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Interplay between Floer homology and Hamiltonian dynamic

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INTERPLAY BETWEEN FLOER HOMOLOGY AND HAMILTONIAN DYNAMICS

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Abstract

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by

Mita Banik

In this thesis we primarily focus on the interplay between Floer homology and Hamiltonian dynamics. This has been an active area of research since the late 1980s with the introduction of pseudo-holomorphic curves by Gromov and Floer homology by Floer. The Floer Homology has been a very powerful tool to study dynamics on a symplectic or contact manifold and the subject is very broad.

Here primarily we concentrate on three aspects of the connections of Floer Homology and dynamics. Firstly, the connection between Hamiltonian dynamics and symplectic topology of the underlying manifold by studying special kind of Hamiltonians such as toric pseudo-rotations. We further study two Floer-theoretic invariants of symplectic and contact dynamics: "barcode entropy" and symplectic capacities. We use these invariants to understand various Hamiltonian dynamics behaviours such as pseudo-rotations (Hamiltonians with finitely many orbits) or other extremes (Hamiltonians with infinitely many orbits or very chaotic dynamics).

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Chapter 1

Introduction

In this thesis we primarily concentrate on the interplay between Floer homology and Hamiltonian dynamics. This has been an active area of research since the late 1980s with the introduction of pseudo-holomorphic curves by Gromov and Floer homology by Floer. The Floer Homology has been a very powerful tool to study dynamics on a symplectic or contact manifold and the subject is very broad. Here primarily we focus on three aspects of the connections of Floer Homology and dynamics.

In the first chapter we delve into the connection between a special type of pseudo-rotations called toric pseudo-rotations and the topology of a symplectic manifold. Pseudo-rotations, roughly speaking, are Hamiltonian diffeomorphisms with finite and minimal number of periodic orbits. Pseudo-rotations have a special place in dynamics and several properties of their dynamics have been studied by Floer theoretic methods. Notable examples include Bramham's results on C^0-rigidity of pseudo-rotations with Lioville rotation number, [Bra15] and Ginzburg-Gürel's results on C^0-rigidity of pseudo-rotations of $\mathbb{CP}^n$ with Lioville mean index vector, [GG18].

Recently Çineli-Ginzburg-Gürel [ÇGG20] and independently Skelukhin [She20] studied connections of dynamics to symplectic topology. Before their work the connection was mostly explored in one direction: symplectic topology to dynamics. One of the interesting problems studied in this direction is the Conley Conjecture. The Conley Conjecture roughly states that for many symplectic manifolds every Hamiltonian diffeomorphism has infinitely many periodic orbits. The conjecture has been proved in many cases and the state of the art result is that it holds for M unless there exists $A \in \pi_2(M)$ such that $\langle \omega, A \rangle > 0$ and $\langle c_1(TM), A \rangle > 0$, [GG15; GG19] that it holds for M unless there exists . In particular, the conjecture holds whenever M is symplectically aspherical or negative monotone or $\omega \mid_{\pi_2(M)} = 0$. Çineli-Ginzburg-Gürel, [ÇGG20], showed that a symplectic manifold admitting a Hamiltonian pseudo-rotation must have non-vanishing Gromov-Witten invariants and moreover its quantum product is deformed. Further they and Shelukhin, [She20], independently showed that if M admits a pseudo-rotation, then $N \leq 2n$, where N is the minimal Chern number.

In [Ban20], we studied the connection between toric pseudo-rotations and the quantum homology of the underlying manifold. By definition, a pseudo-rotation φ is toric if at one of its fixed points the eigen-values of $D\varphi$ satisfy no resonance relations, beyond the conditions that they come in complex conjugation pairs. For instance, pseudo-rotations obtained by the conjugation method from toric symplectic manifolds are toric. While the toric condition might appear generic, in fact the very existence of a toric pseudo-rotation φ imposes strong restrictions on the symplectic topology of the manifold M. (For example, when φ is a toric true rotation, essentially by definition M is toric). In [Ban20], we showed that a closed weakly-monotone symplectic manifold M^{2n}, which has minimal Chern number $N \leq n+1$ and admits a Hamiltonian toric pseudo-rotation, is necessarily monotone and its quantum homology is isomorphic to that of $\mathbb{CP}^n$.

In Chapter 1, we recall the definition of Hamiltonian Floer Homology - one of the essential tools to prove the above result which would also be an important tool for the later chapters. Then we recall the construction of the pair-of-pants product and the quantum product on Floer Homology and the quantum homology respectively and finally, the PSS isomorphism between them. Another key ingredient in proving the main result above and relating the dynamics of pseudo-rotations is the machinery of extremal partitions. This is a combinatorial tool that allows us to detect non-vanishing Gromov-Witten invariants by studying energy zero regular pseudo-holomorphic curves. In the final section of this chapter we prove the main result, [Ban20], from the previous paragraph. Furthermore using the results of Ohta-Ono, [OO96; OO97], we prove that if a closed symplectic manifold M has dimension 4 with minimal Chern number $N \geq n+1$ and admits a toric pseudo-rotation, then M is symplectomorphic to $\mathbb{CP}^n$.

In the second chapter we study another aspect of the Floer homology and its impact on dynamics. Using Hamiltonian Floer theory we define a numerical invariant, called “barcode entropy”, for compactly supported Hamiltonian diffeomorphisms of open symplectic manifolds convex at infinity. Furthermore we prove an inequality between barcode entropy and topological entropy, the latter coming from dynamics of the Hamiltonian diffeomorphism of the manifolds. Our main result in this chapter is closely related to the Çineli-Ginzburg-Gürel’s, see [ÇGG21], recent results on connecting topological entropy with barcode entropy on closed symplectic manifolds. However, here we mainly concentrate on open symplectic manifolds and developing the machinery for studying barcode entropy for these manifolds.

Topological entropy is studied widely in the context of dynamical systems. It measures the evolution of distinguishable orbits over time, thereby providing an idea of how complex the orbit structure of a system is. Dynamical systems with positive topological entropy are often considered chaotic, such as the Smale horseshoe map. The horseshoe map is a hallmark of chaos: it has infinitely many periodic orbits with arbitrarily long periods and the number of periodic orbits grows exponentially with the period. Dynamical systems with this kind of behaviour are rather ubiquitous in nature and topological entropy is an useful machinery to detect this phenomenon.

Barcode entropy can be thought of as a Floer theoretic counterpart of topological entropy which computes the rate of exponential growth under iterations of the number of bars (of length greater than $\epsilon > 0$) in the barcode of the Floer complex. The construction relies on persistent homology, which is a well-established tool in itself. It is primarily used in topological data analysis to understand the structure of high-dimensional data in real world applications, see [Car09]. A close analog of persistent homology in understanding topology of a manifold is Morse theory, where we detect changes of homology as we vary the level sets, giving us a lower bound on the number of critical points of a function on a manifold.

The crux of this chapter lies in defining the barcode entropy for a compactly-supported Hamiltonian on an open symplectic manifold convex at infinity. Our definition is motivated and very similar to that [ÇGG21] in the closed case where the authors connect barcode entropy to topological entropy. Defining Floer Homology on open symplectic manifold encounters many problems, the foremost being that the compactness for the Gromov's pseudo-holomorphic curves is not satisfied. We therefore consider open symplectic manifolds convex at infinity which gives us enough control on the periodic orbits beyond a compact set and the compactness condition is satisfied. Another important part of the barcode entropy definition is that the barcode entropy is also well-defined for degenerate Hamiltonians. The numbers of bars in a barcode is finite only when the Hamiltonian is non-degenerate. In the degenerate case, the barcode entropy is well-defined since we work here with bars of length greater than $\epsilon > 0$ (and this is always finite).

Summarizing, in the second chapter we briefly recall the setup of Floer homology for open symplectic manifolds. Using the natural action filtration we define the filtration on the Floer complex. Then we count the number of bars (of length greater than $\epsilon > 0$), denoted by $b_\epsilon(\varphi^k)$, of the k-th iteration of the Floer complex where φ is a compactly suported Hamiltonian diffeomorphism. The barcode entropy is defined as

$$\hbar(\varphi) := \lim_{\epsilon \searrow 0} \hbar_\epsilon(\varphi) \in [0, \infty],$$

where

$$\hbar_\epsilon(\varphi) := \limsup_{k \to \infty} \frac{\log^+ b_\epsilon\left(\varphi^k\right)}{k}.$$

To connect the barcode entropy with the topological entropy we construct tomographs, following [ÇGG20], and use the Crofton's inequality to prove the final result:

$$\hbar(\varphi) \leq h_{top}(\varphi),$$

where $h_{top}(\varphi)$ is the topological entropy of the Hamiltonian diffeomorphism φ.

In the third chapter we explore the connection of Floer theory with dynamics of Reeb flows on hypersurfaces on $\mathbb{R}^{2n}$. Another computational tool we use here to study dynamics is symplectic capacities. These invariants have been used to prove C^0-rigidity of flows, multiplicity results (the minimal number of periodic orbits of Reeb flows on a hypersurface) and finding obstructions to embedding in symplectic geometry, see e.g., [HZ90; EH90; GH18; Cri19] for various types of symplectic capacities and their applications. In this chapter we focus on how the existence of pseudo-rotations on hypersurfaces on $\mathbb{R}^{2n}$ impact these capacities. Here by pseudo-rotations we refer to Reeb flows with finite and the hypothetically minimal number of periodic orbits.

There are various types of symplectic capacities, e.g., the Gromov width, the Hofer-Zehnder capacity, the ECH capacity in four dimensions, the Ekeland-Hofer capacity, the Gutt-Hutchings capacity. Symplectic capacities have been useful not only to answer important questions in dynamics, see [HZ90; CH16; CHP19], but also in addressing many embedding questions such as symplectically embedding a ball into an ellipsoid or an ellipsoid into a polydisk, etc. Here we primarily focus on the Gutt-Hutchings capacities, [GH18]. These capacities are motivated by the ECH capacities in four dimensions and they were defined in order to generalise the ECH capacities to higher dimensions. These capacities are conjecturally equal to the historical Ekeland-Hofer capacities, [EH90], and they agree with them in simple cases such as the ellipsoids, polydisks etc. The Ekeland-Hofer capacities are very interesting in their own rights. They provide a sequence of numbers associated to a convex hypersurface and sometimes even provide sharper obstructions to embeddings than the ECH capcities in four dimensions, e.g., to embedding of a four-dimensional polydisk into an ellipsoid. However, the Ekeland-Hofer capacities have been incredibly difficult to compute beyond the simple cases. The Gutt-Hutchings capacities however have been calculated for a wider class such as concave and convex toric domains in $\mathbb{R}^{2n}$.

Another sequence of invariants which appear to agree with the Gutt-Hutchings capacities are the spectral invariants defined by Ginzburg-Gürel, [GG20], using equivariant Floer theory. These invariants were used to prove multiplicity results for convex hypersurfaces on $\mathbb{R}^{2n}$. For example, they showed that for a closed contact type, dynamically convex hypersurface in $\mathbb{R}^{2n}$ there are least r simple closed characteristics $x_1, \dots, x_r$ where $r = \lceil n/2 \rceil + 1$ in general and $r = n$ in the non-degenerate case, see also [Lon02; LZ02]. Using their index-recurrence relations (this is a huge field of study by itself, see e.g. [Lon02; LZ02; DLW16]) we prove in this chapter that in the non-degenerate case, when the contact form α is dynamically convex and has exactly n Reeb orbits, then

$$c_{k_i+n-1} - c_{k_i} \to 0,$$

for some sequence $k_i \to \infty$, where c_k's are spectral invariants. Computationally, it was known that the above behavior failed only for polydisks, using the Ekeland-Hofer capacities. The primary reason is that the these capacities/spectral invariants previously couldn't be calculated for any other examples. However, using such invariants, various characterizations of the Besse property of contact manifolds have been obtained. A closed connected contact manifold is called Besse when all its Reeb orbits are closed, and in such a case the Reeb orbits admit a common period by a theorem of Wadsley, [Wad75]. For a convex hypersurface, it roughly says that n-consecutive spectral numbers are same (i.e, $c_i(\Sigma) = c_{n-i+1}(\Sigma)$) for some $i \in \mathbb{N}$ if and only if Σ is Besse and $c_i(\Sigma)$ is a common period of all its Reeb orbits; see [CM20; GGM21].

In this chapter we show computationally that for some hypersurfaces such as concave and convex toric domains when they are not ellipsoids the above behavior fails. Just to emphasize we only computed some examples of concave and convex toric domains and presented them in this chapter and the list of examples is in no way expansive of all the convex/concave toric domains. We use the

Gutt-Hutchings capacities on convex/concave toric domains to demonstrate this behavior. We can do this mainly because the Gutt-Hutchings capacities for such domains are combinatorial in nature and therefore somewhat computationally tractable. Such computations were not previously possible using similar numbers such as the Ginzburg-Gürel's spectral invariants or the Ekeland-Hofer capacities. The behaviour is rather surprising since the dynamics on the convex and concave toric domains is rather simple and in fact the underlying Reeb flows are integrable systems.

Chapter 2

Dynamics characterization of complex projective spaces

In this chapter we study the connection between the dynamics of toric pseudo-rotations and the quantum homology of complex projective spaces. We recall from the introduction that pseudo-rotations are Hamiltonian diffeomorphisms with finite and minimal possible number of periodic orbits. The connections between their dynamics and the topology of the underlying symplectic manifold have been recently studied in [GG18; ÇGG20; She20], to name a few.

By definition, a pseudo-rotation φ is toric if at one of its fixed points the eigenvalue of $D\varphi$ satisfy no resonance relations beyond the conditions that they come in complex conjugation pairs. To be more precise, the requirement is that the semi-simple part of $D\varphi$ topologically generates an n-dimensional torus in $Sp(2n)$. For instance, pseudo-rotations obtained by the conjugation method from toric symplectic manifolds are toric, see [AK70; LS22].

Let us outline the sections of this chapter briefly. In the first section we lay the foundations for Floer theoretic tools, recalling Novikov rings and quantum homology, the pair-of-pants product for Floer homology. Then we discuss pseudo-rotations and toric rotations with the known examples from the literature. In the following section we introduce our main combinatorial tool: extremal partitions, [ÇGG20], and recall their applications in finding regular energy-zero curves. In the final section we prove our main theorem 2.2.9 which says that a closed weakly-monotone symplectic manifold M^{2n}, which has minimal Chern number $N \geq n+1$ and admits a Hamiltonian toric pseudo-rotation is necessarily monotone and its quantum homology is isomorphic to the quantum homology of $\mathbb{C}P^n$, [Ban20].

