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HILBERTIAN SEMINORMS AND LOCAL ORDER IN JB^*-TRIPLES*

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1. Introduction and background

In this note, we wish to investigate the local theory in JB^*-triples arising from a single linear functional.

For C^*-algebras, the important functionals are the states, giving rise in the commutative case to probability measures. These can be decomposed into discrete and continuous parts, and further into absolutely continuous and singular parts. It is this kind of phenomenon that we wish to consider in a setting in which positivity, commutativity, associativity, and even the binary product are absent.

For von Neumann algebras, the use of the trace in the semifinite cases, and of extreme points in the atomic cases has facilitated their study. In the purely infinite cases, where no trace is present, the Tomita-Takesaki theory showed how to effectively use non-tracial normal states. For JBW^*-triples, the Hilbertian seminorms introduced below will be shown to be useful for obtaining structural information on the triple and the functional.

JB^*-triples have been proposed as a framework for modelling the observables of a quantum mechanical system. In this model, states become arbitrary unit vectors of the dual space, since no global order is present in general. Thus our study will have relevance to the application of JB^*-triples to quantum mechanics.

Before going into the background and motivation for this paper, here is a summary of its contents. Our main theorem (Theorem 2.13 in section 2) gives a fundamental relation between two basic kinds of sesquilinear forms (called OP for operator positivity, and RN for Radon-Nikodym). To appreciate the level of abstraction in this theorem, it is necessary to first review some basic concepts and results for JBW^*-triples. These include the polar decomposition of a normal functional, the algebraic inner product arising from a normal functional, and theorems of Grothendieck type in C^*-algebras and JB^*-triples. This section also includes versions of the Radon-Nikodym theorem and the existence of the self-polar form for a normal functional on a JBW^*-triple, as well as a

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definition of a trace. One of the two corollaries of Theorem 2.13 is an new inequality of Grothendieck type.

In Section 3 we begin by giving two examples of the sesquilinear form and Schur multiplier arising from an atomic functional. The first example is an associative one and is included for comparison only. The second one involves the joint Peirce decomposition arising from the spectral decomposition of the atomic functional and shows that the Schur multipliers corresponding to the algebraic inner product and the self-polar form are related to the arithmetic and geometric (numerical) means, respectively. The section concludes with an analog of the Schur product theorem and the construction of Schur multipliers in the non-atomic case.

We now summarize some fundamental facts about JB^*-triples. A JB^*-triple is a complex Banach space A, equipped with a (Jordan) triple product $A \times A \times A \to A$ denoted by $(x, y, z) \to \{x, y, z\}$, with the following properties:

(1) $\{x, y, z\}$ is bilinear and symmetric in x, z and conjugate-linear in y;
(2) $\|\{x, y, z\}\| \leq \|x\| \, \|y\| \, \|z\|$, for any $x, y, z \in A$;
(3) for every $a \in A$, the operator $D(a)$ defined by $D(a)x = \{a, a, x\}$ is Hermitian (i.e. $\|e^{itD(a)}\| = 1$, $t \in \mathbf{R}$), its spectrum is non-negative and $\|D(a)\| = \|a\|^2$;
(4) for every $a \in A$, the operator $\delta(a) := iD(a)$ is a *derivation* of the triple product:

$$\delta(a)\{x, y, z\} = \{\delta(a)x, y, z\} + \{x, \delta(a)y, z\} + \{x, y, \delta(a)z\}.$$

By polarizing the last identity, one obtains the so called *Main Identity* of the triple product

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

The concrete subclass of JB^*-triples are the JC^*-triples, defined as follows: A *JC^*-triple* is a norm closed subspace U of $B(H)$ such that $xx^*x \in U$ whenever $x \in U$. This notion was introduced by L. Harris **[11]**, where the terminology "J^*-algebras" was used. By polarization, we see that a closed subspace U of $B(H)$ is a JC^*-triple if and only if it is closed under the *triple product*

$$\{x, y, z\} := (xy^*z + zy^*x)/2. \tag{1}$$

The JC^*-triples are precisely the JB^*-subtriples of $B(H)$ for some Hilbert space H, i.e., properties (1)–(4) of the above definition hold for the triple product (1). JBW^*-triples are the JB^*-triples which are dual spaces. A JBW^*-triple A has a unique predual, denoted by A_* (**[15]**, **[2]**). The general references for JB^*-triples are **[27]**, **[28]**, **[20]** and **[21]**. The category of JB^*-triples is equivalent to the category of bounded symmetric domains in complex Banach spaces **[20]**.

The building blocks of the algebraic structure of JB^*-triples are the *tripotents* ("triple idempotents"), i.e. elements $u \in A$ satisfying $\{u, u, u\} = u$. Clearly, in a JC^*-triple the tripotents are precisely the partial isometries. Two tripotents u, v are *orthogonal* if $u+v$ and $u-v$ are tripotents. This is denoted by $u \perp v$.

On the set of tripotents in the JB^*-triple A we define a *partial ordering* as follows: $u \leq v$ if and only if $v = u + w$ where w is a tripotent in A and $w \perp u$. Note that $u \leqslant v$ if and only if $u = \{uvu\}$, by [8, Cor. 1.7]. A tripotent u is *minimal* in A if for any tripotent $v \in U$, $v \leqslant u$ implies either $v = 0$ or $v = u$.

To each element x in the JB^*-triple A we associate a conjugate linear map $Q(x)$: $A \to A$, via

$$Q(x)y = \{x, y, x\}; \qquad y \in A.$$

If $v \in A$ is a tripotent, then the operator $D(v) = \{v, v, \cdot\}$ satisfies:

$$D(v)(D(v) - I)(2D(v) - I) = 0.$$

Thus, the spectrum of $D(v)$ consists of eigenvalues only and is contained in the set $\{0, 1/2, 1\}$. Let $A_k(v)$ be the eigenspace of $D(v)$ corresponding to the eigenvalue $k/2$, and let $P_k(v)$ be the corresponding spectral projection onto $A_k(v)$, $k = 0, 1, 2$. $A_k(v)$ are the *Peirce subspaces* of A associated with v, and $P_k(v)$ are the corresponding *Peirce projections*. Clearly, the $P_k(v)$ can be expressed as polynomials in $D(v)$, and $P_2(v) = Q(v)^2$.

Since for every $x \in A$, $\delta(x) := iD(x)$ is a triple derivation, it follows that for any tripotent $v \in A$,

$$\{A_j(v), A_k(v), A_l(v)\} \subseteq A_{j-k+l}(v) \tag{2}$$

where $A_m(v) = \{0\}$ if $m \notin \{0, 1, 2\}$. The multiplication rules (2) are called the *Peirce calculus*.

Moreover,

$$\{A_0(v), A_2(v), A\} = \{A_2(v), A_0(v), A\} = \{0\}$$

and each of the spaces $A_k(v)$ is a JB^*-subtriple, $k = 0, 1, 2$.

