, \theta = \pi) = 0(s^{1/2} \ln^2 s), \tag{7.23}$$

$$B^{\pm}(s, \theta = \pi) = 0(s^{1/2} \ln^2 s). \tag{7.24}$$

and in the nonforward, nonbackward directions ($\pi - \epsilon > \theta > \epsilon$)

$$A^{\pm}(s,\theta) = 0(s^{1/4} \ln^{3/2} s), \tag{7.25}$$

$$B^{\pm}(s,\theta) = 0(s^{1/4} \ln^{3/2} s). \tag{7.26}$$

Let us mention here that if the domain of meromorphy in the J plane of partial-wave amplitudes in the t channel extended up to Re J = $-\epsilon$, the logarithm factors in these upper bounds would be absent, as s is the momentum transfer squared for the crossed channels, and the asymptotic behavior in the Regge picture is a pure power behavior.

B. The Channel $\pi + \pi \rightarrow N + \overline{N}$

A similar analysis can be carried out for this channel. We have the partial-wave expansions given by (5.1) and (5.2). The unitarity requirement, similar to (7.1), is

$$|S^{\pm}_{\pm J}| \leqslant 1. \tag{7.27}$$

We obtain, finally,

$$A^{\pm}(t, \cos\theta_3 = \pm 1) = 0(t^{1/2} \ln^3 t), \tag{7.28}$$

$$B^{\pm}(t, \cos\theta_3 = \pm 1) = 0(t^{1/2} \ln^3 t), \tag{7.29}$$

and for nonforward, nonbackward directions--i.e. ($\epsilon < \theta_3 < \pi - \epsilon$)--

$$A^{\pm}(t, \cos\theta_3) = 0(t^{1/4} \ln^{3/2} t), \tag{7.30}$$

$$B^{\pm}(t, \cos\theta_3) = 0(t^{-1/4} \ln^{3/2} t). \tag{7.31}$$

VIII. FROISSART ANALYSIS

We shall now prove that the upper bounds obtained in Section VII for different amplitudes are inconsistent with any arbitrariness in the partial waves $J > 1/2$ in the s channel and $J > 1$ in the t channel. In this, we shall be following Froissart,[21] who showed that subtractions corresponding to arbitrarily high angular momentum waves in one channel are not consistent with the unitarity requirements in the crossed channels.

Consider the amplitude $A^+(s,u,t)$ for the sake of definiteness. Let us denote the two different unitarity-satisfying amplitudes corresponding to $A^+(s,u,t)$ by $A'(s,u,t)$ and $A''(s,u,t)$, which differ <u>only</u> because of having different subtraction terms. Their difference $\Delta A^+(s,u,t)$ also has the same unitarity limitations and shall have the following general subtracted expression (assuming that the asymtotic behavior is at most like a polynomial, i.e., the finite number of subtractions):

$$\Delta A^+(s,u,t) = \sum_{p=0}^{M} \frac{t^p s^M}{\pi} \int \frac{\Delta\rho_p^{(s)}(s')\,ds'}{s'^M(s'-s)} + \sum_{p=0}^{M} \frac{u^p t^M}{\pi} \int \frac{\Delta\rho_p^{(t)}(t')\,dt'}{t'^M(t'-t)}$$

$$+ \sum_{p=0}^{M} \frac{s^p u^M}{\pi} \int \frac{\Delta\rho_t^{(u)}(u')\,du'}{u'^M(u'-u)} + \sum_{p,q=0}^{L} \Delta\rho_{p,q}\, s^p t^q, \tag{8.1}$$

where M, L are sufficiently large positive integers.

In the physical region of the s channel, we have

$$\mathrm{Im}\,\Delta A^+(s,t) = \sum_{p=0}^{M} t^p\, \Delta\rho_p^{(s)}(s). \tag{8.2}$$

