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On lattices of convex sets in $\mathbb{R}^n$

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To the memory of Ivan Rival

Abstract. Properties of several sorts of lattices of convex subsets of $\mathbb{R}^n$ are examined. The lattice of convex sets containing the origin turns out, for $n > 1$, to satisfy a set of identities strictly between those of the lattice of all convex subsets of $\mathbb{R}^n$ and the lattice of all convex subsets of $\mathbb{R}^{n-1}$. The lattices of arbitrary, of open bounded, and of compact convex sets in $\mathbb{R}^n$ all satisfy the same identities, but the last of these is join-semidistributive, while for $n > 1$ the first two are not. The lattice of relatively convex subsets of a fixed set $S \subseteq \mathbb{R}^n$ satisfies some, but in general not all of the identities of the lattice of "genuine" convex subsets of $\mathbb{R}^n$.

1. Notation, conventions, remarks

For S a subset of $\mathbb{R}^n$, the convex hull of S will be denoted

$$\mathrm{c.h.}(S) \;=\; \{\Sigma_{i=1}^{m}\, \lambda_i\, p_i \mid m \geq 1,\ p_i \in S,\ \lambda_i \in [0,1],\ \Sigma\, \lambda_i = 1\}. \tag{1}$$

When S is written as a list of elements "$\{\,...\,\}$," we shall in general simplify "c.h.($\{\,...\,\}$)" to "c.h.(...)".

$\mathrm{Conv}(\mathbb{R}^n)$ will denote the lattice of all convex subsets of $\mathbb{R}^n$; its lattice operations are

$$x \wedge y \;=\; x \cap y, \qquad x \vee y \;=\; \mathrm{c.h.}(x \cup y). \tag{2}$$

In any lattice, if a finite family of elements y_i has been specified, then an expression such as $\bigwedge_i\, y_i$ will denote the meet over the full range of the index i, and similarly for joins. Likewise, if we write something like $\bigwedge_{j \neq i}\, y_j$ where i has been quantified outside this expression, then the meet will be over all values of j in the indexing family other than i. If L is any lattice and x an element of L, or, more generally, of an overlattice of L, we define the sublattice

$$L_{\geq x} \;=\; \{y \in L \mid y \geq x\}. \tag{3}$$

In particular, $\mathrm{Conv}(\mathbb{R}^n)_{\geq \{0\}}$ is the lattice of those convex subsets of $\mathbb{R}^n$ that contain the origin.

If A is a class of lattices, $\mathbf{V}(A)$ will denote the variety of lattices generated by A, that is, the class of lattices satisfying all identities (in the binary operations

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$\wedge$ and $\vee$) that hold in all lattices in A. Again, for A given as a list, we will abbreviate $\mathbf{V}(\{\ldots\})$ to $\mathbf{V}(\ldots)$. (Most often A will be a singleton $\{L\}$, so that we will write $\mathbf{V}(L)$.)

Given sets x and y, we will write $x-y$ for their set-theoretic difference, $\{p \mid p\in x,\ p\notin y\}$.

Though I am not an expert either in lattice theory or in convex sets, I know more about the former subject than the latter; hence, I may more often state explicitly facts known to every worker in convex sets than those known to every lattice-theorist. I hope this note will nevertheless be of interest to people in both fields. I have no present plans of carrying these investigations further; others are welcome to do so.

Since obtaining the main results of this paper, I have learned that many of them were already in the literature, and have added references; thus, this is now a hybrid research/survey paper. I am grateful to Kira Adaricheva, J. B. Nation, Marina Semenova, Fred Wehrung, and the referee for corrections, information on the literature, and many other helpful comments.

Whereas this note looks at conditions satisfied *universally* in various lattices of convex sets, the papers [1], [19], and others cited there study sufficient conditions for lattices to be embeddable in such lattices, in other words, *existential* properties of such lattices. (This note also includes one result of that type, in §13.)

2. n-Distributivity

The varieties of lattices we will be examining in the first few sections are those listed in

Lemma 1. *Each lattice in the sequence*

$$\mathrm{Conv}(\mathbb{R}^0)_{\geq\{0\}},\ \mathrm{Conv}(\mathbb{R}^0),\ \mathit{Conv}(\mathbb{R}^1)_{\geq\{0\}},\ \ldots,\ \mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}},\ \mathrm{Conv}(\mathbb{R}^n),\ \ldots$$

is embeddable in the next. Hence

$$\mathbf{V}(\mathrm{Conv}(\mathbb{R}^0)_{\geq\{0\}}) \subseteq \mathbf{V}(\mathrm{Conv}(\mathbb{R}^0)) \subseteq \mathbf{V}(\mathrm{Conv}(\mathbb{R}^1)_{\geq\{0\}}) \subseteq \mathbf{V}(\mathrm{Conv}(\mathbb{R}^1)) \qquad (4)$$

$$\subseteq \cdots \subseteq \mathbf{V}(\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}) \subseteq \mathbf{V}(\mathrm{Conv}(\mathbb{R}^n)) \subseteq \ldots\ .$$

Proof. On the one hand, $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ is a sublattice of $\mathrm{Conv}(\mathbb{R}^n)$; on the other, one can embed $\mathrm{Conv}(\mathbb{R}^n)$ in $\mathrm{Conv}(\mathbb{R}^{n+1})_{\geq\{0\}}$ by sending each $x\in\mathrm{Conv}(\mathbb{R}^n)$ to the cone $\mathrm{c.h.}(\{0\}\cup\{(p_1,\ldots,p_n,1)\mid(p_1,\ldots,p_n)\in x\})$. The inclusions (4) follow from these embeddings. □

Clearly $\mathrm{Conv}(\mathbb{R}^0)_{\geq\{0\}}$ is a trivial lattice with unique element $\{0\}$; on the other hand, $\mathrm{Conv}(\mathbb{R}^0)$ is a two-element lattice, so $\mathbf{V}(\mathrm{Conv}(\mathbb{R}^0))$ is the variety of distributive lattices. Hence the first inclusion of (4) is strict. We shall see below that the next inclusion is an equality, while all subsequent inclusions are again strict.

Let us set up notation for some relations among elements in a lattice.

Definition 2 (after Huhn [11])**.** For each positive integer n, we shall denote by $D_n(x, y_1, \dots, y_{n+1})$ (for "n-distributivity") the lattice-relation in $n{+}2$ arguments $x,\ y_1,\ \dots,\ y_{n+1}$,

$$x \wedge (\bigvee_i y_i) \ = \ \bigvee_i \, (x \wedge \bigvee_{j \neq i} y_j). \tag{5}$$

We shall say that a lattice L satisfies "the identity D_n" if (5) holds for all $x,\ y_1,\ \dots,\ y_{n+1} \in L$.

Thus, D_1 is the ordinary distributivity identity.

The use of the subscript n for an identity in $n{+}2$ variables which is symmetric in $n{+}1$ of these may seem confusing; a useful mnemonic is that D_n is the identity that allows one to "reduce meets of x with larger joins to meets of x with n-fold joins". An additional occasion for confusion will arise when we see that the first of these identities to be satisfied by $\mathrm{Conv}(\mathbb{R}^n)$ is not D_n, but D_{n+1}. This will be a consequence of the fact that an n-dimensional simplex has $n{+}1$ vertices.

Note that the left-hand side of (5) is $\geq$ the right-hand side for any family of elements of any lattice, since each of the $n{+}1$ terms in the outer join on the right is majorized by the left-hand side. So to verify any instance of (5) it suffices to prove "$\leq$".

The pioneering work on identities satisfied by lattices of convex sets was done by A. P. Huhn [10], [11]. The results in this and the next three sections will extend Huhn's by approximately doubling both the family of lattices and the family of identities considered, and formalizing some general techniques. Huhn's results will be recovered along with our new ones.

The key to Huhn's and our results on D_n (and some related identities) is the following standard result in the theory of convex sets. Strictly speaking, it is the first sentence below that is Carathéodory's theorem, while the second is a well-known refinement thereof [6, p.431, line 4]. Intuitively, that second sentence says that from an arbitrary point $p_0 \in S \subseteq \mathbb{R}^n$, we can "see" any other point of $\mathrm{c.h.}(S)$ against a background of (or embedded in) some $(n{-}1)$-simplex with vertices in S.

Carathéodory's Theorem. *If S is a subset of $\mathbb{R}^n$, then each element $q \in \mathrm{c.h.}(S)$ belongs to $\mathrm{c.h.}(p_0, \dots, p_n)$ for some $n{+}1$ points $p_0, \dots, p_n \in S$. Moreover, p_0 can be taken to be any pre-specified element of S.*

In each paragraph of the next lemma, it is the first assertion that is due to Huhn.

Lemma 3 (cf. Huhn [11])**.** *For every natural number n and $(n{+}3)$-tuple of convex sets $x, y_1, \dots, y_{n+2} \in \mathrm{Conv}(\mathbb{R}^n)$, the relation $D_{n+1}(x, y_1, \dots, y_{n+2})$ holds. Moreover, if $n > 0$ and (at least) some two of the sets $y_1, \dots, y_{n+1}$ have nonempty intersection, then $D_n(x, y_1, \dots, y_{n+1})$ holds.*

Hence $\mathrm{Conv}(\mathbb{R}^n)$ satisfies the identity D_{n+1}, and for $n > 0$, $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ satisfies the identity D_n.

Proof. We shall prove the assertions of the first paragraph, which clearly imply those of the second.

To get the first assertion, let $x,\ y_1, \dots, y_{n+2} \in \mathrm{Conv}(\mathbb{R}^n)$, and let p be a point of the convex set described by the left-hand side of (5). Then p belongs to both x

and $\bigvee_i\, y_i = \mathrm{c.h.}(\bigcup_i\, y_i)$. Carathéodory's Theorem now says that p belongs to the convex hull of $n{+}1$ points of $\bigcup_i\, y_i$, hence to the join of at most $n{+}1$ of the y_i, so it belongs to one of the terms on the right-hand side of (5), hence to their join, as required.

To prove the second assertion, let $x,\ y_1,\ \dots,\ y_{n+1} \in \mathrm{Conv}(\mathbb{R}^n)$, and suppose that two of $y_1, \dots, y_{n+1}$ have a point p_0 in common. Given p in the left-hand side of the desired instance of (5), we see from the last sentence of our statement of Carathéodory's Theorem that p will belong to the convex hull of p_0 and some n other points of $\bigcup_i\, y_i$. The latter points will lie in the union of some n of the y's, and since those n of our $n{+}1$ sets leave out only one, they do not leave out both of the sets known to contain p_0. So that family of n y's contains the $n{+}1$ points whose convex hull is known to contain p, and the conclusion follows as before. $\square$

Note that for all $p \in \mathbb{R}^n$, we have $\mathrm{Conv}(\mathbb{R}^n)_{\geq \{p\}} \cong \mathrm{Conv}(\mathbb{R}^n)_{\geq \{0\}}$; moreover, if $p \in S \subseteq \mathbb{R}^n$, then $\mathrm{Conv}(\mathbb{R}^n)_{\geq S} \subseteq \mathrm{Conv}(\mathbb{R}^n)_{\geq \{p\}}$. Thus, any identities proved for the lattice $\mathrm{Conv}(\mathbb{R}^n)_{\geq \{0\}}$ will hold in $\mathrm{Conv}(\mathbb{R}^n)_{\geq S}$ for every nonempty set S.

One may ask whether, in the first assertion of the above lemma, we have failed to use the full strength of Carathéodory's Theorem. That theorem says that the convex hull of any family of $N > n{+}1$ points is the union of the hulls of its $(n{+}1)$-element subfamilies, hence for each such N, we may deduce an identity like D_{n+1}, but with the join of N rather than $n{+}2$ convex sets y_i on the left, and expressions involving all the $(n{+}1)$-fold subjoins thereof on the right. Our D_{n+1} is the case $N = n{+}2$.

But in fact, the identities so obtained are all equivalent to D_{n+1}. To derive them from it, note first that if in D_{n+1} we substitute for y_{n+2} the expression $y_{n+2} \vee y_{n+3}$, then the left-hand side becomes $x \wedge (\bigvee_{i=1}^{n+3} y_i)$, while on the right, some of the joins involve $n{+}1$ of the y's and others involve $n{+}2$. If we again apply D_{n+1} to the latter joins, we get precisely the $N = n{+}3$ case of the class of identities discussed above. The identities with still larger N are gotten by repeating this argument. Conversely, one can get D_{n+1} from any of these identities by substituting for the y_i with $i > n{+}2$ repetitions of y_{n+2}, and discarding from the outer join on the right-hand side joinands majorized by others.

Let us also note that $D_{n+1} \Longrightarrow D_{n+2}$. Indeed, the identity with $n{+}3$ y's that we just showed equivalent to D_{n+1} has the same left-hand side as D_{n+2}, while the right-hand side of D_{n+2} can be seen to lie, in an arbitrary lattice, between the two sides of that identity. In particular, the identities obtained in Lemma 3 are successively weaker for larger n, as is reasonable in view of (4).

The argument proving Lemma 3 (for simplicity let us limit ourselves to the first assertion thereof) can be formulated in a more general context, and the above observations on identities allow us to obtain a converse in that context, which we record below, though we shall not use it. Recall that a closure operator on a set X is called "finitary" (or "algebraic") if the closure of every subset $S \subseteq X$ is the union of the closures of the finite subsets of S.

Lemma 4 (cf. [16]). *Let* cl *be a finitary closure operator on a set* X, *such that every singleton subset of* X *is closed, or more generally, such that the closure of* *every singleton is finitely join-irreducible; and let* n *be a positive integer. Then the lattice of closed subsets of* X *under* cl *satisfies* D_n *if and only if* cl *has the "n-Carathéodory property" that the closure of every set is the union of the closures of its* $\leq n$*-element subsets.*

Proof. "If" is shown exactly as in the proof of Lemma 3.

Conversely, suppose the lattice of cl-closed subsets of X satisfies D_n, and let $p \in \mathrm{cl}(S)$ for some $S \subseteq X$. By the assumption that cl is finitary, $p \in \mathrm{cl}(q_1, \dots, q_N)$ for some $q_1, \dots, q_N \in S$. If $N \leq n$, we are done; if not, let us rewrite the condition $p \in \mathrm{cl}(q_1, \dots, q_N)$ as a lattice relation, $\mathrm{cl}(p) = \mathrm{cl}(p) \wedge (\bigvee_i \mathrm{cl}(q_i))$. By the preceding discussion, D_n implies the identity in N variables which, when applied to the right-hand side of the above relation, turns the relation into

$$\mathrm{cl}(p) \ = \ \bigvee_{|I|=n} \, (\mathrm{cl}(p) \, \wedge \, (\bigvee_{i\in I} \ \mathrm{cl}(q_i))),$$

where the subscript to the outer join means that I ranges over all n-element subsets of $\{1, \dots, N\}$. Now by assumption $\mathrm{cl}(p)$ is finitely join-irreducible, hence it equals one of the joinands on the right, showing p to be in the closure of some set of n q's, as required. □

3. Tools for studying related identities

We also appear to have used less than the full force of the middle sentence of Lemma 3 (the stronger relation holding when at least two of the y's have nonempty intersection) in getting the second assertion of the last sentence of that lemma (the stronger identity for $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$), since the latter assertion concerns the case where not just two, but all the y_i (and also x) have a point in common. There is no evident way to take advantage of the weaker hypothesis of said middle sentence when working in the lattice $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$; but might we be able to use it to prove some new identity for $\mathrm{Conv}(\mathbb{R}^n)$; for example, some relation that we can show holds, on the one hand, whenever a family of elements $x, y_1, \dots, y_{n+1}$ satisfies D_n, and also, for some trivial reason, whenever $y_1 \wedge y_2 = \emptyset$?

