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TEST OF CRANKING PLUS RPA ON AN EXACTLY SOLUBLE BACKBENDING MODEL*

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Self-consistent cranking with additional RPA correlations is tested on the exactly soluble R(5) model of Krumlinde and Szymański. Excellent agreement between approximate and exact solutions is obtained excepting a couple of points very near the critical spin.

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Much attention has recently been focused on the behavior of the nuclear moment of inertia at high spin [1]. The sudden increase in the moment of inertia and the occurrence of "backbending" has been qualitatively accounted for in a nice way through self-consistent Hartree-Bogoliubov cranking calculations [2]. In this connection, it is natural to wonder whether the self-consistent cranking (SCC) model is apriori a sufficiently accurate calculational tool, especially in the critical region. The correspondence with experiment achieved thus far could conceivably be fortuitous since the effective interactions were somewhat crude. Previous theoretical estimates of the accuracy of the SCC model depend on infinite power-series expansions in the angular momentum,

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which cannot be expected to converge in the critical region [3].

Our first aim here is to shed light on this question by applying the SCC approximation to the R(5) model of Krumlinde and Szymański and comparing the results with exact solutions [4,5]. Our second aim is to test the idea of introducing random-phase approximation (RPA) correlations as the next improvement to the SCC calculation of yrast energies. The importance of particle-number and angular-momentum conservation in calculations of yrast levels has recently been emphasized [6]. This is usually accomplished by generator-coordinate techniques. The RPA automatically takes care of the conservation laws within the accuracy of the approximation and in a much more simple way. It can also describe the level structure above the yrast line and provide a simple way to calculate the yrast cascade. As a bonus, the RPA provides a check on the stability of the SCC solution, which is of special interest for the backbending parts of the trajectories.

The R(5) model consists of 2Ω identical fermions interacting via a pairing force, distributed among two 2Ω-fold degenerate single-particle levels separated by an amount 2ε and coupled to an external rotor with fixed moment of inertia a^{-1}. The motion is confined to two dimensions. The Hamiltonian is

$$H = \frac{1}{2} a(I-j_x)^2 + H_{sp} + H_p \quad , \tag{1}$$

where

$$j_x = \frac{1}{2} \sum_{\nu=1}^{\Omega} (a_\nu^\dagger b_\nu - a_{-\nu}^\dagger b_{-\nu} + \text{h.c.}) \quad ,$$

$$H_{sp} = \varepsilon \sum_{\nu=1}^{\Omega} (a_\nu^\dagger a_\nu + a_{-\nu}^\dagger a_{-\nu} - b_\nu^\dagger b_\nu - b_{-\nu}^\dagger b_{-\nu}) \quad ,$$

$$H_p = -G \sum_{\nu=1}^{\Omega} (a_\nu^\dagger a_{-\nu}^\dagger + b_\nu^\dagger b_{-\nu}^\dagger) \sum_{\nu=1}^{\Omega} (a_{-\nu} a_\nu + b_{-\nu} b_\nu) . \qquad (2)$$

The total angular momentum of the system is I, the particle angular momentum is j_x, while H_{sp} is the single-particle Hamiltonian and H_p the pairing force. The operator $a_\nu^\dagger$ creates a fermion in a substate of the upper level and $b_\nu^\dagger$ in the lower level, the indices ν and $-\nu$ distinguishing time-reversal conjugate states. Since (1) is composed of generators of the group R(5), the exact diagonalization is greatly simplified as discussed elsewhere [4,5].

The SCC model is obtained by applying Hartree-Bogoliubov factorization to (1) leading to the approximate Hamiltonian

$$H_\omega = E_c + :H_{sp} - \omega j_x - \Delta \sum_{\nu=1}^{\Omega} (a_\nu^\dagger a_{-\nu}^\dagger + b_\nu^\dagger b_{-\nu}^\dagger + h.c.): \quad , \qquad (3)$$

where E_c is the cranking energy,

$$E_c = \frac{1}{2} a (I - \langle j_x \rangle_\omega)^2 + \langle H_{sp} \rangle_\omega - \Delta^2/G \quad , \qquad (4)$$

and self-consistency requires that

$$\omega = a(I - \langle j_x \rangle_\omega) \quad , \qquad (5)$$

and

$$\Delta = G \sum_{\nu=1}^{\Omega} (\langle a_{-\nu} a_{\nu} \rangle_{\omega} + \langle b_{-\nu} b_{\nu} \rangle_{\omega}) \quad . \tag{6}$$

Here, $\langle\ \rangle_{\omega}$ denotes the expectation value with respect to the ground state of (3) and : : normal ordering with respect to this vacuum.