Letting the scattering angle θ assume (M+1) values $\theta_1, \theta_2, \cdots, \theta_{M+1}$ ($= 0, \pi$) in the physical region in (8.2), we get (M+1) equations

$$\mathrm{Im}\,\Delta A^+(s,\theta_i) = \sum_{p=0}^{M} (\sin(\theta_i/2))^{2p}(-)^p (s^p\, \Delta\rho_p^{(s)}(s), \tag{8.3}$$

where we have used $t \approx -s \sin^2 \theta/2$ for large s. These (M+1) linear equations can be solved for (M+1) quantities $s^p \Delta\rho_p^{(s)}(s)$, $(p = 0, 1, \cdots, M)$. The solution obviously implies that the asymptotic behavior of $s^p \Delta\rho_p^{(s)}(s)$

is at most like that of Im $\Delta A^{+}(s, \theta_i)$. Thus, using (8.3), we have

$$\Delta\rho_p^{(s)}(s) = 0(s^{1/4-p}\ell n^{3/2} s). \tag{8.4}$$

Similarly, using the analog of (7.25) in the crossed π-N channel, and (7.30), we get

$$\Delta\rho_p^{(u)}(u) = 0(u^{1/4-p}\ell n^{3/2} s), \tag{8.5}$$

$$\Delta\rho_p^{(t)}(t) = 0(t^{1/4-p}\ell n^{3/2} s). \tag{8.6}$$

With the asymptotic behaviors (8.4) through (8.6) in mind, we can rewrite ΔA^{+} as

$$\Delta A^{+} = \frac{1}{\pi}\sum_{p=0}^{M}\frac{t^p}{s^{p-1}}\int\frac{\Delta\rho_p^{(s)}(s')\,s'^{p-1}ds'}{s'-s} + \frac{1}{\pi}\sum_{p=0}^{M}\frac{u^p}{t^{p-1}}\int\frac{\Delta\rho_p^{(t)}(t')t'^{p-1}dt'}{t'-t}$$

$$+\frac{1}{\pi}\sum_{p=0}^{M}\frac{s^p}{u^{p-1}}\int\frac{\Delta\rho_p^{(u)}(u')\,u'^{p-1}\,du'}{u'-u} + p(s,u,t), \tag{8.7}$$

where

$$p(s,u,t) = \sum_{p,}^{L}\sum_{q}\Delta\rho_{p,q}\,s^p t^q - \sum_{p=0}^{M}\sum_{q=0}^{M+2-p}\Delta\rho_{p,q}^{(1)}\,t^p\,s^{q+1-p}$$

$$-\sum_{p=0}^{M}\sum_{q=0}^{M+2-p}\Delta\rho_{p,q}^{(2)}\,u^p t^{q+1-p} - \sum_{p=0}^{M}\sum_{q=0}^{M+2-p}\Delta\rho_{p,q}^{(3)}\,s^p u^{q+1-p}, \tag{8.8}$$

and we define

$$\Delta\rho_{p,q}^{(1)} = \frac{1}{\pi}\int\frac{\Delta\rho_p^{(s)}(s')\,ds'}{s'^{q-p+2}}, \text{ etc.} \tag{8.9}$$

In writing (8.7) through (8.9) we have used the identity

$$\left(\frac{x}{y}\right)^M\frac{1}{y-x} \equiv \left(\frac{y}{x}\right)^{p-1}\left[\frac{1}{y-x} - \frac{1}{y} - \frac{x}{y^2} - \cdots - \frac{x^{M+2-p}}{y^{M+3-p}}\right].$$

Now the asymptotic behavior, $s \to +\infty$, in the s physical region ($\epsilon < \theta < \pi - \theta$) of the integral terms on the right-hand side in (8.7) is like $s^{1/4}\ell n^{3/2} s$, using (8.4) through (8.6) and noting that $t \approx s$, $u \approx s$ for large s and $\epsilon < \theta < \pi - \theta$. Thus these terms are unitarity-abiding in the physical region of the s channel. However, the polynomial p(s, u, t) contains arbitrarily high integral powers of s, and would thus conflict with unitarity in the s channel, if this polynomial were not a constant. Hence we can only have

$$\Delta A^+ = \frac{1}{\pi}\sum_{p=0}^{M}\frac{t^p}{s^{p-1}}\int\frac{\Delta\rho_p^{(s)}(s')\,ds'\,s'^{p-1}}{s'-s} + \sum_{p=0}^{M}\frac{u^p}{\pi t^{p-1}}\int\frac{\Delta\rho_p^{(t)}(t')t'^{p-1}\,dt'}{t'-t}$$

$$+\frac{1}{\pi}\sum_{p=0}^{M}\frac{s^p}{u^{p-1}}\int\frac{\Delta\rho_p^{(u)}(u')\,du'\,u'^{p-1}}{u'-u} + \Delta\rho_{00}. \tag{8.10}$$

