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Publication Date

2011-12-01

Peer reviewed

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December 2011

This work was supported by the Director, Office of Science, Office of Fusion Energy Sciences, of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

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This work was supported by the Director, Office of Science, Office of Fusion Energy Sciences, of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

1 **Modeling of 10 GeV-1 TeV laser-plasma accelerators using Lorentz boosted**
2 **simulations**

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10 Modeling of laser-plasma wakefield accelerators in an optimal frame of reference [J.-
11 L. Vay, Phys. Rev. Lett. 98 130405 (2007)] allows direct and efficient full-scale
12 modeling of deeply depleted and beam loaded laser-plasma stages of 10 GeV-1 TeV
13 (parameters not computationally accessible otherwise). This verifies the scaling of
14 plasma accelerators to very high energies and accurately models the laser evolution
15 and the accelerated electron beam transverse dynamics and energy spread. Over
16 4, 5 and 6 orders of magnitude speedup is achieved for the modeling of 10 GeV,
17 100 GeV and 1 TeV class stages, respectively. Agreement at the percentage level
18 is demonstrated between simulations using different frames of reference for a 0.1
19 GeV class stage. Obtaining these speedups and levels of accuracy was permitted by
20 solutions for handling data input (in particular particle and laser beams injection)
21 and output in a relativistically boosted frame of reference, as well as mitigation of a
22 high-frequency instability that otherwise limits effectiveness.

23 PACS numbers: 03.30.+p, 52.38.Kd, 29.20.Ej, 52.65.Rr

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24 I. INTRODUCTION

25 Laser plasma accelerators (LPAs) offer order of magnitude increase in accelerating gradi-
26 ent over standard radio-frequency accelerators (which are limited by electrical breakdown),
27 thus holding the promise of much shorter particle accelerators^{1,2}. High quality electron
28 beams of energy up-to 1 GeV have been produced in just a few centimeters³, with 10 GeV
29 stages being planned as modules of a high energy collider^{4,5}.

30 As a laser propagates through a plasma, it displaces electrons while ions remain essentially
31 static, creating a pocket of positive charges that the displaced electrons rush to fill. The
32 resulting coherent periodic motion of the electrons oscillating around their original position
33 creates a wake (plasma wave) with a periodic structure following the laser. The alternate
34 concentration of positive and negative charges in the wake creates very intense electric fields.
35 An electron (or positron) beam injected with the right phase can be accelerated by those
36 fields to high energy in a much shorter distance than is possible in conventional particle
37 accelerators. The efficiency and quality of the acceleration is governed by several factors
38 which require precise three-dimensional shaping of the plasma column, as well as the laser
39 and particle beams, and understanding of their evolution.

40 Computer simulations have had a profound impact on the design and understanding of
41 past and present LPA experiments⁶⁻⁹, with accurate modeling of wake formation, electron
42 self-trapping and acceleration requiring fully kinetic methods (usually Particle-In-Cell) using
43 large computational resources due to the wide range of space and time scales involved^{7,9}. Fu-
44 ture LPA experiments include those that will be carried out using the BELLA (Berkeley lab
45 laser accelerator) facility at LBNL (Lawrence Berkeley National Laboratory), which will use
46 a 40 J, 1 PW laser system to research the production of 10 GeV electron beams in a meter-
47 length plasma¹⁰. Simulations of parameters relevant to such a 10 GeV stage demand as many
48 as 5000 processor hours for a one-dimensional simulation on a NERSC supercomputer¹¹. Var-
49 ious reduced models have been developed to allow multidimensional simulations at manage-
50 able computational costs: fluid approximation¹²⁻¹⁴, quasistatic approximation^{12,15-18}, laser
51 envelope models^{12,14,16,17,19}, and scaled parameters^{20,21}. However, the various approximations
52 that they require result in a narrower range of applicability. As a result, even using several
53 models concurrently does not usually provide a complete description. For example, scaled
54 simulations of 10 GeV LPA stages do not capture correctly some essential transverse physics,

55 e.g. the laser and beam betatron motion, which can lead to inaccurate beam emittance (a
56 measure of the beam quality). An envelope description using a reduce wave operator can
57 capture these effects correctly at full scale for the early propagation through the plasma
58 but can fail as the laser spectrum broadens due to energy depletion as it propagates further
59 in the plasma^{13,19,22}. However, capturing depletion accurately is essential to the design of
60 efficient stages, in order to optimize the transfer of energy from the laser to the wake and
61 particle bunch.

62 An alternative approach allows for orders of magnitude speedup of simulations, whether
63 at full or reduced scale, via the proper choice of a reference frame moving near the speed
64 of light in the direction of the laser²³. It does so without alteration to the fundamental
65 equations of particle motion or electrodynamics, provided that high-frequency light emitted
66 counter to the direction of propagation of the beam can be neglected. This approach ex-
67 ploits the properties of space and time dilation and contraction associated with the Lorentz
68 transformation. It was shown²³ that the ratio of longest to shortest space and time scales
69 of a system of two or more components crossing at relativistic velocities is not invariant
70 under such a transformation (a laser crossing a plasma is just such a relativistic crossing).
71 Since for simulations based on formulations from first principles, the number of computer
72 operations (e.g., time steps) is proportional to the ratio of the longest to shortest time scale
73 of interest, it follows that such simulations will eventually have different computer runtimes,
74 yet equivalent accuracy, depending solely upon the choice of frame of reference.

75 The procedure appears straightforward: identify the frame of reference which will min-
76 imize the range of space and/or time scales and perform the calculation in this frame.
77 However, several practical complications arise. Most importantly, while the fundamental
78 equations of electrodynamics and particle motion are written in a covariant form, the nu-
79 merical algorithms that are derived from them may not retain this property, and calculations
80 in frames moving at different velocities may not be successfully conducted with the use of
81 the exact same algorithms. For example, it was shown²⁴ that calculating the propagation
82 of ultra-relativistic charged particle beams in an accelerator using standard Particle-In-Cell
83 techniques leads to large numerical errors, which were fixed by developing a new particle
84 pusher. The modeling of a LPA stage in a boosted frame involves the fully electromag-
85 netic modeling of a plasma propagating at near the speed of light, for which Numerical
86 Cerenkov^{25,26} is a potential issue. Second, the input and output data are usually known

87 from, or compared to, experimental data. Thus, calculating in a frame other than the labo-
88 ratory entails transformations of the data between the calculation frame and the laboratory
89 frame. Third, electromagnetic calculations that include wave propagation will include waves
90 propagating forward and backward in any direction. For a frame of reference moving in the
91 direction of the accelerated beam (or equivalently the wake of the laser), waves emitted by
92 the plasma in the forward direction expand while the ones emitted in the backward direction
93 contract, following the properties of the Lorentz transformation. If one is to resolve both
94 forward and backward propagating waves emitted from the plasma, there is no gain in select-
95 ing a frame different from the laboratory frame. However, the physics of interest for a laser
96 wakefield is the laser driving the wake, the wake, and the accelerated beam. Backscatter is
97 weak in the short-pulse regime, and does not interact as strongly with the beam as do the
98 forward propagating waves which stay in phase for a long period. It is thus often assumed
99 that the backward propagating waves can be neglected in the modeling of LPA stages. The
100 accuracy of this assumption has been demonstrated by comparison between explicit codes
101 which include both forward and backward waves and envelope or quasistatic codes which
102 neglect backward waves^{7,21,27}.

103 After the idea and basic scaling for performing simulations of LPAs in a Lorentz boosted
104 frame were published²³, there have been several reports of the application of the technique
105 to various regimes of LPA^{9,11,14,28-34}. Speedups varying between several and a few thousands
106 were reported with various levels of accuracy in agreement between simulations performed
107 in a Lorentz boosted frames and in a laboratory frame. High-frequency instabilities were
108 reported to develop in 2D or 3D calculations, that were limiting the velocity of the boosted
109 frame and thus the attainable speedup^{31,32,35}.

110 We presented elsewhere³⁶ numerical techniques that were implemented in the Particle-
111 In-Cell code Warp³⁷ for mitigating the short wavelength instability, including a solver with
112 tunable coefficients. A detailed study of the application of these techniques to the simulations
113 of scaled LPA stages also revealed that choosing a frame near the frame of the wakefield
114 as the reference frame allows for more aggressive application of filtering or damping for
115 mitigating short wavelength instabilities, than is possible in laboratory frame simulations³⁶.
116 We showed that this is due to hyperbolic rotation of the laser oscillations in space time,
117 which is another beneficial consequence of the Lorentz transformation when transforming
118 the laser from the laboratory to a boosted frame, in particular for frames near the frame of

119 the wakefield³⁸.

120 In the present paper, we present accurate modeling of 10 GeV-1 TeV LPA stages with
121 beam loading relevant to laser driven collider designs and stages for upcoming lasers^{5,20,21},
122 verifying the scaling of efficient, deeply depleted LPAs to very high energies⁴. This is enabled
123 by controlling an instability that develops with high-boost frames by using methods that
124 we developed and presented elsewhere^{36,38}, allowing 2D and 3D simulations of 100 GeV and
125 1 TeV class LPA stages in the wakefield frame, thus achieving the maximum theoretical
126 speedups of over 10^5 and 10^6 , respectively. Accuracy of the method is demonstrated at
127 the percentage level. This method is used for for the numerical exploration at full scale of
128 the performance of a 10 GeV stage with a 40 J laser, taking accurately into account laser
129 depletion and spectrum broadening, as well as the accelerated electron beam energy spread,
130 and the transverse dynamics of both the laser and the electron beam.

131 The theoretical speedup expected for performing the modeling of a LPA stage in a boosted
132 frame is derived in Section 2. Section 3 summarizes the issues that have limited speedups
133 in previous work and solutions. Accurate modeling of full scale and scaled 10 GeV class
134 stages is demonstrated in Section 4, and the method is used to simulate stages in the 100
135 GeV-1 TeV range in Section 5. The evolution of the laser spectrum with respect to the
136 frame boost is given in Appendix A and the consequences on the choice of the optimal boost
137 are discussed. Enabling techniques that were implemented in Warp for input and output of
138 data in a boosted frame are described in Appendix B.

139 II. THEORETICAL SPEEDUP DEPENDENCY WITH THE FRAME 140 BOOST

141 The obtainable speedup is derived as an extension of the formula that was derived
142 earlier²³, taking in addition into account the group velocity of the laser as it traverses the
143 plasma. In our previous work²³, the laser was assumed to propagate at the velocity of light
144 in vacuum during the entire process, which is a good approximation when the relativistic
145 factor of the frame boost γ is small compared to the relativistic factor of the laser wake
146 γ_w in the plasma. The expression is generalized here to higher values of γ , for which the
147 actual group velocity of the laser in the plasma must be taken into account. We shall show
148 that for a 10 GeV class LPA stage, the maximum attainable speedup is above four orders

149 of magnitude.

150 Assuming that the simulation box is a fixed number of plasma periods long, which implies
 151 the use (which is standard) of a moving window following the wake and accelerated beam,
 152 the speedup is given by the ratio of the time taken by the laser pulse and the plasma to
 153 cross each other, divided by the shortest time scale of interest, that is the laser period. To
 154 first order, the wake velocity v_w is set by the 1D group velocity of the laser driver, which in
 155 the linear (low intensity) limit, is given by²:

$$v_w/c = \beta_w = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} \quad (1)$$

156 where $\omega_p = \sqrt{(n_e e^2)/(\epsilon_0 m_e)}$ is the plasma frequency, $\omega = 2\pi c/\lambda$ is the laser frequency, n_e is
 157 the plasma density, λ is the laser wavelength in vacuum, ϵ_0 is the permittivity of vacuum, c
 158 is the speed of light in vacuum, and e and m_e are respectively the charge and mass of the
 159 electron.

160 In the simulations presented herein, the runs are stopped when the last electron beam
 161 macro-particle exits the plasma, and a measure of the total time of the simulation is given
 162 by

$$T = \frac{L + \eta\lambda_p}{v_w - v_p} \quad (2)$$

163 where $\lambda_p \approx 2\pi c/\omega_p$ is the wake wavelength, L is the plasma length, v_w and $v_p = \beta_p c$ are
 164 respectively the velocity of the wake and of the plasma relative to the frame of reference,
 165 and η is an adjustable parameter for taking into account the fraction of the wake which
 166 exited the plasma at the end of the simulation. For a beam injected into the n^{th} bucket, η
 167 would be set to $n - 1/2$. If positrons were considered, they would be injected half a wake
 168 period ahead of the location of the electrons injection position for a given period, and one
 169 would have $\eta = n - 1$. The numerical cost R_t scales as the ratio of the total time to the
 170 shortest timescale of interest, which is the inverse of the laser frequency, and is thus given
 171 by

$$R_t = \frac{Tc}{\lambda} = \frac{(L + \eta\lambda_p)}{(\beta_w - \beta_p)\lambda} \quad (3)$$

172 In the laboratory, $v_p = 0$ and the expression simplifies to

$$R_{lab} = \frac{Tc}{\lambda} = \frac{(L + \eta\lambda_p)}{\beta_w\lambda} \quad (4)$$

173 In a frame moving at βc , the quantities become

$$\lambda_p^* = \lambda_p / [\gamma (1 - \beta_w \beta)] \quad (5)$$

$$L^* = L / \gamma \quad (6)$$

$$\lambda^* = \gamma (1 + \beta) \lambda \quad (7)$$

$$\beta_w^* = (\beta_w - \beta) / (1 - \beta_w \beta) \quad (8)$$

$$v_p^* = -\beta c \quad (9)$$

$$T^* = \frac{L^* + \eta \lambda_p^*}{v_w^* - v_p^*} \quad (10)$$

$$R_t^* = \frac{T^* c}{\lambda^*} = \frac{(L^* + \eta \lambda_p^*)}{(\beta_w^* + \beta) \lambda^*} \quad (11)$$

174 where $\gamma = 1/\sqrt{1 - \beta^2}$.