It is worthwhile to note that in the present case, the cranking potential $-\omega j_x$ need not be added as a Lagrange multiplier term but arises automatically since the rotor cranks the system. In more realistic models, such a rotor could be added as a useful formal device and its moment of inertia equated to zero at the end, or it could replace an inert core to improve empirical fits.

Noting that the chemical potential is always zero in this model, one may diagonalize (3) by a Bogoliubov transformation of the form

$$\begin{pmatrix} \alpha_{\nu}^{\dagger} \\ \beta_{\nu}^{\dagger} \\ \beta_{\bar{\nu}} \\ \alpha_{\bar{\nu}} \end{pmatrix} = \underset{\sim}{U}(\omega) \begin{pmatrix} a_{\nu}^{\dagger} \\ b_{\nu}^{\dagger} \\ b_{-\nu} \\ a_{-\nu} \end{pmatrix} \quad , \tag{7}$$

where $\underset{\sim}{U}$ is a 4x4 matrix independent of the index $\underline{\nu}$, so that (3) takes the form

$$H_{\omega} = E_c + E_{+} \sum_{\nu} (\alpha_{\nu}^{\dagger}\alpha_{\nu} + \alpha_{\bar{\nu}}^{\dagger}\alpha_{\bar{\nu}}) + E_{-} \sum_{\nu} (\beta_{\nu}^{\dagger}\beta_{\nu} + \beta_{\bar{\nu}}^{\dagger}\beta_{\bar{\nu}}) \quad , \tag{8}$$

in terms of the quasiparticle operators $(\alpha, \alpha^{\dagger}, \beta, \beta^{\dagger})$. The quasiparticle energies are given by $E_{\pm} = \sqrt{\varepsilon^2 + (\Delta \pm \frac{1}{2}\omega)^2}$.

The gap equation (6) then takes the form

$$\Delta = G\frac{\Omega}{2}\left[\frac{\Delta + \frac{1}{2}\omega}{E_+} + \frac{\Delta - \frac{1}{2}\omega}{E_-}\right] . \qquad (9)$$

An equation of this type has been previously derived by Valatin [7] and by Krumlinde and Szymański [5]. Solution of (9) for $\Delta(\omega)$ determines everything. The total angular momentum, for example, is obtained from (5) in the form

$$I = \frac{\omega}{a} + \frac{\Omega}{2}\left[\frac{\Delta + \frac{1}{2}\omega}{E_+} - \frac{\Delta - \frac{1}{2}\omega}{E_-}\right] , \qquad (10)$$

and the moment of inertia is given by the usual expression

$$\mathcal{J} = I/\omega . \qquad (11)$$

The correlations in (1) not included by the SCC model can be systematically taken into account by a generalized Holstein-Primakoff boson expansion in powers of Ω^{-1}. This type of expansion for the R(5) algebra has been discussed by Evans and Krauss [8]. Within the representation containing the ground state, everything can be expressed in terms of four pairs of commuting boson creation and annihilation operators, corresponding to quasiparticle pairs.

Through the RPA (formerly called the "quasiboson approximation") order, the Hamiltonian is a quadratic form in these bosons. In diagonalizing it, one must distinguish the two cases when the SCC solution has $\Delta \neq 0$ or $\Delta = 0$. If $\Delta \neq 0$, there is a zero-energy mode corresponding to a pairing rotation. Then, the Hamiltonian can be written in the diagonal form:

$$H_{RPA} = E_c - (E_+ + E_-) + (E_+ + E_-)(C^\dagger C - \tfrac{1}{2}) + \tfrac{1}{2}(\hat{N}^{(1)} - 2\Omega)^2/\mathcal{J}_p$$

$$+ \mathcal{E}_+(D_+^\dagger D_+ + \tfrac{1}{2}) + \mathcal{E}_-(D_-^\dagger D_- + \tfrac{1}{2}) , \qquad (12a)$$

where $C^\dagger, D_+^\dagger, D_-^\dagger$ are independent boson creation operators for the normal modes. The boson operators all commute with $\hat{N}^{(1)}$, which is the linear boson approximation to the particle-number operator, and with a canonically-conjugate phase angle ψ. $\mathcal{J}_p$ is the inertial parameter for pairing rotations given by

$$\mathcal{J}_p = \frac{\Omega[4\Delta^2 - (E_+ - E_-)^2]}{E_+E_-(E_+ + E_-)} \quad . \tag{13}$$