One relation with these properties can be obtained by taking the meet of each side of D_n with $y_1 \wedge y_2$. Unfortunately, this turns out to be the trivial identity: both sides simplify to $x \wedge y_1 \wedge y_2$ in any lattice. However, we can circumvent this by first taking the join of both sides of D_n with a new indeterminate z, and only then taking meets with $y_1 \wedge y_2$.

We shall in fact see that this provides what Lemma 3 failed to: an identity holding in $\mathrm{Conv}(\mathbb{R}^n)$ but not in $\mathrm{Conv}(\mathbb{R}^{n+1})_{\geq\{0\}}$. However, the verification of an example showing the failure of this identity in the larger lattice (and of a similar example we will need later) is messy if done entirely "by hand"; so we shall establish in this section some general criteria for certain sorts of inequalities in lattices of convex sets to be strict.

The last assertions of the next two lemmas clearly imply, for $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathrm{Conv}(\mathbb{R}^{n+1})_{\geq\{0\}}$ respectively, that if we are given expressions u, v and w in some lattice indeterminates such that the inequality $u \geq v$ is known to hold identically in our lattice, then we can write down another inequality holding identically among expressions in a slightly larger set of indeterminates, for which equality will hold whenever either u equals v, or w equals the least lattice-element (i.e., $\emptyset$, respectively $\{0\}$), but which will fail in all other cases. This is what is logically called for by the program sketched above. However the earlier parts of these lemmas give some simpler inequalities for which the same is true if the hypotheses hold "in a sufficiently strong way", and it will turn out that in the applications where we need to show failure of an identity, we will be able to use these simpler formulas.

In the statements of these lemmas, for $p, q \in \mathbb{R}^n$, "the ray drawn from q through p" will mean $\{\lambda p + (1-\lambda) q \mid 0 \leq \lambda < \infty\}$, even in the degenerate case $p = q$, where this set is the singleton $\{p\}$.

Lemma 5. *Let n be any natural number, and let $u \geq v$ and w be three elements of* $\mathrm{Conv}(\mathbb{R}^n)$. *Then*

(i) *The following conditions are equivalent:*

(a) *There exists $z \in \mathrm{Conv}(\mathbb{R}^n)$ such that $(u \vee z) \wedge w > (v \vee z) \wedge w$.*

(b) *There exist a point $p \in u$ and a point $q \in w$ such that the ray drawn from q through p contains no point of v.*

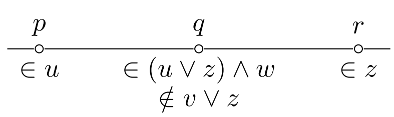
(ii) *The following conditions are equivalent:*

(a) *There exist $z, z' \in \mathrm{Conv}(\mathbb{R}^n)$ such that $((u \wedge z') \vee z) \wedge w > ((v \wedge z') \vee z) \wedge w$.*

(b) *$u > v$, and w is nonempty.*

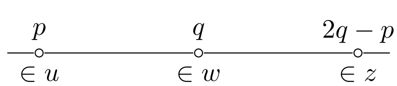
Proof. In view of the hypothesis $u \geq v$, the relation "$\geq$" always holds in the inequalities of (i)(a) and (ii)(a), so in each case, strict inequality is equivalent to the existence of a point q belonging to the left-hand side but not to the right-hand side.

Suppose, first, that (i)(a) holds for some z; thus we get a point q belonging to $u \vee z$ and to w but not to $v \vee z$. Note that the latter condition implies that q does not lie in z. If q lies in u, then (i)(b) is satisfied with $p = q$, so assume $q \notin u$. Hence, being in the convex hull of u and z but in neither set, q must lie on the line-segment $\mathrm{c.h.}(p, r)$ for some $p \in u$, $r \in z$:



Now if a point $s \in v$ lay on the ray drawn from q through p, we would have $q \in \mathrm{c.h.}(s, r) \subseteq v \vee z$, contradicting our assumption that $q \notin v \vee z$. This proves (b).

Conversely, if we are given p and q as in (i)(b), take $z = \{2q - p\}$:



Thus $q \in (u \vee z) \wedge w$, but we claim that $q \notin v \vee z$. Indeed, if q belonged to this set, then v would have to meet the ray from the unique point $2q - p$ of z through

q on the other side of q; i.e., it would meet the ray drawn from q through p, contradicting our choice of p and q as in (b). Hence q belongs to the left-hand but not the right-hand side of the inequality of (a), as required.

Note that (i)(b), and hence (i)(a), holds whenever u and w are nonempty and v is empty.

Now in the situation of (ii)(b), if we take any $p \in u - v$ and let $z' = \{p\}$, then $u \wedge z'$ and $v \wedge z'$ are respectively nonempty and empty, so applying the preceding observation with these two sets in the roles of u and v, we get (ii)(a). The reverse implication is trivial. □

The result we shall prove for $\mathrm{Conv}(\mathbb{R}^n)_{\geq \{0\}}$ is similar. Indeed, in the lemma below, part (i) is exactly as in the preceding lemma; but (ii) becomes two statements, (ii) and (iii), the former having the same "(a)" as in (ii) above but a stronger "(b)", the latter a weaker "(a)" but essentially the same "(b)" as above. Note also the restriction on n (only needed for (iii)).

Lemma 6. *Let $n > 1$, and let $u \geq v$ and w be three elements of* $\mathrm{Conv}(\mathbb{R}^n)_{\geq \{0\}}$. *Then*

(i) *The following conditions are equivalent:*

(a) *There exists $z \in \mathrm{Conv}(\mathbb{R}^n)_{\geq \{0\}}$ such that $(u \vee z) \wedge w \ > \ (v \vee z) \wedge w$.*

(b) *There exist a point $p \in u$ and a point $q \in w$ such that the ray drawn from q through p contains no point of v.*

(ii) *The following conditions are equivalent:*

(a) *There exist $z, z' \in \mathrm{Conv}(\mathbb{R}^n)_{\geq \{0\}}$ such that*

$$((u \wedge z') \vee z) \wedge w \ > \ ((v \wedge z') \vee z) \wedge w.$$

(b) *There exist a point $p \in u$ and a point $q \in w$ such that the ray drawn from q through p contains no point of $v \wedge \mathrm{c.h.}(0, p)$.*

(iii) *The following conditions are equivalent:*

(a) *There exist $z, z', z'', z''' \in \mathrm{Conv}(\mathbb{R}^n)_{\geq \{0\}}$ such that*

$$((((u \wedge z''') \vee z'') \wedge z') \vee z) \wedge w \ > \ ((((v \wedge z''') \vee z'') \wedge z') \vee z) \wedge w.$$

(b) *$u > v$, and $w \neq \{0\}$.*

Proof. (i)(a) $\implies$ (i)(b) holds by the preceding lemma. In proving the reverse implication, we cannot set $z = \{2q - p\}$ as we did there, so let $z = \mathrm{c.h.}(0, 2q - p)$. As before, we have $q \in (u \vee z) \wedge w$ and need to show $q \notin v \vee z$. If the contrary were true, then q would be a convex linear combination of 0, $2q - p$, and a point $r \in v$. This can be rewritten as a convex linear combination of $2q - p$ with a convex linear combination of 0 and r; but the latter combination would also be a point of v, and, as in the previous proof, would lie on the ray drawn from q through p, contradicting our choice of p and q as in (b).

Turning to (ii), if (ii)(a) holds then we can apply (i)(a) $\implies$ (i)(b) with $u \wedge z'$ and $v \wedge z'$ in place of u and v, and the resulting p and q will satisfy (ii)(b) (since $v \wedge \mathrm{c.h.}(0, p) \subseteq v \wedge z'$). Inversely, if (ii)(b) holds, take $z' = \mathrm{c.h.}(0, p)$, and apply (i)(b) $\implies$ (i)(a) with $u \wedge z'$ and $v \wedge z'$ in place of u and v.

In statement (iii), it is clear that (a) implies (b). To prove the converse, let us assume (iii)(b), and consider two cases, according to whether the stronger statement (ii)(b) holds. If it does, we get the inequality of (ii)(a), from which we can immediately get that of (iii)(a) by choosing z''' and z'' to "have no effect" (e.g., by taking them to be u and v respectively). On the other hand, if (ii)(b) fails while (iii)(b) holds, it is easy to see that all elements of w and $u-v$ must lie on a common line x through 0. In that case, we want to use z''' and z'' to "perturb" u and v, so that the modified u has points off the line x, while being careful to preserve the property that u is strictly larger than v. To do this, we begin by taking any point $p \in u-v$, letting $z''' = \mathrm{c.h.}(0,p)$, and noting that the intersections of u and v with this segment are still distinct. Now taking any point r not on the line x (it is for this that we need $n>1$), and letting $z'' = \mathrm{c.h.}(0,r)$, we see that $(u \wedge z''') \vee z''$ and $(v \wedge z''') \vee z''$ remain distinct, and that their difference now has points off x. Hence (ii)(b) holds with these sets in the roles of u and v, and the implication (ii)(b) $\Longrightarrow$ (ii)(a) gives the z and z' needed for (iii)(a). □

Digression: One can get criteria similar to those of the preceding lemmas for other conditions. At the trivial end, given $u \in \mathrm{Conv}(\mathbb{R}^n)$, a condition for u to be nonempty is that there exist a z such that $u \vee z > z$, and likewise the condition for at least one of two elements u and v to be nonempty is that their join have this property. The exercise below offers, for the diversion of the interested reader, some less trivial cases.

Exercise 7. (i) Find an inequality in elements u, v and one or more additional lattice variables $z, \dots,$ which holds identically in lattices, and which, for any $u, v \in \mathrm{Conv}(\mathbb{R}^n)$, is strict for some values of the additional variables if and only if *both* u and v are nonempty.

(ii) Find an inequality in elements u, v, w, x and additional variables which holds in any lattice when $u \geq v$, $w \geq x$, and which, for any such $u, v, w, x \in \mathrm{Conv}(\mathbb{R}^n)$, is strict for some values of the additional variables if and only if $u > v$ and $w > x$.

(iii) Same as (ii), but with "$u > v$ and $w > x$" replaced by "$u > v$ or $w > x$".

(iv)-(vi) Like (i)-(iii), but for $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$, and with "nonzero" in place of "nonempty" in (i).

(vii) In Lemma 6, conditions (i)(a), (ii)(a) and (iii)(a) involved 1, 2 and 4 z's respectively; so the condition that there exist three elements $z, z', z'' \in \mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ such that

$$(((u \vee z'') \wedge z') \vee z) \wedge w > (((v \vee z'') \wedge z') \vee z) \wedge w$$

was skipped. Show by example that this condition is not equivalent to (iii)(b).

(viii) Suppose $u \geq v$ and w are elements of $\mathrm{Conv}(\mathbb{R}^n)$, and consider the conditions dual to (i)(a)-(iii)(a) of Lemma 6, and to the "skipped" condition:

(a) There exists $z \in \mathrm{Conv}(\mathbb{R}^n)$ such that $(u \wedge z) \vee w > (v \wedge z) \vee w$.
(a′) There exist $z, z' \in \mathrm{Conv}(\mathbb{R}^n)$ such that $((u \vee z') \wedge z) \vee w > ((v \vee z') \wedge z) \vee w$.
(a″) There exist $z, z', z'' \in \mathrm{Conv}(\mathbb{R}^n)$ such that

$$(((u \wedge z'') \vee z') \wedge z) \vee w \;>\; (((v \wedge z'') \vee z') \wedge z) \vee w.$$

(a‴) There exist $z, z', z'', z''' \in \mathrm{Conv}(\mathbb{R}^n)$ such that

$$((((u \vee z''') \wedge z'') \vee z') \wedge z) \vee w \ > \ ((((v \vee z''') \wedge z'') \vee z') \wedge z) \vee w.$$

Which of these, if any, are equivalent, for all such u, v and w, to the condition

(b) $u > v$, and $w \neq \mathbb{R}^n$?

Now, back to business.

4. Identities distinguishing our chain of lattices

Given $n > 0$, let us write $((D_n) \vee z) \wedge y_1 \wedge y_2$ for the equation in $n{+}3$ variables $x, y_1, \dots, y_{n+1}, z$ obtained by applying the operator $((-) \vee z) \wedge y_1 \wedge y_2$ to both sides of the relation $D_n(x, y_1, \dots, y_{n+1})$. We can now prove

Theorem 8. *For each positive integer n, $\mathrm{Conv}(\mathbb{R}^n)$ satisfies $((D_n){\vee}z){\wedge}y_1{\wedge}y_2$ but not D_n, while $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ satisfies D_n but, if $n > 1$, not $((D_{n-1}){\vee}z){\wedge}y_1{\wedge}y_2$.*

Hence every inclusion in (4) is strict except the second. Equality holds at that step.

Proof. To see that equality holds at the second inclusion of (4), note that both $\mathrm{Conv}(\mathbb{R}^0)$ and $\mathrm{Conv}(\mathbb{R}^1)_{\geq\{0\}}$ are nontrivial lattices, which by Lemma 3 satisfy D_1, the distributive identity, and that the variety of distributive lattices is known to have no proper nontrivial subvarieties.

The positive assertions of the theorem's first paragraph follow from Lemma 3, combined, in the case of the first of these results, with the observations of the first two paragraphs of §3. (Those observations are equivalent to the contrapositive of the easy implications (i)(a) $\Longrightarrow$ (ii)(a) $\Longrightarrow$ (ii)(b) of Lemma 5.) It remains to give examples showing the negative assertions.

To see that $\mathrm{Conv}(\mathbb{R}^n)$ does not satisfy D_n, let $q_1, \dots, q_{n+1}$ be the vertices of an n-simplex in $\mathbb{R}^n$ and p an interior point of this simplex, and take for the y_i and x the singletons $\{q_i\}$ and $\{p\}$ respectively. Then we see that the left-hand side of $D_n(x, y_1, \dots, y_{n+1})$ gives x, while all the joinands on the right are empty, hence so is the right-hand side itself.

To show that $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ does not satisfy $((D_{n-1}){\vee}z){\wedge}y_1{\wedge}y_2$ when $n > 1$, let $P \subseteq \mathbb{R}^n$ be a hyperplane not passing through 0. The idea will be to mimic the preceding example within P, then replace the resulting singleton sets with the line-segments connecting them with 0, slightly enlarge y_1, so that it has nonzero intersection with y_2, and finally apply part (i) of Lemma 6.

So let $q_1, \dots, q_n$ be the vertices of an $(n{-}1)$-simplex in P, and p a point in the relative interior of that simplex, and let x and the y_i be the line segments $\mathrm{c.h.}(0, p)$ and $\mathrm{c.h.}(0, q_i)$ respectively, except for y_1, which we take to be $\mathrm{c.h.}(0, q_1, q_2/2)$. Note that all of these convex sets lie in the closed half-space H bounded by P and containing 0; hence the intersection with P of any lattice expression in these convex sets can be computed as the corresponding lattice expression in their intersections with P. We see that these intersections are a configuration of the form given in the preceding example, except that the dimension is lower by 1. (Note that the point “ $q_2/2$” in our definition of y_1, not lying in P, does not affect the intersection $y_1 \cap P$.) Hence when we evaluate the two sides of D_{n-1} at these elements, the left-hand side intersects P in the point p, and is thus the whole line-segment x, while the right-hand side does not meet P, hence is a proper subsegment of x.