The *Peirce reflection* associated with a tripotent $v \in A$ is the operator $S(v) = P_2(v) - P_1(v) + P_0(v) = \exp(2\pi i D(v))$. It is the symmetry ($S^2(v) = I$) fixing $A_2(v) \oplus A_0(v)$ element wise and satisfying $S(v)|_{A_1(v)} = -I|_{A_1(v)}$. The operator $Q(v)$ is a conjugate-linear triple automorphism of $A_2(v)$ of period 2, and is used to define there an *involution* via

$$x^{\#} := Q(v)x = \{v, x, v\}, \qquad x \in A_2(v).$$

Given a sequence $\{v_j\}_{j=1}^n (n \leq \infty)$ of orthogonal tripotents in A we define the associated *joint Peirce projections* $\{P_{i,j}\}_{0 \leq i \leq j \leq n}$ by

$$P_{i,i} := P_2(v_i), \qquad 1 \leq i \leq n;$$

$$P_{i,j} := P_1(v_i)P_1(v_j), \qquad 1 \leq i < j \leq n;$$

$$P_{0,j} := P_1(v_j) \prod_{1 \leq i \leq n, i \neq j} P_0(v_i), \qquad 1 \leq j \leq n;$$

$$P_{0,0} := \prod_{1 \leq i \leq n} P_0(v_i).$$

For finite n see **[21]**; for infinite n see **[16]** or **[22]**. The *joint Peirce decomposition* of A is

$$A = \sum_{0 \leq i \leq j \leq n} \oplus A_{i,j}, \qquad \text{where } A_{i,j} = P_{i,j}A.$$

Let e be a tripotent in a JB^*-triple A. The space $A_2(e)$ is a JB^*-algebra (JBW^*-algebra if A is a JBW^*-triple) with identity e, Jordan product $(x, y) \mapsto \{xey\}$, and involution $x \mapsto \{exe\}$. In the Jordan algebra structure of $A_2(e)$, the states of $A_2(e)$ are exactly the elements of the norm exposed face $F_e := \{\rho \in A^*: \rho(e) = 1 = \|\rho\|\}$. The projection $P_2(e)$ is also positive in the sense that $P_2(e)[A_2(e)^+] \subset A_2(e)^+$, where $A_2(e)^+$ denotes the set of positive elements in the JB^*-algebra $A_2(e)$, *i.e.*,

$$A_2(e)^+ = \{x \in A_2(e): \langle x, \rho \rangle \geq 0, \rho \in F_e\}.$$

We close this section with a discussion of some of the motivation for the present paper. The structure of JB^*-triples (cf. **[27]**, **[28]**) is now understood by use of the "Gelfand-Naimark" theorem **[9]** and factor classification of JBW^*-triples **[16]**, **[5]**, **[22]**, **[17]**, **[3]**. Also properties of their preduals are known **[2]**, **[8]**.

Let A denote a JBW^*-triple and A_* its predual. It is known that the predual A_* decomposes into a direct sum of atomic and purely non-atomic parts **[8]**. Any norm one element $f \in A_*$ belonging to the atomic part can therefore be written as a σ-convex combination of extreme points of the unit ball of A_*. But for the non-atomic case there is no natural replacement for extreme points or "good states". In some cases a trace plays this role, but in general even tracial functionals may not exist.

The space A_* is a natural model for describing the states of a quantum mechanical system. The extreme points of the unit ball play the role of the pure states. The time evolution is expressed by a one-parameter group of isometries. Thus it is important to know how the generators of this group act in a neighborhood of any state f. It has been observed in several cases that these generators act like Schur multipliers with respect to the Peirce decomposition of the spectral tripotents of f (f atomic). It is therefore important to obtain a notion of Schur multiplier associated with a non-atomic state. This is done at the end of Section 3.

An example of this kind of Schur multiplier was found in [13] in the formula for the self-polar form, based on Tomita-Takesaki theory for Jordan algebras. As mentioned there, there is no analog for the modular automorphism group of a state in the non-associative case, but a cosine family is shown to exist. It is possible to extend their results to JB^*-triples and to obtain Schur multipliers from non-atomic functionals.

2. Hilbertian seminorms based on algebraic structure and local order

Let A be a JBW^*-triple and A_* its predual. We want to consider local properties of A, A_* starting from a functional $\varphi \in A_*$. Order structure plays a key role when it exists *a priori*. When it does not exist, one can produce a local order structure by means of the following theorem, which is the analog of the polar decomposition of normal functionals on a von Neumann algebra.

THEOREM 2.1 ([8]) *For each $\varphi \in A_*$ of norm 1, there exists a unique $e \in A$ (denoted by $e(\varphi)$ and called the support tripotent of φ) such that*
(a) *e is a tripotent;*
(b) *$\varphi(e) = 1$ and $\overline{\varphi(x)} = \varphi(\{exe\})$, for all $x \in A$ (hence $\varphi = \varphi \circ P_2(e)$);*
(c) *for all tripotents $u \leq e$, $\varphi(u) > 0$.*

For any $\varphi \in A_*$ of norm 1, the face of φ, denoted by Face(φ), is defined as the collection of all $\tau \in A_*$ such that there is a $\sigma \in A_*$ and $0 \leq \lambda \leq 1$ with

$$\varphi = \lambda \frac{\tau}{\|\tau\|} + (1 - \lambda) \frac{\sigma}{\|\sigma\|}.$$

By [6] or [19], the norm closure of Face(φ) is $A_2(e)_*^+$, where $e = e(\varphi)$. The norm exposed face F_φ of φ is defined by

$$F_\varphi := \{\psi \in A_*: \ \psi(e) = 1 = \|\psi\|\}.$$

This is the smallest norm exposed face containing φ (cf. [7]). The cone $(V_\varphi)_* = \mathbf{R}^+ F_\varphi = A_2(e)_*^+$ defines the local order on A_* with respect to φ. Note that (c) of Theorem 2.1 implies that φ is positive and faithful with respect to this order and Face(φ) $= \mathbf{R}^+[0, \varphi]$, where $[0, \varphi]$ is the order interval defined by $\{\psi \in A_2(e)_*: \ 0 \leq \psi \leq \varphi\}$.

Definition 2.2 Let A be any complex vector space. A map b: $A \times A \to \mathbf{C}$ is called a *sesquilinear form* if it is linear in the first argument and conjugate linear in the second one. A sesquilinear form is said to be *positive* if $b(x, x) \geqslant 0$ for all $x \in A$. Since a positive sesquilinear form is automatically hermitian, it defines a seminorm called a *Hilbertian seminorm*. If b is a sesquilinear form, then $b^*(y)$ denotes the linear functional on A defined by $b^*(y) = b(\cdot, y)$:

$$\langle x, b^*(y)\rangle = b(x, y).$$

The following proposition may be used to show that each normal functional φ on a JBW^*-triple A gives rise to a Hilbertian seminorm, which retains much of the information supplied by φ.

PROPOSITION 2.3 ([1]) *Let A be a JBW^*-triple. For $\varphi \in A_*$ of norm 1, let $e = e(\varphi)$. Define a sesquilinear form a_φ by*

$$a_\varphi(x, y) = \varphi(\{xye\}) \qquad x, y \in A. \tag{3}$$

Then a_φ is positive on A, $a_\varphi^*(e) = \varphi$ and

$$\|x\|_\varphi := a_\varphi(x, x)^{1/2}$$

is a Hilbertian seminorm on A. Moreover, $\|x\|_\varphi \leqslant \|\varphi\| \cdot \|x\|$ and e in (3) can be replaced by any $a \in A$ satisfying $\|a\| = 1 = \varphi(a) = \|\varphi\|$.

The a in the notation a_φ for the form is used since it is based on the algebraic structure of A. Let $x = x_2 + x_1 + x_0$ be the Peirce decomposition of x relative to e. From (2) and (b) of Theorem 2.1, it follows that

$$\|x\|_\varphi^2 = \varphi\{xxe\} = \varphi\{x_2x_2e\} + \varphi\{x_1x_1e\} = \|x_2\|_\varphi^2 + \|x_1\|_\varphi^2$$

From Theorem 2.1(c) it follows that $\|x_2\|_\varphi = 0$ only if $x_2 = 0$; and $\|x_1\|_\varphi = 0$ only if $x_1 = 0$. Thus $\|\cdot\|_\varphi$ is a norm on $A_2(e) + A_1(e)$ turning this space into a pre-Hilbert space. One of the places where these seminorms appear is in the Grothendieck inequality for C^*-algebras.

