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Don't teach me 2 + 2 equals 4: Knowledge of arithmetic operations hinders equation learning

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Abstract

This study investigated whether children's knowledge of arithmetic operations hinders their ability to solve novel equations after instruction. Second- and third-grade children completed a timed arithmetic pretest as a means for assessing their proficiency with arithmetic operations. Next, they received lessons on the principle of mathematical equivalence either in a context designed to activate their knowledge of arithmetic operations (e.g., 15 + 13 = 28), or in a context designed to not activate their knowledge of arithmetic operations (e.g., 28 = 28). Then, children completed an equation-solving posttest (e.g., 3 + 9 + 5 = 6 + __). After the posttest, children switched lesson contexts and completed the posttest again. Children solved more equations incorrectly after receiving lessons in the operational context. Additionally, the operational context led children who were most proficient with arithmetic operations to solve more equations using the typical addition strategy of adding up all the numbers. Results highlight that the activation of existing knowledge can interfere with the acquisition of new information.

Some domains of knowledge are particularly difficult for people to learn, even after significant amounts of training or instruction. There are many examples of this in our formal education system, including reading, mathematics, science, and foreign language. Over the past several years, a number of scientists (e.g., Flege, Yeni Komshian, & Liu, 1999; Kuhl, 2000; McNeil & Alibali, 2002; Schauble, 1990; Zevin & Seidenberg, 2002) have begun to consider how existing knowledge may contribute to these difficulties. The general theoretical view is that later learning is strongly constrained by early learning (cf. Tolman, 1948). If this is true, it obviously has implications in domains, like second language learning, where people learn one thing for many years (native language) before switching gears and learning something new, but closely related (second language).

The domain of mathematics is another domain in which people learn one topic for many years before switching gears and learning a new, but closely related, topic. Specifically, in most American mathematics classrooms, children learn arithmetic operations for many years (i.e., grades K-6) before eventually reaching algebra and being introduced formally to equations and the principle of mathematical equivalence. Mathematical equivalence is the principle that the two sides of an equation represent the same quantity.

Elementary school children (ages 7-11) have significant difficulties with equations and the principle of mathematical equivalence (Carpenter & Levi, 2000; Kieran, 1981; Baroody & Ginsburg, 1983). Their difficulties are most apparent when they are presented with equations that have operations on both sides of the equal sign (e.g., 3 + 4 + 5 = 3 + __). In the absence of instruction, approximately 80% of second- through fifth-grade children solve these types of equations incorrectly (Alibali & Goldin-Meadow, 1993; Alibali, 1999; McNeil & Alibali, 2000; NCISLA, 2000; Perry, Church, & Goldin-Meadow, 1988; Rittle-Johnson & Alibali, 1999).

Although there are many possible accounts of children's difficulties, including immature working memory function (Adams & Hitch, 1997; Gathercole & Pickering, 2000) or insufficient knowledge of necessary prerequisite skills (Haverty, 1999), the *change-resistance account* suggests that children's equation-learning difficulties are due, at least in part, to children's existing knowledge (McNeil, 2004). More specifically, the account posits that children construct knowledge on the basis of their early experiences with arithmetic operations and that this knowledge contributes to children's difficulties with more complex equations.

There are at least three knowledge structures that children learn from their early experiences with arithmetic operations that may ultimately hinder the ability to learn complex equations (see McNeil & Alibali, 2002). First, children may learn an operational strategy for solving math problems—perform all the given operations on all the given numbers. For example, in a typical addition problem like 3 + 4 + 5 + 3 = __, a problem solver simply needs to add up all the numbers and put the total in the blank. Second, children may learn an operational perceptual pattern related to the structure of math problems—the traditional "operations = answer" problem structure. For example, in the typical addition problem above, all of the numbers and operations are on the left hand side of the equation, and the answer blank is on the right side of the equation (directly following the equal sign). Third, children may learn an operational concept of the equal sign—the equal sign means "the total." Although these three operational patterns facilitate fast and accurate performance on typical addition problems, they do not map onto more complex equations. For example, when presented with the equation "3 + 4 + 5 = 3 + _", a problem solver cannot just add up all the numbers. He or she cannot assume that the equation will conform to the traditional "operations = answer" problem structure. And, he or she needs to understand that the equal sign denotes an equivalence relationship between the two sides of the equation in order to generate a correct solution.