We can now deduce the failure of $((D_{n-1}) \vee z) \wedge y_1 \wedge y_2$ from the implication (i)(b) $\Longrightarrow$ (i)(a) of Lemma 6, using for u and v respectively the left and right sides of D_{n-1}, and for w the set $y_1 \wedge y_2$. To see that (i)(b) holds for these sets, note that w is the line-segment c.h.$(0, q_2/2)$. Since this lies in a different line through the origin from p, the ray drawn from any nonzero point q of that segment through p does not meet the line-segment x in any point other than p, so in particular, it does not contain any point of v, the right-hand side of D_{n-1}, which we saw was a proper subset of x. □

We remark that by alternately inserting joins and meets with more and more variables $z^{(m)}$ into “$((D_n)\ \dots) \wedge (y_1 \wedge y_2)$” one can get identities that, formally, are successively stronger (though still all implied by D_n), so that the statements that a lattice does not satisfy these identities become successively weaker. Thus, as a stronger version of the above theorem, we could have stated that $\mathrm{Conv}(\mathbb{R}^n)$ satisfies such identities with arbitrarily long strings of inserted terms, while $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ fails to satisfy the particular one given above. But for simplicity, I used just one identity to distinguish the properties of these lattices.

The argument at the beginning of the preceding section showing that $\mathrm{Conv}(\mathbb{R}^n)$ satisfies $((D_n) \vee z) \wedge y_1 \wedge y_2$ also clearly shows that it satisfies the formally stronger identity

$$((D_n) \vee z)\ \wedge\ (\textstyle\bigvee_{i,j;\ i \neq j}\ y_i \wedge y_j). \tag{6}$$

With a little additional work one can get the still stronger identity:

$$((D_n) \vee z)\ \wedge\ \textstyle\bigwedge_i\,(\bigvee_{j \neq i}\ y_j). \tag{7}$$

(Idea: If $\bigwedge_i (\bigvee_{j \neq i}\ y_j)$ is nonempty, use a point thereof as the p_0 in the second sentence of Carathéodory’s Theorem.)

We remark that the fact that $\mathrm{Conv}(\mathbb{R}^n)$ satisfies the identity D_{n+1} says that for any $n{+}2$ convex sets y_i, the union of the $(n{+}1)$-fold joins $\bigvee_{j \neq i} y_j$ $(i = 1, \dots, n{+}2)$ is itself convex, while (7) says essentially that the same holds for the union of the n-fold joins of $n{+}1$ convex sets, if those joins have at least one point in common.

5. Dual n-distributivity

Huhn showed not only that $\mathrm{Conv}(\mathbb{R}^n)$ satisfies D_{n+1}, but also that it satisfies the dual of that identity. Let us write the dual of the relation $D_n(x, y_1, \dots, y_{n+1})$ as

$$D_n^{\mathrm{op}}(x, y_1, \dots, y_{n+1}):\quad x \vee (\textstyle\bigwedge_i\ y_i)\ =\ \bigwedge_i\ (x \vee \bigwedge_{j \neq i}\ y_j). \tag{8}$$

Just as in D_n the direction $\geq$ is automatic, so $\leq$ is automatic in D_n^{op}.

Below, we will strengthen Huhn’s result that D_{n+1}^{op} holds identically in $\mathrm{Conv}(\mathbb{R}^n)$ by showing that it holds in the larger lattice $\mathrm{Conv}(\mathbb{R}^{n+1})_{\geq\{0\}}$, and will again use the method of §3 to manufacture a related identity which holds in $\mathrm{Conv}(\mathbb{R}^n)$ but not in $\mathrm{Conv}(\mathbb{R}^{n+1})_{\geq\{0\}}$. Like Huhn, we start with

Helly's Theorem ([6], p.391). *Let $n \geq 0$. If a finite family of convex subsets of $\mathbb{R}^n$ has the property that every $n{+}1$ of them have nonempty intersection, then the whole family has nonempty intersection.*

We will also use the following observation.

Lemma 9. *Let V be any real vector space. Given a convex set x and a point p in V, there exists a convex set w in V such that for every nonempty convex set y, one has $p \in x \vee y$ if and only if y has nonempty intersection with w.*

Proof. It is straightforward to check that a set w with the required property (in fact, the unique such set) is the cone consisting of the union of all rays from p which meet the central reflection of x through p. (If $p \in x$ then $w = V$; in visualizing the contrary case, it is convenient to assume without loss of generality that $p = 0$.) □

We can now show that certain sorts of families of convex sets satisfy the relation D_n^{op}, from which we will deduce our identities.

Lemma 10. *Let n be a natural number, and $x, y_1, \dots, y_{n+1}$ be $n{+}2$ elements of $\mathrm{Conv}(\mathbb{R}^n)$ such that $y_1 \wedge \dots \wedge y_{n+1} \neq \emptyset$. Then $D_n^{\mathrm{op}}(x, y_1, \dots, y_{n+1})$ holds.*

Proof. Given

$$p \in \textstyle\bigwedge_i \, (x \vee \bigwedge_{j \neq i} y_j) \tag{9}$$

we need to show that

$$p \in x \vee (\textstyle\bigwedge_i y_i). \tag{10}$$

Let w be the set determined by x and p as in Lemma 9. Since by (9), p lies in each of the sets $x \vee \bigwedge_{j \neq i} y_j$, our choice of w shows that for each i, $(\bigwedge_{j \neq i} y_j) \wedge w$ is nonempty. These conditions together with the nonemptiness of $y_1 \wedge \dots \wedge y_{n+1}$ allow us to apply Helly's Theorem to the $n{+}2$ convex sets $y_1, \dots, y_{n+1}, w$, and conclude that $(\bigwedge_i y_i) \wedge w$ is nonempty, which by choice of w is equivalent to (10). □

Now for each positive integer n, let $(((D_n^{\mathrm{op}}) \wedge z') \vee z) \wedge (\bigwedge_i y_i)$ denote the relation in variables x, y_1, $\dots$, y_{n+1}, z, z' obtained by applying the operation $(((-) \wedge z') \vee z) \wedge (\bigwedge_i y_i)$ to both sides of (8). Then we have

Theorem 11 (cf. Huhn [10]). *For each positive integer n, $\mathrm{Conv}(\mathbb{R}^n)$ satisfies the identity $(((D_n^{\mathrm{op}}) \wedge z') \vee z) \wedge (\bigwedge_i y_i)$ but not D_n^{op}, while $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ satisfies the identity D_n^{op} but, if $n > 1$, not $(((D_{n-1}^{\mathrm{op}}) \wedge z') \vee z) \wedge (\bigwedge_i y_i)$.*

Proof. The positive assertions follow from Lemma 10. (Again, the reasoning that obtains the first identity from that lemma can be considered, formally, an application of the contrapositive of an easy direction in of one of our lemmas, in this case the implication (ii)(a) $\Longrightarrow$ (iii)(b) of Lemma 6.)

To get a counterexample to D_n^{op} in $\mathrm{Conv}(\mathbb{R}^n)$, take for x an n-simplex in $\mathbb{R}^n$, let p be a point outside x, let x' be the central reflection of x through p, and let $y_1, \dots, y_{n+1}$ be the $(n-1)$-faces of x'. Then $\bigwedge_i y_i$ is empty, so the left-hand side of (8) is just x, while each of the intersections $\bigwedge_{j \neq i} y_j$ is a nonempty subset of x', so the right-hand side of (8) contains p.

As in the proof of Theorem 8, we begin the counterexample to our more elaborate identity in $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ by taking a copy of our preceding example, for the next lower dimension, in a hyperplane $P \subseteq \mathbb{R}^n$ not containing 0. Let us write p_0 and $x_0,\ x'_0,\ y_{10},\ \dots,\ y_{n0}$ for the point and family of convex subsets of P so obtained. Let us also write q_0 for the vertex of x'_0 opposite to the face y_{10}.

To beef these sets up to the desired members of $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$, we now take

$$x = \mathrm{c.h.}(x_0 \cup \{0\}), \quad y_i = \mathrm{c.h.}(y_{i0} \cup \{0\}) \ \text{ for } i \neq 1, \quad y_1 = \mathrm{c.h.}(y_{10} \cup \{0, q_0/2\}).$$

For the same reason as in the proof of Theorem 8, the operation of intersecting with P commutes with lattice operations on these convex sets; hence when D_{n-1}^{op} is evaluated at the above arguments, the left-hand side meets the plane P only in the set x_0, while the right-hand side will also contain the point p_0. Now observe that $\bigwedge_i y_i$ will be the line-segment $\mathrm{c.h.}(0, q_0/2)$. Letting $p = p_0,\ q = q_0/2$, we see that these lie on different lines through 0, and deduce from the implication (ii)(b) $\implies$ (ii)(a) of Lemma 6 that z and z' can be chosen so that the required inequality holds. $\square$

6. Encore!

Carathéodory's and Helly's Theorems are two members of a well-known triad of results on convex sets in $\mathbb{R}^n$. The third result is

Radon's Theorem ([6], p.391)**.** *Given a natural number* n*, and* $n{+}2$ *points* $p_1, \dots, p_{n+2}$ *in* $\mathbb{R}^n$*, there exists a partition of* $\{1, \dots, n{+}2\}$ *into subsets* I_1 *and* I_2 *such that* $\mathrm{c.h.}(\{p_i \mid i \in I_1\}) \cap \mathrm{c.h.}(\{p_i \mid i \in I_2\}) \neq \emptyset$.

Can we turn this, too, into an identity for lattices of convex sets?

Yes. First let's get rid of reference to points: Clearly an equivalent statement is "Given nonempty convex sets $y_1, \dots, y_{n+2}$ in $\mathbb{R}^n$, there exists a partition of $\{1, \dots, n{+}2\}$ into subsets I_1 and I_2 such that $(\bigvee_{i \in I_1} y_i) \wedge (\bigvee_{i \in I_2} y_i) \neq \emptyset$." Next, the conclusion that such a partition exists can be condensed into the single inequality $\bigvee_{I_1, I_2} ((\bigvee_{i \in I_1} y_i) \wedge (\bigvee_{i \in I_2} y_i)) \neq \emptyset$, where the outer join is over the $2^{n+2} - 2$ partitions of $\{1, \dots, n{+}2\}$ into two nonempty subsets. In the proof of the theorem below, Lemma 9 will be used to turn the above implication between nonemptiness statements into a lattice relation.

In that theorem, I call an inequality $a \leq b$ that always holds an "identity", since it can be rewritten as an equation $a = a \wedge b$.

Theorem 12. *For every natural number* n*, the lattice identity in* $n{+}3$ *variables* $x, y_1, \dots, y_{n+2}$

$$\bigwedge_i (x \vee y_i) \ \leq \ x \ \vee \ \bigvee_{I_1, I_2} ((\bigvee_{i \in I_1} y_i) \wedge (\bigvee_{i \in I_2} y_i)), \tag{11}$$

where "$\bigvee_{I_1,I_2}$" denotes the join over all partitions of $\{1,\dots,n{+}2\}$ *into two nonempty subsets* I_1 *and* I_2, *holds in* $\mathrm{Conv}(\mathbb{R}^n)$, *and indeed in* $\mathrm{Conv}(\mathbb{R}^{n+1})_{\geq\{0\}}$, *but not in* $\mathrm{Conv}(\mathbb{R}^{n+1})$.

Proof. We shall first prove (11) in the simpler case of $\mathrm{Conv}(\mathbb{R}^n)$, then show how to adapt the proof to $\mathrm{Conv}(\mathbb{R}^{n+1})_{\geq\{0\}}$, and finally, give the counterexample in $\mathrm{Conv}(\mathbb{R}^{n+1})$.

Given $x, y_1, \dots, y_{n+2} \in \mathrm{Conv}(\mathbb{R}^n)$ and a point p belonging to the left-hand side of (11), we must show that p also belongs to the right-hand side. Let us choose w as in Lemma 9 for the given x and p. The assumption that p belongs to the left-hand side of (11) says that it belongs to each of the meetands of that expression, which by choice of w means that $w \wedge y_i$ is nonempty for all i. Hence Radon's Theorem applied to those $n{+}2$ sets says that for some I_1, I_2 partitioning $\{1,\dots,n{+}2\}$, we have $\emptyset \neq (\bigvee_{I_1} w \wedge y_i) \wedge (\bigvee_{I_2} w \wedge y_i)$. The latter set is contained in $w \wedge (\bigvee_{I_1} y_i) \wedge (\bigvee_{I_2} y_i)$, and the statement that this is nonempty now translates back to say that $x \vee ((\bigvee_{I_1} y_i) \wedge (\bigvee_{I_2} y_i))$ contains p, whence p belongs to the right-hand side of (11), as required.

If we are given $x, y_1, \dots, y_{n+2} \in \mathrm{Conv}(\mathbb{R}^{n+1})_{\geq\{0\}}$, and a point p on the left-hand side of (11), we begin in the same way, translating the hypothesis and desired conclusion to the same statements about the sets $w \wedge y_i$ (though the convex set w will not in general belong to $\mathrm{Conv}(\mathbb{R}^{n+1})_{\geq\{0\}}$, and so neither will these intersections). This time we apply Radon's Theorem in $n{+}1$ dimensions to the sets $w \wedge y_i$ together with $\{0\}$. In the partition given by that theorem, let us assume without loss of generality that $\{0\}$ goes into the second join; thus we get a relation $\emptyset \neq (\bigvee_{I_1} w \wedge y_i) \wedge (\{0\} \vee \bigvee_{I_2} w \wedge y_i)$. Here the first join is contained in w, hence so is the whole set, so that set is contained in the meet of w with the larger set $(\bigvee_{I_1} y_i) \wedge (\{0\} \vee \bigvee_{I_2} y_i)$. Since all the y's belong to $\mathrm{Conv}(\mathbb{R}^{n+1})_{\geq\{0\}}$, the joinand $\{0\}$ is now redundant, so we have again shown that $w \wedge (\bigvee_{I_1} y_i) \wedge (\bigvee_{I_2} y_i)$ is nonempty for some partition $\{1,\dots,n{+}2\} = I_1 \vee I_2$, which, as before, yields the desired conclusion by choice of w.

To show that (11) does not hold in $\mathrm{Conv}(\mathbb{R}^{n+1})$, we begin essentially as in the first counterexample in the proof of Theorem 11, letting x be an $(n{+}1)$-simplex in $\mathbb{R}^{n+1}$, p a point outside that simplex, and x' the central reflection of x through p. This time, however, we let $y_1, \dots, y_{n+2}$ be singletons, whose unique elements are the vertices of x'. As in the earlier example we find that one side of the identity in question (in this case the left-hand side of (11)) contains p, while the other is simply x, since all joinands in the "big join" on that side are empty; so the right-hand side does not majorize the left-hand side. □

I have not tried to fill in this picture, as I did with the identities D_n and D_n^{op}, by looking for related identities that would separate all distinct terms of (4); but I expect that these exist.

Another observation which I have not followed up on, because it occurred to me late in the preparation of this paper, is that if one defines $\mathrm{Conv}(\mathbb{R}^n, \mathrm{cone}) \subseteq \mathrm{Conv}(\mathbb{R}^n)_{\geq 0}$ to be the sublattice consisting of those convex sets which are unions of rays through 0, then $\mathrm{Conv}(\mathbb{R}^{n-1})$ embeds naturally in $\mathrm{Conv}(\mathbb{R}^n, \mathrm{cone})$, yielding a refinement of the chain of varieties of Lemma 1. It would be interesting to know whether for $n>1$ the varieties so interpolated are distinct from those that precede and follow them.

