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Inquisitive Intuitionistic Logic

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Abstract

Inquisitive logic is a research program seeking to expand the purview of logic beyond declarative sentences to include the logic of *questions*. To this end, inquisitive propositional logic extends classical propositional logic for declarative sentences with principles governing a new binary connective of *inquisitive disjunction*, which allows the formation of questions. Recently inquisitive logicians have considered what happens if the logic of declarative sentences is assumed to be intuitionistic rather than classical. In short, what should inquisitive logic be on an intuitionistic base? In this paper, we provide an answer to this question from the perspective of *nuclear semantics*, an approach to classical and intuitionistic semantics pursued in our previous work. In particular, we show how Beth semantics for intuitionistic logic naturally extends to a semantics for inquisitive intuitionistic logic. In addition, we show how an explicit view of inquisitive intuitionistic logic comes via a translation into *propositional lax logic*, whose completeness we prove with respect to Beth semantics.

Keywords: inquisitive logic, intuitionistic logic, Kripke semantics, Beth semantics, algebraic semantics, Heyting algebra, nucleus, lax logic

1 Introduction

Inquisitive logic is a research program seeking to expand the purview of logic beyond declarative sentences to include the logic of *questions* (see, e.g., [7,12,9,8,10]). While classical logic is based on the idea that any *state of the world* that *makes true* certain declarative sentences also makes true certain other declarative sentences, inquisitive logic is based on the idea that any *state of information* that *answers* certain questions (and incorporates the truth of certain declarative sentences) also answers certain other questions (and incorporates the truth of certain other declarative sentences). Thus, one may study a notion of consequence not only between declarative sentences but also between questions, as well as combinations of declaratives and questions.

To formalize this new notion of consequence, the language of inquisitive propositional logic extends that of classical propositional logic for declarative sentences with a new binary connective of *inquisitive disjunction*, $\mathbin{\mathbb{V}}$, which allows the formation of questions. The formula $p \mathbin{\mathbb{V}} q$ represents the question of *whether* p *or* q, in contrast to the formula $p \vee q$, which represents the declarative sentence p *or* q. To make this distinction with a formal semantics, classical inquisitive semantics evaluates a formula of the inquisitive language at an information state, understood as a set of classical propositional valuations—the states of the world compatible with the information. An information state *answers the question* $p \mathbin{\vee\!\!\!\vee} q$ just in case every valuation in the information state satisfies p or every valuation in the information state satisfies q; by contrast, an information state *supports the declarative* $p \vee q$ just in case every valuation in the information state satisfies p or satisfies q.[1] Thus, while every information state supports the declarative $p \vee \neg p$, not every information state answers the question $p \mathbin{\vee\!\!\!\vee} \neg p$. This gives reasoning with inquisitive disjunction an intuitionistic flavor. Yet the logic of declarative sentences (formulas without $\mathbin{\vee\!\!\!\vee}$) underlying inquisitive logic is classical.

Recently inquisitive logicians have considered what happens if the logic of declarative sentences is assumed to be intuitionistic rather than classical [24,25,11]. In short, what should inquisitive logic be on an intuitionistic base? This is a natural question not only because of the general interest in intuitionistic logic as a formalization of constructive reasoning with declarative sentences, but also because of the affinity between information-state-based semantics and intuitionistic semantics in the style of Beth [1], Grzegorczyk [17], and Kripke [21]. In fact, the classical inquisitive semantics sketched above may be seen as a special case of intuitionistic Kripke semantics, based on restricting to special Kripke models: the underlying poset of the Kripke model must be the set of all nonempty subsets of a set, ordered by reverse inclusion (a "topless Boolean algebra"), and the valuation of each proposition letter in the Kripke model must be a *regular element* of the Heyting algebra of upsets of the poset, i.e., an upset U such that $U = U^{**}$, where * is the pseudocomplement operation in the Heyting algebra of upsets, which is used in Kripke semantics to interpret the intuitionistic negation connective $\neg$. Restricting the valuation of proposition letters to regular elements, the usual Kripke clauses for $\neg$ and $\wedge$, plus the classical definition of $\vee$ in terms of $\neg$ and $\wedge$, yields classical logic for the declarative fragment of the inquisitive language; then interpreting $\mathbin{\vee\!\!\!\vee}$ as the standard Kripke disjunction—as the join (union) in the Heyting algebra of upsets—is responsible for the intuitionistic flavor of $\mathbin{\vee\!\!\!\vee}$ noted above.

Given this connection between classical inquisitive semantics and intuitionistic Kripke semantics, how should one modify the semantics to obtain an intuitionistic base logic of declaratives? Ciardelli et al. [11] do so by moving up one level set-theoretically in Kripke models: their semantics evaluates a formula at a subset of a Kripke model, called a *team*. As the points in an intuitionistic Kripke model are traditionally thought of as information states, a team may be thought of as a set of information states—and therefore as a kind of higher-order information state.

In this paper, we pursue a different semantic approach to inquisitive logic on an intuitionistic base. In our semantics, we evaluate formulas of the inquisitive

[1] This is the semantics for proposition letters p and q. For the general recursive clause, see any of the cited references on classical inquisitive logic.

language at individual states in a poset, not at sets of states of a poset. We are able to do so by switching from Kripke semantics on posets to *Beth semantics* on posets. The difference between our approach and that of Ciardelli et al. [11] can be traced to different perspectives on *declarative disjunction* in classical inquisitive semantics. As noted above, in the original approach to inquisitive logic (see, e.g., [8, Def. 2.1.2]), the declarative disjunction $\vee$ is defined in terms of $\neg$ and $\wedge$ using the usual classical definition: $\varphi\vee\psi := \neg(\neg\varphi\wedge\neg\psi)$. Since $\neg$ and $\wedge$ are interpreted as the pseudocomplement and meet (intersection) operations in the Heyting algebra of upsets of a Kripke model, the definition of $\vee$ in terms of $\neg$ and $\wedge$ is equivalent to interpreting $\vee$ as the regularization of the join:

$$V(\varphi \vee \psi) = (V(\varphi) \sqcup V(\psi))^{**}.$$

Thus, we see classical inquisitive semantics as follows:

- the semantic values of formulas are elements of a Heyting algebra of upsets of a special kind of poset (a topless Boolean algebra);
- the semantic values of proposition letters must be regular elements of the Heyting algebra;
- $\neg$ and $\wedge$ are interpreted as pseudocomplement and meet, respectively, in the Heyting algebra;
- the inquisitive disjunction $\mathbin{\vee\!\!\!\vee}$ is interpreted as the join in the Heyting algebra;
- the declarative disjunction $\vee$ is interpreted as the regularization of the join in the Heyting algebra.

The regularization operation $(\cdot)^{**}$ is an example of a *nucleus* on the Heyting algebra of upsets of a poset (see Section 5 for a definition). The fixpoints of this nucleus—the regular elements—form a Boolean algebra, in which the join of two elements is the regularization of their join in the Heyting algebra. This explains why standard inquisitive semantics, which interprets proposition letters as regular elements and interprets declarative disjunction as the regularization of the join in the Heyting algebra of upsets, yields classical logic for the declarative fragment of the inquisitive language. It also explains why inquisitive logic is not closed under uniform substitution of complex formulas for proposition letters, e.g., why $\neg\neg p \to p$ is valid while $\neg\neg(q \mathbin{\vee\!\!\!\vee} r) \to (q \mathbin{\vee\!\!\!\vee} r)$ is not. This happens because while proposition letters are interpreted as regular elements, the join operation in the Heyting algebra can take one out of the algebra of regular elements. To summarize:

- the semantic values of declarative formulas live in the algebra of fixpoints of $(\cdot)^{**}$, while the semantic values of arbitrary formulas may live anywhere in the ambient Heyting algebra.

From this perspective, also adopted in [5], there is a natural way of obtaining semantics for inquisitive logic on an intuitionistic declarative base: we may simply switch from the Boolean nucleus $(\cdot)^{**}$ to a non-Boolean nucleus.

To do so, first note that interpreting declarative disjunction as the regularization of the join in the Heyting algbera of upsets is equivalent to using the following semantic clause:

- a state x in a poset forces $\varphi \vee \psi$ iff for every $x' \geq x$ there is an $x'' \geq x'$ such that x'' forces φ or x'' forces ψ.

In place of this classical interpretation of $\vee$, we give an intuitionistic interpretation of $\vee$ as in Beth semantics:

- a state x in a poset forces $\varphi \vee \psi$ iff every maximal chain [2] through x contains a state that forces φ or a state that forces ψ.