The excitation energies $\mathcal{E}_\pm$ are given by

$$\mathcal{E}_\pm^{\,2} = \frac{1}{2}\left[u_+ + u_- \pm \sqrt{(u_+ - u_-)^2 + 4v^2}\right],$$

$$u_\pm = 4\,E_\pm^{\,2} + \varepsilon^2(a-2G)\ \Omega/E_\pm \quad ,$$

$$v = \varepsilon^2(a+2G)\ \Omega/\sqrt{E_+E_-} \quad . \tag{14}$$

If $\Delta = 0$, H_{RPA} can be written in the diagonal form

$$H_{RPA} = E_c - 2\,E + \mathcal{E}_-(A^\dagger A + B^\dagger B + 1) + 2E(C^\dagger C - \tfrac{1}{2}) + \mathcal{E}_+(D_+^\dagger D_+ + \tfrac{1}{2}) \quad , \tag{12b}$$

where the bosons $A^\dagger, B^\dagger$, create particle and hole pairing vibrations, respectively, and $E = E_+ = E_- = \sqrt{\varepsilon^2 + \omega^2/4}$. The pair-vibrational excitation energy $\mathcal{E}_-$ takes the form

$$\mathcal{E}_- = 2E\sqrt{1 - \frac{G\Omega\varepsilon^2}{E^3}} \qquad (\Delta = 0) \quad . \tag{15}$$

The energy of the yrast states, W, is the sum of the cranking energy and the zero-point energy, E_{ZP}, of the RPA model, $W = E_c + E_{ZP}$, which can be read off from Eq. (12). One then obtains a corrected rotational frequency ω(RPA) from the usual relation $\omega(\text{RPA}) = dW/dI$. Inasmuch as the cranking frequency $\omega = dE_c/dI$, one may write

$$\omega(\text{RPA}) = \omega + \delta\omega \quad ,$$

$$\delta\omega = dE_{zp}/dI \quad . \tag{16}$$

The general relation (11) then implies a corrected moment of inertia

$$\mathcal{J}_{(\text{RPA})} \cong (1-\delta\omega/\omega)\,\mathcal{J} \quad . \tag{17}$$

We present the results of calculations, both exact and approximate, for the parameters $\Omega = 6$, $a = .05$ on the one hand, and $\Omega = 14$, $a = .075$ on the other, with $G\Omega = .6$ and $\varepsilon = .1$, .3, and .45 in each case. The degree of backbending is mainly determined by the ratio $\varepsilon/(G\Omega)$: the smaller the ratio, the weaker the band mixing and the greater the tendency to bend back.

Figures 1 and 2 summarize the results as conventional plots of moment of inertia and angular momentum, respectively, vs. the square of the angular velocity. The physical points on the continuous trajectories correspond, of course, to even integral values of I. The trajectories have two segments, one with $\Delta \neq 0$, which may or may not backbend, along which Δ decreases continuously until the intersection with the $\Delta = 0$ segment at some $\omega=\omega^*$ corresponding to the cusp. For $\omega<\omega^*$, the $\Delta = 0$ solution is unstable. At $\omega=\omega^*$, the RPA energy $\mathcal{E}_- = 0$, signaling the transition from a pair-rotational to a pair-vibrational scheme. At this point, $d\mathcal{E}_-/dI$ becomes infinite (Fig. 3) and so does $\delta\omega$. Hence, the RPA corrections breakdown as this point is approached, accounting for the break in the RPA curves.