7. The sublattice of compact convex sets

From Carathéodory's Theorem, we see that

$$\text{For any compact subset } S \subseteq \mathbb{R}^n, \quad \text{c.h.}(S) \text{ is also compact.} \tag{12}$$

Hence the join in $\mathrm{Conv}(\mathbb{R}^n)$ of two compact subsets is compact, hence the set of compact convex subsets of $\mathbb{R}^n$ (often called "convex bodies" in the literature, e.g., [9], [1, §3.1], [19, §12]) is a sublattice $\mathrm{Conv}(\mathbb{R}^n, \mathrm{cpct}) \subseteq \mathrm{Conv}(\mathbb{R}^n)$. An obvious question is how the identities of this sublattice compare with those of $\mathrm{Conv}(\mathbb{R}^n)$; i.e., whether it satisfies any identities that the larger lattice does not. Huhn [11, proof of Lemma 3.1] answered this question in the negative, by showing that the still smaller lattice of polytopes (convex hulls of finite sets) does not. Let me give a slightly different proof of the same result. Huhn used the fact that an intersection of polytopes is a polytope, but the next result is applicable to a finitary closure operator that need not have the property that an intersection of closures of finite sets is again one. (The meaning of "finitary" was recalled in the paragraph preceding Lemma 4; the notation $\mathbf{V}(\dots)$ used below was defined in §1.)

Proposition 13. *Let* cl *be a finitary closure operator on a set* X, *and let* L *be the lattice of subsets of* X *closed under* cl. *Then every lattice relation satisfied by all families of elements of* L *that are closures of finite subsets of* X *is an identity of* L. *In particular, if* L' *is any sublattice of* L *which contains all closures of singleton subsets of* X, *then* $\mathbf{V}(L') = \mathbf{V}(L)$.

Proof. Let us topologize the power set 2^X by taking as a basis of open sets the sets

$$U(A,B) \;=\; \{Y \in 2^X \mid A \subseteq Y \subseteq B\},$$

where A ranges over the finite subsets of X, and B over arbitrary subsets. This is stronger than the usual power-set topology, which only uses the sets of the above form with B cofinite. Thus, our topology is Hausdorff, though not in general compact; hence its restriction to L is also Hausdorff, with basis of open sets given by the sets $U_L(A,B) = U(A,B) \cap L$. (Of course, $U_L(A,B)$ is nonempty only when $\mathrm{cl}(A) \subseteq B$.)

We claim that under this topology, the lattice operations of L are continuous, and the closures of finite subsets of X are dense in L. This will imply that any lattice identity holding on that dense subset must hold on all of L, from which the final conclusion will clearly follow.

To see that closures of finite sets are dense, note that every nonempty basic set $U_L(A,B)$ contains the element $\mathrm{cl}(A)$, which is such a set.

The continuity of the meet operation is also straightforward: If $x, y \in L$ are such that $x \wedge y$, i.e., $x \cap y$, lies in $U_L(A,B)$, then $U_L(A, x \cup B)$ and $U_L(A, y \cup B)$ are neighborhoods of x and y respectively such that the intersection of any member of the first neighborhood and any member of the second lies in $U_L(A,B)$.

Finally, suppose $x, y \in L$ are such that $x \vee y$, i.e., $\mathrm{cl}(x \cup y)$, lies in $U_L(A,B)$. Then the finite set A is contained in $\mathrm{cl}(x \cup y)$, so by finitariness of cl, there is a finite subset of $x \cup y$ whose closure contains all elements of A; let us write this subset as $A_x \cup A_y$ where $A_x \subseteq x$ and $A_y \subseteq y$. Then $U_L(A_x, x)$ and $U_L(A_y, y)$ will be neighborhoods of x and y respectively such that the join of any member of the first neighborhood and any member of the second is a member of $U_L(A,B)$. (In the power-set topology, $\vee$ is generally discontinuous; this is why we needed a different topology.) $\square$

For $X = \mathbb{R}^n$ and $\mathrm{cl} = \mathrm{c.h.}$, the L of the above proposition is $\mathrm{Conv}(\mathbb{R}^n)$. Since the convex hull of a finite set is compact, we can apply the last sentence of the proposition with $L' = \mathrm{Conv}(\mathbb{R}^n, \mathrm{cpct})$, getting the first statement of the next theorem. Taking $\mathrm{cl} = \mathrm{c.h.}(\{0\} \cup -)$ we similarly get the second.

Theorem 14 (Huhn)**.** *For every natural number n,*

$$\mathbf{V}(\mathrm{Conv}(\mathbb{R}^n)) \ = \ \mathbf{V}(\mathrm{Conv}(\mathbb{R}^n,\, \mathrm{cpct})),$$

and

$$\mathbf{V}(\mathrm{Conv}(\mathbb{R}^n)_{\geq \{0\}}) \ = \ \mathbf{V}(\mathrm{Conv}(\mathbb{R}^n,\, \mathrm{cpct})_{\geq \{0\}}).$$

$\square$

However, $\mathrm{Conv}(\mathbb{R}^n, \mathrm{cpct})$ is known also to have interesting elementary properties not possessed by $\mathrm{Conv}(\mathbb{R}^n)$. Let us recall that an *extremal point* of a convex set means a point which is not in the convex hull of any two other points of the set, and the following result ([9, p.276]).

Theorem (Minkowski)**.** *Every compact convex subset of $\mathbb{R}^n$ is the convex hull of its set of extremal points.*

Recall also that a lattice L is called *join semidistributive* if for all $x, y_1, y_2 \in L$ one has

$$x \vee y_1 = x \vee y_2 \quad \Longrightarrow \quad x \vee y_1 = x \vee (y_1 \wedge y_2). \tag{13}$$

Lemma 15 (= [1, Theorem 3.4], generalizing [5, Theorem 15])**.** *For every positive integer n, $\mathrm{Conv}(\mathbb{R}^n, \mathrm{cpct})$ is join semidistributive.*

Proof. Every extremal point of a join $x \vee y$ must belong to x or to y, since by definition it cannot arise as a convex combination of other points of $x \vee y$, hence if $x \vee y_1 = x \vee y_2$ as in the hypothesis of (13), extremal points of this set that do not belong to x must belong to y_1, and likewise to y_2. Thus every extremal point of $x \vee y_1 = x \vee y_2$ belongs to $x \cup (y_1 \cap y_2)$; hence by Minkowski's Theorem the convex hull of the latter set, $x \vee (y_1 \wedge y_2)$, contains the former set. The reverse inclusion is trivial. $\square$

On the other hand, for $n \geq 2$, $\mathrm{Conv}(\mathbb{R}^n)$ is not join semidistributive; indeed, the next result will show the failure in these lattices, as n increases, of successively weaker properties, beginning with join-semidistributivity. Following Geyer [8], let us say that a lattice L is *n-join semidistributive* for a positive integer n if for all $x, y_1, \dots, y_{n+1} \in L$ one has

$$x \vee y_1 = \dots = x \vee y_{n+1} \quad \Longrightarrow \quad x \vee y_1 = x \vee (\bigvee_{i,j;\ i \neq j} y_i \wedge y_j). \tag{14}$$

Thus, join semidistributivity is the $n = 1$ case. One defines *n-meet semidistributivity* dually.

The $(n{+}4)$-element lattice of height 2, M_{n+2}, with least element 0, greatest element 1, and $n{+}2$ incomparable elements $y_1, \dots, y_{n+2}$, is neither n-join semidistributive nor n-meet semidistributive, as may be seen by putting y_{n+2} in the role of x in (14), and in the dual statement. We shall now see that there are several sorts of sublattices with that structure within the lattices $\mathrm{Conv}(\mathbb{R}^n)$. The "open bounded" case of the next result was shown to me by D. Wasserman.

Lemma 16 (Wasserman and Bergman)**.** *For every* $n > 1$, $\mathrm{Conv}(\mathbb{R}^n)$ *contains copies of* M_{n+1} *consisting of open bounded sets and copies consisting of closed unbounded sets, in both cases with least element* $\emptyset$; *and* $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ *contains copies of* M_{n+1} *consisting of bounded sets, with least element* $\{0\}$. *Also,* $\mathrm{Conv}(\mathbb{R}^2)_{\geq\{0\}}$ (*and hence* $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ *for all* $n > 1$) *contains copies of* M_c, *the height-*2 *lattice of continuum cardinality* (*and hence contains copies of its sublattices* M_m *for all natural numbers* m) *consisting of vector subspaces, with least element* $\{0\}$.

In particular, for $n > 1$ *neither* $\mathrm{Conv}(\mathbb{R}^n)$ *nor* $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ *is* N*-join or* N*-meet semidistributive for any* N, *and the sublattices of open bounded sets in* $\mathrm{Conv}(\mathbb{R}^n)$ *and of bounded sets in* $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ *are not* $(n{-}1)$*-join or* $(n{-}1)$*-meet semidistributive.*

Proof. To get a copy of M_{n+1} consisting of open bounded sets, start with any n-simplex x, let its faces be $x_1, \dots, x_{n+1}$, choose an interior point p of x, and for each i let y_i be the interior of the n-simplex $\{p\} \vee x_i$ (or any open convex subset of that n-simplex which has the whole face x_i in its closure). We see that the join of any two of the y's will have in its closure two faces of x, hence all vertices of x, hence its closure must be x, hence being itself open and convex, it must be the interior of x. On the other hand, the pairwise intersections of the y_i are all empty. Hence the lattice generated by these sets is isomorphic to M_{n+1}.

For the closed unbounded example with least element again $\emptyset$, extend each of the sets in the preceding example to an infinite cone with apex p, displace each of these cones away from p (say by translating it by the vector from the opposite vertex of x to p), and take their closures. Then every pairwise join is seen to be the whole of $\mathbb{R}^n$, while every pairwise meet is again empty.

For the bounded example in $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$, take the first example above, assuming $p = 0$, and use as our new y_i the union of the y_i of that example with $\{0\}$. (Thus, 0 will be the unique boundary-point belonging to each of these sets.)

Finally, an M_c in $\mathrm{Conv}(\mathbb{R}^2)_{\geq\{0\}}$ is given by the set of all lines through 0. □

There are also cases where we can show the failure of m-join semidistributivity in a natural lattice of convex sets, but where that lattice probably does not contain a copy of M_{m+2}. To get such an example (for any $m \geq 1$) in the lattice of convex open subsets of $\mathbb{R}^2$ containing 0, take $m{+}2$ distinct lines through 0 and "thicken" these to open sets $x, y_1, \ldots, y_{m+1}$ of width 1. The lattice that these generate will not be M_{m+2}, but clearly fails to satisfy (14). We can also get such examples for bounded open sets in $\mathbb{R}^2$, though in this case they fail to have a common point: Fix a triangle T, and let x be the interior of any triangle lying inside T and sharing one edge with T, but not the opposite vertex p. Then take $m+1$ "small narrow" triangles inside T that have p as a common vertex but no other point in common, and the convex hull of whose union is disjoint from x, and let $y_1, \ldots, y_{m+1}$ be their interiors.

The parenthetical comment at the end of the first sentence of the proof of the Lemma 16 shows that the shapes of the convex sets forming a copy of M_{n+1} in the lattice of open bounded convex sets are not unique; but I don't know an example where the top element of such a sublattice is not an open n-simplex. It would also be of interest to know whether that lattice contains copies of M_{n+2}, and if not, whether it is n-join or n-meet semidistributive. We shall obtain a few related results in later sections.

Jónsson and Rival [13, Lemma 2.1] show that a lattice is join and meet semidistributive if and only if two auxiliary overlattices contain no isomorphic copies of any member of a certain list of 6 lattices, beginning with M_3. The above "small narrow triangle" construction gives, when $m=2$, a copy of the lattice L_4 of their list.

Incidentally, Geyer's concept of n-join semidistributivity, which we have been using, does not have any obvious relationship with Huhn's n-distributivity. Although for $n=1$ they give the conditions of join-semidistributivity and distributivity respectively, of which the latter implies the former, no such implication holds for larger n. For instance, the lattice M_c is 2-distributive in Huhn's sense, but it is not N-join semidistributive for any natural number N in Geyer's sense.

8. Open bounded sets do not satisfy additional identities

Let us denote by $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})$ the lattice of open bounded convex subsets of $\mathbb{R}^n$. We shall show that this lattice, like $\mathrm{Conv}(\mathbb{R}^n, \text{cpct})$, satisfies the same identities as $\mathrm{Conv}(\mathbb{R}^n)$. The idea is that compact sets can be approximated by open bounded sets containing them, from which we shall deduce that any identities of $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})$ are also identities of $\mathrm{Conv}(\mathbb{R}^n, \text{cpct})$, and so by Theorem 14 are identities of $\mathrm{Conv}(\mathbb{R}^n)$.

To approximate compact sets by open sets, we need a different topology on $2^{\mathbb{R}^n}$ from the one used earlier; let us again describe this in a general context. If X is any topological space, we may topologize 2^X using a basis of open sets with the same form as before,

$$U(A,B) \;=\; \{Y \in 2^X \mid A \subseteq Y \subseteq B\}, \qquad (15)$$

but where, this time, A ranges over all subsets of X, while B is restricted to open subsets. If X is Hausdorff (or even T1), we see that this family of open sets again includes those defining the power-set topology, so our topology is again Hausdorff. Note that for each $A \in 2^X$, the sets (15) with A as first argument and with second argument containing A form a neighborhood basis of A in 2^X. Thus in proving the next lemma, we shall take it as understood that to be "sufficiently close to" a set A means to contain A and be contained in some specified open neighborhood B of A in X.

In the formulation of that lemma, note that the statement that a function is continuous at arguments with a given property does not simply mean that the restriction of the function to the set of such arguments is continuous, but, more, that such arguments are points of continuity of the whole function.

Lemma 17. *If X is any topological space, then the binary operation $\cup : 2^X \times 2^X \to 2^X$ is continuous in the topology described above; if X is normal* (*i.e., if disjoint closed subsets of X have disjoint open neighborhoods*) *then the binary operation $\cap$ is continuous at arguments given by pairs of closed sets, and if $X = \mathbb{R}^n$ with the usual topology, then the unary convex-hull operation* $\mathrm{c.h.} : 2^X \to 2^X$ *is continuous at compact sets.*

Proof. To show continuity of $\cup$, consider sets A_1, A_2, and an open neighborhood B of $A_1 \cup A_2$ in X. Then we see that the union of any member of $U(A_1, B)$ and any member of $U(A_2, B)$ contains $A_1 \cup A_2$ and is contained in B, as required.

For the case of $\cap$, let A_1 and A_2 be closed sets, and B any open neighborhood of $A_1 \cap A_2$ in X. Then $A_1 - B$ and $A_2 - B$ are disjoint closed sets, hence they have disjoint open neighborhoods C_1 and C_2. We see that $B \cup C_1$ and $B \cup C_2$ will be open neighborhoods of A_1 and A_2 which intersect in B (by distributivity of the lattice 2^X), and it follows that the intersection of a member of $U(A_1, B \cup C_1)$ and a member of $U(A_2, B \cup C_2)$ will belong to $U(A_1 \cap A_2, B)$, as required.

For the final assertion, let A be a compact subset of $\mathbb{R}^n$, and B any open neighborhood of $\mathrm{c.h.}(A)$. By compactness of $\mathrm{c.h.}(A)$, there is some $\epsilon > 0$ such that the set C of all points having distance $< \epsilon$ from $\mathrm{c.h.}(A)$ is contained in B. This set C is a convex open neighborhood of $\mathrm{c.h.}(A)$, hence c.h. carries $U(A, C)$ into $U(\mathrm{c.h.}(A), C) \subseteq U(\mathrm{c.h.}(A), B)$, as required. □

(I played with several topologies before getting the one that made the above result – in particular, continuity of intersection – easy to prove. Some of these might be preferable for other considerations of the same sort. Under the above topology, every open set A is an isolated point, since $U(A, A)$ is a singleton. If one wants to approximate open sets by larger open sets, one might prefer a weaker topology in which the conditions on Y in (15) are, say, strengthened to $A \subseteq Y$, $\mathrm{cl}(Y) \subseteq B$.)