This amounts to interpreting declarative disjunction as the result of applying what we call the *Beth nucleus* to the join in the Heyting algebra of upsets. The Beth nucleus j_b is defined for any upset U of a poset by

$$j_b U = \{x \in X \mid \text{every maximal chain through } x \text{ intersects } U\}.$$

Not only do we interpret declarative disjunction using j_b instead of $(\cdot)^{**}$, but also we require proposition letters to be interpreted as fixpoints of j_b instead of $(\cdot)^{**}$ (i.e., we require that x forces p iff every maximal chain through x contains a state that forces p). Yet the interpretation of inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ as join in the Heyting algebra of upsets remains the same. This Beth semantics for the inquisitive language is a special case of a more general algebraic semantics for the inquisitive language based on Heyting algebras equipped with a nucleus, called *nuclear algebras*. Thus, we are extending to the inquisitive setting the *nuclear semantics* for intuitionistic logic studied in our previous work [4].

The starting point on our road to this nuclear approach to inquisitive semantics was the observation that in the classical semantics for inquisitive logic, the nucleus $(\cdot)^{**}$ is used to constrain the valuation of proposition letters and to interpret the declarative disjunction $\vee$ (just as in the *possibility semantics* of [19,18]). By contrast, Ciardelli et al. [11] have a different starting point. They begin by departing from the original definition of declarative disjunction in classical inquisitive logic as $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$ and by giving a new semantic clause for classical $\vee$ based on team semantics for dependence logic (see, e.g., [26,27]). In team semantics, disjunction is interpreted as follows:

- an information state T (set of classical valuations) supports $\varphi \vee \psi$ iff there are T', T'' such that $T = T' \cup T''$, T' supports φ, and T'' supports ψ.

Already in the classical setting, this semantics for $\varphi \vee \psi$ is not equivalent to defining $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$. For example, with the original definition of $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, the principle

$$((\varphi \vee \varphi) \vee (\varphi \vee \varphi)) \to (\varphi \vee \varphi)$$

is a theorem of inquisitive logic for any φ. Yet with $\vee$ treated as a primitive connective and interpreted using the team semantics above, the principle above

[2] In fact, we will use chains closed under upper bounds, following [4], but this subtlety does not matter here.

is not valid for all φ containing $\vee\!\!\!\vee$ (see Example 4.2). Ciardelli et al. [11] extend the team semantics for $\vee$ to the intuitionistic setting by taking T to be a subset of an intuitionistic Kripke model instead of merely a set of classical valuations.

We do not wish to argue that the nuclear approach to inquisitive logic on an intuitionistic base is superior to the team-based approach. Both are natural from different points of view. Starting from the perspective of team semantics for dependence logic, Ciardelli et al. [11] show how to "intuitionize" the semantics, by moving from teams as sets of classical valuations to teams as subsets of a Kripke model. By contrast, starting from the perspective of Beth semantics for intuitionistic logic, we show how to "inquisitivize" the semantics, by adding the Kripke interpretation for $\vee\!\!\!\vee$ to Beth semantics. This is why we call our resulting logic "inquisitive intuitionitic logic" in contrast to Ciardelli et al.'s "intuitionistic inquisitive logic."

Our main result is a completeness theorem for inquisitive intuitionistic logic with respect to Beth semantics, which is our answer to the question "What should inquisitive logic be on an intuitionistic base?" But even independently of inquisitive logic, it is a natural question whether one can prove a completeness theorem for the propositional language with two disjunctions $\vee$ and $\vee\!\!\!\vee$, with $\vee$ (and proposition letters) interpreted according to Beth semantics, $\vee\!\!\!\vee$ interpreted according to Kripke semantics, and $\neg, \to, \wedge$ having their usual interpretations, which are the same in both semantics.

We prove the completeness theorem using a detour through the intuitionistic modal logic of nuclei [16], known as *propositional lax logic* [15]. Propositional lax logic adds to the signature of intuitionistic propositional logic an operator $\bigcirc$, interpreted using the nucleus in a nuclear algebra. A key step in our proof of completeness of inquisitive intuitionistic logic with respect to Beth semantics is a proof of the completeness of propositional lax logic with respect to Beth semantics, i.e., with $\bigcirc$ interpreted as the Beth nucleus j_b on the Heyting algebra of upsets of a poset. Thus, another contribution of the paper is to provide a new semantics for propositional lax logic.

The paper is organized as follows. In Section 2, we recall the standard language of inquisitive logic and present our new semantic proposal: Beth semantics for inquisitive intuitionistic logic. In Section 3, we define for any superintuitionistic logic L its inquisitive version $\mathsf{Inq}(\mathsf{L})$. In this paper, we concentrate on the case where L is the intuitionistic propositional calculus (IPC). Our completeness theorem states that $\mathsf{Inq}(\mathsf{IPC})$ is sound and complete with respect to the class of all Beth frames (posets) according to Beth semantics. Before proving this result, in Section 4 we compare $\mathsf{Inq}(\mathsf{IPC})$ with the system InqI of Ciardelli et al. [11]. We show that the two logics are incomparable in strength. In Section 5, we develop the nuclear perspective on Beth semantics sketched above, which we turn into explicit translations between the language of inquisitive intuitionistic logic and the language of propositional lax logic in Section 6. This lets us transform the problem of proving the completeness of inquisitive intuitionistic logics with respect to Beth semantics into the problem of proving the completeness of propositional lax logics with respect to Beth semantics. We work up to Beth completeness in three stages:

- in Section 6, we obtain completeness with respect to finite nuclear algebras (proved in Appendix A);
- in Section 7, as an intermediate step, we transfer completeness with respect to finite nuclear algebras to completeness with respect to certain finite relational structures, which we call "S-frames," from [3];
- in Section 8, we transfer completeness with respect to finite S-frames to Beth completeness.

2 Beth Semantics for Inquisitive Logic

The *inquisitive intuitionistic language* $\mathcal{L}_{\vee,\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}}$ is defined as follows, where p belongs to a countably infinite set Prop of proposition letters:

$$\varphi ::= \bot \mid p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi) \mid (\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi).$$

As usual, we define $\neg\varphi := \varphi \to \bot$. Let $\mathcal{L}_{\vee}$ be the fragment without $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, and let $\mathcal{L}_{\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}}$ be the fragment without $\vee$.

Toward introducing our semantics for $\mathcal{L}_{\vee,\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}}$, we need the following notions.

Definition 2.1 Given a poset X, we define:

(i) $\mathsf{Up}(X)$ is the set of all upward closed subsets (upsets) of X, i.e., those $U \subseteq X$ such that if $x \in U$ and $x \leq y$, then $y \in U$;

(ii) a *chain* in X is a $C \subseteq X$ such that for all $x, y \in C$, $x \leq y$ or $y \leq x$;

(iii) a *path* in X is a chain C in X that is closed under upper bounds, i.e., if for all $x \in C$, $x \leq y$, then $y \in C$. If $x \in C$, then C is a *path through* x.

Our proposal is to simply extend Beth semantics [1] for intuitionistic logic (following the presentation in [4]) with $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ interpreted as in Kripke semantics.

Definition 2.2 For any poset X, $x \in X$, valuation $v : \mathsf{Prop} \to \mathsf{Up}(X)$, and $\varphi \in \mathcal{L}_{\vee,\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}}$, we define $X, x \Vdash_v \varphi$ as follows:

(i) $X, x \nVdash_v \bot$; $X, x \Vdash_v p$ iff every path through x intersects $v(p)$;

(ii) $X, x \Vdash_v \varphi \wedge \psi$ iff $X, x \Vdash_v \varphi$ and $X, x \Vdash_v \psi$;

(iii) $X, x \Vdash_v \varphi \vee \psi$ iff every path through x intersects $\{y \in X \mid X, y \Vdash_v \varphi\} \cup \{y \in X \mid X, y \Vdash_v \psi\}$;

(iv) $X, x \Vdash_v \varphi \to \psi$ iff for every $y \geq x$, if $X, y \Vdash_v \varphi$ then $X, y \Vdash_v \psi$;

(v) $X, x \Vdash_v \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$ iff $X, x \Vdash_v \varphi$ or $X, x \Vdash_v \psi$.

A formula φ is *valid* on X according to inquisitive Beth semantics iff for any valuation $v : \mathsf{Prop} \to \mathsf{Up}(X)$, we have $X, x \Vdash \varphi$ for all $x \in X$ (otherwise φ is *refuted*); φ is valid over a class K of posets iff it is valid on every poset in K.

Example 2.3 Fig. 1 shows a poset (the "Beth comb") with a valuation such that according to Beth semantics, the root node forces $p \vee q$ (as every path through the root contains a node that forces p or a node that forces q) but does not force $p \vee\!\!\!\vee q$ (as the root does not force p and does not force q) and does not force $p \vee \neg p$ (as the path consisting of the nodes along the "spine" of the comb does not contain a node that forces p or a node that forces $\neg p$).

