Title
Boundary Controllers and Observers for Korteweg-de Vries, Hyperbolic, and Parabolic PDEs

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Boundary Controllers and Observers for Korteweg-de Vries, Hyperbolic, and Parabolic PDEs

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

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by

Shuxia Tang

Committee in charge:

Professor Miroslav Krstic, Chair
Professor Jorge Cortés
Professor Melvin Leok
Professor Sonia Martinez
Professor Lei Ni

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The dissertation of Shuxia Tang is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2016

DEDICATION

To my family.

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LIST OF ABBREVATIONS

ADRC	Active Disturbance Rejection Control
BMS	Battery Management System
Dom	Domain
DFN	Doyle-Fuller-Newman model for lithium-ion batteries
ECM	Equivalent Circuit Models
EKF	Extended Kalman Filter
ESO	Extended State Observer
EV	Electric Vehicles
fKdV	fifth-order Korteweg-de Vries equation
HEV	Hybrid Electric Vehicles
HUM	Hilbert Uniqueness Method
IDE	Integro-Differential Equation
IEKF	Iterated Extended Kalman Filter
KdV	Korteweg-de Vries equation
KF	Kalman Filter
LKdVAD	Linear Korteweg-de Vries system with Anti-Diffusion
OCP	Open-Circuit Potential
OCV	Open-Circuit Voltage
ODE	Ordinary Differential Equation
P2D	Pseudo Two-Dimensional model
PDE	Partial Differential Equation
PHEV	Plug-in Hybrid Electric Vehicles
SMC	Sliding Mode Control
SoC	State-of-Charge
SoH	State-of-Health

SPM	Single Particle Model
SPM-T	Single Particle Model with a T-equation
UDDS	Urban Dynamometer Driving Schedule
UKF	Unscented Kalman Filter

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A. Diagne, M. Diagne, S.-X. Tang and M. Krstic, "Stabilization of the linearized *Saint-Venant-Exner* model via backstepping boundary control", *Automatica*, to appear.

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A. Diagne, S.-X. Tang*, M. Diagne and M. Krstic, "Output feedback stabilization of the bilayer *Saint-Venant* Model", *Proceedings of the ASME Dynamic Systems and Control Conference*, 2016.

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ABSTRACT OF THE DISSERTATION

Boundary Controllers and Observers for Korteweg-de Vries, Hyperbolic, and Parabolic PDEs

by

Shuxia Tang

Doctor of Philosophy in Engineering Sciences (Mechanical Engineering)

University of California, San Diego, 2016

Professor Miroslav Krstic, Chair

This dissertation presents the study of some advancements in the control theory and application of the systems described by partial differential equations. Three classes of partial differential equations are discussed, mainly concerning three topics: stability analysis, boundary controller design and disturbance rejection.

Stability analysis is a topic of great interest and importance in the control area, for which the Lyapunov method and the operator semigroup theory are used in this dissertation. Stability analysis can serve as a guidance of controller design with the objective of stabilization, and the most frequently employed approach for system controller designs in this work is the backstepping methodology. We design state feedback boundary controllers based on backstepping, and further construct

output feedback controllers in assistance with the state observers designed to recover the full system states. Moreover, the sliding mode control and the active disturbance rejection control techniques are applied for the purpose of disturbance rejection.

In detail, the following three classes of systems are investigated. The first class consists of the Korteweg-de Vries systems. For a (nonlinear) Korteweg-de Vries equation, the asymptotic stability analysis is conducted. Two methods, one relying on normal forms and the other relying on a Lyapunov approach, are employed based on the center manifold theory. Both prove that the equation is (locally) asymptotically stable around the origin. A class of linear Korteweg-de Vries systems with possible anti-diffusion is also discussed, focusing on the backstepping controller designs. The resulting closed-loop control systems exhibit exponential decay with respect to the corresponding state norms.

The second is a general class of coupled bidirectional hyperbolic systems with spatially varying coefficients, concerning stability analysis, controller design and disturbance rejection problems. This study borrows the controller design idea from an existing result for the systems with constant coefficients, and extends it to deal with time-varying coefficients. As a result, finite-time stability is achieved. With regards to the systems running into control matched uncertainties and disturbances, disturbance rejection and disturbance attenuation are achieved for the closed-loop feedback systems with sliding mode control and active disturbance rejection control, respectively.

Furthermore, the application of estimation techniques to a real-world issue is studied, i.e., the state-of-charge estimation problem in lithium-ion batteries. A thermal-compensated electrochemical model of lithium-ion batteries is proposed. Adding thermal dynamics serves a two-fold purpose: improving the accuracy of state-of-charge estimation and keeping track of the average temperature which is critical for battery safety management. This model can be included by the third class, i.e., the parabolic systems with time-varying coefficients. Note that in these systems, the time dependency of the system coefficients makes the observer design problem nontrivial. This dissertation can be conducive to future related endeavors.

Chapter 1

Introduction

1.1 A brief overview of PDE control problems

1.1.1 PDE systems

Most real-world physical systems are modeled by partial differential equations (PDEs), which include fluid dynamics, electrochemical and chemical reactions, large-scale multi-agent systems, etc.

To have a brief overview, the first-order PDEs generally serve as models for water waves, traffic flow [1], gas transportation [2], oil drilling [3] and a wide range of biological systems. Typical first-order PDEs are the transport equation, the inviscid Burger's equation, etc.

Diffusion processes such as thermodynamics and chemical reactions can be governed by the second-order PDE systems. Standard examples of second-order PDEs, to name just a few, are the Laplace's equation, the heat equation, the wave equation and the viscous Burgers' equation.

Third-order PDEs arise in the study of dispersive wave motion, including long shallow water waves, plasma waves, and so on. The Korteweg-de Vries (KdV) equation [4] can serve as a prototype.

The fourth-order PDE systems can be used to model vibration of a uniform elastic beams and plate deviation, the fluctuation of flame fronts [5] and the motion of a fluid falling down a vertical wall. The Euler-Bernoulli beam equation, the

Kuramoto-Sivashinsky equation [6] and the Ginzburg-Landau equation, to name just a few, are all fourth-order PDEs.

The fifth-order Korteweg-de Vries (fKdV) equation [4] is also worth mentioning, which can also be used to model different wave phenomena, such as gravity-capillary waves, shallow water waves over a flat surface and magneto-sound propagation in plasmas. Some well-known equations belong to the families of fKdV, for example, the Kaup–Kupershmidt equation [7], the Lax equation and the Sawada-Kotera equation [8].

This dissertation mainly investigates three class of PDEs: the coupled systems of first-order hyperbolic PDEs, a class of (second-order) parabolic PDEs with time-varying coefficients and the third-order KdV equations.

1.1.2 PDE control problems

Control problems for the PDE systems arise from a wide range of contexts. Four of the main fundamental concepts in modern control theory are controllability, stability, stabilization (also known as "stabilizability") and observability. Roughly speaking, these terminologies can be understood in the following ways.

Controllability. Given two ordered states, is it possible for a controller to steer the control system from the first one to the second one?

Stability. There are several definitions of stability. In this dissertation, the most frequently used ones are asymptotic stability, exponential stability and finite-time stability, for which the meanings should be already clear from the names.

Stabilization. Given a system equilibrium which is not stable without applying any control, is it possible to design a controller such that the controlled closed-loop system is stable (in the sense of the designated state norm) around this equilibrium?

Observability. With only partial measurements, is it possible to recover the full system state?

Moreover, there are other important control terminologies, e.g., reachability. As core subjects of the control systems theory, they maintain a highly active research focus.

This dissertation concerns three control problems: stability analysis, stabilization and observer designs to estimate the full system state.

1.2 A brief introduction of PDE control algorithms

1.2.1 Three tools for stability analysis

A. Center manifold method

A normal process for analyzing the stability of nonlinear systems is to first investigate the corresponding linearized systems. If a linearized system is asymptotically stable, then the original nonlinear system is also asymptotically stable; if the linearized system is unstable, then the nonlinear system is unstable as well. For some "critical situations" when the linearized system is neither asymptotically stable nor unstable, the linearization method unfortunately fails to work. This motivates a need of turning to other approaches such as the center manifold method, which plays an important role in studying the dynamic properties of the nonlinear systems near critical situations.

Back to its history, the center manifold theorem was first proved for finite dimensional systems by Pliss [9] and Kelley [10], and the readers could refer to [11, 12] for more details. Analogous results are also established for infinite dimensional systems, such as PDEs [13, 14] and functional differential equations [15]. The center manifold method usually leads to a dimension reduction of the original problems. Then, in order to derive stability properties of the full nonlinear equations, one only needs to analyze the reduced equation (restricted on the center manifold). When dealing with the infinite dimensional problems, this method can be extremely efficient if the center manifold is finite dimensional.

B. Lyapunov method

The Lyapunov methodology is one of the most frequently used approaches in stability analysis of dynamic systems, which is philosophically meaningful and practically effective. The key idea of this method lies in the seeking process of the Lyapunov function, for which the candidates are usually chosen as the system energy function, its variants or its generalized version. Lyapunov method can be applied onto both linear and nonlinear systems. On one hand, if a Lyapunov function is available, the stability analysis via the Lyapunov techniques would become relatively straightforward compared with other approaches. Researchers can benefit from this fact especially when the system under consideration is nonlinear. On the other hand, this method prototypically associates itself with the Lyapunov function and thus may take too much efforts in finding or constructing an appropriate Lyapunov function.

C. Operator semigroup theory

Centering on the spectrum analysis of the system operator, the operator semigroup theory provides another promising method in the stability analysis of dynamic systems. This method, even though generally considered to be more tangled than other techniques, is powerful in the stability analysis especially for the linear systems. When a suitable Lyapunov function is not available or cannot be easily found, the operator semigroup study can be considered as a direct substitution. Additionally, it has also demonstrated its ability of providing more information of the system nature related to its stability.

1.2.2 A technique for boundary controller design

Originally developed as a systematic feedback control approach for stabilizing nonlinear finite-dimensional systems [16], the backstepping framework has been applied to control ordinary differential equations (ODEs) for decades as integrator backstepping [16, 17].

Backstepping was firstly introduced in [18] to stabilize infinite-dimensional systems. The key point of this technique is the construction of suitable invertible transformations, such as the Volterra integral transformations of which the spatial causality guarantees its invertibility. In general, one would like the to-be-determined transformations to map the possibly unstable original systems into so-called "target systems", which are usually well damped and thus exhibit desirable stability properties, e.g., exponential stability. The gain functions of the transformations are required to satisfy some PDEs, by solving which the backstepping transformation can be determined. In the meantime, the solutions to the gain function PDEs, together with the full state information, can be used to derive the state feedback controllers. From the invertibility and some regularity properties of the constructed transformations, the closed-loop control systems driven by these feedback controllers are then ensured of the same stability properties as the target systems.

In principle, the state feedback controller requires full state information, i.e., full state measurement. However, sometimes full state information is not readily available. For example, the states inside a closed system domain and the ones in some severe environments may not necessarily be measured. Even if one is able to have the complete state information measured, it is still not economic in most circumstances and is thus not recommended. Therefore, state estimation would be needed to determine the system state from the output measurements, and this point of view necessitates a need of designing a state observer to work jointly with the full state feedback controller, called as an output feedback controller.

PDE backsteppping has been applied for the feedback stabilization of various classes of unstable PDEs during recent years. As a result, this method has a rich body of literatures. To name just a few, [19] designs observers for a class of parabolic PDEs, [20] works on the output feedback controller design of a hyperbolic PDE, and [21] builds both state feedback and output feedback control algorithms for the complex-valued Schrödinger PDE. We refer the interested reader to [22] for a more detailed study. One can also refer to [23, 24, 25, 26] for further applications of this approach to other classes of systems including nonlinear PDEs.

1.2.3 Two approaches for system disturbance rejection

System uncertainties and disturbance are common problems, which can sometimes worsen the system performance or even lead to instability and thus need to be taken into account. This motivates the controller designs for rejecting the disturbance. Due to its significant implications, considerable attention has been gained from a wide community of researchers.

A. Sliding mode control

Sliding mode control (SMC) technique has been studied for decades and is characterized by its high simplicity and robustness among the existing methods. Recently, this approach has been generalized to distributed parameter systems. For example, it is used to deal with the boundary input disturbance in the wave equation [27] and the Euler-Bernoulli beam equation [28]. Disturbance rejection and finite time stability is achieved for the resulting closed-loop systems with SMC.

While disturbance rejection is the primary goal and the SMC paradigms do provide a solution to the disturbance rejection problems, the controls community still points out that the SMC strategy generally requires the full state information and sacrifices unnecessary control efforts by always considering the worst case scenario.

B. Active disturbance rejection control

The aforementioned problem coming from applying SMC to deal with the system disturbance can be relieved through other methodologies. For example, the active disturbance rejection control (ADRC) can serve as a potential candidate to deal with the input matched disturbance in the systems.

The proposition of this ADRC design philosophy dates back to the period of 1990s, and [29] is strongly recommended for the uninitiated. ADRC uses the strategy of an estimation-cancellation combination to deal with the system uncertainties and disturbance. In detail, an essential element of ADRC is the extended state observer (ESO), which is used to track the plant dynamics and unknown disturbances in real time. Once estimated, the disturbance can be dynamically compensated and then readily canceled in the closed-loop system with feedback control signal of ADRC. It is also worth noting that different from the output feedback control algorithm, the ADRC works jointly and simultaneously with the ESO instead.

ADRC does not require the full state measurement and asks for relatively less control efforts compared with the SMC. It has been showed to reduce the control energy significantly in practice for ordinary differential equation (ODE) systems [30]. This technique has been generalized to distributed parameter systems very recently, and since then rapidly growing literatures have been dedicated to dealing with boundary input disturbance in PDE systems, see, e.g., [28, 27, 31, 32, 33]. Yet, rather than achieving fully disturbance rejection in a finite time, a price is that it only results in asymptotic disturbance attenuation. This point is not really a surprise. In other words, ADRC provides us a choice of trade-off between the energy cost and the outcomes of the control algorithms.

1.3 Organization

The remainder of the dissertation is structured into three parts. Part I discusses control problems of the KdV systems, including Chapters 2–3. Part II investigates the control problems of the coupled hyperbolic systems, including Chapters 4–6. Part III is concerned about a parabolic PDE with time-varying coefficients, and this part includes Chapter 7.

In Chapter 2, a study of the stability property is first conducted on the KdV equation. Then, a general class of linear Korteweg-de Vries equations with possible anti-diffusion (LKdVAD) is investigated. We conduct a preliminary spectrum analysis of the LKdVAD, which is followed by the stability analysis of a subclass of LKdVAD that are exponentially stable.

With the guidance of results from the stability analysis in Chapter 2, we develop controllers for this class of possibly unstable LKdVAD in Chapter 3. Following the lines of backstepping method, we build both state feedback and observer-based output feedback control algorithms. Exponential stability holds for both the resulting control systems.

In Chapter 4, we consider the stability property of coupled systems of transport PDEs. Starting from a preliminary stability analysis on these coupled systems, we also discuss a class of cascaded transport PDE systems for which the exponential stability is derived.

In Chapter 5, we consider the stabilization problem of the coupled transport PDE systems via backstepping, with the exponentially stable system discussed in the previous chapter as a prototype of the backstepping target system. Designs of both state feedback controller and observer-based output feedback control paradigms are presented, actuating at the system boundary.

In Chapter 6, we extend the controller design results in Chapter 5 and take both the SMC and ADRC methods to deal with input matched disturbances. The backstepping transformation postulated in Chapter 5 is performed on the original system as a simplifying pretreatment. Disturbance is rejected within a finite time from the closed-loop system with the SMC, and is attenuated asymptotically from the closed-loop system driven by the ADRC.

In Chapter 7, we offer an application study of the observer design for a parabolic PDE with time-varying coefficients, which comes from the accurate state-of-charge (SoC) estimation problem of Li-ion batteries. The designed observer converges exponentially to the considered physical model, which helps track the battery SoC and thus also helps monitor the battery status and regulate the charging and discharging processes for operational safety and performance enhancement.

In Chapter 8, we summarize and draw concluding remarks for the dissertation. Suggestions for some future works are presented as well in this chapter.

Part I

Korteweg-de Vries Systems

Chapter 2

Stability of the Korteweg-de Vries Equations

2.1 Introduction

2.1.1 Literature review

The Korteweg-de Vries (KdV) equation

$$y_t + y_x + yy_x + y_{xxx} = 0 \tag{2.1}$$

was first derived by Boussinesq in [34, Equation (283 bis)] and by Korteweg and de Vries in [35], for describing the propagation of small amplitude long water waves in a uniform channel. This equation is now commonly used to model unidirectional propagation of small amplitude long waves in nonlinear dispersive systems, see, [36]. It can also be used to model ion-acoustic waves in plasmas, see, [37]. An excellent reference to help understand both physical motivation and deduction of the KdV equation is the book by Whitham [4].

Rosier studied in [38] the following nonlinear Neumann boundary control problem for the KdV equation with homogeneous Dirichlet boundary conditions, posed on a finite spatial interval:

$$\begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, \ t \in (0, \infty), \ x \in (0, L), \\ y(t, 0) = y(t, L) = 0, \ y_x(t, L) = u(t), \ t \in (0, \infty), \\ y(0, x) = y_0(x), \ x \in (0, L), \end{cases} \tag{2.2}$$

where $L > 0$, the state is $y(t, \cdot) : [0, L] \to \mathbb{R}$, and $u(t) \in \mathbb{R}$ denotes the controller. The equation comes with one boundary condition at the left end-point and two boundary conditions at the right end-point. Rosier first considered the first-order power series expansion of (y, u) around the origin, which gives the following corresponding linearized control system

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \ t \in (0, \infty), \ x \in (0, L), \\ y(t, 0) = y(t, L) = 0, \ y_x(t, L) = u(t), \ t \in (0, \infty), \\ y(0, x) = y_0(x), \ x \in (0, L). \end{cases} \tag{2.3}$$

By means of multiplier technique and the Hilbert Uniqueness Method (HUM) [39], Rosier proved that (2.3) is exactly controllable if and only if the length of the spatial domain is not critical, i.e., $L \notin \mathcal{N}$, where $\mathcal{N}$ denotes the following set of critical lengths

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{j^2 + l^2 + jl}{3}}; \ j, l \in \mathbb{N}^* \right\}. \tag{2.4}$$

Then, by employing the Banach fixed point theorem, he derived that the nonlinear KdV control system (2.2) is locally exactly controllable around 0 provided that $L \notin \mathcal{N}$. In the cases with critical lengths $L \in \mathcal{N}$, Rosier demonstrated in [38] that there exists a finite dimensional subspace M of $L^2(0, L)$ which is unreachable for the linear system (2.3) when starting from the origin. In [40], Coron and Crépeau treated a critical case of $L = 2k\pi$ (i.e., taking $j = l = k$ in $\mathcal{N}$), where k is a positive integer such that (see, [41, Theorem 8.1 and Remark 8.2])

$$\left(j^2 + l^2 + jl = 3k^2 \text{ and } j, l \in \mathbb{N}^*\right) \Rightarrow j = l = k. \tag{2.5}$$

Here, the uncontrollable subspace M for the linear system (2.3) is one-dimensional. However, through a third-order power series expansion of the solution, they showed that the nonlinear term yy_x always allows to “go” in small time into the two directions missed by the linearized control system (2.3), and then, using a fixed point theorem, they deduced the small-time local exact controllability around the origin of the nonlinear control system (2.2). In [42], Cerpa studied the critical case of $L \in \mathcal{N}'$, where

$$\mathcal{N}' := \left\{ 2\pi\sqrt{\frac{j^2+l^2+jl}{3}};\ j,l \in \mathbb{N}^* \text{ satisfying } j > l \text{ and } \right. \\ \left. j^2+jl+l^2 \neq m^2+mn+n^2, \forall m,n \in \mathbb{N}^* \backslash \{j\} \right\}. \tag{2.6}$$

In this case, the uncontrollable subspace M for the linear system (2.3) is of dimension 2, and Cerpa used a second-order expansion of the solution to the nonlinear control system (2.2) to prove the local exact controllability in large time around the origin of the nonlinear control system (2.2) (the local controllability in small time for this length L is still an open problem). Furthermore, Cerpa and Crépeau considered in [41] the cases when the dimension of M for the linear system (2.3) is higher than 2. They implemented a second-order expansion of the solution to (2.2) for the critical lengths $L \neq 2k\pi$ for any $k \in \mathbb{N}^*$, and implemented an expansion to the third order if $L = 2k\pi$ for some $k \in \mathbb{N}^*$. They showed that the nonlinear term yy_x always allows to “go” into all the directions missed by the linearized control system (2.3) and then proved the local exact controllability in large time around the origin of the nonlinear control system (2.2).

2.1.2 Organization

The outline of this chapter is as follows. Section 2.2 is dedicated to the stability analysis for an initial-boundary-value problem of a (nonlinear) KdV equation posed on a finite interval $[0, 2\pi\sqrt{7/3}]$, in which the equation comes with a Dirichlet boundary condition at the left end-point and both the Dirichlet and Neumann homogeneous boundary conditions at the right end-point. Then, Section 2.3 conducts stability analysis for an initial-boundary-value problem of linear KdV systems with possible anti-diffusion (LKdVAD). The systems are posed on a finite interval $[0, L]$, with a Dirichlet boundary condition at the right end. For the left end of the interval, a general combination of Robin and Navier-like boundary conditions is considered. Finally, a conclusion and some possible future works are given in Section 2.4.

2.2 Stability analysis of the initial-boundary-value KdV problem

This section is organized as follows. The initial-boundary-value KdV problem is introduced in Subsection 2.2.1. In Subsection 2.2.2, some basic properties of the linearized KdV equation and the (nonlinear) KdV equation are given. Then, in Subsection 2.2.3, we recall a theorem on the existence of a local center manifold for the (nonlinear) KdV equation and analyze the dynamics on the local center manifold. The main result follows from this analysis and is concluded in Subsection 2.2.4. Moreover, Appendix A.1 contains computations which are important for the study of the dynamics on the center manifold.

2.2.1 Problem statement

In this section, we consider the following initial-boundary-value KdV problem posed on a finite interval $[0, L]$:

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0,\ t \in (0, \infty),\ x \in (0, L), \\ y(t, 0) = y(t, L) = 0,\ y_x(t, L) = 0,\ t \in (0, \infty), \\ y(0, x) = y_0(x),\ x \in (0, L), \end{cases} \tag{2.7}$$

where the boundary conditions are homogeneous. For the Lyapunov function

$$E(t) = \frac{1}{2}\|y(t, \cdot)\|^2_{L^2(0,L)} = \frac{1}{2}\int_0^L y^2(t, x)\,\mathrm{d}x, \tag{2.8}$$

we have

$$\dot{E}(t) = -\int_0^L y(y_x + yy_x + y_{xxx})\,\mathrm{d}x = \int_0^L y_x y_{xx}\,\mathrm{d}x = -\frac{1}{2}y_x^2(t, 0) \leq 0. \tag{2.9}$$

Thus, $0 \in L^2(0, L)$ is stable (see $(\mathcal{P}_1)$ below for the definition of stable) for the KdV equation (2.7). Moreover, it has been proved in [43] that, if $L \notin \mathcal{N}$, where

$\mathcal{N}$ denotes following set of critical lengths

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{j^2 + l^2 + jl}{3}};\ j, l \in \mathbb{N}^* \right\}, \tag{2.10}$$

then 0 is exponentially stable for the corresponding linearized equation around the origin

$$\begin{cases} y_t + y_x + y_{xxx} = 0,\ t \in (0, \infty),\ x \in (0, L), \\ y(t, 0) = y(t, L) = 0,\ y_x(t, L) = 0,\ t \in (0, \infty), \\ y(0, x) = y_0(x),\ x \in (0, L), \end{cases} \tag{2.11}$$

which gives the local asymptotic stability around the origin for the nonlinear equation (2.7). However, when $L \in \mathcal{N}$, Rosier pointed out in [38] that the equation (2.11) is not asymptotically stable. Inspired by the fact that the nonlinear term yy_x introduces the local exact controllability around the origin into the KdV control system (2.2) with $L \in \mathcal{N}$, we would like to discuss whether the nonlinear term yy_x could introduce local asymptotic stability around the origin for (2.7).

This section is devoted to investigating the local asymptotic stability of $0 \in L^2(0, L)$ for (2.7) with the critical length

$$L = 2\pi \sqrt{\frac{7}{3}}, \tag{2.12}$$

corresponding to $j = 1$ and $l = 2$ in (2.10). Let us recall that this local asymptotic stability means that the following two properties are satisfied.

($\mathcal{P}_1$) Stability: for every $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that $\|y_0\|_{L^2(0,L)} < \eta$ implies

$$\|y(t, \cdot)\|_{L^2(0,L)} < \varepsilon, \quad \forall t \geq 0. \tag{2.13}$$

($\mathcal{P}_2$) (Local) attractivity: there exists $\varepsilon_0 > 0$ such that $\|y_0\|_{L^2(0,L)} < \varepsilon_0$ implies

$$\lim_{t \to +\infty} \|y(t, \cdot)\|_{L^2(0,L)} = 0. \tag{2.14}$$

As mentioned above, the stability property ($\mathcal{P}_1$) is implied by (2.9). Our main concern is thus the local attractivity property ($\mathcal{P}_2$).

Our stability analysis relies on the center manifold approach. Following the results on existence, smoothness and attractivity of a center manifold for evolution equations in [44], Chu, Coron and Shang studied in [45] the local asymptotic stability property of (2.7) with the critical length $L = 2k\pi$ for any positive integer k such that (2.5) holds. They proved the existence of a one-dimensional local center manifold. By analyzing the resulting one-dimensional reduced equation, they obtained the local asymptotic stability of 0 for (2.7). For $L = 2\pi\sqrt{7/3}$, we get, following [45], the existence of a two-dimensional local center manifold. It is predictable that the two-dimensional local center manifold introduces more complexity than the one-dimensional local center manifold case.

2.2.2 Preliminaries

Some properties for the linearized equation of (2.7) around the origin

The origin $y = 0$ is an equilibrium of the initial-boundary-value nonlinear KdV problem (2.7). In this subsection, we derive some properties for the linearized KdV equation (2.11) around the origin of (2.7) posed on the finite interval $[0, L]$, where $L = 2\pi\sqrt{7/3} \in \mathcal{N}'$, for which there exists a unique pair $\{j = 2, l = 1\}$ satisfying (2.6).

Let $\mathcal{A} : D(\mathcal{A}) \subset L^2(0, L) \to L^2(0, L)$ be the linear operator defined by

$$\mathcal{A}\varphi := -\varphi' - \varphi''', \tag{2.15}$$

with

$$\mathrm{Dom}(\mathcal{A}) := \left\{\varphi \in H^3(0, L);\ \varphi(0) = \varphi(L) = \varphi'(L) = 0\right\} \subset L^2(0, L), \tag{2.16}$$

then the linearized equation (2.11) can be written as an evolution equation in $L^2(0, L)$:

$$\frac{\mathrm{d}y(t, \cdot)}{\mathrm{d}t} = \mathcal{A}y(t, \cdot). \tag{2.17}$$

The following lemma can be immediately obtained.