It follows from Lemma 17 that in the topology we have defined, the lattice operations of $\mathrm{Conv}(\mathbb{R}^n)$ are continuous at arguments belonging to $\mathrm{Conv}(\mathbb{R}^n, \mathrm{cpct})$. Moreover, $\mathrm{Conv}(\mathbb{R}^n, \mathrm{cpct})$ lies in the closure of $\mathrm{Conv}(\mathbb{R}^n, \mathrm{o.bdd.})$, since every compact convex set A is the limit in this topology, as $\epsilon \to 0$, of the open convex set of points at distance $< \epsilon$ from A. Hence any lattice identities holding in $\mathrm{Conv}(\mathbb{R}^n, \mathrm{o.bdd.})$ also hold in $\mathrm{Conv}(\mathbb{R}^n, \mathrm{cpct})$. The same considerations apply to the pair of lattices $\mathrm{Conv}(\mathbb{R}^n, \mathrm{o.bdd.})_{\geq\{0\}}$ and $\mathrm{Conv}(\mathbb{R}^n, \mathrm{cpct})_{\geq\{0\}}$. In view of Theorem 14, these observations give us

Theorem 18. *For every natural number* n, $\mathbf{V}(\mathrm{Conv}(\mathbb{R}^n)) = \mathbf{V}(\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.}))$, *and* $\mathbf{V}(\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}) = \mathbf{V}(\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq\{0\}})$. □

Let us note that there is an order-preserving bijection between the elements of $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})$ and those compact convex subsets of $\mathbb{R}^n$ which, if nonempty, have nonempty interior, given by the operation of topological closure, with inverse given by topological interior. This is not, however, an isomorphism between sublattices of $\mathrm{Conv}(\mathbb{R}^n)$, because the second of these sets is not closed under intersection. (E.g., consider two adjacent closed polygons in $\mathbb{R}^2$. Nor can we get around this problem by going to a homomorphic image of $\mathrm{Conv}(\mathbb{R}^n, \text{cpct})$ where sets with empty interior are identified with $\emptyset$, since the join of two such sets can have nonempty interior.)

On the other hand, the set of compact convex sets which are neighborhoods of 0 *is* a sublattice of $\mathrm{Conv}(\mathbb{R}^n, \text{cpct})_{\geq\{0\}}$, and the above correspondence gives us an isomorphism between it and $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq\{0\}}$; so properties of $\mathrm{Conv}(\mathbb{R}^n, \text{cpct})_{\geq\{0\}}$ which carry over to sublattices also hold for $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq\{0\}}$. (More generally, any sublattice of $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})$ whose members have a common point p is contained in $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq\{p\}} \cong \mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq\{0\}}$, and so can be studied in the same fashion.) Thus, despite the examples of Lemma 16, we have

Corollary 19 (to Lemma 15)**.** $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq\{0\}}$ *is join semidistributive. Hence, every counterexample to join semidistributivity in* $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})$ *has the property that the intersection of the three sets involved is empty.* □

Since we are considering elementary properties in which the lattices $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})$ agree or differ, we should note the obvious difference, that the former is atomistic (every element is a possibly infinite join of atoms), while the latter has no atoms. Cf. [2] and papers referred to there, in which lattices of convex sets and related structures are characterized in terms of properties of their atoms, and also [1].

We noted above that lattices $\mathrm{Conv}(\mathbb{R}^n)_{\geq S}$ for nonempty S satisfy all identities holding in $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$. Let us end this section by using Theorem 18 to show that for bounded S, the converse is also true.

Corollary 20 (to Theorem 18 and proof of Theorem 14)**.** *For every natural number* n *and every bounded set* $S \subseteq \mathbb{R}^n$,

$$\begin{aligned} &\mathbf{V}(\mathrm{Conv}(\mathbb{R}^n)_{\geq S}) \;=\; \mathbf{V}(\mathrm{Conv}(\mathbb{R}^n,\ \text{cpct})_{\geq S}) \\ =\;& \mathbf{V}(\mathrm{Conv}(\mathbb{R}^n,\ \text{o.bdd.})_{\geq S}) \;=\; \mathbf{V}(\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}). \end{aligned}$$

Proof. By a translation, we can assume without loss of generality that $0 \in S$; thus the last of the above lattices contains all the others, so letting $f(x_1, \dots, x_m) = g(x_1, \dots, x_m)$ be any identity not satisfied there, it suffices to prove that $f = g$ is not satisfied identically in any of the other lattices.

Now Theorem 18 shows that $f = g$ is not an identity of $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq\{0\}}$, hence we can choose open bounded convex sets $x_1, \dots, x_m$ containing 0 which do not satisfy it. The intersection of these sets is a neighborhood of the origin, and dilating the x_i by a large enough real constant, we can assume without loss of generality that this neighborhood contains S. Hence $f = g$ is also not an identity of $\mathrm{Conv}(\mathbb{R}^n,\ \mathrm{o.bdd.})_{\geq S}$, hence not an identity of $\mathrm{Conv}(\mathbb{R}^n)_{\geq S}$ either.

The case of $\mathrm{Conv}(\mathbb{R}^n,\ \mathrm{cpct})_{\geq S}$ is similar. Again take elements $x_1, \dots, x_m \in \mathrm{Conv}(\mathbb{R}^n,\ \mathrm{o.bdd.})_{\geq \{0\}}$ not satisfying $f = g$. We saw in the proof of Theorem 14 that if we approximate $x_1, \dots, x_m$ closely enough from below, in the topology of that proof, by compact convex subsets $y_1, \dots, y_m$, then these approximating sets will also fail to satisfy that identity. Since the intersection of the x_i is a neighborhood of 0, it contains an n-simplex with 0 in its interior; so we can take all the y_i to contain the finitely many vertices of that simplex, hence to be neighborhoods of 0. As before, we may now dilate them so that they all contain S, getting the required elements of $\mathrm{Conv}(\mathbb{R}^n,\ \mathrm{cpct})_{\geq S}$. □

9. The possibility of surface phenomena

The technique by which we just proved Corollary 20 can be inverted to show that if $T \subseteq \mathbb{R}^n$ is any convex set with nonempty interior, then the lattice of convex sets contained in T, and its sublattices of compact convex subsets of T and open bounded convex subsets of T, satisfy the same identities as $\mathrm{Conv}(\mathbb{R}^n)$. Namely, given any identity not holding in $\mathrm{Conv}(\mathbb{R}^n)$, we already know that we can find elements $x_1, \dots, x_m$ not satisfying it in $\mathrm{Conv}(\mathbb{R}^n,\ \mathrm{cpct})$. Assuming without loss of generality that 0 lies in the interior of T, we can shrink $x_1, \dots, x_m$ by a constant so that they are all contained in T; likewise we can, as before, approximate compact convex sets by open bounded convex sets. A similar argument shows that the lattice of convex subsets of T which contain a specified point of the interior of T satisfies the same identities as $\mathrm{Conv}(\mathbb{R}^n)_{\geq \{0\}}$.

However, if we specify two convex sets $S \subseteq T$, say with S compact and T open, and look at the interval $[S,\ T] = \{x \in \mathrm{Conv}(\mathbb{R}^n) \mid S \subseteq x \subseteq T\}$, it is not clear whether, for some choices of S and T, this may satisfy more identities than hold in $\mathrm{Conv}(\mathbb{R}^n)_{\geq \{0\}}$. (Picturing S and T as "very close", e.g., the closed ball of radius 1 and the open ball of radius $1 + \epsilon$, explains, I hope, the title of this section.)

Let us relax our assumptions on S and T for a moment and look at a more extreme example. If we take for S the open unit ball and for T the closed unit ball in $\mathbb{R}^n$, then every set x with $S \subseteq x \subseteq T$ is convex, so in this case $[S,\ T]$ may be identified with the lattice of all subsets of the unit sphere, which is distributive, though we have seen that $\mathrm{Conv}(\mathbb{R}^n)_{\geq \{0\}}$ is not even $(n-1)$-distributive.

In the case where S is compact and T open and nonempty, however, things cannot go that far:

Lemma 21. *Let $S \subseteq T$ be convex subsets of $\mathbb{R}^n$. If S is compact and T is open and nonempty, or more generally, if some hyperplane $P \subseteq \mathbb{R}^n$ disjoint from S intersects T in a set with nonempty relative interior (i.e., is such that $P \cap T$ is $(n-1)$-dimensional), then* $\mathbf{V}([S,T]) \supseteq \mathbf{V}(\mathrm{Conv}(\mathbb{R}^{n-1}))$.

Proof. Clearly, a pair $S \subseteq T$ satisfying the first hypothesis satisfies the second, so let us assume the latter. Let H be the closed half-space in $\mathbb{R}^n$ bounded by P

that contains S (or if S is empty, either of the closed half-spaces bounded by P), and consider the sublattice $[S,\, H\cap T]\subseteq [S,T]$. The operation of intersecting with P can be seen to give a lattice homomorphism $[S,\, H\cap T]\to[\emptyset, P\cap T]$, and this is surjective, since it has the set-theoretic section $x\mapsto S\vee x$. Since $P\cap T$ has nonempty interior in P, the observations of the first paragraph of this section show that $[\emptyset, P\cap T]$ satisfies precisely the identities of $\mathrm{Conv}(\mathbb{R}^{n-1})$. Hence $[S,\, H\cap T]$, since it maps homomorphically onto $[\emptyset, P\cap T]$, cannot satisfy any identities not satisfied by $\mathrm{Conv}(\mathbb{R}^{n-1})$, so neither can the larger lattice $[S,T]$. □

It is not evident whether, for S compact and T open, $\mathbf{V}([S,T])$ can ever be strictly smaller than $\mathbf{V}(\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}})$; nor, for that matter, whether it can ever fail to be strictly smaller, if S has nonempty interior and T is bounded. Another sort of interval $[S,T]\subseteq\mathrm{Conv}(\mathbb{R}^n)$ whose identities it would be interesting to investigate is given by letting $S=\{0\}$ and T be a closed half-space with 0 on its boundary. Again these identities must lie somewhere between those of $\mathrm{Conv}(\mathbb{R}^{n-1})$ and $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$.

In this and preceding sections we have used from time to time the fact that translations and dilations preserve convexity. More generally, if φ is any projective transformation on n-dimensional projective space $\mathbb{P}^n\supseteq\mathbb{R}^n$, then convex subsets of $\mathbb{R}^n$ which do not meet the hyperplane that φ sends to infinity are taken by φ to convex sets. Hence if $S\subseteq T$ are two such convex sets, φ induces a lattice isomorphism $[S,T]\cong[\varphi(S),\varphi(T)]$. This observation might be useful in classifying the varieties determined by such intervals.

Still another class of sublattices of $\mathrm{Conv}(\mathbb{R}^n)$ one might investigate is that of all convex sets that are carried into themselves by a given affine map; e.g., the orthogonal projection of $\mathbb{R}^n$ onto a specified subspace. (If that subspace is $\{0\}$, we get $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ with one additional element $\emptyset$ thrown in.)

10. Dualities

Let us recall the definition of a concept we have referred to a couple of times in passing. A *closed half-space* in $\mathbb{R}^n$ means a set of the form

$$\{p\in\mathbb{R}^n\mid f(p)\ \leq\ \lambda\}, \tag{16}$$

for some nonzero linear functional f on $\mathbb{R}^n$ and some real number λ. It is a standard result that every closed convex subset of $\mathbb{R}^n$ is an intersection of closed half-spaces.

The half-space (16) contains the point 0 if and only if λ is nonnegative, hence closed convex sets containing 0 can be characterized as intersections of half-spaces (16) having $\lambda\geq 0$. Such sets can, in fact, be expressed as intersections of such half-spaces with $\lambda>0$, since a half-space (16) with $\lambda=0$ is the intersection of all the half-spaces with the same f and positive λ. But a half-space (16) with λ positive can be written as $\{p\in\mathbb{R}^n\mid\lambda^{-1}f(p)\leq 1\}$, or, expressing the linear functional $\lambda^{-1}f$ as the dot product with some $q\in\mathbb{R}^n$, as

$$\{p\in\mathbb{R}^n\mid q{\cdot}p\ \leq\ 1\}. \tag{17}$$

Thus, for any subset $S \subseteq \mathbb{R}^n$, if we define

$$S^* \;=\; \{q \in \mathbb{R}^n \mid (\forall p \in S)\ q\cdot p \,\leq\, 1\}, \tag{18}$$

then S^{**} will be the least closed convex set containing $S \cup \{0\}$. Moreover, we see that S^* will also be a closed convex subset containing 0, which uniquely determines and is determined by S^{**}. Thus, the operator * gives a bijection of the family of all closed convex sets containing 0 with itself, which is easily seen to be inclusion-reversing. (This is an example of a Galois connection; cf. [3, §5.5] for a general development of the concept, with many examples.) Let us call two closed convex sets containing 0 that are related in this way dual to one another. (The dual of a convex set is sometimes called its polar set, e.g., in [15].) Examples in $\mathbb{R}^3$ are a cube and an octahedron of appropriate radii centered at the origin, and similarly a dodecahedron and an icosahedron. The unit sphere is self-dual.

The class of closed convex subsets of $\mathbb{R}^n$ forms a lattice (by general properties of Galois connections), which, like $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$, has intersection as its meet operation; but the join operations do not everywhere coincide – a consequence of the fact that for closed sets $u, v \in \mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$, their join in that lattice, $\mathrm{c.h.}(u \cup v)$, may not be closed, so that to get their join as closed convex sets, one must take its topological closure. For example, let $n = 2$, and let u be the closed strip $\{(x, y) \mid -1 \leq y \leq 1\}$ and v the line segment $\mathrm{c.h.}(0,\,(0, 2))$. Then the join of u and v in $\mathrm{Conv}(\mathbb{R}^n)_{\geq 0}$ is

$$\{(x, y) \mid -1 \leq y < 2\} \;\cup\; \{(0, 2)\},$$

while their join in the corresponding lattice of closed convex sets is $\{(x, y) \mid -1 \leq y \leq 2\}$. In view of this difference in operations, care is needed when using our duality on closed convex sets to deduce results about the lattice $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$.

If x and y are mutually dual closed convex sets containing 0 in $\mathbb{R}^n$, it is not hard to see that one of them is bounded (i.e., compact) if and only if 0 is an interior point of the other. It follows that the class of closed bounded convex sets having 0 in their interior is self-dual; moreover, as we noted at the beginning of §7, the join in $\mathrm{Conv}(\mathbb{R}^n)$ of two compact sets is again compact, from which it follows that unlike the lattice of all closed convex sets containing 0, this is a sublattice of $\mathrm{Conv}(\mathbb{R}^n)$. It is easy to see from the equality of the second and fourth varieties in Corollary 20 that this sublattice satisfies the same identities as $\mathrm{Conv}(\mathbb{R}^n)$. Hence the existence of the anti-automorphism just noted gives us

Theorem 22. *The class of lattice identities satisfied by* $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ *is self-dual, i.e., closed under interchanging all instances of* $\vee$ *and* $\wedge$. *Equivalently, the variety* $\mathbf{V}(\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}})$ *is closed under taking dual lattices.* □

And indeed, the identities proved for $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ in Theorems 8 and 11 respectively are dual to one another. This is not true of the identities proved for $\mathrm{Conv}(\mathbb{R}^n)$ in those theorems. In fact, it is easy to verify

Exercise 23. (i) For every positive integer n, show by example that $\mathrm{Conv}(\mathbb{R}^n)$ does not satisfy either of the identities

$$((D_n^{\mathrm{op}}) \wedge z) \vee y_1 \vee y_2 \quad \text{and} \quad (((D_n) \vee z') \wedge z) \vee (\textstyle\bigvee_i\, y_i),$$

dual to the identities proved for that lattice in Theorems 8 and 11 respectively.