Now let s be held fixed at some finite negative value, and let $t \to +\infty$. This means that the angle of scattering in the t channel, i.e., θ_3, goes to zero. The asymptotic behavior of the terms on the right-hand side of (6.10) are respectively (keeping s fixed), t^p, $t^{1/4}\ell n^{3/2}t$, $t^{1/4-p}\ell n^{3/2}t$, t^0. Thus we must have

$$\int\frac{\Delta\rho_p^{(s)}(s')\,s'^{p-1}\,ds'}{s'-s} = 0 \text{ for } p \geqslant 1, \tag{8.11}$$

if we are to respect the $t^{1/2}\ell n^3 t$ behavior of A^+ given by (7.28).

Similarly, holding u fixed at some negative finite value and letting $t \to +\infty$ (i.e., backward direction in the t channel), we get

$$\int\frac{\Delta\rho_p^{(u)}(u')\,u'^{p-1}\,du'}{u'-u} = 0 \text{ for } p \geqslant 1, \tag{8.12}$$

and holding t fixed at some negative vlaue and letting s or u go to $+\infty$ (i.e. forward direction), we get, using (7.20),

$$\int\frac{\Delta\rho_p^{(t)}(t')\,t'^{p-1}\,dt'}{t'-t} = 0 \text{ for } p \geqslant 2. \tag{8.13}$$

We now have

$$\Delta A^+ = \frac{s}{\pi}\int\frac{\Delta\rho_o^{(s)}(s')ds'}{s'(s'-s)} + \frac{u}{\pi}\int\frac{\Delta\rho_o^{(u)}(u')du'}{u'(u'-u)} + \frac{t}{\pi}\int\frac{\Delta\rho_o^{(t)}(t')\,dt'}{t'(t'-t)}$$

$$+\frac{u}{\pi}\int\frac{\Delta\rho_1^{(t)}(t')\,dt'}{t'-t} + \Delta\rho_{00}. \tag{8.14}$$

We have not yet used the crossing symmetry requirement for A^+, which demands

$$\Delta\rho_o^{(s)}(x) = \Delta\rho_o^{(u)}(x) = \Delta\rho_o(x)$$

and

$$\Delta\rho_1^{(t)}(x) = 0. \tag{8.15}$$

Thus, finally, we have

$$\Delta A^+ = \frac{s}{\pi}\int\frac{\Delta\rho_o(s')ds'}{s'(s'-s)} + \frac{u}{\pi}\int\frac{\Delta\rho_o(u')du'}{u'(u'-u)} + \frac{t}{\pi}\int\frac{\Delta\rho_o^{(t)}(t')dt'}{t'(t'-t)} + \Delta\rho_{00}. \tag{8.16}$$

The last expression tells us that the only independent subtractions in A^+ are $J = \frac{1}{2}$ wave subtractions in s and u channels and $J = 0$ wave subtraction in the t channel. A similar analysis for the other three amplitudes A^-, $B^\pm$ leads to the independent subtractions in only $J = \frac{1}{2}$ waves for s, u channels and $J = 0$, 1 for t channel. One might note that if no logarithm factors are present in the asymptotic upper bounds, then the result for arbitrary subtractions can be strengthened to only $J = 0$ amplitudes in the t channel.

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APPENDIX

I. Derivation of the Strip R_2 Double-Spectral Functions

The expressions for the strip R_2 dsf is now calculated by using generalized unitarity, in which the contribution of only the lowest-mass two-particle state, i.e., two-pion state, is retained.

