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THE MULTI-REGGE-POLE HYPOTHESIS*

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ABSTRACT

Toller's group-theoretical analysis of multiparticle scattering amplitudes, which leads naturally to a Regge-pole-type expansion, is extended to give a precise meaning to the multi-Regge pole hypothesis. An essential step in this direction is achieved by the introduction of generalized Toller variables, which are tailored to the description of the asymptotic behavior of a production process in several subenergies.

I. INTRODUCTION

The choice of appropriate variables for the description of a multiparticle amplitude has posed a long-lasting problem to both theoreticians and experimentalists in particle physics. No clear-cut choice seems to have emerged, and it is likely that there is no privileged set of variables in general. However, on the basis of a group theoretical analysis of the production amplitude, Toller[1] was able to single out a definite set of variables which is particularly convenient for the description of a multiparticle process in the limit of the total energy going to infinity while a specified momentum transfer is held fixed. In his analysis, Toller has also reformulated the Regge pole hypothesis without recourse to the analytic continuation of the crossed-channel partial-wave expansion, thus giving an unambiguous meaning to multiparticle Regge pole behavior.

In the present paper we show how the Toller analysis can be extended to the definition of variables which are appropriate to the description of the simultaneous asymptotic behavior of a production amplitude in several independent subenergies.[2] In terms of these variables the formulation of a multi-Regge-pole hypothesis follows naturally from the factorizability of Regge pole residues.

The statement of multi-Regge behavior achieved here is qualitatively similar to conjectures made by a number of authors.[3-8] Some of these earlier conjectures, however, involve vague or inconsistent statements about the complete choice of variables.

Other conjectures employ variables which, even if correct, are not the natural choices on the basis of group theory. We believe that the variables associated with the Toller approach will prove to be the most useful, even if no assumption of Regge asymptotic behavior is made.[9]

In Section II we review the variables defined by Toller for the study of the asymptotic behavior of a scattering or production process in one energy variable. In Section III these results are generalized in the case of spinless particles to the asymptotic behavior in several independent energy variables, with some restriction on the number of outgoing particles. These restrictions are removed in Section IV. Section V contains a precise formulation of the multi-Regge-pole hypothesis, and the generalization to include spin is presented in Section VI. Appendices A and B describe a way of calculating the Toller variables in terms of invariants, and the example of the one-particle production process is worked out in detail in Appendix C.

II. REVIEW OF THE TOLLER VARIABLES FOR TWO CLUSTERS

In order to define a set of variables appropriate to the description of asymptotic behavior in multiparticle production, we first decompose the corresponding amplitude into two clusters A and B, linked by a four-momentum transfer Q_{AB}, as shown in Fig. 1. We label by $p_{1A} \cdots p_{N_A+1,A}$ (or collectively p_A) the four momenta in the cluster A, and by $p_{1B} \cdots p_{N_B+1,B}$ (or p_B) those in cluster B, adopting the convention that particles in group A are outgoing if their energies are positive, with the reverse holding for group B. Thus energy momentum conservation can be written

$$\sum_{i=1}^{N_A+1} p_{iA} = \sum_{i=1}^{N_B+1} p_{iB} = Q_{AB} . \tag{II.1}$$

In particular we are interested in processes where two particles are incoming and $N_A + N_B$ outgoing, so we choose

$$E_{1A} < 0 \quad ; \quad E_{iA} > 0,\ i = 2 \cdots N_A + 1$$

$$E_{1B} > 0 \quad ; \quad E_{iB} < 0,\ i = 2 \cdots N_B + 1 .$$

A Lorentz invariant connected part with $N_A + N_B + 2$ lines depends on $3(N_A + N_B + 2) - 10 = 3(N_A + N_B) - 4$ independent variables. We propose to subdivide these variables into four groups:

(1) The invariant momentum transfer $t_{AB} = Q_{AB}^2$.

(2) A group of $3(N_A + 2) - 10 = 3N_A - 4$ "internal" variables[10] for cluster A, to be designated collectively as V_A.

(3) A group of $3N_B - 4$ internal variables for cluster B, to be designated as V_B.

(4) A set of three variables--two rotation angles and one boost, designated collectively as g^{ab}--which specify those Lorentz transformations of cluster B <u>relative</u> to cluster A (or vice versa) that keep fixed the momentum components "parallel" to Q_{AB}.

The precise meaning of the variables V_A, V_B, and g^{ab} will be elucidated below but the reader may verify immediately that the sum of all four sets is $1 + (3N_A - 4) + (3N_B - 4) + 3 = 3(N_A + N_B) - 4$, the required total number of variables. In the following sections, therefore, when decomposed into two clusters the amplitude will be designated by $f(V_A, g^{ab}, t_{AB}, V_B)$.

We shall be interested mainly in space-like momentum transfers, so to elucidate the meaning of our variables let us go to a Lorentz frame where Q_{AB} points in the positive z direction (in the four-vector sense). To specify the frame completely we further require the three-vector $\overrightarrow{p_{1A}}$ to point in the z direction, while $\overrightarrow{p_{2A}}$ is contained in the yz plane with positive y component. Let this frame of reference be designated as "frame a," four-vectors in this frame carrying a superscript a. In any other frame four-vectors can be expressed through an appropriate Lorentz transformation. In particular we designate by u^a the transformation carrying the four-momenta from frame a to the laboratory frame:

$$p_{iA,B} = L(u^a)\, p^a_{iA,B} \,.$$

The four-vectors p^a_{iA} are thus of the form

$$p^a_{1A} = \left[-(m^2_{1A} + z^2_{1A})^{1/2},\ 0,\ 0,\ z_{1A}\right]$$

$$p^a_{2A} = \left[(m^2_{2A} + z^2_{2A} + y^2_{2A})^{1/2},\ 0,\ y_{2A} \geq 0,\ z_{2A}\right]$$

$$p^a_{N_A+1,A} = \left[(m^2_{N_A+1,A} + z^2_{N_A+1,A} + y^2_{N_A+1,A} + x^2_{N_A+1,A})^{1/2},\ x_{N_A+1,A},\ y_{N_A+1,A},\ z_{N_A+1,A}\right] , \qquad \text{(II.2)}$$

subject to the condition

$$\sum_{i=1}^{N_A+1} p^a_{iA} = Q^a_{AB} = \left[0,\ 0,\ 0, \sqrt{-t_{AB}}\right] .$$

It follows that when t_{AB} is given, the complete specification of the four vectors p^a_{iA} requires $3N_A - 4$ independent variables.[11] This is our set V_A.

In frame a specification of the four-vectors p^a_{iB} requires $3N_B - 1$ independent variables, once the mass shell conditions plus the constraint

$$\sum_{i=1}^{N_B+1} p^a_{iB} = Q^a_{AB}$$

are taken into account. The crucial step in the Toller method is now to split this latter set into $3N_B - 4$ internal variables plus three parameters describing the relative orientation of cluster B with respect to cluster A. The split may be achieved by defining a frame b in which the B four-vectors are specified by $3N_B - 4$ parameters through expressions completely analogous to (II.2), apart from opposite signs for the energy components. This is our variable set V_B. The four-vectors p_{iB}^a are then obtained by applying to p_{iB}^b a Lorentz transformation that preserves $Q_{AB}^a = Q_{AB}^b$, i.e., an element of the little group with respect to Q_{AB}. This transformation may be parametrized by a rotation through an angle $2\nu^{ab}$ around the z axis, a boost of magnitude $2\zeta^{ab}$ in the x direction, and a final rotation around the z axis through an angle $2\mu^{ab}$. Thus we have

$$p_{iB}^a = L(\mu^{ab}, \zeta^{ab}, \nu^{ab})\, p_{iB}^b = L(g^{ab})\, p_{iB}^b \quad .$$

The angle $2\mu^{ab}$ corresponds to a rigid rotation of the set of momenta p_B about the z axis in frame a while $2\nu^{ab}$ is a corresponding rotation of p_A in frame b. By straightforward calculation $\cosh 2\zeta^{ab}$ turns out to be linearly related to the square of the total energy in the barycentric system, i.e., to $S_{AB} = (p_{1B}-p_{1A})^2$, but for our purposes ζ^{ab} will be a more convenient variable than the total energy.[12] Note that the range of the variables g^{ab} is

$$0 \leq \zeta^{ab} < \infty \ , \quad 0 \leq \mu^{ab} < 2\pi, \quad 0 \leq \nu^{ab} < 2\pi, \qquad \text{(II.3)}$$

independently of the values of all the other variables.

III. EXTENSION OF THE TOLLER VARIABLES TO AN ARBITRARY NUMBER OF CLUSTERS

Let us now study the more general multi-cluster grouping of final particles shown in Fig. 2, realizing that the grouping chosen is a matter of convenience. (We shall see in Sec. V that different groupings allow the analysis of different asymptotic limits). Again the arrows represent the convention for the signs of the energy components. We consider the case in which only particles 1A and 1Z are incoming and each cluster contains at least two outgoing particles. Each internal line connecting a pair of clusters I and J carries four momentum Q_{IJ} and we consider a region where all the Q_{IJ} are spacelike.