(ii) Show that these observations together with Theorem 22 yield an alternative way of verifying that in each of Theorems 8 and 11, the identities proved for $\mathrm{Conv}(\mathbb{R}^n)$ are not satisfied by $\mathrm{Conv}(\mathbb{R}^{n+1})_{\geq\{0\}}$.

Theorem 22 also implies that $\mathrm{Conv}(\mathbb{R}^{n+1})_{\geq\{0\}}$, and hence also $\mathrm{Conv}(\mathbb{R}^n)$, satisfies the dual of the identity of Theorem 12, which we had not previously obtained.

Just as every closed convex set is an intersection of closed half-spaces (16), so every open convex set is an intersection of open half-spaces,

$$\{p \in \mathbb{R}^n \mid f(p) < \lambda\}.$$

But here the converse is not true. Indeed, every *closed* half-space is also an intersection of open half-spaces, so the class of intersections of open half-spaces includes both the open and the closed convex sets. If for every subset $S \subseteq \mathbb{R}^n$ one defines

$$S^\diamond \ = \ \{q \in \mathbb{R}^n \mid (\forall p \in S)\ q \cdot p < 1\}, \tag{19}$$

one gets a duality theory for the class of ${}^{\diamond\diamond}$-invariant sets, i.e., sets satisfying $x^{\diamond\diamond} = x$, a larger class than that covered by the preceding duality. It is not hard to see that a necessary and sufficient condition for a set $x \in \mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ to be ${}^{\diamond\diamond}$-invariant is that for every point p of the boundary of x which does not belong to x, there exist a supporting hyperplane of x through p containing no point of x. From this we can see that the union of the open unit ball B in $\mathbb{R}^n$ with any subset of its boundary is ${}^{\diamond\diamond}$-invariant; its ${}^{\diamond}$-dual is the union of B with the complementary subset of the boundary. On the other hand, we find that the union of an open polygonal neighborhood P of 0 in the plane with a nonempty finite (or countable) subset S of its boundary is never ${}^{\diamond\diamond}$-invariant, since by the above characterization of ${}^{\diamond\diamond}$-invariant sets, each point $q \in S$ forces the ${}^{\diamond\diamond}$-closure of $P \cup S$ to contain all points p of the open edge(s) of our polygon containing or adjacent to q.

Like the **-invariant sets (closed convex sets containing 0), the ${}^{\diamond\diamond}$-invariant subsets of $\mathbb{R}^n$ form a lattice by general properties of Galois connections, which has the same meet operation as $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$, but different join operation. For instance, if P is, as above, an open polygonal neighborhood of 0 in $\mathbb{R}^2$, q a point of its boundary, and $Q = \mathrm{c.h.}(0, q)$, then writing $\vee$ for the join operation of $\mathrm{Conv}(\mathbb{R}^2)_{\geq\{0\}}$, we find that $P \vee Q = P \cup Q = P \cup \{q\}$, but as we saw above, this set is not ${}^{\diamond\diamond}$-invariant. Rather, $(P \cup Q)^{\diamond\diamond}$, the join of P and Q in the lattice of ${}^{\diamond\diamond}$-invariant sets, is obtained by attaching to $P \cup \{q\}$ the open edge(s) containing or adjacent to q. Further, the **-invariant subsets of $\mathbb{R}^n$ do not form a sublattice of the the ${}^{\diamond\diamond}$-invariant sets. To see this in $\mathbb{R}^2$, let

$$u = \{(x,\, y) \mid \ |x| < 1,\ y \geq \ \ 1/(1 - x^2) - 2\},$$

$$v = \{(x,\, y) \mid \ |x| < 1,\ y \leq -1/(1 - x^2) + 2\}.$$

We see that the join of u and v in $\mathrm{Conv}(\mathbb{R}^2)_{\geq\{0\}}$ is $\{(x,\, y) \mid |x| < 1\}$, which is open, hence is ${}^{\diamond\diamond}$-invariant, hence is their join in the lattice of ${}^{\diamond\diamond}$-invariant sets; but it is not closed, hence is not their join in the lattice of **-invariant sets.

But again, one can apply $^{\diamond\diamond}$-duality to sublattices of $\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}}$ which consist of $^{\diamond\diamond}$-invariant elements, since for these, the two lattice structures in question must agree. In particular, one can verify that $^{\diamond}$-duality interchanges compact and open sets, giving an anti-isomorphism between the sublattices $\mathrm{Conv}(\mathbb{R}^n,\ \mathrm{cpct})_{\geq\{0\}}$ and $\mathrm{Conv}(\mathbb{R}^n, \mathrm{open})_{\geq\{0\}}$. (This anti-isomorphism can also be obtained by composing the anti-isomorphism "*" between $\mathrm{Conv}(\mathbb{R}^n,\ \mathrm{cpct})_{\geq\{0\}}$ and the lattice of closed convex sets that are neighborhoods of the origin, with the isomorphism noted earlier between that lattice and $\mathrm{Conv}(\mathbb{R}^n, \mathrm{open})_{\geq\{0\}}$.) The existence of this anti-isomorphism gives

Corollary 24 (to Lemma 15)**.** *For every positive integer n, $\mathrm{Conv}(\mathbb{R}^n,\ \mathrm{open})_{\geq\{0\}}$ is meet semidistributive, that is, satisfies the dual of* (13). □

This explains the fact that in Lemma 16 and the discussion that followed, though we saw that $\mathrm{Conv}(\mathbb{R}^n,\ \mathrm{open})_{\geq\{0\}}$ was not join semidistributive, we found no copies of M_3 in it.

Meet-semidistributivity does *not* hold in any of the lattices of convex sets we have considered that are not defined so as to make those sets all have some point (such as 0) in common; for in each of these lattices, it is easy to get examples of a set x having empty intersection with each of two sets y_1 and y_2, but nonempty intersection with their join. It also does not hold in $\mathrm{Conv}(\mathbb{R}^n,\ \mathrm{cpct})_{\geq\{0\}}$ for $n > 1$, since that lattice contains an embedded copy of $\mathrm{Conv}(\mathbb{R}^{n-1},\ \mathrm{cpct})$ (cf. proof of Lemma 1), to which the above observation applies.

While on the topic of join- and meet-semidistributivity, I will note some questions and examples which are easier to state now that we have named several sublattices of $\mathrm{Conv}(\mathbb{R}^n)$. Kira Adaricheva (personal communication) has posed several questions of the following form: If we take one of the sublattices of $\mathrm{Conv}(\mathbb{R}^n)$ that we know to be join- or meet-semidistributive, and extend it by adjoining, within $\mathrm{Conv}(\mathbb{R}^n)$, a single "nice" outside element, is the property in question already lost, and if so, does the resulting lattice in fact contain copies of M_3?

In many cases this does happen. For instance, the sublattice $L \subseteq \mathrm{Conv}(\mathbb{R}^2)$ generated by the join-semidistributive sublattice $\mathrm{Conv}(\mathbb{R}^2,\ \mathrm{cpct})$, and the open disk, which we shall denote B, contains a copy of M_3. To describe it, let p_0, p_1, p_2 be any three distinct points on the unit circle, and for $i = 0, 1, 2$ define the point $q_i = 1/5\, p_i + 2/5\, p_{i+1} + 2/5\, p_{i+2}$ (subscripts evaluated $\mathrm{mod}\ 3$; I suggest making a sketch). Let $x_i = (\mathrm{c.h.}(q_i, p_{i+1}) \wedge B) \vee (\mathrm{c.h.}(q_i, p_{i+2}) \wedge B)$. Then x_1, x_2, x_3 can be seen to generate a copy of M_3 in $\mathrm{Conv}(\mathbb{R}^2)$, resembling, though not identical to, the $n = 2$ case of the first example described in Lemma 16. An analogous construction gives a family of elements of L resembling the example described immediately following that lemma, showing that L is not m-join semidistributive for any m.

As another example let L be the sublattice of $\mathrm{Conv}(\mathbb{R}^3)_{\geq\{0\}}$ generated by $\mathrm{Conv}(\mathbb{R}^3,\ \mathrm{o.bdd.})_{\geq\{0\}}$ and the cube $C = [-1, +1]^3$. Letting B denote the interior of C, I claim that for any triangle T drawn on a face F of C, L contains the union of B with the interior of T relative to F. Indeed, we can find in $\mathrm{Conv}(\mathbb{R}^3,\ \mathrm{o.bdd.})_{\geq\{0\}}$ an open pyramid P (with apex near 0) meeting no face of C

except F, and meeting F in precisely the relative interior of T. Then $(P \wedge C) \vee B$ will be the desired set. If we construct three sets of this sort using as our T's three triangles in a common face F of C, whose relative interiors form a copy of the $n = 2$ case of the first example of Lemma 16, then the sublattice generated by the three resulting sets $(P \wedge C) \vee B$ will have the form M_3 (with B as its least element; cf. the last sentence of Corollary 19). The same trick can again be used to get examples like those of the paragraph following Lemma 16, so this L, too, is not m-join semidistributive for any m.

On the other hand, I do not know what can be said about the lattices gotten by adjoining to $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq\{0\}}$ a closed ball D about 0, respectively an open ball B not containing 0, except that the second of these lattices is not meet-semidistributive, since we can find three "fingers" in $\mathrm{Conv}(\mathbb{R}^n, \text{o.bdd.})_{\geq\{0\}}$ which meet B in disjoint subsets x, y_1, y_2, such that $x \wedge (y_1 \vee y_2)$ is nonempty.

11. Relatively convex sets

Let S be a subset of $\mathbb{R}^n$, in general non-convex. We shall call a subset $x \subseteq S$ convex *relative to* S if $x = \mathrm{c.h.}(x) \cap S$; equivalently, if x is the intersection of S with some convex subset of $\mathbb{R}^n$. Such sets x will form a lattice $\mathrm{RelConv}(S)$, with meet operation given, as in (2), by intersection, but join now given by

$$x \vee y \;=\; \mathrm{c.h.}(x \cup y) \,\cap\, S. \tag{20}$$

Note that the map $x \mapsto \mathrm{c.h.}(x)$ gives a bijection between the relatively convex subsets of S, and the subsets of $\mathbb{R}^n$ which are convex hulls of subsets of S, the inverse map being given by $- \cap S$. Convex sets of the latter sort form a lattice with join as in (2), but with meet operation

$$x \wedge y \;=\; \mathrm{c.h.}(x \cap y \cap S). \tag{21}$$

We observe that Carathéodory's Theorem (with the "refinement" given in the final sentence), regarded as a statement about the closure operator $\mathrm{c.h.}(-)$ on $\mathbb{R}^n$, entails the same properties for the closure operator $\mathrm{c.h.}(-) \cap S$ on S. Hence the proofs of Lemma 3 and of the positive assertions of Theorem 8 immediately yield

Proposition 25 (cf. Huhn [11, Lemma 3.2])**.** *If n is a natural number and S a subset of $\mathbb{R}^n$, then $\mathrm{RelConv}(S)$ satisfies the identity $((D_n) \vee z) \wedge y_1 \wedge y_2$, and if p is a point of S, $\mathrm{RelConv}(S)_{\geq\{p\}}$ satisfies the identity D_n.* □

However, the next lemma shows that the corresponding results fail badly for the identities involving D_n^{op} obtained in Theorem 11. In this lemma, the second assertion embraces the first (plus two obvious intermediate results not stated); however I include the first assertion because both its statement and the example proving it are more transparent than for the second.

Lemma 26. *For S a subset of $\mathbb{R}^2$, $\mathrm{RelConv}(S)$ need not satisfy the identity D_n^{op} for any positive integer n. In fact, for p an element of such an S, $\mathrm{RelConv}(S)_{\geq\{p\}}$ need not satisfy the identity $(((D_n^{\mathrm{op}}) \wedge z') \vee z) \wedge (\bigwedge_i y_i)$.*

Proof. To get the first assertion, let S be the set consisting of the unit circle and its center, 0. Given n, let $q_1, \dots, q_{2n+2}$ be the successive vertices of a regular $(2n{+}2)$-gon on that circle; for $i = 1, \dots, n{+}1$, let $y_i = \{q_1, \dots, q_{n+1}\} - \{q_i\}$, and let $x = \{q_{n+2}, \dots, q_{2n+2}\}$. Observe that each intersection of n of the y's consists of one of the points q_i, and that x contains the antipodal point; hence the join of x with each intersection of n y's contains 0. On the other hand, the intersection of all the y's is empty, hence its join with x is x, which does not contain 0. So D_n^{op} fails for this choice of arguments.

To get an example where all the given sets have a common element p, and where, moreover, the indicated weaker identity fails, let us take for S the same set as in the preceding example, with the addition of one arbitrary point r outside the circle, at distance ≥ 2 from 0. This time, let $q_1, \dots, q_{2n+6}$ be the successive vertices of a regular $(2n{+}6)$-gon on S, placed so that q_{n+2} lies on the line connecting 0 with the external point r. For $i = 1, \dots, n{+}1$, let $y_i = \{q_1, \dots, q_{n+3}\} - \{q_i\}$, and let $x = \{q_{n+3}, \dots, q_{2n+4}\}$. We note that the set of subscripts of the q's occurring in each of these sets lies in an interval of length $< n{+}3$, so the absence of the point 0 does not contradict convexity of these sets relative to S. Taking $p = q_{n+3}$ we see that all of these sets belong to $\mathrm{RelConv}(S)_{\geq\{p\}}$.

As in the previous example, the intersection of any n of the y's contains one of $q_1, \dots, q_{n+1}$, and x contains its antipodal point, so that the join of x with that intersection contains 0, hence so does the intersection of all these joins, i.e., the value of the right-hand side of D_n^{op} at these arguments. On the other hand, the intersection of all the y's is $\{q_{n+2}, q_{n+3}\}$, and the union of this set with x still has all subscripts lying in an interval of length $< n{+}3$, so the join of those two sets, the left-hand side of D_n^{op}, does not contain 0. Hence if we take $z' = \{q_{n+3}, 0\}$, the intersection of the right-hand side of D_n^{op} with z' is $z' = \{q_{n+3}, 0\}$, while intersection of the left-hand side with z' is $\{q_{n+3}\}$. If, finally, we let $z = \{q_{n+3}, r\}$ and take the joins of this element with those two intersections, we see that in the first case the resulting set contains the point q_{n+2}, since we assumed this to lie on the line-segment from 0 to r, while in the second, it does not. Since $q_{n+2} \in \bigwedge_i y_i$, the two sets remain distinct on intersecting with $\bigwedge_i y_i$, showing the failure of $(((D_n^{\mathrm{op}}) \wedge z') \vee z) \wedge (\bigwedge_i y_i)$. $\square$

The above example in $\mathbb{R}^2$ is also an example in $\mathbb{R}^n$ for any $n \geq 2$. For completeness, we should also consider dimensions $n = 0$ and 1. If we look at the chain of varieties corresponding to that of Lemma 1, but with each $\mathbf{V}(\mathrm{Conv}(\mathbb{R}^n))$ replaced by the variety generated by all the lattices $\mathrm{RelConv}(S)$ for subsets $S \subseteq \mathbb{R}^n$, and each $\mathbf{V}(\mathrm{Conv}(\mathbb{R}^n)_{\geq\{0\}})$ by the variety generated by lattices $\mathrm{RelConv}(S)_{\geq\{p\}}$, then we see that the first term of this chain is still the trivial variety and the next two still satisfy the distributive identity (in the last case, by the final assertion of Proposition 25 for $n = 1$). Since the distributive identity implies the identities of every nontrivial variety, we conclude that allowing lattices of relatively convex sets has not enlarged the varieties we get at these three steps. We have just shown the contrary from the fifth step on; this leaves only the fourth step, i.e., the relation between $\mathbf{V}(\mathrm{Conv}(\mathbb{R}^1))$ and the variety generated by all lattices of form $\mathrm{RelConv}(S)$

with $S \subseteq \mathbb{R}^1$. Here again, it turns out that we have equality. This follows from our observation that $\mathrm{RelConv}(S)$ is isomorphic to the lattice of convex hulls of subsets of S, together with

Lemma 27. *Let S be a subset of $\mathbb{R}^1$. Then the convex hulls of subsets of S form a sublattice of* $\mathrm{Conv}(\mathbb{R}^1)$.