175 The expected speedup from performing the simulation in a boosted frame is given by the
176 ratio of R_{lab} and R_t^*

$$S = \frac{R_{lab}}{R_t^*} = \frac{(1 + \beta) (L + \eta \lambda_p)}{(1 - \beta \beta_w) L + \eta \lambda_p} \quad (12)$$

177 We note that assuming that $\beta_w \approx 1$ (which is a valid approximation for most prac-
178 tical cases of interest) and that $\gamma \ll \gamma_w$, this expression is consistent with the expres-
179 sion derived earlier²³ for the LPA case which states that $R_t^* = \alpha R_t / (1 + \beta)$ with $\alpha =$
180 $(1 - \beta + l/L) / (1 + l/L)$, where l is the laser length which is generally proportional to $\eta \lambda_p$,
181 and $S = R_t / R_t^*$. However, higher values of γ are of interest for maximum speedup, as shown
182 below.

183 For intense lasers ($a \sim 1$) typically used for acceleration, the energy gain is limited by
184 dephasing³⁹, which occurs over a scale length $L_d \sim \lambda_p^3 / 2\lambda^2$. Acceleration is compromised
185 beyond L_d and in practice, the plasma length is proportional to the dephasing length, i.e.
186 $L = \xi L_d$. In most cases, $\gamma_w^2 \gg 1$, which allows the approximations $\beta_w \approx 1 - \lambda^2 / 2\lambda_p^2$, and
187 $L = \xi \lambda_p^3 / 2\lambda^2 \approx \xi \gamma_w^2 \lambda_p / 2 \gg \eta \lambda_p$, so that Eq.(12) becomes

$$S = (1 + \beta)^2 \gamma^2 \frac{\xi \gamma_w^2}{\xi \gamma_w^2 + (1 + \beta) \gamma^2 (\xi \beta / 2 + 2\eta)} \quad (13)$$

188 For low values of γ , i.e. when $\gamma \ll \gamma_w$, Eq.(13) reduces to

$$S_{\gamma \ll \gamma_w} = (1 + \beta)^2 \gamma^2 \quad (14)$$

189 Conversely, if $\gamma \rightarrow \infty$, Eq.(13) becomes

$$S_{\gamma \rightarrow \infty} = \frac{4}{1 + 4\eta/\xi} \gamma_w^2 \quad (15)$$

190 Finally, in the frame of the wake, i.e. when $\gamma = \gamma_w$, assuming that $\beta_w \approx 1$, Eq.(13) gives

$$S_{\gamma=\gamma_w} \approx \frac{2}{1 + 2\eta/\xi} \gamma_w^2 \quad (16)$$

191 Since η and ξ are of order unity, and the practical regimes of most interest satisfy $\gamma_w^2 \gg 1$,
 192 the speedup that is obtained by using the frame of the wake will be near the maximum
 193 obtainable value given by Eq.(15).

194 Note that without the use of a moving window, the relativistic effects that are at play
 195 in the time domain would also be at play in the spatial domain²³, and the γ^2 scaling would
 196 transform to γ^4 . In the frame of the wake, there is no need of the moving window, thus
 197 simplifying the procedure, while in a frame traveling faster than the wake in the laboratory,
 198 a moving window propagating in the backward direction would be needed. However, the
 199 scaling shows that there would be very little gain in doing the latter. Furthermore, analysis
 200 presented elsewhere^{36,38} and below show that choosing a frame near the frame of the wake
 201 is optimum for mitigation of a high frequency instability. This point is refined by a detailed
 202 analysis of the laser spectrum on axis in Appendix A which shows that for heavily depleted
 203 lasers where the spectrum red-shifts during propagations, the optimal γ might be at a slightly
 204 lower value, but this does not greatly affect speedup.

205 A. Estimated speedup for 0.1-100 GeV stages

206 Formula (13) is used to estimate the speedup for the calculations of 100 MeV to 1 TeV
 207 class stages, assuming a laser wavelength $\lambda = 0.8\mu\text{m}$. Parameters for the 100 MeV stage are
 208 given in Table 1 below, and parameters for higher energies are derived using scaling laws
 209 from²⁰. The initial plasma densities n_e for the 100 MeV, 1 GeV, 10 GeV, 100 GeV and
 210 1 TeV stages are respectively 10^{19} cm^{-3} , 10^{18} cm^{-3} , 10^{17} cm^{-3} , 10^{16} cm^{-3} and 10^{15} cm^{-3} ,
 211 while the plasma lengths L are 1.5 mm, 4.74 cm, 1.5 m, 47.4 m and 1.5 km if choosing
 212 $\xi \approx 1.63$. For these values, the wake wavelengths λ_p are respectively $10.6\mu\text{m}$, $33.4\mu\text{m}$,
 213 $106.\mu\text{m}$, $334.\mu\text{m}$, 1.06mm , and relativistic factors γ_w are 13.2, 41.7, 132, 417 and 1320. In
 214 the simulations presented in this paper, the beam is injected near the end of the wake period
 215 (first “bucket”). The beam has propagated through about half a wake period to reach full
 216 acceleration (due to dephasing), and we set $\eta \approx 0.5$. For the parameters considered here,
 217 $L \approx \lambda_p/\gamma_w^2$, and (15) gives $S_{\gamma \rightarrow \infty} \approx 2\gamma_w^2$.

218 The speedup versus the relativistic factor of the boosted frame γ is plotted in Fig. 1-a. As
 219 expected, for low values of γ , the speedup scales as (14), and asymptotes to a value slightly
 220 lower than $2\gamma_w^2$ for large values of γ . Calculations using the frame of the wake ($\gamma = \gamma_w$)
 221 attain nearly the maximum speedup. It is of interest to note that the qualitative behavior
 222 is identical to the one obtained in our earlier work²³ (see Fig. 1 and accompanying analysis)
 223 in the analysis of the crossing of two rigid identical beams, confirming the generality of
 224 the generic analysis presented previously²³. For 10 GeV, 100 GeV and 1 TeV class stages,
 225 the maximum estimated speedups are as large as 3×10^4 , 3×10^5 and 3×10^6 respectively.
 226 Estimated computational time without boost scales as λ_p^6 (λ_p^3 volume $\times \lambda_p^3$ long) $\approx E^3$ (where
 227 E is the stage energy) making them harder to model. Fortunately, the boost provides more
 228 computational gain for the higher energy stages, making them accessible.

229 **III. NUMERICAL ISSUES IN PAST BOOSTED FRAME SIMULATIONS** 230 **AND OBSERVED SPEEDUPS**

231 Several numerical limits can restrict the boost performance. Here we review limits in
 232 past simulations and their impact on performance (a short wavelength instability, laser
 233 initialization, statistics), and present methods for circumventing these limits.

234 A violent numerical instability developing at the front of the plasma column for $\gamma \gtrsim$
 235 100 in 2D and $\gamma \gtrsim 50$ in 3D was reported^{31,32,35} using the Particle-In-Cell codes Osiris⁴⁰,
 236 Vorpal⁴¹ and Warp³⁷. The presence and growth rate of the instability is observed to be very
 237 sensitive to the resolution (slower growth rate at higher resolution), and to the amount of
 238 damping of high frequencies and filtering of short wavelengths³⁶. The instability is always
 239 propagating at an angle from the longitudinal axis, and is observed in 2D and 3D runs but
 240 was never observed in any of the 1D runs. When modeling an LPA setup in a relativistically
 241 boosted frame, the background plasma is traveling near the speed of light and it has been
 242 conjectured³² that the observed instability might be caused by numerical Cerenkov effect.
 243 The instability was studied in detail with Warp and effective mitigation was demonstrated
 244 on 10 GeV class LPA stages using newly developed algorithms and results³⁶.

245 Secondly, boosted frame simulations may require larger simulation boxes in the transverse
 246 dimension if the entire laser is to be initialized at $t = 0$, as is common practice for standard
 247 laboratory frame simulations^{11,30,32}. The Rayleigh length of the laser is contracted by γ in

248 the boosted frame, while the laser duration increases by $\gamma(1 + \beta)$, implying an increase of the
 249 entire laser spot size by $\gamma^2(1 + \beta)^{32}$. If the laser is to be initialized entirely in the simulation
 250 box at $t = 0$, then the simulation box transverse surface increases as $\gamma^4(1 + \beta)^2$. Although
 251 the cost of the simulation does not scale linearly with the simulation box transverse surface,
 252 as most of it is used only for laser initialization and does not contain macro-particles, the
 253 scaling is so unfavorable that gains of γ^2 provided by the reduction of time steps can be
 254 overtaken by the γ^4 additional costs in grid size, thus limiting the usefulness of the method
 255 to low values of γ boost. Diagrams of the laser emission procedures used for boosted frame
 256 simulations with the Osiris, Vorpal and Warp are given in Fig. 2. Osiris initializes the entire
 257 laser at once and is thus subject to the abovementioned limitations. To circumvent those,
 258 Vorpal emits the laser from all but one faces of the simulation box¹¹ using total field/scattered
 259 field technique⁴², while Warp emits via a moving planar antenna as described in Appendix
 260 B.

261 Thirdly, for a given number of plasma macro-particles per cell, the total number of macro-
 262 particles in the entire plasma column goes down as $1/\gamma^2$ where γ is the relativistic factor
 263 of the Lorentz boost³². However, simulations of self-injection regimes require a sufficient
 264 number of macro-particles in the plasma column so that adequate statistics ensues in the
 265 number of trapped macro-electrons, imposing a ceiling in the value of γ that can be used.
 266 For a typical scheme, a $\gamma_{max} \simeq 50$ was derived³² using purely statistical arguments assuming
 267 the usage of macro-particles of equal weights. This limit might be relaxed by using varying
 268 macro-particle weights such that regions with high probability of trapping (as determined
 269 from the accumulated knowledge of previous work) are populated with a higher density of
 270 macro-particles of smaller weights. This is already practiced in ordinary runs (i.e. without
 271 boosted-frame) for minimizing the computational cost while maximizing the statistics within
 272 “dynamically interesting” regions⁴³. For instance it is found^{6,44,45} that in the bubble regime,
 273 self-injected particles are initially located within a relatively narrow ring region along the
 274 laser axis whose radius is of the order of the laser waist. Previous simulations can be utilized
 275 to determine exactly the radius and thickness of the ring region. This issue does not affect
 276 the modeling of stages with external injection that will be considered in this paper.

277 Observed speedups from simulations using the Particle-In-Cell codes Osiris, Vorpal and
 278 Warp are plotted for 0.1 GeV to 1 TeV stages in Fig. 1-b and contrasted to the theoretical
 279 speedups from Eq. (13). All three codes were using the same standard Particle-In-Cell

280 method⁴⁶. They all successfully performed 2D and/or 3D calculations with boosts at γ in
281 the range of 20-70, reaching speedups over three orders of magnitude (projected for Osiris
282 assuming no computational cost from laser injection). Without the use of special techniques
283 to mitigate the short wavelength instability, none of the codes could perform successfully
284 2D or 3D simulations for γ boost values over 100. With the use of the special techniques
285 described elsewhere³⁶ and in Appendix B, Warp simulations were successfully performed
286 using γ boost as high as 1,300 in 2D and 400 in 3D for 1 TeV and 100 GeV class stages
287 respectively.

288 It is important to note that observed speedups were obtained from simulations of different
289 setups and thus do not offer a direct comparison of the merits of the different codes with
290 regards to boosted frame simulations: Osiris simulations were of trapped self-injection stages,
291 while Vorpil and Warp simulations were of external injection stages with beam loading.
292 Furthermore, while Vorpil and Warp simulations used special procedures to launch the
293 laser that minimize the transverse grid size, Osiris' did not and used transverse grid sizes
294 that were notably larger (as described above). This made Osiris runs in boosted frames
295 substantially more costly, which does not show in the speedups reported by Osiris as this
296 effect was not factored in. However, it is also important to recognize that the most important
297 limiting factor was the high frequency instability (observed in 2D and 3D), which seems to
298 have affected boosted frame simulations of LPA equally, independently of the code used or
299 the simulated LPA setup, legitimizing the comparison in this respect.

300 The numerical techniques that were developed and implemented in Warp³⁶ [as described
301 in Appendix B] are used in the next sections to demonstrate stability and convergence of
302 the boosted frame method up to the 100 GeV-1 TeV range.

303 **IV. MODELING 10 GEV CLASS LASER PLASMA ACCELERATION** 304 **STAGES**

305 This section presents the modeling of deeply depleted and beam loaded 10 GeV LPA
306 stages at full scale in 2-1/2D and 3D using the new numerical techniques that we imple-
307 mented in Warp in³⁶ and in Appendix B, which has not been done fully self-consistently
308 without the Lorentz boost method.

309 It has been shown that many parameters of high energy LPA stages can be accurately

310 simulated at reduced cost by simulating stages of lower energy gain, with higher density
 311 and shorter acceleration distance, by scaling the physical quantities relative to the plasma
 312 wavelength, and this has been applied to design of 10 GeV LPA stages^{20,21}. The number
 313 of oscillations of a mismatched laser pulse in the plasma channel however depends on stage
 314 energy and does not scale, though this effect is minimized for a channel guided stage^{20,21}.
 315 The number of betatron oscillations of the trapped electron bunch will also depend on the
 316 stage energy, and may affect quantities like the emittance of the beam. For these reasons,
 317 and to prove validity of scaled designs of other parameters, it is necessary to perform full
 318 scale simulations, which is only possible by using reduced models (e.g., Ref.¹⁹) or simulations
 319 in the boosted frame.