The S-matrix element for pion-pion scattering, which we shall need for the application of generalized unitarity in this channel, is taken as follows:

$$\left\langle q_1' q_2' \middle| S \middle| q_1 q_2 \right\rangle = \left\langle q_1' q_2' \middle| q_1 q_2 \right\rangle + \frac{i(2\pi)^4 \delta^4(q_1'+q_2'-q_1-q_2)}{(16\, q_{10}' q_{20}' q_{10} q_{20})^{1/2}} \left\langle q_1' q_2' \middle| 16\pi \mathcal{A} \middle| q_1 q_2 \right\rangle . \tag{A1.1}$$

With this normalization, which agrees with that of Chew and Mandelstam,[2,15] $\mathcal{A}$ has the partial-wave expansion

$$\mathcal{A} = \frac{(q^2+\mu^2)^{1/2}}{q} \sum_\ell (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell\, P_\ell(\hat{q}_1 \cdot \hat{q}_1'). \tag{A1.2}$$

The generalized unitarity condition with only 2π intermediate states then gives

$$\mathrm{Im}\, A(t,\zeta) = \frac{q}{2\pi W_t} \int d\Omega' \left[A^*(t,\zeta'') - \frac{mq}{p} \frac{\zeta''-\zeta\zeta'}{1-\zeta^2} B^*(t,\zeta'')\right] \mathcal{A}(t,\zeta') \tag{A1.3}$$

and

$$\mathrm{Im}\, B(t,\zeta) = \frac{q}{2\pi W_t} \int d\Omega' \left[\frac{\zeta'-\zeta\zeta''}{1-\zeta^2} B^*(t,\zeta'') \mathcal{A}(t,\zeta')\right], \tag{A1.4}$$

where (see Fig. 9)

$$\zeta = (\hat{q}_1 \cdot \hat{p}_1), \quad \zeta' = (\hat{q}_1 \cdot \hat{q}_1'), \quad \zeta'' = (\hat{q}_1' \cdot \hat{p}_1),$$

and

$$d^3 q_1' = q_1'^2\, dq_1'\, d\Omega'.$$

These equations (A1.3) through (A1.4) hold separately and lead to the following expressions for the strip functions $\alpha_2^\pm$, $\beta_2^\pm$:

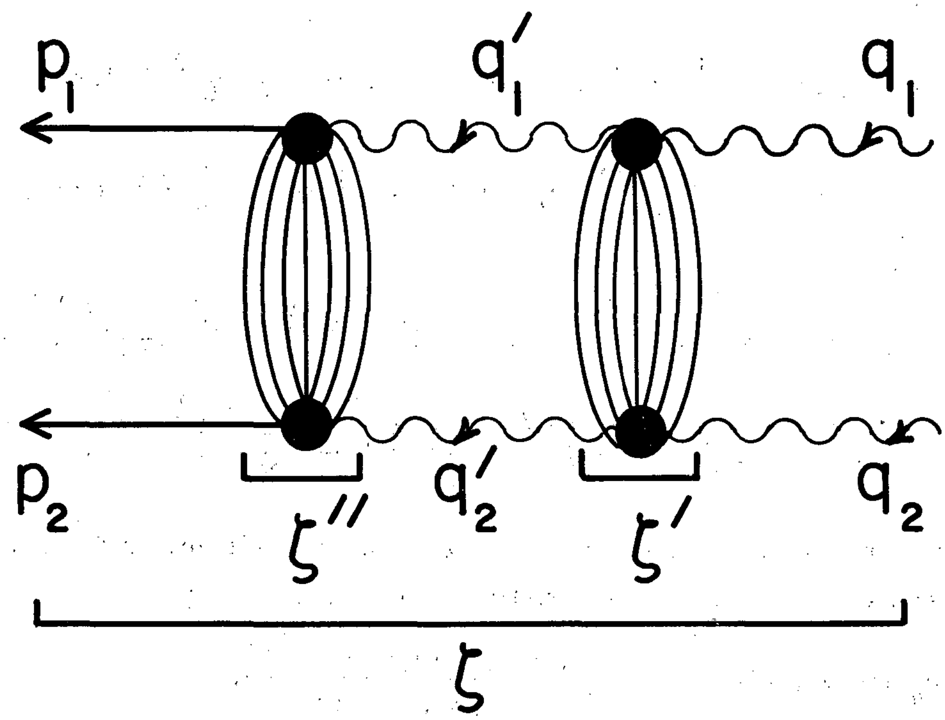

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Fig. 9. The two-particle intermediate state for the $\pi+\pi \rightarrow N+\overline{N}$ channel.