In the present model, the $\Delta \neq 0$ solution can be unstable only along a backbending portion, but this depends on the value of a. For a sufficiently large (small rotor), the backbending region is always stable, as is true in the cases with $\Omega = 14$, a = .075. For sufficiently small values of a (large rotor), $\mathcal{E}_-^{\,2}$ can go through zero and turn negative at any point along the backbending arc. It must return to zero again at $\omega=\omega^*$. This is illustrated by the cases with $\Omega = 6$, a = .05, ε = .1 and .3. In the unstable regions, I decreases with decreasing ω^2 as shown in Fig. 2. There is a competition between the particles, which favor increasing I with decreasing ω^2 in this region, and the rotor, which favors decreasing I with decreasing ω^2, and the latter wins out if sufficiently massive. We conclude that, in practice, stability can always be insured by keeping the rotor small enough. This provides a counter-example to previous claims that backbending per se implies instability of the SCC model [9].

Another interesting situation is illustrated by the case with $\Omega = 6$, and ε = 1, namely, an overlap in I between the lower and upper branches. Thus, there are states with I = 12,14 and 16 lying on stable portions of the superconducting and normal segments. The cranking energy E_c, however, is lower for the superconducting points by a small amount, in agreement with exact solutions.

We see that on the whole, the accuracy of the SCC model is very good for points corresponding to physical values of the spin, and is further improved by the RPA correlations, with the possible exception of a few points in the transition region. It can be shown that SCC plus RPA is exact in the limit $\varepsilon = 0$, which explains why the accuracy is greatest for small ε and diminishes as ε increases. That self-consistent cranking is less accurate with no backbending than with sharp backbending may be a general rule since it is a consequence of the enlargement of the transition region in which the two bands are strongly mixed and zero-point oscillations become important. In general, the RPA then provides a significant improvement, except near the cusp. For $\varepsilon = .45$, the effects of the cusp shift to low ω^2 so that the RPA correction is in the wrong direction on the lower segment, although it is excellent on the upper segment. In realistic situations, the cusp problem should not arise since Δ does not immediately vanish on the upper branch because only a single pair of nucleons align their spins [2].

The plots show quite clearly the asymptotic accuracy of the SCC model with increasing I, in accordance with its quasiclassical nature. The small quantum fluctuations are nicely taken care of by the RPA. This suggests that SCC plus RPA should provide a good tool for calculating the yrast cascade. In this connection, Fig. 3 shows the energy of the first excited state above the yrast state. The accuracy of the RPA is of the order of $1/\Omega$ except at the cusp, which is what one would expect. The pairing vibrations built on the yrast band present a fascinating possibility, but no comparison with exact solutions is made since these were only available for $N = 2\Omega$ particles.