320 The basic prescription for scaling a LPA simulation to lower plasma density and higher
 321 electron energy gain can be briefly summarized as follows (for additional details see Refs^{20,21}).
 322 First, a fully resolved simulation (i.e., sufficient number of grid points per laser wavelength
 323 λ) is performed at a relatively low value of $\lambda_p/\lambda = k/k_p$ (i.e., at a relatively high plasma
 324 density for a fixed laser wavelength). Next, the simulation results are scaled to a higher
 325 value of k/k_p (i.e., lower density) by keeping the normalized laser and beam parameters
 326 fixed (constant a_0 , $k_p L$, $k_p \sigma$, $k_p \sigma_z$, $k_p \sigma_r$, and n_b/n_0 , where a_0 is the laser normalized vector
 327 potential, k_p is the plasma wavenumber, n_b is the electron beam density, L and σ are
 328 the longitudinal and transverse sizes of the laser, and σ_z and σ_r are the longitudinal and
 329 transverse sizes of the beam), since these normalized parameters determine the structure
 330 of the accelerating and focusing plasma wakefields. The acceleration length L_d (e.g., the
 331 length for the electron beam to reach maximum energy) and the electron energy gain γ_{max}
 332 scale as $L_d k_p^3/k^2 = \text{constant}$ and $\gamma_{max} k_p^2/k^2 = \text{constant}$, since the dephasing and depletion
 333 lengths scale as $L_d \sim k^2/k_p^3$ and the accelerating field scales as $E_z \sim k_p$. As noted above,
 334 some physically relevant quantities do not remain constant when scaled to higher values of
 335 k/k_p , such as the trapping threshold for particles in the wake⁴⁷ as well as the normalized
 336 Rayleigh length, $k_p Z_R = k k_p \sigma^2/2$, which determines for example the number of oscillations
 337 of a mismatched laser pulse in a plasma channel. For this reason, in general, fully resolved
 338 simulations at the correct value of k/k_p are still desirable.

339 For benchmarking purposes, scaled simulations²⁰ are performed, first at a density of
 340 $n_e = 10^{19} \text{ cm}^{-3}$, using various values of the boosted frame relativistic factor γ to show
 341 the accuracy and convergence of the technique. These scaled simulations were shown to

342 efficiently accelerate both electrons and positrons with low energy spread, and predicted
 343 acceleration of hundreds of pC to 10 GeV energies using a 40 J laser. The accuracy of the
 344 boosted frame technique is evaluated by modeling scaled stages^{20,21} at 0.1 GeV, which allows
 345 for a detailed comparison of simulations using a reference frame ranging from the laboratory
 346 frame to the frame of the wake. Excellent agreement is obtained on wakefield histories on
 347 axis, beam average energy and transverse RMS size histories, and momentum spread at
 348 peak energy, with speedup over a hundred, in agreement with the theoretical estimates from
 349 Section 2. The boosted frame technique is then applied in the next section to provide full
 350 scale simulation of high efficiency quasilinear LPA stages at higher energy, verifying the
 351 scaling laws in the 10 GeV-1 TeV range.

352 A. Scaled 10 GeV class stages

353 The main physical and numerical parameters of the simulations are given in Table I.
 354 They were chosen to be close (though not identical) to a case reported elsewhere²⁰ with
 355 $k_p L = 2$ where L is the laser pulse length as defined in Table I, the main differences being
 356 a sinusoidal versus gaussian laser longitudinal profile and a laser spot size larger by $\sqrt{2}$.
 357 These simulations are for a fully resolved 100 MeV stage at a density of 10^{19} cm⁻³, which
 358 can be scaled to describe a 10 GeV stage at a density of 10^{17} cm⁻³, thereby allowing short
 359 run times to permit effective benchmarking between the algorithms^{20,21}. These runs were
 360 done using the standard Yee solver with no damping, and with the 4-pass stride-1 filter
 361 plus compensation³⁶. No signs of detrimental numerical instabilities were observed at the
 362 resolutions reported here with these settings in 2-1/2D or 3D.

363 For the given parameters, the wake relativistic factor $\gamma_w \approx 13.2$. Thus, Warp simulations
 364 were performed using reference frames moving between $\gamma = 1$ (laboratory frame) and 13.
 365 For a boosted frame associated with a value of γ approaching γ_w in the laboratory, the
 366 wake is expected to travel at low velocity, and the physics to appear somewhat different
 367 from that observed in the laboratory frame, in accordance to the properties of the Lorentz
 368 transformation³⁸. Figures 3 and 4 show surface renderings of the transverse and longitudinal
 369 electric fields respectively, as the beam enters its early stage of acceleration by the plasma
 370 wake, from a calculation in the laboratory frame and another in the frame at $\gamma = 13$. The two
 371 snapshots offer strikingly different views of the same physical processes: in the laboratory

TABLE I. List of parameters for a LPA stage simulation at 100 MeV

plasma density on axis	n_e	10^{19} cm^{-3}
plasma longitudinal profile		flat
plasma length	L_p	1.5 mm
plasma entrance ramp profile		half sine
plasma entrance ramp length		20 μm
laser profile		$a_0 \exp(-r^2/2\sigma^2) \sin(\pi z/3L)$
normalized vector potential	a_0	1
laser wavelength	λ	0.8 μm
laser spot size (RMS)	σ	8.91 μm
laser length (HWHM)	L	3.36 μm
normalized laser spot size	$k_p\sigma$	5.3
normalized laser length	k_pL	2
beam profile		$n_{b0} \exp(-r^2/2\sigma_r^2 - z^2/2\sigma_z^2)$
beam transverse size (RMS)	σ_r	165 nm
beam length (RMS)	σ_z	85 nm
normalized beam spot size	$k_p\sigma_r$	0.1
normalized beam length	$k_p\sigma_z$	0.05
beam transverse emittance	ϵ	73.5 nm.mrad
beam total charge (3D)	Q	6.42 pC
beam initial energy	E_0	$1.5\gamma_w m_e c^2$
injection distance after laser max		$0.7\lambda_p$
number of cells in x	N_x	75
number of cells in z	N_z	860 ($\gamma = 13$)-1691 ($\gamma = 1$)
cell size in x	δx	$0.65\mu\text{m}$
cell size in z	δz	$\lambda/64$
time step	δt	at CFL limit
particle deposition order		cubic
# of plasma particles/cell		1 macro-e ⁻ +1 macro-p ⁺

372 frame, the wake is fully formed before the beam undergoes any significant acceleration and
373 the imprint of the laser is clearly visible ahead of the wake; in the boosted frame calculation,
374 the beam is accelerated as the plasma wake develops, and the laser imprint is not visible on
375 the snapshot. Close examination reveals that the short spatial variations which make the
376 laser imprint at the front of the wake are transformed into time variations in the boosted
377 frame of $\gamma = 13$. This effect is due to hyperbolic rotation in Minkowski space of the laser
378 propagation in plasma, as explained in more detail elsewhere³⁸. The imprint of the beam
379 loading is clearly visible on the plot of the longitudinal electric field (wake) in the laboratory
380 frame (top plot of Figure 4).

381 Histories of the perpendicular and longitudinal electric fields recorded at a number of
382 stations at fixed locations in the laboratory offer direct comparison between the simulations
383 in the laboratory frame ($\gamma = 1$) and boosted frames at $\gamma = 2, 5, 10$ and 13 . Figure 5 and
384 6 show respectively the transverse and longitudinal electric fields collected at the positions
385 $z = 0.3$ mm and $z = 1.05$ mm (in the laboratory frame) on axis ($x = y = 0$). The agreement
386 is excellent and confirms that despite the apparent differences from snapshots taken from
387 simulations in different reference frames, the same physics was recovered. The effect of beam
388 loading is visible in Figure 6 at $t \approx 1.15$ ps and $t \approx 3.61$ ps, confirming that the amplitude
389 and phase of beam loading was correctly recovered in all frames. This is further confirmed
390 by the plot of the average scaled beam energy gain and transverse RMS size as a function
391 of position in the laboratory frame, and of relative longitudinal momentum dispersion at
392 peak energy (Fig. 7). These show that the correct laser evolution and electron beam
393 energy, momentum spread and transverse dynamics were modeled in all frames. The small
394 differences observed in the mean beam energy histories and on the longitudinal momentum
395 spread are due to a lack of convergence at the resolution that was chosen, and we have
396 verified that convergence was improving with increasing resolution. The beam was launched
397 with the same phase in the 2-1/2D and the 3D simulations, resulting in lower energy gain
398 in 3D, due to proportionally larger laser depletion effects in 3D than in 2-1/2D.

399 The CPU time recorded as a function of the average beam position in the laboratory
400 frame indicates that the simulation in the frame of $\gamma = 13$ took ≈ 25 s in 2-1/2D and
401 ≈ 150 s in 3D versus $\approx 5,000$ s in 2-1/2D and $\approx 20,000$ s in 3D in the laboratory frame,
402 demonstrating speedups of ≈ 200 in 2-1/2D and ≈ 130 in 3D, between calculations in a
403 boosted frame at $\gamma = 13$ and calculations in the laboratory frame.

404 All the simulations presented so far in this section were using the Yee solver³⁶, for
 405 which the Courant condition is given by $c\delta t < (1/\delta x^2 + 1/\delta z^2)^{-1/2}$ in 2D and $c\delta t <$
 406 $(1/\delta x^2 + 1/\delta y^2 + 1/\delta z^2)^{-1/2}$ in 3D where δt is the time step and δx , δy and δz are the
 407 computational grid cell sizes in x , y and z . As γ was varied, the transverse resolution
 408 was kept constant, while the longitudinal resolution was kept at a constant fraction of the
 409 incident laser wavelength $\delta z = \zeta\lambda$, such that in a boosted frame, $\delta z^* = \zeta\lambda^* = \zeta(1 + \beta)\gamma\lambda$.
 410 As a result, the speedup becomes, when using the Yee solver

$$S_{yee2D} = S \frac{\delta z \sqrt{1/\delta x^2 + 1/\delta z^2}}{\delta_z^* \sqrt{1/\delta x^2 + 1/\delta z^{*2}}} \quad (17)$$

411 in 2D and

$$S_{yee3D} = S \frac{\delta z \sqrt{1/\delta x^2 + 1/\delta y^2 + 1/\delta z^2}}{\delta_z^* \sqrt{1/\delta x^2 + 1/\delta y^2 + 1/\delta z^{*2}}} \quad (18)$$

412 in 3D, where S is given by Eq. (13).

413 The speedup versus relativistic factor of the reference frame is plotted in Fig. 8, from
 414 (13), (17) and (18), and contrasted with measured speedups from 1D, 2-1/2D and 3D Warp
 415 simulations, confirming the scaling obtained analytically.

416 This subsection demonstrated accurate modeling of the evolution of the laser and the
 417 electron beam energy, momentum spread and transverse dynamics with agreement at the
 418 percentage level between simulations using various reference frames. The scaling of the
 419 speedup was also confirmed, validating our understanding of the boosted frame method
 420 scaling with γ boost.

421 B. Full scale 10 GeV class stages

422 The boosted frame technique was next applied to the modeling of 10 GeV stages at full
 423 scale (i.e. at plasma density $n_e = 10^{17} \text{ cm}^{-3}$, with parameters scaled from Table 1). As
 424 noted elsewhere¹¹, full scale simulations using the laboratory frame of 10 GeV stages at
 425 plasma densities of 10^{17} cm^{-3} are not practical on present computers in 2D and 3D. At this
 426 density, the wake relativistic factor $\gamma_w \approx 132$, and 2-1/2D and 3D simulations were done
 427 in boosted frames up to $\gamma = 130$, realizing the maximum theoretical speedup. This section
 428 demonstrates accurate modeling of the particle beam acceleration and transverse dynamics
 429 evolution of full scale beam loaded 10 GeV stages in 2D and 3D for boosted frames up to
 430 $\gamma = 130$.

431 Fig. 9 and Fig. 10 show the average beam energy gain and transverse RMS size versus
 432 longitudinal position from respectively 2-1/2D and 3D simulations in boosted frames at
 433 $\gamma = 30$ to 130 in 2-1/2D and at $\gamma = 60$ to 130 in 3D (runs at $\gamma = 1$ are impractical and
 434 were not performed). All runs gave the same beam energy history within a few percent.
 435 The average energy gain peaks around 10 GeV in 2-1/2D and 8 GeV in 3D, in agreement
 436 with the scaled simulations (see Fig. 7). The abovementioned short wavelength instability
 437 that occurs at high values of γ boost is described elsewhere³⁶ and has been mitigated in the
 438 3D simulations using $\gamma \geq 120$ using a novel electromagnetic solver and time step ($c\delta t/\delta z =$
 439 $1/\sqrt{2}$) for which the instability growth rate is greatly reduced, completed by smoothing
 440 of short wavelengths³⁶. The small discrepancy between the results of the runs at various
 441 γ is due to lack of convergence and difficulty in attaining identical initial conditions (see
 442 Appendix B) at the resolution that was chosen (32 grid cells per laser wavelength in vacuum).
 443 Preliminary scans with varying resolution (not shown here) show that agreement improves
 444 with higher resolution and suggest that boosted frame simulations may converge faster than
 445 laboratory frame simulations.