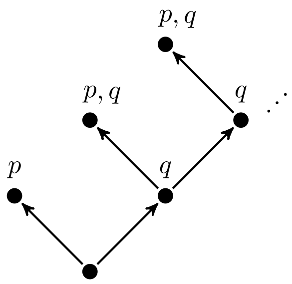

Fig. 1. Beth model for Example 2.3

3 Inquisitive Intuitionistic Logic

We now give a syntactic definition of a family of logical systems, the minimal member of which we will prove complete with respect to Beth semantics.

Definition 3.1 An *inquisitive intuitionistic logic* is a set L of $\mathcal{L}_{\vee,\vee\!\!\!\vee}$ formulas that contains the following formulas and is closed under the following rules, for all $\varphi, \psi, \chi \in \mathcal{L}_{\vee,\vee\!\!\!\vee}$:

- all $\mathcal{L}_{\vee,\vee\!\!\!\vee}$-substitution instances of IPC axioms stated in $\mathcal{L}_{\vee\!\!\!\vee}$;
- $(\alpha \vee \alpha) \to \alpha$ for $\alpha \in \mathcal{L}_{\vee}$;
- $\varphi \to (\varphi \vee \varphi)$; $((\varphi \vee \varphi) \vee (\varphi \vee \varphi)) \to (\varphi \vee \varphi)$; $((\varphi \vee\!\!\!\vee \psi) \vee (\varphi \vee\!\!\!\vee \psi)) \leftrightarrow (\varphi \vee \psi)$;
- $((\varphi \wedge \psi) \vee (\varphi \wedge \psi)) \leftrightarrow ((\varphi \vee \varphi) \wedge (\psi \vee \psi))$;
- rule of modus ponens: if $\varphi \in \mathsf{L}$ and $\varphi \to \psi \in \mathsf{L}$, then $\psi \in \mathsf{L}$;
- rule of replacement of equivalents: if $\varphi \in \mathsf{L}$ and $\psi \leftrightarrow \chi \in \mathsf{L}$, then $\varphi' \in \mathsf{L}$ for any φ' obtained from φ by replacing one or more occurrences of ψ in φ by χ.

The following soundness result is easy to check.

Proposition 3.2 *For any class* K *of posets, the set of* $\mathcal{L}_{\vee,\vee\!\!\!\vee}$*-formulas valid over* K *according to inquisitive Beth semantics is an inquisitive intuitionistic logic.*

One can also consider inquisitive intuitionistic logics based on superintuitionistic logics strictly extending IPC.

Definition 3.3 A *superintuitionistic logic (si-logic) for* $\mathcal{L}_{\vee\!\!\!\vee}$ is a set L of $\mathcal{L}_{\vee\!\!\!\vee}$ formulas that contains the following formulas and is closed under the following rules:

(i) all axioms of IPC stated in $\mathcal{L}_{\vee\!\!\!\vee}$;

(ii) rule of modus ponens: if $\varphi \in \mathsf{L}$ and $\varphi \to \psi \in \mathsf{L}$, then $\psi \in \mathsf{L}$;

(iii) rule of substitution: if $\varphi \in \mathsf{L}$ and φ' is obtained from φ by uniformly substituting formulas for proposition letters in φ, then $\varphi' \in \mathsf{L}$.

Definition 3.4 For any si-logic L for $\mathcal{L}_{\vee\!\!\!\vee}$, let $\mathsf{Inq}(\mathsf{L})$ be the smallest inquisitive intuitionistic logic containing all $\mathcal{L}_{\vee,\vee\!\!\!\vee}$-substitution instances of theorems of L.

In this paper, we concentrate on the smallest inquisitive intuitionistic logic, $\mathsf{Inq}(\mathsf{IPC})$. Our main theorem is the following.

Theorem 3.5 $\mathsf{Inq}(\mathsf{IPC})$ *is sound and complete according to Beth semantics.*

4 Comparison of $\mathsf{Inq}(\mathsf{IPC})$ and InqI

Ciardelli et al. [11] syntactically define a system InqI of intuitionistic inquisitive logic. Below we show that InqI and our $\mathsf{Inq}(\mathsf{IPC})$ are incomparable. We refer the reader to [11] for the full definition of InqI and its team semantics, but we will define as much as we need here to distinguish the two logics.

Example 4.1 The axiom

$$(p \vee (q \vee\!\!\!\vee r)) \to ((p \vee q) \vee\!\!\!\vee (p \vee r))$$

of InqI is not valid according to Beth semantics for inquisitive logic. For example, in the poset with the valuation shown in Figure 2, where p is true only at the top left node, q only at the top middle node, and r only at the top right node, the root node satisfies $p \vee (q \vee\!\!\!\vee r)$ but does not satisfy $(p \vee q) \vee\!\!\!\vee (p \vee r)$.

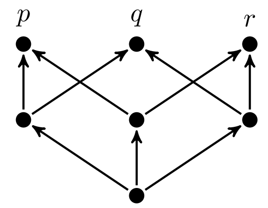

Fig. 2. Beth model for Example 4.1

Example 4.2 The following axiom schemas of $\mathsf{Inq}(\mathsf{IPC})$ have counterexamples according to team semantics for InqI [11]:

(i) $((\varphi \vee \varphi) \vee (\varphi \vee \varphi)) \to (\varphi \vee \varphi)$;

(ii) $((\varphi \wedge \psi) \vee (\varphi \wedge \psi)) \leftrightarrow ((\varphi \vee \varphi) \wedge (\psi \vee \psi))$.

Recall the clauses for $\vee$ and $\vee\!\!\!\vee$ according to team semantics:

- a team T supports $\varphi \vee \psi$ iff there are T', T'' such that $T = T' \cup T''$, T' supports φ, and T'' supports ψ;
- a team T supports $\varphi \vee\!\!\!\vee \psi$ iff T supports φ or T supports ψ.

For (i), take $\varphi := (p_1 \vee\!\!\!\vee p_2) \vee\!\!\!\vee (p_3 \vee\!\!\!\vee p_4)$ and a classical team model (i.e., the Kripke relation R is identity) with $W = \{w_1, w_2, w_3, w_4\}$ and $V(p_i) = \{w_i\}$. Then the team $\{w_1, w_2, w_3, w_4\}$ supports $(\varphi \vee \varphi) \vee (\varphi \vee \varphi)$, because

$\{w_1, w_2, w_3, w_4\} = \{w_1\} \cup \{w_2\} \cup \{w_3\} \cup \{w_4\}$ and each $\{w_i\}$ supports φ. However, $\{w_1, w_2, w_3, w_4\}$ does not support $\varphi \vee \varphi$, because there is no way of writing $\{w_1, w_2, w_3, w_4\}$ as a union of two sets each of which support φ.

For (ii), take $\varphi := p \mathbin{\vee\!\!\!\vee} q$, $\psi := r \mathbin{\vee\!\!\!\vee} s$, and a classical team model with $W = \{w_1, w_2, w_3\}$ such that $V(p) = \{w_1, w_2\}$, $V(q) = \{w_3\}$, $V(r) = \{w_1\}$, and $V(s) = \{w_2, w_3\}$. Then the team $\{w_1, w_2, w_3\}$ supports $\varphi \vee \varphi$, because $\{w_1, w_2, w_3\} = \{w_1, w_2\} \cup \{w_3\}$ and the teams $\{w_1, w_2\}$ and $\{w_3\}$ each support φ; and the team $\{w_1, w_2, w_3\}$ supports $\psi \vee \psi$, because $\{w_1, w_2, w_3\} = \{w_1\} \cup \{w_2, w_3\}$ and the teams $\{w_1\}$ and $\{w_2, w_3\}$ each support ψ. However, the team $\{w_1, w_2, w_3\}$ does not support $(\varphi \wedge \psi) \vee (\varphi \wedge \psi)$, because there is no way of writing $\{w_1, w_2, w_3\}$ as a union of two sets each of which supports $\varphi \wedge \psi$. For the only teams that support $\varphi \wedge \psi$ are the singleton teams.

5 The Nuclear Perspective

The Beth semantics for $\mathcal{L}_{\vee, \mathbin{\vee\!\!\!\vee}}$ in Section 2 can be regarded as a special case of an algebraic semantics based on nuclei.

Definition 5.1 A *nucleus* on a Heyting algebra H is a unary function $j : H \to H$ such that for all $a, b \in H$, $a \leq ja$ (increasing), $jja \leq ja$ (idempotent), and $j(a \wedge b) = ja \wedge jb$ (multiplicative); and j is *dense* if $j0 = 0$.

We denote the meet, join, and implication operations of a Heyting algebra by $\wedge$, $\vee$, and $\to$, trusting that no confusion will arise.

Definition 5.2 A *nuclear algebra* is a pair (H, j) where H is a Heyting algebra and j is a nucleus on H. The algebra is *dense* if j is dense.

The following useful lemma and key theorem are well known.