2.1 Floer homology

In this section firstly we discuss the common conventions and notations in this chapter and then we further discuss the Floer theoretic setup necessary for our purposes, closely following to that in [ÇGG20] and [Ban20].

Notations and Convention

In this chapter, we will assume that (M^{2n}, ω) is a closed symplectic manifold which is weakly monotone in the sense of [HS95]. The minimal Chern number, i.e., the positive generator of the group $\langle c_1(TM), \pi_2(M)\rangle \subset \mathbb{Z}$, is denoted by N. When this group is trivial, $N = \infty$.

A *Hamiltonian diffeomorphism* is the time-one map $\varphi = \varphi_H$ of the time-dependent flow φ_H^t of a 1-periodic in time Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$, where $S^1 = \mathbb{R}/\mathbb{Z}$. The Hamiltonian vector field X_H of H is defined by $i_{X_H}\omega = -dH$. For any two time-one maps φ_H and φ_K, their composition $\varphi_H \circ \varphi_K$ is the time-one map for the Hamiltonian $H\#K$. In fact, the time-one maps form the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms of M.

In this chapter we will only focus on contractible periodic orbits for a Hamiltonian diffeomorphism. Let $x\colon S^1 \to M$ be a contractible loop. A *capping* of x is an equivalence class of maps $A\colon D^2 \to M$ such that $A\mid_{S^1} = x$. Two cappings A and A' of x are equivalent if the integrals of ω and $c_1(TM)$ over the sphere obtained by attaching A to A' are equal to zero. We will equip a closed curve x by an equivalence class of cappings and the corresponding capped closed curve will be denoted $\bar{x}$ throughout this chapter.

The action of a Hamiltonian H on a capped closed curve $\bar{x} = (x, A)$ is

$$\mathcal{A}_H(\bar{x}) = -\int_A \omega + \int_{S^1} H_t(x(t))\, dt.$$

The critical points of action functional $\mathcal{A}_H$ on the space of closed curves with capping are exactly the capped one-periodic orbits of X_H.

An one-periodic orbit x of H is said to be *non-degenerate* if the linearized return map $d\varphi_H\colon T_{x(0)}M \to T_{x(0)}M$ has no eigenvalues equal to one. A Hamiltonian H is non-degenerate if all its one-periodic orbits are non-degenerate. For any non-degenerate capped period orbit $\bar{x}$, we can associate an integer called the *Conley–Zehnder index*. The Conley–Zehnder index $\mu(\bar{x}) \in \mathbb{Z}$ is defined, up to a sign, the rigorous definition can be found in [Sal99; SZ92] . Throughout this chapter, aligning to common conventions, we would normalize μ so that $\mu(\bar{x}) = n$ when x is a non-degenerate maximum (with trivial capping) of an autonomous Hamiltonian with small Hessian. For an uncapped non-degenerate periodic orbit x, the Conley–Zehnder index $\mu(x)$ is well defined as an element of $\mathbb{Z}/2N\mathbb{Z}$.

For a capped periodic orbit $\bar{x}$, even if it is degenerate we can associate a real number so called *mean index* $\hat{\mu}(\bar{x}) \in \mathbb{R}$; see [Lon02; SZ92] for a rigorous definition and properties of the mean index. Roughly, it measures the rotation number of certain (Krein-positive) eigenvalues along the flow of the capped periodic orbit $\bar{x}$. The mean index $\hat{\mu}(\bar{x})$ depends continuously on the Hamiltonian H and the periodic orbit $\bar{x}$. Furthermore for a non-degenerate periodic orbit $\bar{x}$, we have

$$\left|\hat{\mu}(\bar{x}) - \mu(\bar{x})\right| \leq n.$$

Similar to the Conley-Zehnder index, the mean index $\hat{\mu}(x)$ is also well defined as an element of $S^1_{2N} := \mathbb{R}/2N\mathbb{Z}$, however non-degeneracy is not a prior requirement here.

We can also define k-periodic orbits of a Hamiltonian diffeomorphism φ_H. These points are in one-to-one correspondence with the k-periodic *orbits* of H, i.e., of the time-dependent flow φ^t_H. A k-periodic orbit of H is called *simple* or *prime* if it is not iterated. For a capped orbit $\bar{x}$, we denote its k-th iteration as $\bar{x}^k$. The capping of $\bar{x}^k$ is obtained from the capping of $\bar{x}$ by taking its k-fold cover branched at the origin. The action functional is homogeneous with respect to iteration: $\mathcal{A}_{H^{\natural k}}\left(\bar{x}^k\right) = k\mathcal{A}_H(\bar{x})$.

A k-periodic orbit x of H is said to be *non-degenerate* if the linearized return map $d\varphi^k_H \colon T_{x(0)}M \to T_{x(0)}M$ has no eigenvalues equal to one. We call x *strongly non-degenerate* if all iterates x^k are non-degenerate. A Hamiltonian H is strongly non-degenerate if all periodic orbits of H (of all periods) are non-degenerate. Similar to the action functional, the mean index is homogeneous with respect to iteration: $\hat{\mu}\left(\bar{x}^k\right) = k\hat{\mu}(\bar{x})$.

Floer Theoretic Setup

Now, we are finally in a position to establish the setup of the Floer Homology. Let $\varphi = \varphi_H$ be a non-degenerate Hamiltonian diffeomorphism, viewed as an element of the universal cover $\widetilde{Ham}(M)$. The Floer complex and homology of φ will be denoted by $\mathrm{CF}_*(\varphi)$ and $\mathrm{HF}_*(\varphi)$; see, e.g., [HS95; MS12; Sal99], and we will also fix our a ground ring $\mathbb{F}$ to be $\mathbb{Z}_2$ in this chapter. The Floer complex $\mathrm{CF}_*(\varphi)$ is generated by the capped one-periodic orbits $\bar{x}$ of H. It is a filtered complex induced by the natural action filtration of H and graded by the Conley–Zehnder index.

The Floer trajectories are defined as follows. Let $J = J_t$ be a time-dependent almost complex structure on M. A Floer anti-gradient trajectory u is a map $u\colon \mathbb{R} \times S^1 \to M$ satisfying the equation

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} = -\nabla H_t(u).$$

Here the gradient is taken with respect to the time-dependent Riemannian metric $\omega(\cdot, J_t\cdot)$. Denote by $u(s)$ the curve $u(s, \cdot)$.

The energy of u is defined as

$$E(u) = \int_{-\infty}^{\infty} \left\| \frac{\partial u}{\partial s} \right\|_{L^2(S^1)}^2 ds = \int_{-\infty}^{\infty} \int_{S^1} \left\| \frac{\partial u}{\partial t} - J\nabla H(u) \right\|^2 dt\, ds.$$

We say that u is asymptotic to $x^{\pm} \in P_1(H)$ as $s \to \pm\infty$, or connecting x^- and x^+, if $\lim_{s\to\pm\infty} u(s) = x^{\pm}$. In this case

$$A_H(x^-) - A_H(x^+) = E(u).$$

We denote the space of Floer trajectories connecting x^- and x^+, with the topology of uniform C^∞-convergence on compact sets, by $\mathcal{M}_H(x^-, x^+, J)$. This space carries a natural $\mathbb{R}$-action $(\tau \cdot u)(t, s) = u(t, s + \tau)$ and we denote by $\hat{\mathcal{M}}_H(x^-, x^+, J)$ the quotient $\mathcal{M}_H(x^-, x^+, J)/\mathbb{R}$.

2.1.1 Novikov rings and quantum homology

In this chapter we will use the following formulation for Novikov rings (here we take only contractible orbits into account). In chapter 3 we slightly modify the definition of Novikov rings when we also work in the general case for non-contractible orbits.

Let $H_*(M) := H_*(M, \mathbb{Z}_2)$ and $H_2^S(M, \mathbb{Z})$ be the group of integral spherical homology classes, i.e. the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M, \mathbb{Z})$. Set

$$\bar{\pi}_2(M) = H_2^S(M, \mathbb{Z}) / \sim,$$

where by definition $A \sim B$ iff $\omega(A) = \omega(B)$ and $c_1(A) = c_1(B)$. Here $\omega(-)$ and $c_1(-)$ are the integrals of ω and $c_1(TM)$ over the spherical homology classes.

Denote $\Gamma = [\omega](H_2^S(M, \mathbb{Z})) \subset \mathbb{R}$ the subgroup of periods of the symplectic form on M on spherical homology classes. Let s and q be formal variables. Define the field K_Γ whose elements are generalized Laurent series in s of the following form:

$$K_\Gamma = \{ \sum_{\theta \in \Gamma} z_\theta s^\theta,\ z_\theta \in \mathbb{Z}_2,\ \#\{\theta > c \mid z_\theta \neq 0\} < \infty,\ \forall c \in \mathbb{R}\,\}. \tag{2.1.1}$$

Define a graded ring $\Lambda_\Gamma := K_\Gamma[q, q^{-1}]$ by setting the degree of s to be zero and the degree of q to be $2N$. (Note that the latter convention differs from that in [EP08] where the deg of q is 2.)

The (small)*quantum homology* of M is denoted by $HQ_*(M)$. This is a graded algebra over the Novikov ring Λ_Γ ([EP08]) and as a Λ_Γ-module $HQ_*(M) = H_*(M) \otimes_{\mathbb{Z}_2} \Lambda_\Gamma$. The grading on $HQ_*(M)$ is given by the gradings on $H_*(M)$ and Λ_Γ :

$$\deg(a \otimes z_\theta s^\theta q^m) = \deg(a) + 2Nm.$$

The algebra $HQ_*(M)$ is equipped with quantum product: given $a \in H_k(M)$ and $b \in H_l(M)$, their quantum product is a class $a * b \in HQ_{k+l-2n}(M)$ such that

$$a * b = \sum_{A \in \bar{\pi}_2(M)} (a * b)_A \otimes s^{-\omega(A)} q^{-c_1(A)/N}$$

where $(a * b)_A \in H_{k+l-2n+2c_1(A)}(M)$ is defined by

$$(a * b)_A \circ c = GW_A^{\mathbb{Z}_2}(a, b, c) \ , \forall c \in H_*(M).$$

Here $\circ$ is the intersection product and $GW_A^{\mathbb{Z}_2}(a, b, c)$ denotes the Gromov-Witten invariant.

The Floer complex and the Floer homology are denoted by $CF_*(\varphi, \Lambda_\Gamma)$ and $HF_*(\varphi, \Lambda_\Gamma)$ respectively. Let $\tilde{P}(H)$ be the free $\mathbb{Z}_2$-module generated by the set of capped one-periodic orbits $\tilde{x}$ of H. Consider the free Λ_Γ-module $\tilde{P}(H) \otimes_{\mathbb{Z}_2} \Lambda_\Gamma$ and let R be a Λ_Γ-sub module of $\tilde{P}(H) \otimes_{\mathbb{Z}_2} \Lambda_\Gamma$ generated by $A\#\tilde{x} \otimes 1 - \tilde{x} \otimes s^{\omega(A)} q^{c_1(A)/N}$, $A \in \bar{\pi}_2(M)$.

The grading on Λ_Γ and the grading μ on $\tilde{P}(H)$ given by the Conley-Zehnder index give rise to the grading

$$\deg(\tilde{x} \otimes z_\theta s^\theta q^m) = \mu(\tilde{x}) + 2Nm.$$

Then $\deg(A\#\tilde{x} \otimes 1) = \deg(\tilde{x} \otimes s^{\omega(A)} q^{c_1(A)/N}) = \mu(A\#\tilde{x})$. Hence we get the graded Λ_Γ-module $CF_*(\varphi, \Lambda_\Gamma) := \tilde{P}(H) \otimes_{\mathbb{Z}_2} \Lambda_\Gamma / R$ and the Floer homology $HF_*(\varphi, \Lambda_\Gamma)$ is defined as usual. We will denote the Novikov ring by Λ and $HF_*(\varphi, \Lambda_\Gamma)$ by $HF_*(\varphi)$, suppressing the Novikov ring into the notation.

We have the canonical isomorphism between Floer Homology and quantum

Homology

$$\mathrm{HF}_*(\varphi) \cong \mathrm{HQ}_*(M)[-n], \tag{2.1.2}$$

[Sal99; MS12] and references therein.

2.1.2 Pair-of-pants product

For a pair of Hamiltonian diffeomorphisms φ and ψ we have the *pair-of-pants product*

$$\mathrm{HF}_*(\varphi) \otimes \mathrm{HF}_*(\psi) \to \mathrm{HF}_*(\varphi\psi). \tag{2.1.3}$$

This product, which we denote by $*$, has degree $-n$, i.e., $|\alpha * \beta| = |\alpha| + |\beta| - n$. We refer the reader to the standard literature in [AS10; MS12; PSS96] for details on the pair-of-pants product. On the level of Floer complexes, when $\varphi = \varphi_1 \dots \varphi_r$, the product

$$\mathrm{CF}_*(\varphi_1) \otimes \dots \otimes \mathrm{CF}_*(\varphi_r) \to \mathrm{CF}_*(\varphi)$$

"counts" the number of solutions $u\colon \Sigma \to M$ of a Floer equation (suitably defined and when certain regularity conditions are satisfied), where the domain Σ is the $(r+1)$-punctured sphere; see e.g., [AS10; MS12].

Under the identification 2.1.2, the pair-of-pants product turns into the quantum product on $\mathrm{HQ}_*(M)$ (recall from previous section and the PSS isomorphism [PSS96]), which will also be denoted by $*$. The pair-of-pants product 2.1.3 can be generalized by setting $\varphi^k = \varphi^{k_1} \dots \varphi^{k_r}$ where $k_1 + \dots + k_r = k$,

$$\mathrm{HF}_*\left(\varphi^{k_1}\right) \otimes \dots \otimes \mathrm{HF}_*\left(\varphi^{k_r}\right) \to \mathrm{HF}_*\left(\varphi^{k}\right),$$

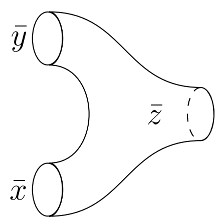
and this product can be identified with the quantum product on $\mathrm{HQ}_*(M)$. Then we have the degree formula,

$$|\alpha_1| + \ldots + |\alpha_r| - |\alpha_1 * \ldots * \alpha_r| = (r-1)n.$$

For Hamiltonians H and K and their composition defined by $H\#K$, and corresponding capped orbits $\bar{x}$, $\bar{y}$ and $\bar{z}$ then,

$$\mathcal{A}_H(\bar{x}) + \mathcal{A}_K(\bar{y}) - \mathcal{A}_{H\#K}(\bar{z}) = E(u) \text{ where } E(u) := \int_\Sigma ||\partial_s u||^2 dsdt.$$

where $u : \Sigma \to M$ is the corresponding pair-of-pants curve (see picture below).