According to the change-resistance account, children learn these operational patterns from their experience with arithmetic operations. They store these operational patterns in memory. Then, when they are presented with a novel equation, their representations of the operational patterns are activated. Once activated, the representations guide attention and can hinder the ability to encode and interpret novel equations that do not directly map onto the patterns (cf. Bruner, 1957; Luchins, 1942; Knoblich, Ohlsson, & Raney, 2001).

In accordance with the change-resistance account, studies have shown that children do, indeed, rely on their knowledge of arithmetic operations when presented with complex equations. For example, when asked to solve the equation "3 + 4 + 5 = 3 + __", most students use their knowledge of the "perform all given operations on all given numbers" strategy and just add up all the numbers and put 15 in the blank (McNeil & Alibali, 2000, 2002, in press b). When asked to reconstruct the equation "3 + 4 + 5 = 3 + __" after viewing it briefly, many students use their knowledge of the traditional "operations = answer" problem structure and write "3 + 4 + 5 + 3 = __" (McNeil & Alibali, 2002, in press b). When asked to define the equal sign, many students use their knowledge of operational symbols (e.g., +) and say that it means, "the total" (McNeil & Alibali, in press a). Thus, children rely on their knowledge of the operational patterns when presented with complex equations.

McNeil and Alibali (2002) provided additional evidence for the change-resistance account by showing that children's reliance on the operational patterns can hinder the ability to learn about equations. In the study, they documented a significant negative linear relationship between children's reliance on the operational patterns on a pretest and the generation of correct equation-solving strategies after a brief lesson on equations. Children who were most reliant on the operational patterns at pretest were the least likely to generate correct equation-solving strategies following a lesson, and children who did not rely on the operational patterns at pretest were the most likely to generate correct equation-solving strategies following a lesson.

Although the results of McNeil and Alibali (2002) support the change-resistance account, they provide only correlational evidence about the relationship between children's knowledge of arithmetic operations and equation-learning difficulties. The change-resistance account argues that the activation of children's knowledge of arithmetic operations *causes* equation-learning difficulties. Thus, in the present study, the activation of children's knowledge of arithmetic operations is manipulated. Children are given lessons about the principle of math equivalence either in a context designed to activate their knowledge of arithmetic operations (operational context), or in a context designed to *not* activate their knowledge of arithmetic operations (non-operational context). If the activation of knowledge of arithmetic operations contributes to difficulties with equations, then the operational lesson context should be inferior to the non-operational lesson context. That is, after receiving lessons in the operational context, children should solve more equations incorrectly, and they should solve more equations with the strategy that is the most often used in the absence of instruction (i.e., they should rely on their knowledge of the operational strategy and just add up all the numbers in the equations).

Additionally, if knowledge of arithmetic operations contributes to difficulties with equation learning, then children who are most proficient with arithmetic operations should be least likely to benefit from the lessons. This, of course, is assuming that children who are most proficient with arithmetic operations have the strongest representations of the operational patterns. Children who are proficient with arithmetic operations should solve more equations incorrectly, and they should solve more equations by just adding up all the numbers in the equations.

Continuing this rationale, the combination of the operational lesson context and proficiency with arithmetic operations should be a "double whammy." That is, proficient children who have just received lessons in the operational context should solve more equations incorrectly, and they should solve more equations by just adding up all the numbers in the equations.

Method

Participants

Ninety-three second- and third-grade children from a public elementary school in Youngsville, North Carolina participated. Eleven children were excluded from the analysis because they were absent on one or more days of the study. Two additional children were excluded because their performance on the equations was three standard deviations away from the mean. The final sample contained eighty children (38 boys and 42 girls).

Measures

Timed Arithmetic Pretest The timed arithmetic pretest was used to assess children's proficiency with arithmetic operations. Participants were given 30 seconds to solve as many arithmetic problems (out of 20) as possible. The problems involved only addition and subtraction (no multiplication or division).

Equation-solving Posttest Participants were given unlimited time to solve twelve equations with operations on both sides of the equal sign (e.g., 5 + 4 + 7 = 5 + __, 6 + 4 + 8 = __ + 3).