THEOREM 2.4 ([25], [12]) *There is a universal constant K such that for any two C^*-algebras A, B, and any bounded bilinear form T: $A \times B \to \mathbf{C}$ there exist states φ on A and ψ on B such that*

$$|T(x, y)| \leqslant K\,\|T\| \left[\varphi\left(\frac{x^*x + xx^*}{2}\right)\right]^{\frac{1}{2}} \left[\psi\left(\frac{y^*y + yy^*}{2}\right)\right]^{\frac{1}{2}} \quad \text{for } x \in A,\ y \in B.$$

The Hilbertian seminorm $[\varphi(x^*x + xx^*/2)]^{\frac{1}{2}}$ is not one of the natural ones associated to C^*-algebras. The natural ones are the ones that occur in the G.N.S. construction, namely

$$\|x\|_\varphi^\# = \varphi(x^*x)^{\frac{1}{2}} \quad \text{and} \quad \|x\|_\varphi^\flat = \varphi(xx^*)^{\frac{1}{2}}.$$

These "associative" seminorms were shown by Pisier to be insufficient for the inequality of Grothendieck. The seminorm that is needed in Grothendieck's inequality for C^*-algebras is the one introduced above, namely

$$\|x\|_\varphi = \left[\varphi\left(\frac{x^*x + xx^*}{2}\right)\right]^{\frac{1}{2}} = \varphi(\{xx1\}),$$

indicating that the inequality does not rely on associativity. This also suggests that the inequality is independent of the order structure and is related to the geometry of the unit ball (which is a bounded symmetric domain). This was confirmed for JB^*-triples (corresponding to arbitrary bounded symmetric domains in complex Banach spaces) in the following result.

THEOREM 2.5 ([**1**]) *There is a universal constant $\tilde{K}$ ($\geq K$) such that for any two JB^*-triples A, B, and any bounded bilinear form T: $A \times B \to \mathbf{C}$ there exist norm one functionals φ on A and ψ on B such that*

$$|T(x, y)| \leq \tilde{K}\ \|T\|\ \|x\|_\varphi\ \|y\|_\psi \qquad x \in A, y \in B.$$

The following Proposition is a reformulation of the Radon-Nikodym theorem for JB^*-triples. The proof is a modification of the proof for von Neumann algebras ([**23**]).

PROPOSITION 2.6 *Let A be a JBW^*-triple and let $\varphi \in A_*$ with support tripotent e. If $0 \leq \psi \leq \varphi$ (in $A_2(e)_*$), then there is a unique y such that $0 \leq y \leq e$ (in $A_2(e)$) and*

$$\psi(x) = \varphi(\{xye\}) \text{ for all } x \in A.$$

Proof. (cf. [**23**, p. 160]) Let $y \in A_2(e)_{s.a.}$, then for all $x \in A_2(e)$,

$$\begin{aligned}\langle a_\varphi^*(y), \{exe\}\rangle &= \varphi\{\{exe\}ye\} = \varphi\{e\{xye\}e\}\\ &= \overline{\varphi\{xye\}} = \overline{\langle a_\varphi^*(y), x\rangle}.\end{aligned}$$

Thus, $a_\varphi^*(y)$ is a self-adjoint functional on $A_2(e)$ when y is self-adjoint. Also note that the map $y \mapsto a_\varphi^*(y)$ is weak*-weak continuous. Hence, $C := \{a_\varphi^*(y)\colon y \in [0, e]\}$ is a weakly compact convex subset of $A_2(e)_{*s.a.}$. Suppose that $\psi \notin C$. By the Hahn-Banach theorem there exist $a \in A_2(e)_{s.a.}$ and $t \in \mathbf{R}$ such that

$$\psi(a) > t \geq \varphi\{aye\} \text{ for all } y \in [0, e].$$

Let $a = a_+ - a_-$ be the Jordan decomposition of a. Then

$$\psi(a_+) \geq \psi(a) > t \geq \varphi\{a, \operatorname{supp}(a_+), e\} = \varphi(a_+),$$

which contradicts $\psi \leq \varphi$. Thus $\psi \in C$, proving existence.

Now let $y_i \in [0, e]$ be such that $\psi = a_\varphi^*(y_i)$. Then $\psi(y_1 - y_2) = \varphi\{y_1 - y_2, y_i, e\}$ which implies that $0 = \varphi\{y_1 - y_2, y_1 - y_2, e\}$ and $y_1 - y_2 = 0$.

Definition 2.7. Let A be a JBW^*-triple and let e be a tripotent. We shall say that a sesquilinear form b satisfies the *Radon-Nikodym property with respect to e* (e-RN property for short) if for any $\psi \in A_2(e)_{*+}$ with $\psi \leq b^*(e)$, there exists a *unique* $h \in [0, e]$ such that $\psi = b^*(h)$. In particular, $[0, b^*(e)] \subset b^*[0, e]$.

Corollary 2.8. Let A be a JBW^*-triple and let $\varphi \in A_*$ with support tripotent e. Then the form a_φ defined by Proposition 2.3 satisfies the e-RN property.

The complementary property to e-RN property for sesquilinear forms is the Order Positivity property defined as follows:

Definition 2.9. Let A be a JBW^*-triple and let e be a tripotent. We shall say that a sesquilinear form b satisfies the *Order Positivity property with respect to e* (e-OP property for short) if $b^* = b^*P_2(e)$ and $b(x, y) \geq 0$ for all $x, y \in A_2(e)^+$.

Note that if $b^* = b^*P_2(e)$, then b is e-OP if and only if $b^*[0, e] \subset [0, b^*(e)]$. Note also that if b is an e-OP form then $b^*(A_2(e)^+) \subset A_2(e)^{*+}$, i.e., b^*: $A_2(e) \to A_2(e)^*$ is a positive map.

Remark 2.10. *Let b be a sesquilinear e-OP form on a JBW^*-triple A. Then*

$$b(x, y) = b(P_2(e)x, P_2(e)y)$$

for any $x, y \in A$. Equivalently $b = bP_2(e)$.

Proof. Note that for any $y \in A_2(e)$ we have $b^*(y) \in A_2(e)^*$. Thus

$$b(P_2(e)x, P_2(e)y) = b^*(P_2(e)y)P_2(e)x = b^*(P_2(e)y)x = b^*(y)x = b(x, y).$$

Recall that a sesquilinear form b (for which $b^* = b^*P_2(e)$) is e-OP if

$$b^*[0, e] \subset [0, b^*(e)]$$

and a sesquilinear form is e-RN if

$$[0, b^*(e)] \subset b^*[0, e].$$

A form that combines both of these properties is the so called self-polar form, defined as follows:

Definition 2.11. A sesquilinear form s on a JB^*-triple A is called *self-polar* (resp. *weakly self-polar*) *relative to the tripotent* e if $s^* = s^*P_2(e)$ and

$$s^*[0, e] = [0, s^*(e)]$$

(resp. $s^*[0, e]$ is $\sigma(A_2(e)^*, A_2(e))$-dense in $[0, s^*(e)]$).

Self-polar forms were introduced by Connes ([4]) and Woronowicz ([29]). They were used by Connes to show that a von Neumann algebra can be represented as the set of derivations of a self-dual cone in a Hilbert space, a result which was generalized to Jordan algebras by Iochum ([19]).