$$\alpha_2^{\pm}(t, s) = \frac{1}{\pi p q^2 W_t} \iint ds' ds'' K_t(t, s, s', s'') \left[A_s^{*(\pm)}(t, s'') - n_\alpha(t; s, s', s'') \times B_s^{*(\pm)}(t, s'') \right] \mathcal{A}_s^{(\pm)}(t, s') \tag{A1.5}$$

and

$$\beta_2^{\pm}(t, s) = \frac{1}{\pi p q^2 W_t} \int ds' \int ds''\, K_t(t, s, s', s'') n_\beta(t; s, s', s'') B_s^{*(\pm)}(t, s'') \times \mathcal{A}_s^{(\pm)}(t, s'), \tag{A1.6}$$

where

$$\mathcal{A}_s^{(+)}(t, s') = \tfrac{1}{3}\mathcal{A}^{(0)}(s, t) + \mathcal{A}^{(1)}(s', t) + \tfrac{5}{3}\mathcal{A}^{(2)}(s', t), \tag{A1.7}$$

$$\mathcal{A}_s^{(-)}(t, s') = \tfrac{1}{3}\mathcal{A}^{(0)}(s', t) + \tfrac{1}{2}\mathcal{A}^{(1)}(s', t) - \tfrac{5}{6}\mathcal{A}^{(2)}(s', t), \tag{A1.8}$$

and

$$K_t(t; x, y, z) = \left\{ \left(\frac{x+p^2+q^2}{2pq} \right) + \left(1 + \frac{y}{2q^2}\right)^2 + \left(\frac{z+p^2+q^2}{2pq}\right)^2 - 1 - 2\left(\frac{x+p^2+q^2}{2pq}\right)\left(1 + \frac{y}{2q^2}\right)\left(\frac{z+p^2+q^2}{2pq}\right) \right\}^{-1/2}, \tag{A1.9}$$

if the quantity under the square root is positive, and zero otherwise and n_α, n_β are the kinematical factors given by

$$n_\alpha(t; s, s', s'') = \frac{m\left[2q^2(s''-s) - s'(s+p^2+q^2)\right]}{4p^2q^2 - (s+p^2+q^2)^2}, \tag{A1.10}$$

$$n_\beta(t; s, s', s'') = \frac{4p^2q^2\left[1+s'/2q^2\right] - \left[ss'' + (p^2+q^2)(s+s'') + (p^2+q^2)^2\right]}{4p^2q^2 - (s+p^2+q^2)^2}. \tag{A1.11}$$

We have also used the crossing symmetry in writing down the expressions (A1.5) and (A1.6).

II. Partial-Wave Analysis of Spin-Zero and Spin-One-Half Particle Scattering with Parity Nonconservation

We shall carry out the decomposition into partial waves of invariant amplitude for spin zero and spin one-half particle scattering for the case in which parity is not conserved.

We write the S-matrix element for scattering of an initial pion of four-momenta k_1, and of an initial nucleon with four-momenta K_1 and spin state r_1 to a final pion with four-momenta k_2 and a final nucleon with four-momenta K_2 and spin state r_2, in terms of invariant amplitudes A, B, C, D as follows:

$$\left\langle K_2, r_2, k_2 \left| S \right| K_1, r_1, k_1 \right\rangle = \left\langle K_2, r_2, k_2 \left| K_1, r_1, k_1 \right.\right\rangle - i(2\pi)^4 \delta^4(K_1+k_1-K_2-k_2)$$

$$\times \sqrt{\frac{m^2}{4K_{10}K_{20}k_{10}k_{20}}}\; T(K_2, r_2 k_2; K_1, r_1, k_1), \tag{A2.1}$$

with

$$T(K_2, r_2, k_2; K_1 r_1, k_1) = \bar{u}_{r_2}(\vec{K}_2)\,[-A + i\gamma \cdot QB + i\gamma_5 \gamma \cdot QC - D\gamma_5]\,u_{r_1}(\vec{K}_1) \tag{A2.2}$$

and

$$Q = \frac{k_1+k_2}{2}.$$

Here u, $\bar{u}$ are appropriate Dirac spinors.

We use Pauli-Dirac representation of the γ matrices, in which we have

$$i\gamma \cdot Q = \begin{pmatrix} -Q_o & \vec{\sigma} \cdot \vec{Q} \\ -\vec{\sigma} \cdot \vec{Q} & +Q_o \end{pmatrix},$$

$$\gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{A2.3}$$

and Dirac spinors take the form

$$u_r(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E+m \\ \sigma \cdot \vec{p} \end{pmatrix} \chi_r, \text{ with } E = +\sqrt{p^2+m^2},$$

$$\bar{u}_s(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}} \chi_s^* \begin{pmatrix} E+m \\ -\vec{\sigma}\cdot\vec{p} \end{pmatrix}, \qquad (A2.4)$$

and $\chi_{r,\,s}$ are Pauli spinors.