Here the domain Σ of a pair-of-pants curve u is treated as a double cover of the cylinder, branching at one point. The domain Σ naturally carries the "coordinates" (s,t) lifted from the cylinder, which are true coordinates on the three open half-cylindrical parts of the domain (as represented in the figure above), and on each of these parts the pair-of-pants curve u satisfies the Floer equation for the corresponding Hamiltonian H or K or $H\#K$.

2.2 Pseudo-rotations

Definition 2.2.1 (Pseudo-rotations)**.** *A Hamiltonian diffeomorphism $\varphi\colon M \to M$ is called a* pseudo-rotation *(over $\mathbb{F}$) if φ is strongly non-degenerate, and the differential in the Floer complex of φ^k over $\mathbb{F}$ vanishes for all $k \in \mathbb{N}$.*

The differential in the Floer complex depends on the almost complex structure, but it is easy to see that its vanishing is a well-defined condition. We can else observe that for a pseudo-rotation all periodic orbits are automatically one-periodic and that an iterate of a pseudo-rotation is again a pseudo-rotation.

Remark 2.2.2. *The definition here 2.2.1 is slightly different from the one in [GG18] although it captures the same phenomenon we discussed above. The reader can refer to [GG18] for detailed discussions of various definitions of a pseudo-rotation.*

Now, we present two well-known examples of pseudo-rotations from the literature. In fact, all known examples of pseudo-rotations are of these types.

Example 2.2.3. *The first example of a pseudo-rotation is when the Hamiltonian diffeomorphism φ is strongly non-degenerate and all its periodic orbits are elliptic.*

Example 2.2.4. *The second example is when the Hamiltonian diffeomorphism φ is a* true rotation*. The Hamiltonian diffeomorphism φ is a true rotation, when, by definition φ generates a compact (but not finite) subgroup G of* $\mathrm{Ham}(M)$*.*

Strongly non-degenerate true rotations are pseudo-rotations. For examples of pseudo-rotations which are obtained from such true rotations by the conjugation method, the reader can refer to [AK70; FK04; LS22]. In fact, pseudo-rotations obtained using conjugation method from toric symplectic manifolds are toric.

The crux of this section is that when φ is a pseudo-rotation we have natural isomorphisms between the Floer complex, the Floer homology and the quantum homology,

$$\mathrm{CF}_*(\varphi) \cong \mathrm{HF}_*(\varphi) \cong \mathrm{HQ}_*(M)[-n].$$

Any iterate φ^k is then also a pseudo-rotation, and hence

$$\mathrm{CF}_*\left(\varphi^k\right) \cong \mathrm{HF}_*\left(\varphi^k\right) \cong \mathrm{HQ}_*(M)[-n]. \tag{2.2.1}$$

This follows by the *PSS* isomorphism [PSS96] between Floer and quantum homology and by the definition of pseudo-rotations 2.2.1. We will be using these isomorphisms ubiquitously in this chapter to obtain some of the main results.

2.2.1 Chance-McDuff conjecture and pseudo-rotations

The Conley Conjecture, roughly states that for many symplectic manifolds every Hamiltonian diffeomorphism has infinitely many periodic orbits. The conjecture has been proved in many cases and currently it holds for M unless there exists $A \in \pi_2(M)$ such that $\langle \omega, A \rangle > 0$ and $\langle c_1(TM), A \rangle > 0$ [GG15; GG19]. In particular, the conjecture holds whenever M is symplectically aspherical or negative monotone or $\omega \mid_{\pi_2(M)} = 0$.

An example when the Conley conjecture fails is : an irrational rotation of S^2 about the z-axis has only two periodic points: these are the fixed points – the Poles. Along the similar lines, the conjecture fails for some other manifolds such as complex projective spaces, Grassmannians and flag manifolds, symplectic toric manifolds, and most of the coadjoint orbits of compact Lie groups. In fact, the conjecture fails for all manifolds admitting a Hamiltonian circle (or torus) action with isolated fixed points – a generic element of the circle or the torus gives rise to a Hamiltonian diffeomorphism with finitely many periodic points. In particular, pseudo-rotations are counterexamples to the Conley conjecture and, in fact, they are the only known counterexamples.

Delving further into the failure of Conley conjecture there is an outstanding problem, referred to as the Chance McDuff Conjecture. It states that whenever the Conley Conjecture fails some Gromov-Witten invariants are non-zero.

It is well-known that there is a strong connection between the symplectic topology of M (e.g., Gromov–Witten invariants or the quantum product) and the dynamics (periodic orbits) of Hamiltonian diffeomorphisms φ of M. However, this connection is explored and usually utilized only in one direction: from symplectic topology to dynamics. The opposite direction was first explored in the work [McD90], where it was shown that a symplectic manifold admitting a Hamiltonian circle action is uniruled, i.e., has a non-zero Gromov–Witten invariant with one of the homology classes being the point class.

Recently variants of Chance-McDuff conjecture for pseudo-rotations have been proved by Çineli-Ginzburg-Gürel [ÇGG20] and independently by Egor Shelukhin [She20]. Namely, [ÇGG20] showed that, under certain additional conditions when a manifold M admits a pseudo-rotation then it must have deformed quantum product and in particular, some non-vanishing Gromov-Witten invariants. Their assumptions were that $N > 1$, where N is the minimal Chern number. We recall their main result here, for the additional conditions in theorem 2.2.5 below the reader can refer to [ÇGG20] .

To state the theorem 2.2.5, let us recall some terminologies. We denote by $\mathrm{HQ}_*(M)$ the (small) quantum homology of M, by $*$ the quantum product, and by $|\alpha|$ the degree of an element $\alpha \in \mathrm{HQ}_*(M)$. The quantum product is said to be *deformed* if it is not equal to the intersection product.

Theorem 2.2.5. *[ÇGG20] Assume that M^{2n} admits a pseudo-rotation φ with an elliptic fixed point x which, for some $r \in \mathbb{N}$, satisfies certain conditions. Then* *there exist r elements $\alpha_1, \ldots, \alpha_r$ in $\mathrm{HQ}_*(M)$ of even degree such that*

$$\alpha_1 * \ldots * \alpha_r \neq 0 \tag{2.2.2}$$

and

$$|\alpha_i| \not\equiv 2n \mod 2N \text{ for all } i = 1, \ldots, r. \tag{2.2.3}$$

Further Çineli-Ginzburg-Gürel and indepedently Shelukhin showed the following topological constraint when a symplectic manifold admits a pseudo-rotation.

Proposition 2.2.6. *[ÇGG20; She20] Assume that M^{2n} admits a pseudo-rotation. Then $N \leq 2n$, where N is the minimal Chern number of M.*

2.2.2 Toric pseudo-rotations

Now, we introduce toric pseudo-rotations, a special kind of pseudo-rotations. By definition, a pseudo-rotation φ is toric if at one of its fixed points the eigenvalue of $D\varphi$ satisfy no resonance relations beyond the conditions that they come in complex conjugation pairs. To be more precise, the requirement is that the semi-simple part of $D\varphi$ topologically generates an n-dimensional torus in $Sp(2n)$. For instance, pseudo-rotations obtained by the conjugation method from toric symplectic manifolds are toric. While the toric condition appear generic, in fact the very existence of a toric pseudo-rotation φ imposes strong restrictions on the symplectic topology of the manifold M. (For example, when φ is a toric true rotation, essentially by definition M is toric). One case when the conditions of Theorem 2.2.5 are automatically satisfied is when a pseudo-rotation behaves as a generic element of the Hamiltonian $\mathbb{T}^n$-action on a toric symplectic $2n$-dimensional manifold.

Definition 2.2.7 (Toric Pseudo-rotations). *A pseudo-rotation φ of a closed symplectic manifold M^{2n} is said to be* toric *if it has a fixed point x with* $\dim \Gamma(x) = n$.

Note that in this case x is necessarily strongly non-degenerate. Here $\Gamma(x)$ is a compact abelian subgroup of $Sp(2n)$ generated by $\tilde{P}$, where $\tilde{P}$ is isospectral to $P = D\varphi|_x$ and semisimple. Alternatively, since P is elliptic all the eigenvalues of P lie on the unit circle. Let $\vec{\theta} := (\theta_1, ..., \theta_n) \in \mathbb{T}^n$ be the collection of Krein-positive eigenvalues of P , ref. [SZ92]. The group $\Gamma(x)$ is naturally isomorphic to the subgroup of the torus $\mathbb{T}^n$ generated by $\vec{\theta}$. Then the above definition of being a toric pseudo-rotation is equivalent to that the sequence $\{k\vec{\theta} \mid k \in \mathbb{N}\} \subset \mathbb{T}^n$ is dense in $\mathbb{T}^n$.

Since conditions of 2.2.5 are automatically satisfied for a toric pseudo-rotation, we have the following corollary.

Corollary 2.2.8. *[ÇGG20] Assume that M admits a toric pseudo-rotation and $N > 1$. Then the quantum product is deformed and, in particular, some Gromov–Witten invariants of M are non-zero.*

In fact, the corollary can be further refined as follows.

Theorem 2.2.9. *[ÇGG20] Assume that M^{2n} admits a toric pseudo-rotation. Then, for every $r \geq 1$, there exists $\alpha \in \mathrm{HQ}_{2n-2}(M)$ such that $\alpha^r \neq 0$.*

The proof of these results rely on the combinatorial tool called extremal partition which help us identifying zero-energy Floer solutions that are regular. We will explore these details in the next section. Before that we recall an example due to [ÇGG20] which motivates the definition of extremal partitions.

Example 2.2.10 (Irrational Rotations of S^2). *Let φ be a Hamiltonian diffeomorphism on S^2 which is an irrational rotation by an angle θ, where $\pi < \theta < 2\pi$.*

Since the rotation is irrational, the only one-periodic orbits of φ *are the North and South Poles, denoted as* y *and* x *respectively. In fact, upon iterations the only* k*-periodic points are* x^k *and* y^k*. To compute the Conley-Zehnder indices, we fix the trivial cappings on the periodic orbits. In that case,* $\mu(x) = -1$, $\mu(y) = 1$ *and* $\mu\left(x^2\right) = -3$*. (this is due to the choice of angle* θ*). Under the identification of the Floer complex* $\mathrm{CF}_*(\varphi)$ *with the quantum homology* $\mathrm{HQ}_*\left(S^2\right)[-1]$ *(ref 2.2.1), the North Pole* y *represents the fundamental class* $[S^2]$*, the South Pole* x *represents* $[pt]$ *and* $[x^2]$ *represents the class* $q[S^2]$*. Here* q *is the generator of the Novikov ring with degree* $|q| = -4$*.*

There is only one pair-of-paints curve from (x, x) *to* x^2 *– the constant curve. Indeed, later we will see that this a zero-index, regular pair-of-pants curve, therefore we have,*

$$x * x = x^2 + \dots,$$

where $*$ *is the pair-of-pants product and the dots stand for capped periodic orbits with action strictly smaller than the action of* x^2*. On the quantum side, then we have that*

$$[pt] * [pt] = q[S^2] + \dots \neq 0.$$

This implies that the quantum product is indeed deformed (i.e, it does not agree with the intersection product). Otherwise, we will have $[pt] * [pt] = 0$*. This also implies that are non-vanishing Gromov-Witten invariants, i.e,*

$$\mathrm{GW}_A\left([pt], [pt], [pt]\right) \neq 0,$$

where A *is the "positive" generator of* $\mathrm{H}_2(S^2)$*.*

Remark 2.2.11. *In this example,* $1 + 1 = 2$ *is an extremal partition of length* 2*,* *with*

$$\mu(x) + \mu(x) - \mu(x^2) = -1 - 1 - (-3) = 1 = (2-1)1 = 1$$

the pair-of-pants curve from (x, x) to x^2 is the constant curve which is regular. As we shall see below, the definition of extremal partition is directly motivated by observing the deformed quantum product in this example.

2.3 Extremal partitions

In this section we will introduce a combinatorial tool called extremal partition central to the proof of the main results in the following section.

2.3.1 Definiton and examples

We fix a path $\Phi \in \widetilde{Sp}(2n)$. For the sake of simplicity, we will assume that Φ is elliptic and strongly non-degenerate, i.e., the iterate end-point $\Phi^k(1)$ is non-degenerate for all $k \in \mathbb{N}$.

Definition 2.3.1 (Extremal Partitions)**.** *A partition $k_1 + \ldots + k_r = k$, $k_i \in \mathbb{N}$, of length r is said to be* extremal *(with respect to Φ) if*

$$\mu\Big(\Phi^{k_1}\Big) + \ldots + \mu\Big(\Phi^{k_r}\Big) - \mu\Big(\Phi^{k}\Big) = (r-1)n. \tag{2.3.1}$$

In the equation 2.3.1 the right hand side can be thought as defect of the left hand side expression. For $\Phi_i \in \widetilde{Sp}(2n)$, we assume that all Φ_i and their partial products $\Phi_1 \cdot \ldots \cdot \Phi_\ell$, $\ell \leq r$, are non-degenerate. The defect is defined as,

$$D = D(\Phi_1, \ldots, \Phi_r) := \sum \mu\Big(\Phi_i\Big) - \mu\Big(\Phi_1 \cdot \ldots \cdot \Phi_r\Big). \tag{2.3.2}$$

Then, as is shown in [DDP08], that,

$$|D| \leq (r-1)n.$$

Just a quick note, the non-degeneracy requirement is essential for the inequality above. The extremal partitions maximize the defect; and therefore they are called extremal.