Lemma 5.3 *For any nuclear algebra (H, j) and $a, b \in H$, we have $j(a \vee b) = j(ja \vee b) = j(a \vee jb) = j(ja \vee jb)$.*

Theorem 5.4 *For any nuclear algebra (H, j), the set $H_j = \{a \in L \mid ja = a\}$ of fixpoints of j is a Heyting algebra, called the* algebra of fixpoints *in (H, j), under the following operations for $a, b \in H_j$: $0_j = j0$, $a \wedge_j b = a \wedge b$, $a \vee_j b = j(a \vee b)$, and $a \to_j b = a \to b$.*

The key idea of the nuclear semantics for $\mathcal{L}_{\vee, \mathbin{\vee\!\!\!\vee}}$ is that the inquisitive disjunction $\mathbin{\vee\!\!\!\vee}$ is interpreted as the join in the nuclear algebra, while the declarative disjunction $\vee$ is interpreted by applying the nucleus to the join.

Definition 5.5 Given a nuclear algebra (H, j) and valuation $v : \mathsf{Prop} \to H_j$, we define $\widehat{v} : \mathcal{L}_{\vee, \mathbin{\vee\!\!\!\vee}} \to H$ by: $\widehat{v}(\bot) = j0$, $\widehat{v}(\varphi \wedge \psi) = \widehat{v}(\varphi) \wedge \widehat{v}(\psi)$, $\widehat{v}(\varphi \vee \psi) = j(\widehat{v}(\varphi) \vee \widehat{v}(\psi))$, $\widehat{v}(\varphi \to \psi) = \widehat{v}(\varphi) \to \widehat{v}(\psi)$, and $\widehat{v}(\varphi \mathbin{\vee\!\!\!\vee} \psi) = \widehat{v}(\varphi) \vee \widehat{v}(\psi)$.

A formula φ is *valid* on (H, j) according to inquisitive nuclear semantics iff for any $v : \mathsf{Prop} \to H_j$, we have $\widehat{v}(\varphi) = 1$ (otherwise φ is *refuted*); and φ is valid over a class K of nuclear algebras iff it is valid on every algebra in K.

The following soundness result is easy to check.

Proposition 5.6 *For any class* K *of nuclear algebras, the set of* $\mathcal{L}_{\vee,\mathbb{V}}$*-formulas valid over* K *according to inquisitive nuclear semantics is an inquisitive intuitionistic logic.*

Beth semantics can be seen as a special case of nuclear semantics using the following intermediate structures from [4].

Definition 5.7 A *nuclear frame* is a triple $(S, \sqsubseteq, j)$ where $(S, \sqsubseteq)$ is a poset and j is a nucleus on the Heyting algebra $\mathsf{Up}(S, \sqsubseteq)$ of upsets of $(S, \sqsubseteq)$.

Example 5.8 The Beth nucleus j_b on $\mathsf{Up}(S, \sqsubseteq)$ is defined by

$$j_b U = \{x \in X \mid \text{every path through } x \text{ intersects } U\}.$$

6 Translation into Lax Logic

The nuclear perspective of the previous section can be made explicit, at the level of the object language, by translating the inquisitive intuitionitic language $\mathcal{L}_{\vee,\mathbb{V}}$ into the language $\mathcal{L}_{\bigcirc}$ of propositional lax logic [15].

Definition 6.1 Let $\mathcal{L}_{\bigcirc}$ be the language defined as follows, where $p \in \mathsf{Prop}$:

$$\varphi ::= \bot \mid p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi) \mid \bigcirc\varphi.$$

Let $\mathcal{L}_{\bigcirc p}$ be the language defined as follows, where $p \in \mathsf{Prop}$:

$$\varphi ::= \bigcirc\bot \mid \bigcirc p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi) \mid \bigcirc\varphi.$$

We first consider the obvious nuclear-algebraic semantics for $\mathcal{L}_{\bigcirc}$, where the sole disjunction $\vee$ of $\mathcal{L}_{\bigcirc}$ is interpreted as the join in the Heyting algebra.

Definition 6.2 Given a nuclear algebra (H, j) and valuation $v : \mathsf{Prop} \to H$, we define $\overline{v} : \mathcal{L}_{\bigcirc} \to H$ by: $\overline{v}(\bot) = 0$, $\overline{v}(\varphi \wedge \psi) = \overline{v}(\varphi) \wedge \overline{v}(\psi)$, $\overline{v}(\varphi \vee \psi) = \overline{v}(\varphi) \vee \overline{v}(\psi)$, $\overline{v}(\varphi \to \psi) = \overline{v}(\varphi) \to \overline{v}(\psi)$, $\overline{v}(\bigcirc\varphi) = j\overline{v}(\varphi)$.

For simplicity, we will drop the overline on v when there is no risk of confusion.

A special case of this nuclear-algebraic semantics for $\mathcal{L}_{\bigcirc}$ is the following *lax Beth semantics* for $\mathcal{L}_{\bigcirc}$.

Definition 6.3 For any poset X, $x \in X$, valuation $v : \mathsf{Prop} \to \mathsf{Up}(X)$, and $\varphi \in \mathcal{L}_{\bigcirc}$, we define $X, x \Vdash_v \varphi$ as follows:

(i) $X, x \nVdash_v \bot$; $X, x \Vdash_v p$ iff $x \in v(p)$;

(ii) $X, x \Vdash_v \varphi \wedge \psi$ iff $X, x \Vdash_v \varphi$ and $X, x \Vdash_v \psi$;

(iii) $X, x \Vdash_v \varphi \vee \psi$ iff $X, x \Vdash_v \varphi$ or $X, x \Vdash_v \psi$;

(iv) $X, x \Vdash_v \varphi \to \psi$ iff for every $y \geq x$, if $X, y \Vdash_v \varphi$ then $X, y \Vdash_v \psi$;

(v) $X, x \Vdash_v \bigcirc\varphi$ iff every path through x intersects $\{y \in X \mid X, y \Vdash_v \varphi\}$.

We now translate $\mathcal{L}_{\vee,\mathbb{V}}$ into $\mathcal{L}_{\bigcirc p}$ as follows.

Definition 6.4 Let ℓ be the translation from $\mathcal{L}_{\vee,\mathbb{V}}$ to $\mathcal{L}_{\bigcirc p}$ defined by: $\ell(\bot) = \bigcirc\bot$, $\ell(p) = \bigcirc p$, $\ell(\varphi \wedge \psi) = \ell(\varphi) \wedge \ell(\psi)$, $\ell(\varphi \vee \psi) = \bigcirc(\ell(\varphi) \vee \ell(\psi))$, $\ell(\varphi \mathbin{\mathbb{V}} \psi) = \ell(\varphi) \vee \ell(\psi)$, and $\ell(\varphi \to \psi) = \ell(\varphi) \to \ell(\psi)$.

It is easy to check that the translation is full and faithful in the following sense.

Lemma 6.5 *Let X be a poset and $\varphi \in \mathcal{L}_{\vee,\mathbin{\rotatebox[origin=c]{180}{$\land$}}\!\!\!\vee}$. Then X validates φ according to inquisitive Beth semantics for $\mathcal{L}_{\vee,\mathbin{\vee\!\!\!\vee}}$ iff X validates $\ell(\varphi)$ according to lax Beth semantics for $\mathcal{L}_{\bigcirc}$.*

We can also define a translation in the other direction as follows.

Definition 6.6 Let ι be the translation from $\mathcal{L}_{\bigcirc}$ to $\mathcal{L}_{\vee,\mathbin{\vee\!\!\!\vee}}$ defined by: $\iota(\bot) = \bot$, $\iota(p) = p$, $\iota(\varphi \wedge \psi) = \iota(\varphi) \wedge \iota(\psi)$, $\iota(\varphi \vee \psi) = \iota(\varphi) \mathbin{\vee\!\!\!\vee} \iota(\psi)$, $\iota(\varphi \to \psi) = \iota(\varphi) \to \iota(\psi)$, and $\iota(\bigcirc\varphi) = \iota(\varphi) \vee \iota(\varphi)$.

For the fragment $\mathcal{L}_{\bigcirc p}$ of $\mathcal{L}_{\bigcirc}$, it is easy to check the following.

Lemma 6.7 *Let X be a poset and $\varphi \in \mathcal{L}_{\bigcirc p}$. Then X validates φ according to lax Beth semantics for $\mathcal{L}_{\bigcirc}$ iff X validates $\iota(\varphi)$ according to inquisitive Beth semantics for $\mathcal{L}_{\vee,\mathbin{\vee\!\!\!\vee}}$.*

However, the lemma does not extend to all $\varphi \in \mathcal{L}_{\bigcirc}$.

Example 6.8 The formula $\bigcirc p \to p$ is not valid according to lax Beth semantics, but $\iota(\bigcirc p \to p) = (p \vee p) \to p$ is valid according to inquisitive Beth semantics.

Next it is easy to check that composing the translations produces a formula provably equivalent to the original input.