In the case A is a JBW^*-algebra (with unit $1 = e$), or a von Neumann algebra, a self-polar form s_ψ with $s_\psi^*(e) = \psi$ exists and is unique for each positive faithful normal functional ψ on A. (See [29] for the von Neumann algebra case and [13] for the JBW-algebra case). For a JBW^*-triple A and $\varphi \in A_*$, let e be the support tripotent of φ and define $\psi = \varphi_{|A_2(e)}$. By Theorem 2.1, ψ is a positive faithful normal functional on the unital JBW^*-algebra $A_2(e)$, and so by use of Remark 2.10 (for the uniqueness part) we have:

PROPOSITION 2.12. *Let A be a JBW^*-triple and let $\varphi \in A_*$ have support tripotent e. Then there is a unique self-polar form s_φ relative to e such that $s_\varphi^*(e) = \varphi$.*

Before giving the main theorem below, we must introduce some terminology. As in [26], we say a sesquilinear form a on a vector space A is *represented* by (π, T) if

$$a(x, y) = (T\pi(x) \mid \pi(y))_H, \qquad x, y \in A.$$

Here π is a linear map of A onto a dense subset of a Hilbert space H and $T \in B(H)$. Obviously, a is positive if and only if T is a positive operator.

Obviously, any positive sesquilinear form can be represented in this way (by the identity operator) and in fact, by [26, Theorem 1.1], any two such forms can be represented on the same Hilbert space by *commuting* positive operators. The *geometric mean* $\sqrt{\alpha\beta}$ of two sesquilinear forms α, β is defined by [26, Theorem 1.2], as follows: if α is represented by (π, S) and β is represented by (π, T), then $\sqrt{\alpha\beta}$ is represented (unambiguously) by $(\pi, (ST)^{1/2})$.

In the context of JBW^*-triples, Woronowicz's maximality principle ([29, Theorem 1.1]) asserts that if s is a self-polar form relative to e, and b is an e-OP form with $b^*(e) \leq s^*(e)$, then $b(x, x) \leq s(x, x)$ for all $x \in A$.

If $T \in B(H)$ then the *pseudo-inverse* $T^{\#}$ is the possibly unbounded but densely defined operator with domain ran $T \oplus (\text{ran } T)^{\perp}$ defined by

$$T^{\#}(\xi \oplus \eta) = \xi'$$

where ξ' is the unique vector in $(\ker T)^{\perp}$ with $T\xi' = \xi$. The pseudo-inverse defined in this way has the following properties:

- $TT^{\#} \subset P_{\overline{\text{ran } T}}$
- $T^{\#}T = P_{(\ker T)^{\perp}}$
- $TT^{\#}T = T$

Note that if T is self-adjoint, then $TT^{\#} = T^{\#}T$, and if $T \geqslant 0$ then $T^{\#} \geqslant 0$.
We are now ready to state and prove the main theorem of this section.

THEOREM 2.13 *Let a and b be positive sesquilinear forms on a JBW*-triple A and let $\varphi \in A_*$ have support tripotent e. Suppose that a and b satisfy the following:*

- $a^*(e) = b^*(e) = \varphi$;
- *a satisfies e-RN;*
- *b satisfies e-OP.*

Then there exists a positive sesquilinear form h with $h^*(e) = \varphi$ which satisfies e-OP, such that b is the geometric mean $\sqrt{ah}$ of a and h. Moreover

$$b(x, x) \leqslant a(x, x) \tag{4}$$

for any $x \in A$.

Equation (4) states that the seminorm defined by any e-RN form is larger than the seminorm defined by an e-OP form. In particular, the seminorm defined by a self-polar form is larger than all seminorms defined by an e-OP form and is smaller than all seminorms defined by an e-RN form.

Proof. For any $x \in [0, e]$ the functional $b^*(x)$ is positive on $A_2(e)$ and therefore by Proposition 2.6 we can define $v(x) \in [0, e]$ such that $b^*(x) = a^*(v(x))$. Extend v by linearity to all of $A_2(e)$ and to A by $v = vP_2(e)$. Thus we have

$$b(x, y) = a(x, v(y)) \quad \text{for} \quad x, y \in A. \tag{5}$$

In particular, this implies (put $x = e$) that $\varphi(y) = \varphi(v(y))$. Now define

$$h(x, y) := a(v(x), v(y)). \tag{6}$$

Then h is easily seen to be sesquilinear. For example,

$$h(x_1+x_2,y)=a(v(x_1+x_2),v(y))=\overline{a(v(y),v(x_1+x_2))}$$
$$=\overline{b(v(y),x_1+x_2)}=h(x_1,y)+h(x_2,y).$$

The positivity of h follows from the positivity of a. Since $h(x,e)=a(v(x),v(e))=b(v(x),e)=\langle b^*(e),v(x)\rangle=\varphi(v(x))=\varphi(x)$ we have $h^*(e)=\varphi$. To show that h satisfies e-OP note first that from the definition of h we have $h^*=h^*P_2(e)$. Let $x,\ y\in A_2(e)^+$. Then $v(x)\in A_2(e)^+$ and therefore $h(x,y)=a(v(x),v(y))=b(v(x),y)\geq 0$ implying that h is e-OP.

As mentioned above, using [26], we can choose representations (π, T) of a and (π, S) of b by commuting bounded positive operators T and S on a Hilbert space H. From (5), we conclude

$$S\pi=T\pi v \text{ and therefore } S(\pi(A))\subset T(\pi(A)). \tag{7}$$

From (7), we have $T^{\#}S\pi=P_{(\ker T)^\perp}\pi v$, and the operator $T^{\#}S^2$ represents a positive sesquilinear form. This form is the one defined in (6) since

$$h(x,y)=a(v(x),v(y))=(T\pi v(x)\mid \pi v(y))$$
$$=(S\pi(x)\mid \pi v(y))=(S\pi(x)\mid P_{(\ker T)^\perp}\pi v(y)+P_{\ker T}\pi v(y))$$
$$=(S\pi(x)\mid P_{(\ker T)^\perp}\pi v(y))=(T^{\#}S^2\pi(x)\mid \pi(y)),$$

where the last step follows by (7). Thus h is represented by $T^{\#}S^2$. Since a is represented by T, $\sqrt{ah}$ is represented by $(TT^{\#}S^2)^{1/2}=P_{ran\,T}S$. By (7), b is also represented by $P_{ran\,T}S$, implying $b=\sqrt{ah}$.

From the Woronowicz maximality principle mentioned above, it is enough to prove (4) for the case when $b=s_\varphi$ is the self-polar form associated with φ. By use of the first part of the theorem for this b we obtain π, h, S, T as above. By the maximality principle, we have $h(x,x)\leq s_\varphi(x,x)$, which asserts that $T^{\#}S^2\leq S$. Since T and S are positive commuting operators it follows that $S\leq T$. To see this, let P denote the projection on the closure of the range of T. Then $S=PS+P_{\ker T}S=PS$ by (7). Since P corresponds to the characteristic function of the range of the function representing T, $PS\leq T$. So $s_\varphi(x,x)=(S\pi(x)\mid\pi(x))\leq (T\pi(x)\mid\pi(x))=a(x,x)$.

COROLLARY 2.14 *Let a be a positive sesquilinear form on a JBW^*-triple A which satisfies e-RN with respect to a tripotent e. Then the seminorm defined by a is a norm on $A_2(e)$. Hence a is non-degenerate on $A_2(e)$ and the map v is uniquely determined by* (5).