Another crucial point to note that the defect D depends only on the end-points $\Phi_1(1), \ldots, \Phi_r(1)$. Therefore recalling the notation from the definition 2.2.9 of a toric pseudo-rotation we set $\Gamma(\Phi) := \Gamma\Big(\Phi(1)\Big)$, where $\Gamma(\Phi)$ is a compact abelian subbgroup of $Sp(2n)$ generated by $\tilde{\Phi}$, where $\tilde{\Phi}$ is isopectral to Φ and semisimple.

We can observe that composing any one of the maps Φ_i with a loop changes both terms in 2.3.2 by the mean index of the loop. As a result, the defect D only depends on the end-points $\Phi_1(1), \ldots, \Phi_r(1)$. More particularly, the chosen partition and the end point $\Phi(1)$ determines the left-hand side of the equation 2.3.1.

Now we present two examples, these examples would serve as important motivations for portions of our proof of the main theorem 2.4.1.

Example 2.3.2. *Assume that $\Phi \in \widetilde{Sp}(2n)$ is the direct sum of n counterclockwise rotations $\exp\Big(2\pi\sqrt{-1}\lambda_i t\Big)$, where $\lambda_i > 0$ are small and $t \in [0, 1]$. Then $\mu\Big(\Phi^r\Big) = n$ as long as $r \max \lambda_i < 1$, and $1 + \ldots + 1 = r$ is an extremal partition with 2.3.1 taking form $rn - (r-1)n = n$.*

Example 2.3.3. *Assume that $\Phi \in \widetilde{Sp}(2n)$ is toric, i.e., $\dim \Gamma(\Phi) = n$. Then Φ admits extremal partitions of arbitrarily large length. We use this notion in the last section, it is not hard to see this as a consequence of our previous example 2.3.2.*

2.3.2 Extremal partitions and zero energy solutions

In this section we will recall the relation between extremal partitions and zero index energy-zero regular curves. These energy-zero regular curves would help us identify non-vanishing Gromov Witten invariants in theorem 2.4.1, connecting the dynamics of toric-pseudo rotations with the topology of the underlying manifold.

Recall that the pair-of-pants product

$$\mathrm{CF}_*(\varphi_1) \otimes \ldots \otimes \mathrm{CF}_*(\varphi_r) \to \mathrm{CF}_*(\varphi)$$

"counts" the number of solutions $u \colon \Sigma \to M$ of the Floer equation, with the domain Σ being the $(r+1)$-punctured sphere; see, e.g., [AS10; MS12].

Let $\mathcal{M}$ be the moduli space of such solutions u "connecting" $\bar{x}_1, \ldots, \bar{x}_r$ to $\bar{y}$. Here $\bar{x}_i$ is a capped one-periodic orbits of φ_i and $\bar{y}$ a capped periodic orbit of φ. The virtual dimension of $\mathcal{M}$ is

$$\dim \mathcal{M} = \mu(\bar{x}_1) + \ldots + \mu(x_r) - \mu(\bar{y}) - (r-1)n. \tag{2.3.3}$$

Assume that this dimension is zero and the regularity conditions are met. Then the coefficient of $\bar{y}$ in the product $\bar{x}_1 * \ldots * \bar{x}_r$ equals to the number of points (here, we take $\mathbb{F} = \mathbb{Z}_2$) in the moduli space of such u "connecting" $\bar{x}_1, \ldots, \bar{x}_r$ to $\bar{y}$.

Even when the regularity condition is not satisfied, we necessarily have

$$\mathcal{A}_{H_1}(\bar{x}_1) + \ldots + \mathcal{A}_{H_r}(\bar{x}_r) - \mathcal{A}_H(\bar{y}) = E(u) \geq 0,$$

where $E(u)$ is the energy of u, see [AS10]. The energy $E(u) = 0$ if and only if

$$\mathcal{A}_{H_1}(\bar{x}_1) + \ldots + \mathcal{A}_{H_r}(\bar{x}_r) = \mathcal{A}_H(\bar{y}). \tag{2.3.4}$$

This happens, if and only if $x_1(0) = \ldots = x_r(0)$. Without loss of generality we may assume that the orbits x_i are constant; see, e.g., [Gin10, Sect. 2.3]. Then 2.3.4 holds if and only $E(u) = 0$ and if and only if u is a constant map. Now, we move onto the discussion of the regularity of these energy zero curves with zero index.

Regularity for zero-energy solutions

Our goal in this section is to show that zero index, zero energy pair-of-pants solutions of the Floer equation are automatically regular. Let x be a strongly non-degenerate one-periodic orbit of H and let $u\colon \Sigma \to M$ be a zero index and energy-zero solution asymptotic to $\bar{x}^{k_1} \ldots \bar{x}^{k_r}$ and $\bar{x}^k$ where $k_1 + \ldots k_r = k$. As discussed above, we may assume that x is a constant one-periodic orbit, and hence u is a constant solution of the Floer equation mapping Σ to x.

Recall that the $\mathcal{E}^1$ is the space of $W^{1,p}$-sections of u^*TM with $p > 1$ and $\mathcal{E}^0$ is the space of L^p-sections.

Proposition 2.3.4. *[ÇGG20] Let $D\colon \mathcal{E}^1 \to \mathcal{E}^0$ be the linearized Floer equation along u. Then we have,*

$$\ker D = 0.$$

Here $u\colon \Sigma \to M$ be a pair-of-pants curve asymptotic to $\bar{x}^{k_1} \ldots \bar{x}^{k_r}$ and $\bar{x}^k$ where $k_1 + \ldots k_r = k$.

This proposition is quite standard and variants of this proposition for Floer cylinders is established in [Sal99, Sect. 2.3] and for closed holomorphic curves in [MS12, Lemma 6.7.6].

The operator D has the form $\bar{\partial} + S$, where S is an automorphism of u^*TM, and is Fredholm due to the non-degeneracy assumption.

We omit the details of the proof the proposition 2.3.4 here. It can be shown by using graph construction passing to Lagrangian Floer theory following an argument by [Sei15].

We will now employ proposition 2.3.4 to identify regular zero-energy pseudo-holomorphic curves. Indeed, by theory of pseudo-holomorphic curves, a curve u is regular when D is onto, i.e., $\operatorname{coker} D = 0$. The Fredholm index of the operator D is given by 2.3.3,

$$\dim\ker D - \dim\operatorname{coker} D = \mu\left(\bar{x}^{k_1}\right) + \ldots + \mu\left(x^{k_r}\right) - \mu\left(\bar{x}^{k}\right) - (r-1)n.$$

Thus, in our situation since $\ker D = 0$, $\operatorname{coker} D = 0$ whenever the index of D is zero. Therefore, the following useful result on extremal partitions can be obtained.

Corollary 2.3.5. *[ÇGG20] Assume that*

$$\mu\left(\bar{x}^{k_1}\right) + \ldots + \mu\left(x^{k_r}\right) - \mu\left(\bar{x}^{k}\right) - (r-1)n = 0,$$

i.e., $k_1 + \ldots + k_r = k$ is an extremal partition (see Definition 2.3.1). Then the zero energy solution is automatically regular.

Summarizing these discussions, we have the following theorem, due to [ÇGG20], relating the quantum product with extremal partitions.

Theorem 2.3.6. *[ÇGG20] Let $\tilde{x}$ be a capped one-periodic orbit of a pseudo-rotation φ, and let $k = k_1 + k_2 + \cdots + k_r$ be an extremal partition of length r*

with respect to $\Phi = D\varphi^t|_{\tilde{x}}$*. Set* $\alpha_i = [\tilde{x}_i^k \otimes 1] \in HQ_*(M)$*. Then* $|\alpha_i| = n + \mu(\Phi^{k_i})$ *and the following holds*

$$\alpha_1 * \cdots * \alpha_r \neq 0.$$

In fact when Φ is toric, i.e. $\dim \Gamma(\Phi) = n$, then we have a extremal partition for every $r \in \mathbb{N}$ and thus we have ubiquitous number of non-vanishing Gromov-Witten invariants.

Lemma 2.3.7. *[ÇGG20] Assume* Φ *is toric. Then for every* $r \geq 1$*, there exists an extremal partition* $m + m + \cdots + m = k$ *of length* r *(i.e.,* $m.r = k$*) such that*

$$\mu(\Phi^m) \equiv n - 2 \mod 2N$$

This lemma essentially proves Theorem 2.2.9 in the essence of Theorem 2.3.6.

Now we move onto the next section, where we state the main results of this chapter connecting dynamics of a toric pseudo-rotation with the topology of its underlying manifold.

2.4 Main results

In this section we present the proof of our main theorem where we relate the dynamics of a toric pseudo-rotation and the quantum homology of the underlying manifold. For the statement of the theorem, we fix the base field $\mathbb{F} = \mathbb{Z}_2$ for our coefficients.

Theorem 2.4.1. *[Ban20] Assume that a weakly-monotone symplectic manifold* M^{2n} *admits a toric pseudo-rotation with minimal Chern number* $N \geq n+1$*. Then* $N = n+1$*,* M *is monotone and the quantum homology* $HQ_*(M)$ *is isomorphic to* $HQ_*(\mathbb{C}P^n)$*.*

Remark 2.4.2. *Note that $N \geq n+1$ in all known examples of closed monotone manifolds and $\mathbb{C}P^n$ is the only such known manifold with $N = n+1$. (However proving this in the symplectic setting for $n > 2$ appears to be currently out of reach). However, in Thm 2.4.1 the manifold M is not a priori assumed to be monotone and thus can have large N. For monotone manifolds admitting a pseudo-rotation, $N \leq 2n$ ([ÇGG20], [She20]), and $N \leq n+1$ in all known examples of weakly monotone manifolds with pseudo-rotations. Thus Thm 2.4.1 establishes in particular the latter fact for toric pseudo-rotations.*

In [OO96], [OO97] Ohta-Ono proved proved that the diffeomorphism type of any closed monotone symplectic 4-manifold is $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ for $0 \leq k \leq 8$ based on the work of McDuff [McD90] and Taubes [Tau00]. We also have the uniqueness of monotone symplectic structures on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ for $0 \leq k \leq 8$ (see the survey [Sal99]). The quantum homology $HQ_*(M)$ is isomorphic to $HQ_*(\mathbb{C}P^n)$. Thus, as a consequence of our result and the references presented above, we have the following corollary.

Corollary 2.4.3. *[Ban20] Assume that a closed connected symplectic 4-manifold with minimal Chern number $N \geq 3$ admits a toric pseudo-rotation. Then M is symplectomorphic to $\mathbb{C}P^2$.* □

The first step in proving theorem 2.4.1 is showing the invertibility of the $[pt] \otimes 1$ class with respect to the quantum product in $HQ_*(M)$. We begin by establishing the following lemma.

Lemma 2.4.4. *[Ban20] Let φ be a toric pseudo-rotation of M^{2n} with minimal Chern number $N \geq n+1$. Then $([pt] \otimes 1)^r \neq 0$ for all $r \geq 1$.*

Proof. Let Φ be the linearized flow $D\varphi^t|_{\tilde{x}}$ along a capped orbit $\tilde{x}$ such that $\dim \Gamma(x) = n$, i.e. $\Gamma(x) = \mathbb{T}^n$.

When $\Phi(1)$ is semi-simple we can decompose Φ as a product $\phi\xi$ where ϕ is a loop and ξ is a direct sum of n "short paths" $t \mapsto exp(\pi\sqrt{-1}\lambda t)$ where $t \in [0,1)$ and $|\lambda| < 1$, ref, [[GG22], Sect. 4]. We set $loop(\Phi) := \hat{\mu}(\phi)$ where $\hat{\mu}(\phi)$ is the mean index, which is twice the Maslov index of ϕ. For any iteration $\Phi^k(1)$ we have $\mu(\Phi^k) = k\, loop(\Phi) + \mu(\xi^k)$.

All the eigenvalues of $\Phi(1)$ are necessarily distinct and in particular $\Phi(1)$ is semi-simple. Therefore, $\Phi(1)^k$ $k \in \mathbb{N}$ is dense in $\mathbb{T}^n$. For some $l \in \mathbb{N}$, we can choose $\Phi(1)^l$ to be sum of small rotations $exp(\pi\sqrt{-1}\theta_i)$ such that $\theta_i < 0$ for all $i = 1, ..., n$ and θ_i's are very close to each other. Iterating again to bring $exp(\pi\sqrt{-1}m\theta_i)$ close to $1 \in S^1$. The choice of "m" is such that the $loop(\Phi^m) = -2n + d$ (with $2N|d$) and if λ_i's be the end points of the "short paths" in ξ^m then we have $\lambda_i > 0$ and small. We have

$$\mu(\Phi^m) = -2n + d + n = -n + d.$$

Also ensuring $r\max|\lambda_i| < 2$ we have,

$$\mu(\Phi^{rm}) = r(-2n + d) + n.$$

And $m + ... + m = rm$ is an extremal partition since

$$r\mu(\Phi^m) - (r-1)n = r(-n + d) - (r-1)n = r(-2n + d) + n = \mu(\Phi^{rm}).$$

Since $\mu(\Phi^m) = \mu(\tilde{x}^m) = -n \pmod{2N}$ therefore for some element $\lambda \in K_\Gamma$, $\tilde{x}^m \otimes \lambda = [pt] \otimes 1$ (since $N \geq n+1$). Thus $([pt] \otimes 1)^r \neq 0$ by Theorem 2.3.6. $\square$

Corollary 2.4.5. *[Ban20] Assume that a weakly-monotone symplectic manifold M^{2n} admits a toric pseudo-rotation and $N \geq n+1$, then $[pt \otimes 1]$ is invertible and the $[pt \otimes 1]$ class satisfies the following conditions:*

$$([pt] \otimes 1)^N = [M] \otimes \alpha, \tag{2.4.1}$$

where α is invertible in K_Γ and $deg(\alpha) = -2Nn$.

Proof. By the previous lemma we have $([pt] \otimes 1)^N \neq 0$. We have $\deg([pt] \otimes 1)^N$ $=-2n(N-1)$ and since $N \geq n+1$, therefore $([pt] \otimes 1)^N = [M] \otimes \alpha$ where $deg(\alpha) = -2Nn$. The class α is of the form $(\sum_{\theta\in\Gamma} z_\theta s^\theta)q^{-n}$. Therefore α is invertible since K_Γ is a field. □

The invertibility of the $[pt] \otimes 1$ class is crucial to the arguments below. By doing a simple degree analysis we immediately get some obstructions to the minimal Chern number. For the later part of the theorem invertibility gives us uniqueness of the homology classes.