446 The boosted frame technique was applied to the direct simulation of a 10 GeV stage
 447 ($n_e = 10^{17} \text{ cm}^{-3}$) in which the accelerated charge was sufficiently high so that the effects of
 448 beam loading were readily evident. These parameters are relevant to experiments that will
 449 be carried out on new lasers, such as the BELLA facility at LBNL¹⁰, and to LPA stages that
 450 can serve as the basis for high energy collider modules^{4,5}. In this simulation, a laser pulse
 451 with intensity $a_0 = 1.414$, wavelength $\lambda = 0.8$ microns and RMS duration of $L/c = 40$ fs
 452 (i.e. $k_p L = 1/\sqrt{2}$), where L is the longitudinal RMS size of the gaussian laser pulse profile
 453 $a(r, z) = a_0 \exp(-r^2/2\sigma^2 - z^2/2L^2)$ was focused to a RMS gaussian transverse spot size of
 454 63 microns (i.e. $k_p \sigma = 3.75$) at the channel entrance. The plasma channel had an on-axis
 455 density $n_0 = 10^{17} \text{ cm}^{-3}$, a length of 0.75 m with a parabolic channel (factor=0.6) and a
 456 longitudinal taper^{20,48} of the form $n(x) = n_0(1.32x + 1)$. An electron beam with a gaussian
 457 profile and 17 microns size (i.e. $k_p \sigma_r = 1$) and 8.5 microns length (i.e. $k_p \sigma_z = 0.5$) was
 458 externally injected at a distance of $1.53\lambda_p$ behind the location of maximum laser intensity
 459 with an initial energy of 100 MeV and an initial emittance of 60 mm-mrad. The calculation
 460 was in 2-1/2D, and the beam charge density corresponded to a total charge of ~ 52 pC in
 461 3D. The large input emittance was chosen to maximize the beam radius for efficient beam
 462 loading and for emittance matching to the wakes focusing fields²⁰. Figure 11 shows the

463 density wake excited by an intense laser pulse and the externally injected electron beam
 464 accelerated by the wake. The color coding indicates the energy reached by the electrons.
 465 The depression in the density wake is due to self-consistent beam loading of the injected
 466 electron bunch. The histories of electron beam mean energy, emittance and RMS size are
 467 given in Figure 12, while the longitudinal momentum distribution is given in Figure 13. At
 468 the exit of the structure, electrons with energy of up to 11 GeV were observed. The time
 469 projected energy spread and normalized emittance when exiting the plasma channel were
 470 15% and 61 mm-mrad, respectively. The slice energy spread and emittance of a slice at
 471 9.5 GeV were 1% and 54 mm-mrad. Whereas these values are larger than acceptable for
 472 collider and light source applications, it has been shown that lower emittance bunches can be
 473 accelerated by using high order laser modes to control the transverse focusing forces⁴⁹ and
 474 lower energy spread by controlling beam loading^{20,50}. Future work will aim at optimizing
 475 the phase space properties of the bunch including optimization of taper⁴⁸ and use of higher
 476 order laser modes to minimize emittance.

477 The present work demonstrates the ability to simulate at full scale a 10 GeV stage that
 478 exhibits significant laser depletion and beam loading. It confirms that the electron beam
 479 acceleration and energy gain is very well predicted by scaled simulations, and shows that
 480 emittance conservation is obtained through good matching, which is only accurately ac-
 481 counted for at full scale.

482 V. FULL SCALE 100 GEV-1 TEV STAGES

483 The numerical techniques that we developed³⁶ and successfully applied to the modeling
 484 of 10 GeV class stages in the preceding section are applied in this section to the modeling of
 485 stages in the 0.1 GeV-1 TeV range in 2-1/2D and in the 0.1-100 GeV range in 3D, showing
 486 scaling of LPAs to high energies. The plasma density n_e scales inversely to the energy gain,
 487 from 10^{19} cm^{-3} down to 10^{15} cm^{-3} in the 0.1 GeV-1 TeV range. These simulations used
 488 the parameters given in Table I scaled appropriately to higher energies²⁰, and used the high
 489 speed of the boosted simulations to allow fast turnaround improvement of the stage design
 490 presented in^{20,21}.

491 The average beam energy gain history is plotted in Fig. 14, scaling the 0.1-100 GeV
 492 runs to the 1 TeV run in 2-1/2D, and the 0.1-10 GeV runs to the 100 GeV run in 3D. The

493 differences at 10^{19} cm^{-3} of the scaled beam energy gain history can be attributed to the
494 effects from having only a few laser oscillations per pulse.

495 Using (13), the speedup of the full scale 100 GeV class run, which used a boosted frame
496 of $\gamma = 400$ as frame of reference, is shown to be over 100,000, as compared to a run using the
497 laboratory frame. Assuming the use of a few thousands of CPUs, a simulation that would
498 require several decades to complete using standard PIC techniques in the laboratory frame,
499 was completed in four hours using 2016 CPUs of the Cray system at NERSC. Also using
500 (13), the speedup of the 2-1/2D 1 TeV stage is shown to be over a million.

501 This section demonstrated the scaling of highly depleted beam loaded stages up to 1 TeV
502 in 2D and 100 GeV in 3D, providing greater credibility for evaluation of various LPA based
503 collider options⁴.

504 VI. CONCLUSIONS

505 Calculations using an optimal boosted frame of 10 GeV, 100 GeV and 1 TeV class stages
506 including beam loading were presented, with speedups over 4, 5 and 6 orders of magnitude
507 respectively over what would be required by “standard” laboratory frame calculations, which
508 are impractical for such stages due to computational requirements. Our previous theoretical
509 speedup estimate²³ was extended to high boost values, while complications associated with
510 the handling of input and output data between a boosted frame and the laboratory frame
511 were discussed. Practical solutions were implemented, including a technique for injecting
512 the laser that is simpler and more efficient than methods proposed previously.

513 The boosted frame Particle-In-Cell technique has been shown to accurately model the
514 laser evolution and resolve the wavelength shifting and broadening (as described in Appendix
515 A) that occurs as the laser depletes, offering advantages over other models (for example
516 envelope, quasistatic) while providing the speed required for direct simulation of 10 GeV and
517 beyond laser plasma accelerators to accurately model laser and beam transverse oscillations.
518 It has been shown to also model accurately the electron beam acceleration, longitudinal
519 and transverse dynamics. The results are within a few percent of those from ‘standard’
520 laboratory frame simulations, which is within acceptable range for the design of proof-of-
521 principle experiments. The boosted frame technique is being applied to the direct simulation
522 of 10 GeV beam loaded stages, which is relevant to experiments on new lasers (e.g., the

523 BELLA facility at LBNL), as well as next generation controlled laser plasma accelerator
524 stages and collider modules.

525 VII. ACKNOWLEDGMENTS

526 We are thankful to C. Benedetti, D. L. Bruhwiler, J. R. Cary, B. Cowan, A. Fried-
527 man, C. Huang, S. F. Martins, W. B. Mori, and B. A. Shadwick for insightful discussions.
528 Work supported by US-DOE Contracts DE-AC02-05CH11231 and DE-AC52-07NA27344,
529 and US-DOE SciDAC program COMPASS. Used resources of NERSC, supported by US-
530 DOE Contract DE-AC02-05CH11231.

531 Appendix A: Laser spectrum on axis and optimal frame

532 In this section, we discuss in more detail how the choice of the optimal frame for smoothing
533 is determined by the laser spectrum. It was shown previously³⁸ that choosing γ boost near
534 γ of the wakefield is a possible option. We extend the discussion to consider depletion of
535 the laser and show that in this case a lower value of γ boost might be desirable.

536 The spectrum history of the laser field on axis is given in Fig. 15 for selected values of γ
537 between 1 and 135. The history is given up to the time of the electron beam peak energy. In
538 the laboratory frame at $\gamma = 1$, the initial ($t=0$) spectral content is very localized in a narrow
539 band around the laboratory frame vacuum laser wavelength λ_0 , spreading and redshifting
540 as the laser propagates and depletes its energy into the wake². Although it is not visible in
541 the spectrum, the laser waves propagate in the positive direction in the laboratory frame.

542 At higher values of γ frame, the initial spectral content of the laser shifts to longer wave-
543 lengths relative to the boosted frame vacuum laser wavelength λ'_0 . As the frame approaches
544 the wake frame $\gamma_w \approx 132$, the initial spectrum is displaced toward very long wavelengths
545 (standing waves), because the frame is moving near the laser group velocity. At later times,
546 the high γ frame spectra show content repopulating progressively shorter wavelengths. This
547 corresponds to the redshifting observed in the lab frame; with the calculation frame match-
548 ing the initial laser group velocity, the redshifted light which propagates slower now slips
549 backward in the moving frame. As γ frame rises, eventually all waves propagate in the
550 negative direction for $\gamma \geq \gamma_w$.

551 Mitigation of the short wavelength instability necessitates higher amounts of smoothing
 552 at higher γ , and smoothing is most effective (and has minimal effect on simulation physics)
 553 when spectral content is confined to long wavelengths³⁶. This occurs for $\gamma \approx \gamma_w$ initially, and
 554 Fig. 15 indicates how for strongly depleted stages the optimum γ may be adjusted slightly
 555 below γ_w , in order to maximize the wavelengths of the average spectral content over the
 556 propagation length rather than only at the start. The plasma column also usually exhibits
 557 a parabolic transverse profile so as to provide transverse focusing of the laser, which also
 558 slightly reduces group velocity². These corrections to the optimal gamma are relatively small
 559 and thanks to the weak dependency of the speedup with γ near γ_w (cf Fig. 1), simulations
 560 with γ approaching γ_w offer speedups that are very near the maximum attainable, thus
 561 offering in practice the maximum benefit of the boosted frame technique while maintaining
 562 the highest level of accuracy. Other effects such as tapering of the plasma density may
 563 further decrease the optimal γ for smoothing. A large value of γ boost can still be used for
 564 high energy stages, thus achieving orders of magnitude speedups in practice.

565 At higher resolution, the instability level is reduced³⁶, and so is the amount of smoothing
 566 that is necessary to control it. Furthermore, the instability spectrum is confined to a very
 567 narrow band located near the Nyquist cutoff of the simulation grid³⁶, and thus separates
 568 further from the spectrum with physical content of interest as resolution increases. Hence
 569 high resolution simulations may use γ boost near γ_w and achieve maximum speedup
 570 even for runs using a tapered plasma. Simulations have been conducted at up to three times
 571 the base resolution, where use of $\gamma = \gamma_w$ is possible even including the above effects. The
 572 boosted frame speedup allows such high resolutions which may be important for evaluation
 573 of future low emittance stage concepts.

574 **Appendix B: Input and output to and from boosted frame simulations in Warp**

575 This section describes the procedures that have been implemented in the Particle-In-Cell
 576 framework Warp³⁷ to handle the input and output of data between the frame of calculation
 577 and the laboratory frame. Simultaneity of events between two frames is valid only for a
 578 plane that is perpendicular to the relative motion of the frame. As a result, the input/output
 579 processes involve the input of data (particles or fields) through a plane, as well as output
 580 through a series of planes, all of which are perpendicular to the direction of the relative

581 velocity between the frame of calculation and the other frame of choice.

582 1. Input

583 a. *Particles*

584 Particles are launched through a plane using a technique which applies to many calcu-
585 lations in a boosted frame, including LPA, and is illustrated using the case of a positively
586 charged particle beam propagating through a background of cold electrons in an assumed
587 continuous transverse focusing system, leading to a growing transverse instability²³. In the
588 laboratory frame, the electron background is initially at rest and a moving window is used
589 to follow the beam progression. Traditionally, the beam macroparticles are initialized all
590 at once in the window, while background electron macroparticles are created continuously
591 in front of the beam on a plane that is perpendicular to the beam velocity. In a frame
592 moving at some fraction of the beam velocity in the laboratory frame, the beam initial
593 conditions at a given time in the calculation frame are generally unknown and one must
594 initialize the beam differently. However, it can be taken advantage of that the beam initial
595 conditions are often known for a given plane in the laboratory, either directly, or via sim-
596 ple calculation or projection from the conditions at a given time. Given the position and
597 velocity $\{x, y, z, v_x, v_y, v_z\}$ for each beam macroparticle at time $t = 0$ for a beam moving at
598 the average velocity $v_b = \beta_b c$ (where c is the speed of light) in the laboratory, and using
599 the standard synchronization ($z = z' = 0$ at $t = t' = 0$) between the laboratory and the
600 calculation frames, the procedure for transforming the beam quantities for injection in a
601 boosted frame moving at velocity βc in the laboratory is as follows (the superscript $'$ relates
602 to quantities known in the boosted frame while the superscript $*$ relates to quantities that
603 are know at a given longitudinal position z^* but different times of arrival):

- 604 1. project positions at $z^* = 0$ assuming ballistic propagation

$$t^* = (z - \bar{z}) / v_z \tag{B1}$$

$$x^* = x - v_x t^* \tag{B2}$$

$$y^* = y - v_y t^* \tag{B3}$$

$$z^* = 0 \tag{B4}$$

605 the velocity components being left unchanged,

606 2. apply Lorentz transformation from laboratory frame to boosted frame

$$t'^* = -\gamma t^* \quad (\text{B5})$$

$$x'^* = x^* \quad (\text{B6})$$

$$y'^* = y^* \quad (\text{B7})$$

$$z'^* = \gamma\beta ct^* \quad (\text{B8})$$

$$v_x'^* = \frac{v_x^*}{\gamma(1 - \beta\beta_b)} \quad (\text{B9})$$

$$v_y'^* = \frac{v_y^*}{\gamma(1 - \beta\beta_b)} \quad (\text{B10})$$

$$v_z'^* = \frac{v_z^* - \beta c}{1 - \beta\beta_b} \quad (\text{B11})$$

607 where $\gamma = 1/\sqrt{1 - \beta^2}$. With the knowledge of the time at which each beam macropar-
 608 ticle crosses the plane into consideration, one can inject each beam macroparticle in
 609 the simulation at the appropriate location and time.

610 3. synchronize macroparticles in boosted frame, obtaining their positions at a fixed t' (=
 611 0) which is before any particle is injected

$$z' = z'^* - \bar{v}_z'^* t'^* \quad (\text{B12})$$

612 This additional step is needed for setting the electrostatic or electromagnetic fields
 613 at the plane of injection. In a Particle-In-Cell code, the three-dimensional fields are
 614 calculated by solving the Maxwell equations (or static approximation like Poisson,
 615 Darwin or other²⁴) on a grid on which the source term is obtained from the macropar-
 616 ticles distribution. This requires generation of a three-dimensional representation of
 617 the beam distribution of macroparticles at a given time before they cross the injection
 618 plane at z'^* . This is accomplished by expanding the beam distribution longitudinally
 619 such that all macroparticles (so far known at different times of arrival at the injection
 620 plane) are synchronized to the same time in the boosted frame. To keep the beam
 621 shape constant, the particles are “frozen” until they cross that plane: the three veloc-
 622 ity components and the two position components perpendicular to the boosted frame
 623 velocity are kept constant, while the remaining position component is advanced at the

624 average beam velocity. As particles cross the plane of injection, they become regular
 625 “active” particles with full 6-D dynamics.