Moreover, if $e=e(\varphi)$, then a^{*-1}, initially defined on $[0,\varphi]$, *extends to a positive linear map of the complex linear span of Face* (φ) *into $A_2(e)$.*

The Cauchy-Schwarz inequality holds for any positive sesquilinear form, and by Corollary 2.8, a_φ is e-RN. Therefore we can obtain another inequality of Grothendieck type for JB^*-triples. Note that this version has the constant 1 and involves one functional and one JBW^*-triple.

COROLLARY 2.15 *Let b be a positive sesquilinear form on a JBW^*-triple A which is e-OP and satisfies $b^*(e) = \varphi$. Then*

$$|b(x, y)| \leq \|x\|_\varphi \, \|y\|_\varphi$$

for any $x, y \in A$, where the norm $\|\cdot\|_\varphi$ is defined as in Proposition 2.3.

If we interpret (5) in the case of a von Neumann algebra (resp. JBW^*-algebra), the map v emerges as the analog of the square root of the modular operator (resp. integral of the cosine group).

Indeed, if φ is a normal state on a von Neumann algebra, its self-polar form is given by (cf. [4])

$$s_\varphi(x, y) = \langle \pi_\varphi(x)\Omega \mid \Delta^{1/2}\pi_\varphi(y)\Omega\rangle \tag{8}$$

where π_φ is the cyclic representation associated with φ and Ω is its cyclic vector. In this context, $a_\varphi(x, y)$ is given by $\varphi(y^*x)$, so that by (5),

$$\begin{aligned} s_\varphi(x, y) &= a_\varphi(x, v(y)) = \varphi(v(y)^*x) = \langle \pi_\varphi(v(y)^*x)\Omega \mid \Omega\rangle \\ &= \langle \pi_\varphi(x)\Omega \mid \pi_\varphi(v(y))\Omega\rangle = \langle \pi_\varphi(x)\Omega \mid \tilde{v}(\pi_\varphi(y)\Omega\rangle. \end{aligned}$$

Moreover, the operator v can be identified with a positive norm 1 operator on $H_\varphi^\#$, the completion of A with respect to the inner product a_φ. Writing $H_\varphi^\natural$ for the completion of A with respect to the inner product s_φ, it can be shown that $v^{1/2}$ determines a unitary mapping of $H_\varphi^\natural$ onto $H_\varphi^\#$, analogously to $\Delta^{1/4}$.

On the other hand, if φ is a normal state on a JBW^*-algebra, its self-polar form is expressed in terms of a hyperbolic cosine group as

$$s_\varphi(x, y) = \int_{-\infty}^{\infty} \varphi(\rho_t(x) \circ y^*)(\cosh \pi t)^{-1}\, dt, \tag{9}$$

where ρ_t is the one-parameter group constructed in [13]. In this context, $a_\varphi(x, y) = \varphi\{x, y^*, e\} = \varphi(x \circ y^*)$. By (5), v is self-adjoint with respect to a_φ. Thus

$$s_\varphi(x, y) = a_\varphi(x, v(y)) = a_\varphi(v(x), y) = \varphi(v(x) \circ y^*),$$

and (9) suggests that $v(x) = \int_{-\infty}^{\infty} \rho_t(x)(\cosh \pi t)^{-1}\, dt.$

The sesquilinear form $a_\varphi P_2(e)$ is not e-OP in general. The following proposition gives several equivalent conditions when this holds. The proof is a modification of the proof for Jordan algebras ([**24**]).

PROPOSITION 2.16. *Let A be a JBW^*-triple and let $\varphi \in A_*$ have norm 1 and support tripotent e. The following are equivalent, and each one serves as the definition of a trace on a JBW^*-triple.*

1. $\varphi(Q(x)(\{yye\})) = \varphi(Q(y)(\{xxe\}))\forall x, y \in A_2(e)_{s.a.}$;
2. *for any tripotent $v \leqslant e$, we have $P_1(v)\varphi = 0$, i.e.*

$$\varphi(x) = \varphi(S(v)x) = \varphi(P_2(v)x - P_1(v)x + P_0(v)x), \qquad \forall x \in A;$$

3. $\varphi(\{xye\}) \geqslant 0 \forall x, y \in A_2(e)^+$, *i.e.* $a_\varphi P_2(e)$ *is e-OP*;
4. $\varphi(\{xye\}) \leqslant \|x\|\ \varphi(|y|)\forall x,\ y \in A_2(e)_{s.a.}$;
5. $\varphi(\{xyz\}) = \varphi(\{xzy\})\forall x \in A_2(e), \forall y, z \in A_2(e)_{s.a.}$;
6. $\varphi(Q(x)y) = \varphi(\{xxy\})\forall x, y \in A_2(e)_{s.a.}$;
7. $a_\varphi P_2(e)$ *is the self-polar form associated to* φ.

Proof. 1$\Rightarrow$2 Let $v \leqslant e$, $\tilde{v} = e - v$. It is easily checked that $Q(v - \tilde{v})Q(e) = S(v)|_{A_2(e)}$. Now let $x \geqslant 0$, say $x = z^2$. Then

$$\begin{aligned}\langle \varphi, Q(v - \tilde{v})x\rangle &= \langle \varphi, Q(v - \tilde{v})Q(z)e\rangle = \langle \varphi, Q(z)\{v - \tilde{v}, v - \tilde{v}, e\}\rangle \\ &= \langle \varphi, Q(z)e\rangle = \varphi(x).\end{aligned}$$

Hence, $\varphi(S(v)x) = \langle \varphi, Q(v - \tilde{v})Q(e)x\rangle = \varphi(x)$ for all $x \in A_2(e)^+$, and by linearity $\varphi(S(v)x) = \varphi(x)$ for all $x \in A_2(e)$. Since $\varphi(x) = \varphi(P_2(e)x)$ this holds for all $x \in A$.
2$\Rightarrow$3 Let $p \leqslant e$ be a projection, $x \geqslant 0$. Then by 2.,

$$\begin{aligned}\varphi\{xpe\} &= \varphi(P_2(p)\{xpe\} + P_0(p)\{xpe\}) = \varphi(P_2(p)\{xpe\}) \\ &= \varphi(P_2(p)x) \geqslant 0.\end{aligned}$$

Note that if $v \leqslant e$ and $x \geqslant 0$, then $x = \{zze\}$ for some z and $P_2(v)\{zze\} = P_2(v)\{zzv\} = \{z_2z_2v\} + \{z_1z_1v\} \geqslant 0$, where $z_j = P_j(v)z$ (cf. [**8**, p. 73]). Therefore, $P_2(v)x \geqslant 0$. It follows that $\varphi\{xye\} \geqslant 0$ whenever $y = \sum t_ip_i$ for $t_i > 0$ and p_i are orthogonal projections. By spectral theory and weak*-continuity, $\varphi\{xye\} \geqslant 0$ for all $x \geqslant 0$, $y \geqslant 0$.
3$\Rightarrow$4 Let x be self-adjoint and y positive. Then $\|x\|\ e - x \geqslant 0$, so by 3., $\varphi\{\|x\|\ e - x,\ y,\ e\} \geqslant 0$ and thus $\varphi\{xye\} \leqslant \|x\|\ \varphi(y)$. For y self-adjoint, say $y = y_+ - y_-$,

$$\varphi\{xye\} = \varphi\{xy_+e\} - \varphi\{xy_-e\} \leqslant \|x\|\ \varphi(y_+ + y_-) = \|x\|\ \varphi(|y|).$$

4$\Rightarrow$5 In the JB-algebra $A_2(e)_{s.a.}$, $\{xye\} = \{xey\}$, so by [**24**, (iv)$\Rightarrow$(vi)] we have

$$\varphi(x \circ (y \circ z)) = \varphi(z \circ (y \circ x))\forall x, y, z \in A_2(e)_{s.a.}.$$

Since $\{xyz\} = (x \circ y) \circ z + (z \circ y) \circ x - (x \circ z) \circ y$, 5. follows.
5 ⇒ 6 Trivial.
6 ⇒ 1 Since $\{xxy\} = x^2 \circ y$, from [24, (vii) ⇒ (i)] we have $\varphi\{x, y^2, x\} = \varphi\{y, x^2, y\}$ for all $x, y \in A_2(e)_{s.a.}$. But $y^2 = \{yye\}$, so 1. follows.
3 ⇔ 7 By Corollary 2.8, a_φ is a e-RN form and therefore it is self-polar if and only if it is e-OP.