Procedure

The study was conducted over a two-week period in children's regular mathematics classrooms. Children's mathematics teachers collected the measures and administered the lessons in the classroom setting. Children first completed the timed arithmetic pretest. Then, teachers taught a set of lessons about the principle of mathematical equivalence. The following is an excerpt from the spoken lesson script: "The correct answer is 28! That's because whatever is on one side of the equal sign has to be **the same amount as** [teachers were told to stress words in bold] whatever is on the other side of the equal sign."

Because the lessons were scripted, all children received the same *spoken* lessons. Children were randomly assigned to lesson contexts through the use of individual booklets. The booklets enabled children to follow along with the spoken lessons. Children received lessons in one of two contexts. In the operational context, booklets contained problems designed to activate children's knowledge of arithmetic operations (e.g., 15 + 13 = 28). In the non-operational context, booklets contained problems designed to *not* activate children's knowledge of arithmetic operations (e.g., 28 = 28). Children received two days of lessons (approximately 15 minutes per day) before they completed the first equation-solving posttest. After the first posttest, children received lessons in the other context (e.g., children who had already received lessons in the operational context now received lessons in the non-operational context). Finally, children once again completed the equation-solving posttest.

Coding

Proficiency with Arithmetic Operations Children were categorized as proficient on the timed arithmetic test if they both solved three (median) or more arithmetic problems correctly, *and* solved one (median) or fewer arithmetic problems incorrectly. This coding system led to approximately equal numbers in the proficient (N = 37) and not proficient (N = 43) groups.

Equation-solving Performance Children's strategies were coded using a system developed by Perry, Church and Goldin-Meadow (1988). Strategies were assigned based on the solutions that children wrote in the answer blank. Examples are presented in Table 1. Solutions were coded as reflecting a particular strategy as long as they were within ±1 of the solution that would be achieved with that particular strategy. Again, we were especially interested in children's use of the add-all strategy (see Table 1) because it is the most commonly used strategy in the absence of instruction.

Table 1: Example solutions and corresponding strategy codes for the given equation.

5 + 4 + 7 = 5 + __	
Solution	Strategy
11	Correct
21	Add all
16	Add to equal sign
4	Carry
1	Idiosyncratic

Results

Number of Incorrect Solutions

Overall, performance on the equation-solving posttest was abysmal. Children solved 11.27 (SD = 0.98) equations incorrectly (out of 12). We performed a 2 (proficiency with arithmetic operations: proficient or not proficient) x 2 (lesson context: operational context or non-operational context) ANOVA with repeated measures on lesson context and number incorrect on the equation-solving posttest (out of 12) as the dependent measure. As expected, the analysis revealed a significant main effect of lesson context, $F(1, 78) = 5.24$, $p = .025$. Children solved more equations incorrectly after receiving lessons in the operational context (M = 11.44, SD = 0.90) than after receiving lessons in the non-operational context (M = 11.11, SD = 1.03). Neither the main effect of proficiency nor the interaction of proficiency and lesson context was significant. Although, as mentioned, children's performance was very poor overall, so there was not a great deal of variability on the dependent measure to predict.

Number of Add-all Solutions

Consistent with prior work, the add-all strategy was the most popular strategy. On average, children solved 5.29 (SD = 3.67) equations (out of 12) by just adding up all the numbers in the equations. We performed a 2 (proficiency with arithmetic operations: proficient or not proficient) x 2 (lesson context: operational context or non-operational context) ANOVA with repeated measures on lesson context and number of equations solved with the add-all strategy (out of 12) as the dependent measure. As expected, the analysis revealed a significant interaction of proficiency and lesson context, $F(1, 78) = 4.90$, $p = .03$. As shown in Figure 1, the children who solved the most equations using the add-all strategy were the ones who were proficient with arithmetic operations and had just received lessons in the operational context ($M = 6.11$, $SD = 3.60$).