626 Figure 16 (top) shows a snapshot of a beam that has passed partly through the injection
 627 plane. As the frozen beam macroparticles pass through the injection plane (which moves
 628 opposite to the beam in the boosted frame), they are converted to “active” macroparticles.
 629 The charge or current density is accumulated from the active and the frozen particles, thus
 630 ensuring that the fields at the plane of injection are consistent.

631 ***b. Laser***

632 Similarly to the particle beam, the laser is injected through a plane perpendicular to the
 633 axis of propagation of the laser (by default z). The electric field E_{\perp} that is to be emitted is
 634 given by the formula

$$E_{\perp}(x, y, t) = E_0 f(x, y, t) \sin[\omega t + \phi(x, y, \omega)] \quad (\text{B13})$$

635 where E_0 is the amplitude of the laser electric field, $f(x, y, t)$ is the laser envelope, ω is the
 636 laser frequency, $\phi(x, y, \omega)$ is a phase function to account for focusing, defocusing or injection
 637 at an angle, and t is time. By default, the laser envelope is a three dimensional gaussian of
 638 the form

$$f(x, y, t) = e^{-(x^2/2\sigma_x^2 + y^2/2\sigma_y^2 + c^2 t^2/2\sigma_z^2)} \quad (\text{B14})$$

639 where σ_x , σ_y and σ_z are the dimensions of the laser pulse; or it can be defined arbitrarily
 640 by the user at runtime. If $\phi(x, y, \omega) = 1$, the laser is injected at a waist and parallel to the
 641 axis z .

642 If, for convenience, the injection plane is moving at constant velocity $\beta_s c$, the formula is
 643 modified to take the Doppler effect on frequency and amplitude into account and becomes

$$E_{\perp}(x, y, t) = (1 - \beta_s) E_0 f(x, y, t) \times \sin[(1 - \beta_s) \omega t + \phi(x, y, \omega)]. \quad (\text{B15})$$

644 The injection of a laser of frequency ω is considered for a simulation using a boosted
 645 frame moving at βc with respect to the laboratory. Assuming that the laser is injected at a

646 plane that is fixed in the laboratory, and thus moving at $\beta_s = -\beta$ in the boosted frame, the
 647 injection in the boosted frame is given by

$$E_{\perp}(x', y', t') = (1 - \beta_s) E'_0 f(x', y', t') \times \sin[(1 - \beta_s) \omega' t' + \phi(x', y', \omega')] \quad (\text{B16})$$

$$= (E_0/\gamma) f(x', y', t') \times \sin[\omega t'/\gamma + \phi(x', y', \omega')] \quad (\text{B17})$$

648 since $E'_0/E_0 = \omega'/\omega = 1/(1 + \beta)\gamma$.

649 The electric field is then converted into currents that get injected via a 2D array of
 650 macro-particles, with one positive and one dual negative macro-particle for each array cell
 651 in the plane of injection, whose weights and motion are governed by $E_{\perp}(x', y', t')$. Injecting
 652 using this dual array of macroparticles offers the advantages of automatically including the
 653 longitudinal component which arise from emitting into a boosted frame, and to automatically
 654 verify the discrete Gauss' law thanks to using the Esirkepov current deposition scheme⁵¹.

655 As discussed in section III, the technique implemented in Warp presents several advan-
 656 tages over other procedures that have been proposed elsewhere^{11,32}. The method presented
 657 here avoids the caveat of the broadening of the transverse size of the laser while retaining
 658 simplicity and versatility by injecting through one plane rather than several faces of the box.

659 2. Output

660 Some quantities, e.g. charge or dimensions perpendicular to the boost velocity, are
 661 Lorentz invariant. Those quantities are thus readily available from standard diagnostics
 662 in the boosted frame calculations. Quantities which do not fall in this category are recorded
 663 at a number of regularly spaced “stations”, immobile in the laboratory frame, at a succession
 664 of time intervals to record data history, or averaged over time. A visual example is given
 665 on Fig. 16 (bottom). Since the space-time locations of the diagnostic grids in the labo-
 666 ratory frame generally do not coincide with the space-time positions of the macroparticles
 667 and grid nodes used for the calculation in a boosted frame, some interpolation is performed
 668 at runtime during the data collection process. As a complement or an alternative, selected
 669 particle or field quantities are dumped at regular interval for post-processing. The choice of
 670 the methods depends on the requirements of the diagnostics and particular implementations.

671 **REFERENCES**

- 672 ¹T. TAJIMA and J. DAWSON, PHYSICAL REVIEW LETTERS **43**, 267 (1979), ISSN
673 0031-9007.
- 674 ²E. Esarey, C. B. Schroeder, and W. P. Leemans, Rev. Mod. Phys. **81**, 1229 (2009), ISSN
675 0034-6861.
- 676 ³W. P. Leemans, B. Nagler, A. J. Gonsalves, C. Toth, K. Nakamura, C. G. R. Geddes,
677 E. Esarey, C. B. Schroeder, and S. M. Hooker, NATURE PHYSICS **2**, 696 (2006), ISSN
678 1745-2473.
- 679 ⁴C. B. Schroeder, E. Esarey, C. G. R. Geddes, C. Benedetti, and W. P. Leemans, Phys.
680 Rev. Spec. Topics Accel. Beams. **13**, 101301 (2010).
- 681 ⁵W. Leemans and E. Esarey, PHYSICS TODAY **62**, 44 (2009), ISSN 0031-9228.
- 682 ⁶F. Tsung, W. Lu, M. Tzoufras, W. Mori, C. Joshi, J. Vieira, L. Silva, and R. Fonseca,
683 PHYSICS OF PLASMAS **13**, 056708 (2006), ISSN 1070-664X.
- 684 ⁷C. Geddes, D. Bruhwiler, J. Cary, W. Mori, J.-L. Vay, S. Martins, T. Katsouleas,
685 E. Cormier-Michel, W. Fawley, C. Huang, et al., in *Journal of Physics: Conference Series*
686 (2008), vol. 125, p. 012002 (11 pp.), ISSN 1742-6596.
- 687 ⁸C. Geddes, *et al.*, in *SciDAC Review 13* (2009), p. 13.
- 688 ⁹C. Huang, W. An, V. Decyk, W. Lu, W. Mori, F. Tsung, M. Tzoufras, S. Morshed,
689 T. Antonsen, B. Feng, et al., Journal of Physics: Conference Series **180**, 012005 (11 pp.)
690 (2009), ISSN 1742-6596.
- 691 ¹⁰W. P. Leemans, R. Duarte, E. Esarey, S. Fournier, C. G. R. Geddes, D. Lockhart, C. B.
692 Schroeder, C. Toth, J.-L. Vay, and S. Zimmermann, AIP Conference Proceedings **1299**, 3
693 (2010).
- 694 ¹¹D. Bruhwiler, J. Cary, B. Cowan, K. Paul, C. Geddes, P. Mullaney, P. Messmer,
695 E. Esarey, E. Cormier-Michel, W. Leemans, et al., in *AIP Conference Proceedings* (2009),
696 vol. 1086, pp. 29–37, ISSN 0094-243X.
- 697 ¹²J. KRALL, A. TING, E. ESAREY, and P. SPRANGLE, Physical Review E **48**, 2157
698 (1993).
- 699 ¹³B. A. Shadwick, C. B. Schroeder, and E. Esarey, PHYSICS OF PLASMAS **16**, 056704
700 (2009), ISSN 1070-664X.

- 701 ¹⁴C. Benedetti, C. B. Schroeder, E. Esarey, C. G. R. Geddes, and W. P. Leemans, AIP
702 Conference Proceedings **1299**, 250 (2010).
- 703 ¹⁵P. SPRANGLE, E. ESAREY, and A. TING, PHYSICAL REVIEW LETTERS **64**, 2011
704 (1990), ISSN 0031-9007.
- 705 ¹⁶T. M. ANTONSEN and P. MORA, Physical Review Letters **69**, 2204 (1992).
- 706 ¹⁷C. Huang, V. K. Decyk, C. Ren, M. Zhou, W. Lu, W. B. Mori, J. H. Cooley, T. M.
707 Antonsen, Jr., and T. Katsouleas, JOURNAL OF COMPUTATIONAL PHYSICS **217**,
708 658 (2006), ISSN 0021-9991.
- 709 ¹⁸B. Feng, C. Huang, V. Decyk, W. Mori, P. Muggli, and T. Katsouleas, Journal of Com-
710 putational Physics **228**, 5340 (2009), ISSN 0021-9991.
- 711 ¹⁹B. M. Cowan, D. L. Bruhwiler, E. Cormier-Michel, E. Esarey, C. G. Geddes, P. Messmer,
712 and K. M. Paul, Journal of Computational Physics **230**, 61 (2011), ISSN 0021-9991.
- 713 ²⁰E. Cormier-Michel, C. Geddes, E. Esarey, C. Schroeder, D. Bruhwiler, K. Paul, B. Cowan,
714 and W. Leemans, in *AIP Conference Proceedings* (2009), vol. 1086, pp. 297–302, ISSN
715 0094-243X.
- 716 ²¹C. G. R. Geddes, *et al.*, in *Proc. Particle Accelerator Conference* (Vancouver, Canada,
717 2009), wE6RFP075.
- 718 ²²S. V. BULANOV, I. N. INOVENKOV, V. I. KIRSANOV, N. M. NAUMOVA, and A. S.
719 SAKHAROV, Physics of Fluids B-plasma Physics **4**, 1935 (1992).
- 720 ²³J.-L. Vay, Physical Review Letters **98**, 130405/1 (2007), ISSN 0031-9007.
- 721 ²⁴J. L. Vay, Physics of Plasmas **15**, 056701 (2008).
- 722 ²⁵J. BORIS and R. LEE, JOURNAL OF COMPUTATIONAL PHYSICS **12**, 131 (1973),
723 ISSN 0021-9991.
- 724 ²⁶I. Haber, R. Lee, H. Klein, and J. Boris, in *Proc. Sixth Conf. Num. Sim. Plasmas* (Berkeley,
725 CA, 1973), pp. 46–48.
- 726 ²⁷B. Cowan, D. Bruhwiler, E. Cormier-Michel, E. Esarey, C. Geddes, P. Messmer, and
727 K. Paul, in *AIP Conference Proceedings* (2009), vol. 1086, pp. 309–14, ISSN 0094-243X.
- 728 ²⁸J.-L. Vay, *et al.*, in *Proc. Particle Accelerator Conference* (Vancouver, Canada, 2009),
729 tU1PBI04.
- 730 ²⁹S. F. Martins, *et al.*, in *Proc. Particle Accelerator Conference* (Vancouver, Canada, 2009),
731 tH4GBC05.

- 732 ³⁰J.-L. Vay, D. Bruhwiler, C. Geddes, W. Fawley, S. Martins, J. Cary, E. Cormier-Michel,
733 B. Cowan, R. Fonseca, M. Furman, et al., *Journal of Physics: Conference Series* **180**,
734 012006 (5 pp.) (2009), ISSN 1742-6596.
- 735 ³¹J.-L. Vay, *et al.*, arXiv:0909.5603 (2009).
- 736 ³²S. F. Martins, R. A. Fonseca, L. O. Silva, W. Lu, and W. B. Mori, *COMPUTER PHYSICS*
737 *COMMUNICATIONS* **181**, 869 (2010), ISSN 0010-4655.
- 738 ³³S. F. Martins, R. A. Fonseca, W. Lu, W. B. Mori, and L. O. Silva, *NATURE PHYSICS*
739 **6**, 311 (2010), ISSN 1745-2473.
- 740 ³⁴S. F. Martins, R. A. Fonseca, J. Vieira, L. O. Silva, W. Lu, and W. B. Mori, *PHYSICS*
741 *OF PLASMAS* **17**, 056705 (2010), ISSN 1070-664X.
- 742 ³⁵D. L. Bruhwiler (2008), private Communication.
- 743 ³⁶J. L. Vay, C. G. R. Geddes, E. Cormier-Michel, and D. P. Grote, *Journal of Computational*
744 *Physics* **230**, 5908 (2011).
- 745 ³⁷D. Grote, A. Friedman, J.-L. Vay, and I. Haber, in *AIP Conference Proceedings* (2005),
746 749, pp. 55–8, ISSN 0094-243X.
- 747 ³⁸J. Vay, C. G. R. Geddes, E. Cormier-Michel, and D. P. Grote, *Physics of Plasmas* **18**,
748 030701 (2011).
- 749 ³⁹C. B. Schroeder, C. Benedetti, E. Esarey, and W. P. Leemans, *Physical Review Letters*
750 **106**, 135002 (2011).
- 751 ⁴⁰R. A. Fonseca, *et al.*, *Lec. Notes in Comp. Sci.* **2329**, 342 (2002).
- 752 ⁴¹C. Nieter and J. R. Cary, *J. Comput. Phys.* **196**, 448 (2004).
- 753 ⁴²Taflove and Hagness, *Computational Electrodynamics: The Finite-Difference Time-*
754 *Domain Method* (MA: Artech House, Norwood, 2000), 2nd ed.
- 755 ⁴³C. Benedetti, *et al.*, *Nuclear Inst. and Methods in Physics Research A* **608**, 94 (2009).
- 756 ⁴⁴C. G. R. Geddes, *Plasma Channel Guided Laser Wakefield Accelerator* (Ph.D. dissertation,
757 University of California, Berkeley, Berkeley CA, 2005).
- 758 ⁴⁵S. Kalmykov, *et al.*, *Phys. Rev. Lett.* **103**, 135004 (2009).
- 759 ⁴⁶C. Birdsall and A. Langdon, *Plasma physics via computer simulation* (Adam-Hilger, 1991),
760 ISBN 0 07 005371 5.
- 761 ⁴⁷C. B. Schroeder, E. Esarey, B. A. Shadwick, and W. P. Leemans, *Physics of Plasmas* **13**,
762 033103 (2006).

763 ⁴⁸W. Rittershofer, C. B. Schroeder, E. Esarey, F. J. Gruner, and W. P. Leemans, Physics of
764 Plasmas **17**, 063104 (2010).