3. Schur multipliers and Hilbertian seminorms of an atomic functional

Recall ([8, Theorems 1 and 2]) that every JBW^*-triple and its predual have atomic decompositions. Let A be an atomic JBW^*-triple and let φ be a normal functional on A. Then

$$\varphi = \sum_j s_j f_j,$$

where the f_j's form an orthogonal family of extreme points of the unit ball of A_* and the s_j's are nonnegative scalars with $\sum s_j = \|\varphi\|$.

There is ([8, Lemma 2.11]) a contractive conjugate linear map π: $A_* \to A$ defined by

$$\pi\Big(\sum \alpha_j f_j\Big) = \sum \overline{\alpha_j} v_j$$

for each finite linear combination $\sum \alpha_j f_j$ of extreme points f_j of the unit ball of A_*, where $v_j = e(f_j)$ is the support tripotent of f_j. The map π is injective and gives rise in turn to a sesquilinear form $\langle f \mid g \rangle_\pi := \langle f, \pi(g) \rangle$ which is positive definite.

Because of the importance of the map π, we indicate the proof of the last two statements. If $f \in A_*$ then $f \in F_e$ where e is the support tripotent of f, and therefore f is the norm limit of a convex combination of extreme points of F_e (see [10, Proposition 3.4] for example). Then $\langle f \mid f \rangle_\pi = \lim \langle f, \sum \alpha_j e_j \rangle \geqslant 0$ since $e_j \leqslant e$ implies $\langle f, e_j \rangle \geqslant 0$. Thus, $\langle \cdot \mid \cdot \rangle_\pi$ is positive semi-definite. Because of the existence of grids (cf. [5]) generating the atomic JBW^*-triple A, $\langle \cdot \mid \cdot \rangle_\pi$ is positive definite. If $\pi(f) = 0$, then $\langle f \mid f \rangle_\pi = 0$ so $f = 0$.

We shall call this inner product the *tracial* inner product. For x, $y \in \pi(A_*)$, which is known to be weak*-dense in A, define $\langle x \mid y \rangle_\pi$ to be $\langle \pi^{-1} x \mid \pi^{-1} y \rangle_\pi$. This is well defined since π is injective. A large number of Hilbertian seminorms associated to a given functional can be obtained in the following way:

Definition 3.1. Let m: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a function of two variables s, t

and P: $\mathbf{R}\times\mathbf{R}\to\mathscr{L}(A)$ be a projection-valued function of the same variables. We shall call a linear map

$$\mu=\sum_{s,t} m(s,t)P(s,t)$$

on A (when defined) a *Schur multiplier* associated to the pair (m, P). We also define a sesquilinear form on the dense subset $\pi(A_*)$ of A by

$$\langle x \mid y\rangle_\mu=\sum_{s,t} m(s,t)\langle P(s,t)x \mid P(s,t)y\rangle_\pi=\langle \mu x \mid y\rangle_\pi \textit{ for } x, y\in\pi(A_*).$$

Note that a Schur multiplier is positive if and only if $m(s,t)\geq 0$ whenever $P(s,t)\neq 0$.

We now consider several examples illustrating this construction. In these examples we shall have $\mu(\pi(A_*))\subset\pi(A_*)$ and in fact $\mu(A)\subset \pi(A_*)$, which gives rise to a map $\tilde{\mu}:=\pi^{-1}\circ\mu$ of A into A_*.

Example 3.2. Let A be an atomic von Neumann algebra and let φ be a positive normal functional on A given by a positive trace class operator $a=\sum s_j e_{s_j}$. Define $P(s,t)x=e_s x e_t$, where $e_s=e_{s_j}$ if $s=s_j$ and $e_s=0$ otherwise.

a) if $m_1(s,t):=s$ and μ corresponds to (m_1, P), then

$$\langle x \mid y\rangle_\mu=\langle x \mid y\rangle^\flat=\varphi(xy^*);$$

b) if $m_2(s,t):=t$ and μ corresponds to (m_2, P), then

$$\langle x \mid y\rangle_\mu=\langle x \mid y\rangle^\#=\varphi(y^*x).$$

Proof. We use the notation $e_j=e_{s_j}$ and $P(i,j)=P(s_i,s_j)$. a) Recall that $\langle x \mid y\rangle^\flat=\varphi(xy^*)$ and that $\varphi=\operatorname{Tr}(a\cdot)$. Thus

$$\begin{aligned}\langle x \mid y\rangle^\flat=\varphi(xy^*)&=\operatorname{Tr}\Big(\sum_j s_j e_j x y^*\Big)\\&=\sum s_j\operatorname{Tr}[e_j x y^* e_j]=\sum_j\sum_i s_j\operatorname{Tr}[e_j x e_i y^* e_j]\\&=\sum_{i,j} m_1(s_j,s_i)\operatorname{Tr}[P(j,i)x(P(j,i)y)^*]\\&=\sum_{i,j} m_1(s_j,s_i)\langle P(j,i)x \mid P(j,i)y\rangle_\pi.\end{aligned}$$

Statement b) is proved similarly.

Example 3.3. Let A be an atomic JBW^*-triple and let $\varphi=\sum_j s_j f_j$ be a normal functional on A given by an orthogonal family f_j of extreme points of the unit ball of A_* and scalars $s_j \geq 0$ with $\sum s_j = \|\varphi\| = 1$. Let v_j denote the support tripotent of f_j and $e = \sum v_j$. Define the projection valued function $P(s, t)$ by the joint Peirce decomposition relative to the family $\{v_j\}$ as follows

$$P(s,t) = \begin{cases} P_2(v_j) & \text{if } s = t = s_j \\ P_1(v_i)P_1(v_j) & \text{if } s = s_i \neq s_j = t \\ P_1(e)P_1(v_j) & \text{if } s = 0,\ t = s_j \\ P_0(e) & \text{if } s = t = 0. \end{cases}$$

Then we have three examples corresponding to the choice of m.

a) if $m(s, t) = (s + t)/2$ is the arithmetic mean of s and t, then

$$\langle x \mid y \rangle_\mu = \varphi(\{xye\}) = a_\varphi(x, y)$$

(where a_φ is defined as in Proposition 2.3);

b) if $m(s, t) = \sqrt{st}$ is the geometric mean of s and t, then

$$\langle x \mid y \rangle_\mu = s_\varphi(x, y)$$

is the self-polar form of φ;

c) if $m(s, t) = 2(s^{-1} + t^{-1})^{-1}$ is the harmonic mean of s and t, then

$$\langle x \mid y \rangle_\mu = h_\varphi(x, y)$$

(where h_φ is the sesquilinear form h defined in Theorem 2.13).

Moreover, in all cases we have

$$\langle x \mid x \rangle_\mu \leq \|x\|^2 \tag{10}$$

and therefore the sesquilinear form defined by Definition 3.1 has a continuous extension from $\pi(A_*)$ to its norm closure[1].