In the previous section it was helpful to introduce two different frames, in both of which Q_{AB} points along the z axis. These two frames differ by an element g^{ab} of the little group with respect to Q_{AB}. In frame a the four vector set p_A is in conventional form while in frame b it is p_B that is conventional. In a similar fashion, for the multi-cluster grouping, two frames differing by an element g^{ij} of the little group with respect to Q_{IJ} may be defined for each internal momentum transfer. Therefore there will be two frames associated with each internal cluster J which will be labeled frame j_ℓ, if cluster J occurs on the "left" side of the z-pointing momentum transfer, and j_r if cluster J occurs on the "right" side. The Lorentz transformation carrying us from frame j_ℓ

to frame j_r will be denoted by q^j. To summarize,

$$p^{j_\ell} = L(g^{ik})\, p^{k_r}$$
$$p^{j_r} = L(q^j)\, p^{j_\ell}, \quad p^{i_\ell} = L(g^{ij})\, p^{j_r}, \quad \text{etc.} \tag{III.1}$$

It will be shown that the transformations q^j are completely determined by t_{IJ}, t_{JK} and the internal J-cluster variables.

A subset V_J of $3N_J - 4$ internal variables for cluster J may be defined essentially as in the previous section. To make the choice concrete we begin by taking V_A to be the parameters specifying the four-vector set p_A in frame a_ℓ, exactly as before. The set V_B presents a slightly different problem because the spacelike vector Q_{BC} fulfills the role played by the timelike p_{1B} in Fig. 1. However, given $t_{BC} < 0$, and the constraint

$$Q_{BC} = Q_{AB} - \sum_{i=2}^{N_B+1} p_{iB}, \tag{III.2}$$

with the p_{iB} all negative timelike, we can find a frame b_r in which Q_{AB} points in the positive z direction while Q_{BC} has its spacelike part pointing in the same direction. That is

$$Q_{AB}^{b_r} = \left[0,\ 0,\ 0,\ \sqrt{-t_{AB}}\right] \tag{III.3}$$
$$Q_{BC}^{b_r} = \left\{\left[t_{BC} + (z_{BC}^{b_r})^2\right]^{1/2},\ 0,\ 0,\ z_{BC}^{b_r} > 0\right\}. \tag{III.4}$$

The frame b_r is further defined by the conventional form

$$p_{2B}^{b_r} = \left\{ - \left[m_{2B}^2 + (y_{2B}^{b_r})^2 + (z_{2B}^{b_r})^2 \right]^{1/2}, 0, y_{2B}^{b_r} \geq 0, z_{2B}^{b_r} \right\} ,$$
$$p_{3B}^{b_r} = \left\{ - \left[m_{3B}^2 + (x_{3B}^{b_r})^2 + (y_{3B}^{b_r})^2 + (z_{3B}^{b_r})^2 \right]^{1/2}, x_{3B}^{b_r}, y_{3B}^{b_r}, z_{3B}^{b_r} \right\} ,$$

leading to a set of $3N_B - 4$ parameters constituting V_B, just as before.[11]

By corresponding criteria the frame b_ℓ next may be defined, in which Q_{BC} points in the positive z direction:

$$Q_{BC}^{b_\ell} = \left[0, 0, 0, \sqrt{-t_{BC}} \right] , \qquad \text{(III.5)}$$

while Q_{AB} has its spacelike part pointing in the positive z direction, and the outgoing four vectors p_B maintain conventional form. Evidently the frame b_ℓ is reached from the frame b_r by a pure boost in the z direction--of magnitude determined by $z_{BC}^{b_r}$ and t_{BC} from Eqs. (III.4) and (III.5). Thus, the Lorentz transformation q^b is completely determined[13] by V_B, together with t_{AB} and t_{BC}.

From this point on the analysis proceeds in the same fashion; through successive applications of $L(g^{ij})$ and $L(q^j)$ we eventually reach the final frame z_r. The complete collection of variables is $3N_J - 4$ for each cluster, plus four for each internal line (one from t_{JK} and three from g^{jk}). Since there is one more cluster than internal line, the total number of variables is $3N - 4$, if N is the total number of outgoing particles. This is the well known result, usually expressed as $3(N+2) - 10$, where $N+2$ is the total number of lines in the connected part. In appendices A and B we show how to relate these generalized Toller variables to Lorentz invariants.

To summarize, the momenta $p_A, p_B, \cdots p_Z$ in the laboratory frame are obtained from our variables, $V_A, V_B \cdots V_Z$, $t_{AB}, t_{BC} \cdots t_{YZ}$, $g^{ab}, g^{bc}, \cdots g^{yz}$, by

$$
\begin{aligned}
p_A &= L(u^a)\, p_A^{a_\ell}\, (V_A, t_{AB}) \\
p_B &= L(u^a)\, L(g^{ab})\, p_B^{b_r}\, (t_{AB}, V_B, t_{BC}) \\
p_C &= L(u^a)\, L(g^{ab})\, L\left[q^b(t_{AB}, V_B, t_{BC})\right] L(g^{bc})\, p_C^{c_r}(t_{BC}, V_C, t_{CD}) \qquad \text{(III.6)} \\
&\;\;\vdots \\
p_Z &= L(u^a)\, L(g^{ab})\, L\left[q^b(t_{AB}, V_B, t_{BC})\right] \cdots \\
&\quad \cdots L(g^{xy})\, L\left[q^y\, (t_{XY}, V_Y, t_{YZ})\right] L(g^{yz})\, p_Z^{z_r}\, (t_{YZ}, V_Z).
\end{aligned}
$$

This prescription is reproduced in tabular form in Table I.

Table I. Notation for the four-momenta of the external particles in the different frames used to describe a multi-cluster diagram, and for the Lorentz transformations connecting these frames.

Transformation	Frame \ Cluster	A	B	C	. . .	Y	Z
	Lab.	p_A	p_B	p_C	. . .	p_Y	p_Z
u^a							
	a_ℓ	$p_A^{a_\ell}$	$p_B^{a_\ell}$	$p_C^{a_\ell}$	. . .	$p_Y^{a_\ell}$	$p_Z^{a_\ell}$
g^{ab}							
	b_r		$p_B^{b_r}$	$p_C^{b_r}$	. . .	$p_Y^{b_r}$	$p_Z^{b_r}$
q^b							
	b_ℓ		$p_B^{b_\ell}$	$p_C^{b_\ell}$	. . .	$p_Y^{b_\ell}$	$p_Z^{b_\ell}$
g^{bc}							
	c_r			$p_C^{c_r}$	. . .	$p_Y^{c_r}$	$p_Z^{c_r}$
q^c							
	c_ℓ			$p_C^{c_\ell}$	. . .	$p_Y^{c_\ell}$	$p_Z^{c_\ell}$
g^{cd}							
	⋮					⋮	⋮
	y_r					$p_Y^{y_r}$	$p_Z^{y_r}$
q^y							
	y_ℓ					$p_Y^{y_\ell}$	$p_Z^{y_\ell}$
g^{yz}							
	z_r						$p_Z^{z_r}$

IV. CLUSTERS WITH ONE OUTGOING PARTICLE

Let us now consider the special case of a cluster with only three lines, one outgoing and the remaining two either both internal or one internal and one (ingoing) external.

Starting with the second possibility, suppose that cluster A of Fig. 2 has one outgoing line, i.e., $N_A = 1$. The momenta p_A in frame a_ℓ are then given by

$$\begin{aligned} p_{1A}^{a_\ell} &= \left\{ -\left[m_{1A}^2 + (z_{1A}^{a_\ell})^2 \right]^{1/2}, 0, 0, z_{1A}^{a_\ell} \right\} \\ p_{2A}^{a_\ell} &= \left\{ \left[m_{2A}^2 + (z_{2A}^{a_\ell})^2 \right]^{1/2}, 0, 0, z_{2A}^{a_\ell} \right\} , \end{aligned} \tag{IV.1}$$

with

$$p_{1A}^{a_\ell} + p_{2A}^{a_\ell} = Q_{AB}^{a_\ell} = \left[0, 0, 0, \sqrt{-t_{AB}} \right] , \tag{IV.2}$$

a constraint which determines both $z_{1A}^{a_\ell}$ and $z_{2A}^{a_\ell}$ in terms of t_{AB}. There are thus <u>no</u> free internal parameters V_A. The amplitude furthermore does not depend on the angle μ^{ab}. Changing this angle amounts to a rigid rotation of the momenta $p_B^{a_\ell}$, $p_C^{a_\ell} \cdots p_Z^{a_\ell}$ around the z axis. Formula (IV.1) shows that $p_A^{a_\ell}$ with $N_A = 1$ would be left unchanged by such a rotation, so changing μ^{ab} amounts to a rotation of <u>all</u> momenta--an operation that leaves the amplitude invariant.[14] It is amusing to note that the formula $3N_A - 4$, for the number of variables in V_A, yields -1 when applied unthinkingly to $N_A = 1$, and that this result might be interpreted as the subtraction of one parameter from the set specifying the little group g^{ab} .