765 ⁴⁹E. Cormier-Michel, E. Esarey, C. G. R. Geddes, C. B. Schroeder, K. Paul, P. J. Mullaney,
766 J. R. Cary, and W. P. Leemans, Physical Review Special Topics-accelerators and Beams
767 **14**, 031303 (2011).

768 ⁵⁰M. Tzoufras, W. Lu, F. S. Tsung, C. Huang, W. B. Mori, T. Katsouleas, J. Vieira, R. A.
769 Fonseca, and L. O. Silva, Physical Review Letters **101**, 145002 (2008).

770 ⁵¹T. Esirkepov, COMPUTER PHYSICS COMMUNICATIONS **135**, 144 (2001), ISSN 0010-
771 4655.

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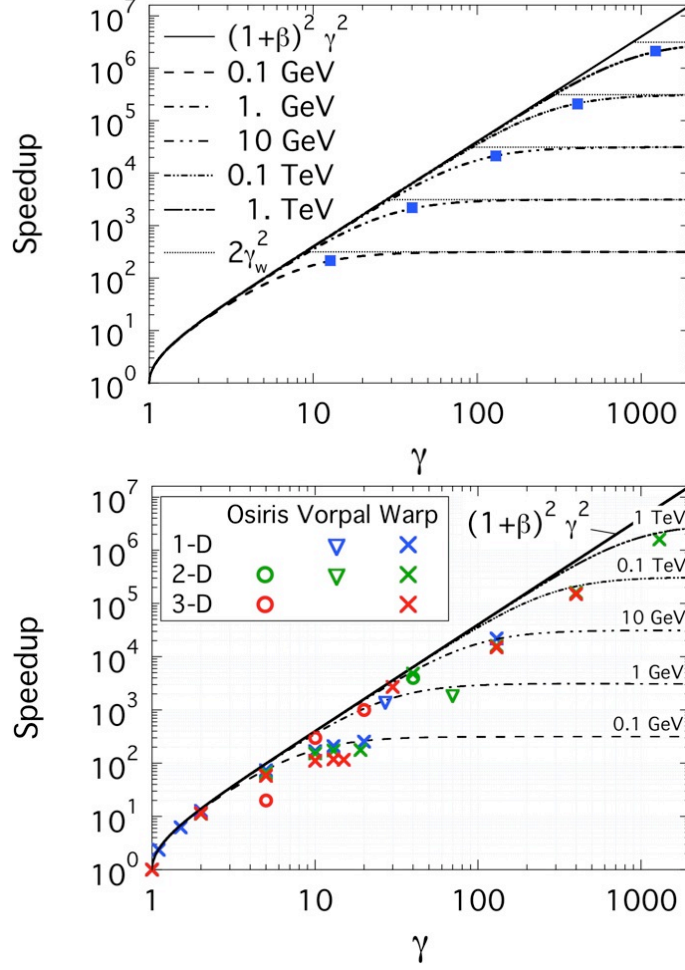


FIG. 1. (top) Theoretical speedup from Eq.(13) versus relativistic factor of the boosted frame for 0.1 GeV - 1 TeV LPA class stages (squares indicate speedup obtained using the frame of the wake $\gamma = \gamma_w$); (bottom) observed speedups from simulations using the code Osiris (circles), Vorpak (triangles) and Warp (crosses) and theoretical speedups (lines) for 0.1 GeV to 1 TeV stages. Vorpak reported speedups courtesy of D. L. Bruhwiler, Tech-X Corp., USA. Osiris reported speedups courtesy of S. F. Martins, IST, Portugal, and W. B. Mori, UCLA, USA.

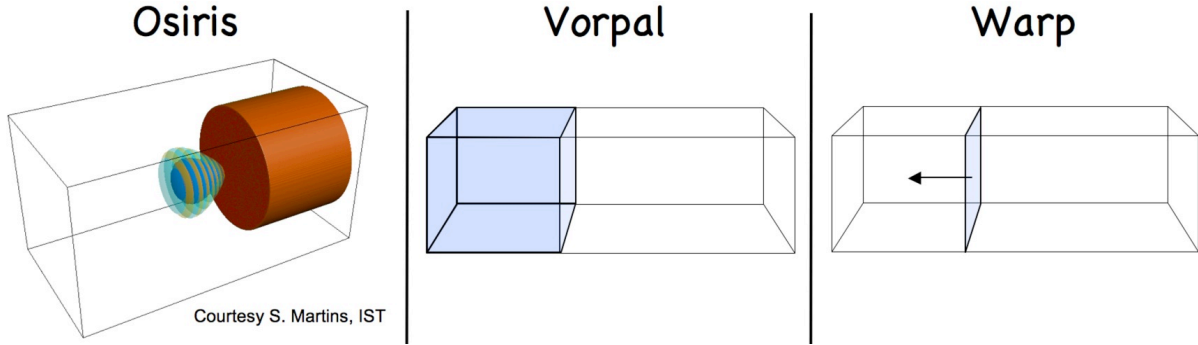


FIG. 2. (color online) Diagrams of laser emission procedures in the Particle-In-Cell codes Osiris (left), Vorpall (middle) and Warp (right) for Lorentz boosted frame simulations. Osiris initializes the entire laser at once. Vorpall emits the laser from all but one faces (blue) of the simulation box. Warp emits through a moving plane (blue). For all three diagrams, the laser propagates from left to right. Reprinted with permission from J.-L. Vay et al., *Proc. 14th Workshop Advanced Accelerator Concepts*. Copyright 2010 American Institute of Physics.

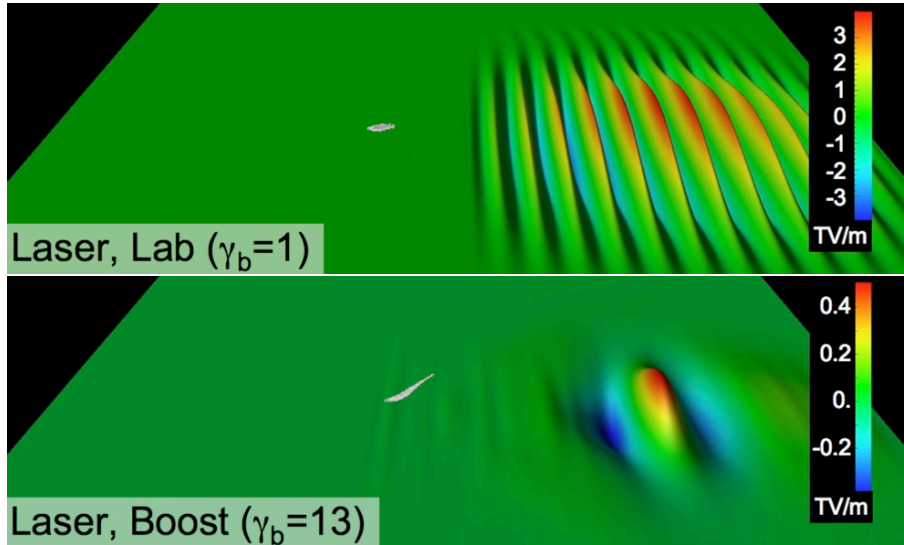


FIG. 3. (color online) Colored surface rendering of the transverse electric field from a 2-1/2D Warp simulation of a laser wakefield acceleration stage in the laboratory frame (top) and a boosted frame at $\gamma = 13$ (bottom), with the beam (white) in its early phase of acceleration. The laser and the beam are propagating from left to right.

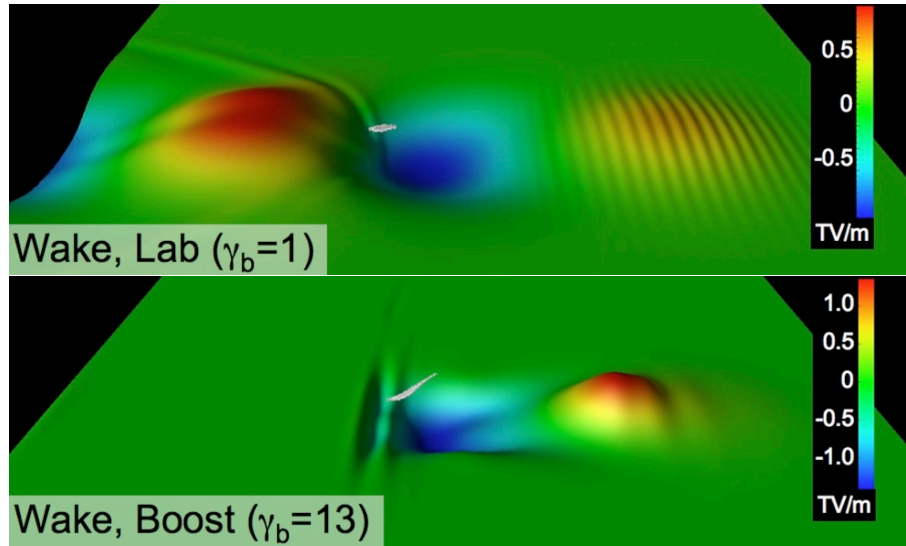


FIG. 4. (color online) Colored surface rendering of the longitudinal electric field from a 2-1/2D Warp simulation of a laser wakefield acceleration stage in the laboratory frame (top) and a boosted frame at $\gamma = 13$ (bottom), with the beam (white) in its early phase of acceleration. The laser and the beam are propagating from left to right.

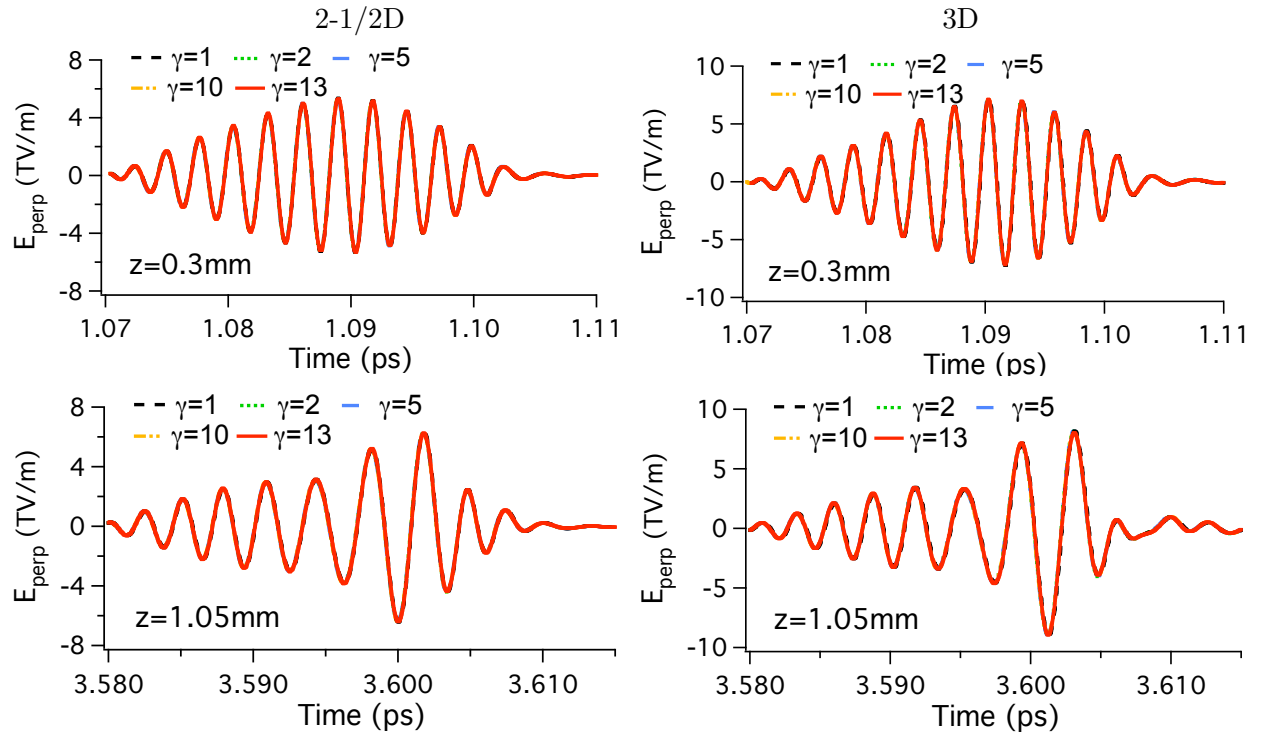


FIG. 5. (color online) History of transverse electric field at the position $x = y = 0$, $z = 0.3\text{ mm}$ and $z = 1.05\text{ mm}$ (in the laboratory frame) from simulations in the laboratory frame ($\gamma = 1$) and boosted frames at $\gamma = 2, 5, 10$ and 13 .