Lemma 2.1. *$\mathcal{A}^{-1}$ exists and is compact on $L^2(0,L)$. Hence, $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues only: $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$, where $\sigma_p(\mathcal{A})$ denotes the set of eigenvalues of $\mathcal{A}$.*

Proof. By calculation, we get

$$\mathcal{A}^{-1}\varphi = \psi, \ \forall \varphi \in L^2(0,L), \tag{2.18}$$

with

$$\psi := -\frac{1-\cos(x-L)}{1-\cos L}\int_0^L (1-\cos y)\varphi(y)\,\mathrm{d}y + \int_x^L (1-\cos(x-y))\varphi(y)\,\mathrm{d}y. \tag{2.19}$$

Hence we get the existence of $\mathcal{A}^{-1}$ and that, by the Sobolev embedding theorem, this operator is compact on $L^2(0,L)$. Therefore, $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues only. □

The following proposition is proved.

Proposition 1. ([38, Proposition 3.1]). $\mathcal{A}$ generates a C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$ on $L^2(0,L)$, that is, for any given initial data $y_0 \in L^2(0,L)$, $S(t)y_0$ is the mild solution of the linearized equation (2.11), and

$$\|S(t)y_0\|_{L^2(0,L)} \leq \|y_0\|_{L^2(0,L)}, \quad \forall t \geq 0. \tag{2.20}$$

Moreover, $\mathrm{Re}\,(\lambda) \leq 0$ for every $\lambda \in \sigma(\mathcal{A})$.

If $\mathrm{Re}\,(\lambda) < 0$, $\forall \lambda \in \sigma(\mathcal{A})$, then it follows directly from the ABLP (Arendt-Batty-Lyubich-Phong) theorem [46] that the semigroup $S(t)$ is asymptotically stable on $L^2(0,L)$. Since we only have $\mathrm{Re}\,(\lambda) \leq 0$, $\forall \lambda \in \sigma(\mathcal{A})$, the main concern needs to be put on the eigenvalues on the imaginary axis and their corresponding eigenfunctions. Following the proofs of [45, Lemma 2.6] and [38, Lemma 3.5] yields the next lemma.

Lemma 2.2. *There exists a unique pair of conjugate eigenvalues of $\mathcal{A}$ on the imaginary axis, that is,*

$$\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \left\{\lambda = \pm iq;\ q = \frac{20}{21\sqrt{21}}\right\}. \tag{2.21}$$

Moreover, the corresponding eigenfunctions of $\mathcal{A}$ *with respect to* $\lambda = \pm iq$ *are*

$$\varphi := C\left(\varphi_1 \mp i\varphi_2\right), \tag{2.22}$$

respectively, where C *is an arbitrary constant, and* φ_1, φ_2 *are two nonzero real-valued functions:*

$$\varphi_1(x) = \Theta\left(\cos\left(\frac{5}{\sqrt{21}}x\right) - 3\cos\left(\frac{1}{\sqrt{21}}x\right) + 2\cos\left(\frac{4}{\sqrt{21}}x\right)\right), \tag{2.23}$$

$$\varphi_2(x) = \Theta\left(-\sin\left(\frac{5}{\sqrt{21}}x\right) - 3\sin\left(\frac{1}{\sqrt{21}}x\right) + 2\sin\left(\frac{4}{\sqrt{21}}x\right)\right), \tag{2.24}$$

with

$$\Theta := \frac{1}{\sqrt{14\pi}}\sqrt[4]{\frac{3}{7}}. \tag{2.25}$$

Remark 2.1. The equations satisfied by φ_1 and φ_2 are

$$\begin{cases} \varphi_1' + \varphi_1''' = -q\varphi_2, \\ \varphi_1(0) = \varphi_1(L) = 0, \\ \varphi_1'(0) = \varphi_1'(L) = 0, \end{cases} \tag{2.26}$$

and

$$\begin{cases} \varphi_2' + \varphi_2''' = q\varphi_1, \\ \varphi_2(0) = \varphi_2(L) = 0, \\ \varphi_2'(0) = \varphi_2'(L) = 0. \end{cases} \tag{2.27}$$

Remark 2.2. We have

$$\int_0^L \varphi_1(x)\varphi_2(x)\,\mathrm{d}x = 0, \tag{2.28}$$

and, with the definition of Θ given in (2.25),

$$\|\varphi_1\|_{L^2(0,L)} = \|\varphi_2\|_{L^2(0,L)} = 1. \tag{2.29}$$

From the results in Lemma 2.1, Proposition 1 and Lemma 2.2, we obtain the following corollary.

Corollary 1. $\lambda = \pm i\frac{20}{21\sqrt{21}}$ is the unique eigenvalue pair of $\mathcal{A}$ on the imaginary axis, and all the other eigenvalues of $\mathcal{A}$ have negative real parts which are uniformly bounded away from the imaginary axis, i.e., there exists $r > 0$ such that any of the nonzero eigenvalues of $\mathcal{A}$ has a real part which is less than $-r$.

Let us define

$$M := \operatorname{span}\{\varphi_1, \varphi_2\} = \{m_1\varphi_1 + m_2\varphi_2;\ \mathbf{m} = (m_1, m_2) \in \mathbb{R}^2\} \subset L^2(0, L), \quad (2.30)$$

where $\varphi_1,\ \varphi_2$ are defined in (2.23), (2.24) and (2.25). Then the following decomposition holds:

$$L^2(0, L) = M \oplus M^\perp, \quad (2.31)$$

with

$$M^\perp := \left\{\varphi \in L^2(0, L);\ \int_0^L \varphi(x)\varphi_1(x)\,\mathrm{d}x = 0, \int_0^L \varphi(x)\varphi_2(x)\,\mathrm{d}x = 0\right\}. \quad (2.32)$$

Some properties of the KdV equation (2.7)

By considering the equation (2.7) as a special case (with $f = 0$ and $u = 0$) of the equation (4.6)–(4.8) in [41], we give the following definition for a solution to the equation (2.7), which follows from [41, Definition 4.1].

Definition 1. Let $T > 0$ and $y_0 \in L^2(0, L)$. A solution to the Cauchy problem (2.7) on $[0, T]$ is a function

$$y \in \mathcal{B} := C^0([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L)) \quad (2.33)$$

such that, for every $\tau \in [0, T]$ and for every $\phi \in C^3([0, \tau] \times [0, L])$ satisfying

$$\phi(t, 0) = \phi(t, L) = \phi_x(t, 0) = 0,\ \forall t \in [0, \tau], \quad (2.34)$$

one has

$$-\int_0^\tau \int_0^L (\phi_t + \phi_x + \phi_{xxx})y\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_0^L \phi y y_x\,\mathrm{d}x\,\mathrm{d}t + \int_0^L y(\tau, x)\phi(\tau, x)\,\mathrm{d}x - \int_0^L y_0(x)\phi(0, x)\,\mathrm{d}x = 0. \quad (2.35)$$

A solution to the Cauchy problem (2.7) on $[0, +\infty)$ is a function

$$y \in C^0([0, +\infty); L^2(0, L)) \cap L^2_{\mathrm{loc}}([0, +\infty); H^1(0, L)) \quad (2.36)$$

such that, for every $T > 0$, y restricted to $[0, T] \times (0, L)$ is a solution to (2.7) on $[0, T]$.

Then by considering equation (2.7) as a special case (with $f = 0$ and $u = 0$), of the equation $(A.1)$ in [40], the following two propositions about the existence and uniqueness of the solutions to (2.7) follow directly from [40, Proposition 14 and Proposition 15].

Proposition 2. Let $T \in (0, +\infty)$. There exist $\varepsilon = \varepsilon(T) > 0$ and $C = C(T) > 0$ such that, for every $y_0 \in L^2(0, L)$ with $\|y_0\|_{L^2(0,L)} < \varepsilon(T)$, there exists at least one solution y to the equation (2.7) on $[0, T]$ which satisfies

$$\begin{aligned} \|y\|_{\mathcal{B}} &:= \max_{t\in[0,T]} \|y(t,\cdot)\|_{L^2(0,L)} + \left(\int_0^T \|y(t,\cdot)\|^2_{H^1(0,L)}\,\mathrm{d}t\right)^{1/2} \\ &\leq C(T)\|y_0\|_{L^2(0,L)}. \end{aligned} \tag{2.37}$$

Proposition 3. Let $T \in (0, +\infty)$. There exists $C > 0$ such that, for every solutions y_1 and y_2, corresponding to every initial conditions $(y_{10}, y_{20}) \in (L^2(0, L))^2$ respectively, to the equation (2.7) on $[0, T]$, one has the following inequalities:

$$\begin{aligned} \int_0^T\int_0^L (y_{1x}(t,x) - y_{2x}(t,x))^2\,\mathrm{d}x\,\mathrm{d}t \leq{}& \int_0^L (y_{10}(x) - y_{20}(x))^2\,\mathrm{d}x \\ &\times \exp\left(C\left(1 + \|y_1\|^2_{L^2(0,T;H^1(0,L))} + \|y_2\|^2_{L^2(0,T;H^1(0,L))}\right)\right), \end{aligned} \tag{2.38}$$

$$\begin{aligned} \int_0^L (y_1(t,x) - y_2(t,x))^2\,\mathrm{d}x \leq{}& \int_0^L (y_{10}(x) - y_{20}(x))^2\,\mathrm{d}x \\ &\times \exp\left(C\left(1 + \|y_1\|^2_{L^2(0,T;H^1(0,L))} + \|y_2\|^2_{L^2(0,T;H^1(0,L))}\right)\right), \end{aligned} \tag{2.39}$$

for all $t \in [0, T]$.

Let us also mention that for every solution y to (2.7) on $[0, T]$ or on $[0, +\infty)$,

$$t \mapsto \|y(t,\cdot)\|^2_{L^2(0,L)} \text{ is a non-increasing function.} \tag{2.40}$$

This can be easily seen by multiplying the first equation of (2.7) with y, integrating on $[0, L]$ and performing integration by parts. One then gets, if y is smooth enough,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L y(t,x)^2\,\mathrm{d}x = -y_x(t,0)^2, \tag{2.41}$$

which gives (2.40). The general case follows from a smoothing argument. As a consequence of Proposition 2, Proposition 3 and (2.40), one sees that (2.7) has one and only one solution defined on $[0, +\infty)$ if $\|y_0\|_{L^2(0,L)} < \varepsilon(1)$.

2.2.3 Existence of a center manifold and dynamics on this manifold

Existence of a local center manifold

In [45, Theorem 3.1], following [44], the existence of a center manifold for (2.7) was proved for the first critical length, i.e., $L = 2\pi$. The same proof applies for our L (i.e., the L defined by (2.12)) and allows us to get the following theorem.

Theorem 2.1. *There exist $\delta \in (0, \varepsilon(1))$, $K > 0$, $\omega > 0$ and a map $g : M \to M^{\perp}$ satisfying*

$$g \in C^3(M; M^{\perp}), \tag{2.42}$$

$$g(0) = 0,\ \ g'(0) = 0, \tag{2.43}$$

such that, with G defined by

$$G := \{m + g\,(m)\,;\ m \in M\} \subset L^2(0, L), \tag{2.44}$$

the following two properties hold for every solution $y(t, x)$ to (2.7) with $\|y_0\|_{L^2(0,L)} < \delta$,

1. *(Local exponential attractivity of G.)*

$$d(y(t, \cdot), G) \leq K e^{-\omega t} d(y_0, G),\ \ \forall t > 0, \tag{2.45}$$

where $d(\chi, G)$ denotes the distance between $\chi \in L^2(0, L)$ and G:

$$d(\chi, G) := \inf\{\|\chi - \psi\|_{L^2(0,L)};\ \psi \in G\}. \tag{2.46}$$

2. *(Local invariance of G.)*

$$\text{If } y_0 \in G, \text{ then } y(t, \cdot) \in G,\ \forall t \geq 0. \tag{2.47}$$

Dynamics on the local center manifold

In this subsection we study the dynamics of (2.7) on G_δ with

$$G_\delta := \{\zeta(x) \in G;\ \|\zeta\|_{L^2(0,L)} < \delta\}. \tag{2.48}$$

Let

$$\Omega := \{(m_1, m_2) \in \mathbb{R}^2;\ m_1\varphi_1 + m_2\varphi_2 + g(m_1\varphi_1 + m_2\varphi_2) \in G_\delta\}, \tag{2.49}$$

then Ω is a bounded open subset of $\mathbb{R}^2$ which contains $(0,0) \in \mathbb{R}^2$. Let $\mathbf{m}^0 = (m_1^0, m_2^0) \in \Omega$, and let y be the solution of (2.7) on $[0, +\infty)$ for the initial data $y_0 := m_1^0\varphi_1 + m_2^0\varphi_2 + g(m_1^0\varphi_1 + m_2^0\varphi_2)$. It follows from (2.40) and Theorem 2.1 that $y(t, \cdot) \in G_\delta$ for every $t \in [0, +\infty)$. Hence we can define, for $t \in [0, +\infty)$, $\mathbf{m}(t) = (m_1(t), m_2(t)) \in \Omega$ by requiring that

$$y(t, \cdot) = m_1(t)\varphi_1 + m_2(t)\varphi_2 + g(m_1(t)\varphi_1 + m_2(t)\varphi_2). \tag{2.50}$$

Since $y \in C^0([0, +\infty); L^2(0, L))$, we have $\mathbf{m} \in C^0([0, +\infty); \mathbb{R}^2)$. Let $T > 0$ and $u \in C_0^\infty(0, T)$. We apply (2.35) with $\tau = T$ and $\phi(t, x) := u(t)\varphi_1(x)$ (note that, by (2.26), (2.34) holds). We get

$$\begin{aligned} -\int_0^T \int_0^L (\dot{u}(t)\varphi_1(x) + u(t)\varphi_1'(x) + u(t)\varphi_1'''(x)) y(t, x)\, \mathrm{d}x\, \mathrm{d}t& \\ + \int_0^T \int_0^L u(t)\varphi_1(x)(yy_x)(t, x)\, \mathrm{d}x\, \mathrm{d}t = 0.& \end{aligned} \tag{2.51}$$

From (2.26), (2.32), (2.50) and (2.51), we have

$$-\int_0^T (m_1(t)\dot{u}(t) - qm_2(t)u(t))\, \mathrm{d}t - \frac{1}{2}\int_0^T \int_0^L y^2(t, x)\varphi_1'(x)u(t)\, \mathrm{d}x\, \mathrm{d}t = 0. \tag{2.52}$$

Hence, in the sense of distributions on $(0, T)$,

$$\dot{m}_1 = -qm_2 + \frac{1}{2}\int_0^L (m_1\varphi_1 + m_2\varphi_2 + g\,(m_1\varphi_1 + m_2\varphi_2))^2\, \varphi_1'\, \mathrm{d}x. \tag{2.53}$$

Similarly, in the sense of distributions on $(0, T)$,

$$\dot{m}_2 = qm_1 + \frac{1}{2}\int_0^L (m_1\varphi_1 + m_2\varphi_2 + g\,(m_1\varphi_1 + m_2\varphi_2))^2\, \varphi_2'\, \mathrm{d}x. \tag{2.54}$$

Hence, if we define $F : \Omega \to \mathbb{R}^2$, $\mathbf{m} = (m_1, m_2) \mapsto F(\mathbf{m})$, by

$$F(\mathbf{m}) := \begin{pmatrix} -qm_2 + \dfrac{1}{2}\displaystyle\int_0^L (m_1\varphi_1 + m_2\varphi_2 + g(m_1\varphi_1 + m_2\varphi_2))^2\varphi_1'\, \mathrm{d}x \\ qm_1 + \dfrac{1}{2}\displaystyle\int_0^L (m_1\varphi_1 + m_2\varphi_2 + g(m_1\varphi_1 + m_2\varphi_2))^2\varphi_2'\, \mathrm{d}x \end{pmatrix}, \tag{2.55}$$

then

$$\dot{\mathbf{m}} = F(\mathbf{m}). \tag{2.56}$$

Note that, by (2.42) and (2.55),

$$F \in C^3(\Omega; \mathbb{R}^2), \tag{2.57}$$

which, together with (2.56), implies that

$$\mathbf{m} \in C^4([0, +\infty); \mathbb{R}^2). \tag{2.58}$$

We now estimate g close to $0 \in M$. Let $\psi \in C^3([0, L])$ be such that

$$\psi(0) = \psi(L) = \psi'(0) = 0. \tag{2.59}$$

Using Definition 1 with $\phi(t, x) := \psi(x)$, (2.59) and integration by parts, we get

$$\begin{aligned} -\frac{1}{\tau}\int_0^\tau \int_0^L (\psi' + \psi''')y \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2\tau}\int_0^\tau \int_0^L \psi' y^2 \,\mathrm{d}x\,\mathrm{d}t& \\ + \int_0^L \frac{1}{\tau}\left(y(\tau, x) - y_0(x)\right)\psi(x)\,\mathrm{d}x = 0.& \end{aligned} \tag{2.60}$$

Letting $\tau \to 0^+$ in (2.60), and using (2.55), (2.56) and (2.58), we get

$$\begin{aligned} -\int_0^L (\psi' + \psi''')y_0 \,\mathrm{d}x - \frac{1}{2}\int_0^L \psi' y_0^2 \,\mathrm{d}x + \int_0^L \Big(\dot{m}_1(0)\varphi_1(x) + \dot{m}_2(0)\varphi_2(x)& \\ + \frac{\partial g}{\partial m_1}(\mathbf{m}^0)\dot{m}_1(0) + \frac{\partial g}{\partial m_2}(\mathbf{m}^0)\dot{m}_2(0)\Big)\psi \,\mathrm{d}x = 0.& \end{aligned} \tag{2.61}$$

We expand g in a neighborhood of $0 \in M$. Using (2.42) and (2.43), there exist

$$a \in M^\perp, b \in M^\perp, c \in M^\perp \tag{2.62}$$

such that

$$g(\alpha\varphi_1 + \beta\varphi_2) = \alpha^2 a + \alpha\beta b + \beta^2 c + o(\alpha^2 + \beta^2) \text{ in } L^2(0, L) \text{ as } \alpha^2 + \beta^2 \to 0, \tag{2.63}$$

$$\frac{\partial g}{\partial m_1}(\alpha\varphi_1 + \beta\varphi_2) = 2\alpha a + \beta b + o(|\alpha| + |\beta|) \text{ in } L^2(0, L) \text{ as } |\alpha| + |\beta| \to 0, \tag{2.64}$$

$$\frac{\partial g}{\partial m_2}(\alpha\varphi_1 + \beta\varphi_2) = \alpha b + 2\beta c + o(|\alpha| + |\beta|) \text{ in } L^2(0, L) \text{ as } |\alpha| + |\beta| \to 0. \tag{2.65}$$

As usual, by (2.63), we mean that, for every $\varsigma_1 > 0$, there exists $\varsigma_2 > 0$ such that

$$\begin{aligned} &\alpha^2+\beta^2 \leq \varsigma_1 \\ &\Rightarrow \|g(\alpha\varphi_1+\beta\varphi_2) - \big(\alpha^2 a + \alpha\beta b + \beta^2 c\big)\|_{L^2(0,L)} \leq \varsigma_2(\alpha^2+\beta^2). \end{aligned} \tag{2.66}$$

Similar definitions are used in (2.64), (2.65) and later on. We now expand the left hand side of (2.61) in terms of m_1^0, m_2^0, $(m_1^0)^2$, $m_1^0 m_2^0$ and $(m_2^0)^2$ as $|m_1^0|+|m_2^0| \to 0$.

For the functions φ_1 and φ_2 defined by (2.23), (2.24) and (2.25), the following equalities can be derived from (2.26), (2.27) and using integrations by parts:

$$\int_0^L \varphi_1(x)\varphi_2'(x)\,\mathrm{d}x = \frac{10}{7\sqrt{21}}, \qquad \int_0^L \varphi_2(x)\varphi_1'(x)\,\mathrm{d}x = -\frac{10}{7\sqrt{21}}, \tag{2.67}$$

$$\int_0^L \varphi_1^2(x)\varphi_1'(x)\,\mathrm{d}x = 0, \qquad \int_0^L \varphi_2^2(x)\varphi_2'(x)\,\mathrm{d}x = 0, \tag{2.68}$$

$$\int_0^L \varphi_1^2(x)\varphi_2'(x)\,\mathrm{d}x = -2c_1, \qquad \int_0^L \varphi_2^2(x)\varphi_1'(x)\,\mathrm{d}x = 2\sqrt{3}c_1, \tag{2.69}$$

$$\int_0^L \varphi_1(x)\varphi_2(x)\varphi_1'(x)\,\mathrm{d}x = c_1, \quad \int_0^L \varphi_1(x)\varphi_2(x)\varphi_2'(x)\,\mathrm{d}x = -\sqrt{3}c_1, \tag{2.70}$$

where the constant c_1 is defined by

$$c_1 := \frac{177147}{392392\pi}\sqrt{\frac{1}{2\pi}}\sqrt[4]{\frac{3}{7}}. \tag{2.71}$$

Looking successively at the terms in $(m_1^0)^2$, $m_1^0 m_2^0$ and $(m_2^0)^2$ in (2.61) as $|m_1^0|+|m_2^0| \to 0$, we get, using (2.55), (2.56), (2.63) – (2.65) as well as (2.67)–(2.70),

$$-\int_0^L (\psi_x+\psi_{xxx})a\,\mathrm{d}x - \frac{1}{2}\int_0^L \psi_x\varphi_1^2\,\mathrm{d}x + \int_0^L (-c_1\varphi_2+qb)\,\psi\,\mathrm{d}x = 0, \tag{2.72}$$

$$\begin{aligned} -\int_0^L (\psi_x+\psi_{xxx})b\,\mathrm{d}x &- \int_0^L \psi_x\varphi_1\varphi_2\,\mathrm{d}x \\ &+ \int_0^L \Big(c_1\varphi_1 - \sqrt{3}c_1\varphi_2 - 2qa + 2qc\Big)\,\psi\,\mathrm{d}x = 0, \end{aligned} \tag{2.73}$$

$$-\int_0^L (\psi_x+\psi_{xxx})c\,\mathrm{d}x - \frac{1}{2}\int_0^L \psi_x\varphi_2^2\,\mathrm{d}x + \int_0^L \Big(\sqrt{3}c_1\varphi_1 - qb\Big)\,\psi\,\mathrm{d}x = 0. \tag{2.74}$$

Since (2.72) – (2.74) must hold for every $\psi \in C^3([0,L])$ satisfying (2.59), one gets that a, b and c are of class C^∞ on $[0,L]$ and satisfy

$$\begin{cases} a' + a''' + \varphi_1\varphi_1' - c_1\varphi_2 + qb = 0, \\ a(0) = a(L) = 0,\ a'(L) = 0, \end{cases} \tag{2.75}$$

$$\begin{cases} b' + b''' + \varphi_1\varphi_2' + \varphi_1'\varphi_2 + c_1\varphi_1 - \sqrt{3}c_1\varphi_2 - 2qa + 2qc = 0, \\ b(0) = b(L) = 0,\ b'(L) = 0, \end{cases} \tag{2.76}$$

$$\begin{cases} c' + c''' + \varphi_2\varphi_2' + \sqrt{3}c_1\varphi_1 - qb = 0, \\ c(0) = c(L) = 0,\ c'(L) = 0. \end{cases} \tag{2.77}$$

In Appendix A.1, we derive the unique functions $a : [0, L] \to \mathbb{R}$, $b : [0, L] \to \mathbb{R}$ and $c : [0, L] \to \mathbb{R}$ which are solutions to (2.75), (2.76) and (2.77). From (2.55) and (2.63), we get that, as $\mathbf{m} \to \mathbf{0} \in \mathbb{R}^2$,

$$F(\mathbf{m}) = \begin{pmatrix} -qm_2 + \sqrt{3}c_1m_2^2 + c_1m_1m_2 + A_1m_1^3 + B_1m_1^2m_2 + C_1m_1m_2^2 + D_1m_2^3 \\ qm_1 - c_1m_1^2 - \sqrt{3}c_1m_1m_2 + A_2m_1^3 + B_2m_1^2m_2 + C_2m_1m_2^2 + D_2m_2^3 \end{pmatrix} + o(|\mathbf{m}|^3), \tag{2.78}$$

with

$$A_1 := \int_0^L a\varphi_1\varphi_1' \,\mathrm{d}x, \tag{2.79}$$

$$B_1 := \int_0^L b\varphi_1\varphi_1' \,\mathrm{d}x + \int_0^L a\varphi_2\varphi_1' \,\mathrm{d}x, \tag{2.80}$$

$$C_1 := \int_0^L c\varphi_1\varphi_1' \,\mathrm{d}x + \int_0^L b\varphi_2\varphi_1' \,\mathrm{d}x, \tag{2.81}$$

$$D_1 := \int_0^L c\varphi_2\varphi_1' \,\mathrm{d}x, \tag{2.82}$$

$$A_2 := \int_0^L a\varphi_1\varphi_2' \,\mathrm{d}x, \tag{2.83}$$

$$B_2 := \int_0^L b\varphi_1\varphi_2' \,\mathrm{d}x + \int_0^L a\varphi_2\varphi_2' \,\mathrm{d}x, \tag{2.84}$$

$$C_2 := \int_0^L c\varphi_1\varphi_2' \,\mathrm{d}x + \int_0^L b\varphi_2\varphi_2' \,\mathrm{d}x, \tag{2.85}$$

$$D_2 := \int_0^L c\varphi_2\varphi_2' \,\mathrm{d}x. \tag{2.86}$$

Let us now study the local asymptotic stability property of $\mathbf{0} \in \mathbb{R}^2$ for (2.56). We propose two methods for that. The first one is a more direct one, which relies on normal forms for dynamical systems on $\mathbb{R}^2$. The second one, which relies on a Lyapunov approach related to the physics of (2.7), is less direct. However, there is a reasonable hope that this second method can be applied to other critical lengths $L \in \mathcal{N} \setminus 2\pi\mathbb{N}$ for which the dimension of M is larger than 2.