Now consider an internal cluster with $N_J = 1$. The frame j_r is such that

$$Q_{JK}^{j_r} = \left\{ \left[t_{JK} + (z_{JK}^{j_r})^2 \right]^{1/2}, 0, 0, z_{JK}^{j_r} > 0 \right\}$$
$$p_{2J}^{j_r} = \left\{ -\left[m_{2J}^2 + (z_{2J}^{j_r})^2 \right]^{1/2}, 0, 0, z_{2J}^{j_r} \right\} , \tag{IV.3}$$

with

$$Q_{JK}^{j_r} + p_{2J}^{j_r} = Q_{IJ}^{j_r} = \left[0, 0, 0, \sqrt{t_{IJ}} \right] , \tag{IV.4}$$

a constraint that determines $z_{JK}^{j_r}$ and $z_{2J}^{j_r}$ in terms of t_{IJ} and t_{JK}. Once again there are no free variables in V_J, and one may anticipate that one of the little group parameters will become superfluous. What happens is that the amplitude depends on the sum of the two angles ν_{ij} and μ_{jk} but not on these angles separately. This fact is implicit in the results of Chan, Kajantie, and Ranft, who based their analysis on variables especially suited to single-particle clusters.[7]

To understand the foregoing point, observe from Eq. (III.6) that the momenta $p_K, \cdots p_Z$ depend only on this sum of angles because the z-direction boost $L(q^j)$ commutes with either of the adjacent rotations about the z axis. In other words,

$$L(\mu^{ij}, \zeta^{ij}, \nu^{ij})\, L(q^j)\, L(\mu^{jk}, \zeta^{jk}, \nu^{jk})$$

$$= L(\mu^{ij}, \zeta^{ij}, \nu^{ij} + \mu^{jk})\, L(q^j)\, L(0, \zeta^{jk}, \nu^{jk}). \qquad \text{(IV.5)}$$

From Formula (IV.3) we see that p_J with $N_J = 1$ is independent of rotations about the z axis, while the momenta $p_A \cdots p_I$ never depend on variables g^{ij}. Of course, as soon as cluster J contains more than one outgoing particle, non-vanishing y components will appear in the momenta $p_J^{j_r}$ and there will be a consequent dependence of p_J on ν^{ij}.

The analysis of this section covers any number of single (outgoing) particle clusters. Note that the total number of variables is still $3N - 4$, with N the total number of outgoing particles.

V. ASYMPTOTIC BEHAVIOR OF MULTI-PARTICLE AMPLITUDES

Let us now exploit the variables defined in the previous section to extend the Regge-pole hypothesis to the asymptotic behavior in more than one variable of the multi-particle production amplitudes. Consider first the case where only two clusters, A and B', have been defined, B' comprising clusters B through Z of Fig. 2. As shown by Toller,[1] this two-cluster form can be expanded in terms of its projection onto the unitary irreducible representations of the little group with respect to Q_{AB}. Writing this projection as

$$f^{\ell}_{mn}(V_A, t_{AB}, V_{B'}) = \int dg^{ab} \exp(-2im\mu^{ab})\, d^{\ell}_{mn}(\zeta^{ab}) \exp(-2in\nu^{ab}) \times f(V_A, g^{ab}, t_{AB}, V_{B'}) , \qquad (V.1)$$

where $\exp(-2im\mu)\, d^{\ell}_{mn}(\zeta)\exp(-2in\nu)$ is a unitary irreducible representation of the little group of Q_{AB}. The inverse formula for the amplitude is

$$f(V_A, g^{ab}\ t_{AB}, V_{B'}) = \sum_{m,n=-\infty}^{\infty} \exp(2im\mu^{ab})\, f_{mn}(V_A, \zeta^{ab}, t_{AB}, V_{B'})\exp(2in\nu^{ab}) \qquad (V.2)$$

with

$$f_{mn}(V_A,\zeta^{ab},t_{AB}, V_{B'}) = i \int_C d\ell\, \frac{2\ell+1}{\mathrm{tg}\,\pi\ell}\, a_{mn}^{-\ell-1}(\zeta^{ab}) f_{mn}^{\ell}(V_A, t_{AB}, V_{B'}) + \sum_{\ell_i} D_{mn}^{\ell_i} \quad . \tag{V.3}$$

Here $D_{mn}^{\ell_i}$ stands for the contributions from the discrete representations at integer values ℓ_i, and the contour C runs along the line $\mathrm{Re}\,\ell = -\frac{1}{2}$ from $-\frac{1}{2} - i\infty$ to $-\frac{1}{2} + i\infty$. The functions a_{mn}^{ℓ} can be easily related to d_{mn}^{ℓ} and play a role analogous to the Legendre functions of the second kind.[1]

Assuming f_{mn}^{ℓ} to be meromorphic in the ℓ plane, the leading term in the asymptotic expansion of f_{mn} is controlled by the position $\alpha_1(t_{AB})$ and the residue of the leading pole in f_{mn}^{ℓ}. The poles in $(\mathrm{tg}\,\pi\ell)^{-1}$ not cancelled by zeros of $a_{mn}^{-\ell-1}$ are cancelled by the discrete terms in the expansion.

If we now assume that the residues are factorizable, we have

$$f_{mn}(V_A,\zeta^{ab},t_{AB},V_{B'}) \underset{\zeta^{ab}\to\infty}{\sim} \rho_m^A(V_A,t_{AB})\, [\cosh 2\zeta^{ab}]^{\alpha_1(t_{AB})}\, \rho_n^{B'}(t_{AB},V_{B'}) \tag{V.4a}$$

or, alternatively

$$f(V_A,g^{ab},t_{AB},V_{B'}) \underset{\zeta^{ab}\to\infty}{\sim} \phi^A(V_A,t_{AB},\mu^{ab})\, [\cosh 2\zeta^{ab}]^{\alpha_1(t_{AB})} \phi^{B'}(\nu^{ab},t_{AB},V_{B'}) \quad , \tag{V.4b}$$

with

$$\phi^A(V_A, t_{AB}, \mu^{ab}) = \sum_{m=-\infty}^{\infty} e^{2im\mu^{ab}} \rho_m^A(V_A, t_{AB}) \tag{V.5}$$

and correspondingly for $\phi^{B'}$. Next consider cluster B' separated into two clusters, B and C', with C' comprising all clusters C through Z. The set of variables $V_{B'}$ becomes V_B, g^{bc}, t_{BC} and $V_{C'}$, while V_A, g^{ab}, and t_{AB} remain unchanged. The last factor in Eq. (V.4) can thus be reexpressed as

$$\phi^{B'}(\nu^{ab}, t_{AB}, V_B, g^{bc}, t_{BC}, V_{C'}) ,$$

and we can expand the dependence in g^{bc} just as before. The leading term in the asymptotic expansion of $\phi^{B'}$ for large ζ^{bc}, due to a pole at $\alpha_2(t_{BC})$ in the corresponding complex ℓ plane, may thus be written

$$\phi^{B'}(\nu^{ab}, t_{AB}, V_B, g_{bc}, t_{BC}, V_{C'}) \underset{\zeta^{bc}\to\infty}{\sim}$$

$$\phi^B(\nu^{ab}, t_{AB}, V_B, t_{BC}, \mu^{bc}) \, [\cosh 2\zeta^{ab}]^{\alpha_2(t_{BC})} \, \phi^{C'}(\nu^{bc}, t_{BC}, V_{C'})$$

This procedure can evidently be pursued indefinitely. The general result for the multi-Regge asymptotic behavior of an amplitude consisting of clusters A, $B, \cdots, Z$, becomes

$$f(V_A, g^{ab}, t_{AB}, V_B, g^{bc}, t_{BC}, V_C, \cdots, V_Y, g^{yz}, t_{YZ}, V_Z)_{\zeta^{ab}\to\infty,\ \zeta^{bc}\to\infty \cdots \zeta^{yz}\to\infty}$$

$$\sim \phi^A(V_A, t_{AB}, \mu^{ab})\ [\cosh 2\zeta^{ab}]^{\alpha_1(t_{AB})} \phi^B(\nu^{ab}, t_{AB}, V_B, t_{BC}, \mu^{bc})\ [\cosh 2\zeta^{bc}]^{\alpha_2(t_{BC})}$$

$$\times\ \phi^C(\nu^{bc}, t_{BC}, V_C, t_{CD}, \mu^{cd}) \cdots \phi^Y(\nu^{xy}, t_{XY}, V_Y, t_{YZ}, \mu^{yz})\ [\cosh 2\zeta^{yz}]^{\alpha_L(t_{YZ})}$$

$$\times\ \phi^Z(\nu^{yz}, t_{YZ}, V_Z)\ . \tag{V.6}$$

The initial and final "vertex functions" each depend on a single angle and a single momentum transfer, in addition to the variables of the corresponding clusters, while the intermediate vertices each depend on two angles, two momentum transfers and the internal variables of the intervening cluster. For the special case of a single-outgoing-particle cluster, an initial or final vertex becomes independent of its angle, while an intermediate vertex depends only on the sum of its two angles, as was shown in Sec. IV.