REFERENCES

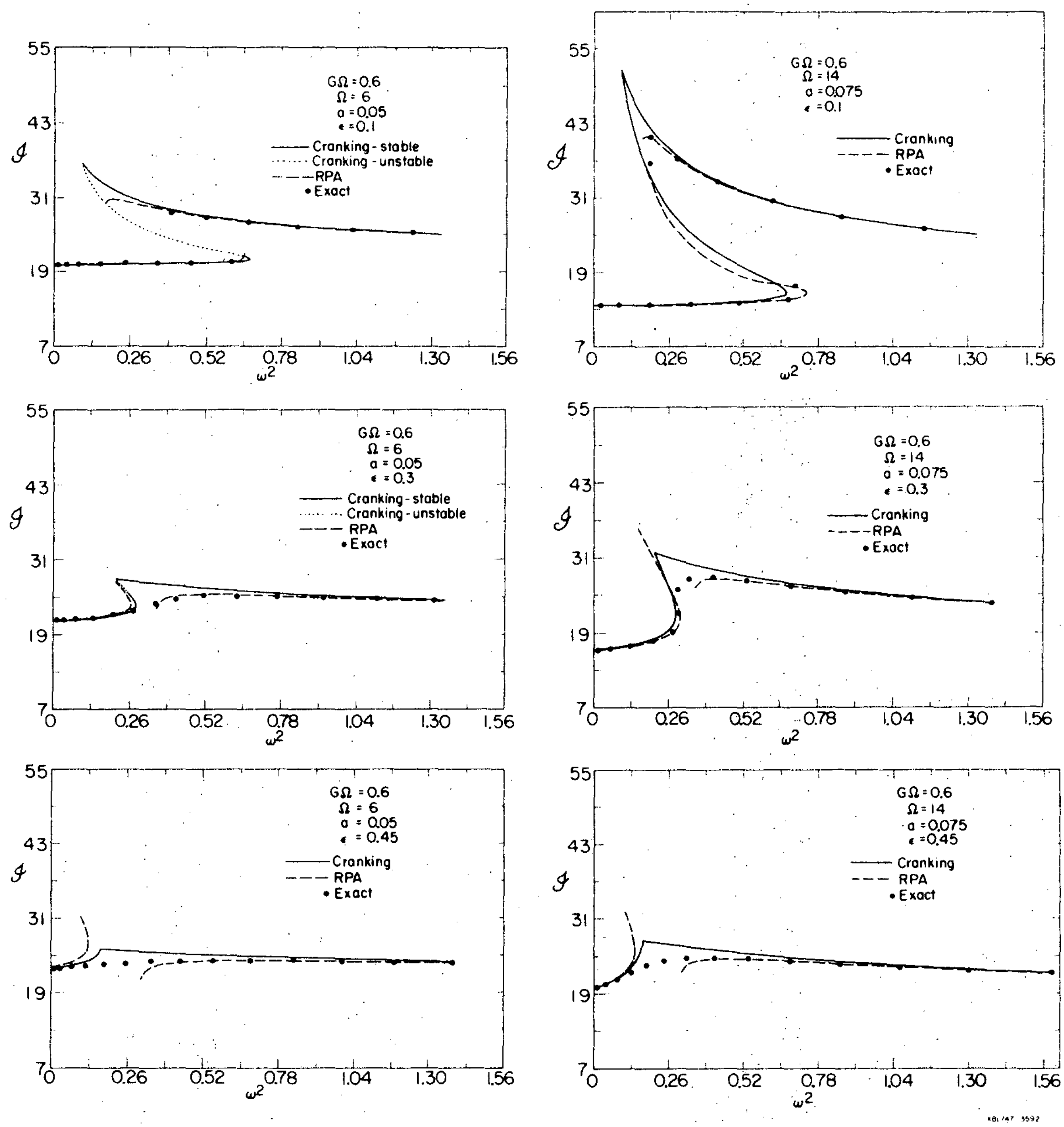
1. See for example the review of R. A. Sorensen, Rev. Mod. Phys. 45 (1973) 353.
2. B. Banerjee, H. J. Mang, and P. Ring, Nucl. Phys. A215 (1973) 366.
3. E. R. Marshalek and J. Weneser, Phys. Rev. C2 (1970) 1682.
4. J. Krumlinde and Z. Szymański, Phys. Lett. 36B (1971) 157.
5. J. Krumlinde and Z. Szymański, Ann. Phys. 79 (1973) 201.
6. A. Faessler, F. Grümmer, L. Lin and J. Urbano 48B (1974) 87.
7. J. G. Valatin, Lectures in Theoretical Physics, Vol. IV (University of Colorado, Boulder, Colorado, 1971) p. 1.
8. J. Evans and N. Krauss, Phys. Lett. 37B (1971) 455.
9. M. Sano, T. Takemasa and M. Wakai, Nucl. Phys. A190 (1972) 471.

FIGURE CAPTIONS

Fig. 1. Moment of inertia $\mathcal{J}$ vs. square of angular velocity ω^2.

Fig. 2. Angular momentum $\underline{I}$ vs. square of angular velocity ω^2.