We work in the center-of-mass system in which

$$\vec{k}_1 + \vec{k}_1 = \vec{k}_2 + \vec{k}_2 = 0, \qquad (A2.5)$$

and define

$$\vec{k}_1 = -\vec{K}_1 = \hat{k}_i k,$$

$$\vec{k}_2 = -\vec{K}_2 = \hat{k}_f k,$$

$$E = +\sqrt{k^2 + m^2},$$

$$W = s^{1/2} = +\sqrt{k^2+m^2} + \sqrt{k^2 + 1}. \qquad (A2.6)$$

We have

$$Q_o = W - E,$$

$$\vec{Q} = \frac{1}{2} k(\hat{k}_i + \hat{k}_f). \qquad (A2.7)$$

Using (A2.3) through (A2.7), we can reduce T to the form

$$T = -\frac{4\pi W}{m} \chi_{r_2}^* f \chi_{r_1}, \qquad (A2.8)$$

where

$$f = f_1 + f_2 \sigma\cdot\hat{k}_f \sigma\cdot\hat{k}_i + f_3 \sigma\cdot(\hat{k}_f + \hat{k}_i) + f_4 \sigma\cdot(\hat{k}_f - \hat{k}_i) \qquad (A2.9)$$

and

$$f_1 = (8\pi W)^{-1} [A + (W-m)B] (E+m), \qquad (A2.10)$$

$$f_2 = (8\pi W)^{-1} (E-m) \quad [-A + (W+m)B], \qquad (A2.11)$$

$$f_3 = -kC/8\pi, \qquad (A2.12)$$

$$f_4 = -kD/8\pi W. \qquad (A2.13)$$

It is appropriate before going further to point out that, if time-reversal invariance alone did hold, then we would have

$$D = 0,$$

$$f_4 = 0.$$

This is easy to see, since under time-reversal operation,

$$\sigma \to \sigma' = -\sigma,$$
$$\vec{k}_f \to \vec{k}'_f = -\vec{k}_i,$$
$$\vec{k}_i \to \vec{k}_i' = -\vec{k}_f,$$

and therefore

$$f \to f' = f_1 + f_2\, \sigma\cdot \hat{k}_f\, \sigma \cdot \hat{k}_i + f_3 \sigma\cdot (\hat{k}_f + \hat{k}_i) - f_4 \sigma\cdot (\hat{k}_f - \hat{k}_i),$$

and thus we must have $f_4 = 0$.

On the other hand, parity conservation alone would give

$$f_3 = f_4 = 0,$$
$$C = D = 0,$$

since, under space reflection,

$$\sigma \to +\sigma,$$
$$k_i \to -k_i,$$
$$k_f \to -k_f.$$

We now proceed to angular-momentum decomposition of the f's. First we note that we have for the differential cross section the expression

$$\frac{d\sigma}{d\Omega} = \Sigma\, |f|^2. \tag{A2.14}$$

Therefore, if the incident wave ψ_{inc} is

$$\psi_{inc} = e^{i\vec{k}_i \cdot \vec{r}}\, \chi_r, \tag{A2.15}$$

the scattered wave is given by

$$\psi_{scatt} \underset{r \to \infty}{\approx} f\, \frac{e^{ikr}}{r}\, \chi_r. \tag{A2.16}$$