Proof. We have noted, for arbitrary n and $S \subseteq \mathbb{R}^n$, that the convex hulls of subsets of S are closed under the join operation of the full lattices of convex sets, so we need only show them closed under meets, i.e., intersections, when $n = 1$. This comes down to showing that if $p \in \mathrm{c.h.}(x) \cap \mathrm{c.h.}(y)$, where x and y are relatively convex subsets of S, then $p \in \mathrm{c.h.}(x \cap y)$. The fact that $p \in \mathrm{c.h.}(x)$ means that $q_1 \leq p \leq q_2$ for some $q_1, q_2 \in x$; similarly, $q_3 \leq p \leq q_4$ for some $q_3, q_4 \in y$. From the relative convexity of x and y and the order-relations of these elements, we now see that $\max(q_1, q_3) \in x \cap y$ and $\min(q_2, q_4) \in x \cap y$. Hence $p \in \mathrm{c.h.}(\max(q_1, q_3), \min(q_2, q_4)) \subseteq \mathrm{c.h.}(x \cap y)$. □

For further results on $\mathbf{V}(\mathrm{Conv}(\mathbb{R}^1))$ and its subvarieties, see [17].

Incidentally, the lattice $\mathrm{RelConv}(S)$, for S the set used in the first part of the proof of Lemma 26 (consisting of the unit circle and its center) shows that none of the identities $(((D_m^{\mathrm{op}}) \wedge z') \vee z) \wedge (\bigwedge_i y_i)$ implies any of the identities D_n^{op}. Indeed, we showed that $\mathrm{RelConv}(S)$ satisfies none of the latter identities; let us now show that (unlike the lattice considered in the second part of that proof) it satisfies all of the former. It suffices to verify the strongest of these, the case $m = 1$. Writing $S = C \cup \{0\}$, where C is the unit circle, we see that the map $-\cap C : \mathrm{RelConv}(S) \to \mathrm{RelConv}(C) = 2^C$ is a homomorphism; so if two lattice expressions in x, y_1, y_2, z, z' are identically equal in distributive lattices, their values in $\mathrm{RelConv}(S)$ will always agree except, perhaps, as to whether they contain 0. We now consider separately the cases $0 \in y_1 \wedge y_2$ and $0 \notin y_1 \wedge y_2$. In the former case, the two sides of $D_1^{\mathrm{op}}(x, y_1, y_2)$ agree in containing 0, hence are equal, so a fortiori the two sides of $(((D_1^{\mathrm{op}}) \wedge z') \vee z) \wedge (y_1 \wedge y_2)$ are equal. In the latter case, neither side of the latter relation can contain 0, hence again they are equal.

12. The snowflake

Let us look at a particularly neat example of a lattice of relatively convex subsets of a set S.

Let p_1, p_2, $p_3, -p_1, -p_2, -p_3$ be the successive vertices of a regular hexagon in $\mathbb{R}^2$ centered at 0. Let

$$S_1 = \mathrm{c.h.}(p_1, -p_1), \quad S_2 = \mathrm{c.h.}(p_2, -p_2), \quad S_3 = \mathrm{c.h.}(p_3, -p_3),$$

$$S \;=\; S_1 \cup S_2 \cup S_3,$$

and let L be the sublattice of $\mathrm{RelConv}(S)_{\geq \{0\}}$ generated by the three line-segments S_1, S_2 and S_3. (In view of the form of S, I think of this example as “the snowflake”.)

Every element of L will clearly be centrally symmetric and topologically closed; hence every such element has the form

$$\lambda_1 S_1 \cup \lambda_2 S_2 \cup \lambda_3 S_3 \quad (\lambda_1, \lambda_2, \lambda_3 \in [0,1]). \tag{22}$$

The join of S_1 and S_2 in this lattice must have the form $S_1 \cup S_2 \cup \lambda S_3$ for some $\lambda \in (0,1)$. (Elementary geometry shows that $\lambda = 1/2$; but we don't need to know this now, and will get it from a general formula soon.) Intersecting this join with S_3, we see that $\lambda S_3 \in L$. By symmetry we also have $\lambda S_1, \lambda S_2 \in L$, and we see that these together generate a proper sublattice of L isomorphic to the whole lattice. In particular, L has infinite descending chains of elements, e.g., $S_1 > \lambda S_1 > \lambda^2 S_1 > \dots$.

Let's figure out how to calculate in L. The first thing we should find are the conditions on λ_1, λ_2 and λ_3 for a set (22) to be relatively convex. Calculation shows that for λ_1, λ_2 and λ_3 nonzero, the points $\lambda_1 p_1$, $\lambda_2 p_2$ and $\lambda_3 p_3$ are collinear if and only if $\lambda_2^{-1} = \lambda_1^{-1} + \lambda_3^{-1}$. (In verifying this, the key relation is $p_2 = p_1 + p_3$.) Hence we see that one necessary condition for the convexity of (22) is $\lambda_2^{-1} \leq \lambda_1^{-1} + \lambda_3^{-1}$; by symmetry, the remaining conditions are $\lambda_1^{-1} \leq \lambda_2^{-1} + \lambda_3^{-1}$ and $\lambda_3^{-1} \leq \lambda_1^{-1} + \lambda_2^{-1}$. It is easy to verify that if the λ_i are also allowed to be zero, and we write $0^{-1} = \infty$ and consider ∞ greater than all real numbers, then these three conditions continue to be necessary and sufficient for convexity.

Let us therefore index elements (22) by the three parameters λ_1^{-1}, λ_2^{-1}, λ_3^{-1}, defining

$$[a_1, a_2, a_3] \;=\; a_1^{-1} S_1 \cup a_2^{-1} S_2 \cup a_3^{-1} S_3 \quad (a_1, a_2, a_3 \in [1, \infty]), \tag{23}$$

so that the lattice of topologically closed centrally symmetric elements of $\mathrm{RelConv}(S)_{\geq\{0\}}$ consists of the sets $[a_1, a_2, a_3]$ with

$$a_1 \leq a_2 + a_3, \qquad a_2 \leq a_1 + a_3, \qquad a_3 \leq a_1 + a_2. \tag{24}$$

Note that the ordering of this lattice is by reverse componentwise comparison of expressions $[a_1, a_2, a_3]$, and that S_1, S_2, S_3 are the elements $[1, \infty, \infty]$, $[\infty, 1, \infty]$, $[\infty, \infty, 1]$. Lattice-theoretic meet is clearly given by componentwise supremum, while the lattice-theoretic join of two elements is gotten by first taking their componentwise infimum, which represents their set-theoretic union, then getting its relative convex hull by reducing the largest entry to the sum of the other two if it exceeds this. So, for instance, $S_1 \vee S_2 = [1, \infty, \infty] \vee [\infty, 1, \infty]$ is gotten by forming the componentwise infimum, $[1, 1, \infty]$, and then decreasing the last component to the sum of the first two, getting $[1, 1, 2]$ (confirming the value $\lambda = 1/2$ in our earlier description of this element). We can now calculate, e.g., the meet (componentwise supremum) of this with $S_3 = [\infty, \infty, 1]$, namely $[\infty, \infty, 2]$; and take the join of this meet with $S_2 = [\infty, 1, \infty]$ by forming the componentwise infimum $[\infty, 1, 2]$, and adjusting the first component as above, getting $[3, 1, 2]$.

Note that the only way these lattice operations yield components in their values that did not occur as components in their arguments is by addition; hence, as L was defined to be generated by S_1, S_2 and S_3, all finite components a_i that occur in the expressions (23) for elements of L are positive integers. It is not hard to verify that all positive integers indeed occur, and that the elements of L are all the elements (23) satisfying (24) with $a_1,\, a_2,\, a_3 \in \{1,2,3,\dots,\infty\}$. So we have gotten a very arithmetic description of this lattice.

Though we have seen that L contains an infinite descending chain, it is interesting to note that the sublattice generated by any finite set of elements (23), none of which have any infinite components, is finite; for the lattice operations will not produce, in any position, entries larger than the corresponding entries of their arguments.

It would be of interest to examine lattices of the form $\mathrm{RelConv}(S)_{\geq\{0\}}$ for more general sets S which are unions of finitely many line-segments (or rays) through the origin in $\mathbb{R}^n$. In this situation, the conditions for convexity are always given by linear inequalities in the "λ_i^{-1}"; let me sketch why.

First, some general observations. Suppose $p_1,\dots,p_m$ $(m\geq 3)$ are a minimal linearly dependent family of vectors in $\mathbb{R}^n$; thus they satisfy a nontrivial linear relation $\sum c_i\, p_i = 0$, unique up to scalars. Given positive real numbers $\lambda_1,\dots,\lambda_m$, how can we tell whether the points $\lambda_1 p_1,\dots,\lambda_m p_m$ lie in an $(m{-}1)$-dimensional affine subspace of their m-dimensional span? This will hold if and only if the linear relation satisfied by these modified elements is an *affine relation*; i.e., has coefficients summing to 0. Those coefficients are $c_i\,\lambda_i^{-1}$, so the condition is

$$\textstyle\sum c_i\,\lambda_i^{-1} = 0.$$

In this situation, is the affine relation among the $\lambda_i p_i$ equivalent to a relation expressing one of them, say $\lambda_j p_j$, as a *convex* linear combination of the others? One sees that this is so if and only if the coefficient c_j is opposite in sign to all the other c_i; in that situation, let us rewrite the expression satisfied by the λ_i^{-1} as

$$\lambda_j^{-1} \;=\; -c_j^{-1} \textstyle\sum_{i\neq j}\, c_i\,\lambda_i^{-1}.$$

Note that the abovementioned condition on signs can be satisfied by at most one j, and can be looked at as saying that the ray determined by the corresponding vector p_j is in the convex hull of the rays determined by the other p_i. When it is satisfied, one finds that a union of intervals $\bigcup_i\,[0,\,\lambda_i p_i]$ is relatively convex in the union of the rays determined by the p_i if and only if λ_j has at least the value given by the above formula, i.e., if and only if

$$\lambda_j^{-1} \;\leq\; -c_j^{-1} \textstyle\sum_{i\neq j}\, c_i\,\lambda_i^{-1}. \tag{25}$$

If the c_i do not consist of one of one sign and the rest of the opposite sign, then none of the rays determined by the p_i is in the convex hull of the rest, and every set $\bigcup_i\,[0,\,\lambda_i p_i]$ $(\lambda_1,\dots,\lambda_m > 0)$ is relatively convex. (An example of this situation is given for $n=3$ by letting $p_1,\dots,p_4$ be the vertices of a convex quadrilateral lying in a plane not containing 0.)

Let us now drop the condition that the p_i are minimal among linearly dependent families, assuming only that none of them is a nonnegative multiple of another (i.e., that they determine distinct rays), and let S denote the union of the rays through 0 that they determine. Then one can show that a union of intervals $\bigcup_i\,[0,\,\lambda_i p_i]$ is relatively convex in S if and only if (25) holds for each linearly dependent family of ≥ 3 of the p_i which is minimal among linearly dependent families, and which has the property that the unique linear relation it satisfies has exactly one coefficient c_j of different sign from the rest. (The reduction to the minimal-linearly-independent-family case can be gotten by a recursive application of Carathéodory's Theorem, with $p_0=0,$ within subspaces spanned by successively smaller subsets of $\{p_1,\ \dots\ ,\ p_n\}$.)

As an example of the sort of lattice one gets, let us drop the central-symmetry and length ≤ 1 conditions from our snowflake construction, writing $-p_1,\ -p_2,\ -p_3$ as $p_4,\ p_5,\ p_6,$ and considering general sets

$$\lambda_1\,\mathrm{c.h.}(0,p_1)\,\cup\,\dots\,\cup\,\lambda_6\,\mathrm{c.h.}(0,p_6)\quad(\lambda_1,\dots,\lambda_6\in[0,\infty]).$$

Then setting $a_i=\lambda_i^{-1},$ we get six conditions for relative convexity, namely

$$a_2\leq a_1+a_3,\quad a_3\leq a_2+a_4,\quad \dots,\quad a_1\leq a_6+a_2,$$

each corresponding to the fact that one of our six rays lies in the cone spanned by its two immediate neighbors.

Our snowflake example showed that a lattice $\mathrm{RelConv}(S)_{\geq\{0\}}$ could contain a 3-generator sublattice with an infinite descending chain. Let me sketch an example with an infinite ascending chain. In $\mathbb{R}^2,$ let

$$p_1=(0,3),\quad p_2=(1,2),\quad p_3=(2,1),\quad p_4=(3,0).$$

(Or for a more abstract description, take any four points of $\mathbb{R}^n$ in arithmetic progression, on a line not passing through the origin.) Define

$$S_i=\mathrm{c.h.}(0,p_i)\quad\text{and}\quad S=S_1\cup S_2\cup S_3\cup S_4.$$

Now let L be the sublattice of $\mathrm{RelConv}(S)_{\geq\{0\}}$ generated by

$$x_1=S_1\cup(S_2/2),\quad y=S_2\cup S_3,\quad x_2=(S_3/2)\cup S_4.$$

It is easy to see from a sketch that, starting with $x_1\wedge y,$ if we alternately apply $(-\vee x_2)\wedge y$ and $(-\vee x_1)\wedge y,$ we obtain an infinite ascending chain of sets.

Is the existence of infinite chains in sublattices of $\mathrm{RelConv}(S)$ generated by few elements limited to cases where $S\neq\mathbb{R}^n,$ or does it also occur in lattices $\mathrm{Conv}(\mathbb{R}^n)$? To get a large part of the answer without any computation, recall that $\mathrm{RelConv}(S)$ is isomorphic to the lattice of convex hulls in $\mathbb{R}^n$ of subsets of $S.$ This lattice, which we shall here denote $L_S,$ is a subset but not a sublattice of $\mathrm{Conv}(\mathbb{R}^n);$ however, in cases like those considered above, where S is a finite union of convex sets, $S=S_1\cup\dots\cup S_m,$ we can write the operations of this lattice as "polynomial operations" in those of $\mathrm{Conv}(\mathbb{R}^n).$ Namely, temporarily writing $\wedge_S$ and $\vee_S$ for the operations of $L_S,$ and $\wedge$ and $\vee$ for those of $\mathrm{Conv}(\mathbb{R}^n),$ we get, for $x,y\in L_S,$

$$x\vee_S y=x\vee y,\qquad x\wedge_S y=\bigvee\nolimits_i\,(x\wedge y\wedge S_i).\tag{26}$$

Hence if the sublattice of L_S generated by elements $y_1,\dots,y_k$ has an infinite ascending or descending chain (or any other specified join-sublattice), so will the sublattice of $\mathrm{Conv}(\mathbb{R}^n)$ generated by $y_1,\dots,y_k,S_1,\dots,S_m.$ In the case of our "snowflake lattice", the y's and the S's happen to be the same, so we immediately conclude that the sublattice of $\mathrm{Conv}(\mathbb{R}^2)_{\geq\{0\}}$ generated by those three elements has an infinite descending chain. From our example with an ascending chain, the best conclusion this general argument gives is that the sublattice of $\mathrm{Conv}(\mathbb{R}^2)_{\geq\{0\}}$ generated by the six elements S_1, S_2, S_3, S_4, $S_2/2$ and $S_3/2$ has an infinite ascending chain. Nevertheless, a little diagram-drawing shows that in this case, the sublattice of $\mathrm{Conv}(\mathbb{R}^2)_{\geq\{0\}}$ generated by the three elements $\mathrm{c.h.}(x_1)$, $\mathrm{c.h.}(x_2)$ and $\mathrm{c.h.}(y)$ shows essentially the same behavior as our lattice of relatively convex subsets.