Fig. 3. Energy of first excited state above yrast line ΔE plotted as a function of angular momentum I.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Ω = 6
a = 0.05
ε = 0.1
Cranking - stable
Cranking - unstable
RPA
Exact
GΩ = 0.6
Ω = 14
a = 0.075
ε = 0.1
Cranking
RPA
Exact
GΩ = 0.6
Ω = 6
a = 0.05
ε = 0.3
GΩ = 0.6
Ω = 14
a = 0.075
ε = 0.3
GΩ = 0.6
Ω = 6
a = 0.05
ε = 0.45
GΩ = 0.6
Ω = 14
a = 0.075
ε = 0.45
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31
43
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0
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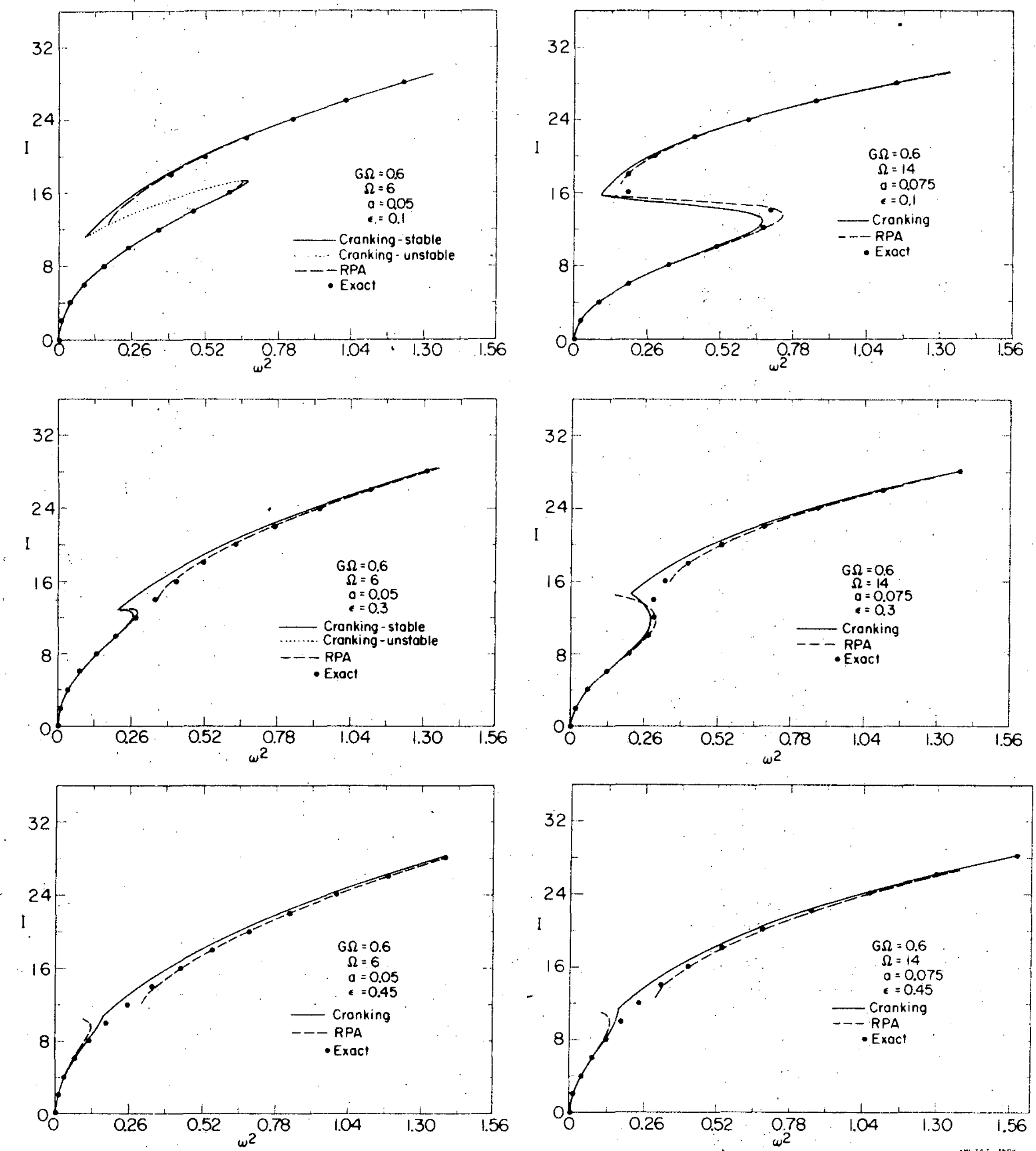
0.52
0.78
1.04
1.30
1.56

Fig. 1


GΩ = 0.6
Ω = 6
a = 0.05
ε = 0.1
Cranking - stable
Cranking - unstable
RPA
Exact
GΩ = 0.6
Ω = 14
a = 0.075
ε = 0.1
Cranking
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GΩ = 0.6
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ε = 0.3
Cranking - stable
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Ω = 14
a = 0.075
ε = 0.45
Cranking
RPA
Exact
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32
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0.52
0.78
1.04
1.30
1.56

