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A Characterization of Robust Sunspot Equilibria[¤]

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Abstract

In nonconvex environments, a sunspot equilibrium can sometimes be destroyed by the introduction of new extrinsic information. We provide a simple test for determining whether or not a particular equilibrium survives, or is robust to, all possible re...nements of the state space. We use this test to provide a characterization of the set of robust sunspot-equilibrium allocations of a given economy: it is equivalent to the set of equilibrium allocations of the associated lottery economy.

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1. Introduction

In sunspots economies, as introduced in Shell [8] and Cass and Shell [2] , prices and allocations are based on an extrinsic randomizing device that is typically taken to be a parameter of the economy. This raises the question of how the set of equilibria depends on the choice of this randomizing device. To address this issue, Goenka and Shell [5] introduce the concept of *robustness* of sunspot equilibria. A particular equilibrium allocation is said to be *robust to refinements* if it is also an equilibrium allocation of an economy based on any refinement of the original randomizing device. Robust equilibrium allocations necessarily "survive" the introduction of new information about the extrinsic state space, while nonrobust allocations may not.[1]

As an example, consider an economy where there are originally two possible extrinsic events (either there is sunspot activity or there is not) and look at a particular equilibrium allocation of this economy. Suppose the consumers in this economy start to entertain the possibility that the thickness of the briefcase carried by the chairperson of the central bank on a given day (either thick or thin) also affects prices, and hence that markets open for briefcase-contingent trade.[2] There are now four possible events (sunspots and thick briefcase, sunspots and thin briefcase, etc.), and the question is whether the original equilibrium allocation (which is briefcase-independent) is also an equilibrium allocation of the new economy. If so, then the allocation is robust to this particular refinement of the set of extrinsic events. Goenka and Shell [5] establish that for convex economies with restricted participation, sunspot equilibria based on a finite number of events are always robust to refinements. In other words, there always exists an equilibrium of the refined economy where consumers ignore the new information. However, they demonstrate through examples that this is not true for nonconvex economies. In such environments, some sunspot equilibrium allocations are robust to refinements but others are not; it may be the case that every equilibrium of the refined economy depends on the new information in a nontrivial way.[3]

In this paper, we provide a simple test for identifying robust sunspot equilibria in nonconvex economies with perfect markets. Our results are an application of a recent paper by Garratt *et al.*

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[1]   A related concept is the *strong sunspot immunity* of Antinolfi and Keister [1] , which requires that the set of equilibrium allocations be independent of the randomizing device.

[2]   We assume, of course, that briefcase thickness has no real effects.

[3]   See also Garratt [3] , which considers nonconvex economies and establishes the robustness (although not by that name) of degenerate sunspot equilibrium allocations that are also deterministic lottery equilibrium allocations.

[4] , which establishes a relationship between the equilibrium allocations of a sunspots economy[4] based on a continuous random variable and those of a lottery economy of the type introduced by Prescott and Townsend [6] , [7] . There is a precise sense in which the sets of equilibrium allocations generated by these two models are identical. We show how this result greatly simplifies the procedure for checking whether or not a given sunspot equilibrium allocation is robust to refinements. In particular, we show that it is not necessary to check whether an equilibrium allocation is robust to *every* possible refinement (a difficult task, to be sure); it suffices to check whether it is robust to the refinement to (any) one continuous randomizing device. The allocation is robust to every refinement if and only if it is robust to a continuous one.

In the process of proving this result, we derive an alternate characterization of robust sunspot equilibria: The set of robust sunspot equilibrium allocations is equivalent to the set of equilibrium allocations of the corresponding lottery economy. This helps clarify the relationship between the lottery model and the finite-event sunspots model. Garratt [3] demonstrates that every lottery equilibrium has a corresponding sunspot equilibrium, but that there exist sunspot equilibria that have no lottery equilibrium counterpart. Our result demonstrates that these additional equilibria are exactly the nonrobust equilibria of the sunspots model.

We also provide a test for robustness based on support prices: If a sunspot equilibrium allocation cannot be supported by prices that, when adjusted for the probabilities of the extrinsic states, are constant across states, then it is not robust to refinements. Although it is only a necessary condition for robustness, this test has the advantage of being based solely on the original randomizing device.