It is clear from the above discussion that for a process of given initial and final particles, different asymptotic formulae will be obtained depending on the way in which the outgoing particles are assigned to individual clusters. In each case the asymptotic behavior will be controlled by the leading Regge poles of the appropriate quantum numbers.

VI. GENERALIZATION TO INCLUDE PARTICLES WITH SPIN

In this section we shall generalize the previous discussions to include particles with spin. This objective can readily be achieved in the language of M-functions, for which Lorentz invariance reads

$$M_{s_A\, s_B \cdots s_Z}[L(u)\, p_A,\, L(u)\, p_B, \cdots,\, L(u)\, p_Z] = \sum_{s'_A, s'_B, \cdots, s'_Z} D^A_{s_A\, s'_A}(u)\, D^B_{s_B\, s'_B}(u) \cdots D^Z_{s_Z\, s'_Z}$$

$$\times\, M_{s'_A\, s'_B \cdots s'_Z}(p_A,\, p_B, \cdots,\, p_Z)\ . \qquad \text{(VI.1)}$$

Here we have indicated collectively by s_J the spin indices of all external particles connected to cluster J, and by $D^J_{s_J\, s'_J}(u)$ the direct product of the corresponding spin transformation matrices, which are finite dimensional representations of the homogeneous Lorentz group.

The amplitude introduced by Toller in the two-cluster case is, in our notation,

$$f_{s_A s_B}(V_A, g^{ab}, t_{AB}, V_B) = \sum_{s'_B} D^B_{s_B s'_B}\,[(g^{ab})^{-1}]$$

$$\times\, M_{s_A s'_B}\,[p_A^{a_\ell},\, L(g^{ab})\, p_B^{b_r}]\ , \qquad \text{(VI.2)}$$

and the M-function describing the process in an arbitrary frame can be written

$$M_{s_A s_B}(p_A, p_B) = M_{s_A s_B}\,[L(u^a)\ p_A^{a_\ell},\ L(u^a g^{ab})\ p_B^{b_r}]$$

$$= \sum_{s'_A s'_B} D^A_{s_A s'_A}(u^a)\, D^B_{s_B s'_B}(u^a g^{ab})\, f_{s'_A s'_B}(V_A, g^{ab}, t_{AB}, V_B) \tag{VI.3}$$

The essential property of $f_{s_A s_B}(V_A, g^{ab}, t_{AB}, V_B)$ is that the spin indices are independent of the exhibited continuous variables. We want to generalize this definition to the multi-cluster case, and it is convenient to introduce a more compact notation for the product of transformations leading from a frame j_r to the frame a_ℓ. We call this product u^{aj} and thus have

$$u^{ab} = g^{ab}$$
$$u^{ac} = g^{ab}\, q^b\, g^{bc}$$
$$\vdots$$
$$u^{az} = g^{ab}\, q^b\, g^{bc} \cdots g^{xy}\, q^y\, g^{yz}\ . \tag{VI.4}$$

We now are in position to define the generalization of (VI.2), namely

$$f_{s_A s_B \cdots s_Z}(V_A, g^{ab}, t_{AB}, V_B, \cdots, g^{yz}, t_{YZ}, V_Z)$$

$$= \sum_{s'_B, s'_C, \cdots s'_Z} D^B_{s_B s'_B}[(u^{ab})^{-1}]\, D^C_{s_C s'_C}[(u^{ac})^{-1}] \cdots D^Z_{s_Z s'_Z}[(u^{az})^{-1}]$$

$$\times\, M_{s_A s'_B \cdots s'_Z}(p_A^{a}{}^{\ell}, p_B^{a}{}^{\ell}, \cdots, p_Z^{a}{}^{\ell}) \quad . \qquad \text{(VI.5)}$$

In terms of this amplitude, the M function can be written

$$M_{s_A s_B \cdots s_Z}(p_A, p_B, \cdots, p_Z)$$

$$= M_{s_A s_B \cdots s_Z}[L(u^a)\, p_A^{a}{}^{\ell},\, L(u^a\, u^{ab})\, p_B^{b}{}^{r}, \cdots,\, L(u^a\, u^{az})\, p_Z^{z}{}^{r}]$$

$$= \sum_{s'_A, s'_B, \cdots, s'_Z} D^A_{s_A s'_A}(u^a)\, D^B_{s_B s'_B}(u^a\, u^{ab}) \cdots D^Z_{s_Z s'_Z}(u^a\, u^{az})$$

$$\times\, f_{s'_A s'_B \cdots s'_Z}(V_A, g^{ab}, t_{AB}, V_B, \cdots, g^{yz}, t_{YZ}, V_Z) \quad . \qquad \text{(VI.6)}$$

The analysis of the asymptotic behavior of the amplitude given in Eq. (VI.5) proceeds for $f_{s_A s_B \cdots s_Z}$ along the same steps used in the previous section for the spinless case, with only the addition of appropriate spin indices; each ρ^J and ϕ^J acquires the index s_J, whereas f, f^{ℓ}_{mn} and f_{mn} depend on all indices s_A to s_Z. With these trivial additions all equations in Sec. V remain valid in the presence of spin.[15]

VII. CONCLUSION

From the Toller point of view, the multi-Regge pole hypothesis is a natural and unambiguous extension of the single-pole hypothesis--and physically just as plausible. Since experiments indicate the single-pole model to have approximate validity, the multi-pole model necessarily is of great physical interest. The asymptotic form (V.6) embodies many striking experimental predictions, particularly if supplemented by the "peripheral assumption" that all vertex functions decrease rapidly with the increase of the adjacent momentum transfers. A number of such predictions have been discussed by previous authors, using variables that are less systematically chosen than ours and less general in one respect or another. We hope that the kinematic analysis given here will provide a basis for extension of such work and will increase confidence in its significance.

An indirect aspect of the considerations treated in this paper is the unambiguous meaning achieved for "addition" of complex angular momenta. Consider a three-cluster decomposition, for example, with the associated graph drawn as in Fig. 3. Suppose that when we make the ℓ^{ab} projection of g^{ab} and the ℓ^{bc} projection of g^{bc}, we find it possible to identify a pole in each of these complex variables, say at $\ell^{ab} = \alpha_1(t_{AB})$ and $\ell^{bc} = \alpha_2(t_{BC})$; then the residue of the pole (product) leads by factorization to the definition of a quantity

$$f^{\alpha_1(t_{AB}),\alpha_2(t_{BC})}_{m^{ab},n^{bc}}(t_{AB}, V_B, t_{BC}) \ , \qquad \text{(VII.1)}$$

Which may be described as the amplitude for the Regge pole $\alpha_1(t_{AB})$ to interact with the Regge pole $\alpha_2(t_{BC})$ to produce the physical particles in cluster B. The indices m^{ab} and n^{bc} describe the "spin states" of the "Reggeons."

Note that although we have not succeeded in defining an amplitude <u>all</u> of whose external particles have complex angular momentum, heretofore an unambiguous meaning has been lacking for more than one Reggeized external particle. One may hope that dynamical models based on combinations of "Reggeons" now may be constructed, since discontinuity formulas will always involve physical particles in intermediate states and thus never require that <u>all</u> external angular momenta be complex.

APPENDIX A

EXPRESSION OF THF TOLLER VARIABLES IN TERMS OF INVARIANTS

The Toller variables μ^{ij}, ζ^{ij}, ν^{ij}, have been shown to be most natural ones to describe multi-cluster processes. However, they have the apparent disadvantage of being defined in different Lorentz frames. Nevertheless, we shall show they can be easily expressed in terms of invariant scalar products which can be computed in any frame, including the laboratory frame.

We start from a multi-cluster diagram, as is shown in Fig. 2, and the following steps lead us to the desired variables.

(i) From

$$Q_{IJ} = \sum_{i=1}^{N_A+1} p_{iA} - \sum_{G=B}^{I} \sum_{i=2}^{N_G+1} p_{iG} \tag{A.1}$$

we can compute all $t_{IJ} = Q_{IJ}^2$.