Method 1: normal form. Let

$$z := m_1 + im_2 \in \mathbb{C}. \tag{2.87}$$

Then

$$m_1 = \frac{z+\overline{z}}{2}, m_2 = \frac{z-\overline{z}}{2i}, \tag{2.88}$$

and it follows from (2.56) and (2.78) that, as $|z| \to 0$,

$$\dot{z} = (iq)\, z + P_2(z,\overline{z}) + P_3(z,\overline{z}) + o(|z|^3), \tag{2.89}$$

where $P_j(z,\overline{z})$ are polynomials in $z, \overline{z}$ of degree j. To be more precise, we have

$$\begin{aligned} P_2(z,\overline{z}) \;&:=\; \left(\sqrt{3}c_1 m_2^2 + c_1 m_1 m_2\right) + i\left(-c_1 m_1^2 - \sqrt{3}c_1 m_1 m_2\right) \\ &=\; -\frac{c_1}{2}\left(\sqrt{3}+i\right) z^2 + \frac{c_1}{2}\left(\sqrt{3}-i\right) z\overline{z}, \end{aligned} \tag{2.90}$$

and

$$\begin{aligned} P_3(z,\overline{z}) := (A_1 + iA_2)\left(\frac{z+\overline{z}}{2}\right)^3 + (B_1 + iB_2)\left(\frac{z+\overline{z}}{2}\right)^2\left(\frac{z-\overline{z}}{2i}\right) \\ + (C_1 + iC_2)\left(\frac{z+\overline{z}}{2}\right)\left(\frac{z-\overline{z}}{2i}\right)^2 + (D_1 + iD_2)\left(\frac{z-\overline{z}}{2i}\right)^3. \end{aligned} \tag{2.91}$$

We can rewrite (2.89) as

$$\dot{z} = (iq)\, z + \sum_{i+j=2}^{3} \frac{1}{i!j!} g_{ij} z^i \overline{z}^j + o(|z|^3), \tag{2.92}$$

and it is known from [47, page 45 and page 47] that (2.92) has the following Poincaré normal form

$$\dot{\xi} = (iq)\, \xi + \rho \xi^2 \overline{\xi} + o(|\xi|^3), \tag{2.93}$$

where

$$\rho = \frac{i}{2q}\left(g_{20}g_{11} - 2\left|g_{11}\right|^2 - \frac{1}{3}\left|g_{02}\right|^2\right) + \frac{g_{21}}{2}. \tag{2.94}$$

According to (2.90) and (2.91), through a simple computation, we have

$$g_{20} = -c_1\left(\sqrt{3}+i\right), g_{11} = \frac{c_1}{2}\left(\sqrt{3}-i\right), g_{02} = 0, \tag{2.95}$$

$$g_{21} = \frac{1}{4}\left(3A_1 + i3A_2 - iB_1 + B_2 + C_1 + iC_2 + -i3D_1 + 3D_2\right). \tag{2.96}$$

Using (2.95) and (2.96), the formula of ρ provided by (2.94) gives

$$\rho = \rho_1 + i\rho_2, \tag{2.97}$$

with

$$\rho_1 := \frac{1}{8}\left(3A_1 + C_1 + B_2 + 3D_2\right), \tag{2.98}$$

$$\rho_2 := -2\frac{c_1^2}{q} + \frac{1}{8}\left(-B_1 - 3D_1 + 3A_2 + C_2\right). \tag{2.99}$$

It follows that we can derive the Poincaré normal form of the reduced equation on the local center manifold (2.93). Moreover, in Cartesian coordinates, (2.93) is

$$\dot{\xi}_1 = -q\xi_2 + (a\xi_1 - b\xi_2)\left(\xi_1^2 + \xi_2^2\right) + o(|\xi_1|^3 + |\xi_2|^3), \tag{2.100}$$

$$\dot{\xi}_2 = q\xi_1 + (a\xi_2 + b\xi_1)\left(\xi_1^2 + \xi_2^2\right) + o(|\xi_1|^3 + |\xi_2|^3), \tag{2.101}$$

where

$$\xi = \xi_1 + i\xi_2. \tag{2.102}$$

In polar coordinates, set

$$r = \sqrt{\xi_1^2 + \xi_2^2}, \theta = \arctan\frac{\xi_2}{\xi_1}. \tag{2.103}$$

We have, as $r \to 0$,

$$\dot{r} = \rho_1 r^3 + o(r^3),\ \dot{\theta} = q + \rho_2 r^2 + o(r^2). \tag{2.104}$$

Now it is clear to see from (2.104) that the origin $\mathbf{0} \in \mathbb{R}^2$ is asymptotically stable for (2.56) if $\rho_1 < 0$ and is not stable if $\rho_1 > 0$. From (2.23) – (2.25), (2.79)–(2.86) and Appendix A.1, we can obtain all the coefficients A_i, B_i, C_i, D_i $(i = 1, 2)$. Then, using Matlab, it follows that

$$\rho_1 := \frac{1}{8}\left(A_1 + C_1 + B_2 + 3D_2\right) = -0.014325 < 0. \tag{2.105}$$

A straightforward computation leads to the existence of $C > 0$ such that, at least if $r(0) \in [0, +\infty)$ is small enough, one has for the solution to (2.104),

$$r(t) \leq \frac{Cr(0)}{\sqrt{1 + tr(0)^2}}, \quad \forall t \in [0, +\infty). \tag{2.106}$$

Method 2: Lyapunov function. Let us start with a formal motivation. Recall that, by (2.9) and with E defined in (2.8), we have, along the trajectories of (2.7),

$$\dot{E} = -\frac{1}{2}K^2, \tag{2.107}$$

with

$$K := y_x(0). \tag{2.108}$$

It is therefore natural to consider the following candidate for a Lyapunov function

$$V := E - \mu K\dot{K}, \tag{2.109}$$

where $\mu > 0$ is small enough. Indeed, one then gets

$$\dot{V} := -\frac{1}{2}K^2 - \mu\left(\dot{K}\right)^2 - \mu K\ddot{K}, \tag{2.110}$$

and one may hope to absorb $-\mu K\ddot{K}$ with $-(1/2)K^2 - \mu\left(\dot{K}\right)^2$ and get $\dot{V} < 0$ on $G \setminus \{0\}$, at least in a neighborhood of 0.

We follow this strategy together with the approximation of g previously found. For $\mathbf{m} = (m_1, m_2) \in \Omega$, let (see (2.63))

$$\tilde{y} = m_1\varphi_1 + m_2\varphi_2 + m_1^2 a + m_1 m_2 b + m_2^2 c \in C^\infty([0, L]), \tag{2.111}$$

$$\tilde{E} := \frac{1}{2}\int_0^L \tilde{y}^2 \,\mathrm{d}x. \tag{2.112}$$

Then, using (2.26), (2.27) and (2.75) – (2.77) (compare with (2.61)), one gets that, along the trajectories of (2.56), for $\mathbf{m} \in \Omega$ and $\psi \in C^3([0, L])$ satisfying

$$\psi(0) = \psi(L) = 0, \tag{2.113}$$

one has

$$-\int_0^L (\psi' + \psi''')\tilde{y}\,\mathrm{d}x + \psi'(0)\left(m_1^2 a'(0) + m_1 m_2 b'(0) + m_2^2 c'(0)\right)$$
$$-\frac{1}{2}\int_0^L \psi_x \tilde{y}^2\,\mathrm{d}x + \int_0^L \left(\dot{m}_1\varphi_1 + \dot{m}_2\varphi_2 + \frac{\partial\tilde{g}}{\partial m_1}\dot{m}_1 + \frac{\partial\tilde{g}}{\partial m_2}\dot{m}_2\right)\psi\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^L (\tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} + \tilde{y}\tilde{y}_x)\psi \,\mathrm{d}x \\
&= \int_0^L \Big[m_1^3 (A_1\varphi_1 + A_2\varphi_2 - bc_1 + \varphi_1 a' + a\varphi_1') \\
&\quad + m_1^2 m_2 (B_1\varphi_1 + B_2\varphi_2 + 2ac_1 - b\sqrt{3}c_1 - 2cc_1 + \varphi_1 b' + \varphi_2 a' + a\varphi_2' + b\varphi_1') \\
&\quad + m_1 m_2^2 (C_1\varphi_1 + C_2\varphi_2 + 2a\sqrt{3}c_1 + bc_1 - 2c\sqrt{3}c_1 + \varphi_1 c' + \varphi_2 b' + b\varphi_2' + c\varphi_1') \\
&\quad + m_2^3 (D_1\varphi_1 + D_2\varphi_2 + b\sqrt{3}c_1 + \varphi_2 c' + c\varphi_2') + o(|\mathbf{m}|^3) \Big] \psi \,\mathrm{d}x \text{ as } |\mathbf{m}| \to 0.
\end{aligned}
\tag{2.114}
$$

Then, using (2.114) with $\psi := \tilde{y}$ (which, by (2.26), (2.27), (2.75), (2.76), (2.77) and (2.111), satisfies (2.113)), along the trajectories of (2.56), we have from (2.28), (2.29), (2.62) and (2.78)–(2.86) that the right hand side of (2.114) is $o(|\mathbf{m}|^4)$, and

$$
\dot{\tilde{E}} = -\frac{1}{2}\tilde{K}^2 + o(|\mathbf{m}|^4) \text{ as } |\mathbf{m}| \to 0, \tag{2.115}
$$

with $\tilde{K} : \Omega \to \mathbb{R}$ defined by

$$
\tilde{K} := a'(0)m_1^2 + b'(0)m_1 m_2 + c'(0)m_2^2. \tag{2.116}
$$

Let us emphasize that, even if "along the trajectories of (2.56)" might be misleading, $\dot{\tilde{E}}$ is just a function of $\mathbf{m} \in \Omega$. It is the same for $\dot{\tilde{V}}$, $\dot{\tilde{K}}$, $\ddot{\tilde{K}}$ which appear below. Using (2.43) and (2.55), we have, along the trajectories of (2.56),

$$
\dot{\tilde{K}} = qb'(0)m_1^2 + 2q(c'(0) - a'(0))m_1 m_2 - qb'(0)m_2^2 + o(|\mathbf{m}|^2). \tag{2.117}
$$

Using (2.55), we get the existence of $C > 0$ such that, along the trajectories of (2.56),

$$
\left|\ddot{\tilde{K}}\right| \leq C|\mathbf{m}|^2, \quad \forall \mathbf{m} \in \Omega. \tag{2.118}
$$

We can now define our Lyapunov function $\tilde{V}$. Let $\mu \in (0, 1/4]$. Let $\tilde{V} : \Omega \to \mathbb{R}$ be defined by

$$
\tilde{V} := \tilde{E} - \mu \tilde{K}\dot{\tilde{K}}. \tag{2.119}
$$

From (2.119), we have the existence of $\eta_0 > 0$ such that, for every $\mathbf{m} \in \mathbb{R}^2$ satisfying $|\mathbf{m}| < \eta_0$ and along the trajectories of (2.56),

$$
\dot{\tilde{V}} \;=\; -\frac{1}{2}\tilde{K}^2 - \mu\left(\dot{\tilde{K}}\right)^2 - \mu\tilde{K}\ddot{\tilde{K}} + o(|\mathbf{m}|^4)
$$

$$
\begin{aligned}
&\leq -\frac{1}{4}\tilde{K}^2 - \mu\left(\dot{\tilde{K}}\right)^2 + \mu^2\left(\ddot{\tilde{K}}\right)^2 + o(|\mathbf{m}|^4) \\
&\leq -\frac{1}{4}\tilde{K}^2 - \mu\left(\dot{\tilde{K}}\right)^2 + 2\mu^2C^2|\mathbf{m}|^4 \\
&\leq -\mu\left(\tilde{K}^2 + \left(\dot{\tilde{K}}\right)^2 - 2\mu C^2|\mathbf{m}|^4\right). \qquad (2.120)
\end{aligned}
$$

Let us assume for the moment that, for every $\mathbf{m} = (m_1, m_2) \in \mathbb{R}^2$,

$$
\left\{\begin{array}{l} a'(0)m_1^2 + b'(0)m_1m_2 + c'(0)m_2^2 = 0, \\ qb'(0)m_1^2 + 2q(c'(0) - a'(0))m_1m_2 - qb'(0)m_2^2 = 0, \end{array}\right. \Rightarrow \mathbf{m} = \mathbf{0}. \qquad (2.121)
$$

Then, by homogeneity, there exists $\eta_1 > 0$ such that

$$
\begin{aligned}
&\left(a'(0)m_1^2 + b'(0)m_1m_2 + c'(0)m_2^2\right)^2 \\
&+ \left(qb'(0)m_1^2 + 2q(c'(0) - a'(0))m_1m_2 - qb'(0)m_2^2\right)^2 \\
&\geq 2\eta_1|\mathbf{m}|^4, \ \forall \mathbf{m} = (m_1, m_2) \in \mathbb{R}^2. \qquad (2.122)
\end{aligned}
$$

From (2.116), (2.117) and (2.122), we get the existence of $\eta_2 > 0$ satisfying

$$
\tilde{K}^2 + \left(\dot{\tilde{K}}\right)^2 \geq \eta_1|\mathbf{m}|^4, \quad \forall \mathbf{m} \in \mathbb{R}^2 \text{ such that } |\mathbf{m}| < \eta_2. \qquad (2.123)
$$

From (2.120) and (2.123), we get the existence of $\eta_3 > 0$ such that, for every $\mu \in (0, \eta_3)$,

$$
\dot{\tilde{V}} \leq -\frac{\mu}{2}\eta_1|\mathbf{m}|^4, \quad \forall \mathbf{m} \in \mathbb{R}^2 \text{ such that } |\mathbf{m}| < \eta_3. \qquad (2.124)
$$

Moreover, straightforward estimates show that there exists $\eta_4 > 0$ such that, for every $\mu \in (0, \eta_4)$,

$$
\eta_4|\mathbf{m}|^2 \leq \tilde{V} \leq \frac{1}{\eta_4}|\mathbf{m}|^2, \quad \forall \mathbf{m} \in \mathbb{R}^2 \text{ such that } |\mathbf{m}| < \eta_4, \qquad (2.125)
$$

which, together with (2.124), proves the existence of $C > 0$ such that, at least if $\mathbf{m}^0 \in \mathbb{R}^2$ is small enough, the solution to (2.56) satisfies

$$
|\mathbf{m}(t)| \leq \frac{C|\mathbf{m}^0|}{\sqrt{1 + t|\mathbf{m}^0|^2}}, \quad \forall t \geq 0. \qquad (2.126)
$$

It only remains to prove (2.121). From Appendix A.1, one gets that $c'(0) \approx 0.0118 \neq 0$, then (2.121) holds if $m_1 = 0$. Let us now deal with the case $m_1 \neq 0$. If we divide both polynomials on the two equations on the left hand side of (2.121) by m_1^2, then the two resulting polynomials have a common root if and only if their resultant is zero. This resultant is the determinant of the Sylvester matrix S:

$$S := \begin{pmatrix} c'(0) & b'(0) & a'(0) & 0 \\ 0 & c'(0) & b'(0) & a'(0) \\ -b'(0) & -2(a'(0)-c'(0)) & b'(0) & 0 \\ 0 & -b'(0) & -2(a'(0)-c'(0)) & b'(0) \end{pmatrix}. \tag{2.127}$$

Straightforward computations show that

$$\begin{aligned} \det(S) =& a'(0)^3[b'(0)+4c'(0)] + a'(0)^2[-2b'(0)^2 + b'(0)c'(0) - 8c'(0)^2] \\ &+ a'(0)[5b'(0)^2c'(0) + 4c'(0)^3] - b'(0)^2c'(0)^2 - b'(0)^4. \end{aligned} \tag{2.128}$$

From (2.128) and Appendix A.1 (see in particular (A.42) – (A.44)), we have

$$\det(S) \approx -0.0197 \neq 0. \tag{2.129}$$

Hence, the two resulting polynomials do not have a common root. Thus, (2.121) is proved.

2.2.4 Main result

Thus, the following theorem concludes the results in this section.

Theorem 2.2. *Consider the KdV equation* (2.7) *with* $L = 2\pi\sqrt{7/3}$. *There exist* $\delta \in (0, +\infty)$, $K > 0$, $\omega > 0$ *and a map* $g : M \to M^\perp$, *where* $M^\perp \subset L^2(0, L)$ *is the orthogonal of* M *for the* L^2*-scalar product, satisfying*

$$g \in C^3(M; M^\perp), \tag{2.130}$$

$$g(0) = 0, \; g'(0) = 0, \tag{2.131}$$

such that, with

$$G := \{m + g\,(m)\,;\; m \in M\} \subset L^2(0, L), \tag{2.132}$$

the following three properties hold for every solution y *to* (2.7) *with* $\|y_0\|_{L^2(0,L)} < \delta$,

1. *(Local exponential attractivity of G.)*

$$d(y(t,\cdot),G) \leq Ke^{-\omega t} d(y_0,G), \ \forall t > 0, \tag{2.133}$$

where $d(\chi, G)$ *denotes the distance between* $\chi \in L^2(0,L)$ *and* G*:*

$$d(\chi, G) := \inf\{\|\chi - \psi\|_{L^2(0,L)}; \ \psi \in G\}. \tag{2.134}$$

2. *(Local invariance of G.)*

$$\text{If } y_0 \in G, \text{ then } y(t,\cdot) \in G, \ \forall t \geq 0. \tag{2.135}$$

3. *If* y_0 *is in* G*, then there exists* $C > 0$ *such that*

$$\|y(t,\cdot)\|_{L^2(0,L)} \leq \frac{C\|y_0\|_{L^2(0,L)}}{\sqrt{1 + t\|y_0\|^2_{L^2(0,L)}}}, \quad \forall t \geq 0. \tag{2.136}$$

In particular, $0 \in L^2(0,L)$ *is locally asymptotically stable in the sense of the* $L^2(0,L)$*-norm for* (2.7).

Remark 2.3. It follows from our derivation of Theorem 2.2 that the decay rate stated in (2.136) is optimal in the following sense: there exists $\varepsilon > 0$ such that, for every $y_0 \in G$ with $\|y_0\|_{L^2(0,L)} \leq \varepsilon$,

$$\|y(t,\cdot)\|_{L^2(0,L)} \geq \frac{\varepsilon\|y_0\|_{L^2(0,L)}}{\sqrt{1 + t\|y_0\|^2_{L^2(0,L)}}}. \tag{2.137}$$

$\Big($For the Lyapunov approach, let us point out that, decreasing if necessary $\eta_3 > 0$, one has, for every $\mu \in (0, \eta_3)$,

$$\dot{\tilde{V}} \geq -\frac{1}{\eta_3}|\mathbf{m}|^4, \quad \forall \mathbf{m} \in \mathbb{R}^2 \text{ such that } |\mathbf{m}| < \eta_3.\Big) \tag{2.138}$$

Remark 2.4. It can be derived from [48, Theorem 1 and Comments] that, for every $L > 0$, there are non-zero stationary solutions with the period of L to the following ordinary differential equation (ODE):

$$\begin{cases} f' + ff' + f''' = 0 \text{ in } [0,L], \\ f(0) = f(L) = 0, \\ f'(L) = 0. \end{cases} \tag{2.139}$$

That is, besides the origin, there also exist other steady states of the nonlinear KdV equation (2.7). Therefore, $0 \in L^2(0,L)$ is not globally asymptotically stable for (2.7): Property $(\mathcal{P}_2)$ does not hold for arbitrary $\varepsilon_0 > 0$.

Let us recall that we need to check whether the local attractivity property $(\mathcal{P}_2)$ holds. Let $y_0 \in L^2(0,L)$ be such that $\|y_0\|_{L^2(0,L)} < \delta$ and let y be the solution to (2.7). It suffices to check that

$$y(t,\cdot) \to 0 \text{ in } L^2(0,L) \text{ as } t \to +\infty. \tag{2.140}$$

By (2.40), (2.45) and the fact that M is of finite dimension, there exists an increasing sequence of positive real numbers $(t_n)_{n\in\mathbb{N}}$ and $z_0 \in L^2(0,L)$ such that

$$t_n \to +\infty \text{ as } n \to +\infty, \tag{2.141}$$

$$y(t_n,\cdot) \to z_0 \text{ in } L^2(0,L) \text{ as } n \to +\infty, \tag{2.142}$$

$$z_0 \in G \text{ and } \|z_0\|_{L^2(0,L)} < \delta. \tag{2.143}$$

Let $z : [0,+\infty) \times (0,L) \to \mathbb{R}$ be the solution to (2.7) satisfying the initial condition $z(0,\cdot) = z_0$. It follows from (2.136) and (2.143) that

$$z(t,\cdot) \to 0 \text{ in } L^2(0,L) \text{ as } t \to +\infty. \tag{2.144}$$

Let $\eta > 0$. By (2.144), there exists $\tau > 0$ such that

$$\|z(\tau,\cdot)\|_{L^2(0,L)} \le \frac{\eta}{2}. \tag{2.145}$$

By Proposition 3 and (2.142),

$$y(t_n+\tau,\cdot) \to z(\tau,\cdot) \text{ in } L^2(0,L) \text{ as } n \to +\infty. \tag{2.146}$$

By (2.145) and (2.146), there exists $n_0 \in \mathbb{N}$ such that

$$\|y(t_{n_0}+\tau,\cdot)\|_{L^2(0,L)} < \eta, \tag{2.147}$$

which, together with (2.40), implies that

$$\|y(t,\cdot)\|_{L^2(0,L)} < \eta, \quad \forall t \ge t_{n_0}+\tau, \tag{2.148}$$

which concludes the proof of (2.140).

2.3 Stability analysis of the initial-boundary-value LKdVAD problem

When anti-diffusion exists in systems such as Kuramoto-Sivashinsky equation [49, 50], Ginzburg-Landau equation [51], it can make significant influence on the system stability. Moreover, the effect of anti-diffusion term in some KdV-type equations is dicussed in [52].

In this section, we give a brief discussion about stability analysis of the initial-boundary-value LKdVAD problem introduced in Subsection 2.3.1. A preliminary spectrum analysis of the LKdVAD equations is presented in Subsection 2.3.2. Then, Subsection 2.3.3 discusses the exponential stability of a class of LKdVAD equations. Moreover, Appendix A.2 shows that the derived decay rates in Subsection 2.3.3 are not necessarily optimal.

2.3.1 Problem statement

Consider the stability property of the following initial-boundary-value problem of the LKdVAD equations:

$$u_t(t,x) = u_{xxx}(t,x) + \lambda_2 u_{xx}(t,x) + \lambda_1 u_x(t,x) + \lambda_0 u(t,x), x \in (0,L), \tag{2.149}$$

$$u_x(t,0) = q_1 u(t,0),\ u_{xx}(t,0) = q_2 u(t,0),\ u(t,L) = 0, \tag{2.150}$$

where $u(t,x) \in \mathbb{R}$ is the system state; $\lambda_2, \lambda_1, \lambda_0, q_1, q_2$ are known constants which can take any values except infinity. When $\lambda_2 < 0$, the second-order term acts as an anti-diffusion or backwards heat operator, at least locally [52], and may cause destabilizing effects.

2.3.2 Spectrum analysis of the LKdVAD equations

Consider the energy state space $\mathcal{H} = L^2(0,L)$. Define the system operator $\mathbf{A} : \mathrm{Dom}(\mathcal{A})(\subset \mathcal{H}) \to \mathcal{H}$ as follows:

$$\mathbf{A}f = f''' + \lambda_2 f'' + \lambda_1 f' + \lambda_0 f, \forall f \in \mathrm{Dom}(\mathbf{A}), \tag{2.151}$$

$$\mathrm{Dom}(\mathbf{A}) = \left\{ f \in H^3(0,L) \mid f'(0) = q_1 f(0),\ f''(0) = q_2 f(0), f(L) = 0 \right\}, \tag{2.152}$$

then the system (2.149)–(2.150) can be written as an evolution equation in $\mathcal{H}$:

$$\frac{\mathrm{d}u(t,\cdot)}{\mathrm{d}t} = u(t,\cdot). \tag{2.153}$$

Lemma 2.3. *If* $\begin{pmatrix} 1 & q_1 & q_2 \end{pmatrix} e^{DL} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ *is nonzero, where*

$$D = \begin{pmatrix} 0 & 0 & -\lambda_0 \\ 1 & 0 & -\lambda_1 \\ 0 & 1 & -\lambda_2 \end{pmatrix}, \tag{2.154}$$

then $\mathbf{A}^{-1}$ *exists and is compact on* $\mathcal{H}$. *Hence,* $\sigma(\mathbf{A})$*, the spectrum of* $\mathbf{A}$*, consists of isolated eigenvalues only:* $\sigma(\mathbf{A}) = \sigma_p(\mathbf{A})$*, where* $\sigma_p(\mathbf{A})$ *denotes the set of eigenvalues of* $\mathbf{A}$*. Moreover, each* $\lambda \in \sigma(\mathbf{A})$ *is geometrically simple and satisfies the characteristic equation*

$$\begin{aligned} 0 =& e^{\sigma_1 L}(\sigma_2 - \sigma_3)\left(\sigma_2\sigma_3 - q_1(\sigma_2+\sigma_3) + q_2\right) \\ &+ e^{\sigma_2 L}(\sigma_3 - \sigma_1)\left(\sigma_3\sigma_1 - q_1(\sigma_3+\sigma_1) + q_2\right) \\ &+ e^{\sigma_3 L}(\sigma_1 - \sigma_2)\left(\sigma_1\sigma_2 - q_1(\sigma_1+\sigma_2) + q_2\right), \end{aligned} \tag{2.155}$$

where

$$\sigma_1 = -\frac{\lambda_2}{3} + \alpha + \beta,\ \sigma_2 = -\frac{\lambda_2}{3} + \omega\alpha + \omega^2\beta,\ \sigma_3 = -\frac{\lambda_2}{3} + \omega^2\alpha + \omega\beta, \tag{2.156}$$

and

$$\alpha = \sqrt[3]{\tau_1 + \sqrt{\tau_1^2 + \tau_2^3}},\ \beta = \sqrt[3]{\tau_1 - \sqrt{\tau_1^2 + \tau_2^3}},\ \omega = e^{2/3\pi i}, \tag{2.157}$$

$$\tau_1 = \frac{\lambda_1\lambda_2}{6} - \frac{\lambda_2^3}{27} - \frac{\lambda_0 - \lambda}{2},\ \tau_2 = \frac{\lambda_1}{3} - \frac{\lambda_2^2}{9}. \tag{2.158}$$

An eigenfunction f *corresponding to* λ *is*

$$\begin{aligned} f(x) =& (\sigma_2 - \sigma_3)\left(\sigma_2\sigma_3 - q_1(\sigma_2+\sigma_3) + q_2\right) e^{\sigma_1 x} \\ &+ (\sigma_3 - \sigma_1)\left(\sigma_3\sigma_1 - q_1(\sigma_3+\sigma_1) + q_2\right) e^{\sigma_2 x} \\ &+ (\sigma_1 - \sigma_2)\left(\sigma_1\sigma_2 - q_1(\sigma_1+\sigma_2) + q_2\right) e^{\sigma_3 x}. \end{aligned} \tag{2.159}$$

Proof. (Part 1) Solve

$$\mathbf{A}f_1 = f,\ \ f_1 \in \mathrm{Dom}(\mathbf{A}), \tag{2.160}$$

then we get

$$\mathbf{A}^{-1}f = f_1,\ \forall f \in \mathcal{H}, \tag{2.161}$$

$$f_1(x) = f_1(0)\begin{pmatrix} 1 & q_1 & q_2 \end{pmatrix} e^{Dx}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \int_0^x f(\tau)\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} e^{D(x-\tau)}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathrm{d}\tau, \tag{2.162}$$

where

$$f_1(0) = -\int_0^L f(\tau)\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} e^{D(L-\tau)}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathrm{d}\tau \left(\begin{pmatrix} 1 & r_1 & r_2 \end{pmatrix} e^{DL}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)^{-1}. \tag{2.163}$$

Hence we get the unique $f_1 \in \mathrm{Dom}(\mathcal{A})$ and thus $\mathbf{A}^{-1}$ exists and is compact on $\mathbf{H}$ by the Sobolev embedding theorem. Therefore, $\sigma(\mathbf{A})$, the spectrum of $\mathbf{A}$, consists of isolated eigenvalues only.