In the next section, we describe the sunspots economy, paying particular attention to the formulation of the randomizing device. We also briefly describe the corresponding lottery economy and the equivalence result of Garratt *et al.* [4] . In section 3 we then present our results on testing for and characterizing robust sunspot equilibria.

2.   The Model

The model we consider is essentially that of Garratt *et al.* [4] , a pure exchange economy with perfect markets and no intrinsic uncertainty. There are arbitrary, finite numbers of consumers and

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[4]   As introduced in Shell [8] and Cass and Shell [2] .

commodities, and all consumers have access to complete markets for state-contingent commodities. The only way in which this model differs from the canonical general equilibrium model is that the consumption set need not be convex. This nonconvexity may lead consumers to desire random allocations, and thereby introduces a role for extrinsic uncertainty.

2.1 Extrinsic Uncertainty

Extrinsic uncertainty is represented by a probability space (S, Σ, μ), where S is the unit interval $[0, 1)$, Σ is a σ-algebra of subsets of S, and μ is a probability measure on (S, Σ). We assume that all sets in Σ are Borel-measurable (*i.e.*, that Σ is a sub-σ-field of the Borel sets). The probability space associated with a particular economy is called the *randomizing device* (RD) for that economy. Given an equilibrium allocation, we are interested in whether or not it survives refinement of the randomizing device. We take the definition of refinement from Goenka and Shell [5] .

Definition (Goenka and Shell [5] **):** A randomizing device $\left(S, \hat{\Sigma}, \hat{\mu} \right)$ is said to be a *refinement* of the randomizing device (S, Σ, μ) if we have

(i) $\Sigma \subseteq \hat{\Sigma}$; that is, every Σ-measurable set is $\hat{\Sigma}$ measurable

and

(ii) If A is a Σ-measurable set with measure $\mu(A)$, then $\hat{\mu}(A) = \mu(A)$.

The set of all refinements of (S, Σ, μ) is denoted by $Ref(S, \Sigma, \mu)$. Our test for robustness applies only to economies where the RD can be refined to a continuous one, that is, where $Ref(S, \Sigma, \mu)$ contains some element with $\hat{\Sigma}$ equal to the Borel sets and $\hat{\mu}$ absolutely continuous with respect to Lebesgue measure. While this rules out economies based on certain types of mixed distributions, it includes economies based on any discrete distribution (as well as any continuous one, of course). Following Goenka and Shell [5] , a discrete distribution is defined slightly differently from the typical approach, which uses a countable state space and assigns probability p_j to point j in that space. In the setup here, a discrete distribution is a σ-algebra generated by a partition of $[0, 1)$ into a countable number of disjoint intervals, with probability p_j assigned to interval j. Such a discrete RD can then easily be refined to a continuous one by, say, "spreading" the probability evenly over each interval.

One natural way for a countable RD to be refined is by the introduction of a new extrinsic random variable, as in our example in the previous section involving sunspots and briefcase thickness. In that example, the original RD depends only on sunspots and can be represented by the σ-algebra generated by, say, the intervals $\left[0; \frac{1}{2}\right)$ and $\left[\frac{1}{2}; 1\right)$; with μ assigning the appropriate probability to each interval. When markets open for briefcase-contingent trade, these intervals can be divided, or refined, as follows: $\left[0; \frac{1}{4}\right)$ now represents sunspots and thick briefcase, $\left[\frac{1}{4}; \frac{1}{2}\right)$ sunspots and thin briefcase, $\left[\frac{1}{2}; \frac{3}{4}\right)$ no sunspots and thick briefcase, and $\left[\frac{3}{4}; 1\right)$ no sunspots and thin briefcase. These intervals generate the new σ-algebra $\mathcal{S}$; and the new probability measure ν assigns the appropriate joint probability to each interval. If the two variables are independent, for example, then the value assigned by ν to the first interval is simply the probability of sunspot activity multiplied by the probability of a thick briefcase. It is easy to see that the RD so defined is a refinement of the original one.