(ii) Introducing the squared invariant mass of cluster I as

$$s_I = (Q_{IJ} - Q_{HI})^2 = t_{IJ} + t_{HI} - 2(Q_{IJ} \cdot Q_{HI}) \tag{A.2}$$

we can write

$$z_{IJ}^{i_r} = \frac{(s_I - t_{HI} - t_{IJ})}{2\sqrt{-t_{HI}}} \qquad z_{IJ}^{j_\ell} = \frac{(s_J - t_{IJ} - t_{JK})}{2\sqrt{-t_{JK}}} \quad . \tag{A.3}$$

(iii) We introduce the squared invariant of combined cluster I and J as

$$s_{IJ} = (Q_{HI} - Q_{JK})^2 \quad . \tag{A.4}$$

Then ζ^{ij} can be readily expressed in terms of this invariant and the t's and z's defined above as

$$\cosh \zeta^{ij} = -\frac{t_{HI} + t_{JK} - s_{IJ} + 2\, z_{HI}^{i_\ell}\, z_{JK}^{j_r}}{2\left[t_{HI} + \left(z_{HI}^{i_\ell}\right)^2\right]^{\frac{1}{2}}\left[t_{JK} + \left(z_{JK}^{j_r}\right)^2\right]^{\frac{1}{2}}} \tag{A.5}$$

(iv) We now choose one particle in cluster I to be particle 2I, and we compute the scalar product $(Q_{IJ} \cdot p_{2I})$ in frame i_ℓ so that

$$z_{2I}^{i_\ell} = -\frac{Q_{IJ} \cdot p_{2I}}{\sqrt{-t_{IJ}}} \quad . \tag{A.6}$$

The $y_{2I}^{i_\ell}$ component of particle 2I can then be calculated from the scalar product $(Q_{HI} \cdot p_{2I})$

$$y_{2I}^{i_\ell} = \left\{\frac{\left[(p_{2I} \cdot Q_{HI}) + z_{2I}^{i_\ell}\, z_{HI}^{i_\ell}\right]^2}{t_{HI} + \left(z_{HI}^{i_\ell}\right)^2} - m_{2I}^2 - \left(z_{2I}^{i_\ell}\right)^2\right\}^{\frac{1}{2}} \quad , \tag{A.7}$$

while $x_{2I}^{i_\ell}$ is of course zero. The angle ν^{ij} can be evaluated in terms of the above quantities and the scalar product $(Q_{JK} \cdot p_{2I})$

$$\sin 2\nu^{ij} = - \frac{Q_{JK}\cdot p_{2J} + z_{2I}^{i_\ell} z_{JK}^{j_r} + \epsilon\left[t_{JK} + \left(z_{JK}^{j_r}\right)^2\right]^{\frac{1}{2}}\left[m_{2I}^2 + \left(z_{2I}^{i_\ell}\right)^2 + \left(y_{2I}^{i_\ell}\right)^2\right]^{\frac{1}{2}} \cosh 2\zeta^{ij}}{y_{2I}^{i_\ell}\left[t_{JK} + \left(z_{JH}^{j_r}\right)^2\right]^{\frac{1}{2}} \sinh 2\zeta^{ij}}, \tag{A.8}$$

where $\epsilon = -1$ if $i = a$ and $+1$ otherwise.

(v) To compute μ^{ij} we follow a procedure analogous to that of (iv). Choosing a particle in cluster J to be particle $2J$, we have

$$z_{2J}^{j_r} = - \frac{Q_{IJ} \cdot p_{2J}}{\sqrt{-t_{IJ}}}, \tag{A.9}$$

and

$$y_{2J}^{j_r} = \left\{ \frac{\left[(p_{2J} \cdot Q_{JK}) + z_{2J}^{j_r} z_{JK}^{j_r}\right]^2}{t_{JK} + \left(z_{JK}^{i_r}\right)^2} - m_{2J}^2 - \left(z_{2J}^{j_r}\right)^2 \right\}^{\frac{1}{2}} \tag{A.10}$$

so that

$$\sin 2\mu^{ij} = \frac{Q_{HI}\cdot p_{2J} + z_{2J}^{j_r} z_{HI}^{i_\ell} - \left[t_{HI} + \left(z_{HI}^{i_\ell}\right)^2\right]^{\frac{1}{2}}\left[m_{2J}^2 + \left(z_{2J}^{j_r}\right)^2 + \left(y_{2J}^{j_r}\right)^2\right]^{\frac{1}{2}} \cosh 2\zeta^{ij}}{y_{2J}^{j_r} \sinh 2\zeta^{ij}\left[t_{HI} + \left(z_{HI}^{i_\ell}\right)^2\right]^{\frac{1}{2}}} \tag{A.11}$$

Following the above procedure, all the variables of the little groups involved can be obtained from scalar products of the particle momenta.

In the above formulae, for the case $I = A$, we see that p_{1A} and m^2_{1A} replace Q_{HI} and t_{HI}, and frame a_r does not exist. Correspondingly, when $J = Z$, p_{1Z} and m^2_{1Z} substitute for Q_{JK} and t_{JK} and there is no frame z_ℓ.

Once the parameters of all the little groups are known, two more steps are required to be able to specify the conventional sets, and thus all $V_A \cdots V_B$, from the laboratory momenta. The first is to determine the overall Lorentz transformation $(u^a)^{-1}$ which takes the laboratory momenta to a frame a_ℓ, in which set P_A is conventional. The calculation of this transformation if straightforward. The second is to compute the boost parameter of the transformations q^j. This can readily be seen to be

$$\sinh 2q^j = \frac{\left[t_{JK} + \left(z^{i_r}_{JK}\right)^2\right]^{\frac{1}{2}}}{\sqrt{-t_{JK}}} = \frac{\left[t_{IJ} + \left(z^{j_\ell}_{IJ}\right)^2\right]^{\frac{1}{2}}}{\sqrt{-t_{IJ}}} . \qquad (A.12)$$

In Appendix B we shall discuss the determination of the conventional sets from the four-momenta of the reaction.

APPENDIX B

RELATION OF THE INTERNAL CLUSTER VARIABLES TO INVARIANTS

Appendix A gives recipes in terms of scalar invariants for the parameters of the successive Lorentz transformations connecting frames a_ℓ through z_r. The present appendix will show how the internal variables V_I for cluster I can be related to appropriate Lorentz scalars. Except for cluster A these internal variables have been introduced previously as the independent components of the four-vectors p_I and Q_{IJ} in frame i_r, where these vectors are constrained to a certain conventional form. (The special case of V_A refers to frame a_ℓ.) These momentum components can of course be related to lab momenta by direct application to the latter of the appropriate sequence of Lorentz transformations, but this somewhat laborious process may be avoided through the introduction of Lorentz invariants. We consider separately the cases $N_I = 1, 2, 3$ and $N_I \geq 4$.

(i) $N_I = 1$.

As pointed out in Sec. IV, there are no free internal variables for a one-particle cluster. All components of the four-vectors $Q_{IJ}^{i_r}$ and $p_{2I}^{i_r}$ and determined by a knowledge of t_{HI} and t_{IJ}:

$$Q_{IJ}^{i_r} = \left\{ \left[t_{IJ} + \left(z_{IJ}^{i_r} \right)^2 \right]^{\frac{1}{2}} , 0, 0, z_{IJ}^{i_r} > 0 \right\}$$

$$p_{2I}^{ir} = \left\{ -\left[m_{2I}^2 + \left(z_{2I}^{i_r} \right)^2 \right]^{\frac{1}{2}} \; 0, 0, z_{2I}^{i_r} \right\} , \tag{B.1}$$

with

$$Q_{IJ}^{i_r} + p_{2I}^{i_r} = Q_{HI}^{i_r} = (0, 0, 0, \sqrt{-t_{HI}}) . \tag{B.2}$$

An elementary calculation thus yields

$$z_{IJ}^{i_r} = \frac{m_{2I}^2 - t_{HI} - t_{IJ}}{2\sqrt{-t_{HI}}} , \tag{B.3}$$

$$z_{2I}^{i_r} = \frac{-m_{2I}^2 - t_{HI} + t_{IJ}}{2\sqrt{-t_{HI}}} . \tag{B.4}$$

(ii) $N_I = 2.$

Here there are two independent variables constituting V_I. A natural choice is any two of the three scalars, $Q_{IJ} \cdot Q_{HI}$, $Q_{HI} \cdot p_{2I}$ and $Q_{IJ} \cdot p_{2I}$, which obey the linear constraint

$$p_{3I}^2 = m_{3I}^2 = (Q_{HI} - Q_{IJ} - p_{2I})^2 . \tag{B.5}$$

The conventional set now reads

$$
\begin{aligned}
Q_{IJ}^{i_r} &= \left\{ \left[t_{IJ} + \left(z_{IJ}^{i_r} \right)^2 \right]^{\frac{1}{2}}, \ 0, \ 0, \ z_{IJ}^{i_r} > 0 \right\} , \\
p_{2I}^{i_r} &= \left\{ -\left[m_{2I}^2 + \left(y_{2I}^{i_r} \right)^2 + \left(z_{2I}^{i_r} \right)^2 \right]^{\frac{1}{2}}, \ 0, \ y_{2I}^{i_r} > 0, \ z_{2I}^{i_r} \right\} , \\
p_{3I}^{i_r} &= \left\{ -\left[m_{3I}^2 + \left(y_{3I}^{i_r} \right)^2 + \left(z_{3I}^{i_r} \right)^2 \right]^{\frac{1}{2}}, \ 0, \ y_{3I}^{i_r}, \ z_{3I}^{i_r} \right\} ,
\end{aligned}
\tag{B.6}
$$

with

$$
Q_{IJ}^{i_r} + p_{2J}^{i_r} + p_{3J}^{i_r} = Q_{HI}^{i_r} = (0, \ 0, \ 0, \sqrt{-t_{HI}}) \ . \tag{B.7}
$$