(Part 2) For any $\lambda \in \sigma_p(\mathbf{A})$, we have

$$\mathbf{A}f = f''' + \lambda_2 f'' + \lambda_1 f' + \lambda_0 f = \lambda f, \tag{2.164}$$

$$f'(0) = q_1 f(0), f''(0) = q_2 f(0), f(L) = 0, \tag{2.165}$$

which has at least one nonzero solution. If it has two linearly independent solutions f_1, f_2, then there exists constants a, b $(a^2 + b^2 \neq 0)$ such that

$$af_1(0) + bf_2(0) = 0. \tag{2.166}$$

Thus,

$$f = af_1 + bf_2 \tag{2.167}$$

satisfies

$$\mathbf{A}f = f''' + \lambda_2 f'' + \lambda_1 f' + \lambda_0 f = \lambda f, \tag{2.168}$$

$$f(0) = f'(0) = f''(0) = f(L) = 0, \tag{2.169}$$

which has only zero solution. Hence, $af_1 + bf_2 \equiv 0$, which contradicts with the assumption. Therefore, each $\lambda \in \sigma_p(\mathbf{A})$ is geometrically simple.

(Part 3) For any $\lambda \in \sigma_p(\mathbf{A})$, from (2.164), we have

$$f(x) = c_1 e^{\sigma_1 x} + c_2 e^{\sigma_2 x} + c_3 e^{\sigma_3 x} \ (c_1^2 + c_2^2 + c_3^2 \neq 0). \tag{2.170}$$

From (2.165), we get

$$\begin{vmatrix} \sigma_1 - q_1 & \sigma_2 - q_1 & \sigma_3 - q_1 \\ \sigma_1^2 - q_2 & \sigma_2^2 - q_2 & \sigma_3^2 - q_2 \\ e^{\sigma_1 L} & e^{\sigma_2 L} & e^{\sigma_3 L} \end{vmatrix} = 0 \tag{2.171}$$

and the characteristic equation is (2.155). We can also derive the corresponding eigenfunction (2.159). □

Remark 2.5. Assume that all the other parameters $\lambda_2, \lambda_1, q_1, q_2$ are fixed, then by letting $\tilde{\lambda} = \lambda - \lambda_0$, it can be observed from (2.155)–(2.158) that the RHS of characteristic equation (2.155) is a function depending only on variable λ_0. After solving this equation, the eigenvalues are calculated from $\lambda = \lambda_0 + \tilde{\lambda}$, which means that the locations of all the eigenvalues shift uniformly with choice of λ_0.

2.3.3 Exponential stability of a class of LKdVAD equations

Assume that the parameters λ_0 and q_1 can take arbitrary values, and that the parameters λ_2 and q_2 satisfy

$$\lambda_2 \geqslant 0 \text{ and } \lambda_2 > 4L^2\lambda_0, \ q_2 \geqslant \frac{1}{2}(q_1^2 - \lambda_1 - 2\lambda_2 q_1), \tag{2.172}$$

then the following result can be proved for the equation (2.149)–(2.150).

Lemma 2.4. $\mathbf{A}$ *is dissipative in* $\mathcal{H}$, *and* $\mathbf{A}$ *generates a* C_0*-semigroup* $e^{\mathbf{A}t}$ *of contractions in* $\mathcal{H}$.

Proof. Let $f \in \mathrm{Dom}(\mathbf{A})$, then

$$< \mathbf{A}f, f >= -(\lambda_2 q_1 + q_2)|f(0)|^2 - \int_0^L f''\bar{f}' \,\mathrm{d}x - \lambda_2 \|f'\|^2$$

$$+ \lambda_1 \int_0^L f' \bar{f} \, \mathrm{d}x + \lambda_0 \|f\|^2, \tag{2.173}$$

and

$$Re < \mathbf{A}f, f >= - \left(\frac{\lambda_1}{2} + \lambda_2 q_1 - \frac{q_1^2}{2} + q_2 \right) |f(0)|^2 - \frac{1}{2} |f'(L)|^2 - \lambda_2 \|f'\|^2 + \lambda_0 \|f\|^2$$
$$\leq - \rho_u \|f\|^2 < 0, \tag{2.174}$$

where

$$\rho_u = \frac{1}{4L^2} \lambda_2 - \lambda_0 > 0. \tag{2.175}$$

Hence, $\mathbf{A}$ is dissipative in $\mathcal{H}$, and $\mathbf{A}$ generates a C_0-semigroup $e^{\mathbf{A}t}$ of contractions in $\mathcal{H}$ by the Lumer-Philips theorem [53]. □

Indeed, the system (2.149)–(2.150) is exponentially stable.

Lemma 2.5. *For each $\lambda \in \sigma(\mathbf{A})$, $Re\lambda < 0$. $\mathbf{A}$ generates an exponentially stable C_0-semigroup on $\mathcal{H}$. For any initial value $u(0, \cdot) \in \mathcal{H}$, there exists a unique (mild) solution to* (2.149)–(2.150) *with* (2.172)*, such that*

$$u(t, \cdot) \in C\left([0, \infty); \mathcal{H}\right), \tag{2.176}$$

and

$$\|u(t, \cdot)\| \leq e^{-\rho_u t} \|u(0, \cdot)\|, \tag{2.177}$$

where ρ_u is defined in (2.175)*. Moreover, if $u(0, \cdot) \in \mathrm{Dom}(\mathbf{A})$, then*

$$u(t, \cdot) \in C^1\left([0, \infty); \mathcal{H}\right) \tag{2.178}$$

is the classical solution to (2.149)–(2.150) *.*

Proof. From the proof of Lemma 2.4, we have

$$Re < \mathbf{A}f, f > \leq -\rho_u \|f\|^2, \ \forall f \in \mathrm{Dom}(\mathbf{A}). \tag{2.179}$$

Define a Lyapunov function

$$V(t) = \frac{1}{2} \|u(t, \cdot)\|^2, \tag{2.180}$$

then we can get

$$\dot{V}(t) \leq -2\rho_u V(t) \Rightarrow V(t) \leq V(0)e^{-2\rho_u t}, \tag{2.181}$$

and thus, (2.177) is obtained. Because $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$, this semigroup must be exponentially stable. □

Remark 2.6. As can be inferred from Appendix A.2, exponential stability still holds in some special cases if the strict inequality $\lambda_2 > 4L^2\lambda_0$ in (2.172) is relaxed to a nonstrict one. Another important fact, as can be also seen from Appendix A.2, is that the exponential decay rate ρ_u is not necessarily equal to the optimal decay rate.

2.4 Notes and references

In this chapter, we have proved that for the critical case of $L = 2\pi\sqrt{7/3}$, $0 \in L^2(0, L)$ is locally asymptotically stable for the (nonlinear) KdV equation (2.7). First, we recalled that the equation has a two-dimensional local center manifold. Next, through a second-order power series approximation at $0 \in M$ of the function g defining the local center manifold, we derived the local asymptotic stability of $0 \in L^2(0, L)$ on the local center manifold and obtained a polynomial decay rate for the solution to the KdV equation (2.7) on the center manifold. Since the KdV equation (2.7) also has other (periodic) steady states than the origin (see Remark 2.4), it remains an open and interesting problem to consider the (local) stability property of these steady states for the KdV equation (2.7). Furthermore, it remains to consider all the other critical cases with a two-dimensional (local) center manifold as well as all the last remaining critical cases, i.e., when the equation has a (local) center manifold with a dimension larger than 2.

We have also conducted a preliminary stability analysis for the LKdVAD problem, which can serve as a guidance for the controller design and other related control problems.

Chapter 2 contains reprints or adaptions of the following papers: 1. S.-X. Tang, J.-X. Chu, P.-P. Shang and J.-M. Coron, "Local asymptotic stability of a

KdV equation with a two-dimensional center manifold", *Advances in Nonlinear Analysis*, to appear. 2. S.-X. Tang and M. Krstic, "Stabilization for linearized Kortweg-de Vries systems with anti-diffusion", in *Proceedings of the American Control Conference*, pp. 3302-3307, Washington, D.C., USA, June 17-19, 2013. 3. S.-X. Tang and M. Krstic, "Stabilization of an anti-diffusive linear Korteweg-de Vries PDE," *IEEE Transactions on Automatic Control*, to be submitted. The dissertation author is the primary investigator and author of these papers, and would like to thank Jixun Chu, Jean-Michel Coron, Miroslav Krstic and Peipei Shang for their contributions.

Chapter 3

Backstepping Control of the Korteweg-de Vries Equations

3.1 Introduction

3.1.1 Problem statement

This chapter addresses the problem of stabilizing the following class of LKdVAD posed on a finite interval $[0, L]$:

$$u_t(t,x) = u_{xxx}(t,x) + \lambda_2 u_{xx}(t,x) + \lambda_1 u_x(t,x) + \lambda_0 u(t,x), x \in (0,L), \tag{3.1}$$

$$u_x(t,0) = q_1 u(t,0),\ u_{xx}(t,0) = q_2 u(t,0),\ u(t,L) = U(t), \tag{3.2}$$

where $u(t,x) \in \mathbb{R}$ is the system state; the external forcing term $U(t)$ is the to-be-designed control input acting on the right Dirichlet boundary condition. At the left end, a general case of boundary conditions is considered: a Robin boundary condition and a Navier-like boundary condition. $\lambda_2, \lambda_1, \lambda_0, q_1, q_2$ are known constants which can take any values except infinity. λ_2 is allowed to be negative.

The control objective is to exponentially stabilize the system state to zero in energy state space $\mathcal{H} = L^2(0,L)$, or more specifically, to force the solutions of these systems to achieve exponential decay by choosing the control input $U(t)$.

Remark 3.1. The critical cases ($q_1 = \infty$ or $q_2 = \infty$) of boundary conditions are considered in Appendix B.2.

3.1.2 Literature review

Research on the control problems of KdV systems has drawn a large amount of effort during the past decades. There are mainly three cases in the study of KdV systems from control point of view, based on the domain/interval of interest. The first class is posed on a periodic domain with internal control added it to the equation [54, 55]; the second class is posed on the half real line $\mathbb{R}^+$ [56, 57]; and the third one is posed on a bounded interval with boundary controller(s) (see, e.g., [38, 58]). For KdV sytem control and stabilization, [59] covers a detailed review of some recent results and open problems, and it could serve as a good reference.

For the KdV systems on a bounded interval such as $(0, L)$, boundary control properties vary greatly depending on the boundary conditions, see, [59]. A KdV system with two homogeneous left-end boundary conditions (one Dirichlet, the other Neumann) and one right-end Dirichlet boundary control input is considered in [60]. The authors firstly stabilize the corresponding linearized KdV system with state feedback control, and then extend the results to the nonlinear case. A locally stabilizing result is obtained as a result. KdV system. Based on this, they derived a local stabilizing result for the nonlinear system.

3.1.3 Organization

In this section, we employ the backstepping method to design feedback controllers for the LKdVAD. The remainder of this section is organized as follows. The state feedback stabilizing problem is discussed in Section 3.2. Section 3.3 consists of two parts. In Subsection 3.3.1, a Luenberger-type observer is designed, and it is then used in Subsection 3.3.2 to help construct a stabilizing output feedback controller. Both the state and output feedback closed-loop control systems can achieve arbitrarily prescribed exponential decay rates. Some concluding remarks and possible future work are given in Section 3.4. In addition, an example is presented in Appendix B.1 to illustrate effectiveness of the designed state feedback controllers. Furthermore, two critical cases regarding the left-end boundary conditions of the LKdVAD are discussed and stabilized in Appendix B.2.

3.2 State feedback stabilization

Before proceeding, we summarize the state feedback control design result for the system (3.1)–(3.2).

3.2.1 Main result

We propose the state feedback boundary controller as

$$U(t) = \int_0^L \kappa(L,y)u(y,t)e^{\frac{\lambda_2-\mu_2}{3}(y-L)}dy, \tag{3.3}$$

where the function $\kappa(x,y) \in \mathbb{R}$ satisfies the following PDE:

$$\begin{aligned}
&\kappa_{xxx}(x,y) + \kappa_{yyy}(x,y) + \mu_2\left(\kappa_{xx}(x,y) - \kappa_{yy}(x,y)\right) \\
&\quad + \left(\frac{\mu_2^2-\lambda_2^2}{3} + \lambda_1\right)\left(\kappa_x(x,y) + \kappa_y(x,y)\right) \\
&\quad - \left[\frac{\lambda_2-\mu_2}{3}\left(\frac{(\lambda_2-\mu_2)(2\lambda_2+\mu_2)}{9} - \lambda_1\right) + \lambda_0 - \mu_0\right]\kappa(x,y) = 0,
\end{aligned} \tag{3.4}$$

which is defined on the triangle $\mathcal{T} = \{(x,y)|0 < y < x < 1\}$ and satisfies the boundary conditions as follows:

$$\kappa(x,x) = \frac{\lambda_2-\mu_2}{3} + q_1 - r_1, \tag{3.5}$$

$$\begin{aligned}
\kappa_x(x,x) = &\left(\frac{\mu_0-\lambda_0}{3} + \lambda_1\frac{\lambda_2-\mu_2}{9} - \frac{2\lambda_2+\mu_2}{9}\left(\frac{\lambda_2-\mu_2}{3}\right)^2\right)x \\
&+ r_1\frac{\lambda_2-\mu_2}{3} - q_1(q_1-r_1) + q_2 - r_2,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\kappa_{yy}(x,0) - \left(\frac{\lambda_2+2\mu_2}{3} + q_1\right)\kappa_y(x,0) \\
&\quad + \left(\lambda_1 + \frac{2\lambda_2+\mu_2}{3}\left(\frac{\mu_2-\lambda_2}{3} + q_1\right) + q_2\right)\kappa(x,0) = 0.
\end{aligned} \tag{3.7}$$

The parameters μ_0 and r_1 are arbitrary, and the parameters μ_2 and r_2 satisfy

$$\mu_2 \geqslant 0 \text{ and } \mu_2 > 4L^2\mu_0, \tag{3.8}$$

$$r_2 > \frac{1}{2}(2r_1^2 + \frac{\lambda_2^2-\mu_2^2}{3} - \lambda_1 - 2\mu_2 r_1) \triangleq r_2^*. \tag{3.9}$$

With this controller, exponential stability with an arbitrarily prescribed decay rate holds for the resulting closed-loop control system, which is stated in the following theorem.

Theorem 3.1. *For any initial value $u(0,\cdot) \in \mathcal{H}$, the closed-loop system* (3.1)–(3.2) *with the controller* (3.3) *determined by* (3.4)–(3.9) *has a unique (mild) solution*

$$u(t,\cdot) \in C\left([0,\infty);\mathcal{H}\right), \tag{3.10}$$

and there exists a positive constant M_u such that

$$\|u(t,\cdot)\|_{\mathcal{H}} \leq M_u e^{-\rho_u t}\|u(0,\cdot)\|_{\mathcal{H}},\ \ \rho_u = \frac{1}{4L^2}\mu_2 - \mu_0 > 0. \tag{3.11}$$

Moreover, if $u(\cdot,0)$ satisfies a boundary compatibility condition, then

$$u(t,\cdot) \in C^1\left([0,\infty);\mathcal{H}\right) \tag{3.12}$$

is the classical solution.

In the sequel, we denote the coefficient of first order partial derivative terms in the gain PDE (3.4) as

$$\frac{\mu_2^2 - \lambda_2^2}{3} + \lambda_1 \triangleq \mu_1 \tag{3.13}$$

for simplicity, and we drop the subscripts from the norms and inner products when they are clear from the context.

Remark 3.2.

1. For state feedback stabilization only, the choice of r_2 can be relaxed to the following non-strict inequality

$$r_2 \geqslant \frac{1}{2}(r_1^2 - \mu_1 - 2\mu_2 r_1) = r_2^* - \frac{r_1^2}{2}. \tag{3.14}$$

Moreover, even this relaxed choice of control parameters (3.8) and (3.14) is a sufficient but not necessary condition to achieve exponential stability of the closed-loop control system.

2. The exponential decay rate ρ_u can be arbitrarily large by choosing μ_2 large enough.

3.2.2 Proofs

We first prove existence, uniqueness and regularity of the function $\kappa(x,y)$. Second, by employing a PDE backstepping transformation, the closed-loop u-system with the designed controller is converted into an exponential stable target system. Third, we derive the existence, uniqueness and regularity of the inverse transformation. Last, we prove well-posedness and exponential stability of the closed-loop control u-system, which completes the proof of Theorem 3.1.

A. Existence, uniqueness and regularity of the function $\kappa(x,y)$

We analyze next the existence, uniqueness and regularity of solution to the $\kappa(x,y)$-system (3.4)–(3.7).

Let

$$\kappa(x,y) = p(x,y)e^{c(x-y)}, \tag{3.15}$$

where

$$c = -\frac{1}{2}\left(\frac{\lambda_2+2\mu_2}{3}+q_1\right), \tag{3.16}$$

then the system (3.4)–(3.7) is equivalent to the following $p(x,y)$-system:

$$\begin{aligned}
&p_{xxx}(x,y)+p_{yyy}(x,y)-\frac{1}{2}(\lambda_2+3q_1)\left(p_{xx}(x,y)-p_{yy}(x,y)\right)\\
&+\left(\mu_1+\frac{1}{4}\left(\frac{\lambda_2+2\mu_2}{3}+q_1\right)(\lambda_2-2\mu_2+3q_1)\right)\left(p_x(x,y)+p_y(x,y)\right)\\
&-\left[\frac{\lambda_2-\mu_2}{3}\left(\frac{(\lambda_2-\mu_2)(2\lambda_2+\mu_2)}{9}-\lambda_1\right)+\lambda_0-\mu_0\right]p(x,y)=0,
\end{aligned} \tag{3.17}$$

$$p(x,x) = \frac{\lambda_2-\mu_2}{3}+q_1-r_1, \tag{3.18}$$

$$\begin{aligned}
p_x(x,x) =& \left(\frac{\mu_0-\lambda_0}{3}+\lambda_1\frac{\lambda_2-\mu_2}{9}-\frac{2\lambda_2+\mu_2}{9}\left(\frac{\lambda_2-\mu_2}{3}\right)^2\right)x\\
&+\left(\frac{\lambda_2+2\mu_2}{6}+\frac{1}{2}q_1+r_1\right)\frac{\lambda_2-\mu_2}{3}+\left(\frac{\lambda_2+2\mu_2}{6}-\frac{1}{2}q_1\right)(q_1-r_1)+q_2-r_2,
\end{aligned} \tag{3.19}$$

$$p_{yy}(x,0)+\left[\lambda_1-\frac{1}{4}\left(\frac{\lambda_2+2\mu_2}{3}+q_1\right)^2\right.$$

$$+\frac{2\lambda_2+\mu_2}{3}\left(\frac{\mu_2-\lambda_2}{3}+q_1\right)+q_2\Bigg]p(x,0)=0. \tag{3.20}$$

Introduce a change of variables as follows

$$\xi = x+y,\ \ \eta = x-y, \tag{3.21}$$

and denote

$$G(\xi,\eta)=p(x,y), \tag{3.22}$$

then the problem (3.17)–(3.20) is transformed into the following IPDE:

$$\begin{aligned}
&2G_{\xi\xi\xi}(\xi,\eta)+6G_{\xi\eta\eta}(\xi,\eta)-2(\lambda_2+3q_1)G_{\xi\eta}(\xi,\eta)\\
&+2\left(\mu_1+\frac{1}{4}\left(\frac{\lambda_2+2\mu_2}{3}+q_1\right)(\lambda_2-2\mu_2+3q_1)\right)G_\xi(\xi,\eta)\\
&-\left[\frac{\lambda_2-\mu_2}{3}\left(\frac{(\lambda_2-\mu_2)(2\lambda_2+\mu_2)}{9}-\lambda_1\right)+\lambda_0-\mu_0\right]G(\xi,\eta)=0
\end{aligned} \tag{3.23}$$

for $(\xi,\eta)\in\mathcal{T}_1=\{(\xi,\eta)|0<\xi<2, 0<\eta<\min(\xi,2-\xi)\}$ with boundary conditions

$$G(\xi,0)=\frac{\lambda_2-\mu_2}{3}+q_1-r_1, \tag{3.24}$$

$$\begin{aligned}
G_\eta(\xi,0)=&\ \frac{1}{2}\left[\frac{\mu_0-\lambda_0}{3}+\lambda_1\frac{\lambda_2-\mu_2}{9}-\frac{2\lambda_2+\mu_2}{9}\left(\frac{\lambda_2-\mu_2}{3}\right)^2\right]\xi\\
&+\left(\frac{\lambda_2+2\mu_2}{6}+\frac{1}{2}q_1+r_1\right)\frac{\lambda_2-\mu_2}{3}\\
&+\left(\frac{\lambda_2+2\mu_2}{6}-\frac{1}{2}q_1\right)(q_1-r_1)+q_2-r_2,
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
G_{\xi\xi}(\xi,\xi)-2G_{\xi\eta}(\xi,\xi)+G_{\eta\eta}(\xi,\xi)+\Bigg[\lambda_1-\frac{1}{4}\left(\frac{\lambda_2+2\mu_2}{3}+q_1\right)^2\\
+\frac{2\lambda_2+\mu_2}{3}\left(\frac{\mu_2-\lambda_2}{3}+q_1\right)+q_2\Bigg]G(\xi,\xi)=0,
\end{aligned} \tag{3.26}$$

that is,

$$G_{svv}(s,v)=d_1G(s,v)+d_2G_s(s,v)+d_3G_{sv}(s,v)+d_4G_{sss}(s,v), \tag{3.27}$$

$$G(s,0)=d_5, \tag{3.28}$$

$$G_t(s,0)=d_6s+d_7, \tag{3.29}$$

$$G_{ss}(s,s)-2G_{sv}(s,s)+G_{vv}(s,s)+d_8G(s,s)=0, \tag{3.30}$$

where

$$d_1 = \frac{1}{6}\left[\frac{\lambda_2-\mu_2}{3}\left(\frac{(\lambda_2-\mu_2)(2\lambda_2+\mu_2)}{9}-\lambda_1\right)+\lambda_0-\mu_0\right], \tag{3.31}$$

$$d_2 = -\frac{1}{3}\left[\mu_1+\frac{1}{4}\left(\frac{\lambda_2+2\mu_2}{3}+q_1\right)(\lambda_2-2\mu_2+3q_1)\right], \tag{3.32}$$

$$d_3 = \frac{1}{3}(\lambda_2+3q_1), \tag{3.33}$$

$$d_4 = -\frac{1}{3}, \tag{3.34}$$

$$d_5 = \frac{\lambda_2-\mu_2}{3}+q_1-r_1, \tag{3.35}$$

$$d_6 = \frac{1}{2}\left[\frac{\mu_0-\lambda_0}{3}+\lambda_1\frac{\lambda_2-\mu_2}{9}-\frac{2\lambda_2+\mu_2}{9}\left(\frac{\lambda_2-\mu_2}{3}\right)^2\right], \tag{3.36}$$

$$d_7 = \left(\frac{\lambda_2+2\mu_2}{6}+\frac{1}{2}q_1+r_1\right)\frac{\lambda_2-\mu_2}{3}+\left(\frac{\lambda_2+2\mu_2}{6}-\frac{1}{2}q_1\right)(q_1-r_1)+q_2-r_2, \tag{3.37}$$

$$d_8 = \lambda_1-\frac{1}{4}\left(\frac{\lambda_2+2\mu_2}{3}+q_1\right)^2+\frac{2\lambda_2+\mu_2}{3}\left(\frac{\mu_2-\lambda_2}{3}+q_1\right)+q_2. \tag{3.38}$$

Integrating (3.27) with respect to υ from 0 to τ, we get

$$\begin{aligned} G_{s\upsilon}(s,\tau) =& d_6+\int_0^\tau\Bigg[d_1G(s,\upsilon)+d_2G_s(s,\upsilon)+d_3G_{s\upsilon}(s,\upsilon)+d_4G_{sss}(s,\upsilon) \\ &+\frac{1}{6}\int_{\frac{s-\upsilon}{2}}^{\frac{s+\upsilon}{2}}G\left(\frac{s+\upsilon}{2}+\zeta,\frac{s+\upsilon}{2}-\zeta\right)e^{\frac{\lambda_2+q_1}{2}\left(\zeta-\frac{s-\upsilon}{2}\right)}f\left(\zeta,\frac{s-\upsilon}{2}\right)\mathrm{d}\zeta \\ &-\frac{1}{6}f\left(\frac{s+\upsilon}{2},\frac{s-\upsilon}{2}\right)e^{\frac{\lambda_2+q_1}{2}\upsilon}\Bigg]\,\mathrm{d}\upsilon. \end{aligned} \tag{3.39}$$

Integrating (3.39) with respect to τ from 0 to η, we get

$$\begin{aligned} &G_s(s,\eta) = d_6\eta \\ &+\int_0^\eta\int_0^\tau\Big[d_1G(s,\upsilon)+d_2G_s(s,\upsilon)+d_3G_{s\upsilon}(s,\upsilon)+d_4G_{sss}(s,\upsilon)\Big]\,\mathrm{d}\upsilon\,\mathrm{d}\tau. \end{aligned} \tag{3.40}$$

Integrating now (3.40) with respect to s from η to ξ, we get

$$\begin{aligned} &G(\xi,\eta) = G(\eta,\eta)+d_6\eta(\xi-\eta) \\ &+\int_\eta^\xi\int_0^\eta\int_0^\tau\Big[d_1G(s,\upsilon)+d_2G_s(s,\upsilon)+d_3G_{s\upsilon}(s,\upsilon)+d_4G_{sss}(s,\upsilon)\Big]\,\mathrm{d}\upsilon\,\mathrm{d}\tau\,\mathrm{d}s. \end{aligned} \tag{3.41}$$

To find $G(\eta,\eta)$, using (3.26),

$$\frac{d^2}{d\xi^2}G(\xi,\xi) = 4G_{\xi\eta}(\xi,\xi) - d_8 G(\xi,\xi). \tag{3.42}$$

Using (3.39) with $s=\xi, \tau=\xi$, (3.42) can be written in the form of an integro-differential equation (IDE) for $G(\xi,\xi)$ as

$$\begin{aligned}\frac{d^2}{d\xi^2}G(\xi,\xi) &= -d_8 G(\xi,\xi) + 4d_6 \\ &+ 4\int_0^\xi \Big[d_1 G(\xi,\upsilon) + d_2 G_\xi(\xi,\upsilon) + d_3 G_{\xi\upsilon}(\xi,\upsilon) + d_4 G_{\xi\xi\xi}(\xi,\upsilon)\Big] d\upsilon.\end{aligned} \tag{3.43}$$

Thus,

$$\begin{aligned}G(\xi,\xi) = &\begin{pmatrix} d_5 & d_7 \end{pmatrix} e^{E\xi} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4d_6 \int_0^\xi \begin{pmatrix} 0 & 1 \end{pmatrix} e^{E(\xi-\sigma)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}\sigma \\ &+ 4\int_0^\xi \int_0^\sigma \Big[d_1 G(\sigma,\upsilon) + d_2 G_\sigma(\sigma,\upsilon) + d_3 G_{\sigma\upsilon}(\sigma,\upsilon) \\ &+ d_4 G_{\sigma\sigma\sigma}(\sigma,\upsilon)\Big] d\upsilon \begin{pmatrix} 0 & 1 \end{pmatrix} e^{E(\xi-\sigma)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}\sigma,\end{aligned} \tag{3.44}$$

where

$$E = \begin{pmatrix} 0 & -d_8 \\ 1 & 0 \end{pmatrix}. \tag{3.45}$$

From (3.41) and (3.44), an IDE for G is obtained:

$$G(\xi,\eta) = G^0(\xi,\eta) + F[G](\xi,\eta), \tag{3.46}$$

where G_0 and $F[G]$ are defined by

$$\begin{aligned}G^0(\xi,\eta) = d_6\eta(\xi-\eta) &+ \begin{pmatrix} d_5 & d_7 \end{pmatrix} e^{E\eta} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &+ 4d_6 \int_0^\eta \begin{pmatrix} 0 & 1 \end{pmatrix} e^{E(\eta-\sigma)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}\sigma\end{aligned} \tag{3.47}$$

and

$F[G](\xi,\eta)$

$$= \int_{\eta}^{\xi} \int_{0}^{\eta} \int_{0}^{\tau} \Big[d_1 G(s,\upsilon) + d_2 G_s(s,\upsilon) + d_3 G_{s\upsilon}(s,\upsilon) + d_4 G_{sss}(s,\upsilon))\Big] \,\mathrm{d}\upsilon\, \mathrm{d}\tau\, \mathrm{d}s$$
$$+ 4 \int_{0}^{\eta} \int_{0}^{\sigma} \Big[d_1 G(\sigma,\upsilon) + d_2 G_\sigma(\sigma,\upsilon) + d_3 G_{\sigma\upsilon}(\sigma,\upsilon) + d_4 G_{\sigma\sigma\sigma}(\sigma,\upsilon)\Big] d\upsilon$$
$$\times \begin{pmatrix} 0 & 1 \end{pmatrix} e^{E(\eta-\sigma)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}\sigma. \tag{3.48}$$

The integral equation (3.46) is equivalent to the system (3.23)–(3.26).