It is convenient to choose the incident wave vector $\hat{k}_i$ along the positive z axis, i.e.,

$$\hat{k}_i = (0, 0, 1).$$

We further define θ, ϕ through

$$\hat{k}_f = (\sin\theta \cos\phi,\ \sin\theta \sin\phi,\ \cos\theta).$$

Now let

$$\psi_{inc} = e^{i\hat{k}_i \cdot r} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$r \underset{\to \infty}{\approx} \sum_{\ell=0}^{\infty} (2\ell+1) \left(\frac{e^{ikr} - e^{-ikr}}{2\,ikr} \right) P_\ell(\hat{k}_i \cdot \hat{r}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$r \underset{\to \infty}{\approx} \sum_{\ell=0}^{\infty} \left(\frac{e^{ikr} - e^{ikr}}{2\,ikr} \right) \sqrt{4\pi} \left[\sqrt{\ell+1} \left| \ell + \frac{1}{2}, \frac{1}{2} \right\rangle - \sqrt{\ell} \left| \ell - \frac{1}{2}, -\frac{1}{2} \right\rangle \right], \qquad \text{(A2.17)}$$

where $|J, M>$ is the state total angular momentum J, and J_z=M. We use the angular momentum conventions of Rose.[23] We have thus re-expressed the incident wave in total angular momentum representation. This is useful, since angular momentum is conserved. As a result of interaction the outgoing part of the incident wave shall be modified as to its amplitude, which is unity.

Consider $\left|\ell + \frac{1}{2}, \frac{1}{2}\right\rangle$ in (A2.17). We have

$$\left|\ell + \frac{1}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{\ell+1}{2\ell+1}}\, Y_\ell^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{\ell}{2\ell+1}}\, Y_\ell^1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ with parity } (-)^{\ell+1},$$

$$= -\sqrt{\frac{\ell+1}{2\ell+3}}\, Y_{\ell+1}^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{\ell+2}{2\ell+3}}\, Y_{\ell+1}^1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ with parity } (-)^{\ell+2}.$$

Therefore, the part

$$\frac{e^{ikr}}{2ikr} \sqrt{4\pi} \quad \Sigma \sqrt{\ell+1} \left| \ell + \frac{1}{2}, \frac{1}{2} \right\rangle$$

of the incident wave would give rise to the following part in the scattered wave:

$$\frac{e^{ikr}}{2\,ikr} \sqrt{4\pi} \sum_\ell \sqrt{\ell+1} \left\{ a_{\ell,\ell}^{\ell+1/2} \left(\sqrt{\frac{\ell+1}{2\ell+1}}\, Y_\ell^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{\ell}{2\ell+1}}\, Y_\ell^1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right.$$
$$\left. + a_{\ell+1,\ell}^{\ell+1/2} \left(-\sqrt{\frac{\ell+1}{2\ell+3}}\, Y_{\ell+1}^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{\ell+2}{2\ell+3}}\, Y_{\ell+1}^1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right\},$$

where $a_{L'', L'}$ deonte the amplitude for transition between two states which have total angular momentum J, and where the initial and final states have orbital angular momentum L equal to L' and L'' respectively.

Similarly, one can treat the rest of the incident wave (A2.17) to get finally

$$\psi_{scatt} \xrightarrow[r\to\infty]{} \frac{e^{ikr}}{r} \cdot \sum_{\ell} \left\{ \left\{ \left[(\ell+1)\, a^{\ell+1/2}_{\ell,\ell} + \ell\, a^{\ell-1/2}_{\ell,\ell} \right] P_\ell - (\ell+1)\, a^{\ell+1/2}_{\ell+1,\ell} P_{\ell+1} - \ell\, a^{\ell-1/2}_{\ell-1,\ell} P_{\ell-1} \right\} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left\{ \left[a^{\ell-1/2}_{\ell,\ell} - a^{\ell+1/2}_{\ell,\ell} \right] P'_\ell - a^{\ell+1/2}_{\ell+1,\ell} P'_{\ell+1} + a^{\ell-1/2}_{\ell-1,\ell} P'_{\ell-1} \right\} \sin\theta e^{i\phi} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (A2.18)$$

On the other hand, if the incident wave were

$$\psi_{inc} = e^{i\vec{k}\cdot\vec{r}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (A2.19)$$

one would get

$$\psi_{scatt} \xrightarrow[r\to\infty]{} \frac{e^{ikr}}{r} \sum_{\ell=0} \left\{ \left(\left[(\ell+1) a^{\ell+1/2}_{\ell,\ell} + \ell a^{\ell-1/2}_{\ell,\ell} \right] P_\ell + (\ell+1)\, a^{\ell+1/2}_{\ell+1,\ell} P_{\ell+1} + \ell\, a^{\ell-1/2}_{\ell-1,} P_{\ell-1} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left(\left[-a^{\ell-1/2}_{\ell,\ell} + a^{\ell+1/2}_{\ell,\ell} \right] P'_\ell - a^{\ell+1/2}_{\ell+1,\ell} P'_{\ell+1} + a^{\ell-1/2}_{\ell-1,\ell} P'_{\ell-1} \right) \sin\theta\, e^{-i\phi} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}. \quad (A2.20)$$