2.2   Consumers

There is a finite set of consumers $N = \{1; \ldots; h; \ldots; n\}$. In each state of nature, the commodity bundle c chosen by a consumer must be contained in the set C; a Borel set in $\mathbb{R}_+^\ell$: We do *not* assume that C is convex; this is what introduces the possibility of nondegenerate sunspot equilibria in this model. As in Garratt *et al.* [4] , in cases where C is bounded our results apply only to equilibria in which no consumer receives her most-preferred allocation in every state. The commodity space is the set of $\mathcal{S}$-measurable functions $x_h : S \to \mathbb{R}_+^\ell$ that are bounded in the essential supremum norm. The consumption set X is the subset of functions such that $x_h(s) \in C$ for every state s: A price system is a function $p : S \to \mathbb{R}_+^\ell$ that is $\mathcal{S}$-measurable and bounded in the L_1 norm.

Since all uncertainty is extrinsic, the endowment e_h of consumer h is independent of the state of nature: Each consumer is assumed to be globally nonsatiated, and consumer h's preferences are represented by a von Neumann-Morgenstern utility function $u_h : C \to \mathbb{R}$: The consumer's problem is to

$$\max_{x_h} \int_S u_h(x_h(s))\mu(ds)$$

$$\text{subject to} \quad \int_S p(s) \cdot x_h(s)\mu(ds) \le \int_S p(s) \cdot e_h\,\mu(ds)$$

$$x_h \in X:$$

Since the integration in the budget constraint is with respect to the probability measure $\frac{1}{4}$; the price function p represents probability-adjusted prices.

An aggregate allocation $x = (x_h)_{h2N}$ is feasible if markets clear almost surely. Let $a = (a_h)_{h2N}$ denote a pure (nonstochastic) allocation with $a_h \, 2 \, C$ for every h: The set of feasible pure allocations is then given by

$$F = \left(a \, 2 \, C^n : \, \underset{h}{\times} a_h \cdot \underset{h}{\times} e_h \right) : \tag{1}$$

Feasibility of x requires that $x(s) \, 2 \, F$ hold for almost every s: We denote an economy by $E \, (S; \, \S; \, \frac{1}{4}) :^5$

2.3 Equilibrium

A *sunspot equilibrium* consists of a price function $p^{\text{¤}}$ and an allocation $x^{\text{¤}} \, 2 \, X^n$ such that (i) given $p^{\text{¤}}$; $x_h^{\text{¤}}$ solves the consumer's maximization problem for each $h \, 2 \, N$; and (ii) $x^{\text{¤}}$ is feasible. We denote the set of equilibrium allocations of the economy $E \, (S; \, \S; \, \frac{1}{4})$ by $E \, (S; \, \S; \, \frac{1}{4})$: We next give the precise definition of what it means for an equilibrium to be robust to all refinements of the RD.

Definition (Goenka and Shell [5]): An equilibrium allocation $x^{\text{¤}}$ of the economy $E \, (S; \, \S; \, \frac{1}{4})$ is said to be *robust to refinements* if for every $\left(S; \, \hat{\S}; \, \hat{\frac{1}{4}} \right) \, 2 Ref \, (S; \, \S; \, \frac{1}{4})$ we have $x^{\text{¤}} \, 2 \, E \, \left(S; \, \hat{\S}; \, \hat{\frac{1}{4}} \right) .^6$

2.4 The Corresponding Lottery Economy

Let $\mathbb{C} \, (C)$ denote the set of probability measures over the set C: An individual lottery for consumer h is an element of $\mathbb{C} \, (C)$; denoted $\pm_h$: Let $q \, 2 \, R_+$ denote a vector of prices. Consumer h's lottery-choice problem is given by

$$\max_{\pm_h} \int_C u_h(c)\pm_h(dc)$$
$$\text{subject to } q \, \mathbb{C} \int_C c \, \pm_h(dc) \, \cdot \, q \, \mathbb{C} \, e_h$$
$$\pm_h \, 2 \, \mathbb{C}(C):$$

Feasibility of an allocation requires that each individually-demanded lottery be the marginal of

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[5] For futher details on and motivation for this setup, the reader is referred to Garratt *et al.* [4] .