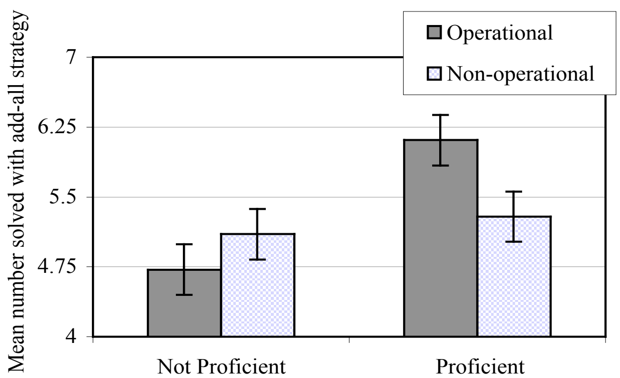

Figure 1: Mean number of equations solved with the add-all strategy (out of 12) as a function of arithmetic proficiency and lesson context. Error bars represent the pooled standard error.

Discussion

Consistent with the change-resistance account of children's equation-learning difficulties, results of the present study suggest that children's knowledge of arithmetic operations hinders their ability to learn about more complex equations. Children solved the fewest equations correctly after receiving lessons in contexts designed to activate their knowledge of arithmetic operations. Moreover, children who were most proficient with arithmetic operations and had just received lessons in the operational context solved the most problems by just adding up all the numbers, which is the most common strategy used by children who have not received any instruction at all.

Results suggest that it is vital to consider the state of children's existing knowledge when theorizing about children's learning difficulties. Thus, prevailing theories that focus on children's immature working memory system or their lack of prerequisite knowledge are missing a layer of complexity. More generally, results contribute to a growing body of work that suggests that knowledge can be detrimental to learning in some cases (Adelson, 1984; Flege et al., 1999; Kuhl, 2000; Schauble, 1990). Because knowledge typically facilitates learning, cases like these in which knowledge hinders learning can provide a unique window onto how the mind works (cf. Luchins, 1942).

Although the present study suggests that knowledge of arithmetic operations hinders equation learning. The results are not definitive. Performance on the equations with operations on both sides of the equal sign was abysmal, even after four, fifteen-minute classroom-based lessons. Thus, most children in the study had difficulties learning from the lessons on the principle of math equivalence. However, this is not surprising when viewed from the perspective of the change-resistance account. Children in the study are learning math on a day-to-day basis from a traditional, skills-based mathematics curricula. Thus, they are deeply entrenched in the operational patterns that are predicted to hinder learning. It is not that surprising that the brief lessons in the present study were not able to override this deeply entrenched way of thinking.

In terms of educational implications, results conflict with both intuition, and traditional mathematics practices. Intuition suggests that the children who are best at one topic in math should be best at another topic in math. And indeed, most schools assign, or track, children to algebra based on their performance in elementary school with arithmetic operations. This policy certainly makes sense if children need to be highly proficient with arithmetic operations before they are able to learn algebraic equations. However, it makes less sense if arithmetic proficiency hinders equation learning. Thus, our schools may be holding back children who would thrive in an early algebra course.

Equally important, American schools often implement spiral curricula in which old information is reintroduced year after year. The idea is that old information provides a framework within which new material can be introduced. In relation to mathematics instruction, this means that basic arithmetic operations are reintroduced year after year. Indeed, data from the Third International Mathematics and Science Study (Beaton et al., 1996) show that, unlike students from higher-achieving countries, students in American mathematics classrooms spend substantial amounts of class time practicing and reviewing basic arithmetic skills throughout the elementary and middle school years, when they should be concentrating on more advanced topics.

This type of spiral, review-based instruction has received some support from the scientific community. For example, Nathan et al. (2004) argue that the most effective instructions are the ones that "bridge" from children's existing knowledge to the new material. However, results of the present study suggest that teachers need to be careful about what they are trying to build bridges between. In some case, to-be-learned information does not map well onto existing knowledge, and in these cases, bridging might not be the most effective instructional strategy.

Instead of reintroducing basic arithmetic facts year after year, mathematics educators may wish to develop creative ways to integrate more algebraic ways of thinking into the math curricula as early as possible. One recommended strategy for integrating algebraic thinking into the earlier grades is to focus on equality and the equal sign (e.g., Carpenter, Franke, & Levi, 2003). For example, instead of simply reviewing and practicing basic arithmetic "facts" such as "3 + 4 = 7" year after year, young students can learn "3 + 4 = 7," "7 = 3 + 4," "3 + 4 = 5 + 2," and "7 = 7." Instructional strategies such as this may prevent the entrenchment of operational patterns and facilitate the notoriously difficult transition from arithmetic to algebra.

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