Lemma 6.9 *For any $\varphi \in \mathcal{L}_{\vee,\mathbin{\vee\!\!\!\vee}}$, the formula $\varphi \leftrightarrow \iota(\ell(\varphi))$ is a theorem of $\mathsf{Inq}(\mathsf{IPC})$.*

Proof. By induction on φ. In the base case, $\bot \leftrightarrow (\bot \vee \bot)$ and $p \leftrightarrow (p \vee p)$ are provable using the axioms of $\mathsf{Inq}(\mathsf{IPC})$. The $\wedge$, $\to$, and $\mathbin{\vee\!\!\!\vee}$ cases use the inductive hypothesis and replacement of equivalents. For the $\vee$ case, proving

$$(\varphi \vee \psi) \leftrightarrow ((\iota(\ell(\varphi)) \mathbin{\vee\!\!\!\vee} \iota(\ell(\psi))) \vee (\iota(\ell(\varphi)) \mathbin{\vee\!\!\!\vee} \iota(\ell(\psi))))$$

uses the inductive hypothesis, replacement of equivalents, and an axiom. □

Though in [15] 'propositional lax logic' (PLL) refers to a single system, we can define a family of lax logics, of which PLL is the smallest.

Definition 6.10 A *propositional lax logic* is a set L of $\mathcal{L}_{\bigcirc}$ formulas that contains the following formulas and is closed under the following rules for all $\varphi, \psi \in \mathcal{L}_{\bigcirc}$:

- all $\mathcal{L}_{\bigcirc}$-substitution instances of IPC axioms stated in $\mathcal{L}_{\vee}$;
- $\varphi \to \bigcirc\varphi$, $\bigcirc\bigcirc\varphi \to \bigcirc\varphi$, and $\bigcirc(\varphi \wedge \psi) \leftrightarrow (\bigcirc\varphi \wedge \bigcirc\psi)$;
- rules of modus ponens and replacement of equivalents.

A *dense* propositional lax logic is a propositional lax logic containing $\bigcirc\bot \to \bot$.

Again soundness is easy to check.

Proposition 6.11 *For any class K of posets, the set of $\mathcal{L}_{\bigcirc}$-formulas valid over K according to lax Beth semantics is a dense propositional lax logic.*

Remark 6.12 To obtain Beth semantics for lax logics without the density axiom, one needs to consider Beth semantics with "strange" or "exploded" worlds that force $\bot$ (see, e.g., [14]), but we leave this for future work.

Here we will consider the (dense) lax logic based on IPC, but the same idea applies to any si-logic.

Definition 6.13 For any si-logic L, let $\mathsf{Lax}(\mathsf{L})$ and $\mathsf{Lax_d}(\mathsf{L})$ be the smallest propositional lax logic and the smallest dense propositional lax logic, respectively, containing all $\mathcal{L}_{\bigcirc}$-substitution instances of L axioms stated in $\mathcal{L}_{\bigvee}$.

In the Appendix, we prove the following algebraic completeness results.

Theorem 6.14

(i) *If* $\mathsf{Lax}(\mathsf{IPC}) \nvdash \varphi$*, then there is a finite nuclear algebra that refutes* φ*.*

(ii) *If* $\mathsf{Lax_d}(\mathsf{IPC}) \nvdash \varphi$*, then there is a finite dense nuclear algebra that refutes* φ*.*

7 S-Frame Completeness

We now transfer the completeness result of Theorem 6.14 to completeness with respect to certain finite relational structures, which we call "S-frames," from [3]. Proofs of the two lemmas and proposition below can be extracted from [3].

Definition 7.1 An *S-frame* is a triple $\mathfrak{S} = (X, \sqsubseteq, S)$ where $(X, \sqsubseteq)$ is a poset and $S \subseteq X$. $\mathfrak{S}$ is *cofinal* if S is cofinal in $(X, \sqsubseteq)$, i.e., for all $x \in X$ there is a $y \in S$ such that $x \sqsubseteq y$.

S-frames can be constructed from nuclear algebras as follows.

Definition 7.2 Given a nuclear algebra $\mathfrak{A} = (H, j)$, define $\mathfrak{A}_\star := (X, \sqsubseteq, S)$ as follows: X is the set of all prime filters of H; $\sqsubseteq$ is the inclusion order on X; and $S = \{F \in X \mid j^{-1}[F] = F\}$.

Lemma 7.3 *For any nuclear algebra* $\mathfrak{A}$*:*

(i) $\mathfrak{A}_\star$ *is an S-frame;*

(ii) *if* $\mathfrak{A}$ *is dense, then* $\mathfrak{A}_\star$ *is cofinal.*

Conversely, we construct a nuclear algebra from an S-frame as follows.

Definition 7.4 Given an S-frame $\mathfrak{S} = (X, \sqsubseteq, S)$, define the algebra $\mathfrak{S}^\star := (H, j_S)$ as follows: $H = \mathsf{Up}(X)$; for $U \in H$, $j_S U = \{x \in X \mid {\uparrow}x \cap S \subseteq U\}$.

Lemma 7.5 *For any S-frame* $\mathfrak{S}$*,* $\mathfrak{S}^\star$ *is a nuclear algebra.*

Proposition 7.6 *If* $\mathfrak{A}$ *is a nuclear algebra, then* $\mathfrak{A}$ *embeds into* $(\mathfrak{A}_\star)^\star$*. Moreover, if* $\mathfrak{A}$ *is finite, then the embedding is an isomorphism.*

Say that an S-frame $\mathfrak{S}$ validates/refutes a formula $\varphi \in \mathcal{L}_{\bigcirc}$ just in case $\mathfrak{S}^\star$ validates/refutes φ according to Definition 5.5. Then we obtain the following completeness result from Theorem 6.14, Lemma 7.3, and Proposition 7.6.

Corollary 7.7 *If* $\mathsf{Lax_d}(\mathsf{IPC}) \nvdash \varphi$*, then there is a finite cofinal S-frame that refutes* φ*.*

8 Beth Completeness

In this section, we first prove the Beth completeness of our lax logics and then the Beth completeness of our inquisitive intuitionistic logics.

8.1 Beth Completeness of $\mathsf{Lax_d}(\mathsf{IPC})$

Our strategy is to turn any finite cofinal S-frame refuting a non-theorem φ of $\mathsf{Lax_d}(\mathsf{IPC})$ into a poset refuting φ according to Beth semantics. For this purpose, we use the following key construction.

Definition 8.1 Given an S-frame $(X, \sqsubseteq, S)$, define $(X^\star, \sqsubseteq^\star)$ by:

(i) $X^\star := X \times \mathbb{N}$:

(ii) $\langle x, t\rangle \sqsubseteq^\star \langle x', t'\rangle$ iff one of the following holds:
  (a) $x \sqsubseteq x'$ and $t = t'$;
  (b) $x = x'$, $x \in S$, and $t < t'$;
  (c) $x \sqsubset x'$ and $t < t'$.

Lemma 8.2 *The relation $\sqsubseteq^\star$ is a partial order.*

There are two key properties of $(X^\star, \sqsubseteq^\star)$. First, if $(X, \sqsubseteq, S)$ is finite and cofinal, then every path eventually reaches a pair whose first coordinate is in S.

Lemma 8.3 *Let $(X, \sqsubseteq, S)$ be a finite cofinal S-frame. If C is a path in $(X^\star, \sqsubseteq^\star)$ through $\langle x, t\rangle$, then there is an $\langle x', t'\rangle$ such that $\langle x, t\rangle \sqsubseteq^\star \langle x', t'\rangle \in C$ and $x' \in S$.*

Proof. Let C be a path in $(X^\star, \sqsubseteq^\star)$ through $\langle x, t\rangle$. Thus, $C_\star = \{x' \in X \mid \langle x', t'\rangle \in C\}$ is a chain in $(X, \sqsubseteq)$, which is finite since $(X, \sqsubseteq)$ is finite. Hence $C_\star$ has a maximum, x_{max}. Suppose for contradiction that $C_\star \cap S = \varnothing$, so $x_{max} \notin S$. Since $x_{max} \in C_\star$, there is a t_{max} such that $\langle x_{max}, t_{max}\rangle \in C$. Then $\langle x_{max}, t_{max}\rangle$ is the maximum of C, for if $\langle x', t'\rangle \in C$ and $\langle x_{max}, t_{max}\rangle \sqsubseteq^\star \langle x', t'\rangle$, then since x_{max} is the maximum of $C_\star$, we have $x_{max} = x'$, in which case $x_{max} \notin S$ implies $t_{max} = t'$ by Definition 8.1. Since $(X, \sqsubseteq)$ is finite, $C_\star$ has an upper bound y that is maximal in $(X, \sqsubseteq)$, so $y \in S$ by the cofinality of the S-frame. As $\langle x_{max}, t_{max}\rangle \sqsubseteq^\star \langle y, t_{max}\rangle$ and $\langle x_{max}, t_{max}\rangle$ is the maximum of C, $\langle y, t_{max}\rangle$ is an upper bound of C. Then since C is a path, $\langle y, t_{max}\rangle \in C$, which with $y \in S$ implies $C_\star \cap S \neq \varnothing$, a contradiction. □

Second, we can always create a path in which the first coordinate of the pairs remains forever stuck at some element of S, as follows.