Lemma 3.1. *Any* $G(\xi,\eta)$ *satisfying* (3.23)–(3.26) *with* (3.8), (3.9), (3.13) *also satisfy* (3.46) *with* (3.47)–(3.48) *and* (3.31)–(3.38), (3.45)*; and vice versa.*

Next, by the method of successive approximation, we show existence, uniqueness and regularity of solution to the equation (3.46) with (3.47)–(3.48) and (3.31)–(3.38), (3.45).

(Existence and regularity) In fact, set

$$G^{n+1}(\xi,\eta) = F[G^n](\xi,\eta), n = 0, 1, 2, \cdots, \tag{3.49}$$

and denote

$$M = 2\,|d_6| + \sup_{0\le\eta\le1} \left| \begin{pmatrix} d_5 & d_7 \end{pmatrix} e^{E\eta} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4d_6 \int_0^{\eta} \begin{pmatrix} 0 & 1 \end{pmatrix} e^{E(\eta-\sigma)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}\sigma \right|, \tag{3.50}$$

$$\begin{aligned} N =& 2\,(e_1+e_2+e_3+e_4) \left(1 + 2 \sup_{0\le\sigma\le\eta\le1} \left| \begin{pmatrix} 0 & 1 \end{pmatrix} e^{E(\eta-\sigma)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| \right) \\ =& 2\,(e_1+e_2+e_3+e_4) \left(1 - 2d_8^{-1/2} i \sup_{0\le\varsigma\le1} \sinh\left(d_8^{1/2} i\varsigma\right)\right), \end{aligned} \tag{3.51}$$

where i denotes the imaginary unit and

$$e_1 = |d_1|,\ e_2 = |d_2|,\ e_3 = |d_3|,\ e_4 = |d_4|, \tag{3.52}$$

then it can be proved that

$$\left|G^0(\xi,\eta)\right| \le M, \tag{3.53}$$
$$\left|G^0_\xi(\xi,\eta)\right| = |d_6|\eta \le |d_6| \le 2|d_6|(\xi+\eta)^{-1} \le M(\xi+\eta)^{-1}, \tag{3.54}$$

$$\left|G^0_{\xi\eta}(\xi,\eta)\right| = |d_6| \le M(\xi+\eta)^{-1}, \tag{3.55}$$

$$G^0_{\xi\xi}(\xi,\eta) = 0, \tag{3.56}$$

and

$$\begin{aligned}
\left|G^1(\xi,\eta)\right| \le & \int_\eta^\xi \int_0^\eta \int_0^\tau \left(e_1\left|G^0(s,\upsilon)\right| + e_2\left|G^0_s(s,\upsilon)\right| + e_3\left|G^0_{st}(s,\upsilon)\right|\right) \mathrm{d}t\,\mathrm{d}\tau\,\mathrm{d}s \\
& + 4\int_0^\eta \int_0^\sigma \left(e_1\left|G^0(\sigma,t)\right| + e_2\left|G^0_\sigma(\sigma,t)\right| + e_3\left|G^0_{\sigma t}(\sigma,t)\right|\right) dt \\
& \times \left|\begin{pmatrix} 0 & 1 \end{pmatrix} e^{E(\eta-\sigma)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right| \mathrm{d}\sigma \\
\le & M(\xi+\eta)\Bigg(\frac{e_1}{2} + e_2 + e_3 \\
& \qquad + (e_1 + 4e_2 + 4e_3) \sup_{0\le\sigma\le\eta\le 1} \left|\begin{pmatrix} 0 & 1 \end{pmatrix} e^{E(\eta-\sigma)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|\Bigg) \\
\le & MN(\xi+\eta),
\end{aligned} \tag{3.57}$$

$$\begin{aligned}
\left|G^1_\xi(\xi,\eta)\right| \le & \int_0^\eta \int_0^\tau \left(e_1\left|G^0(\xi,t)\right| + e_2\left|G^0_\xi(\xi,t)\right| + e_3\left|G^0_{\xi t}(\xi,t)\right|\right) \mathrm{d}t\,\mathrm{d}\tau \\
\le & \left(\frac{e_1}{2} + 2e_2 + 2e_3\right) M \le MN,
\end{aligned} \tag{3.58}$$

$$\begin{aligned}
\left|G^1_{\xi\eta}(\xi,\eta)\right| \le & \int_0^\eta \left(e_1\left|G^0(\xi,t)\right| + e_2\left|G^0_\xi(\xi,t)\right| + e_3\left|G^0_{\xi t}(\xi,t)\right|\right) \mathrm{d}t \\
\le & (e_1 + e_2 + e_3)M \le MN,
\end{aligned} \tag{3.59}$$

$$\left|G^1_{\xi\xi}(\xi,\eta)\right| \le \int_0^\eta \int_0^\tau e_1\left|G^0_\xi(\xi,t)\right| \mathrm{d}t\,\mathrm{d}\tau \le 2e_1 M \le MN, \tag{3.60}$$

$$\left|G^1_{\xi\xi\eta}(\xi,\eta)\right| \le \int_0^\eta e_1\left|G^0_\xi(\xi,t)\right| \mathrm{d}t \le e_1 M \le MN, \tag{3.61}$$

$$G^1_{\xi\xi\xi}(\xi,\eta) = 0. \tag{3.62}$$

For $n \ge 1$, $\xi + \eta \le n+1$, and thus it holds that

$$\frac{(\xi+\eta)^{n+1}}{(n+1)!} \le \frac{(\xi+\eta)^n}{n!}. \tag{3.63}$$

Suppose that for $n \ge 1$,

$$|G^n(\xi,\eta)| \le MN^n \frac{(\xi+\eta)^n}{n!}, \tag{3.64}$$

$$\left|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,\eta)\right| \le MN^n \frac{(\xi+\eta)^{n-1}}{(n-1)!},\ m = \overline{1, n+1}, \tag{3.65}$$

$$\left|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}\eta}(\xi,\eta)\right| \le MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!}, \tag{3.66}$$

$$G^n_{\underbrace{\xi\xi\cdots\xi}_{n+2}}(\xi,\eta) = 0, \tag{3.67}$$

then we have

$$\begin{aligned}\left|G^{n+1}(\xi,\eta)\right| \le &\int_\eta^\xi\int_0^\eta\int_0^\tau \Big[e_1\left|G^n(s,\upsilon)\right| + e_2\left|G^n_s(s,\upsilon)\right| \\ &\qquad + e_3\left|G^n_{s\upsilon}(s,\upsilon)\right| + e_4\left|G^n_{sss}(s,\upsilon)\right|\Big]\,\mathrm{d}\upsilon\,\mathrm{d}\tau\,\mathrm{d}s \\ &+ 4\int_0^\eta\int_0^\sigma \Big[e_1\left|G^n(\sigma,\upsilon)\right| + e_2\left|G^n_\sigma(\sigma,\upsilon)\right| + e_3\left|G^n_{\sigma\upsilon}(\sigma,\upsilon)\right| \\ &\qquad + e_4\left|G^n_{\sigma\sigma\sigma}(\sigma,\upsilon)\right|\Big]d\upsilon \times \left|\begin{pmatrix} 0 & 1\end{pmatrix} e^{E(\eta-\sigma)}\begin{pmatrix} 1 \\ 0\end{pmatrix}\right|\mathrm{d}\sigma \\ \le &MN^{n+1}\frac{(\xi+\eta)^{n+1}}{(n+1)!}. \end{aligned}\tag{3.68}$$

For $m=\overline{1,n+2}$, we have

$$\begin{aligned}\left|G^{n+1}_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,\eta)\right| \le &\int_0^\eta\int_0^\tau \Bigg[e_1\left|G^n_{\underbrace{\xi\xi\cdots\xi}_{m-1}}(\xi,t)\right| + e_2\left|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,t)\right| \\ &\qquad + e_3\left|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}t}(\xi,t)\right| + e_4\left|G^n_{\underbrace{\xi\xi\cdots\xi}_{m+2}}(\xi,t)\right|\Bigg]\,\mathrm{d}t\,\mathrm{d}\tau \\ \le &MN^{n+1}\frac{(\xi+\eta)^n}{n!}, \end{aligned}\tag{3.69}$$

and

$$\begin{aligned}\left|G^{n+1}_{\underbrace{\xi\xi\cdots\xi}_{m}\eta}(\xi,\eta)\right| \le &\int_0^\eta \Bigg(e_1\left|G^n_{\underbrace{\xi\xi\cdots\xi}_{m-1}}(\xi,t)\right| + e_2\left|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,t)\right| \\ &\qquad + e_3\left|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}t}(\xi,t)\right| + e_4\left|G^n_{\underbrace{\xi\xi\cdots\xi}_{m+2}}(\xi,t)\right|\Bigg)\,\mathrm{d}t \\ \le &MN^{n+1}\frac{(\xi+\eta)^n}{n!}. \end{aligned}\tag{3.70}$$

Also, we get

$$G^{n+1}_{\underbrace{\xi\xi\cdots\xi}_{n+3}}(\xi,\eta) = 0. \tag{3.71}$$

Thus, by induction, the estimates (3.64)–(3.67) are proved for any $n \geq 1$. Moreover, from (3.53), the estimate (3.64) holds for any $n \geq 0$, which shows that the series

$$G(\xi,\eta) = \sum_{n=0}^{\infty} G^n(\xi,\eta) \tag{3.72}$$

converges absolutely and uniformly in $\mathcal{T}_1$. From (3.46)–(3.48), we can get that $G^n(\xi,\eta)$ is C^∞, and thus $G(\xi,\eta)$ is C^∞ with a bound

$$|G(\xi,\eta)| \leq Me^{N(\xi+\eta)}. \tag{3.73}$$

Remark 3.3. From (3.49) and (3.48), it is reasonable to deduce that we might need all the partial derivative estimation $\left|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,\eta)\right|, \left|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}\underbrace{\eta}_{m'}}(\xi,\eta)\right|$, the integers $m = \overline{0,\infty}$, $m' = 0$ or 1, (infinite terms for each n) to conduct successive approximation. However, we note from (3.47) that the starting point of iteration G^0 is linear in ξ, and thus it is possible to simplify the proof as above (finite number of terms for each iteration).

(Uniqueness) Consider any two solutions $G_1(\xi,\eta), G_2(\xi,\eta)$ to (3.46), then $\Delta G(\xi,\eta) = G_1(\xi,\eta) - G_2(\xi,\eta)$ satisfies

$$\Delta G(\xi,\eta) = F[\Delta G](\xi,\eta). \tag{3.74}$$

From the result of boundedness (3.73), we have

$$|\Delta G(\xi,\eta)| \leq 2Me^{2N} \leq 2MCe^{2N}, \tag{3.75}$$

where

$$C = 2N + 1. \tag{3.76}$$

Also, we derive from (3.74) that

$$
\begin{aligned}
|G_\xi(\xi,\eta)| &\le \left|G_\xi^0(\xi,\eta)\right| + \sum_{n=1}^{\infty}\left|G_\xi^n(\xi,\eta)\right| \le M(\xi+\eta)^{-1} + \sum_{n=1}^{\infty} MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!} \\
&\le M(\xi+\eta)^{-1}\left(1+2Ne^{N(\xi+\eta)}\right) \le MCe^{N(\xi+\eta)}(\xi+\eta)^{-1}, \qquad (3.77)\\
|G_{\xi\eta}(\xi,\eta)| &\le \left|G_{\xi\eta}^0(\xi,\eta)\right| + \sum_{n=1}^{\infty}\left|G_{\xi\eta}^n(\xi,\eta)\right| \le M(\xi+\eta)^{-1} + \sum_{n=1}^{\infty} MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!} \\
&\le MCe^{N(\xi+\eta)}(\xi+\eta)^{-1}, \qquad (3.78)
\end{aligned}
$$

and for $m \geqslant 2$,

$$
\begin{aligned}
\Bigg|G_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,\eta)\Bigg| &\le \sum_{n=0}^{m-2}\Bigg|G^0_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,\eta)\Bigg| + \sum_{n=m-1}^{\infty}\Bigg|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,\eta)\Bigg| \\
&= \sum_{n=m-1}^{\infty}\Bigg|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,\eta)\Bigg| \\
&\le \sum_{n=m-1}^{\infty} MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!} \le \sum_{n=1}^{\infty} MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!} \\
&= MNe^{N(\xi+\eta)} \le MCe^{N(\xi+\eta)}(\xi+\eta)^{-1}, \qquad (3.79)
\end{aligned}
$$

$$
\begin{aligned}
\Bigg|G_{\underbrace{\xi\xi\cdots\xi}_{m}\eta}(\xi,\eta)\Bigg| &\le \sum_{n=0}^{m-2}\Bigg|G^0_{\underbrace{\xi\xi\cdots\xi}_{m}\eta}(\xi,\eta)\Bigg| + \sum_{n=m-1}^{\infty}\Bigg|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}\eta}(\xi,\eta)\Bigg| \\
&= \sum_{n=m-1}^{\infty}\Bigg|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}\eta}(\xi,\eta)\Bigg| \\
&\le \sum_{n=m-1}^{\infty} MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!} \le MCe^{N(\xi+\eta)}(\xi+\eta)^{-1}. \qquad (3.80)
\end{aligned}
$$

Thus, from (3.77)–(3.80), for $m \geqslant 1$,

$$
\Bigg|\Delta G_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,\eta)\Bigg| \le 2MCe^{2N}(\xi+\eta)^{-1}, \qquad (3.81)
$$

$$
\Bigg|\Delta G_{\underbrace{\xi\xi\cdots\xi}_{m}\eta}(\xi,\eta)\Bigg| \le 2MCe^{2N}(\xi+\eta)^{-1}. \qquad (3.82)
$$

With (3.75), (3.81), (3.82), we calculate once again based on (3.74) and get

$$|\Delta G| \leq 2MCe^{2N}N(\xi+\eta), \tag{3.83}$$

$$\Big|\Delta G_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,\eta)\Big| \leq 2MCe^{2N}N, \tag{3.84}$$

$$\Big|\Delta G_{\underbrace{\xi\xi\cdots\xi}_{m}\eta}(\xi,\eta)\Big| \leq 2MCe^{2N}N. \tag{3.85}$$

Based on (3.74), we repeat the above procedure for deriving (3.83)–(3.85). Then, the following general form of upper bound estimates is derived: for any integer $j \geqslant 1$,

$$|\Delta G| \leq 2MCe^{2N}N^j\frac{(\xi+\eta)^j}{j!}, \tag{3.86}$$

$$\Big|\Delta G_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,\eta)\Big| \leq 2MCe^{2N}N^{j-1}\frac{(\xi+\eta)^{j-1}}{(j-1)!}, \tag{3.87}$$

$$\Big|\Delta G_{\underbrace{\xi\xi\cdots\xi}_{m}\eta}(\xi,\eta)\Big| \leq 2MCe^{2N}N^{j-1}\frac{(\xi+\eta)^{j-1}}{(j-1)!}. \tag{3.88}$$

Note that the induction method is employed in order to derive this general form. From (3.86),

$$|\Delta G| \to 0 \text{ as j} \to \infty. \tag{3.89}$$

Thus, $\Delta G \equiv 0$, which gives uniqueness of the solution (3.72) to (3.46).

We have derived the unique C^∞ function $G(\xi,\eta)$, and thus existence, uniqueness, and regularity of the function $p(x,y)$ and kernel $\kappa(x,y)$ follows, which gives the following lemma.

Lemma 3.2. *The kernel $\kappa(x,y)-$system* (3.4)–(3.7) *with* (3.8), (3.9) *has a unique $C^\infty(\mathcal{T})$ solution with a bound*

$$|\kappa(x,y)| \leq Me^{2Nx}, \tag{3.90}$$

where M, N are given by (3.50), (3.51) *and* (3.52), (3.31)–(3.38), (3.45).

B. Well-posedness and stability analysis of transformed systems

Through a PDE backstepping transformation $u \mapsto w$:

$$w(t,x) = u(t,x)e^{\frac{\lambda_2-\mu_2}{3}x} - \int_0^x \kappa(x,y)u(y,t)e^{\frac{\lambda_2-\mu_2}{3}y}dy, \tag{3.91}$$

where the kernel function $\kappa(x,y)$ satisfies equation (3.4)–(3.7), the class of closed-loop systems (3.1)–(3.2) with controller (3.3) is mapped into the following class of stable target transformed systems:

$$w_t(t,x) = w_{xxx}(t,x) + \mu_2 w_{xx}(t,x) + \mu_1 w_x(t,x) + \mu_0 w(t,x), x \in (0,L), \tag{3.92}$$

$$w_x(t,0) = r_1 w(t,0),\ w_{xx}(t,0) = r_2 w(t,0),\ w(t,L) = 0. \tag{3.93}$$

Remark 3.4. The transformation $u \mapsto w$ is derived by composition of transformation $u \mapsto v$:

$$v(t,x) = u(t,x)e^{\frac{\lambda_2-\mu_2}{3}x} \tag{3.94}$$

and transformation $v \mapsto w$:

$$w(t,x) = v(t,x) - \int_0^x \kappa(x,y)v(y,t)dy. \tag{3.95}$$

Define the system operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A})(\subset \mathcal{H}) \to \mathcal{H}$ as follows:

$$\mathcal{A}f = f''' + \mu_2 f'' + \mu_1 f' + \mu_0 f, \forall f \in \mathrm{Dom}(\mathcal{A}), \tag{3.96}$$

$$\mathrm{Dom}(\mathcal{A}) = \left\{ f \in H^3(0,L) \mid f'(0) = r_1 f(0), f''(0) = r_2 f(0), f(L) = 0 \right\}, \tag{3.97}$$

then the system (3.92)–(3.93) can be written as an evolution equation in $\mathcal{H}$:

$$\frac{\mathrm{d}w(\cdot,t)}{\mathrm{d}t} = \mathcal{A}w(\cdot,t). \tag{3.98}$$

Then, the following lemma follows directly from Lemma 2.5.

Lemma 3.3. *For each $\lambda \in \sigma(\mathcal{A})$, $Re\lambda < 0$. $\mathcal{A}$ generates an exponentially stable C_0-semigroup on $\mathcal{H}$. For any initial value $w(0,\cdot) \in \mathcal{H}$, there exists a unique (mild) solution to* (3.92)–(3.93) *with* (3.8), (3.9), (3.13), *such that*

$$w(t,\cdot) \in C\left([0,\infty);\mathcal{H}\right), \tag{3.99}$$

and

$$\|w(t,\cdot)\| \leq e^{-\rho_u t}\|w(0,\cdot)\|, \tag{3.100}$$

where ρ_u *is defined in* (3.11). *Moreover, if* $w(\cdot,0) \in \mathrm{Dom}(\mathcal{A})$, *then*

$$w(t,\cdot) \in C^1\left([0,\infty);\mathcal{H}\right) \tag{3.101}$$

is the classical solution to (3.92)–(3.93).

C. Existence, uniqueness and regularity of the inverse transformation

The backstepping transformation (3.91) is invertible, and the inverse transformation $w \mapsto u$ can also be as follows:

$$u(t,x) = w(t,x)e^{-\frac{\lambda_2-\mu_2}{3}x} - \int_0^x \iota(x,y)w(y,t)dye^{-\frac{\lambda_2-\mu_2}{3}x}, \tag{3.102}$$

which satisfies

$$\begin{aligned}
&\iota_{xxx}(x,y) + \iota_{yyy}(x,y) + \mu_2\left(\iota_{xx}(x,y) - \iota_{yy}(x,y)\right) + \mu_1\left(\iota_x(x,y) + \iota_y(x,y)\right) \\
&= \left[\frac{\lambda_2-\mu_2}{3}\left(\lambda_1 - \frac{(\lambda_2-\mu_2)(2\lambda_2+\mu_2)}{9}\right) - \lambda_0 + \mu_0\right]\iota(x,y), && (3.103)\\
&\iota(x,x) = r_1 - q_1 - \frac{\lambda_2-\mu_2}{3}, && (3.104)\\
&\iota_x(x,x) = \left[\frac{\lambda_0-\mu_0}{3} - \lambda_1\frac{\lambda_2-\mu_2}{9} + \left(\frac{\lambda_2-\mu_2}{3}\right)^2\frac{2\lambda_2+\mu_2}{9}\right]x \\
&\qquad + \frac{\lambda_2-\mu_2}{3}\left(r_1 - \frac{\lambda_2-\mu_2}{3} - 2q_1\right) + r_1(q_1-r_1) + r_2 - q_2, && (3.105)\\
&\iota_{yy}(x,0) - (\mu_2+r_1)\iota_y(x,0) + (\mu_1+\mu_2 r_1 + r_2)\iota(x,0) = 0. && (3.106)
\end{aligned}$$

Similar results about existence, uniqueness and regularity of the kernel $\iota(x,y)$ can be proved in a similar way as proving for the kernel $\kappa(x,y)$.

D. Well-posedness and stability analysis of closed-loop control systems

Because the transformation (3.91) is continuous, then there exists a positive constant C_κ such that

$$\|w\| \leq C_\kappa\|u\|. \tag{3.107}$$

The inverse transformation (3.102) is also continuous and thus there exists a positive constant C_ι such that

$$\|u\| \leq C_\iota \|w\|. \tag{3.108}$$

From (3.100), (3.107), (3.108), we then have

$$\|u(t,\cdot)\| \leq C_\kappa C_\iota \|u(0,\cdot)\| e^{-\rho_u t}, \tag{3.109}$$

which proves the exponential decay at rate ρ_u for the class of closed-loop control systems (3.1)–(3.2) with controllers (3.3), and Theorem 3.1 is thus proved with equivalence between (3.92)–(3.93) and the closed-loop systems from invertibility of transformation.

3.3 Output feedback stabilization

For the output feedback problem, we consider the non-collocated case here. The objective is stabilizing the class of LKdVAD (3.1)–(3.2) with boundary control and non-collocated observation: $y(t) = u(t, 0)$. A state observer is first designed, and then an output feedback controller is constructed based on the recovered full state information from the observer.