By straightforward calculation from (B.6) and (B.7),

$$
z_{IJ}^{i_r} = - \frac{Q_{HI} \cdot Q_{IJ}}{\sqrt{-t_{HI}}} \tag{B.8}
$$

$$
z_{2I}^{i_r} = - \frac{Q_{HI} \cdot p_{2I}}{\sqrt{-t_{HI}}}
$$

$$
z_{3I}^{i_r} = \sqrt{-t_{HI}} - z_{2I}^{i_r} - z_{IJ}^{i_r} , \tag{B.9}
$$

$$
y_{3I}^{i_r} = - y_{2I}^{i_r} , \tag{B.10}
$$

and as in Eq. (A.10)

$$
y_{2I}^{i_r} = \left\{ \frac{\left(Q_{IJ} \cdot p_{2I} + z_{2I}^{i_r} z_{IJ}^{i_r} \right)^2}{t_{IJ} + \left(z_{IJ}^{i_r} \right)^2} - m_{2I}^2 - z_{2I}^{i_r} \right\}^{\frac{1}{2}} . \tag{B.11}
$$

(iii) $N_I = 3$.

There are now 5 variables in the set V_I. It is convenient to consider six invariants, $Q_{HI} \cdot Q_{IJ}$, $Q_{HI} \cdot p_{2I}$, $Q_{HI} \cdot p_{3I}$, $Q_{IJ} \cdot p_{2I}$, $Q_{IJ} \cdot p_{3I}$ and $p_{2I} \cdot p_{3I}$, with the single linear constraint

$$p_{4I}^2 = m_{4I}^2 = (Q_{HI} - Q_{IJ} - p_{2I} - p_{3I})^2 . \qquad \text{(B.12)}$$

The conventional set is

$$\begin{aligned}
{}^{i}_{r}Q_{IJ} &= \left\{ \left[t_{IJ} + \left({}^{i}_{r}z_{IJ}\right)^2 \right]^{\frac{1}{2}} , 0, 0, {}^{i}_{r}z_{IJ} > 0 \right\} \\
{}^{i}_{r}p_{2I} &= \left\{ - \left[m_{2I}^2 + \left({}^{i}_{r}z_{2I}\right)^2 + \left({}^{i}_{r}y_{2I}\right)^2 \right]^{\frac{1}{2}} , 0, {}^{i}_{r}y_{2I} > 0, {}^{i}_{r}z_{2I} \right\} \qquad \text{(B.13)} \\
{}^{i}_{r}p_{3I} &= \left\{ - \left[m_{3I}^2 + \left({}^{i}_{r}x_{3I}\right)^2 + \left({}^{i}_{r}y_{3I}\right)^2 + \left({}^{i}_{r}z_{3I}\right)^2 \right]^{\frac{1}{2}} , {}^{i}_{r}x_{3I}, {}^{i}_{r}y_{3I}, {}^{i}_{r}z_{3I} \right\} \\
{}^{i}_{r}p_{4I} &= \left\{ - \left[m_{4I}^2 + \left({}^{i}_{r}x_{4I}\right)^2 + \left({}^{i}_{r}y_{4I}\right)^2 + \left({}^{i}_{r}z_{4I}\right)^2 \right]^{\frac{1}{2}} , {}^{i}_{r}x_{4I}, {}^{i}_{r}y_{4I}, {}^{i}_{r}z_{4I} \right\} ,
\end{aligned}$$

with

$${}^{i}_{r}Q_{IJ} + {}^{i}_{r}p_{2I} + {}^{i}_{r}p_{3I} + {}^{i}_{r}p_{4I} = {}^{i}_{r}Q_{HI} = (0, 0, 0, \sqrt{-t_{HI}}) . \qquad \text{(B.14)}$$

The various z components are easily related to our scalars:

$${}^{i}_{r}z_{IJ} = - \frac{Q_{HI} \cdot Q_{IJ}}{\sqrt{-t_{HI}}} , \qquad \text{(B.15a)}$$

$$z_{nI}^{i_r} = - \frac{Q_{HI}\cdot p_{nI}}{\sqrt{-t_{HI}}} \quad , \quad n = 2, 3, \tag{B.15b}$$

while $y_{2I}^{i_r}$ is still given by Eq. (B.11). Next, forming the two scalars $Q_{IJ}\cdot p_{3I}$ and $p_{2I}\cdot p_{3I}$, we get two relations which may be solved for $x_{3I}^{i_r}$ and $y_{3I}^{i_r}$:

$$y_{3I}^{i_r} = - \frac{1}{y_{2I}^{i_r}} \left\{ p_{2I}\cdot p_{3I} + z_{2I}^{i_r} z_{3I}^{i_r} + (Q_{IJ}\cdot p_{3I} + z_{IJ}^{i_r} z_{3I}^{i_r}) \times \left[\frac{m_{2I}^2 + \left(z_{2I}^{i_r}\right)^2 + \left(y_{2I}^{i_r}\right)^2}{t_{IJ} + \left(z_{IJ}^{i_r}\right)^2} \right]^{\frac{1}{2}} \right. \tag{B.16}$$

$$x_{3I}^{i_r} = \pm \left\{ \frac{\left(Q_{IJ}\cdot p_{3I} + z_{IJ}^{i_r} z_{3I}^{i_r}\right)^2}{t_{IJ} + z_{IJ}^{i_r\,2}} - m_{3I}^2 - \left(y_{3I}^{i_r}\right)^2 - \left(z_{3I}^{i_r}\right)^2 \right\}^{\frac{1}{2}} \tag{B.17}$$

when the $\pm$ sign may be determined by requiring that the pseudoscalar $\epsilon_{\lambda\mu\nu\rho}\ Q_{HI}^{\lambda}\ Q_{IJ}^{\mu}\ p_{2I}^{\nu}\ p_{3I}^{\rho}$ have the same sign when computed in the lab and i_r frames. Finally, the components of p_{4I} are obtained from (B.14).

(iv) $N_I \geq 4$.

The total number of variables is $3N_I - 4$ and it is convenient to work with $3N_I - 3$ scalars obeying one constraint. A possible choice is the set of N_I dot products $Q_{HI}\cdot Q_{IJ}$, $Q_{HI}\cdot p_{2I} \cdots Q_{HI}\cdot p_{N_I,I}$, the set of N_I-1 dot products, $Q_{IJ}\cdot p_{2I} \cdots Q_{IJ}\cdot p_{N_I,I}$, plus the set of N_I-2 dot products, $p_{2I}\cdot p_{3I} \cdots p_{2I}\cdot p_{N_I,I}$. The constraint is unfortunately non-linear, a fact well known for connected parts with more than 5 lines.

The procedure for determining momentum components in frame i_r is a straightforward extension of that for $N_I=3$. The z components of all vectors except $p^{i_r}_{N_I+1,I}$ are given as in Eqs. (B.15) by the N_I dot products involving Q_{HI}. The component $y^{i_r}_{2I}$ is calculated from $Q_{IJ} \cdot p_{2I}$ just as before, whereas the pair of components $x^{i_r}_{nI}$ and $y^{i_r}_{nI}$, with $n = 3 \cdots N_I$, can, for each n, be related to the pair of scalars $Q_{IJ} \cdot p_{nI}$ and $p_{2I} \cdot p_{nI}$ by equations of the form (B.16) and (B.17). Finally the components of $p^{ir}_{N_I+1,\ I}$ may be obtained from the equation

$$p^{i_r}_{N_I+1,I} = Q^{i_r}_{HI} - Q^{i_r}_{IJ} - p^{i_r}_{2I} - p^{i_r}_{3I} - \cdots - p^{i_r}_{N_I,I}. \tag{B.18}$$

(v) The Special Case of End Clusters

When the above procedure is applied to the case $I = Z$, the only required changes are to replace $Q^{i_r}_{IJ}$ by $p^{z_r}_{1Z}$, and t_{IJ} by m^2_{1Z}. With cluster A, on the other hand, the following changes are necessary.

(a) Superscript i_r becomes a_ℓ.

(b) $Q^{i_r}_{IJ}$ is replaced by the negative timelike $p^{a_\ell}_{1A}$, and t_{IJ} by m^2_{1A}.