[6] This definition differs slightly from that used by Goenka and Shell [5] in that it applies to a specific equilibrium allocation instead of the entire set of equilibrium allocations of an economy. This is solely a matter of terminology. We use this definition because our test for robustness below applies to individual allocations.

some joint probability distribution over feasible pure (nonstochastic) allocations of the endowments (see Garratt [3]). Let $\mathbb{C}(F)$ denote the set of probability measures over the set F defined in (1); elements L of this set are called joint lotteries: A *lottery equilibrium* is a price vector $q^* \in \mathbb{R}_+$ and an allocation $\mu^* = (\mu_h^*)_{h \in N}$ such that (i) given q^*; μ_h^* solves the consumer's lottery-choice problem for each $h \in N$; and (ii) there exists a joint lottery $L^* \in \mathbb{C}(F)$ such that μ_h^* is the marginal distribution of L^* for each $h \in N$.

For any sunspots economy, there is a corresponding lottery economy with the same set of consumers and the same set of feasible consumption bundles C: Since a lottery allocation is an element of $\mathbb{C}(C)^n$ and a sunspot allocation is an element of X^n; however, we must perform a translation in order to compare the equilibria of the two models. Note that every sunspot allocation induces a probability distribution over F; and therefore induces a lottery allocation. At the individual level, the consumption plan x_h induces the individual lottery μ_h through the equation

$$\mu_h = \frac{1}{4} \pm x_h^{i\,-1}:$$

A sunspot allocation x then induces a lottery allocation μ simply by inducing the individual lottery μ_h for each consumer h. The reverse of this procedure is slightly more complicated, since many different sunspot allocations can induce the same lottery allocation. Garratt, *et al.* [4] show that every individual lottery μ_h is induced by an equivalence class of sunspot consumption plans. Using this fact, the equivalence result from that paper can be stated in terms of equilibrium allocations as follows.

Result (Garratt *et al.*[4]): Let x^* be an allocation in the sunspots economy $E(S; \S; \frac{1}{4})$: If $(S; \S; \frac{1}{4})$ is a continuous randomizing device, then x^* is an equilibrium allocation if and only if the lottery allocation μ^* that it induces is an equilibrium allocation of the corresponding lottery economy.

3.   Characterization of Robust Equilibria

In principle, establishing the robustness of an equilibrium allocation requires showing that it is also an equilibrium allocation for *every* possible refinement of the probability space. This is clearly a difficult task. In this section, we show that only *one* refinement needs to be checked, the refinement

to any one continuous RD. We then state a pair of implications of this result as corollaries.

Theorem: A sunspot equilibrium allocation x^π is robust to refinements if and only if it is robust to the refinement to some continuous randomizing device.

Proof. In one direction the statement is trivial: if x^π is robust to any refinement, then it is robust to the refinement to any continuous RD. For the other direction, suppose that x^π is robust to the refinement to some continuous RD. Then, from the result of Garratt *et al.*[4] stated above, we know that the lottery allocation which has

$$\pm_h^\pi (A) = \frac{1}{4} \, {}^i x_h^{\pi_i \, 1} (A)^\complement;$$

for every Borel subset A of C and for every h; is an equilibrium allocation of the associated lottery economy. This implies that there exists a price vector $q^\pi \, 2 \, R_+$ such that, for every consumer h; the measure $\pm_h^\pi$ solves

$$\max_{\pm_h} \int_C u_h(c) \pm_h(dc) \tag{2}$$
$$\text{subject to } q^\pi \, \complement \int_C c \pm_h(dc) \; \cdot \; q^\pi \, \complement \, e_h$$
$$\pm_h \; 2 \; \complement(C):$$

The corresponding price system in a sunspots economy is the constant function $p^\pi(s) = q^\pi$ for all s: We now show that this price system supports the allocation x^π as an equilibrium of the economy based on any refinement of the original RD.