Proof. a) If $x = x_2 + x_1 + x_0$ is the Peirce decomposition of $x \in \pi(A_*)$ with respect to e, then by Peirce calculus and the fact that φ vanishes on $A_1(e) \oplus A_0(e)$,

$$\varphi(x, y) = \langle \varphi, \{xye\} \rangle = \langle \varphi, \{x_2 y_2 e\} \rangle + \langle \varphi, \{x_1 y_1 e\} \rangle. \tag{11}$$

On the other hand,

$$\langle x \mid y \rangle_\mu = \sum m(s, t) \langle P(s, t)x \mid P(s, t)y \rangle_\pi \tag{12}$$

$$= \sum_j s_j \langle P_2(v_j)x \mid P_2(v_j)y \rangle_\pi \tag{13}$$

$$+ \sum_{i<j} \frac{s_i + s_j}{2} \langle P_1(v_i)P_1(v_j)x \mid P_1(v_i)P_1(v_j)y \rangle_\pi \tag{14}$$

$$+ \sum_j \frac{s_j}{2} \langle P_1(e)P_1(v_j)x \mid P_1(e)P_1(v_j)y \rangle_\pi \tag{15}$$

[1] In fact, in these three examples, the sesquilinear form $\langle x \mid y \rangle_\mu$ extends to all of A.

We shall show that the first term on the right side of (11) is the sum of (13) and (14), and the second term equals (15). To do this we shall need the following fact about spin factors (cf. [10]).

LEMMA 3.4. *Let u_i, $\tilde{u}_i$ be a spin grid for a spin factor U. For x, $y \in U_1(u_1)$*

$$\langle \hat{u}_1, \{xyu_1\}\rangle = \tfrac{1}{2}\langle x \mid y\rangle_\pi.$$

Proof. Writing $x = \sum_{i\neq 1} (x_i u_i + \tilde{x}_i \tilde{u}_i)$ and $y = \sum_{i\neq 1} (y_i u_i + \tilde{y}_i \tilde{u}_i)$, we have

$$\{xyu_1\} = \sum_{i\neq 1}\sum_{j\neq 1} [x_i \bar{y}_j \{u_i u_j u_1\} + \tilde{x}_i \bar{y}_j \{\tilde{u}_i u_j u_1\} + x_i \bar{\tilde{y}}_j \{u_i \tilde{u}_j u_1\} + \tilde{x}_i \bar{\tilde{y}}_j \{\tilde{u}_i \tilde{u}_j u_1\}].$$

By Peirce calculus this reduces to

$$\{xyu_1\} = \frac{1}{2}\sum_{k\neq 1} [x_k \bar{y}_k + \tilde{x}_k \bar{\tilde{y}}_k] u_1$$

and therefore $\langle \hat{u}_1, \{xyu_1\}\rangle = \frac{1}{2}\sum_{k\neq 1} [x_k \bar{y}_k + \tilde{x}_k \bar{\tilde{y}}_k] = \frac{1}{2}\langle x \mid y\rangle_\pi$.

Returning to the proof of Example 3.3, if we write $x_2 = \sum_{1\leq j\leq k} x_{jk}$ for the joint Peirce decomposition of x_2, and similarly for y_2, we have

$$\langle \varphi, \{x_2 y_2 e\}\rangle = \sum_i s_i \Big\langle f_i, \sum_{j\leq k}\sum_{p\leq q} \{x_{jk} y_{pq} v_i\}\Big\rangle.$$

By Peirce calculus, any nonzero term in this sum must satisfy $i \in \{j, k\} \cap \{p, q\}$. Now $x_{ii} = \langle \hat{v}_i, x_2\rangle v_i$, so $\langle f_i, \{x_{ii} y_{ii} v_i\}\rangle = \overline{\langle \hat{v}_i, x_2\rangle}\langle \hat{v}_i, y_2\rangle = \langle P_2(v_i)x \mid P_2(v_i)y\rangle_\pi$. On the other hand, by Lemma 3.4, for $i \neq k$, $\langle f_i, \{x_{ik} y_{ik} v_i\}\rangle = \frac{1}{2}\langle P_1(v_i)P_1(v_k)x \mid P_1(v_i)P_1(v_k)y\rangle_\pi$. This shows that the first term on the right side of (11) is the sum of (13) and (14).

Finally, writing $x_1 = \sum x_j$ with $x_j \in A_1(e) \cap A_1(v_j)$, then

$$\langle \varphi, \{x_1 y_1 e\}\rangle = \sum_{p,j,k} s_p \langle f_p, \{x_j y_k v_p\}\rangle = \sum_p s_p \langle f_p, \{x_p y_p v_p\}\rangle.$$

It suffices to verify that $\langle f_p, \{x_p y_p v_p\}\rangle = \frac{1}{2}\langle x_p \mid y_p\rangle$.

b) Since φ restricts to a faithful normal positive functional on the *atomic JBW^*-algebra* $A_2(e)$, from [13] we know that there is an element $h = \sum s_j v_j \in A_2(e)^+$ and a faithful trace $\tau = \sum f_j$ such that the self-polar form associated with φ is given by

$$s_\varphi(a, b) = \tau(\{h^{\frac{1}{2}} a h^{\frac{1}{2}}\} \circ b^*).$$

That is, $s_\varphi^*(e) = \varphi$ and $s_\varphi^*[0, e] = [0, \varphi]$.

It is easy to verify the following by using the Peirce calculus.

1. $\tau(x) := \langle x \mid e \rangle_\pi$ is a faithful normal trace on the JBW^*-algebra $A_2(e)$.
2. $\tau(x \circ h) = \varphi(x)$.
3. $s_\varphi(x, y) = \langle x \mid y \rangle_\mu$.

c) The proof of c) follows directly from the proof of Theorem 2.13.

By use of Theorem 2.13 and the definition of a_φ for any $x \in A$ we have:

$$h_\varphi(x, x) \leq s_\varphi(x, x) \leq a_\varphi(x, x) \leq \|\varphi\| \, \|x\|^2 \leq \|x\|^2$$

proving (10).

Note that in Example 3.3, the operator v is also a Schur multiplier, given by

$$m(s, t) = \frac{2\sqrt{st}}{s + t}.$$

Indeed, in (5)

$$s_\varphi(x, y) = \sum_{s,t} \sqrt{st} \langle P(s, t)x \mid y \rangle_\pi,$$

and setting $v' = \sum_{s,t} \frac{2\sqrt{st}}{s + t} P(s, t)$, we have

$$a_\varphi(x, v'(y)) = \sum_{s,t} \frac{s + t}{2} \langle P(s, t)x \mid v'(y) \rangle_\pi = s_\varphi(x, y).$$

Now use the uniqueness of v.

In Example 3.3, three widely used means have appeared (geometric, arithmetic, harmonic). Two of them correspond to sesquilinear forms which can be described intrinsically (a_φ by the algebraic structure, s_φ by the order structure).

Problem 1. *Can h_φ be described and constructed by some intrinsic properties?*

A positive constructive answer would lead to a constructive method of obtaining the self-polar form.

In Example 3.3, the operator $\mu = \sum_{s,t} m(s, t)P(s, t)$ acts as a Schur multiplier relative to the joint Peirce decomposition $A_2(e) = \sum \oplus A_{ij}$ of $A_2(e)$ with respect to the orthogonal family of tripotents $\{v_j\}$. In this connection, we have the following analog of the Schur product theorem, [**18**, 5.2.1]. To prove it we need the following lemma.