Lemma 8.4 *Let $(X, \sqsubseteq, S)$ be an S-frame, $\langle x, t\rangle \in X^\star$, and $x \sqsubseteq x' \in S$. Then*

$$C := \{\langle x, t\rangle\} \cup \{\langle x', t'\rangle \mid t' > t\}$$

is a path in $(X^\star, \sqsubseteq^\star)$ through $\langle x, t\rangle$.

Proof. Clearly C is a chain in $(X^\star, \sqsubseteq^\star)$. It is also easy to see that C has no upper bound and hence is closed under upper bounds. Thus, C is a path. □

Lemmas 8.3 and 8.4 are the key ingredients for the proposition to follow. First, a *nuclear p-morphism* [4] between nuclear frames $(S, \sqsubseteq, j)$ and $(S', \sqsubseteq', j')$

is a p-morphism f from $(S, \sqsubseteq)$ to $(S', \sqsubseteq')$ such that for $U \in \mathsf{Up}(S', \sqsubseteq')$, $f^{-1}[j'U] = jf^{-1}[U]$; this ensure that f^{-1} is a nucleus-preserving homomorphism from the nuclear algebra $(\mathsf{Up}(S', \sqsubseteq'), j')$ to the nuclear algebra $(\mathsf{Up}(S, \sqsubseteq), j)$; and if f is onto, then f^{-1} is an embedding, which implies that the fixpoint algebra $\mathsf{Up}(S', \sqsubseteq')_{j'}$ embeds into the fixpoint algebra $\mathsf{Up}(S, \sqsubseteq)_j$.

Proposition 8.5 *Let $(X, \sqsubseteq, S)$ be a finite S-frame. The function f defined by $f(\langle x, t\rangle) = x$ is a nuclear p-morphism from the nuclear frame $(X^\star, \sqsubseteq^\star, j_b)$ onto the nuclear frame $(X, \sqsubseteq, j_S)$.*

Proof. First, we check that f is a p-morphism. It is immediate from Definition 8.1 that if $\langle x, t\rangle \sqsubseteq^\star \langle x', t'\rangle$, then $x \sqsubseteq x'$ and hence $f(\langle x, t\rangle) \sqsubseteq f(\langle x', t'\rangle)$. For the back condition, if $f(\langle x, t\rangle) = x \sqsubseteq y$, then we have $\langle x, t\rangle \sqsubseteq^\star \langle y, t\rangle$ and $f(\langle y, t\rangle) = y$. Next we check that for all $U \in \mathsf{Up}(X, \sqsubseteq)$,

$$j_b f^{-1}[U] = f^{-1}[j_S U].$$

Suppose $\langle x, t\rangle \notin j_b f^{-1}[U]$, so there is a path C through $\langle x, t\rangle$ such that $C \cap f^{-1}[U] = \varnothing$. By Lemma 8.3, there is an $\langle x', t'\rangle$ such that $\langle x, t\rangle \sqsubseteq^\star \langle x', t'\rangle \in C$ and $x' \in S$. From $\langle x', t'\rangle \in C$ and $C \cap f^{-1}[U] = \varnothing$, we have $\langle x', t'\rangle \notin f^{-1}[U]$, so $f(\langle x', t'\rangle) = x' \notin U$. From $\langle x, t\rangle \sqsubseteq^\star \langle x', t'\rangle$, we have $x \sqsubseteq x'$. Since $x \sqsubseteq x' \in S \setminus U$, we have $x \notin j_S U$, so $f(\langle x, t\rangle) \notin j_S U$ and hence $\langle x, t\rangle \notin f^{-1}[j_S U]$.

Now suppose $\langle x, t\rangle \notin f^{-1}[j_S U]$, so $f(\langle x, t\rangle) = x \notin j_S U$. Thus, there is an x' such that $x \sqsubseteq x' \in S \setminus U$, which also implies $x \notin U$. By Lemma 8.4, $C := \{\langle x, t\rangle\} \cup \{\langle x', t'\rangle \mid t' > t\}$ is a path through $\langle x, t\rangle$, and it follows from our choice of x' that C does not intersect $f^{-1}[U]$. Hence $\langle x, t\rangle \notin j_b f^{-1}[U]$. □

We now obtain a completeness result for dense lax logic with respect to Beth semantics that is of interest independently of its application to inquisitive logic.

Theorem 8.6 *For any $\varphi \in \mathcal{L}_{\bigcirc}$, if φ is valid on all posets according to lax Beth semantics, then φ is a theorem of $\mathsf{Lax_d}(\mathsf{IPC})$.*

Proof. Suppose φ is not a theorem of $\mathsf{Lax_d}(\mathsf{IPC})$. Then by Corollary 7.7, φ can be refuted according to S-frame semantics on a finite cofinal S-frame $(X, \sqsubseteq, S)$. Then it follows by Lemma 8.5 that φ can be refuted according to lax Beth semantics on a poset. □

8.2 Beth Completeness of Inq(IPC)

In this section, we transfer the completeness result in Theorem 8.6 to Beth completeness for $\mathsf{Inq}(\mathsf{IPC})$. To do so, we use the translation ι of Definition 6.6. Since Lemma 6.7 for ι only applied to the fragment $\mathcal{L}_{\bigcirc p}$ of $\mathcal{L}_{\bigcirc}$, we will also use the following preliminary translation.

Definition 8.7 Let ξ be the translation from $\mathcal{L}_{\bigcirc}$ to $\mathcal{L}_{\bigcirc p}$ defined by: $\xi(\bot) = \bigcirc\bot$; $\xi(p) = \bigcirc p$; $\xi(\varphi \# \psi) = \xi(\varphi) \# \xi(\psi)$ for $\# \in \{\wedge, \vee, \to\}$; $\xi(\bigcirc\varphi) = \bigcirc\xi(\varphi)$.

Definition 8.8 Let $\mathsf{Lax_d}(\mathsf{L})_{\bigcirc p}$ be the logic for $\mathcal{L}_{\bigcirc p}$ whose axioms are all axioms of $\mathsf{Lax_d}(\mathsf{L})$ that belong to $\mathcal{L}_{\bigcirc p}$ and whose rules are modus ponens and replacement of equivalents.

Lemma 8.9 *For any* $\varphi \in \mathcal{L}_{\bigcirc p}$*, if* φ *is a theorem of* $\mathsf{Lax_d(L)}$*, then* φ *is a theorem of* $\mathsf{Lax_d(L)}_{\bigcirc p}$*.*

Proof. Suppose φ is a theorem of $\mathsf{Lax_d(L)}$, so there exists a proof $\langle \varphi_1, \dots, \varphi_n \rangle$ in $\mathsf{Lax_d(L)}$ with $\varphi_n = \varphi$. Then $\langle \xi(\varphi_1), \dots, \xi(\varphi_n)\rangle$ is a proof in $\mathsf{Lax_d(L)}_{\bigcirc p}$; for if φ_i is an axiom of $\mathsf{Lax_d(L)}$, then $\xi(\varphi_i)$ is an axiom of $\mathsf{Lax_d(L)}_{\bigcirc p}$, and if φ_i is obtained from φ_j and φ_k by modus ponens, then clearly $\xi(\varphi_i)$ is obtained from $\xi(\varphi_j)$ and $\xi(\varphi_k)$ by modus ponens. Let $\bigcirc a_1, \dots, \bigcirc a_k$ be the atomic formulas occurring in φ, so a_i is either a proposition letter or $\bot$. Then $\xi(\varphi_n)$ is obtained from φ_n by replacing each $\bigcirc a_i$ by $\bigcirc\bigcirc a_i$. Thus, by extending $\langle \xi(\varphi_1), \dots, \xi(\varphi_n)\rangle$ with the axioms $\bigcirc\bigcirc a_i \leftrightarrow \bigcirc a_i$ to

$$\langle \xi(\varphi_1), \dots, \xi(\varphi_n), \bigcirc\bigcirc a_1 \leftrightarrow \bigcirc a_1, \dots, \bigcirc\bigcirc a_k \leftrightarrow \bigcirc a_k\rangle$$

and then repeatedly applying replacement of equivalents starting with $\xi(\varphi_n)$, we finally obtain a proof in $\mathsf{Lax_d(L)}_{\bigcirc p}$ of φ_n. □