Proof of Theorem 2.4.1

Proof. Let us begin by proving that when a weakly-monotone M^{2n} admits a toric pseudo-rotation, then $N \leq n+1$. We recall from [ÇGG20] that when M admits a toric pseudo-rotation there is a non zero class $u \in H_{2n-2}(M)$ such that $([u] \otimes 1)^r \neq 0$ for every $r \geq 1$. Since $[pt] \otimes 1$ satisfies (2), $([pt] \otimes 1) * ([u] \otimes 1) \neq 0$. By doing degree computation we see $deg(([pt] \otimes 1) * ([u] \otimes 1)) = -2$ thus if $N > n+1$, then $([pt] \otimes 1) * ([u] \otimes 1) = 0$ which is a contradiction.

For the final part of the proof we will show that when $N = n+1$, then

$$\dim(H_{2n-2i}(M)) = 1 \text{ for } 1 \leq i \leq n.$$

Let us first establish the result for $i = n-1$.

Let u_1, u_2 be two non-zero classes in $H_2(M)$, consider $([pt] \otimes 1)^{n-1} * ([u_i] \otimes 1)$. Now $deg(([pt]\otimes 1)^{n-1} * ([u_i]\otimes 1)) = 2n + -2(n+1)(n-1)$ and $([pt]\otimes 1)^{n-1} * ([u_i]\otimes 1) \neq$

0 since $[pt] \otimes 1$ satisfies (2). Therefore $([pt] \otimes 1)^{n-1} * ([u_i] \otimes 1) = [M] \otimes \lambda_i$ for some invertible element $\lambda_i \in K_\Gamma$.

Multiplying $[pt]^2 \otimes 1$ with both sides and using invertibility of α we obtain $[u_1] \otimes 1 = [u_2] \otimes \lambda_2^{-1}\lambda_1$. This shows the classes $[u_1]$ and $[u_2]$ are linearly dependent, hence the dimension of $H_2(M)$ is 1. This also implies that $\dim H_2(M, \mathbb{Z}) = 1$ and thus M is monotone.

Let class A_0 be the obvious generator for $H_2^S(M, \mathbb{Z})$. We set $q' = (s^{-\omega(A_0)}q^{-1})$ with $deg(q') = -2(n+1)$ We will rename q' by q which is the generator of the Novikov ring and denote $A \otimes \alpha$ by αA for $\alpha \in K_\Gamma$ and $A \in H_*(M)$.

Now let us prove the result for $i > 1$. We have $u^i \neq 0$ with $deg(u^i) = 2n - 2i$. Let β be another non-zero class in $H_{2n-2i}(M)$, then $deg([pt]^i * \beta) = 2n - 2i - 2ni = 2n - 2i(n+1)$ and thus $[pt]^i * \beta = \lambda q^i M$. By similar arguments as above and using (2), β and u^i are linearly dependent and hence the dimension of $H_{2n-2i}(M)$ is 1.

So by above arguments it follows that $HQ_*(M)$ is generated by $u \in H_{2n-2}(M)$. The identity $u^{n+1} = q[M]$ readily follows since $deg(u^{n+1}) = -2 = -2(n+1) + 2n$, and the theory of extremal partition asserts the coefficient is 1. This establishes the isomorphism with that of the quantum homology of $\mathbb{C}P^n$. $\square$

Chapter 3

Topological entropy

In this chapter we will concentrate on another aspect of the interplay between Floer homology and Hamiltonian dynamics. Following [ÇGG21], we introduce a Floer-theoretic invariant called barcode entropy of compactly-supported Hamiltonian diffeomorphisms, and show that it is bounded above by topological entropy for *open* symplectic manifolds convex at infinity. Our setup is quite different to what we had encountered the last chapter. We will concentrate on a special kind of symplectic manifolds which are not closed, namely they are convex at infinity and we will further assume that they are either atoroidal or toroidically monotone.

To be more specific, a cohomology class ω is *atorodial* if and only if for every map $v : \mathbb{T}^2 \to M$ the integral of ω over v vanishes: $\langle \omega, [v] \rangle = 0$. A symplectic manifold (M, ω) is said to be toroidally monotone when for some constant $\lambda \geq 0$ the class $\omega = [\omega] - \lambda c_1(TM)$ is atoroidal. The constant λ is referred to as the *toroidal monotonicity constant*. We call the positive generator N_T of the group generated by the integrals $\langle c_1(TM), [v] \rangle$ for all tori the *minimal toroidal Chern number* of M. We set $N_T = \infty$ when this group is $\{0\}$, i.e., $c_1(TM)$ is atoroidal.

The assumption that our underlying symplectic manifold (M, ω) is convex at infinity means that there is a compact domain $M_1 \subset M$ such that

$$(M \setminus M_1, \omega) \cong (\Sigma \times (1, \infty), d(r^2\alpha))$$

where $(\Sigma = \partial M_1, \alpha)$ is a closed contact manifold and r denotes the coordinate on $[1, \infty)$. Then one works with Hamiltonians H_t on M that are of the form

$$H_t(x, r) = ar^2 + b,$$

for large r. Here and throughout this chapter we assume to be $a > 0$ and small. We make this assumption so that the Hamiltonian flow φ_H^1 does not have any fixed points outside the compact domain M_1. Also, for such Hamiltonians the r-component of a solution of the Floer equation is necessarily subharmonic (i.e. the maximum principle is satisfied, [Wen13]), and this prevents such solutions from escaping to infinity.

The main result of this chapter is the following: let $\varphi : M \to M$ be a compactly supported Hamiltonian diffeomorphism on an open symplectic manifold convex at infinity, then

$$\hbar(\varphi) \le h_{top}(\varphi).$$

Here $\hbar(\varphi)$ is the barcode entropy defined below and $h_{top}(\varphi)$ is the topological entropy.

Let us now briefly outline the sections in this chapter. In the first section we recall the construction of Floer homology from the first chapter with some slight modifications of the Novikov ring. We then introduce the filtered-version of Floer homology that would be necessary to define the barcode entropy. In the second section we briefly recall the notion of persistent homology and the necessary tools for defining barcodes on the filtered Floer complex and finally define barcode entropy. In this section we further setup the construction of the tomographs and recall Crofton's inequality: necessary tools for proving our main theorem 3.3.1. In the last section we prove our main theorem 3.3.1 connecting barcode entropy with the topological entropy.

3.1 Floer theoretic setup

In this section we recall the setup of Hamiltonian Floer homology necessary for our purposes. The setup is similar to that in first chapter except we use the "universal" Novikov ring to take into account the non-contractible orbits. We then define the filtered version for this homology: a necessary tool to define the barcodes in the following section.

Hamiltonian Floer Homology

Let $H : S^1 \times M \to \mathbb{R}$ be a compactly supported Hamiltonian on the symplectic manifold M convex at infinity. Consider a Hamiltonian Q_{r_0} such that for $r \geq r_0$ we have $Q_{r_0}(r) = ar^2 + b$ and Q_{r_0} is zero otherwise. We also require Q_{r_0} to vanish on $\operatorname{supp} H$. We consider the new Hamiltonian $H = H_Q = H + Q$ and take a small non-degenerate perturbation of it.

We denote by $\tilde{\pi}_1(M)$ be the set of homotopy classes of free loops in M. The free homotopy class of a loop x is denoted by $[x]$. The Hamiltonian vector field X_H is defined by the equation $i_{X_H}\omega = -dH$. The time-dependent flow of X_H is denoted by φ_H^t and in particular, the time-one map, a *Hamiltonian diffeomorphism* is denoted by $\varphi_H = \varphi_H^1$.

For a class $\mathfrak{f} \in \tilde{\pi}_1(M)$, let us fix a reference loop $z \in \mathfrak{f}$. A capping of $x \colon S^1 \to M$

with free homotopy class $\mathfrak{f}$ is a cylinder (i.e., a homotopy) $\Pi\colon [0,1]\times S^1 \to M$ connecting x and z taken up to a certain equivalence relation. Namely, two cappings Π and Π' are equivalent if the integral of $c_1(TM)$, and hence of ω, over the torus obtained by attaching Π' to Π is equal to zero.

The *action* of H on a capped loop $\bar{x} = (x, \Pi)$ is

$$\mathcal{A}_H(\bar{x}) = -\int_\Pi \omega + \int_{S^1} H_t(x(t))\, dt.$$

Clearly, $\mathcal{A}_H(\bar{x})$ is well defined. Moreover, the critical points of $\mathcal{A}_H$ are exactly the capped one-periodic orbits of H in the homotopy class $\mathfrak{f}$. The action spectrum $\mathcal{S}(H, \mathfrak{f})$ is the set of critical values of $\mathcal{A}_H$. It has zero measure.

For the reference loop $z \in \mathfrak{f}$ we fix a trivialization of $TM|_z$. Using this trivialization, we can define the *Conley–Zehnder index* $\mu_{\mathrm{CZ}}(H, \bar{x}) \in \mathbb{Z}$ of a capped *non-degenerate* orbit $\bar{x}$. Similarly to the contractible case, the Conley-Zehnder $\mu_{\mathrm{CZ}}(H, x)$ of an orbit without capping is defined only modulo $2N_T$.

The constructions from this section readily carry over to the case where a single free homotopy class $\mathfrak{f}$ is replaced by a collection of free homotopy classes. For instance, one can specify the collection of free homotopy classes of loops by prescribing a homology class.

Replacing the one-periodic orbits of H by the capped one-periodic orbits, one could define the Floer complex and the homology of H as an ungraded module over the Novikov ring Λ. By our choice of Hamiltonian, the Floer complex is a finite-dimensional vector space over the "universal" *Novikov field* Λ formed by formal sums

$$\lambda = \sum_{j\geq 0} f_j T^{a_j},$$

where $f_j \in \mathbb{F}$ and $a_j \in \mathbb{R}$ and the sequence a_j (with $f_j \neq 0$) is either finite or $a_j \to \infty$. The possible recapping contributes to the term $T^{\omega(v)}$ by the torus $v : \mathbb{T}^2 \to M$ with the symplectic area $\omega(v)$.

To be more precise if we denote the generators of the Floer complex by x_i, then

$$CF(H) = \oplus \Lambda x_i.$$

This complex is not graded due to the choice of the Novikov ring, or only $\mathbb{Z}_2$-graded. We will denote the Floer homology, i.e, the homology of $CF(H)$ by $HF(H)$. This is also a finite dimensional vector space over Λ.

3.1.1 Filtered floer homology

The Hamiltonian action of a capped loop $\bar{x} = (x, \Pi)$ is given by

$$\mathcal{A}_H(\bar{x}) = -\int_{\Pi} \omega + \int_{S^1} H_t(x(t))\, dt.$$

For $\lambda \in \Lambda$, given by $\lambda = \sum_{j \geq 0} f_j T^{a_j}$, we define

$$\nu(\lambda) := \min a_j, \text{ where } f_j \neq 0.$$

The Floer differential is strictly action-decreasing, and we obtain the required action filtration on the complex $CF(H)$ by defining

$$\mathcal{A}(\sum \lambda_i x_i) := \max_i \mathcal{A}(\lambda_i x_i), \text{ where } \mathcal{A}(\lambda_i x_i) := \mathcal{A}(x_i) - \nu(\lambda_i).$$

The Floer homology $HF(H)$, as a vector space over the field $\mathbb{F}$, inherits this action filtration and therefore we will call this as Filtered Floer Homology.

Remark 3.1.1. *As we can see, the action filtration depends on the choice of the reference paths from the previous section. A change of the reference paths would shift the filtration by a constant. Thus the filtration is well-defined only up to these shifts. We will see in the next section, the number of bars of a given length in the barcode of the Floer homology $HF(H)$ does depend on the reference paths and therefore $b_\epsilon(\varphi_H)$ is well defined.*

3.2 Persistent homology and barcodes

Persistent homology is a well-established tool in the field of topological data analysis. In the algebraic setup, one starts with "persistent modules", i.e. structures $\mathbb{V}$ consisting of a module V_t associated to each $t \in \mathbb{R}$ with homomorphisms $\sigma_{st} : V_s \to V_t$ whenever $s \leq t$ satisfying the functoriality properties that $\sigma ss = I_{V_s}$, the idenitity map on module V_s, and $\sigma_{su} = \sigma_{tu} \circ \sigma_{st}$. One natural example of persistent modules in topology arises when one considers a continuous map $f : X \to \mathbb{R}$ on a topological space X. For a field κ one can then consider the homology of the t-sublevel set by $V_t = H_*(\{f \leq t\}; \kappa)$, with the σ_{st} being the inclusion-induced maps. In real-word situations, see e.g. Carlsson [Car09], if $X = \mathbb{R}^n$ and the function $f : \mathbb{R}^n \to \mathbb{R}$ is given by the minimal distance to a finite collection of points samples from one subset $S \subset \mathbb{R}^n$, then V_t is the homology of the union if balls of radius t around the points of the sample.

Under the finiteness hypotheses on the modules V_t, provided the coefficient ring is a field κ, it can be shown that the persistence module $\mathbb{V}$ is isomorphic to a direct sum of "interval modules" κ_I, where $I \subset \mathbb{R}$ is an interval and by defintion $(\kappa_I)_t = \kappa$ for $t \in I$ and $\{0\}$ otherwise. The *barcode* of $\mathbb{V}$ is then defined to be the multiset of intervals appearing in this direct sum decomposition. We can similarly extract barcode from filtered Morse homology groups defined using a Morse function $f : X \to \mathbb{R}$ on X which is a smooth compact manifold. For compact symplectic manifold M, when it is aspherical we can also extract the barcode of the filtered Floer complex for a non-degenerate Hamiltonian diffeormorphism φ_H since the finiteness hypothesis holds.

However, in our situation while the filtered Floer homology gives a persistence module, the finiteness hypothesis is not longer satisfied. So the above discussion for defining barcodes does not translate here. We will therefore slightly vary and follow closely Usher-Zhang, [UZ16], and Çineli-Ginzburg-Gürel, [ÇGG21], for defining barcodes in the general Floer theoretic setting over Novikov rings. We briefly recall Singular Value Decomposition for the Floer chain complex and then define barcodes in our case.