Recall that in a JB^*-algebra, if a is a self-adjoint element, then $Q(a)$ is a *positive* conjugate linear operator. For non-self-adjoint elements, we have the following result.

LEMMA 3.5. *Let a be a* normal *element in a JB^*-algebra A (meaning* *that a and a^* operator commute). Then $Q(a)^2$ is a positive (linear) operator. Also, for arbitrary a, $Q(a, a^*)$ is a positive conjugate linear operator on A.*

Proof. Writing $a = x + iy$ with x, y self-adjoint, it follows that x and y operator commute in the JB-algebra $A_{s.a.}$. Identifying $A_{s.a.}$ with its canonical image in the JBW^*-algebra A^{**}, x, y together generate an associative JBW-subalgebra of A^{**} (cf. [**14**, page 44]). So for a given $\epsilon > 0$, there exist projections $e_1, \dots, e_n \in A^{**}$ and real numbers λ_j, μ_j such that

$$\left\| a - \sum (\lambda_j + i\mu_j)e_j \right\| > \epsilon.$$

Set $\alpha_j = \lambda_j + i\mu_j$ and $b = \sum \alpha_j e_j$. As operators on A^{**}, $Q(b)^2 = Q(|b|^2)$, where $|b| = \sum |\alpha_j|\, e_j$. Indeed, $Q(b)x = \{bxb\} = \sum \alpha_j\alpha_k\{e_j x e_k\}$, so that

$$Q(b)^2x = \{b, Q(b)x, b\} = \sum \alpha_p\alpha_q\{e_p, Q(b)x, e_q\}$$

$$= \sum \alpha_p\alpha_q\overline{\alpha_j\alpha_k}\{e_p, \{e_j x e_k\}, e_q\} = \sum |\alpha_j\alpha_k|^2\, \{e_j\{e_j x e_k\}e_k\},$$

since $\{A_{pp}, A_{jk}, A_{qq}\} = 0$ unless $\{j, k\} = \{p, q\}$.

Finally,

$$Q(a)^2x = \{a, \{axa\}, a\} = \lim_{\epsilon\to 0} \{b_\epsilon, \{b_\epsilon x b_\epsilon\}, b_\epsilon\}$$

$$= \lim_{\epsilon\to 0} Q(b_\epsilon)^2x = \lim_{\epsilon\to 0} Q(|b_\epsilon|^2)x \geqslant 0,$$

if $x \geqslant 0$.

The last statement of the lemma follows from [**14**, 3.3.6], since with $a = x + iy$, where x and y are self-adjoint, $Q(a, a^*) = Q(x) + Q(y)$.

PROPOSITION 3.6. *Let A, φ, P be as in Example* 3.3 *and let $P_{ij} = P(s_i, s_j)$. Let $[m_{ij}]$ be a matrix (possibly infinite) such that $m_{0j} = m_{i0} = 0$. Then $\mu = \sum_{i,j} m_{ij}P_{i,j}$ is a positive operator on the JB^*-algebra $A_2(e)$,* i.e.,

$$\mu(A_2(e))^+ \subset A_2(e)^+$$

(equivalently, $\langle x \mid y\rangle_\mu$ is e-OP) if and only if $[m_{ij}]$ is a positive semi-definite matrix.

Proof. Let T be the operator represented by the matrix $[m_{ij}]$. If T is a one-dimensional projection, then $\mu_{ij} = a_i\bar{a}_j$ with $\sum |a_j|^2 < \infty$ and thus $\mu(\sum x_{ij}) = \sum a_i\bar{a}_j x_{ij}$. Since $x_{ij} = 4\{v_i, \{v_i x v_j\}, v_j\}$, and since $\{a_j\}$ is bounded, $a := \sum a_j v_j$ converges in the weak*-topology. By joint weak*-continuity of the triple product on bounded sets, this sum equals $4Q(e)Q(a, a^*)x$ (cf. [**2**]). Thus μ is positive by Lemma 3.5.

By the spectral theorem for positive operators, it suffices to prove the Proposition in the case when T is a projection, and it is trivial by the previous paragraph in case the projection is of finite rank. Now any projection T on a separable Hilbert space is the strong limit of a sequence of finite rank projections T_n with positive Schur multiplier μ_n. If $x = \sum x_{ij} \geq 0$, then $P_2(v_1 + \cdots v_m)x \geq 0$ for every $m \geq 1$, and so $\mu x = \lim_n \mu_n x = \lim_n \lim_m \mu_n P_2(v_1 + \cdots v_m)x \geq 0$.

We now prove the converse. For $n \geq 1$, $v_1, \ldots, v_n$ is a finite family of orthogonal minimal tripotents. Hence there exist elements v_{ij} with $i<j$ such that, setting $v_{ii} = v_i$, the family $\{v_{ij}\}$ forms a symplectic grid, that is, the span of $\{v_{ij}\}$ is isometrically isomorphic to the JBW^*-triple $S_n(\mathbf{C})$ of all n by n symmetric matrices. Let $w := \sum v_{ij}$, which corresponds to a positive multiple of a minimal tripotent in $S_n(\mathbf{C})$ (namely, the matrix with a 1 in every entry). Then μw is positive and corresponds to $[m_{ij}]$.

We now indicate how to define a large family of Schur multipliers on an arbitrary JBW^*-triple A from a given functional $\varphi \in A_*$. We shall simply apply the functional calculus for sesquilinear forms ([**26**, Theorem 1.2]) to a_φ and s_φ. That is, for each f in the class J defined in [**26**], we obtain, by [**26**, Theorem 1.2], a sesquilinear form $b_f = f(a_\varphi, s_\varphi)$. We can then determine the conjugate linear map $\tilde{\mu}$: $A \to A_*$ by the rule $\langle x, \tilde{\mu} y\rangle = b_f(x, y)$. The class J consists of all Borel measurable functions on $[0, \infty) \times [0, \infty)$ which are homogeneous ($f(\lambda r, \lambda s) = \lambda f(r, s)$, λ, r, $s \in [0, \infty)$) and bounded on compact sets.

Note that this definition is consistent with Definition 3.1. Indeed, if φ is an atomic functional and the $P(s, t)$ are chosen as in Example 3.3, then (a) and (b) of Example 3.3 give rise to Schur multipliers

$$\mu_a = \sum_{i,j} \frac{s_i + s_j}{2} P_{ij} \quad \text{and} \quad \mu_s = \sum_{i,j} \sqrt{s_i s_j} P_{ij} \quad \text{where} \quad P_{ij} = P(s_i, s_j).$$

Any $f \in J$ gives rise to a sesquilinear form $b_f = f(a_\varphi, s_\varphi)$ as follows:

$$b_f(x, y) = (f(S, T)h(x) \mid h(y))_H \tag{16}$$

where (h, S) and (h, T) represent a_φ and s_φ respectively. It follows from (16) and the commutativity of S and T that b_f determines a Schur multiplier of the form

$$\mu_f = \sum_{i,j} f\left(\frac{s_i + s_j}{2}, \sqrt{s_i s_j}\right) P_{ij}.$$

Note that any $g \in J$ also determines a Schur multiplier of the form $\mu_g = \sum_{i,j} g(s_i, s_j) P_{ij}$. This amounts to solving the pair of equations

$$2\sigma = s + t, \qquad \tau^2 = st.$$

The representation (16) and the closed graph and uniform boundedness theorems show that the sesquilinear form b_f is bounded and thus determines a map $\tilde{\mu}_f$: $A \to A^*$.

Problem 2. If b_f is e-OP, does $\tilde{\mu}_f$ map A into its predual?

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