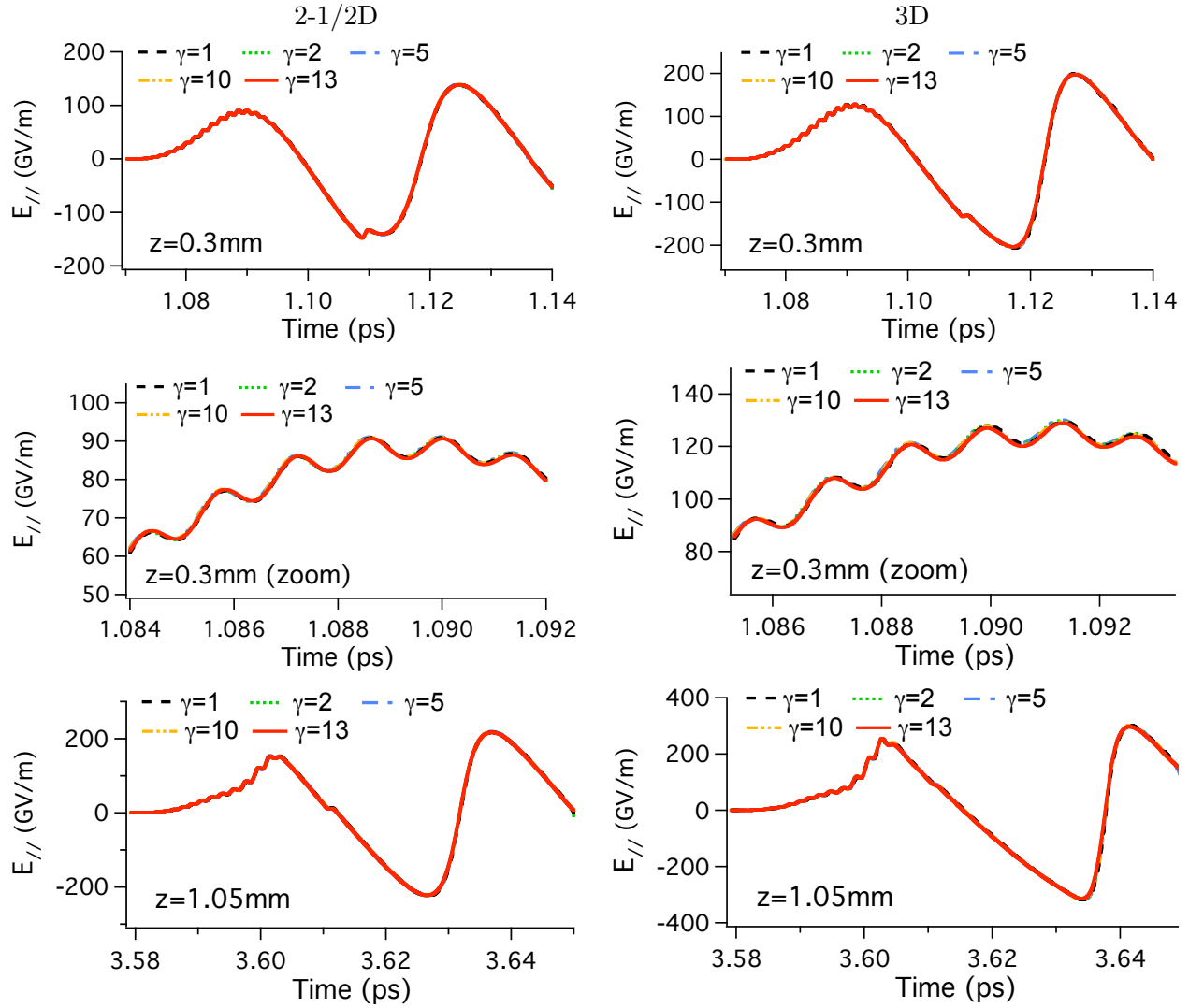


FIG. 6. (color online) History of longitudinal electric field at the position $x = y = 0$, $z = 0.3$ mm and $z = 1.05$ mm (in the laboratory frame) from simulations in the laboratory frame ($\gamma = 1$) and boosted frames at $\gamma = 2, 5, 10$ and 13 .

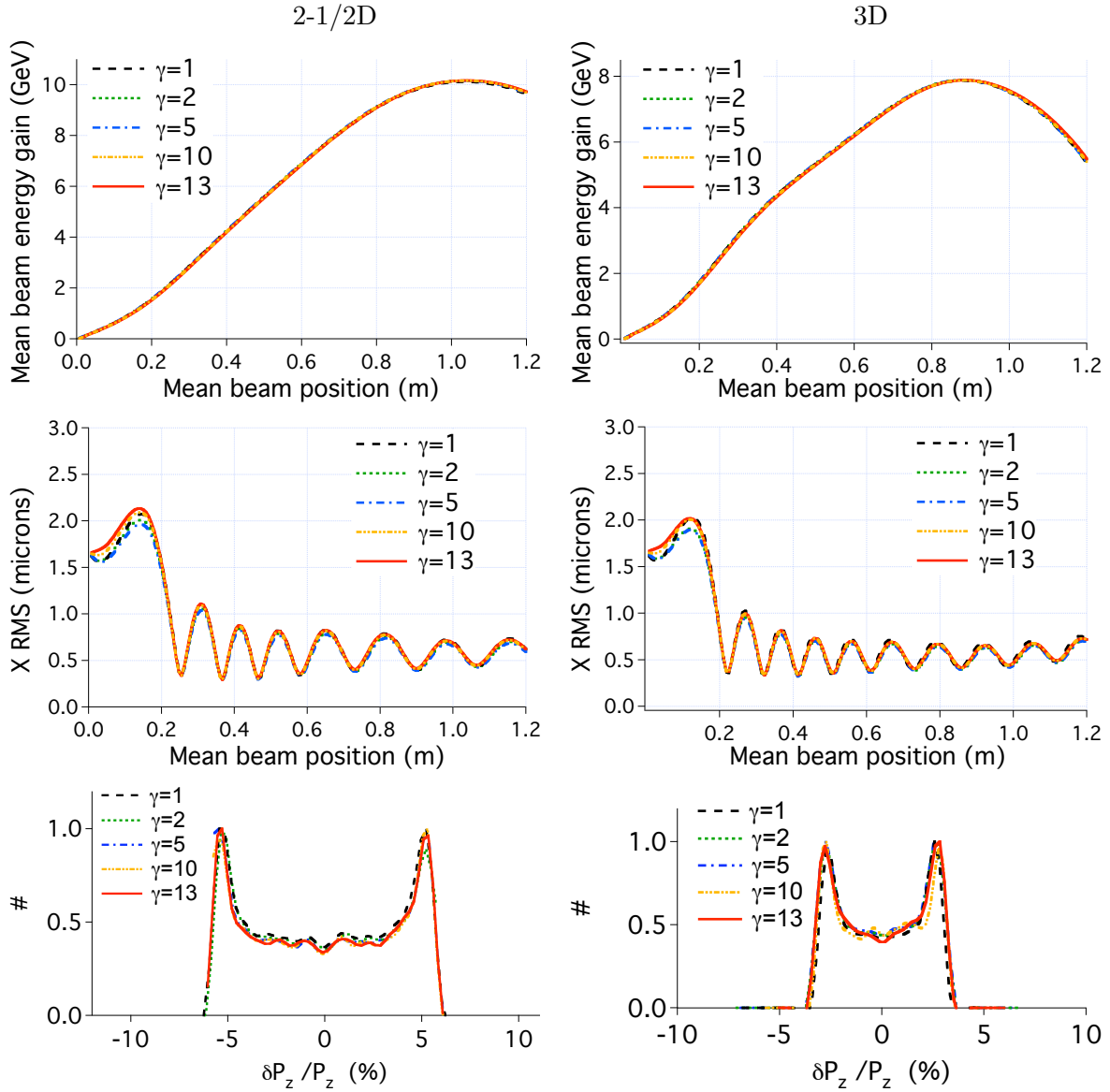


FIG. 7. (color online) Average scaled beam energy gain (top) and beam RMS transverse size (middle) versus longitudinal position in the laboratory frame from simulations; (bottom) distribution of relative longitudinal momentum dispersion at peak energy, in the laboratory frame ($\gamma = 1$) and boosted frames at $\gamma = 2, 5, 10$ and 13 .

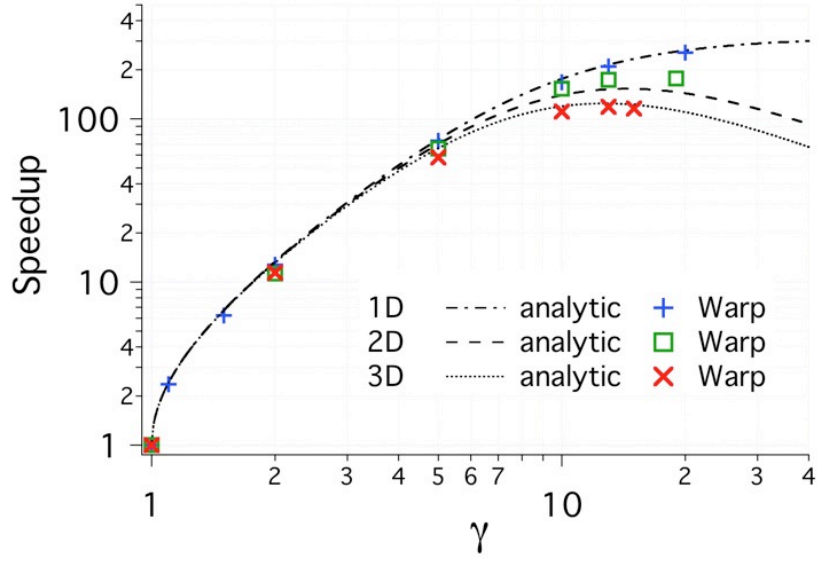


FIG. 8. (color online) Speedup versus relativistic factor of the boosted frame in 1D, 2D and 3D from theoretical estimates (Eq. (13), (17), (18)), and Warp simulations.

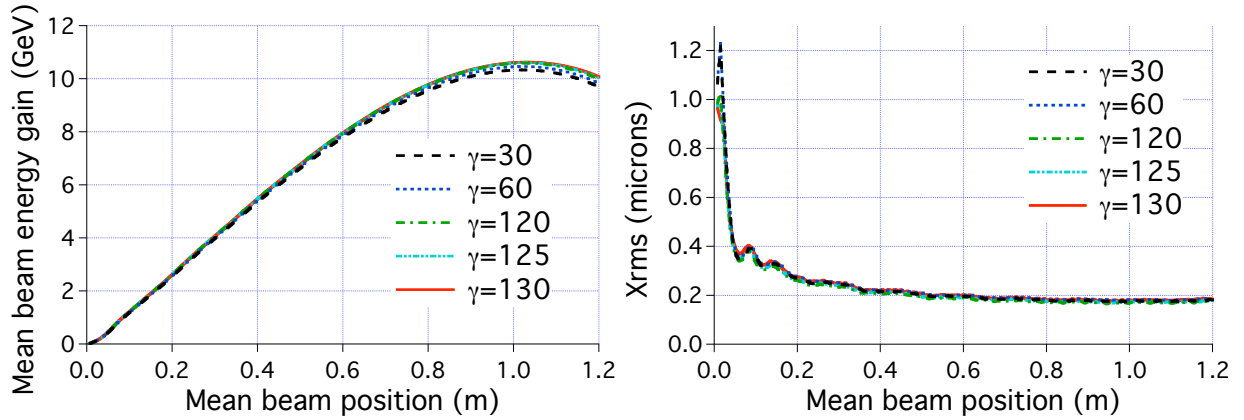


FIG. 9. (color online) Average beam energy gain and transverse size versus longitudinal position (in the laboratory frame) from 2D-1/2 simulations of a full scale 10 GeV LPA in a boosted frame at $\gamma = 30, 60$ and 130 , using the Yee solver.

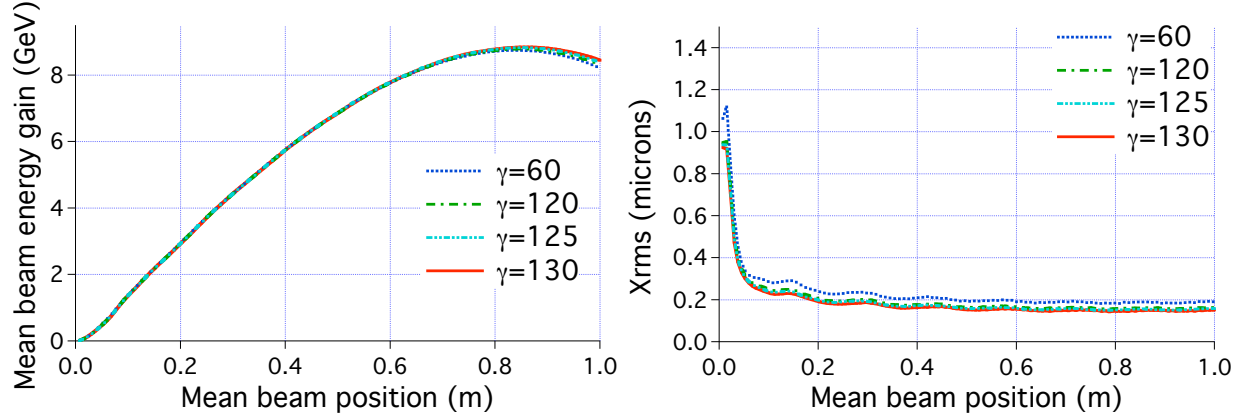


FIG. 10. (color online) Average beam energy gain and transverse size versus longitudinal position (in the laboratory frame) from 3D simulations of a full scale 10 GeV LPA in a boosted frame at $\gamma = 30, 60, 120, 125$ and 130 , using the Yee solver ($\gamma = 30$ and 60) and the CK2 solver ($\gamma = 120 - 130$), with digital filter S(1) and with the time step set by $c\delta t/\delta z = 1/\sqrt{2}$ for stability³⁶.

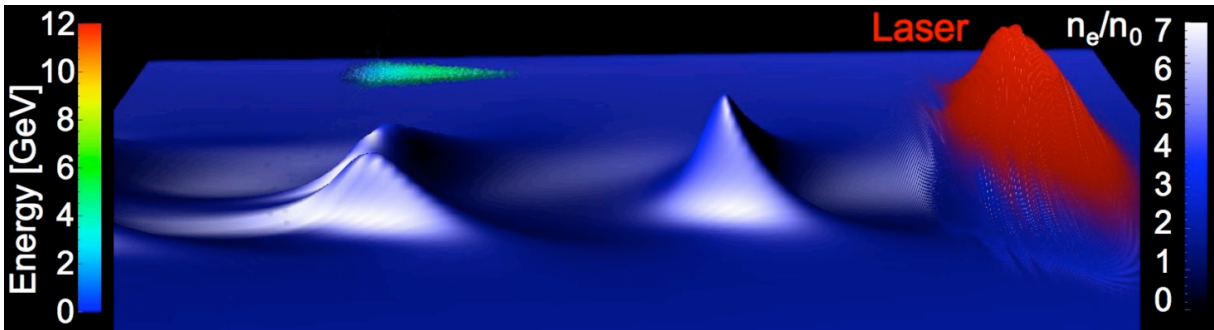


FIG. 11. (color online) Snapshot from a 2-1/2D 10 GeV LPA stage boosted frame simulation as the beam is halfway through acceleration. The image shows an externally injected electron bunch (middle) riding a density wake excited by an intense laser pulse (right), propagating in a 0.75 m long plasma channel.