Singular value decomposition

We consider the filtered Floer complex $CF(H)$ with the action filtration,

$$\mathcal{A}(\sum \lambda_i x_i) := \max_i \mathcal{A}(\lambda_i x_i).$$

Indeed the complex $CF(H)$ forms a non-Archimedean normed vector space over Λ, see [UZ16; ÇGG21]. It is an orthogonalizable Λ-space and an orthogonal set is necessarily linearly independent over Λ. Thus, (see [UZ16]) we can define a singular-value decompositon of $CF(H)$ over Λ.

Definition 3.2.1. *[UZ16] A basis $\Sigma = \{\alpha_i, \eta_i, \gamma_i\}$ of $C_*(H)$ over Λ is said to be a singular value decomposition if :*

- $\partial_H \alpha_i = 0$.

- $\partial_H \gamma_i = \eta_i$.
- *the basis is orthogonal (in the non-Archimedean sense).*

We can order the *finite bars* as

$$A_H(\gamma_1) - A_H(\eta_1) \leq A_H(\gamma_2) - A_H(\eta_2) \leq \dots$$

The infinite bars correspond to the $\dim_\Lambda HF_*(H)$. Together they form the *barcode* of $HF_*(H)$.

Persistence and Hofer's metric

For any compactly supported $\phi = \phi_H^1 \in \mathrm{Ham}(M, \omega)$ the Hamiltonian diffeomorphism group we have the well-known Hofer's metric d_H. First, we define the Hofer norm,

$$||\phi||_H = \inf\{\int_0^1 (\max_X H(t,.) - \min_X H(t,.))dt \mid \phi = \phi_H\}.$$

Suppose we have two Hamiltonians H_1 and H_2 with respective Hamiltonian diffeomorphisms ϕ, $\psi \in \mathrm{Ham}(M, \omega)$ such that for large $r > 0$, $H_1(x, r) = H_2(x, r) = ar^2 + b$. Then we define the Hofer's metric for ϕ, $\psi \in \mathrm{Ham}(M, \omega)$ by,

$$d_H(\phi, \psi) = ||\phi^{-1} \circ \psi||.$$

This is a bi-invariant metric on $\mathrm{Ham}(M, \omega)$ (see [Pol01]) with fixed choices of r, a and b leading to Hofer's geometry.

Barcode distance

We introduce a distance on barcodes, in the spirit of Gromov-Hausdroff metric. A barcode consists of a collection of finite intervals and infinite intervals. We will denote our barcodes as $([0], L)$ where $L \in [0, \infty]$ is the length of the bar. Here the bars are as in [ÇGG21] which are not pinned.

Definition 3.2.2. *(δ- matching) For two barcodes (viewed as multisets) S and T, we say that an δ-matching between S and T consist of:*

- *(Short multisets S_{short} and T_{short}) consisting of finite bars, where the length $L \leq 2\delta$.*
- *A bijection $\sigma : S \setminus S_{short} \to T \setminus T_{short}$ such that for each $\sigma([0], L) = ([0], L')$ where for all $\epsilon > 0$ the representative $([0], L')$ can be chosen, such that either $L = L' = \infty$ or $\mid L - L' \mid \leq \delta + \epsilon$.*

In order S and T to be δ-matched, there should be equal number of infinite-length bars from S and T.

Definition 3.2.3. *(Bottle-Neck distance) For two barcodes S and T, the bottle-neck distance between S and T is,*

$$d_B(S,T) = \inf\{\delta \geq 0 \mid \textit{there is a } \delta\textit{-matching between } S \textit{ and } T\}.$$

We have the following theorem which is a Lipschitz-comparison between the bottle-neck distance and the Hofer's norm for Hamiltonian Floer Homology.

Theorem 3.2.4. *[UZ16] For any two Hamiltonians H_1 and H_2 with barcodes B_{H_1} and B_{H_2} respectively, we have*

$$d_B(B_{H_1}, B_{H_2}) \leq d_H(\phi^1_{H_1}, \phi^1_{H_2})$$

3.2.1 Barcode entropy

Let M be an open symplectic manifold which is convex at infinity and either atoroidal or toroidically monotone.

Let $H : S^1 \times M \to \mathbb{R}$ be a compactly supported Hamiltonian on M and the Hamiltonian diffeomorphism is denoted by $\varphi = \varphi_H$. Consider a fixed Hamiltonian Q_{r_0} such that for $r \geq r_0$, $Q_{r_0}(r) = ar^2 + b$ and otherwise Q_{r_0} is zero. Here r_0 is determined by the support of H and b is a constant determined by r_0 such that $Q_{r_0}(r_0) = 0$.

Let $\epsilon > 0$, and $\psi_{(\epsilon,k)}$ be a non-degenerate $\frac{\epsilon}{2}$-perturbation of $\varphi_Q \varphi^k$. Denote $B(\psi_{(\epsilon,k)})$ be the barcode of the Floer complex of $\mathrm{CF}(\psi_{(\epsilon,k)})$ over the Novikov ring Λ. Then denote,

$$b_\epsilon(\varphi^k) = |\{\text{bars of length greater than } \epsilon \text{ in } B(\psi_{(\epsilon,k)})\}|.$$

It is easy to see that $b_\epsilon(\varphi^k)$ is independent of the perturbation as long as it is small.

Definition 3.2.5 (Barcode Entropy)**.** *The* ϵ-barcode entropy *of* φ *is*

$$\hbar_\epsilon(\varphi) := \limsup_{k\to\infty} \frac{\log^+ b_\epsilon\big(\varphi^k\big)}{k}$$

and the barcode entropy *of* φ *is*

$$\hbar(\varphi) := \lim_{\epsilon \searrow 0} \hbar_\epsilon(\varphi) \in [0,\, \infty]$$

Here again $\hbar_\epsilon(\varphi)$ is increasing as $\epsilon \searrow 0$, and hence the limit in the definition of $\hbar(\varphi)$ exists.

Remark 3.2.6. *One can show that the barcode entropy does not depend on the choice of Q_{r_0} when $a > 0$ is small.*

3.2.2 Crofton's inequality

In this section we will discuss the setup for Crofton's inequality. The proof follows similar reasoning as in [ÇGG21], so we will omit it here.

Setup:

Here we describe the setup for Crofton's inequality necessary for our purposes. Let L be a manifold and B be an open ball in $\mathbb{R}^m$ for some m with finite radius. We denote the projection into the first component by $\pi : E = B \times L \to B$. Let

$$\Psi : E \to P$$

be a smooth map to some manifold P. Suppose there exists a compact submanifold C of L with boundary ∂C such that Ψ is a submersion on $B \times C^\circ$, where $C^\circ := C \setminus \partial C$. We also require that the map $\Psi_s = \Psi|_{s\times C}$ be an embedding for all s. The images of C under the embedding are denoted by $\Psi_s(C) = C_s$ which are also compact submanifolds with boundary. Further the images of L under the map Ψ_s is denoted by L_s.

Now, we let L' be a closed submanifold of P with

$$\operatorname{codim} L' = \dim L$$

and $L' \cap (\Psi(B \times (L \setminus C^\circ))) = \phi$. Since Ψ is a submersion in $B \times C^\circ$, we have $\Psi \pitchfork L'$.

This implies $\Psi_s \pitchfork L'$ for almost all $s \in B$ and

$$N(s) = \mid L_s \cap L' \mid$$

is a finite number and N is an integrable function on B. Using arguments from [ÇGG21] one can show that the following inequality holds.

Lemma 3.2.7. *(Crofton's Inequality) [ÇGG21] We have,*

$$\int_B N(s)ds \ \leq \ c \cdot vol(L'),$$

where c is a constant which depends on ds, Ψ and the metric on P, but doesn't depend on L'.

Construction of the tomograph

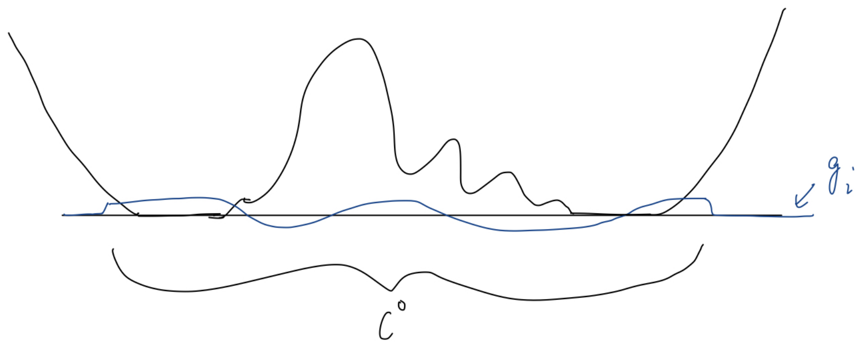
We are now in a position to construct a partial "tomograph" of the diagonal $\Delta = \{(x,x) \mid x \in M\}$ submanifold of the symplectic manifold $M \times M$. In our setup M is convex at infinity, which means that there is a compact domain M_1 and a contact manifold (Σ, α) with contact form α, such that $(M \setminus M_1, \omega) \cong (\Sigma \times (1, \infty), d(r^2\alpha))$, where $\Sigma = \partial M_1$.

We will work with Hamiltonians $H : S^1 \times M \to \mathbb{R}$ such that $H_t(x,r) = ar^2 + b$ when r is large and $(x,r) \notin M_1$. We further can choose a, b such that ϕ_H^1 has no fixed points outside M_1.

We choose a collection of functions $\{g_i\}_{i=1,\ldots,n}$ and a compact submanifold C with boundary ∂C such that $M_1 \subset C$ such that,

- $\{g_i\}$ generate T_pM for all $p \in C^\circ = C \setminus \partial C$.

- $g_i \equiv 0$, for all points outside C°.



We consider functions with the form

$$f_s = s_1 g_1 + \cdots + s_n g_n$$

where $s_i, \ldots, s_n \in B = B^n(0, d)$ for some d. We can squeeze the ball B such that $\Psi : B \times M \to M \times M$ defined by

$$\Psi(s, x) = df_s(x)$$

where df_s is the graph of the function f_s. The function Ψ is a submersion on $B \times C^\circ$, for our choice of functions $\{g_l\}$ and Hamiltonian H with $\phi_H^1(M) \cap (\Psi(B \times M \setminus C^\circ)) = \emptyset$.

To apply Crofton's inequality we will replace L' by graph of Hamiltonian diffeomorphism ϕ_H, P will be replaced by $M \times M$ and L_s would be replaced by $\Psi_s(M)$. In this case the function $N(s)$ is integrable and the above lemma 3.2.7 is satisfied.

3.3 Main theorem

The main goal of this section is to prove the following theorem.

Theorem 3.3.1. *Let $\varphi : M \to M$ be a compactly supported Hamiltonian diffeomorphism on an open symplectic manifold convex at infinity, then*

$$\hbar(\varphi) \leq h_{top}(\varphi).$$

We fix a autonomous Hamiltonian Q such that for $r \geq r_0$ where r_0 is determined by the support of $\varphi = \varphi_H$ and $Q \equiv 0$ in the support of H. Denote by the graphs of diffeomorphisms by $\varphi_Q \varphi^k$ by ,

$$L_k = \{(x, \varphi_Q \varphi^k(x)) \mid x \in M\} \subset M \times M.$$

Before we prove the above theorem; we prove the following lemmas necessary to prove the theorem.

Lemma 3.3.2. *Let f be a Hamiltonian such that $f \equiv 0$ outside a compact set containing the support of H and the graph of its time-one map $L_f = \{(x, \varphi_f(x)) \mid x \in M\}$ intersects L_k transversely,*

$$\mid L_f \cap L_k \mid \geq b_\epsilon(\varphi_f^{-1} \varphi_Q \varphi_H^k)$$

for any $\epsilon > 0$.

Proof. $\mid L_f \cap L_k \mid$ is the number of one-periodic orbits of $\varphi_f^{-1}\varphi_Q\varphi$. Since L_f intersects L_k the Hamiltonian diffeomorphism $\varphi_f^{-1}\varphi_Q\varphi_H$ is non degenerate and since f is zero outside a compact set the Hamiltonian Floer homology $HF_*(\varphi_f^{-1}\varphi_Q\varphi_H)$ is well-defined.

Let $b(\varphi_f^{-1}\varphi_Q\varphi_H)$ be the total number of finite bars in the Floer homology $HF_*(\varphi_f^{-1}\varphi_Q\varphi_H)$. Then,

$$\begin{aligned} \mid L_f \cap L_k \mid &= dim_\Lambda CF(\varphi_f^{-1}\varphi_Q\varphi_H) \\ &= 2b(\varphi_f^{-1}\varphi_Q\varphi_H) - \dim_\Lambda HF(\varphi_f^{-1}\varphi_Q\varphi_H) \\ &\geq b(\varphi_f^{-1}\varphi_Q\varphi_H). \end{aligned}$$

This implies that for any $\epsilon > 0$, $\mid L_f \cap L_k \mid \geq b_\epsilon(\varphi_f^{-1}\varphi_Q\varphi_H^k)$. □

Proof of 3.3.1. Fix $\epsilon > 0$ and assume that

$$h_{2\epsilon}(\varphi) \geq \alpha \text{ for some } \alpha > 0.$$

This implies there is a sequence $k_i \to \infty$, such that,

$$b_{2\epsilon}(\varphi^{k_i}) \ \geq \ \text{const}\, 2^{\alpha k_i}.$$

We need to show that $\mathrm{vol}\,(L_{k_i}) \geq \text{const}\, 2^{\alpha k_i}$. This implies that $\alpha \leq h_{top}(\varphi)$. Passing to the limit as $\epsilon > 0$, we have $\alpha \mapsto \hbar(\varphi)$.