3.3.1 Observer design

A. Main result

We propose the following Luenberger-type observer, which is a "copy of the plant plus output injection terms":

$$\begin{aligned}\hat{u}_t(t,x) =& \hat{u}_{xxx}(t,x) + \lambda_2 \hat{u}_{xx}(t,x) + \lambda_1 \hat{u}_x(t,x) + \lambda_0 \hat{u}(t,x) \\ & - b_0(x)(u(t,0) - \hat{u}(t,0)), x \in (0,L), \end{aligned} \tag{3.110}$$

$$\hat{u}_x(t,0) = q_1 u(t,0) - b_1(u(t,0) - \hat{u}(t,0)), \tag{3.111}$$

$$\hat{u}_{xx}(t,0) = q_2 u(t,0) - b_2(u(t,0) - \hat{u}(t,0)), \tag{3.112}$$

$$\hat{u}(t,L) = U(t) \tag{3.113}$$

with the function

$$b_0(x) = \Bigg[\tilde{\kappa}_{yy}(x,0) - (\tilde{\mu}_2 + c_1)\tilde{\kappa}_y(x,0) + \left(\frac{\tilde{\mu}_2^2 - \lambda_2^2}{3} + \lambda_1 + \tilde{\mu}_2 c_1 + c_2\right)\tilde{\kappa}(x,0)\Bigg] e^{-\frac{\lambda_2 - \tilde{\mu}_2}{3}x} \tag{3.114}$$

and the constants

$$b_1 = c_1 - \frac{\lambda_2 - \tilde{\mu}_2}{3}, \tag{3.115}$$

$$b_2 = c_2 + \frac{\lambda_2 - \tilde{\mu}_2}{3}\left(\frac{\lambda_2 - \tilde{\mu}_2}{3} - 2c_1\right) + \left(\left(\frac{\lambda_2 - \tilde{\mu}_2}{3}\right)^2 \frac{2\lambda_2 + \tilde{\mu}_2}{9} - \lambda_1 \frac{\lambda_2 - \tilde{\mu}_2}{9} + \frac{\lambda_0 - \tilde{\mu}_0}{3}\right) L, \tag{3.116}$$

where the function $\tilde{\kappa}(x,y) \in \mathbb{R}$ satisfies

$$\tilde{\kappa}_{xxx}(x,y) + \tilde{\kappa}_{yyy}(x,y) + \tilde{\mu}_2\left(\tilde{\kappa}_{xx}(x,y) - \tilde{\kappa}_{yy}(x,y)\right) + \left(\frac{\tilde{\mu}_2^2 - \lambda_2^2}{3} + \lambda_1\right)\left(\tilde{\kappa}_x(x,y) + \tilde{\kappa}_y(x,y)\right) = \left[\frac{\lambda_2 - \tilde{\mu}_2}{3}\left(\lambda_1 - \frac{(\lambda_2 - \tilde{\mu}_2)(2\lambda_2 + \tilde{\mu}_2)}{9}\right) + \tilde{\mu}_0 - \lambda_0\right]\tilde{\kappa}(x,y), \tag{3.117}$$

$$\tilde{\kappa}(x,x) = 0, \tag{3.118}$$

$$\tilde{\kappa}_x(x,x) = \left[\frac{\lambda_0 - \tilde{\mu}_0}{3} - \lambda_1 \frac{\lambda_2 - \tilde{\mu}_2}{9} + \left(\frac{\lambda_2 - \tilde{\mu}_2}{3}\right)^2 \frac{2\lambda_2 + \tilde{\mu}_2}{9}\right](x - L), \tag{3.119}$$

$$\tilde{\kappa}(L,y) = 0. \tag{3.120}$$

The parameters $\tilde{\mu}_0$ and c_1 are arbitrary, and the parameters $\tilde{\mu}_2$ and c_2 satisfy design parameters

$$\tilde{\mu}_2 \geq 0 \text{ and } \tilde{\mu}_2 > 4L^2\tilde{\mu}_0, \tag{3.121}$$

$$c_2 > \frac{1}{2}\left(c_1^2 - 2\tilde{\mu}_2 c_1 - \frac{\tilde{\mu}_2^2 - \lambda_2^2}{3} - \lambda_1\right) = c_2^*. \tag{3.122}$$

Denote

$$\tilde{u}(t,x) = u(t,x) - \hat{u}(t,x), \tag{3.123}$$

then the observer error system is

$$\tilde{u}_t(t,x) = \tilde{u}_{xxx}(t,x) + \lambda_2 \tilde{u}_{xx}(t,x) + \lambda_1 \tilde{u}_x(t,x) + \lambda_0 \tilde{u}(t,x) + b_0(x)\tilde{u}(t,0), x \in (0,L), \tag{3.124}$$

$$\tilde{u}_x(t,0) = b_1 \tilde{u}(t,0), \tilde{u}_{xx}(t,0) = b_2 \tilde{u}(t,0), \tilde{u}(t,L) = 0. \tag{3.125}$$

The following theorem can be obtained to state the effectiveness of the observer.

Theorem 3.2. *For any initial data* $u(\cdot,0), \hat{u}(\cdot,0) \in \mathcal{H}$*, the observer* (3.110)–(3.113)*, in which the function* $b_0(x)$ *and constants* b_1, b_2 *are chosen from* (3.114)–(3.122)*, guarantees that the observer error* $\tilde{u}$*-system* (3.124)–(3.125) *has a unique (mild) solution*

$$\tilde{u}(t,\cdot) \in C\left([0,\infty);\mathcal{H}\right), \tag{3.126}$$

and there exist a positive constant $M_{\tilde{u}}$ *such that*

$$\|\tilde{u}(t,\cdot)\| \leq M_{\tilde{u}} e^{-\rho_{\tilde{u}} t} \|\tilde{u}(\cdot,0)\|, \ \rho_{\tilde{u}} = \frac{1}{4L^2}\tilde{\mu}_2 - \tilde{\mu}_0 > 0. \tag{3.127}$$

Moreover, if $\tilde{u}(\cdot,0)$ *satisfies boundary compatibility condition, then*

$$\tilde{u}(t,\cdot) \in C^1\left([0,\infty);\mathcal{H}\right) \tag{3.128}$$

is the classical solution to (3.124)–(3.125)*. That is, the designed observer* (3.110)–(3.113) *exponentially tracks the system* (3.1)–(3.2) *in the sense of state norm.*

We denote in the sequel the coefficient of first order partial derivative terms in the PDE (3.117) as

$$\frac{\tilde{\mu}_2^2 - \lambda_2^2}{3} + \lambda_1 \triangleq \tilde{\mu}_1 \tag{3.129}$$

for simplicity.

Remark 3.5.

1. For observer design only, the choice of c_2 can be relaxed to the following non-strict inequality

$$c_2 \geq \frac{1}{2}\left(c_1^2 - 2\tilde{\mu}_2 c_1 - \tilde{\mu}_1\right) = c_2^*. \tag{3.130}$$

2. Similarly as in Remark 3.2, the exponential decay rate $\rho_{\tilde{u}}$ can be arbitrarily large by choosing $\tilde{\mu}_2$ large enough. On the other hand, similarly as in Remark 2.6, $\rho_{\tilde{u}}$ is not necessarily equal to the optimal decay rate of the observer error system, and in some special cases, exponential convergence of the observer to the original system still holds if the strict inequality $\tilde{\mu}_2 > 4L^2\tilde{\mu}_0$ is relaxed to a nonstrict one.

B. Proofs

1. Existence, uniqueness and regularity of the function $\tilde{\kappa}(x, y)$

Let

$$\tilde{\kappa}(x, y) = \bar{\kappa}(\bar{x}, \bar{y}), \tag{3.131}$$

where

$$\bar{x} = L - y,\ \bar{y} = L - x, \tag{3.132}$$

then $\bar{\kappa}(\bar{x}, \bar{y})$ satisfies

$$\begin{aligned}
&\bar{\kappa}_{\bar{x}\bar{x}\bar{x}}(\bar{x}, \bar{y}) + \bar{\kappa}_{\bar{y}\bar{y}\bar{y}}(\bar{x}, \bar{y}) + \tilde{\mu}_2\left(\bar{\kappa}_{\bar{x}\bar{x}}(\bar{x}, \bar{y}) - \bar{\kappa}_{\bar{y}\bar{y}}(\bar{x}, \bar{y})\right) + \tilde{\mu}_1\left(\bar{\kappa}_{\bar{x}}(\bar{x}, \bar{y}) + \bar{\kappa}_{\bar{y}}(\bar{x}, \bar{y})\right) \\
&\qquad = \left[\frac{\lambda_2 - \tilde{\mu}_2}{3}\left(\frac{(\lambda_2 - \tilde{\mu}_2)(2\lambda_2 + \tilde{\mu}_2)}{9} - \lambda_1\right) + \lambda_0 - \tilde{\mu}_0\right]\bar{\kappa}(\bar{x}, \bar{y}), \qquad (3.133)\\
&\bar{\kappa}(\bar{x}, \bar{x}) = 0, \qquad (3.134)\\
&\bar{\kappa}_{\bar{x}}(\bar{x}, \bar{x}) = \left[\frac{\tilde{\mu}_0 - \lambda_0}{3} + \lambda_1\frac{\lambda_2 - \tilde{\mu}_2}{9} - \left(\frac{\lambda_2 - \tilde{\mu}_2}{3}\right)^2\frac{2\lambda_2 + \tilde{\mu}_2}{9}\right]\bar{x} + c_2 - b_2, \qquad (3.135)\\
&\bar{\kappa}(\bar{x}, 0) = 0. \qquad (3.136)
\end{aligned}$$

The above PDE is in the class of (B.18)–(B.21) from Appendix B.2 (with μ_2, μ_1, μ_0 replaced by $\tilde{\mu}_2, \tilde{\mu}_1, \tilde{\mu}_0$, respectively). Thus, existence, uniqueness, regularity and invertibility of $\bar{\kappa}(\bar{x}, \bar{y})$ and also $\tilde{\kappa}(x, y)$ follow from Appendix B.2.

2. Well-posedness and stability analysis of the closed-loop control system

A backstepping transformation

$$\tilde{u}(t, x) = \tilde{w}(t, x)e^{\frac{\tilde{\mu}_2 - \lambda_2}{3}x} - \int_0^x \tilde{\kappa}(x, y)\tilde{w}(y, t)dye^{\frac{\tilde{\mu}_2 - \lambda_2}{3}x} \tag{3.137}$$

transforms the system (3.124)–(3.125) into the following exponentially stable system:

$$\tilde{w}_t(t,x) = \tilde{w}_{xxx}(t,x) + \tilde{\mu}_2 \tilde{w}_{xx}(t,x) + \tilde{\mu}_1 \tilde{w}_x(t,x) + \tilde{\mu}_0 \tilde{w}(t,x), x \in (0,L), \quad (3.138)$$

$$\tilde{w}_x(t,0) = c_1 \tilde{w}(t,0),\ \tilde{w}_{xx}(t,0) = c_2 \tilde{w}(t,0),\ \tilde{w}(t,L) = 0. \quad (3.139)$$

Remark 3.6. Exponential stability of system (3.138)–(3.139) with (3.121), (3.122) and (3.129) can be proved in the same way as Lemma 3.3 in Section 3.2.2, with an exponential decay rate $\rho_{\tilde{u}}$.

We can also prove invertibility of the transformation (3.137), and we can prove the uniqueness and regularity of its inverse as well. Hence, Theorem 3.2 is proved.

3.3.2 Output feedback stabilization

A. Main result

Consider the observer (3.110)–(3.113) with

$$U(t) = \int_0^L \kappa(L,y)\hat{u}(y,t)e^{\frac{\lambda_2-\mu_2}{3}(y-L)}dy, \quad (3.140)$$

where the constant μ_2 and function $\kappa(x,y)$ are determined from Subsection 3.2.1. Applying the backstepping transformation $\hat{u} \mapsto \hat{w}$:

$$\hat{w}(t,x) = \hat{u}(t,x)e^{\frac{\lambda_2-\mu_2}{3}x} - \int_0^x \kappa(x,y)\hat{u}(y,t)e^{\frac{\lambda_2-\mu_2}{3}y}dy, \quad (3.141)$$

we get

$$\begin{aligned}
&\hat{w}_t(t,x) = \hat{w}_{xxx}(t,x) + \mu_2 \hat{w}_{xx}(t,x) + \mu_1 \hat{w}_x(t,x) + \mu_0 \hat{w}(t,x) \\
&- \left[(q_1 - b_1)\kappa_y(x,0) - \left((q_1 - b_1)\frac{2\lambda_2 + \mu_2}{3} + q_2 - b_2 \right) \kappa(x,0) + b_0(x)e^{\frac{\lambda_2-\mu_2}{3}x} \right. \\
&\left. - \int_0^x \kappa(x,y)b_0(y)e^{\frac{\lambda_2-\mu_2}{3}y}dy \right] \tilde{w}(t,0), x \in (0,L), \quad (3.142) \\
&\hat{w}_x(t,0) = r_1 \hat{w}(t,0) + d_1 \tilde{w}(t,0), \quad (3.143)
\end{aligned}$$

$$\hat{w}_{xx}(t,0) = r_2\hat{w}(t,0) + d_2\tilde{w}(t,0), \tag{3.144}$$

$$\hat{w}(t,L) = 0, \tag{3.145}$$

where

$$d_1 = q_1 - b_1, \tag{3.146}$$

$$d_2 = (q_1 - b_1)\left(\frac{\lambda_2 - \mu_2}{3} - q_1 + r_1\right) + q_2 - b_2. \tag{3.147}$$

The $\tilde{w}$-system (3.138)–(3.139) and the homogenous part of the $\hat{w}$-system (3.142)–(3.145) are both exponentially stable. Interconnection of the two systems $(\hat{w}, \tilde{w})$ is a cascade, and the combined system (3.142)–(3.145), (3.138)–(3.139) is exponentially stable. Hence, exponential stability of the closed-loop $(u, \hat{u})$-system with output feedback controller (3.140) can be proved, and it is stated in the following theorem.

Theorem 3.3. *For any initial data* $(u(0,\cdot), \hat{u}(0,\cdot)) \in (L^2(0,L))^2$*, the closed-loop* $(u,\hat{u})$*-system* (3.1)–(3.2), (3.110)–(3.113)*, in which the function* $b_0(x)$ *and the constants* b_1, b_2 *are chosen from* (3.114)–(3.122)*, with the controller* (3.140) *determined by* (3.4)–(3.9) *has a unique (mild) solution*

$$(u(t,\cdot), \hat{u}(\cdot,t)) \in C\left([0,\infty); \left(L^2(0,L)\right)^2\right), \tag{3.148}$$

and there exists a positive constant $M_{u\hat{u}}$ *such that*

$$\|(u(t,\cdot), \hat{u}(\cdot,t))\| \le M_{u\hat{u}} e^{-\rho_{u\hat{u}} t}\|(u(\cdot,0), \hat{u}(\cdot,0))\|, \tag{3.149}$$

$$\rho_{u\hat{u}} = \min\left\{\frac{1}{4L^2}\mu_2 - \mu_0 - \frac{\theta^2 L}{4m_1},\ \frac{1}{4L^2}\tilde{\mu}_2 - \tilde{\mu}_0\right\} > 0, \tag{3.150}$$

where

$$m_1 > \frac{\theta^2 L^3}{\mu_2 - 4\mu_0 L^2}, \tag{3.151}$$

and

$$\theta = \sup_{0\le x\le L}\Bigg|(q_1 - b_1)\kappa_y(x,0) - \left((q_1 - b_1)\frac{2\lambda_2 + \mu_2}{3} + q_2 - b_2\right)\kappa(x,0) + b_0(x)e^{\frac{\lambda_2-\mu_2}{3}x} - \int_0^x \kappa(x,y)b_0(y)e^{\frac{\lambda_2-\mu_2}{3}y}dy\Bigg|. \tag{3.152}$$

Moreover, if $(u(0,\cdot),\hat{u}(\cdot,0))$ satisfies boundary compatibility condition, then

$$(u(t,\cdot),\hat{u}(\cdot,t)) \in C^1\left([0,\infty);\left(L^2(0,L)\right)^2\right) \tag{3.153}$$

is the classical solution.

Remark 3.7. Exponential decay rate $\rho_{u\hat{u}}$ can be arbitrarily large by first choosing $\mu_2, \tilde{\mu}_2$ large enough and then choosing m_1 large enough.

B. Proofs

Consider the Hilbert space

$$\mathbb{H} = \left(L^2(0,L)\right)^2 \tag{3.154}$$

with inner product

$$< (f_1,g_1),(f_2,g_2) >_{\mathbb{H}} = \int_0^L a f_1(x)\overline{f_2(x)} + g_1(x)\overline{g_2(x)}dx,\ \forall (f_1,g_1),(f_2,g_2) \in \mathbb{H}, \tag{3.155}$$

where $a > 0$ is a constant to be determined. Define the system operator $\mathbb{A}$: $\text{Dom}(\mathbb{A})(\subset \mathbb{H}) \to \mathbb{H}$ as follows:

$$\begin{aligned}\mathbb{A}(f,g) = \Bigg(& f''' + \mu_2 f'' + \mu_1 f' + \mu_0 f \\ & - \left[(q_1-b_1)\kappa_y(x,0) - \left((q_1-b_1)\frac{2\lambda_2+\mu_2}{3} + q_2 - b_2\right)\kappa(x,0)\right. \\ & \left. + b_0(x)e^{\frac{\lambda_2-\mu_2}{3}x} - \int_0^x \kappa(x,y)b_0(y)e^{\frac{\lambda_2-\mu_2}{3}y}dy\right]g(0), \\ & g''' + \tilde{\mu}_2 g'' + \tilde{\mu}_1 g' + \tilde{\mu}_0 g\Bigg), \forall (f,g) \in \text{Dom}(\mathbb{A}),\end{aligned} \tag{3.156}$$

$$\begin{aligned}\text{Dom}(\mathbb{A}) = \big\{ & (f,g) \in H^3(0,L) \times H^3(0,L) | f(L) = g(L) = 0, \\ & f'(0) = r_1 f(0) + d_1 g(0), g'(0) = c_1 g(0), \\ & f''(0) = r_2 f(0) + d_2 g(0), g''(0) = c_2 g(0)\big\},\end{aligned} \tag{3.157}$$

then the $(\hat{w},\tilde{w})$-system (3.142)–(3.145), (3.138)–(3.139) can be written as an evolution equation in $\mathbb{H}$:

$$\frac{d(\hat{w}(t,\cdot),\tilde{w}(\cdot,t))}{\mathrm{d}t} = \mathbb{A}(\hat{w}(\cdot,t),\tilde{w}(\cdot,t)). \tag{3.158}$$

By following the similar way as proving Lemma 2.3, this following lemma is derived. In the sequel we omit the proofs which are straightforward by following similar procedures as those in Section 2.3.

Lemma 3.4. *If*

$$\begin{pmatrix} 1 & r_1 & r_2 \end{pmatrix} e^{D_1 L} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq 0, \tag{3.159}$$

$$\begin{pmatrix} 1 & c_1 & c_2 \end{pmatrix} e^{D_2 L} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq 0, \tag{3.160}$$

where

$$D_1 = \begin{pmatrix} 0 & 0 & -\mu_0 \\ 1 & 0 & -\mu_1 \\ 0 & 1 & -\mu_2 \end{pmatrix}, \ D_2 = \begin{pmatrix} 0 & 0 & -\tilde{\mu}_0 \\ 1 & 0 & -\tilde{\mu}_1 \\ 0 & 1 & -\tilde{\mu}_2 \end{pmatrix}, \tag{3.161}$$

then $\mathbb{A}^{-1}$ exists and is compact on $\mathbb{H}$. Hence, $\sigma(\mathbb{A})$, the spectrum of $\mathbb{A}$, consists of isolated eigenvalues only. Moreover, each $\lambda \in \sigma(\mathbb{A})$ is geometrically simple.

Lemma 3.5. *$\mathbb{A}$ is dissipative in $\mathbb{H}$, and $\mathbb{A}$ generates a C_0-semigroup $e^{\mathbf{A}t}$ of contractions in $\mathbb{H}$. For each $\lambda \in \sigma(\mathbb{A})$, $Re\lambda < 0$.*

Proof. Let $(f, g) \in \text{Dom}(\mathbb{A})$, then

$$\begin{aligned} & Re < \mathbb{A}(f,g),(f,g) >_{\mathbb{H}} \\ & \leq a\left(\mu_0 - \frac{1}{4L^2}\mu_2 + \frac{\theta^2 L}{4m_1}\right)\|f\|^2_{\mathcal{H}} + \left(\tilde{\mu}_0 - \frac{1}{4L^2}\tilde{\mu}_2\right)\|g\|^2_{\mathcal{H}} \\ & - a\left(r_2 + \mu_2 r_1 - r_1^2 + \frac{1}{2}\mu_1 - (d_2 + \mu_2 d_1)^2 m_2\right)|f(0)|^2 \\ & - \left(c_2 + \tilde{\mu}_2 c_1 - \frac{1}{2}c_1^2 + \frac{1}{2}\tilde{\mu}_1 - a\left(d_1^2 + m_1 + \frac{1}{4m_2}\right)\right)|g(0)|^2, \end{aligned} \tag{3.162}$$

where constants θ and $m_1 > 0$ are given by (3.151), (3.152), and $m_2 > 0$ is to be determined.

Choose

$$0 < m_2 \leq \frac{r_2 + \mu_2 r_1 - r_1^2 + \frac{1}{2}\mu_1}{(d_2 + \mu_2 d_1)^2}, \tag{3.163}$$

and

$$0 < a \leq \frac{c_2 + \tilde{\mu}_2 c_1 - \frac{1}{2}c_1^2 + \frac{1}{2}\tilde{\mu}_1}{d_1^2 + m_1 + \frac{1}{4m_2}}, \tag{3.164}$$

then

$$Re <\mathbb{A}(f,g),(f,g)> \leq -\rho_{u\hat{u}}\|(f,g)\|^2, \ \forall (f,g) \in \mathrm{Dom}(\mathbb{A}), \tag{3.165}$$

where $\rho_{u\hat{u}}$ is defined as (3.150). □

Lemma 3.6. $\mathbb{A}$ *generates an exponentially stable* C_0 *semigroup on* $\mathbb{H}$*. For any initial data* $(\hat{w}(0,\cdot),\tilde{w}(0,\cdot)) \in \mathbb{H}$*, the transformed* $(\hat{w},\tilde{w})$*-system* (3.142)–(3.147), (3.138)–(3.139) *with* (3.8), (3.9), (3.13), (3.114)–(3.122) *and* (3.129) *has a unique (mild) solution*

$$(\hat{w}(t,\cdot),\tilde{w}(\cdot,t)) \in C\left([0,\infty);\mathbb{H}\right), \tag{3.166}$$

and

$$\|(\hat{w}(t,\cdot),\tilde{w}(\cdot,t))\| \leq e^{-\rho_{u\hat{u}}t}\|(\hat{w}(\cdot,0),\tilde{w}(\cdot,0))\|. \tag{3.167}$$

Moreover, if $(\hat{w}(0,\cdot),\tilde{w}(\cdot,0)) \in \mathrm{Dom}(\mathbb{A})$*, then*

$$(\hat{w}(t,\cdot),\tilde{w}(\cdot,t)) \in C^1\left([0,\infty);\mathbb{H}\right) \tag{3.168}$$

is the classical solution.

Proof. Define a Lyapunov function

$$L(t) = \frac{1}{2}\|\left(\hat{w}(\cdot,t),\ \tilde{w}(\cdot,t)\right)\|_{\mathbb{H}}^2 = \frac{a}{2}\|\hat{w}(\cdot,t)\|_{\mathcal{H}}^2 + \frac{1}{2}\|\tilde{w}(\cdot,t)\|_{\mathcal{H}}^2, \tag{3.169}$$

then we can get

$$\dot{L}(t) \leq -2\rho_{u\hat{u}}L(t). \tag{3.170}$$

Because $\mathbb{A}$ generates a C_0-semigroup $e^{\mathbf{A}t}$, this semigroup must be exponentially stable. □

Exponential stability in $(\hat{w}, \tilde{w})$-variables is obtained. Then, from regularity of the transformations (3.137), (3.141) and their inverses, we obtain exponential stability in $(\hat{u}, \tilde{u})$-variables, and therefore in stability in $(u, \hat{u})$-variables. Hence, Theorem 3.3 holds.

3.4 Notes and references

In this chapter, state feedback boundary controllers and output feedback boundary controllers by means of non-collocated observed state are designed for stabilizing a class of LKdVAD systems by applying the PDE backstepping method. The resulting closed-loop control systems are exponentially stable with arbitrarily prescribed decay rates. Note that the results derived in this chapter can be extended to the nonlinear KdV equations and derive locally stabilizing results. In the meantime, a challenging problem is to consider stabilizing the nonlinear KdV equations.

The exponential decay rates for the closed-loop control systems derived in this paper are not always optimal. Thus, for future work, another open problem is to derive optimal decay rates for the systems. Stabilization of LKdVAD systems with spatially varying coefficients or with unknown parameters can also be an interesting problem. Moreover, an ongoing work is to consider control design for cascaded/coupled LKdVAD-ODE systems, such as

$$\dot{X}(t) = AX(t) + Bu(t,0), \tag{3.171}$$

$$u_t(t,x) = u_{xxx}(t,x) + \lambda_2 u_{xx}(t,x) + \lambda_1 u_x(t,x) + \lambda_0 u(t,x), x \in (0,L), \tag{3.172}$$

$$u_x(t,0) = q_1 u(t,0) + CX(t), \tag{3.173}$$

$$u_{xx}(t,0) = q_2 u(t,0), \tag{3.174}$$

$$u(t,L) = U(t). \tag{3.175}$$

Chapter 3 contains reprints or adaptions of the following papers: 1. S.-X. Tang and M. Krstic, “Stabilization for linearized Korteweg-de Vries systems with anti-diffusion”, in *Proceedings of the American Control Conference*, pp. 3302-3307, Washington, D.C., USA, June 17-19, 2013. 2. S.-X. Tang and M. Krstic, “Stabi-

lization of linearized Korteweg-de Vries systems with anti-diffusion by boundary feedback with non-collocated observation", in *Proceedings of the American Control Conference*, pp. 1959-1964, Chicago, IL, USA, July 1-3, 2015. 3. S.-X. Tang and M. Krstic, "Stabilization of an anti-diffusive linear Korteweg-de Vries PDE," *IEEE Transactions on Automatic Control*, to be submitted. The dissertation author is the primary investigator and author of these papers, and would like to thank Miroslav Krstic for his contributions.

Part II

Coupled Hyperbolic Systems with Spatially Dependent Coefficients

Chapter 4

Stability of the Coupled Hyperbolic Systems with Spatially Dependent Coefficients

4.1 Introduction

4.1.1 Literature review

Coupled hyperbolic partial differential equations (PDE) systems have applications in a wide range of physical systems spanning fluid dynamics [61], population dynamics [62], structure dynamics [63], etc. Specifically, the coupled systems of first-order linear hyperbolic PDEs can serve as models for the applications of oil wells [3], transmission lines [2], road traffic [1], open channels [64], the dynamics of open-channel hydraulic systems [65, 66], and so on.

We consider in this paper a coupled system of n leftwards and m rightwards convecting transport PDEs with spatially varying coefficients. This system can be used to model the dynamics of open-channel hydraulic systems, e.g., estuaries, rivers and irrigation canals.

4.1.2 Organization

The outline of this chapter is as follows. Section 4.2 is dedicated to a preliminary stability analysis of a coupled system which consists of n leftwards and m rightwards first-order PDEs with spatially varying coefficients. Then, Section 4.3 investigates an exponentially stable cascaded system which also consists of $n+m$ transport PDEs with spatially varying coefficients. Finally, some conclusion remarks are given in Section 4.4.