Let $\, S; \hat{S}; \mathbb{1}\,$ be an arbitrary refinement of $(S; S; \frac{1}{4})$; and consider the sunspots economy $E \, S; \hat{S}; \mathbb{1} \,$: If prices are given by the constant price function $p^\pi(s) = q^\pi$ for all s, then the problem of consumer h is to

$$\max_{x_h} \int_S u_h(x_h(s)) \mathbb{1}(ds) \tag{3}$$
$$\text{subject to } q^\pi \, \complement \int_S x_h(s) \mathbb{1}(ds) \; \cdot \; q^\pi \, \complement \, e_h$$
$$x_h \; 2 \; X:$$

Note that, because $\left(S, \hat{S}, \hat{\mu}\right)$ is a refinement of (S, S, μ), we have that

$$\hat{\mu}\left(x_h^{\pi i\,-1}(A)\right)^{\complement} = \mu\left(x_h^{\pi i\,-1}(A)\right)^{\complement} = \mu_h^\pi(A)$$

holds for every Borel subset A of C and for every h; that is, x_h^π generates the same probability distribution over C in both economies. Changing variables to integrate over the consumption set rather than over states shows that this problem is equivalent to

$$\max_{\mu_h} \int_C u_h(c)\,\mu_h(dc) \tag{4}$$

$$\text{subject to } q^\pi \cdot \int_C c\,\mu_h(dc) \cdot q^\pi \cdot e_h$$

$$\mu_h \in \mathcal{C}^0(C):$$

where $\mathcal{C}^0(C)$ is the set of all probability distributions that can be generated by some $x_h \in X$ for the economy $E\left(S, \hat{S}, \hat{\mu}\right)$: Note that we have the natural inclusion relation

$$\mathcal{C}^0(C) \subseteq \mathcal{C}(C);$$

that is, the set of probability distributions generated by elements of X is contained in the set of all probability distributions over C: We know that μ_h^π solves (2) and is in the constraint set of (4), since it is generated by $x_h^\pi \in X$: It follows that μ_h^π also solves (4). Since problem (4) is equivalent to problem (3), any x_h that generates μ_h^π; including our original allocation x_h^π, is a solution to (3): Hence, at prices p^π; x_h^π is optimal for every consumer h in economy $E\left(S, \hat{S}, \hat{\mu}\right)$: This demonstrates that $(x^\pi; p^\pi)$ is an equilibrium of $E\left(S, \hat{S}, \hat{\mu}\right)$: Since $\left(S, \hat{S}, \hat{\mu}\right)$ is an arbitrary element of (S, S, μ); we have shown that x^π is robust to any refinement. ∎

The theorem shows that robustness to any refinement is equivalent to robustness to the refinement to a continuous RD. This, in turn, is equivalent to the induced lottery allocation being an equilibrium of the lottery economy. Thus we have the following corollary for any economy whose RD can be refined to a continuous one.

Corollary 1: A sunspot equilibrium allocation x^π is robust to refinements if and only if the associated lottery allocation is an equilibrium of the lottery economy.

The proof of the theorem also points out a partial characterization of robust equilibria that can be made in terms of supporting prices. Suppose an equilibrium allocation x^α based on the RD $(S; \S; \frac{1}{4})$ is robust to the refinement to some continuous sunspot variable. Then the induced lottery allocation is an equilibrium of the lottery economy supported by some prices q^α: The proof shows how this implies that the sunspot allocation x^α is supported by the constant price function $p^\alpha(s) = q^\alpha$ for all s in the economy based on *any* element of *Ref* $(S; \S; \frac{1}{4})$; including $E\ (S; \S; \frac{1}{4})$ itself. This means that any robust allocation can be supported (in the original economy) by a constant price function, and hence gives us the following test for robustness.

Corollary 2: If a sunspot equilibrium allocation cannot be supported by a price function that is constant across states, then it is not robust to refinements.

An example of a sunspot equilibrium allocation that cannot be supported by a constant price function is given in section 2.3 of Garratt, *et al.* [4] . Note that, unlike the theorem and the first corollary, this test only requires examination of the original economy. It does not require evaluation of a continuum-of-states economy or of the associated lottery economy. It gives us a necessary condition for robustness, but it is not sufficient. It is not difficult to construct examples of equilibria that are supported by constant price functions but are not robust.

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