Lemma 8.10 *For any* $\varphi \in \mathcal{L}_{\bigcirc p}$*, if* φ *is a theorem of* $\mathsf{Lax_d(L)}_{\bigcirc p}$*, then* $\iota(\varphi)$ *is a theorem of* $\mathsf{Inq(L)}$*.*

Proof. It suffices to show that for any axiom φ of $\mathsf{Lax_d(L)}_{\bigcirc p}$, $\iota(\varphi)$ is a theorem of $\mathsf{Inq(L)}$. The axioms of $\mathsf{Lax_d(L)}_{\bigcirc p}$ are of two kinds: (i) all $\mathcal{L}_{\bigcirc p}$-substitution instances of L-axioms stated in $\mathcal{L}_{\vee}$, and (ii) the axioms for $\bigcirc$. For (i), each $\mathcal{L}_{\bigcirc p}$-substitution instance φ of an L-axiom stated in $\mathcal{L}_{\vee}$ translates to a formula $\iota(\varphi)$ that is also an $\mathcal{L}_{\vee,\mathbb{V}}$-substitution instance of an L-axiom stated in $\mathcal{L}_{\mathbb{V}}$, so $\mathsf{Inq(L)}$ contains $\iota(\varphi)$. For (ii), in each case the ι-translation of an axiom for $\bigcirc$ is an axiom of $\mathsf{Inq(L)}$:

$$\begin{aligned}
\iota(\varphi \to \bigcirc\varphi) &= \iota(\varphi) \to \iota(\bigcirc\varphi)\\
&= \iota(\varphi) \to (\iota(\varphi) \vee \iota(\varphi)), \text{ an axiom of } \mathsf{Inq(L)}\\
\iota(\bigcirc\bigcirc\varphi \to \bigcirc\varphi) &= \iota(\bigcirc\bigcirc\varphi) \to \iota(\bigcirc\varphi)\\
&= (\iota(\bigcirc\varphi) \vee \iota(\bigcirc\varphi)) \to (\iota(\varphi) \vee \iota(\varphi))\\
&= ((\iota(\varphi) \vee \iota(\varphi)) \vee (\iota(\varphi) \vee \iota(\varphi))) \to (\iota(\varphi) \vee \iota(\varphi)),\\
&\quad \text{an axiom of } \mathsf{Inq(L)}\\
\iota(\bigcirc(\varphi \wedge \psi) \leftrightarrow (\bigcirc\varphi \wedge \bigcirc\psi)) &= \iota(\bigcirc(\varphi \wedge \psi)) \to \iota(\bigcirc\varphi \wedge \bigcirc\psi)\\
&= (\iota(\varphi \wedge \psi) \vee \iota(\varphi \wedge \psi)) \to (\iota(\bigcirc\varphi) \wedge \iota(\bigcirc\psi))\\
&= ((\iota(\varphi) \wedge \iota(\psi)) \vee (\iota(\varphi) \wedge \iota(\psi))) \to\\
&\quad ((\iota(\varphi) \vee \iota(\varphi)) \wedge (\iota(\psi) \vee \iota(\psi))),\\
&\quad \text{an axiom of } \mathsf{Inq(L)}.
\end{aligned}$$

Finally, for the dense axiom: $\iota(\bigcirc\bot \to \bot) = (\bot \vee \bot) \to \bot$, an axiom of $\mathsf{Inq(L)}$. □

We can now put everything together to prove our main result: completeness of inquisitive intuitionistic logic with respect to Beth semantics.

Theorem 8.11 $\mathsf{Inq}(\mathsf{IPC})$ *is sound and complete with respect to all posets according to Beth semantics.*

Proof. Soundness is easy. For completeness, we have:

φ is valid in Beth semantics for $\mathcal{L}_{\vee,\mathbin{\rotatebox[origin=c]{0}{$\vee\!\!\!\vee$}}}$

$\Rightarrow$ $\ell(\varphi)$ is valid in Beth semantics for $\mathcal{L}_{\bigcirc}$ by Lemma 6.5

$\Rightarrow$ $\ell(\varphi)$ is a theorem of $\mathsf{Lax_d}(\mathsf{IPC})$ by Theorem 8.6

$\Rightarrow$ $\ell(\varphi)$ is a theorem of $\mathsf{Lax_d}(\mathsf{IPC})_{\bigcirc p}$ by Lemma 8.9

$\Rightarrow$ $\iota(\ell(\varphi))$ is a theorem of $\mathsf{Inq}(\mathsf{IPC})$ by Lemma 8.10

$\Rightarrow$ φ is a theorem of $\mathsf{Inq}(\mathsf{IPC})$ by Lemma 6.9.

□

9 Conclusion

We have shown the viability of an approach to inquisitive logic on an intuitionistic base using Beth semantics rather than team semantics. As noted, there are two other motivations for this work, independent of inquisitive logic:

- it is natural to consider adding a "Kripke disjunction" $\vee\!\!\!\vee$ to Beth semantics and to axiomatize the resulting logic, as we have done with our $\mathsf{Inq}(\mathsf{IPC})$;
- this study unearthed the fact that an old semantics for intuitionistic logic, Beth semantics, can provide a new semantics for (dense) lax logic.

A natural next step, given our general definition of the inquisitive version $\mathsf{Inq}(\mathsf{L})$ of a superintuitionistic logic L, is to investigate the completeness of $\mathsf{Inq}(\mathsf{L})$ for some well-motivated choices of L. One of the axiom schemas of classical inquisitive logic [7] is the schema

$$(\neg\varphi \to (\psi \vee\!\!\!\vee \chi)) \to ((\neg\varphi \to \psi) \vee\!\!\!\vee (\neg\varphi \to \chi))$$

of the superintuitionistic Kreisel-Putnam logic (KP), which is valid on the special Kripke models used for classical inquisitive logic (recall Section 1). Since we have considered Beth semantics over arbitrary posets, we can refute the KP axiom, but we could also consider restricting to posets satisfying the first-order property corresponding to the KP axiom in Kripke semantics (see, e.g., [6, p. 55]). In fact, in their intuitionistic inquisitive logic, Ciardelli et al. [11] include the schema

$$(\alpha \to (\psi \vee\!\!\!\vee \chi)) \to ((\alpha \to \psi) \vee\!\!\!\vee (\alpha \to \chi))$$

for α a formula without $\vee\!\!\!\vee$, which is equivalent to having the Kreisel-Putnam schema in classical inquisitive logic but not in the intuitionistic setting (see endnote 4 of [11]). We leave the Beth completeness of inquisitive intuitionistic logics with these additional schemas as an open problem.

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Appendix

A Proof of Theorem 6.14

In this appendix, we prove the algebraic completeness and finite model property of the lax logics considered in this paper. We first recall the two strategies that have been previously employed to the prove the finite model property for superintuitionistic logics (cf. [2]). The one strategy uses the local finiteness of bounded distributive lattices, while the other that we will build on uses the local finiteness of implicative semilattices, as in the following standard lemmas.

Lemma A.1 *Let $\mathfrak{A}$ be a Heyting algebra, X a finite subset of $\mathfrak{A}$, and $\mathfrak{C}$ the bounded sublattice of $\mathfrak{A}$ generated by X. Then:*

(i) *$\mathfrak{C}$ is a finite Heyting algebra with implication $\to_{\mathfrak{C}}$ given by $a \to_{\mathfrak{C}} b = \bigvee\{x \in \mathfrak{C} \mid x \leq a \to b\}$;*

(ii) *for any $a, b \in \mathfrak{C}$, we have $a \to_{\mathfrak{C}} b \leq a \to b$;*

(iii) *for any $a, b \in \mathfrak{C}$ such that $a \to b \in \mathfrak{C}$, we have $a \to_{\mathfrak{C}} b = a \to b$.*

First proof of FMP of IPC. The approach via Lemma A.1 was utilized by McKinsey and Tarski [22] to prove that IPC has the finite model property. If $\mathsf{IPC} \nvdash \varphi$, then there is a Heyting algebra $\mathfrak{A}$, e.g., the Lindenbaum algebra of IPC, and valuation $v_{\mathfrak{A}}$ on $\mathfrak{A}$ refuting φ. Let $X = \{\overline{v_{\mathfrak{A}}}(\psi) \mid \psi \in Sub(\varphi)\}$, where $Sub(\varphi)$ is the set of subformulas of φ, and generate the finite $\mathfrak{C}$ by X as in Lemma A.1.i. Then $v_{\mathfrak{A}}$ restricts to a valuation $v_{\mathfrak{C}}$ on $\mathfrak{C}$, and for all $\psi \in Sub(\varphi)$, we have $\overline{v_{\mathfrak{A}}}(\psi) = \overline{v_{\mathfrak{C}}}(\psi)$, where the key step of the inductive proof uses the fact about $\to_{\mathfrak{C}}$ in Lemma A.1.iii. Hence $\overline{v_{\mathfrak{A}}}(\varphi) \neq 1$ implies $\overline{v_{\mathfrak{C}}}(\varphi) \neq 1$.