4.2 Stability analysis of a coupled hyperbolic system

4.2.1 Problem statement

Consider the following coupled system of n leftwards and m rightwards transport PDEs (see, Figure 4.1)[1] with spatially varying coefficients:

$$\boldsymbol{u}_t(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{u}_x(t,x) = \boldsymbol{S}^{uu}(x)\boldsymbol{u}(t,x) + \boldsymbol{S}^{uv}(x)\boldsymbol{v}(t,x), \tag{4.1}$$

$$\boldsymbol{v}_t(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{v}_x(t,x) = \boldsymbol{S}^{vu}(x)\boldsymbol{u}(t,x) + \boldsymbol{S}^{vv}(x)\boldsymbol{v}(t,x), \tag{4.2}$$

$$\boldsymbol{u}(t,0) = \boldsymbol{Q}\boldsymbol{v}(t,0), \tag{4.3}$$

$$\boldsymbol{v}(t,1) = \boldsymbol{R}\boldsymbol{u}(t,1), \tag{4.4}$$

$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), \tag{4.5}$$

$$\boldsymbol{v}(0,x) = \boldsymbol{v}_0(x), \tag{4.6}$$

where

$$\boldsymbol{u}(t,x) = \Big[u_1(t,x),\, u_2(t,x),\, \ldots,\, u_n(t,x)\Big]^T, \tag{4.7}$$

$$\boldsymbol{v}(t,x) = \Big[v_1(t,x),\, v_2(t,x),\, \ldots,\, v_m(t,x)\Big]^T \tag{4.8}$$

are the system states with $t > 0,\ x \in [0,1]$. The following spatially varying matrices stand for the state transport speeds:

$$\boldsymbol{\Lambda}^{\mathbf{r}}(x) = \operatorname{diag}\Big[\lambda^{\mathrm{r}}_1(x)\,,\, \lambda^{\mathrm{r}}_2(x)\,,\, \cdots\,,\, \lambda^{\mathrm{r}}_n(x)\Big] \in \mathcal{M}_n(\mathbb{R}), \tag{4.9}$$

[1]The term of $\boldsymbol{v}(x,t)$ in the $\boldsymbol{v}$-equation (4.2) can also be removed easily, see, [67, Remark 1].

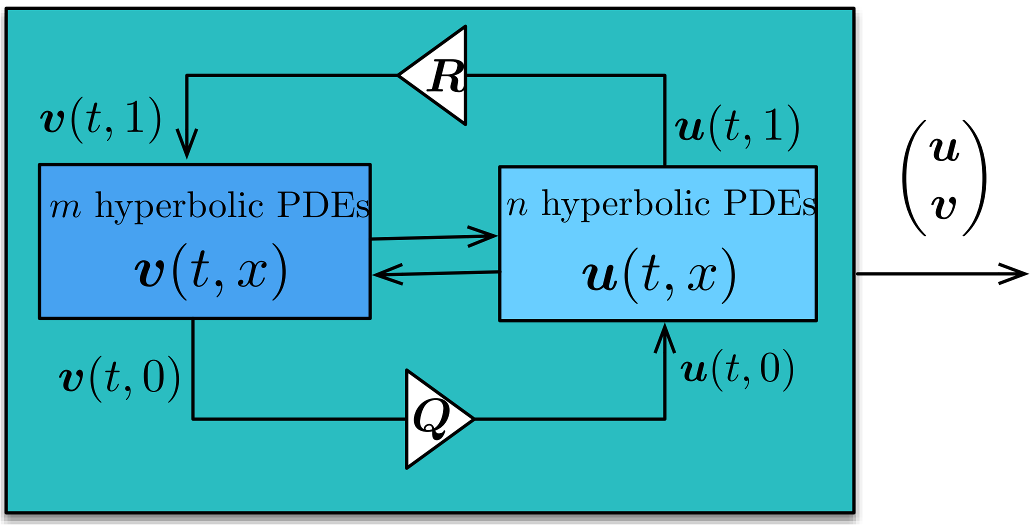

Figure 4.1: Block diagram of the coupled hyperbolic systems.

$$\mathbf{\Lambda^l}(x) = \operatorname{diag}\left[\lambda_1^{\mathrm{l}}(x)\,,\,\lambda_2^{\mathrm{l}}(x)\,,\,\cdots\,,\,\lambda_m^{\mathrm{l}}(x)\right] \in \mathcal{M}_m(\mathbb{R}), \tag{4.10}$$

where[2]

$$0 < \lambda_1^{\mathrm{r}}(x) < \lambda_2^{\mathrm{r}}(x) < \cdots < \lambda_n^{\mathrm{r}}(x), \tag{4.11}$$

$$-\lambda_m^{\mathrm{l}}(x) < -\lambda_2^{\mathrm{l}}(x) \cdots < -\lambda_1^{\mathrm{l}}(x) < 0,\ \forall x \in [0,1]. \tag{4.12}$$

And the in-domain parameters are also organized in spatially varying matrices:

$$\boldsymbol{S}^{uu}(x) = \left(S_{ij}^{uu}(x)\right)_{1\le i\le n, 1\le j\le n}, \tag{4.13}$$

$$\boldsymbol{S}^{uv}(x) = \left(S_{ij}^{uv}(x)\right)_{1\le i\le n, 1\le j\le m}, \tag{4.14}$$

$$\boldsymbol{S}^{vu}(x) = \left(S_{ij}^{vu}(x)\right)_{1\le i\le m, 1\le j\le n}, \tag{4.15}$$

$$\boldsymbol{S}^{vv}(x) = \left(S_{ij}^{vv}(x)\right)_{1\le i\le m, 1\le j\le m}. \tag{4.16}$$

The independent matrices $\boldsymbol{Q}$, $\boldsymbol{R}$ of boundary parameters are given as

$$\boldsymbol{Q} = \{q_{ij}\} \in \mathcal{M}_{n,m}(\mathbb{R}), \tag{4.17}$$

$$\boldsymbol{R} = \{r_{ij}\} \in \mathcal{M}_{m,n}(\mathbb{R}). \tag{4.18}$$

All the investigation will be conducted under the following assumption.

Assumption 4.1.

1. $\mathbf{\Lambda^r}(x) \in C^1\left(0,1;\mathcal{M}_n(\mathbb{R})\right), \mathbf{\Lambda^l}(x) \in C^1\left(0,1;\mathcal{M}_m(\mathbb{R})\right)$, and we denote

$$\underline{\lambda} := \min\left\{\lambda_i^r(x), \lambda_j^l(x);\ x \in [0,1],\ i = \overline{1,n},\ j = \overline{1,m}\right\}, \tag{4.19}$$

$$\overline{\lambda} := \max\left\{\lambda_i^r(x), \lambda_j^l(x);\ x \in [0,1],\ i = \overline{1,n},\ j = \overline{1,m}\right\}. \tag{4.20}$$

2. $\boldsymbol{S}^{uu}(x) \in L^\infty\left(0,1;\mathcal{M}_n(\mathbb{R})\right)$, $\boldsymbol{S}^{uv}(x) \in L^\infty\left(0,1;\mathcal{M}_{n,m}(\mathbb{R})\right)$,
$\boldsymbol{S}^{vu}(x) \in L^\infty\left(0,1;\mathcal{M}_{m,n}(\mathbb{R})\right)$, $\boldsymbol{S}^{vv}(x) \in L^\infty\left(0,1;\mathcal{M}_m(\mathbb{R})\right)$, and we denote

$$M_{\boldsymbol{S}} := \max_{x\in[0,1]}\{\|\boldsymbol{S}^{uu}(x)\|, \|\boldsymbol{S}^{uv}(x)\|\}, \tag{4.21}$$

3. $\boldsymbol{Q} \in \mathcal{M}_{n,m}(\mathbb{R})$, $\boldsymbol{R} \in \mathcal{M}_{m,n}(\mathbb{R})$, and we denote

$$\bar{q} := \|\boldsymbol{Q}^T\boldsymbol{Q}\|. \tag{4.22}$$

Here and in the sequel, $\|\cdot\|$ denotes the 2−norm of a matrix.

[2]Isotachic cases (i.e., $\lambda_i^r = \lambda_j^r$, or, $\lambda_i^l = \lambda_j^l$ for some $i \neq j$) can also be considered, see, [67, Remark 2].

4.2.2 Spectrum analysis

To analyze the system (4.1)–(4.6), we consider the state Hilbert space $\boldsymbol{H} = (L^2(0,1))^{n+m}$. Define an operator $\mathbf{A} : \mathrm{Dom}(\mathbf{A})(\subset \boldsymbol{H}) \to \boldsymbol{H}$ as follows:

$$\begin{aligned}
&\mathbf{A}(\boldsymbol{f}^T(x), \boldsymbol{g}^T(x))^T \\
&= \Big([-\boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{f}'(x) + \boldsymbol{S}^{uu}(x)\boldsymbol{f}(x) + \boldsymbol{S}^{uv}(x)\boldsymbol{g}(x)]^T, \\
&\quad \big[\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{g}'(x) + \boldsymbol{S}^{vu}(x)\boldsymbol{f}(x) + \boldsymbol{S}^{vv}(x)\boldsymbol{g}(x)\big]^T \Big)^T, \ \forall (\boldsymbol{f}^T, \boldsymbol{g}^T)^T \in \mathrm{Dom}(\mathbf{A}), \quad (4.23)
\end{aligned}$$

$$\mathrm{Dom}(\mathbf{A}) = \{(\boldsymbol{f}^T, \boldsymbol{g}^T)^T \in \big(H^1(0,1)\big)^{n+m};\ \boldsymbol{f}(0) = \boldsymbol{Q}\boldsymbol{g}(0), \boldsymbol{g}(1) = \boldsymbol{R}\boldsymbol{f}(1)\}. \quad (4.24)$$

Lemma 4.1. *If*

$$\begin{pmatrix} \boldsymbol{R} & -\boldsymbol{I}_m \end{pmatrix} e^{\int_0^1 \begin{pmatrix} [\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uu}(s) & [\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uv}(s) \\ -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vu}(s) & -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vv}(s) \end{pmatrix} \mathrm{d}s} \begin{pmatrix} \boldsymbol{Q} \\ \boldsymbol{I} \end{pmatrix}$$

is nonsingular, where $\boldsymbol{I}$ denotes the unit matrix, then $\mathbf{A}^{-1}$ exists and is compact on $\mathcal{H}$. Hence, $\sigma(\mathbf{A})$, the spectrum of $\mathbf{A}$, consists of isolated eigenvalues only: $\sigma(\mathbf{A}) = \sigma_p(\mathbf{A})$, where $\sigma_p(\mathbf{A})$ denotes the set of eigenvalues of $\mathbf{A}$. Moreover, each $\lambda \in \sigma(\mathbf{A})$ is geometrically simple and satisfies the characteristic equation

$$\begin{pmatrix} \boldsymbol{R} & -\boldsymbol{I}_m \end{pmatrix} e^{\int_0^1 \begin{pmatrix} [\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}[\boldsymbol{S}^{uu}(s) - \lambda] & [\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uv}(s) \\ -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vu}(s) & -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}[\boldsymbol{S}^{vv}(s) - \lambda] \end{pmatrix} \mathrm{d}s} \begin{pmatrix} \boldsymbol{Q} \\ \boldsymbol{I} \end{pmatrix} = 0, \quad (4.25)$$

An eigenfunction f corresponding to λ is

$$\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix}(x) = e^{\int_0^x \begin{pmatrix} [\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}[\boldsymbol{S}^{uu}(s) - \lambda] & [\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uv}(s) \\ -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vu}(s) & -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}[\boldsymbol{S}^{vv}(s) - \lambda] \end{pmatrix} \mathrm{d}s} \begin{pmatrix} \boldsymbol{Q} \\ \boldsymbol{I} \end{pmatrix} \boldsymbol{g}(0). \quad (4.26)$$

Proof. (Part 1) By solving

$$\mathbf{A}\begin{pmatrix} \boldsymbol{f}_1^T & \boldsymbol{g}_1^T \end{pmatrix}^T = \begin{pmatrix} \boldsymbol{f}^T & \boldsymbol{g}^T \end{pmatrix}^T, \ \begin{pmatrix} \boldsymbol{f}_1^T & \boldsymbol{g}_1^T \end{pmatrix}^T \in \mathrm{Dom}(\mathbf{A}), \quad (4.27)$$

we get

$$\mathbf{A}^{-1}\begin{pmatrix}\boldsymbol{f}^T & \boldsymbol{g}^T\end{pmatrix}^T = \begin{pmatrix}\boldsymbol{f}_1^T & \boldsymbol{g}_1^T\end{pmatrix}^T, \ \forall \begin{pmatrix}\boldsymbol{f}^T & \boldsymbol{g}^T\end{pmatrix}^T \in \mathbf{H}, \tag{4.28}$$

$$\begin{aligned}\begin{pmatrix}\boldsymbol{f}_1\\ \boldsymbol{g}_1\end{pmatrix}(x) &= e^{\int_0^x \begin{pmatrix}[\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uu}(s) & [\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uv}(s)\\ -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vu}(s) & -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vv}(s)\end{pmatrix}\mathrm{d}s}\begin{pmatrix}\boldsymbol{Q}\\ \boldsymbol{I}\end{pmatrix}\boldsymbol{g}_1(0)\\ &\quad + \int_0^x e^{\int_\tau^x \begin{pmatrix}[\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uu}(s) & [\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uv}(s)\\ -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vu}(s) & -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vv}(s)\end{pmatrix}\mathrm{d}s}\begin{pmatrix}\boldsymbol{f}(\tau)\\ \boldsymbol{g}(\tau)\end{pmatrix}\mathrm{d}\tau,\end{aligned} \tag{4.29}$$

where

$$\begin{aligned}\boldsymbol{g}_1(0) &= -\left[\begin{pmatrix}\boldsymbol{R} & -\boldsymbol{I}_m\end{pmatrix} e^{\int_0^1 \begin{pmatrix}[\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uu}(s) & [\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uv}(s)\\ -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vu}(s) & -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vv}(s)\end{pmatrix}\mathrm{d}s}\begin{pmatrix}\boldsymbol{Q}\\ \boldsymbol{I}\end{pmatrix}\right]^{-1}\\ &\times \begin{pmatrix}\boldsymbol{R} & -\boldsymbol{I}_m\end{pmatrix}\int_0^1 e^{\int_\tau^1 \begin{pmatrix}[\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uu}(s) & [\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uv}(s)\\ -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vu}(s) & -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vv}(s)\end{pmatrix}\mathrm{d}s}\begin{pmatrix}\boldsymbol{f}(\tau)\\ \boldsymbol{g}(\tau)\end{pmatrix}\mathrm{d}\tau.\end{aligned} \tag{4.30}$$

That is, we derive the unique solution $(\boldsymbol{f}_1^T, \boldsymbol{g}_1^T)^T \in \mathrm{Dom}(\mathbf{A})$, and thus, $\mathbf{A}^{-1}$ exists and is compact on $\mathbf{H}$ by the Sobolev embedding theorem. Therefore, $\sigma(\mathbf{A})$, the spectrum of $\mathbf{A}$, consists of isolated eigenvalues only.

(Part 2) For any $\lambda \in \sigma_p(\mathbf{A})$, we have

$$\mathbf{A}\begin{pmatrix}\boldsymbol{f}(x)\\ \boldsymbol{g}(x)\end{pmatrix} = \begin{pmatrix}-\boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{f}'(x) + \boldsymbol{S}^{uu}(x)\boldsymbol{f}(x) + \boldsymbol{S}^{uv}(x)\boldsymbol{g}(x)\\ \boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{g}'(x) + \boldsymbol{S}^{vu}(x)\boldsymbol{f}(x) + \boldsymbol{S}^{vv}(x)\boldsymbol{g}(x)\end{pmatrix} = \lambda\begin{pmatrix}\boldsymbol{f}(x)\\ \boldsymbol{g}(x)\end{pmatrix}, \tag{4.31}$$

$$\boldsymbol{f}(0) = \boldsymbol{Q}\boldsymbol{g}(0), \boldsymbol{g}(1) = \boldsymbol{R}\boldsymbol{f}(1), \tag{4.32}$$

which has at least one nonzero solution $(\boldsymbol{f}^T, \boldsymbol{g}^T)^T$. If it has two linearly independent solutions $(\boldsymbol{f}_1^T, \boldsymbol{g}_1^T)^T, (\boldsymbol{f}_2^T, \boldsymbol{g}_2^T)^T$, then there exist two non-singular constant matrices A, B such that

$$A\boldsymbol{g}_1(0) + B\boldsymbol{g}_2(0) = 0. \tag{4.33}$$

Thus,

$$\begin{pmatrix} \boldsymbol{f}(x) \\ \boldsymbol{g}(x) \end{pmatrix} = A \begin{pmatrix} \boldsymbol{f}_1(x) \\ \boldsymbol{g}_2(x) \end{pmatrix} + B \begin{pmatrix} \boldsymbol{f}_1(x) \\ \boldsymbol{g}_2(x) \end{pmatrix} \tag{4.34}$$

satisfies

$$\mathbf{A} \begin{pmatrix} \boldsymbol{f}(x) \\ \boldsymbol{g}(x) \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{f}'(x) + \boldsymbol{S}^{uu}(x)\boldsymbol{f}(x) + \boldsymbol{S}^{uv}(x)\boldsymbol{g}(x) \\ \boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{g}'(x) + \boldsymbol{S}^{vu}(x)\boldsymbol{f}(x) + \boldsymbol{S}^{vv}(x)\boldsymbol{g}(x) \end{pmatrix} = \lambda \begin{pmatrix} \boldsymbol{f}(x) \\ \boldsymbol{g}(x) \end{pmatrix}, \tag{4.35}$$

$$\boldsymbol{f}(0) = \mathbf{0}, \boldsymbol{g}(0) = \mathbf{0}, \boldsymbol{g}(1) = \boldsymbol{R}\boldsymbol{f}(1), \tag{4.36}$$

which has only zero solution. Hence,

$$A \begin{pmatrix} \boldsymbol{f}_1(x) \\ \boldsymbol{g}_2(x) \end{pmatrix} + B \begin{pmatrix} \boldsymbol{f}_1(x) \\ \boldsymbol{g}_2(x) \end{pmatrix} \equiv 0, \tag{4.37}$$

which contradicts with the assumption. Therefore, each $\lambda \in \sigma_p(\mathcal{A})$ is geometrically simple.

(Part 3) For any $\lambda \in \sigma_p(\mathcal{A})$, from (4.31), we have

$$\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix}(x) = e^{\int_0^x \begin{pmatrix} [\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}[\boldsymbol{S}^{uu}(s) - \lambda] & [\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{S}^{uv}(s) \\ -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}\boldsymbol{S}^{vu}(s) & -[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]^{-1}[\boldsymbol{S}^{vv}(s) - \lambda] \end{pmatrix} \mathrm{d}s} \begin{pmatrix} \boldsymbol{f}(0) \\ \boldsymbol{g}(0) \end{pmatrix}. \tag{4.38}$$

From (4.32), we get the characteristic equation (4.25). We can also derive the corresponding eigenfunction (4.26). □

It can be concluded from Lemma 4.1 that the spectrum of the system (4.1)–(4.6) depends on the system matrices, and that the system is not necessarily stable in the general cases. In what follows we give an example of exponentially stable systems, which is a non-local cascaded system of n leftwards and m rightwards transport PDEs with spatially varying coefficients. The exponential stability of this cascaded system will provide some insights for controller designs of the coupled system.

4.3 Exponential stability of a cascaded hyperbolic system

4.3.1 Problem statement

Consider the following system (see, Figure 4.2):

$$\begin{aligned}\partial_t \boldsymbol{u}(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x \boldsymbol{u}(t,x) &= \boldsymbol{S}^{uu}(x)\boldsymbol{u}(t,x) + \boldsymbol{S}^{uv}(x)\boldsymbol{v}(t,x)\\ &+ \int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)\boldsymbol{u}(t,\xi)\,\mathrm{d}\xi + \int_0^x \boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{v}(t,\xi)\,\mathrm{d}\xi, \end{aligned} \tag{4.39}$$

$$\partial_t \boldsymbol{v}(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x \boldsymbol{v}(t,x) = \boldsymbol{\Delta}(x)\boldsymbol{v}(t,0), \tag{4.40}$$

$$\boldsymbol{u}(t,0) = \boldsymbol{Q}\boldsymbol{v}(t,0), \tag{4.41}$$

$$\boldsymbol{v}(t,1) = \boldsymbol{0}, \tag{4.42}$$

where the matrix

$$\boldsymbol{\Delta}(x) = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \delta_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \delta_{m,1}(x) & \cdots & \delta_{m,m-1}(x) & 0 \end{bmatrix} \tag{4.43}$$

guarantees that

$$\int_0^1 [\boldsymbol{\Lambda}^{\mathrm{l}}(\xi)]^{-1}\boldsymbol{\Delta}(\xi)\,\mathrm{d}\xi - \boldsymbol{I}$$

is nonsingular. The matrices of functions $\boldsymbol{\Lambda}^{\mathrm{r}}(x)$, $\boldsymbol{\Lambda}^{\mathrm{l}}(x)$, $\boldsymbol{S}^{uu}(x)$, $\boldsymbol{S}^{uv}(x)$ and the constant matrix $\mathbf{Q}$ satisfy Assumption 4.1. $\boldsymbol{C}^{\mathrm{r}}$, $\boldsymbol{C}^{\mathrm{l}}$ are L^∞ matrices of functions defined on the triangular domain

$$\mathbb{T} = \Big\{(x,\xi) \in \mathbb{R}^2 |\ 0 \le \xi \le x \le 1\Big\},$$

and we denote

$$M_{\boldsymbol{C}} := \max_{(x,\xi)\in\mathbb{T}} \{\|\boldsymbol{C}^r(x,\xi)\|_{L^\infty(\mathbb{T})}, \|\boldsymbol{C}^l(x,\xi)\|_{L^\infty(\mathbb{T})}\}, \tag{4.44}$$

where

$$\|\cdot\|_{L^\infty(\mathbb{T})} = \operatorname{ess}\sup_{\mathbb{T}} |\cdot|. \tag{4.45}$$

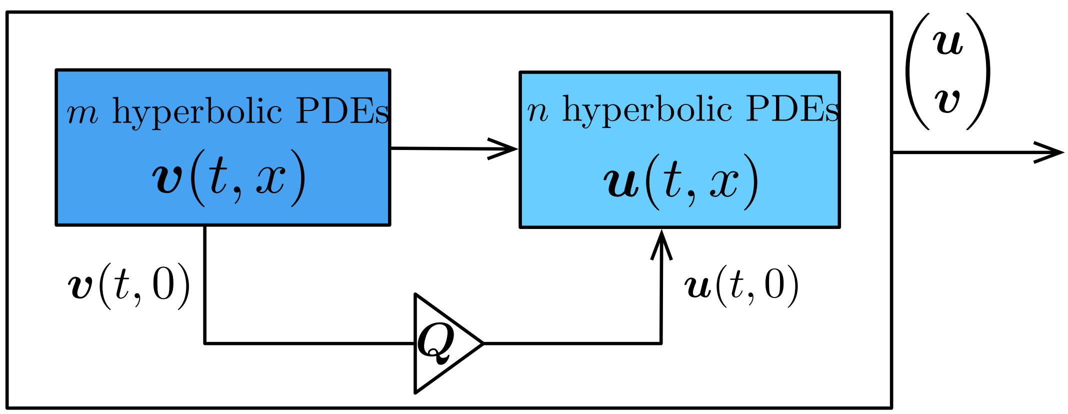

Figure 4.2: Block diagram of the cascaded hyperbolic systems.

Note that compared with the system (4.1)–(4.6), this system (4.39)–(4.42) is cascaded in the sense that $\boldsymbol{v}(t,x)$-system evolves by itself and then the state information goes into the $\boldsymbol{u}(t,x)$-system.