(c) $Q^{i_r}_{HI}$ is replaced by $Q^{a_\ell}_{AB}$, and t_{HI} by t_{AB}.

(d) The signs of the time components of $p_{2A}, p_{3A}, \cdots, p_{N_A+1,A}$ are positive.

APPENDIX C

THE SPECIAL CASE OF A THREE-PARTICLE FINAL STATE

To illustrate the techniques of this paper, we here work out, for the one-particle production shown in Fig. 4, the detailed relation between the three Toller variables ζ^{ab}, ζ^{bc}, $\omega^b = \nu^{ab} + \mu^{bc}$ and the three more familiar variables, s_{AB}, s_{BC}, s, with s_{IJ} being the invariant mass squared of the IJ cluster combination and s being the square of the total center-of-mass evergy. The remaining two variables t_{AB} and t_{BC} are common to almost all variable sets.

We begin by computing $s_{AB} = (p_{1A} - Q_{BC})^2$ in the frame a_ℓ, where p_{1A} is in conventional form while Q_{BC} is to be obtained by the transformation $L(g^{ab})$ acting on the conventional form which Q_{BC} assumes in frame b_r. The result is

$$s_{AB} = t_{BC} + m_{1A}^2 + 2\left[m_{1A}^2 + \left(z_{1A}^{a_\ell}\right)^2\right]^{\frac{1}{2}}\left[t_{BC} + \left(z_{BC}^{b_r}\right)^2\right]^{\frac{1}{2}}$$

$$\times \cosh 2\zeta^{ab} + 2\, z_{BC}^{b_r}\, z_{1A}^{a_\ell} \quad , \tag{C.1}$$

where

$$z_{1A}^{a_\ell} = \frac{m_{2A}^2 - t_{AB} - m_{1A}^2}{2\sqrt{-t_{AB}}} \, ,$$

$$z_{BC}^{b_r} = \frac{m_{2B}^2 - t_{AB} - t_{BC}}{2\sqrt{-t_{AB}}} \, . \tag{C.2}$$

Next we compute $s_{BC} = (Q_{AB} - p_{1C})^2$ in the frame b_ℓ, where Q_{AB} is in conventional form while p_{1C} is to be obtained by the transformation $L(g^{bc})$ acting on $p_{1C}^{c_r}$ [the form assumed by p_{1C} in frame c_r.] The result is

$$s_{BC} = t_{AB} + m_{1C}^2 + 2\left[m_{1C}^2 + \left(z_{1C}^{c_r}\right)^2\right]^{\frac{1}{2}} \left[t_{AB} + \left(z_{AB}^{b_\ell}\right)^2\right]^{\frac{1}{2}}$$

$$\times \cosh 2\zeta^{bc} + 2\, z_{AB}^{b_\ell}\, z_{1C}^{c_r} \quad , \tag{C.3}$$

where

$$z_{1C}^{c_r} = \frac{m_{2C}^2 - t_{BC} - m_{1C}^2}{2\sqrt{-t_{BC}}} \quad ,$$

$$z_{AB}^{b_\ell} = \frac{m_{2B}^2 - t_{BC} - t_{AB}}{2\sqrt{-t_{BC}}} \quad . \tag{C.4}$$

Note that s_{IJ} increases linearly with $\cosh 2\zeta^{ij}$ when the momentum transfers are held fixed. Large values of $\cosh 2\zeta^{ij}$ thus imply large values of s_{IJ} at fixed values of t_{AB}, t_{BC} and ω^b. It is dangerous, however, to interchange the variables ω^b and $s = (p_{1A} - p_{1C})^2$ in asymptotic analysis. It is true, as we shall see, that s is a linear function of $\cos \omega^b$ but it turns out to be impossible to keep s fixed as s_{AB} and (or) s_{BC} tend to infinity.

To establish the foregoing we compute $s = (p_{1A} - p_{1C})^2$ in the a_ℓ frame, where p_{1A} is in conventional form and where

$$p_{1C}^{a_\ell} = L(\mu^{ab}, \zeta^{ab}, \omega^b)\, L(q^b)\, L(0, \zeta^{bc}, \nu^{bc})\, p_{1C}^{c_r} .$$

We find

$$\begin{aligned} s = {} & 2\left[m_{1C}^2 + \left(z_{1C}^{c_r}\right)^2\right]^{\frac{1}{2}} \left[m_{1A}^2 + \left(z_{1A}^{a_\ell}\right)^2\right]^{\frac{1}{2}} \\ & \times \left[\frac{z_{AB}^{b_\ell}}{\sqrt{-t_{AB}}} \cosh 2\zeta^{ab} \cosh 2\zeta^{bc} + \cos 2\omega^b \sinh 2\zeta^{ab} \sinh 2\zeta^{bc}\right] \\ & + 2\left[\frac{t_{AB} + \left(z_{AB}^{b_\ell}\right)^2}{-t_{AB}}\right]^{\frac{1}{2}} \left\{ z_{1A}^{a_\ell} \left[m_{1C}^2 + \left(z_{1C}^{c_r}\right)^2\right]^{\frac{1}{2}} \cosh 2\zeta^{bc} \right. \\ & \left. + z_{1C}^{c_r} \left[m_{1A}^2 + \left(z_{1A}^{a_\ell}\right)^2\right]^{\frac{1}{2}} \cosh 2\zeta^{ab} \right\} \\ & + 2\,\frac{z_{1A}^{a_\ell}\, z_{1C}^{c_r}\, z_{AB}^{b_\ell}}{\sqrt{-t_{AB}}} + m_{1A}^2 + m_{1C}^2 . \end{aligned} \tag{C.5}$$

Thus regardless of the value of the angle ω^b, s increases without limit as ζ^{ab} and (or) ζ^{bc} increase.

In terms of the Toller variables, Regge asymptotic behavior of the amplitude means

$$f(\zeta^{ab}, t_{AB}, \omega^b, t_{BC}, \zeta^{bc}) \underset{\substack{\zeta^{ab}\to\infty \\ \zeta^{bc}\to\infty}}{\sim} ,$$

$$\sim \phi^A(t_{AB}) \left[\cosh 2\zeta^{ab}\right]^{\alpha_1(t_{AB})} \phi^B(t_{AB}, \omega^b, t_{BC}) \left[\cosh 2\zeta^{bc}\right]^{\alpha_2(t_{BC})} \phi^C(t_{BC}) , \tag{C.6}$$

with t_{AB}, t_{BC}, and ω^b fixed.

We have noted that replacement of $\cosh 2\zeta^{ab}$ by s_{AB} and of $\cosh 2\zeta^{bc}$ by s_{BC} maintains the structure of this limiting form, but introduction of the variable s would require a complicated prescription, through Formulas (C.1), (C.3), and (C.5), as to how s is supposed to vary when s_{AB} and s_{BC} increase. The advantage of the Toller variables is evident.

FOOTNOTES AND REFERENCES

* This work was supported in part by the U.S. Atomic Energy Commission.

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1. M. Toller, Nuovo Cimento 37, 631 (1965).

2. A definite set of Toller variables is prescribed for each of the different ways that the final group of particles can be successively subdivided.

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6. R. G. Roberts and G. M. Fraser, Regge Poles and High Energy Single Particle Production (Imperial College Preprint, 1967).

7. H. Chan, K. Kajantie and G. Ranft, High-Energy Collisions Producing Three Final Particles (CERN preprint 66/1260/5-TH.719).

8. F. Zachariasen and G. Zweig, Bounded Momentum Transfer Restrictions on High-Energy Interactions (California Institute of Technology preprint CALT-68-116) and High-Energy Interactions and a Multi-Regge Pole Hypothesis (California Institute of Technology preprint CALT-68-117).

9. The Toller variables, in a special case discussed in Appendix C, are the analytic continuation of variables introduced by Wick to describe three-particle configurations. Wick-type variables form the basis for the analysis of Chan, Kajantie and Ranft (loc.cit.) as well as that of Razmi (loc.cit.).

10. We limit ourselves here to $N_{A,B} > 1$. The special case of $N_{A,B} = 1$ will be considered in Section IV.

11. See Appendix B for a particular choice of internal cluster variables.

12. The relation is

$$s_{AB} = m_{1A}^2 + m_{1B}^2 + 2 \cosh 2\zeta^{ab}(m_{1A}^2 + z_{1A}^2)^{1/2}(m_{1B}^2 + z_{1B}^2)^{1/2} .$$

13. See Appendix A for details.

14. A kinematical dependence on μ^{ab} appears if particles 1A or 2A have spin.

15. When spin is included, discrete representations corresponding to half-integer values of ℓ_i may be present in Eq. (V.3).

FIGURE CAPTIONS

Fig. 1. Decomposition of a multi-particle production amplitude into two clusters, cluster A containing $N_A + 1$ particles, and cluster B containing $N_B + 1$ particles.

Fig. 2. A multi-cluster grouping for the final particles in a production amplitude.