We construct our tomograph such that the functions $\{g_i\}_{i=1,2,\ldots,n}$ and ball $B^n(r), r > 0$ are dependent on the choice of Q and further satisfies the additional condition that

$$d_H(\varphi_{f_s}, id) < \epsilon, \ \text{ where } \ f_s = s_1 g_1 + \cdots + s_n g_n$$

for $(s_1, \ldots, s_n) \in B^n(r)$. Then $L_s = \{(x, \varphi_{f_s}^1(x)) \mid x \in M\}$ is transversal to L_k for all k and for all almost all $s \in B^n(r)$. Therefore

$$N_k(s) := |L_s \cap L_k|$$

the number of intersections between L_s and L_k are finite for almost all s and all k. By Crofton's Inequality we already have

$$\int_B N_k(s)\, ds \;\leq\; c \cdot \mathrm{vol}(L^k)$$

for all $k \in \mathbb{N}$. From previous lemma we have $N_k(s) \geq b_\epsilon(\varphi_{f_s}^{-1}\varphi_Q\varphi_H^k)$. Since $d_H(\varphi_{f_s}, id) < \epsilon$,

$$d_B(B(\varphi_{f_s}^{-1}\varphi_Q\varphi_H^k), B(\varphi_Q\varphi_H^k)) \leq d_H(\varphi_{f_s}, id) < \epsilon$$

we have a ϵ-matching between bars of length $L \geq 2\epsilon$. Therefore $b_\epsilon(\varphi_{f_s}^{-1}\varphi_Q\varphi_H^k)) \geq b_{2\epsilon}(\varphi_{f_s}^{-1}\varphi_Q\varphi_H^k)) = b_{2\epsilon}(\varphi_Q\varphi_H^k)) = b_{2\epsilon}(\varphi_H^k)$ (by definition, here one might have to take a small perturbation of $\varphi_Q\varphi_H^k$ so that barcode entropy is well defined). This completes the proof. □

Chapter 4

Toric domains and equivariant capacities

In this chapter we present examples of integrable Reeb flows where the "spectral gaps" are bounded away from zero . These Reeb flows are not pseudo-rotations (i.e the number of periodic orbits are more than the hypothetical minimal number possible). In fact in our examples we have infinitely many periodic orbits.

A contact manifold is a pair (Σ, λ), where Σ is a manifold of dimension $2n-1$, and λ is a 1-form on Σ such that $\lambda \wedge (\lambda)^{n-1}$ is nowhere vanishing. Throughout this chapter, all contact manifolds are assumed to be closed and connected. For a contact form λ, we denote by R the Reeb vector field on (Σ, λ), defined by $\lambda(R) \equiv 1$ and $d\lambda(R, \cdot) \equiv 0$.

Using symplectic capacities such as the Ekeland-Hofer and the ECH capacities, various characterizations of the *Besse* property of contact manifolds have been shown, see [CM20; GGM21]. A closed connected contact manifold is called Besse when all its Reeb orbits are closed, and in such a case the Reeb orbits admit a common period by a theorem of Wadsley, [Wad75]. For a convex hypersurface, it roughly says that n-consecutive spectral numbers are same (i.e, $c_i(\Sigma) = c_{n-i+1}(\Sigma)$) for some $i \in \mathbb{N}$ if and only if Σ is Besse and $c_i(\Sigma)$ is a common period of all its Reeb orbits, see [CM20; GGM21]. The Ekeland-Hofer capacities are not easy to calculate, they are only known for ellipsoids and polydisks. In the case of rational ellipsoids there is a subsequence for which the consecutive differences (spectral gaps) are zero. However for polydisks they are always bounded below away from zero.

In this chapter, we show that for Reeb flows on star-shaped domains in $\mathbb{R}^n$ which are pseudo-rotations and dynamically convex, there is a subsequence of spectral gaps converging to zero, using index recurrence and resonance relations. In this result we do not have any information about the periods of the Reeb flows, we only assume that there are finitely many orbits (with the minimal number possible). As mentioned above the Ekeland Hofer capacities only provide examples such as polydisks for which the spectral gaps are bounded below away from zero. In this chapter, we also present some computational examples using the Gutt-Hutchings capacities for which the spectral gaps are bounded away from zero. These capacities are conjecturally equal to the Ekeland Hofer capacities. These examples along with polydisks suggest that indeed when the Reeb flow fails to be a pseudo-rotation then we should no longer expect the spectral gaps converging to zero.

4.1 Symplectic capacities

In this section we begin by defining some S^1-equivariant symplectic capacities due to [GH18], see also [GG20]. These capacities were introduced by Gutt-Hutchings using the idea to imitate the ECH capacities, [Hut14], for higher dimensions. Ginzburg-Gürel also introduced similar capacities which they treated as certain spectral invariants and these numbers appear to be equal. A very interesting aspect of the Gutt-Hutchings capacties is that they equal the well-known historical Ekeland-Hofer capacities for simple cases such as ellipsoids and polydisks and it is conjectured they are equal for compact star-shaped domains. The Ekeland-Hofer capacities are very difficult to calculate for other cases but the Gutt-Hutchings capacities were calculated for convex and concave toric domains and we will use these numbers in the later sections.

We will digress a little bit and talk about another form of capacities that have been studied in the last decade: the ECH capacities. The ECH capacities are only defined for four-dimensional manifolds, for example they have been computed for concave toric domains, [Cho+14], and convex toric domains, [Cri19]. The ECH capacities, among other applications in understanding dynamics of Reeb flows, see [CH16], give sharp obstructions to symplectically embedding a four-dimensional ellipsoid into an ellipsoid, [McD11], or polydisk, [Hut11], or more generally a four-dimensional concave toric domain into a convex toric domain, [Cri19]. In some other situations, such as for some cases of symplectically embedding a four-dimensional polydisk into an ellipsoid, the ECH capacities do not give sharp obstructions, and the Ekeland-Hofer capacities are better. Therefore when the Gutt-Hutchings capacities are shown to be equal to the Ekeland-Hofer capacities, we can expect many more interesting embedding obstructions.

Nice star shaped domains

We define a *nice star-shaped domain* in $\mathbb{R}^{2n}$ to be a compact $2n$-dimensional submanifold X of $\mathbb{R}^{2n} = \mathbb{C}^n$ with smooth boundary Y, such that Y is transverse to the radial vector field

$$\rho = \frac{1}{2}\sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right).$$

In this case, the 1-form

$$\lambda_0 = \frac{1}{2}\sum_{i=1}^{n}(x_i\,dy_i - y_i\,dx_i)$$

on $\mathbb{C}^n$ restricts to a contact form on Y. If γ is a Reeb orbit of $\lambda_0|_Y$, define its *symplectic action* by

$$A(\gamma) = \int_\gamma \lambda_0 \in (0,\infty).$$

If we further assume that $\lambda_0|_Y$ is nondegenerate, i.e. each Reeb orbit of $\lambda_0|_Y$ is nondegenerate, then each Reeb orbit γ has a well-defined Conley-Zehnder index $CZ(\gamma) \in \mathbb{Z}$.

We state the properties of equivariant capacities introduced by Gutt-Hutchings, [GH18], see also Ginzburg-Gürel, [GG20]. These capacities are similar and conjecturally agree with the Ekeland-Hofer [EH90] capacities.

4.1.1 Equivariant capacities

The capacities c_k for nice star-shaped domains in $\mathbb{R}^{2n}$ satisfy the following axioms:

(Conformality) If X is a nice star-shaped domain in $\mathbb{R}^{2n}$ and r is a positive real number, then $c(rX) = r^2c(X)$.

(Increasing) $c_1(X) \le c_2(X) \le \cdots < \infty$.

(Monotonicity) If X and X' are nice star-shaped domains in $\mathbb{R}^{2n}$, and if there exists a symplectic embedding $X \to X'$, then $c_k(X) \leq c_k(X')$ for all k.

(Reeb Orbits) If $\lambda_0|_{\partial X}$ is nondegenerate, then $c_k(X) = A(\gamma)$ for some Reeb orbit γ of $\lambda_0|_{\partial X}$ with $CZ(\gamma) = 2k + n - 1$.

The capacities c_k can extended to star-shaped domains which are not necessarily nice (such as polydisks) as follows: If X is a star-shaped domain in $\mathbb{R}^{2n}$, then

$$c_k(X) = \sup\{c_k(X')\},$$

where the supremum is over nice star-shaped domains X' in $\mathbb{R}^{2n}$ such that there exists a symplectic embedding $X' \to X$.

Remark 4.1.1. *The capacities c_k were first introduced by Ekeland-Hofer in, [EH90], using min-max formulation for the gradient flow on $H^{1/2}$ space and Fadell-Raboniwitz index. They calculated the capacities explicitly for polydisks and ellipsoids.*

Remark 4.1.2. *The numbers c_k were introduced in, [GH18], using the idea to imitate the ECH capacities for higher dimensions and they were calculated them for convex and concave toric domains in $\mathbb{R}^{2n}$. We will use these capacities for our computational examples to non-vanishing spectral gaps.*

Remark 4.1.3. *The capacities c_k were discussed in [GG20] as spectral invariants with applications to multiplicity results for simple Reeb orbits. They are equal to the Gutt-Hutching's capacities for ellipsoids and appear to be equal for other star shaped domains.*

4.2 Examples

In this section we present some examples of star-shaped domains, namely ellipsoids, polydisks and concave and convex toric domains and their corresponding Gutt-Hutchings capacities. We recall the necessary details of the Gutt-Hutchings capacities for these manifolds. We will use these capacities for convex and concave toric domains in the last section to present examples of non-vanishing spectral gaps.

4.2.1 Ellipsoids

We begin by the simplest case of star-shaped domains, the ellipsoids. They are as follows: If $a_1, \ldots, a_n > 0$, define the *ellipsoid*

$$E(a_1, \ldots, a_n) = \left\{ z \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} \leq 1 \right\}$$

and the *polydisk*

$$P(a_1, \ldots, a_n) = \left\{ z \in \mathbb{C}^n \;\middle|\; \pi |z_i|^2 \leq a_i, \quad \forall i = 1, \ldots, n \right\}.$$

The *ball* is defined as $B(a) = E(a, \ldots, a)$.

The Gutt-Hutchings symplectic capacities for these manifolds agree with the Ekeland-Hofer capacities and were calculated by Ekelend and Hofer using calculus of variations for the symplectic action functional on the loop space of $\mathbb{R}^{2n}$. To state the computations we introduce the following sequence, if $a_1, \ldots, a_n > 0$, let $(M_k(a_1, \ldots, a_n))_{k=1,2,\ldots}$ denote the sequence of positive integer multiples of $a_1, \ldots, a_n$, arranged in nondecreasing order with repetitions. We then have:

- [GH18; EH90] The equivariant capacities of an ellipsoid are given by

$$c_k(E(a_1, \ldots, a_n)) = M_k(a_1, \ldots, a_n).$$

- [GH18; EH90] The equivariant capacities of a polydisk are given by

$$c_k(P(a_1, \ldots, a_n)) = k \cdot \min(a_1, \ldots, a_n).$$

4.2.2 Toric domains

The capacities c_k were explicitly computed by Gutt-Hutchings using S^1-equivariant symplectic homology for convex and concave toric domains.

We define the moment map $\mu : \mathbb{C}^n \to \mathbb{R}^n_{\geq 0}$ by

$$\mu(z_1, \ldots, z_n) = \pi(|z_1|^2, \ldots, |z_n|^2).$$

If Ω is a domain in $\mathbb{R}^n_{\geq 0}$, define the *toric domain*

$$X_\Omega = \mu^{-1}(\Omega) \subset \mathbb{C}^n,$$

and some special toric domains defined as $\Omega \subset \mathbb{R}^n_{\geq 0}$ given by,

$$\widehat{\Omega} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; (|x_1|, \ldots, |x_n|) \in \Omega \right\}.$$

Definition 4.2.1. *A* convex toric domain *is a toric domain X_Ω such that $\widehat{\Omega}$ is a compact convex domain in $\mathbb{R}^n$.*

Definition 4.2.2. *A* concave toric domain *is a toric domain X_Ω such that Ω is* *a compact and* $\mathbb{R}^n_{\geq 0} \setminus \Omega$ *is convex.*

The dynamics on the convex and concave toric domains are relatively simple, in fact they are integrable systems. Particularly, when X_Ω is a nice star-shaped compact toric domain, i.e. X_Ω is compact with smooth boundary. We give a short description of their Reeb dynamics when $n = 1$. The boundary $\overline{\partial_+ \Omega}$ is a smooth arc from some point $(0, b)$ with $b > 0$ to some point $(a, 0)$ with $a > 0$. There are three types of simple Reeb orbits on ∂X_Ω, see [GHR22]:

- There is a simple Reeb orbit corresponding to $(a, 0)$, whose image is the circle in ∂X_Ω with $\pi|z_1|^2 = a$ and $z_2 = 0$.
- Similarly, there is a simple Reeb orbit corresponding to $(0, b)$, whose image is the circle in ∂X_Ω with $z_1 = 0$ and $\pi|z_2|^2 = b$.
- For each point $\mu \in \partial_+ \Omega$ where $\partial_+ \Omega$ has rational slope, there is an S^1 family of simple Reeb orbits whose images sweep out the torus in ∂X_Ω where $\pi(|z_1|^2, |z_2|^2) = \mu$.

Now we recall the calculations of the Gutt-Hutchings capacities c_k for convex and concave toric domains as follows.

For convex toric domain X_Ω in $\mathbb{R}^{2n}$, we define the dual norm for a vector $v \in \mathbb{R}^n_{\geq 0}$ with all components non-negative as

$$\|v\|^*_\Omega = \max\{\langle v, w \rangle \mid w \in \Omega\},$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. Let $\mathbb{N}$ denote the set of non-negative integers.

Suppose that X_Ω is a convex toric domain in $\mathbb{R}^{2n}$. Then the Gutt-Hutchings capacities are

$$c_k(X_\Omega) = \min\left\{ \|v\|_\Omega^* \;\middle|\; v = (v_1, \ldots, v_n) \in \mathbb{N}^n,\ \sum_{i=1}^n v_i = k \right\}.$$

In the similar vein, suppose that X_Ω is a concave toric domain. Let Σ denote the closure of the set $\partial\Omega \cap \mathbb{R}^n_{>0}$. Similarly to the dual-norm, if $v \in \mathbb{R}^n_{\geq 0}$, define the anti-norm by

$$[v]_\Omega = \min\left\{ \langle v, w\rangle \;\middle|\; w \in \Sigma \right\}.$$

Then the Gutt-Hutchings capacities c_k for convex toric domain X_Ω are

$$c_k(X_\Omega) = \max\left\{ [v]_\Omega \;\middle|\; v \in \mathbb{N}^n_{>0},\ \sum_i v_i = k + n - 1 \right\}.$$

Note that in the above equation all components of v are required to be positive, while in the convex toric domain case, we only require that all components of v be non-negative.

4.3 Pseudo-rotations in the Reeb case

The Gutt-Hutchings capacities and the Ginzburg-Gürel spectral invariants have very similar constructions. They have been shown to be equal for ellipsoids and appear to be equal for other star-shaped domains in $\mathbb{R}^{2n}$. In the case of Ginzburg-Gürel capacities, we can show that for pseudo-rotations (i.e, Reeb flows with finitely many periodic orbits) that are dynamically convex, the "spectral gaps" converge to zero. In this section we will prove this theorem.