Fig. 2

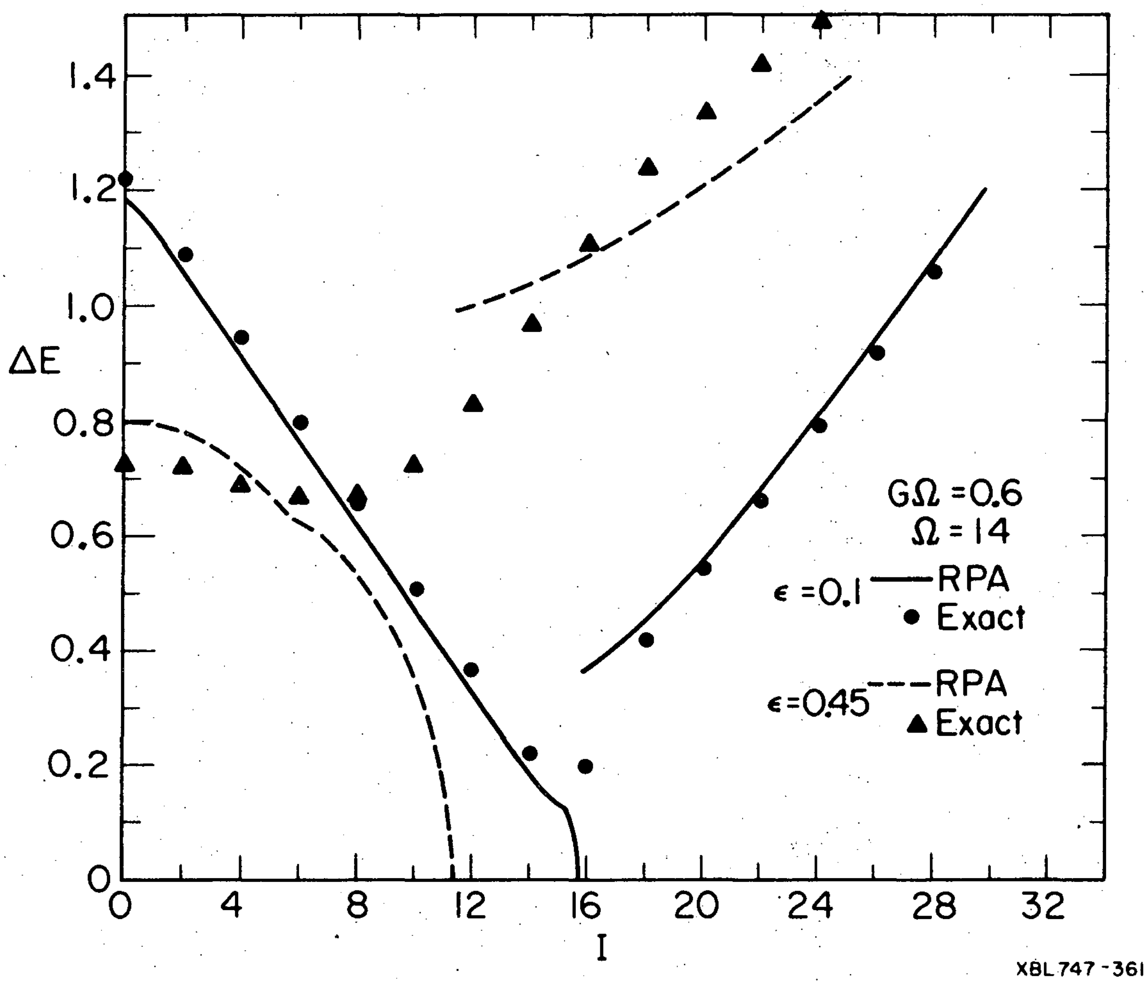
1.4
1.2
1.0
ΔE
0.8
0.6
0.4
0.2
0
0
4
8
12
16
20
24
28
32
I
GΩ =0.6
Ω = 14
ε =0.1 RPA
Exact
ε=0.45 RPA
Exact
XBL 747-3611

Fig. 3

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