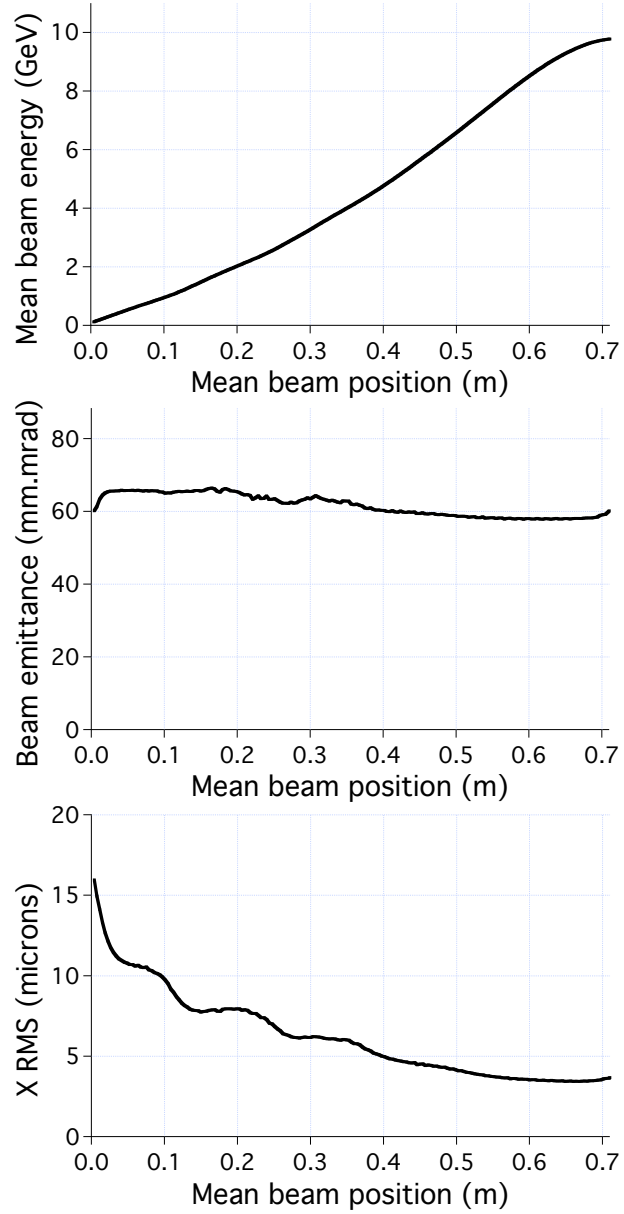


FIG. 12. (top) Average electron beam energy gain, (middle) beam emittance, and (bottom) beam RMS size, versus longitudinal position (in the laboratory frame) from a 2D 10 GeV LPA stage boosted frame simulation.

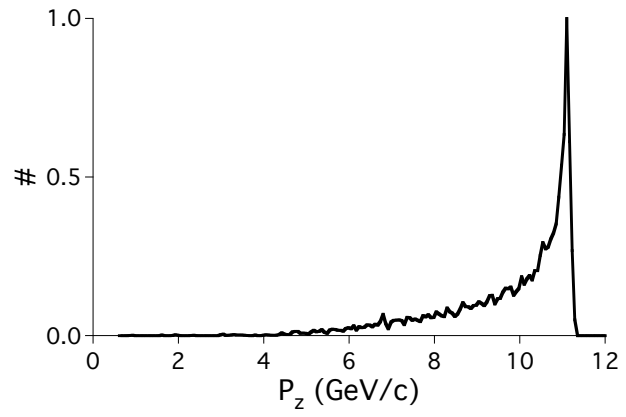


FIG. 13. Longitudinal momentum distribution of the electron beam at maximum energy ($z=0.7$ m) from a 2D 10 GeV LPA stage boosted frame simulation.

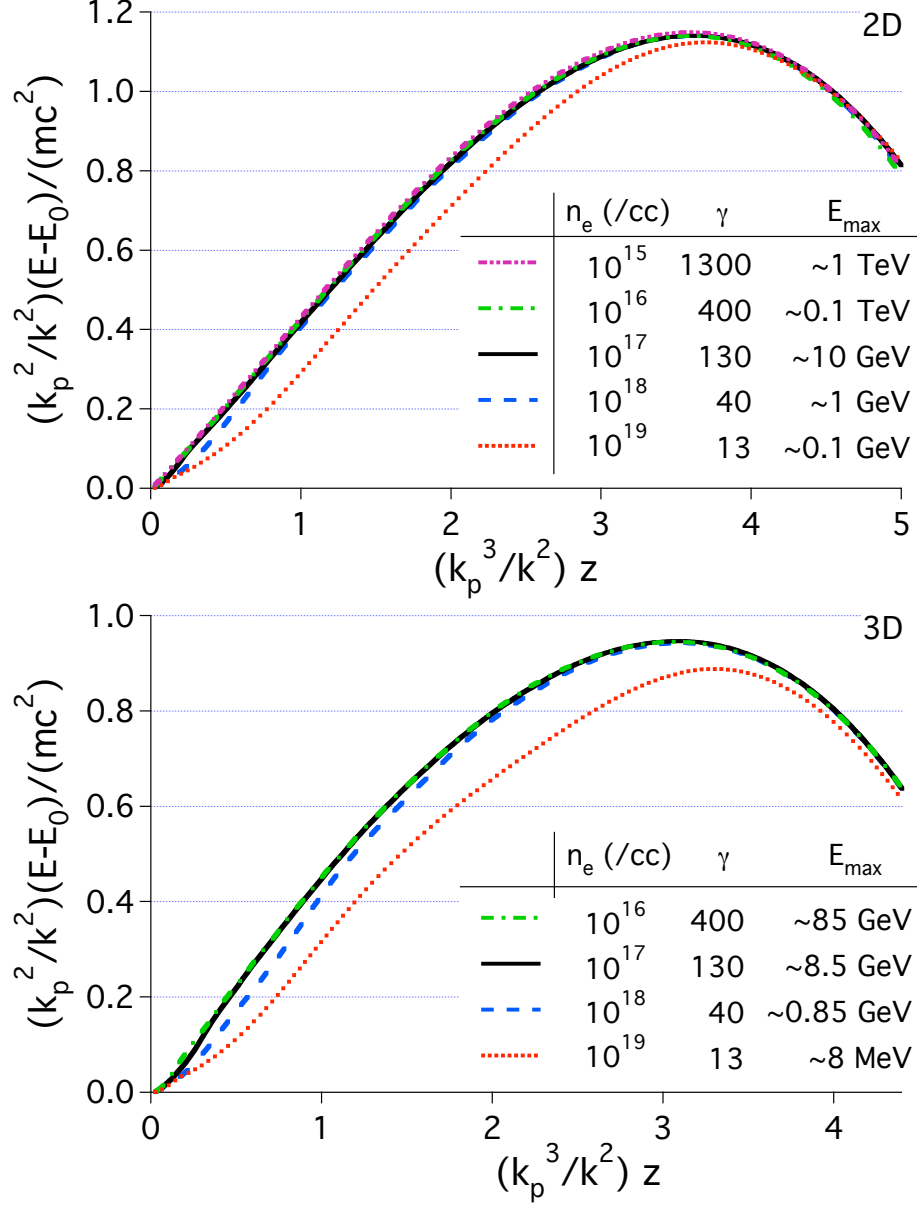


FIG. 14. (color online) Average beam energy gain versus longitudinal position (in the laboratory frame) for simulations at $n_e = 10^{19} \text{ cm}^{-3}$ down to 10^{15} cm^{-3} , using frames of reference between $\gamma = 13$ and $\gamma = 1300$, in 2-1/2D (top) and 3D (bottom).

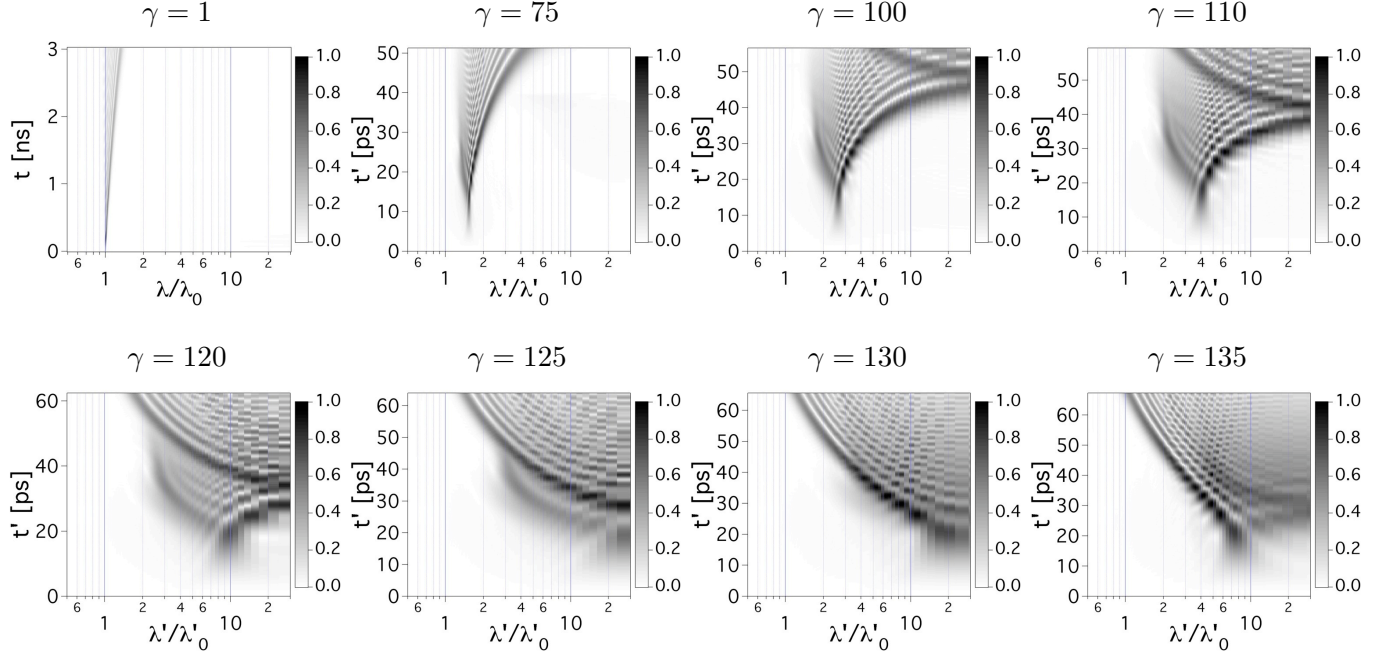


FIG. 15. Spectrum history of the laser field on axis of a 10 GeV stage for selected values of γ between 1 and 135, given up to the time of the electron beam peak energy. The length scale (horizontal axis) is normalized relative to the vacuum laser wavelength as given in each respective frame.

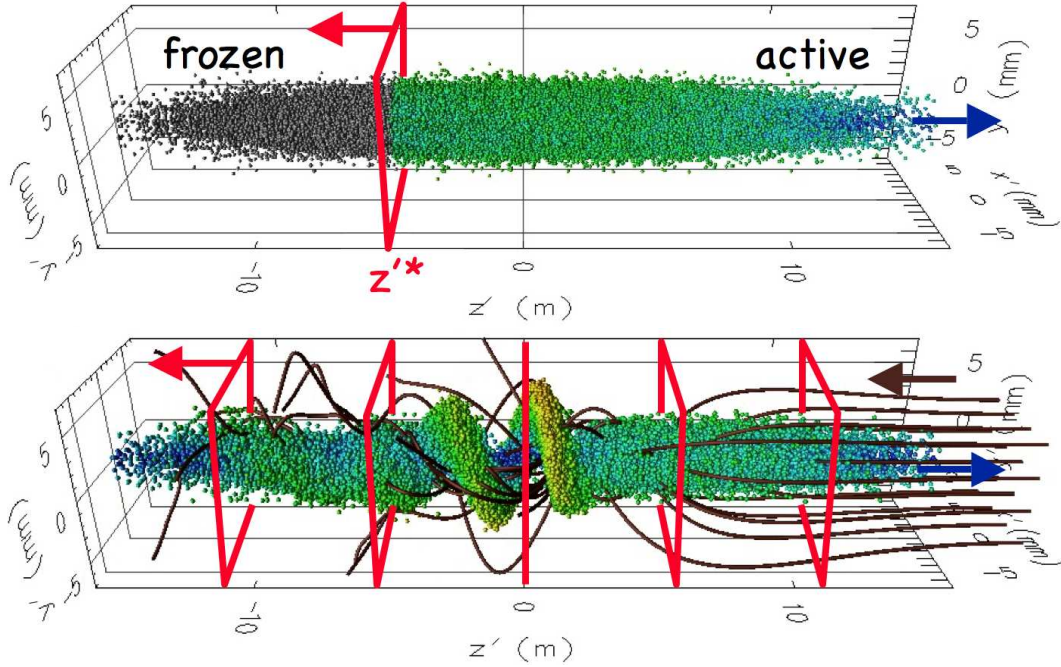


FIG. 16. (color online) (top) Snapshot of a particle beam showing “frozen” (grey spheres) and “active” (colored spheres) macroparticles traversing the injection plane (red rectangle). (bottom) Snapshot of the beam macroparticles (colored spheres) passing through the background of electrons (dark brown streamlines) and the diagnostic stations (red rectangles). The electrons, the injection plane and the diagnostic stations are fixed in the laboratory plane, and are thus counterpropagating to the beam in a boosted frame.