Definition 4.3.1. *The Reeb flow on a* $(2n-1)$*-dimensional contact manifold is* *said to be* dynamically convex *if every closed Reeb orbit is dynamically convex, i.e.* $\mu_{CZ}(x) \geq n+1$ *for all closed Reeb orbits* x*.*

Remark 4.3.2. *The Reeb flow on a strictly convex hypersurface in* $\mathbb{R}^{2n}$ *is dynamically convex, see [HWZ98].*

Using the index-recurrence theorem for non-degenerate case and resonance relations from [GG20], we will prove the following theorem in this section.

Theorem 4.3.3. *Let* (M^{2n-1}, α) *be a closed contact-type dynamically convex hypersurface in* $\mathbb{R}^{2n}$ *bounding a Liouville domain. Suppose that* α *is non-degenerate, carrying exactly* n *periodic orbits then* $\exists\, k_i \to \infty$*, such that*

$$c_{k_i+n-1} - c_{k_i} \to 0.$$

Remark 4.3.4. *We will refer to* $c_{k+n-1} - c_k$ *as the kth* spectral gap*. It has been shown in [GGM21], that when the kth spectral gap vanishes, the Reeb flow is Besse (i.e. all Reeb orbits are closed) and* c_k *is the common period of its Reeb orbits.*

We will use the following index recurrence result from [GG20], to prove the theorem. Studying index recurrence relations is a huge field in itself, we refer the reader to [Lon02; LZ02; DLW16] for further details and references. We begin by recalling that the mean index of a continuous path $\Phi : [0,1] \to Sp(2m)$ beginning at $\Phi(0) = I$, denoted by $\hat{\mu}(\Phi) \in \mathbb{R}$, is a homotopy invariant of the path with fixed end-points. Roughly the mean index measures the total rotation angle of certain unit eigenvalues of $\Phi(t)$, we omit the details here.

Theorem 4.3.5. *[GG20] Consider a finite collection of strongly non-degenerate elements* $\Phi_1, \ldots, \Phi_r$ *in* $\widetilde{\mathrm{Sp}}(2m)$ *such that all mean indices* $\Delta_i := \hat{\mu}(\Phi_i)$ *are non-zero and positive. Then for any* $\epsilon > 0$ *and any* $\ell_0 \in \mathbb{N}$*, there exists an integer* *sequence* $d_j \to \infty$ *and* r *integer sequences* k_{ij}, $i = 1, \ldots, r$, *at least one of which goes to infinity, such that for all* i *and* j, *and all* $\ell \in \mathbb{Z}$ *in the range* $1 \le |\ell| \le \ell_0$, *we have*

(i) $\left|\hat{\mu}(\Phi_i^{k_{ij}}) - d_j\right| < \epsilon$, *and*

(ii) $\mu_{CZ}(\Phi_i^{k_{ij}+\ell}) = d_j + \mu_{CZ}(\Phi_i^{\ell})$.

We can also ensure that $k_{ij} \to \pm\infty$ *as* $j \to \infty$ *for all* i, *and that for any* $N \in \mathbb{N}$ *we can make all* d_j *and* k_{ij} *divisible by* N.

Further when the paths $\Phi_1, \ldots, \Phi_r$ *are dynamically convex* $d_j \to \infty$ *and* $k_{ij} \to \infty$ *as* $j \to \infty$ *, and for all* $l \in \mathbb{N}$,

- $\mu_{CZ}(\Phi_i^{k_{ij}+\ell}) \ge d_j + 2 + m$,
- $\mu_{CZ}(\Phi_i^{k_{ij}-\ell}) \le d_j - 2 - m$.

This theorem is proved in Ginzburg-Gürel [GG20], where they obtain multiplicity results for dynamically convex hypersurfaces in $\mathbb{R}^{2n}$. They prove that when (M^{2n-1}, α) is a closed contact type, dynamically convex hypersurface in $\mathbb{R}^{2n}$ bounding a simply connected Liouville domain, then M carries at least r simple closed characteristics $x_1, \ldots, x_r$ when $r = \lceil n/2 \rceil + 1$ in general and $r = n$ when α is non-degenerate.

Proof of 4.3.3. We assume that the Reeb flow of (M^{2n-1}, α) is non-degenerate and has only finitely many simple closed orbits denoted by $x_1, \ldots, x_n$. The set of closed Reeb orbits $\mathcal{P}(\alpha)$ comprises all iterations x_i^k, $k \in \mathbb{N}$, of the orbits x_i.

Consider the map

$$\psi \colon \mathcal{I} = \{n+1, n+3, n+5, \ldots\} \to \mathcal{P}(\alpha), \quad d \mapsto y_d,$$

which denotes the action selectors of the capacities c_k, where we relabeled the domain of ψ by the index. (In other words, this map is obtained by composing the map in the corollary with the bijection $d \mapsto (d+1-n)/2$ from $\mathcal{I}$ to $\mathbb{N}$.) Thus the orbits denoted by $y_1, y_2, \ldots$, where $c_k = A_\alpha(y_k)$ are now $y_{n+1}, y_{n+3}, \ldots$. Since the orbits are non-degenerate

$$\mu_{CZ}(y_d) = d.$$

Let $\Phi_i \in \widetilde{\mathrm{Sp}}(2m)$, $m = n-1$, be the linearized Poincaré return map along x_i; without loss of generality we can assume that the paths Φ_i are parametrized by $[0, 1]$. Fixing a small parameter $\epsilon > 0$ and a sufficiently large $\ell_0 \in \mathbb{N}$, let us apply theorem 4.3.5 to the paths Φ_i. Since our orbits x_i are strongly non-degenerate, for all $\ell \in \mathbb{N}$, we have

- $\mu_{CZ}(x_i^{k_{ij}+\ell}) \geq d_j + n + 1$,
- $\mu_{CZ}(x_i^{k_{ij}-\ell}) \leq d_j - n - 1$, and
- $\left|\hat{\mu}(x_i^{k_{ij}}) - d_j\right| < \epsilon$

Let us denote L the index interval $[d_j - n,\, d_j + n] \cap \mathcal{I}$. Then for any $d \in L$ the orbit y_d must have the form $x_i^{k_{ij}}$, and therefore at most one iteration of x_i can occur as y_d with $d \in L$.

Using the resonance relations [GG20], since the set $\{\frac{\mathcal{A}_\alpha(x)}{\hat{\mu}(x)}\}$ is discrete and all the orbits are reoccurring (i.e the iterations of a orbits occur infinitely many times in the image of the injection ψ), for two Reeb orbits x and y we have

$$\frac{\mathcal{A}_\alpha(x)}{\hat{\mu}(x)} = \frac{\mathcal{A}_\alpha(y)}{\hat{\mu}(y)}.$$

This implies that for any $a_1, a_2 \in L$, $\left|A_\alpha(y_{a_1}) - A_\alpha(y_{a_2})\right| < 2C\epsilon$, where C is a constant. Since for any epsilon $\epsilon > 0$ we can choose a very large d_j such that the above reasoning work, this implies that there exists a sequence $k_i \to \infty$ such that $\left| c_{k_i+n-1} - c_{k_i} \right| \to 0$. □

4.4 Examples of toric domains with non-zero spectral gaps at infinity

In this section we present some computational examples of convex and concave toric domain with non-zero spectral gaps at infinity, in the spirit of theorem 4.3.3. These examples illustrate the fact that the finiteness of the number of Reeb orbits is probably necessary, otherwise we cannot expect theorem 4.3.3 to hold true.

Recall that the Gutt-Hutchings capacities introduced in Section 2 are equal to the Ginzburg-Gürel spectral invaraints for ellipsoids and appear to be equal for other star shaped domains. Here we use Python computing to calculate the spectral gaps for the Gutt-Hutchings capacities for convex and concave domains.

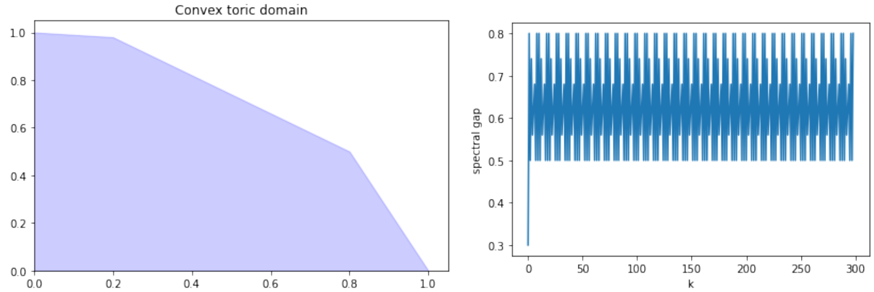
- **Example 1 :** Convex toric domain with non-smooth boundary. For $n = 2$, X_Ω is a convex toric domain if and only if

  $$\Omega = \{(x_1, x_2) \mid \ 0 \le x_1 \le A, \ 0 \le x_2 \le g(x_1) \ \}, \qquad (4.4.1)$$

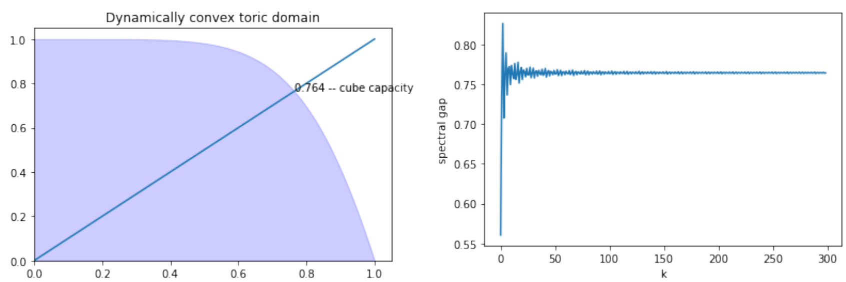
  where

  $$g : [0, A] \to \mathbb{R}_{\ge 0}$$

  is a concave function (there is a general notion of convex toric domain due to [Cri19] where Ω is convex and $\hat{\Omega}$ is not required to be convex).



- **Example 2 :** Convex toric domain with smooth boundary, in fact in this case the Reeb flow is also dynamically convex.



By computing examples in many smooth cases we can see that the spectral gaps would converge to the *cube capacity* as $k \to \infty$.

Cube capacity

We will digress slightly into the discussion of the cube capacity and show their significance with respect to convex and concave domains and more generally star-shaped domains.

Given $\delta > 0$, define the cube

$$\square_n(\delta) = P(\delta, \dots, \delta) \subset \mathbb{C}^n,$$

or equivalently

$$\square_n(\delta) = \{z \in \mathbb{C}^n \mid \max_{i=1,\dots,n} \{\pi |z_i|^2\} \leq \delta\}.$$

Definition 4.4.1. *Given a $2n$-dimensional symplectic manifold (X, ω), define the cube capacity as*

$$c_\square(X, \omega) = \sup\{\delta > 0 \mid \textit{there exists a symplectic embedding } \square_n(\delta) \to (X, \omega)\}.$$

Immediately from the definition, it is clear that the cube capacity $c_\square$ is a symplectic capacity.

Remark 4.4.2. *For a convex or concave toric domain $X_\Omega \subset \mathbb{C}^n$, the cube capacity is the largest δ such that $\square_n(\delta)$ is a subset of X_Ω; one cannot do better than this obvious symplectic embedding by inclusion. Indeed we observe this is in the Example 2 and 4 respectively.*

For the convex or concave toric domain (see [GH18]), we have

$$c_\square(X_\Omega) = \max\{\delta \mid (\delta, \dots, \delta) \in \Omega\}.$$

The cube capacity also has a nice relationship with the Gutt-Hutchings capacities when the following condition is satisfied. To state the condition we first need to conisder the non-disjoint union of n symplectic cylinders,

$$L_n(\delta) = \{z \in \mathbb{C}^n \mid \min_{i=1,\dots,n} \{\pi |z_i|^2\} \leq \delta\}.$$

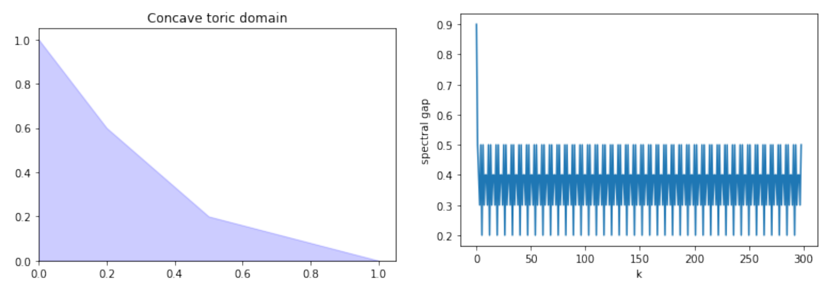
For a star-shaped domains, the cube capacity is equal (see [GH18]) to

$$c_{\square}(X) = \lim_{k \to \infty} \frac{c_k(X)}{k}$$

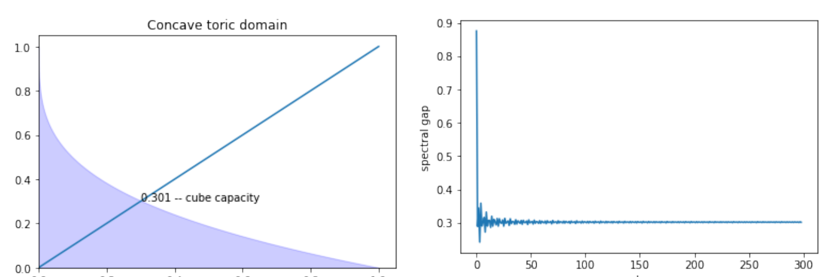
when the following condition is satisfied

$$\square_n(\delta) \subset X \subset L_n(\delta).$$

- **Example 3 :** Concave toric domain with non-smooth boundary. For a concave toric domain X_Ω is a concave toric domain if and only if Ω is given by (4.4.1) where $g : [0, A] \to \mathbb{R}_{\geq 0}$ is a convex function where $g(A) = 0$.



- **Example 4 :** Concave toric domain with smooth boundary. Here also as in Example 2 we can computationally see that the spectral gaps would converge to the cube capacity.



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