Fig. 3. A three-cluster decomposition which allows a definition of the interaction of one Regge pole with another.

Fig. 4. The two particle-three particle reaction, with single-particle clusters.

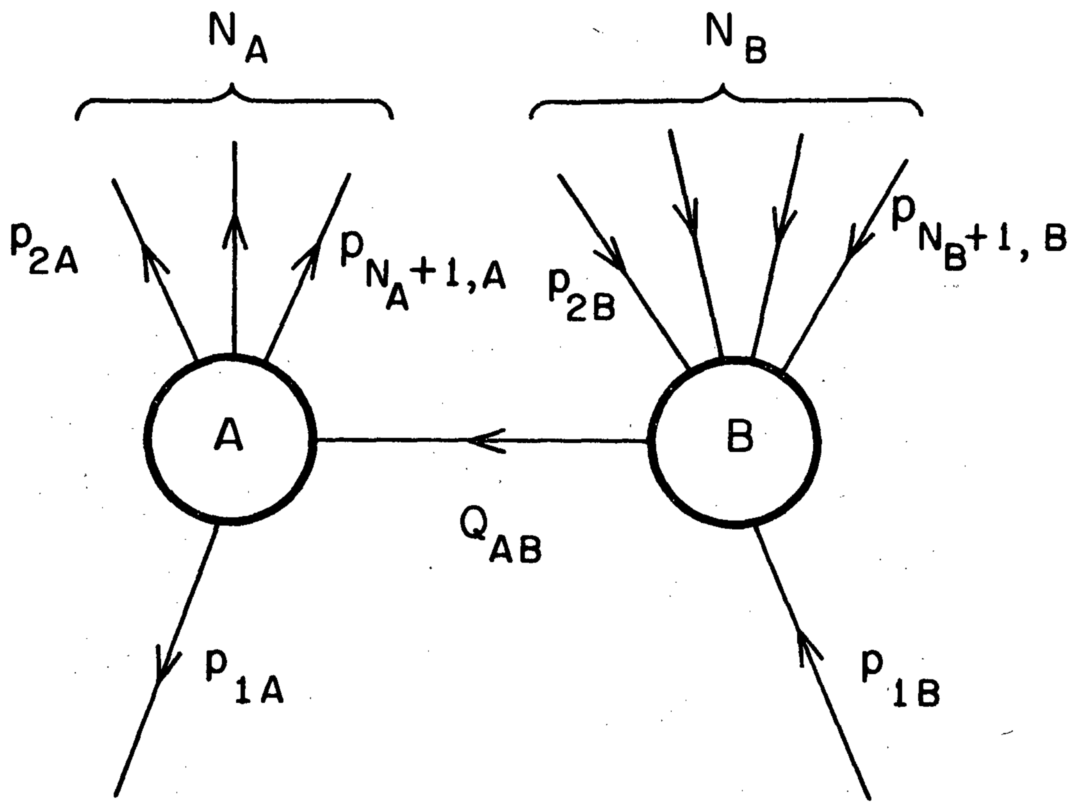

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Fig. 1

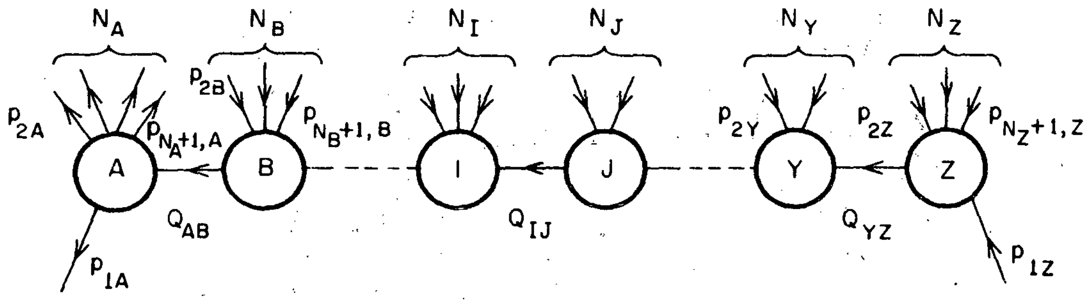

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Fig. 2

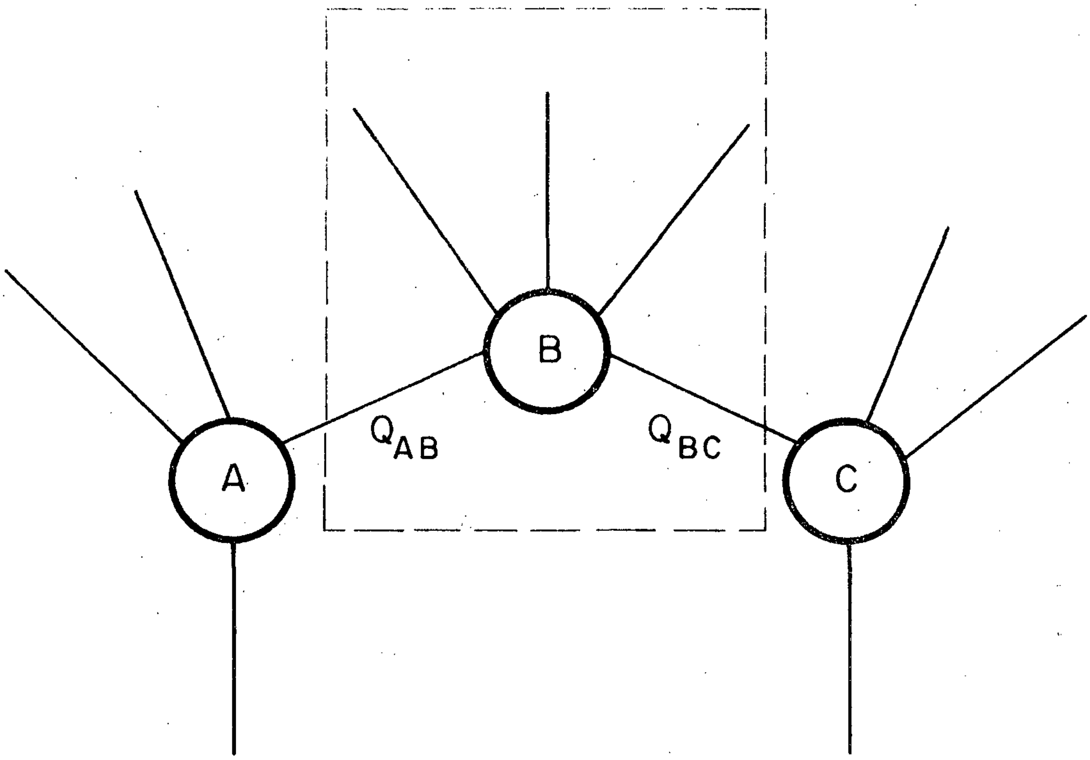

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Fig. 3

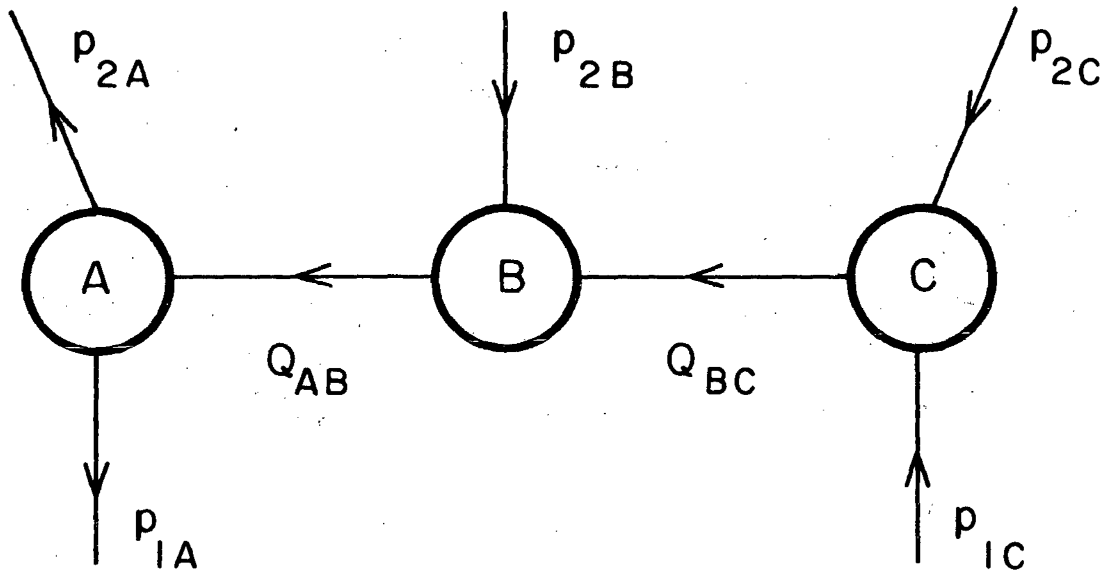

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Fig. 4