Lemma A.2 *Let $\mathfrak{A}$ be a Heyting algebra, X a finite subset of $\mathfrak{A}$, and $\mathfrak{B}$ the $\{\wedge, \to, 0\}$-subalgebra of $\mathfrak{A}$ generated by X. Then:*

(i) *$\mathfrak{B}$ is a finite Heyting algebra with join $\vee_{\mathfrak{B}}$ given by $a \vee_{\mathfrak{B}} b = \bigwedge\{x \in \mathfrak{B} \mid a \vee b \leq x\}$;*

(ii) *for any $a, b \in \mathfrak{B}$, we have $a \vee b \leq a \vee_{\mathfrak{B}} b$;*

(iii) *for any $a, b \in \mathfrak{B}$ such that $a \vee b \in \mathfrak{B}$, we have $a \vee_{\mathfrak{B}} b = a \vee b$.*

Second proof of FMP of IPC. The approach via Lemma A.2 is due to Diego [13]. One can prove the finite model property of IPC by using the same strategy as above but generating $\mathfrak{B}$ instead of $\mathfrak{C}$ from X. Again one proves that for all $\psi \in Sub(\varphi)$, we have $\overline{v_{\mathfrak{A}}}(\psi) = \overline{v_{\mathfrak{B}}}(\psi)$, but now the key step of the inductive proof uses the fact about $\vee_{\mathfrak{B}}$ in Lemma A.2.iii.

To obtain the finite model property for our lax logics, we need to incorporate nuclei into the above constructions. For this we require the following definition.

Definition A.3

(i) Given posets P and Q, $r : P \to Q$, and $\ell : Q \to P$, we say that (r, ℓ) forms an *adjoint pair* iff for all $p \in P$ and $q \in Q$, $\ell(q) \leq p$ iff $q \leq r(p)$. Then r is the *right adjoint* and ℓ is the *left adjoint*.

(ii) If P and Q are monoids, we say that ℓ is *left exact* if in addition ℓ preserves finite meets.

(iii) A *localization* of a monoid M is a pair (L, ℓ) where L is a submonoid of M and $\ell : M \to L$ is a left exact left adjoint to the inclusion $L \to M$.

The following lemma is well known (see [3, p. 88] and references therein).

Lemma A.4 *There is a one-to-one correspondence between nuclei on a monoid M and its localizations: for any nucleus j on M, we have that (M_j, j) is a localization of M; and for any localization (L, ℓ), we have that ℓ is a nucleus on M such that $M_\ell = L$.*

If the monoid M is in addition a Brouwerian semilattice, then M_j is not only a subalgebra of M but also satisfies the following stronger condition.

Definition A.5 [[23,20]] Let A be a Brouwerian semilattice. A subalgebra T of A is *total* if for any $a \in A$ and $t \in T$, we have that $a \to t \in T$.

The next two lemmas are known, but we include short proofs for the reader's convenience.

Lemma A.6 *Let A be a Brouwerian semilattice and j a nucleus on A.*

(i) *A_j is a total subalgebra of A;*

(ii) *if B is a subalgebra of A, then $A_j \cap B$ is a total subalgebra of B.*

Proof. Part (i) follows from the fact that $j(a \to jb) = a \to jb$, and part (ii) follows from part (i). □

Lemma A.7 *Let A be a Brouwerian semilattice and T a total subalgebra of A. If the inclusion $T \to A$ has a left adjoint ℓ, then (T, ℓ) is a localization of A.*

Proof. Since ℓ is left adjoint to the inclusion, we have $\ell(a) \leq b$ iff $a \leq b$ for all $a \in A$ and $b \in T$. From this it follows that ℓ is order preserving, increasing, and idempotent. To see that it is left exact, let $x, y \in A$. Since ℓ is order preserving, we have $\ell(x \wedge y) \leq \ell(x) \wedge \ell(y)$. For the converse, since ℓ is increasing, we have $x \wedge y \leq \ell(x \wedge y)$, so $x \leq y \to \ell(x \wedge y)$. Since T is a total subalgebra, $y \to \ell(x \wedge y) \in T$, so the adjunction property gives $\ell(x) \leq y \to \ell(x \wedge y)$. Therefore, $\ell(x) \wedge y \leq \ell(x \wedge y)$. From this it follows that $y \leq \ell(x) \to \ell(x \wedge y)$. Since T is a subalgebra, $\ell(x) \to \ell(x \wedge y) \in T$, so applying the adjunction property again yields $\ell(y) \leq \ell(x) \to \ell(x \wedge y)$. Thus, $\ell(x) \wedge \ell(y) \leq \ell(x \wedge y)$. Therefore, (T, ℓ) is a localization of A. □

We now extend Lemma A.2 to the setting of nuclear algebras.

Lemma A.8 *Let $\mathfrak{A}$ be a nuclear algebra, X a finite subset of $\mathfrak{A}$, and $\mathfrak{B}$ the $\{\wedge, \rightarrow, 0\}$-subalgebra of $\mathfrak{A}$ generated by X. Then:*

(i) *$\mathfrak{B}$ is a finite nuclear algebra with nucleus $j_{\mathfrak{B}}$ given by $j_{\mathfrak{B}}(a) = \bigwedge\{x \in \mathfrak{A}_j \cap \mathfrak{B} \mid a \leq x\}$;*

(ii) *for any $a \in \mathfrak{B}$, we have $j(a) \leq j_{\mathfrak{B}}(a)$;*

(iii) *for any $a \in \mathfrak{B}$ such that $j(a) \in \mathfrak{B}$, we have $j_{\mathfrak{B}}(a) = j(a)$.*

Proof. For part (i), it follows from Lemma A.2 that $\mathfrak{B}$ is a finite Heyting algebra. By Lemma A.6, $\mathfrak{A}_j \cap \mathfrak{B}$ is a total $\{\wedge, \rightarrow, 0\}$-subalgebra of $\mathfrak{B}$. Since $\mathfrak{B}$ is finite, $\mathfrak{A}_j \cap \mathfrak{B}$ is finite, so the inclusion $\mathfrak{A}_j \cap \mathfrak{B} \rightarrow \mathfrak{B}$ has a left adjoint ℓ given by $\ell(a) = \bigwedge\{b \in \mathfrak{A}_j \cap \mathfrak{B} \mid a \leq b\}$. Hence by Lemmas A.7 and A.4, ℓ is a nucleus on $\mathfrak{B}$. Thus, $(\mathfrak{B}, \ell)$ is a finite nuclear algebra. In addition, for any $a \in \mathfrak{B}$, we have $j(a) \leq \ell(a)$, and if $j(a) \in \mathfrak{B}$, then $j(a) = \ell(a)$, which yields parts (ii)-(iii). □

We can now prove the FMP for $\mathsf{Lax}(\mathsf{IPC})$ and $\mathsf{Lax_d}(\mathsf{IPC})$.

Theorem A.9

(i) *If $\mathsf{Lax}(\mathsf{IPC}) \nvdash \varphi$, then there is a finite nuclear algebra that refutes φ.*

(ii) *If $\mathsf{Lax_d}(\mathsf{IPC}) \nvdash \varphi$, then there is a finite dense nuclear algebra that refutes φ.*

Proof. For part (i), suppose $\mathsf{Lax}(\mathsf{IPC}) \nvdash \varphi$. Then there is a valuation v on the Lindenbaum algebra $\mathfrak{A}$ of $\mathsf{Lax}(\mathsf{IPC})$ such that $v(\varphi) \neq 1$. Let $S = \{v(\psi) \mid \psi \text{ is a subformula of } \varphi\}$. Let $\mathfrak{B}$ be the $\{\wedge, \rightarrow, 0\}$-subalgebra of $\mathfrak{A}$ generated by S. By Lemma A.2, $\mathfrak{B}$ is finite Heyting algebra such that

$$\text{if } a \vee b \in S, \text{ then } a \vee b = a \vee_{\mathfrak{B}} b. \tag{A.1}$$

By Lemma A.8, $(\mathfrak{B}, j_{\mathfrak{B}})$ is a finite nuclear algebra such that

$$\text{if } j(a) \in S, \text{ then } j(a) = j_{\mathfrak{B}}(a). \tag{A.2}$$

Let v' be any valuation on $\mathfrak{B}$ such that for all proposition letters $p \in S$, we have $v'(p) = v(p)$. Then an easy induction using (A.1) and (A.2) shows that for all $\psi \in S$, $v(\psi) = v'(\psi)$, whence $v(\varphi) \neq 1$ in $\mathfrak{A}$ implies $v'(\varphi) \neq 1$ in $\mathfrak{B}$.

For part (ii), observe that if $j(0) = 0$, then $j_{\mathfrak{B}}(0) = 0$. The rest of the proof is the same as for part (i). □

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