Using (A2.9) and (A2.15) and (A2.16) and comparing them with (A2.17) through (A2.20), we get

$$f_1 + f_2 \cos\theta = \sum_{J=1/2} \left[(J+1/2)\, a^J_{J-1/2,J-1/2} P_{J-1/2} + (J+1/2)\, a^J_{J+1/2,J+1/2} P_{J+1/2} \right],$$

$$f_2 = \sum_{J=1/2} \left[a^J_{J+1/2,J+1/2} P'_{J+1/2} - a^J_{J-1/2,J-1/2} P'_{J-1/2} \right],$$

$$f_3 + f_4 = \sum_{J=1/2} \left[a^J_{J-1/2,J+1/2} P'_{J-1/2} - a^J_{J+1/2,J-1/2} P'_{J+1/2} \right]$$

$$-(f_3+f_4)\cos\theta + (f_4 - f_3) = \sum_{J=1/2} \left[a^J_{J-1/2,J+1/2} (J+1/2) P_{J-1/2} + (J+1/2)\, a^J_{J+1/2,J-1/2} P_{J+1/2} \right].$$

Solving for f_i, one gets

$$f_1 = \sum_J (a^J_{J-1/2,J-1/2} P'_{J+1/2} - a^J_{J+1/2,J+1/2} P'_{J-1/2}), \tag{A2.21}$$

$$f_2 = \sum_J (a^J_{J+1/2,J+1/2} P'_{J+1/2} - a^J_{J-1/2,J-1/2} P'_{J-1/2}), \tag{A2.22}$$

$$f_3 = \frac{1}{2} \sum_J (a^J_{J-1/2,J+1/2} + a^J_{J+1/2,J-1/2})(P'_{J-1/2} - P'_{J+1/2}), \tag{A2.23}$$

$$f_4 = \frac{1}{2} \sum_J (a^J_{J-1/2,J+1/2} - a^J_{J+1/2,J-1/2})(P'_{J-1/2} + P'_{J+1/2}), \tag{A2.24}$$

where we have used the identities

$$(1 - \cos\theta) P'_\ell(\cos\theta) - (\ell+1) P_\ell(\cos\theta) = P'_\ell(\cos\theta) - P'_{\ell+1}(\cos\theta),$$

$$(1 + \cos\theta) P'_\ell(\cos\theta) + (\ell+1) P_\ell(\cos\theta) = P'_\ell(\cos\theta) + P'_{\ell+1}(\cos\theta),$$

$$P'_{\ell+1} - \cos\theta\, P'_\ell = (\ell+1) P_\ell.$$

The projection formulas for different $a^J_{L'',L'}$ have to be worked out now. They are known in the literature for $a^J_{L,L}$ where $L = J \pm 1/2$, and are given by

$$a^J_{J+1/2,J+1/2} = 1/2 \int_{-1}^{+1} (f_1 P_{J+1/2} + f_2 P_{J-1/2})\, dz, \tag{A2.25}$$

$$a^J_{J-1/2,J-1/2} = 1/2 \int_{-1}^{+1} (f_1 P_{J-1/2} + f_2 P_{J+1/2})\, dz. \tag{A2.26}$$

By use of the integral formula

$$1/2 \int_{-1}^{+1} (P'_\ell \mp P'_{\ell+1})(P_m \pm P_{m+1})\, dz = \mp \delta_{\ell,m},$$

they can be worked out for $a^J_{J-1/2,J+1/2}$ and $a^J_{J+1/2,J-1/2}$, and are given by

$$a^J_{J-1/2,J+1/2} + a^J_{J+1/2,J-1/2} = \int_{-1}^{+1} f_3 (P_{J-1/2} + P_{J+1/2})\, dz, \tag{A2.27}$$

$$a^J_{J-1/2,J+1/2} - a^J_{J+1/2,J-1/2} = \int_{-1}^{+1} f_4 (P_{J-1/2} - P_{J+1/2})\, dz. \tag{A2.28}$$

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