I do not know any examples of 3-generator lattices of convex sets (relative or absolute) that have both infinite ascending and descending chains, or that have infinite antichains. On the other hand, it is not hard to show that the 4-generator sublattice of $\mathrm{Conv}(\mathbb{R}^2)$ generated by the diameters of a regular octagon has all three.

13. Notes on related work on relatively convex sets, and some further observations

The lattices $\mathrm{RelConv}(S)$ are examples of what are known as convex geometries; for the definition, and for results on these, see [7], [1].

Huhn [11] looked briefly at lattices of relatively convex sets determined by finite sets $S \subseteq \mathbb{R}^n$ for the purpose of "approximating" $\mathrm{Conv}(\mathbb{R}^n)$ by finite lattices, and for the same purpose he considered in [10, §2] the dual construction, namely the lattice of those subsets of $\mathbb{R}^n$ which can be represented as intersections of members of a given finite set of closed half-spaces. Not surprisingly in view of the dual natures of these two sorts of relativization, he found that lattices of the latter sort satisfied the identities of the form D_n^{op} that he had obtained in the nonrelativized lattice, but not those of the form D_n (compare Proposition 25 and Lemma 26 above).

In [19] it is shown that for every finite lattice L which is "lower bounded" (a strengthening of join semidistributive), there exist an n and a finite subset $S \subseteq \mathbb{Q}^n$ such that L is embeddable in the sublattice of $\mathrm{Conv}(\mathbb{R}^n)$ generated by $\{\{p\} \mid p \in S\}$. Clearly, such an embedding will send all members of L to convex polytopes; if we let S' denote the union of the vertex-sets of this finite set of polytopes, it is not hard to see that we get an embedding of L in $\mathrm{RelConv}(S')$. Whether embeddability in the lattice of relatively convex sets of a finite subset of $\mathbb{R}^n$ holds not only for lower bounded lattices, but for all join-semidistributive lattices, is an open question [19, Problem 1], cf. [1, Problem 3]. It is also shown in [19] that every lattice can be embedded in the lattice of convex subsets of some infinite-dimensional vector space (over an arbitrary totally ordered division ring). Hence, in particular, lattices of the latter sort need not satisfy any nontrivial lattice identities. In contrast, the lattice of *subspaces* of any vector space satisfies the modular identity and others.

Here is another embedding result using not necessarily finite-dimensional vector spaces, although it may be seen that the connection with convexity is somewhat artificial, based on the fact that subspaces are in particular convex sets; it is essentially a result on lattices of "relative subspaces". (Since most of the results in this note concerned subspaces of $\mathbb{R}^n$, we gave our basic definitions for that case only, but we shall use them here without that restriction.)

Lemma 28. *Let V be a real vector space and B a basis of V. For each pair of distinct elements $a, b \in B$, let $S_{a,b}$ be the subspace of V spanned by $a-b$. Define the set*

$$S \;=\; \bigcup_{a,b} S_{a,b} \;\subseteq\; V.$$

Then the following lattices are isomorphic:

(a) *The lattice* $\mathrm{Equiv}(B)$ *of all equivalence relations on B (ordered by inclusion).*

(b) *The lattice* $\mathrm{RelSubsp}(S)$ *consisting of all sets of the form $S \cap U$ where U is a subspace of V. (Here meet is given by intersection; join by taking the subspace spanned by the union of the sets in question and intersecting it with S.)*

(b′) *The lattice of all subspaces of V spanned by subsets of S. (The join of two such subspaces W_1, W_2 is $W_1 + W_2$, the meet is the span of $W_1 \cap W_2 \cap S$.)*

(c) *The sublattice of* $\mathrm{RelConv}(S)$ *consisting of all elements thereof that are unions of subspaces $S_{a,b}$.*

(c′) *The lattice of all subsets of V which are convex hulls of unions of subspaces $S_{a,b}$. (Join as in* $\mathrm{Conv}(V)$*, while the meet of x_1, x_2 is* $\mathrm{c.h.}(x_1 \cap x_2 \cap S)$*).*

Proof. That the sets described in (b), (b′), (c) and (c′), ordered by inclusion, form lattices, with meet and join as described, is immediate.

Note that the convex hull of a union of subspaces of V is the sum of those subspaces, and that the intersection of S with a subspace is always a union of certain of the $S_{a,b}$. From this it is easily seen that (b) and (c) are not merely, as asserted, isomorphic, but equal, and likewise (b′) and (c′). It is also clear that (b) is isomorphic to (b′), via the "span of" map in one direction and the operator $- \cap S$ in the other. So these four lattices are isomorphic; to complete the proof we shall describe an isomorphism between the lattice $\mathrm{Equiv}(B)$ of (a) and the lattice of (b′).

Given an equivalence relation $R \in \mathrm{Equiv}(B)$, let $\varphi(R)$ be the subspace of V spanned by all elements $a-b$ with $(a,b) \in R$, which by definition belongs to (b′), while given a subspace $W \subseteq V$ spanned by a subset of S, let $\psi(W) = \{(a,b) \mid a, b \in B,\ a-b \in W\}$, which it is easy to check is an equivalence relation on B. The maps φ and ψ are clearly isotone, and from the definition of (b′), we see that $\varphi\psi$ is the identity function thereof; moreover, for any $R \in \mathrm{Equiv}(B)$ it is clear that $\psi(\varphi(R)) \geq R$, so it remains to prove the reverse inequality.

So suppose that $(a,b) \in \psi(\varphi(R))$, i.e., that $a-b \in \varphi(R)$. From the definition of $\varphi(R)$ it is easy to see that for each R-equivalence class $C \subseteq B$, the sum of the coefficients of all members of C in any element of $\varphi(R)$ is zero. But the only way this can hold for $a-b$ is if a and b are in the same equivalence class, i.e., $(a,b) \in R$, as required to complete our proof. □

Pudlák and Tůma [18] have shown that any finite lattice L embeds in $\mathrm{Equiv}(X)$ for some finite set X. Hence by the above lemma, for every such L one can find an n and a subset $S \subseteq \mathbb{R}^n$ such that L is embeddable in $\mathrm{RelConv}(S)$. It would be interesting to know whether this same conclusion can be proved without using the deep result of [18]. So far as is known, the embedding of L in a lattice $\mathrm{Equiv}(X)$ may require a set X whose cardinality is enormous compared with that of L (though [14] and [12] somewhat improve the upper bound of [18]); but it is conceivable that one can do better with embeddings in lattices of relatively convex sets.

Our final remark will concern the observations at the end of the preceding section on the form of the lattice operations of $\mathrm{RelConv}(S)$ when S is a finite union of convex sets $S_1 \cup \cdots \cup S_m$. Let us put these in a more general context. (Readers allergic to category theory may ignore this discussion.)

Let **Lattice** denote the category of all lattices, objects of which we will here write $L = (|L|,\ \vee,\ \wedge)$, distinguishing between the lattice L and its underlying set $|L|$. For m a natural number, let $\mathbf{Lattice}^{m\text{-pt}}$ denote the category of lattices with m distinguished elements, i.e., of systems $(|L|, \vee, \wedge, S_1, \dots, S_m)$ such that $(|L|,\ \vee,\ \wedge)$ is a lattice and $S_1, \dots, S_m \in |L|$, and where a morphism between such systems means a lattice homomorphism which respects the ordered m-tuple of distinguished elements. Let us, finally, write $\mathbf{Lattice}^{\mathrm{int}}$ for the category of objects $(|L|,\ \vee,\ \wedge,\ \mathrm{int})$, where $(|L|,\ \vee,\ \wedge)$ is a lattice, and int is an "interior operator" (the dual of a closure operator), that is, a map $|L| \to |L|$ satisfying

$$\mathrm{int}(x) \leq x, \qquad x \leq y \implies \mathrm{int}(x) \leq \mathrm{int}(y), \qquad \mathrm{int}(\mathrm{int}(x)) = \mathrm{int}(x).$$

(Here by "$u \leq v$" we of course mean $u = u \wedge v$, equivalently, $u \vee v = v$.) The morphisms of $\mathbf{Lattice}^{\mathrm{int}}$ will be the lattice homomorphisms respecting this additional operation.

We can define a functor $\mathbf{Lattice}^{m\text{-pt}} \to \mathbf{Lattice}^{\mathrm{int}}$ taking each object $(|L|, \vee, \wedge, S_1, \dots, S_m)$ to the object $(|L|, \vee, \wedge, \mathrm{int}_{(S_i)})$, with interior operator defined by

$$\mathrm{int}_{(S_i)}(x) \ = \ \textstyle\bigvee_i (x \wedge S_i),$$

and another functor $\mathbf{Lattice}^{\mathrm{int}} \to \mathbf{Lattice}$, taking each object $(|L|,\ \vee,\ \wedge,\ \mathrm{int})$ to the object $(|L|^{\mathrm{int}},\ \vee,\ \wedge^{\mathrm{int}})$, where

$$|L|^{\mathrm{int}} = \{x \in |L| \mid x = \mathrm{int}(x)\} \quad \text{and} \quad x \wedge^{\mathrm{int}} y \ = \ \mathrm{int}(x \wedge y).$$

(It is not hard to verify that the join operation of L carries $|L|^{\mathrm{int}}$ into itself.)

We now see that if we take $L = \mathrm{Conv}(\mathbb{R}^n)$, and let $S_1, \dots, S_m$ be any m elements of this lattice, then the composite of the above two functors, applied to $(|L|,\ \vee,\ \wedge,\ S_1, \dots, S_m)$, gives precisely the lattice we named $L_S \cong \mathrm{RelConv}(S)$, for $S = S_1 \cup \cdots \cup S_m$.

The constructions given by the above functor $\mathbf{Lattice}^{\mathrm{int}} \to \mathbf{Lattice}$, and its dual, with a closure operator replacing the interior operator, are well-known, if not in this functorial form. I do not know whether the construction $\mathbf{Lattice}^{m\text{-pt}} \to \mathbf{Lattice}^{\mathrm{int}}$ (and its variant with lattices replaced by complete lattices and the specified finite family by an arbitrary family) has been considered.

Stepping back a little further, we may observe that the lattice $\mathrm{Conv}(\mathbb{R}^n)$ arises as the fixed set of the closure operator $\mathrm{c.h.}(-)$ on the lattice $2^{\mathbb{R}^n}$ of subsets of $\mathbb{R}^n$, so that the construction of the mutually isomorphic lattices L_S and $\mathrm{RelConv}(S)$

can be seen as arising from the interaction of the closure operator c.h.$(-)$ and the interior operator $-\cap S$ on $2^{\mathbb{R}^n}$. Again, this situation can be made into a general construction.

The reader familiar with the concept of representable algebra-valued functors ([3, Chapter 9] or [4, §1, §8]) will be happy to observe that all the functors of the above discussion are representable.

14. A question

The referee has pointed out that some properties of the lattice of convex sets are known to change if the base field $\mathbb{R}$ is replaced by another ordered field (e.g., $\mathbb{Q}$), but that the arguments of §§1-6 look as though they should work over any ordered field; perhaps even any ordered division ring. I have the same feeling, but as an amateur in the area, I will leave this question to others. (I do not know whether the theorems of Helly, Carathéodory and Radon hold in that context, nor how much of what I have justified as geometrically evident may rely on properties of the reals.)

A straightforward generalization of these results could not, of course, extend to §7, on compact convex sets, since over an ordered field which is not locally compact, the only nonempty compact convex sets are the singletons. One might be able to prove results like those of that section with "compact" replaced by "closed and bounded", but new proofs would be needed, since the theorem of Minkowski we used there is not true in that context. Later sections depend to varying degrees on that one.

References

[1] K. V. Adaricheva, V. A. Gorbunov and V. I. Tumanov, *Join-semidistributive lattices and convex geometries,* Adv. Math. **173** (2003) 1–49. MR**2003m**:06004.

[2] Mary K. Bennett, *Biatomic lattices,* Algebra Universalis **24** (1987) 60–73. MR**89a**:06021.

[3] George M. Bergman, *An Invitation to General Algebra and Universal Constructions,* Henry Helson, Berkeley, 1998. ii + 398 pp.; available online at http://math.berkeley.edu/~gbergman/245. MR**99h**:18001.

[4] George M. Bergman and Adam O. Hausknecht, *Cogroups and co-rings in categories of associative rings,* American Mathematical Society, Mathematical Surveys and Monographs series, v.45, ix + 388 pp., 1996. MR**97k**:16001.

[5] Garrett Birkhoff and Mary K. Bennett, *The convexity lattice of a poset,* Order **2** (1985) 223–242. MR**87d**:06004.

[6] Jürgen Eckhoff, *Helly, Radon, and Carathéodory type theorems,* Ch.2.1 (pp.389–448) of [9]. MR**94k**:52010.

[7] Paul H. Edelman and Robert E. Jamison, *The theory of convex geometries,* Geom. Dedicata **19** (1985), 247–270. MR**87f**:52002.

[8] Winfried Geyer, *Generalizing semidistributivity,* Order **10** (1993) 77–92. MR**95a**:06009.

[9] P. M. Gruber and J. M. Wills, ed., *Handbook of Convex Geometry,* vol. A, North-Holland 1993. MR**94e**:52001.

[10] András P. Huhn, *On nonmodular n-distributive lattices: the decision problem for identities in finite n-distributive lattices,* Acta Sci. Math. (Szeged) **48** (1985) 215–219. MR**87d**:06027.

[11] András P. Huhn, *On nonmodular n-distributive lattices. I. Lattices of convex sets,* Acta Sci. Math. (Szeged) **52** (1988) 35–45. MR**89j**:06006.

[12] T. Ihringer, *Congruence Lattices of Finite Algebras: the Characterization Problem and the Role of Binary Operations,* Algebra Berichte v.53, Verlag Reinhard Fischer, München, 1986. MR**87c**:08003.

[13] Bjarni Jónsson and Ivan Rival, *Lattice varieties covering the smallest nonmodular variety,* Pacific J. Math. **82** (1979) 463–478. MR**81j**:06007.

[14] Hans Kurzweil, *Endliche Gruppen mit vielen Untergrupppen,* J. reine angew. Math. **356** (1985) 140–160. MR**86f**:20024.

[15] Jim Lawrence, *Clustered families of convex sets,* Houston J. Math. **15** (1989) 235–243. MR**90k**:52010.

[16] L. Libkin, *n-distributivity, dimension and Carathéodory's theorem,* Algebra Universalis **34** (1995) 72–95. MR**96h**:06020.

[17] Marina Semenova and Friedrich Wehrung, *Sublattices of lattices of order-convex sets, III. The case of totally ordered sets,* Internat. J. Algebra Comput. **14**, (2004) 357–387. MR**2005d**:06008.

[18] Pavel Pudlák and Jiří Tůma, *Every finite lattice can be embedded in a finite partition lattice,* Algebra Universalis **10** (1980) 74–95. MR**81e**:06013.

[19] Friedrich Wehrung and Marina Semenova, *Sublattices of lattices of convex subsets of vector spaces,* Algebra Logic **43**, no. 3, (2004) 145–161.

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