4.3.2 Stability analysis

Consider the inner product given by

$$\begin{aligned}
&< (\boldsymbol{f}_1^T,\ \boldsymbol{g}_1^T)^T, (\boldsymbol{f}_2^T,\ \boldsymbol{g}_2^T)^T >_{\boldsymbol{H}} \\
&= \int_0^1 \left(e^{-\nu x} \boldsymbol{f}_1(x)^T \left[\boldsymbol{\Lambda}^{\mathrm{r}}(x)\right]^{-1} \overline{\boldsymbol{f}_2(x)} + (1+x)\boldsymbol{g}_1(x)^T \boldsymbol{D} \left[\boldsymbol{\Lambda}^{\mathrm{l}}(x)\right]^{-1} \overline{\boldsymbol{g}_2(x)} \right) \mathrm{d}x, \\
&\quad \forall\ (\boldsymbol{f}_1^T,\ \boldsymbol{g}_1^T)^T, (\boldsymbol{f}_2^T,\ \boldsymbol{g}_2^T)^T \in \boldsymbol{H},
\end{aligned} \tag{4.46}$$

where the parameter ν is chosen as a positive constant large enough to satisfy the following inequality:

$$\nu - \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 - \left(\frac{M_{\boldsymbol{C}}}{\underline{\lambda}}\right)^2 \left(\frac{1}{\nu} + 1\right) - 3 > 0, \tag{4.47}$$

and the matrix

$$\boldsymbol{D} := \mathrm{diag}\{d_1\,,\ d_2\,,\ \cdots\,,\ d_{m-1}\,,\ d_m\} > \boldsymbol{0} \tag{4.48}$$

contains the positive weight coefficients $d_1,\ d_2\,,\ \cdots,\ d_{m-1},\ d_m$ which are chosen successively as follows:

$$d_m \geq \max\left\{\bar{q}, \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + \frac{1}{\nu} + 2(m-1)\right\}, \tag{4.49}$$

$$d_{m-1} \geq \max\left\{\bar{q} + \int_0^1 (1+x) d_m^2 \frac{1}{\lambda_m^{\mathrm{l}}(x)^2} \delta_{m,m-1}^2(x)\,\mathrm{d}x, \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + \frac{1}{\nu} + 2(m-1)\right\}, \tag{4.50}$$

$$\begin{aligned}
d_{m-2} \geq \max\Bigg\{&\bar{q} + \int_0^1 (1+x) \sum_{i=m-1}^{m} d_i^2 \frac{1}{\lambda_i^{\mathrm{l}}(x)^2} \delta_{i,m-2}^2(x)\,\mathrm{d}x, \\
&\left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + \frac{1}{\nu} + 2(m-1)\Bigg\},
\end{aligned} \tag{4.51}$$

$$\vdots$$

$$d_1 \geq \max \left\{ \bar{q} + \int_0^1 (1+x) \sum_{i=2}^{m} d_i^2 \frac{1}{\lambda_i^{\mathrm{l}}(x)^2} \delta_{i,1}^2(x)\, \mathrm{d}x, \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}} \right)^2 + \frac{1}{\nu} + 2(m-1) \right\}. \tag{4.52}$$

Denote $\|\cdot\|_{\boldsymbol{H}}$ as the norm induced by the inner product (4.46), then a simple derivation gives the following inequality stating the equivalence between the $\|\cdot\|_{\boldsymbol{H}}$-norm and the usual L^2-norm:

$$\begin{aligned} C_1 \|(\boldsymbol{f}^T(t,\cdot),\ \boldsymbol{g}^T(t,\cdot))^T\|^2_{(L^2(0,1))^{n+m}} \\ \leq \left\| \left(\boldsymbol{f}^T(t,\cdot),\ \boldsymbol{g}^T(t,\cdot) \right)^T \right\|^2_{\boldsymbol{H}} \\ \leq C_2 \|(\boldsymbol{f}^T(t,\cdot),\ \boldsymbol{g}^T(t,\cdot))^T\|^2_{(L^2(0,1))^{n+m}}, \end{aligned} \tag{4.53}$$

where

$$\|(\boldsymbol{f}^T,\ \boldsymbol{g}^T)^T\|^2_{(L^2(0,1))^{n+m}} := \int_0^1 \left(\boldsymbol{f}^T \overline{\boldsymbol{f}} + \boldsymbol{g}^T \overline{\boldsymbol{g}} \right) \mathrm{d}x, \tag{4.54}$$

and

$$C_1 := \frac{1}{\overline{\lambda}} \min \left\{ e^{-\nu},\ d_i; i = \overline{1,m} \right\} > 0, \tag{4.55}$$

$$C_2 := \frac{1}{\underline{\lambda}} \max \left\{ 1,\ 2d_i; i = \overline{1,m} \right\} > 0. \tag{4.56}$$

Define an operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A})(\subset \boldsymbol{H}) \to \boldsymbol{H}$ as follows:

$$\begin{aligned} \mathcal{A}(\boldsymbol{f}^T(x), \boldsymbol{g}^T(x))^T = \Big([-\boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{f}'(x) + \boldsymbol{S}^{uu}(x)\boldsymbol{f}(x) + \boldsymbol{S}^{uv}(x)\boldsymbol{g}(x) \\ + \int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)\boldsymbol{f}(\xi)\,\mathrm{d}\xi + \int_0^x \boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{g}(\xi)\,\mathrm{d}\xi \Big]^T, \\ \left[\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{g}' + \boldsymbol{\Delta}(x)\boldsymbol{g}(0) \right]^T \Big)^T,\ \forall (\boldsymbol{f}, \boldsymbol{g})^T \in \mathrm{Dom}(\mathcal{A}), \end{aligned} \tag{4.57}$$

$$\mathrm{Dom}(\mathcal{A}) = \{ (\boldsymbol{f}^T, \boldsymbol{g}^T)^T \in \left(H^1(0,1) \right)^{n+m};\ \boldsymbol{f}(0) = \boldsymbol{Q}\boldsymbol{g}(0), \boldsymbol{g}(1) = \boldsymbol{0} \}, \tag{4.58}$$

for which the adjoint operator is

$$\begin{aligned} &\mathcal{A}^*(\boldsymbol{\phi}^T(x), \boldsymbol{\psi}^T(x))^T \\ &= \Bigg(\Big[\boldsymbol{\Lambda}^{\mathrm{r}}(x)\{ -\nu\boldsymbol{\phi}(x) + \boldsymbol{\phi}'(x) + [\boldsymbol{S}^{uu}(x)]^T [\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\boldsymbol{\phi}(x) \} \\ &\qquad + e^{\nu x}\boldsymbol{\Lambda}^{\mathrm{r}}(x) \int_x^1 e^{-\nu\xi} [\boldsymbol{C}^{\mathrm{r}}(\xi,x)]^T [\boldsymbol{\Lambda}^{\mathrm{r}}(\xi)]^{-1} \boldsymbol{\phi}(\xi)\,\mathrm{d}\xi \Big]^T, \end{aligned}$$

$$\left[\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{D}^{-1}\frac{e^{-\nu x}}{1+x}\{[\boldsymbol{S}^{uv}(x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\boldsymbol{\phi}(x)\}\right.$$

$$+\frac{1}{1+x}\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{D}^{-1}\int_x^1 e^{-\nu\xi}[\boldsymbol{C}^{\mathrm{l}}(\xi,x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(\xi)]^{-1}\boldsymbol{\phi}(\xi)\,\mathrm{d}\xi$$

$$\left.\left.-\frac{1}{1+x}\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{\psi}(x)-\boldsymbol{\Lambda}^{\mathbf{l}}(x)\boldsymbol{\psi}'(x)\right]^T\right)^T,\ \forall(\boldsymbol{\phi},\boldsymbol{\psi})^T\in\mathrm{Dom}(\mathcal{A}^*), \tag{4.59}$$

$$\mathrm{Dom}(\mathcal{A}^*)=\left\{\left(\boldsymbol{\phi}^T,\boldsymbol{\psi}^T\right)^T\in\left(H^1(0,1)\right)^2;\right.$$

$$\left.\boldsymbol{Q}^T\boldsymbol{\phi}(0)=\boldsymbol{D}\boldsymbol{\psi}(0)-\int_0^1(1+x)\boldsymbol{\Delta}(x)^T\boldsymbol{D}[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\boldsymbol{\psi}(x)\,\mathrm{d}x,\ \boldsymbol{\phi}(1)=\mathbf{0}\right\}. \tag{4.60}$$

Taking the inner product with $(\boldsymbol{\phi}^T,\boldsymbol{\psi}^T)^T\in\mathrm{Dom}(\mathcal{A}^*)$ on both sides of (4.39)–(4.42), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\begin{pmatrix}\boldsymbol{u}(t,\cdot)\\\boldsymbol{v}(t,\cdot)\end{pmatrix},\begin{pmatrix}\boldsymbol{\phi}\\\boldsymbol{\psi}\end{pmatrix}\right\rangle_{\boldsymbol{H}}=\left\langle\begin{pmatrix}\boldsymbol{u}\\\boldsymbol{v}\end{pmatrix},\mathcal{A}^*\begin{pmatrix}\boldsymbol{\phi}\\\boldsymbol{\psi}\end{pmatrix}\right\rangle_{\boldsymbol{H}}, \tag{4.61}$$

then the system (4.39)–(4.42) can be written into the following abstract form in $\boldsymbol{H}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{u}\\\boldsymbol{v}\end{pmatrix}=\mathcal{A}\begin{pmatrix}\boldsymbol{u}\\\boldsymbol{v}\end{pmatrix}. \tag{4.62}$$

Lemma 4.2. *Assume that the constant*

$$C_0:=\int_0^1\int_0^x\left\|\int_s^x[\boldsymbol{\Lambda}^{\mathrm{r}}(\xi)]^{-1}\boldsymbol{C}^{\mathrm{r}}(\xi,s)\,\mathrm{d}\xi e^{\int_x^s[\boldsymbol{\Lambda}^{\mathrm{r}}(\eta)]^{-1}\boldsymbol{S}^{\mathrm{r}}(\eta)\,\mathrm{d}\eta}\right\|^2\mathrm{d}s\,\mathrm{d}x<1, \tag{4.63}$$

then $\mathcal{A}^{-1}$ exists and is compact on $\boldsymbol{H}$.

Proof. For any given $(\boldsymbol{f}^T,\boldsymbol{g}^T)^T\in\boldsymbol{H}$, we consider the existence of $(\boldsymbol{f}_1^T,\boldsymbol{g}_1^T)^T\in\mathrm{Dom}(\mathcal{A})$ such that

$$\mathcal{A}(\boldsymbol{f}_1^T,\boldsymbol{g}_1^T)^T=(\boldsymbol{f}^T,\boldsymbol{g}^T)^T. \tag{4.64}$$

First, for the $\boldsymbol{g}_1(x)$-equation:

$$\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{g}_1'+\boldsymbol{\Delta}(x)\boldsymbol{g}_1(0)=\boldsymbol{g}, \tag{4.65}$$

$$\boldsymbol{g}_1(1)=\mathbf{0}, \tag{4.66}$$

it can be derived that there exists a unique solution:

$$\boldsymbol{g}_1(x) = \int_x^1 [\boldsymbol{\Lambda}^{\mathrm{l}}(\xi)]^{-1}[\boldsymbol{\Delta}(\xi)\boldsymbol{g}_1(0) - \boldsymbol{g}(\xi)]\,\mathrm{d}\xi, \tag{4.67}$$

where

$$\boldsymbol{g}_1(0) = \left[\int_0^1 [\boldsymbol{\Lambda}^{\mathrm{l}}(\xi)]^{-1}\boldsymbol{\Delta}(\xi)\,\mathrm{d}\xi - \boldsymbol{I}\right]^{-1} \int_0^1 [\boldsymbol{\Lambda}^{\mathrm{l}}(\xi)]^{-1}\boldsymbol{g}(\xi)\,\mathrm{d}\xi. \tag{4.68}$$

Second, $\boldsymbol{f}_1(x)$ needs to satisfy the following equation:

$$\begin{aligned} & -\boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{f}_1'(x) + \boldsymbol{S}^{uu}(x)\boldsymbol{f}_1(x) + \boldsymbol{S}^{uv}(x)\boldsymbol{g}_1(x) \\ & + \int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)\boldsymbol{f}_1(\xi)\,\mathrm{d}\xi + \int_0^x \boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{g}_1(\xi)\,\mathrm{d}\xi = \boldsymbol{f}, \quad (4.69) \\ & \boldsymbol{f}_1(0) = \boldsymbol{Q}\boldsymbol{g}_1(0). \quad (4.70) \end{aligned}$$

Let

$$\boldsymbol{F}_1(x) = e^{-\int_0^x [\boldsymbol{\Lambda}^{\mathrm{r}}(\xi)]^{-1}\boldsymbol{S}^{uu}(\xi)\,\mathrm{d}\xi}\boldsymbol{f}_1(x), \tag{4.71}$$

then $\boldsymbol{F}_1(x)$ satisfies

$$\begin{aligned} & -\boldsymbol{\Lambda}^{\mathrm{r}}(x)e^{\int_0^x [\boldsymbol{\Lambda}^{\mathrm{r}}(\xi)]^{-1}\boldsymbol{S}^{uu}(\xi)\,\mathrm{d}\xi}\boldsymbol{F}_1'(x) + \int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)e^{\int_0^\xi [\boldsymbol{\Lambda}^{\mathrm{r}}(\eta)]^{-1}\boldsymbol{S}^{uu}(\eta)\,\mathrm{d}\eta}\boldsymbol{F}_1(\xi)\,\mathrm{d}\xi \\ & = \boldsymbol{f}(x) - \boldsymbol{S}^{uv}(x)\boldsymbol{g}_1(x) - \int_0^x \boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{g}_1(\xi)\,\mathrm{d}\xi \triangleq \boldsymbol{h}(x), \quad (4.72) \\ & \boldsymbol{F}_1(0) = \boldsymbol{Q}\boldsymbol{g}_1(0). \quad (4.73) \end{aligned}$$

This integro-differential equation (IDE) can be easily written into a Volterra integral:

$$\boldsymbol{F}_1(x) = \boldsymbol{Q}\boldsymbol{g}_1(0) + \int_0^x \left[(T\boldsymbol{F}_1)(s) - e^{-\int_0^s [\boldsymbol{\Lambda}^{\mathrm{r}}(\xi)]^{-1}\boldsymbol{S}^{uu}(\xi)\,\mathrm{d}\xi}[\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{h}(s)\right]\mathrm{d}s, \tag{4.74}$$

where

$$(T\boldsymbol{F}_1)(x) = [\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)e^{\int_x^\xi [\boldsymbol{\Lambda}^{\mathrm{r}}(\eta)]^{-1}\boldsymbol{S}^{uu}(\eta)\,\mathrm{d}\eta}\boldsymbol{F}_1(\xi)\,\mathrm{d}\xi. \tag{4.75}$$

Consider the operator $P : (L^2(0,1))^n \to (L^2(0,1))^n$ defined by

$$(P\cdot)(x) = \boldsymbol{Q}\boldsymbol{g}_1(0) + \int_0^x \left[(T\cdot)(s) - e^{-\int_0^s [\boldsymbol{\Lambda}^{\mathrm{r}}(\xi)]^{-1}\boldsymbol{S}^{uu}(\xi)\,\mathrm{d}\xi}[\boldsymbol{\Lambda}^{\mathrm{r}}(s)]^{-1}\boldsymbol{h}(s)\right]\mathrm{d}s, \tag{4.76}$$

then it is easy to get that

$$\|P\boldsymbol{F}_1^1 - P\boldsymbol{F}_1^2\|_{(L^2(0,1))^n} \leq \sqrt{C_0}\|\boldsymbol{F}_1^1 - \boldsymbol{F}_1^2\|_{(L^2(0,1))^n}. \tag{4.77}$$

Since $C_0 < 1$, the Banach fixed point theorem implies that P has a unique fixed point $\boldsymbol{F}_1$ in $(L^2(0,1))^n$, and this $\boldsymbol{F}_1$ is the unique solution to the initial-value problem (4.72)–(4.73). Therefore, (4.69)–(4.70) admits a unique solution $\boldsymbol{f}_1$ and (4.64) admits a unique solution $(\boldsymbol{f}_1^T,\ \boldsymbol{g}_1^T)^T$. Hence, $\mathcal{A}^{-1}$ exists and is compact on $\boldsymbol{H}$ by the Sobolev embedding theorem. □

Lemma 4.3. *$\mathcal{A}$ and $\mathcal{A}^*$ are dissipative in $\boldsymbol{H}$, and $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions in $\boldsymbol{H}$.*

Proof. Let $(\boldsymbol{f}^T,\ \boldsymbol{g}^T)^T \in \mathrm{Dom}(\mathcal{A})$, then

$$\begin{aligned}
&\mathrm{Re}\left\langle \mathcal{A}(\boldsymbol{f}^T,\ \boldsymbol{g}^T)^T,\ \ (\boldsymbol{f}^T,\ \boldsymbol{g}^T)^T\right\rangle_{\boldsymbol{H}} \\
&= -\frac{1}{2}e^{-\nu}|\boldsymbol{f}(1)|^2 + \frac{1}{2}\int_0^1 -\nu e^{-\nu x}|\boldsymbol{f}(x)|^2\,\mathrm{d}x \\
&\quad + \mathrm{Re}\int_0^1 e^{-\nu x}\boldsymbol{f}(x)^T[\boldsymbol{S}^{uu}(x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\overline{\boldsymbol{f}(x)}\,\mathrm{d}x \\
&\quad + \mathrm{Re}\int_0^1 e^{-\nu x}\boldsymbol{g}(x)^T[\boldsymbol{S}^{uv}(x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\overline{\boldsymbol{f}(x)}\,\mathrm{d}x \\
&\quad + \mathrm{Re}\int_0^1 e^{-\nu x}\int_0^x \boldsymbol{f}(\xi)^T[\boldsymbol{C}^{\mathrm{r}}(x,\xi)]^T\,\mathrm{d}\xi[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\overline{\boldsymbol{f}(x)}\,\mathrm{d}x \\
&\quad + \mathrm{Re}\int_0^1 e^{-\nu x}\int_0^x \boldsymbol{g}(\xi)^T[\boldsymbol{C}^{\mathrm{l}}(x,\xi)]^T\,\mathrm{d}\xi[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\overline{\boldsymbol{f}(x)}\,\mathrm{d}x \\
&\quad - \frac{1}{2}\boldsymbol{g}(0)^T(\boldsymbol{D} - \boldsymbol{Q}^T\boldsymbol{Q})\overline{\boldsymbol{g}(0)} \\
&\quad + \mathrm{Re}\int_0^1 (1+x)\boldsymbol{g}(0)^T\boldsymbol{\Delta}(x)^T\boldsymbol{D}[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\overline{\boldsymbol{g}(x)}\,\mathrm{d}x \\
&\quad - \frac{1}{2}\mathrm{Re}\int_0^1 \boldsymbol{g}(x)^T\boldsymbol{D}\overline{\boldsymbol{g}(x)}\,\mathrm{d}x,
\end{aligned} \tag{4.78}$$

where $|\cdot|$ denotes the Euclidean norm of a vector, and the equalities in (4.57)–(4.58) have been used.

It can be derived that

$$\mathrm{Re}\int_0^1 e^{-\nu x}\boldsymbol{f}(x)^T[\boldsymbol{S}^{uu}(x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\overline{\boldsymbol{f}(x)}\,\mathrm{d}x$$

$$\le \frac{1}{2}\int_0^1 e^{-\nu x}\left[\boldsymbol{f}^T(x)[\boldsymbol{S}^{uu}(x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-2}\boldsymbol{S}^{uu}(x)\overline{\boldsymbol{f}(x)} + \boldsymbol{f}^T(x)\overline{\boldsymbol{f}(x)}\right]\mathrm{d}x$$

$$\le \frac{1}{2}\int_0^1 e^{-\nu x}\left[\left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + 1\right]|\boldsymbol{f}(x)|^2\,\mathrm{d}x, \tag{4.79}$$

$$\mathrm{Re}\int_0^1 e^{-\nu x}\boldsymbol{g}^T(x)[\boldsymbol{S}^{uv}(x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\overline{\boldsymbol{f}(x)}\,\mathrm{d}x$$

$$\le \frac{1}{2}\int_0^1 e^{-\nu x}\left[\boldsymbol{g}^T(x)[\boldsymbol{S}^{uv}(x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-2}\boldsymbol{S}^{uv}(x)\overline{\boldsymbol{g}(x)} + \boldsymbol{f}^T(x)\overline{\boldsymbol{f}(x)}\right]\mathrm{d}x$$

$$\le \frac{1}{2}\left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2\int_0^1 e^{-\nu x}|\boldsymbol{g}(x)|^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 e^{-\nu x}|\boldsymbol{f}(x)|^2\,\mathrm{d}x, \tag{4.80}$$

$$\mathrm{Re}\int_0^1 e^{-\nu x}\int_0^x \boldsymbol{f}(\xi)^T[\boldsymbol{C}^{\mathrm{r}}(x,\xi)]^T\,\mathrm{d}\xi[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\overline{\boldsymbol{f}(x)}\,\mathrm{d}x$$

$$\le \frac{1}{2}\int_0^1 e^{-\nu x}\int_0^x\left[f(\xi)^T[\boldsymbol{C}^{\mathrm{r}}(x,\xi)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-2}\boldsymbol{C}^{\mathrm{r}}(x,\xi)\overline{\boldsymbol{f}(\xi)} + \boldsymbol{f}(x)^T\overline{\boldsymbol{f}(x)}\right]\mathrm{d}\xi\,\mathrm{d}x$$

$$\le \frac{1}{2}\left(\frac{M_{\boldsymbol{C}}}{\underline{\lambda}}\right)^2\int_0^1 e^{-\nu x}\int_0^x|\boldsymbol{f}(\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}x + \frac{1}{2}\int_0^1 e^{-\nu x}\int_0^x|\boldsymbol{f}(x)|^2\,\mathrm{d}\,\mathrm{d}\xi\,\mathrm{d}x$$

$$\le \frac{1}{2}\left(\frac{M_{\boldsymbol{C}}}{\underline{\lambda}}\right)^2\frac{1}{\nu}\int_0^1 e^{-\nu x}|\boldsymbol{f}(x)|^2\,\mathrm{d}\,\mathrm{d}x + \frac{1}{2}\int_0^1 e^{-\nu x}x|\boldsymbol{f}(x)|^2\,\mathrm{d}x$$

$$\le \frac{1}{2}\int_0^1 e^{-\nu x}\left[x + \left(\frac{M_{\boldsymbol{C}}}{\underline{\lambda}}\right)^2\frac{1}{\nu}\right]|\boldsymbol{f}(x)|^2\,\mathrm{d}x, \tag{4.81}$$

and

$$\mathrm{Re}\int_0^1 e^{-\nu x}\int_0^x \boldsymbol{g}(\xi)^T[\boldsymbol{C}^{\mathrm{l}}(x,\xi)]^T\,\mathrm{d}\xi[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\overline{\boldsymbol{f}(x)}\,\mathrm{d}x$$

$$\le \frac{1}{2}\int_0^1 e^{-\nu x}\int_0^x\left[\boldsymbol{f}^T(x)[\boldsymbol{C}^{\mathrm{l}}(x,\xi)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-2}\boldsymbol{C}^{\mathrm{l}}(x,\xi)\overline{\boldsymbol{f}(x)} + \boldsymbol{g}^T(\xi)\overline{\boldsymbol{g}(\xi)}\right]\mathrm{d}\xi\,\mathrm{d}x$$

$$\le \frac{1}{2}\left(\frac{M_{\boldsymbol{C}}}{\underline{\lambda}}\right)^2\int_0^1 e^{-\nu x}x|\boldsymbol{f}(x)|^2\,\mathrm{d}x + \frac{1}{2}\frac{1}{\nu}\int_0^1 e^{-\nu x}|\boldsymbol{g}(x)|^2\,\mathrm{d}x. \tag{4.82}$$

Moreover, we have

$$(1+x)\boldsymbol{g}^T(0)\boldsymbol{\Delta}(x)^T\boldsymbol{D}[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\overline{\boldsymbol{g}(x)}$$

$$= (1+x)\Big\{g_1(0)\delta_{2,1}(x)d_2\frac{1}{\lambda_2^{\mathrm{l}}(x)}\overline{g_2(x)}$$

$$+ [g_1(0)\delta_{3,1}(x) + g_2(0)\delta_{3,2}(x)]\,d_3\frac{1}{\lambda_3^{\mathrm{l}}(x)}\overline{g_3(x)}$$

$$
\begin{aligned}
&+\cdots \\
&+\left[g_1(0)\delta_{m-1,1}(x)+g_2(0)\delta_{m-1,2}(x)+\cdots\right. \\
&\qquad\left.+g_{m-2}(0)\delta_{m-1,m-2}(x)\right] d_{m-1}\frac{1}{\lambda^1_{m-1}(x)}\overline{g_{m-1}(x)} \\
&+\left[g_1(0)\delta_{m,1}(x)+g_2(0)\delta_{m,2}(x)+\cdots+g_{m-1}(0)\delta_{m,m-1}(x)\right] d_m\frac{1}{\lambda^1_m(x)}\overline{g_m(x)}\Bigg\}.
\end{aligned} \tag{4.83}
$$

By rearranging the terms, we have

$$
\begin{aligned}
I(t) &:= \operatorname{Re}\int_0^1 (1+x)\boldsymbol{g}^T(0)\boldsymbol{\Delta}(x)^T\boldsymbol{D}[\boldsymbol{\Lambda}^1(x)]^{-1}\overline{\boldsymbol{g}(x)}\,\mathrm{d}x \\
&\le \int_0^1 \frac{(1+x)}{2}\left\{\sum_{i=2}^m\left[|g_1(0)|^2\delta^2_{i,1}(x)d_i^2\frac{1}{\lambda^1_i(x)^2}+|g_i(x)|^2\right]\right. \\
&\quad+\sum_{i=3}^m\left[|g_2(0)|^2\delta^2_{i,2}(x)d_i^2\frac{1}{\lambda^1_i(x)^2}+|g_i(x)|^2\right]+\cdots \\
&\quad+\sum_{i=m-1}^m\left[|g_{m-2}(0)|^2\delta^2_{i,m-2}(x)d_i^2\frac{1}{\lambda^1_i(x)^2}+|g_i(x)|^2\right] \\
&\quad\left.+\left[|g_{m-1}(0)|^2\delta^2_{m,m-1}(x)d_m^2\frac{1}{\lambda^1_m(x)^2}+|g_m(x)|^2\right]\right\}\mathrm{d}x \\
&\le (m-1)\int_0^1\frac{(1+x)}{2}|\boldsymbol{g}(x)|^2\,\mathrm{d}x \\
&\quad+|g_1(0)|^2\int_0^1\frac{(1+x)}{2}\sum_{i=2}^m\delta^2_{i,1}(x)d_i^2\frac{1}{\lambda^1_i(x)^2}\,\mathrm{d}x \\
&\quad+|g_2(0)|^2\int_0^1\frac{(1+x)}{2}\sum_{i=3}^m\delta^2_{i,2}(x)d_i^2\frac{1}{\lambda^1_i(x)^2}\,\mathrm{d}x \\
&\quad+\cdots \\
&\quad+|g_{m-2}(0)|^2\int_0^1\frac{(1+x)}{2}\sum_{i=m-1}^m\delta^2_{i,m-2}(x)d_i^2\frac{1}{\lambda^1_i(x)^2}\,\mathrm{d}x \\
&\quad+|g_{m-1}(0)|^2\int_0^1\frac{(1+x)}{2}\delta^2_{m,m-1}(x)d_m^2\frac{1}{\lambda^1_m(x)^2}\,\mathrm{d}x.
\end{aligned} \tag{4.84}
$$

Then, by plugging (4.79)–(4.82) and (4.84) into (4.78), the following estimate can be obtained:

$$
\operatorname{Re}\left\langle \mathcal{A}(\boldsymbol{f}^T,\ \boldsymbol{g}^T)^T,\ \ (\boldsymbol{f}^T,\ \boldsymbol{g}^T)^T\right\rangle_{\boldsymbol{H}}
$$

$$\leq -\frac{1}{2}e^{-\nu}|\boldsymbol{f}(1)|^2 - \frac{1}{2}\zeta_1(\nu)\int_0^1 e^{-\nu x}|\boldsymbol{f}(t,x)|^2\,\mathrm{d}x - \frac{1}{2}\zeta_2(\nu)\int_0^1 |\boldsymbol{g}(t,x)|^2\,\mathrm{d}x \tag{4.85}$$

$$\leq 0, \tag{4.86}$$

where the inequalities (4.49)–(4.52) are used in the derivation of (4.85) and we have from (4.47), (4.49)–(4.52) that

$$\zeta_1(\nu) := \nu - \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 - \left(\frac{M_{\boldsymbol{C}}}{\underline{\lambda}}\right)^2\left(\frac{1}{\nu}+1\right) - 3 > 0, \tag{4.87}$$

$$\zeta_2(\nu) := \min\left\{d_i; i = \overline{1,m}\right\} - \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 - \frac{1}{\nu} - 2(m-1) > 0. \tag{4.88}$$

Let $(\boldsymbol{\phi}^T,\ \boldsymbol{\psi}^T)^T \in \mathrm{Dom}(\mathcal{A}^*)$, then it can be derived similarly that

$$\begin{aligned}
&\mathrm{Re}\left\langle(\boldsymbol{\phi}^T,\ \boldsymbol{\psi}^T)^T,\ \ \mathcal{A}^*(\boldsymbol{\phi}^T,\ \boldsymbol{\psi}^T)^T\right\rangle_{\boldsymbol{H}}\\
&\leq -\boldsymbol{\psi}^T(1)\boldsymbol{D}\overline{\boldsymbol{\psi}(1)} - \frac{1}{2}\zeta_1(\nu)\int_0^1 e^{-\nu x}|\boldsymbol{\phi}(x)|^2\,\mathrm{d}x\ - \frac{1}{2}\zeta_2(\nu)\int_0^1|\boldsymbol{\psi}(x)|^2\,\mathrm{d}x\\
&\leq 0.
\end{aligned} \tag{4.89}$$

Hence, $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative in $\boldsymbol{H}$. According to the Lumer-Phillips theorem [53, Corollary 4.4], $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions in $\boldsymbol{H}$. □

The following lemma can thus be proved, see, [65].

Lemma 4.4. *The C_0-semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ is exponentially stable. More precisely, consider the following system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix} = \mathcal{A}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}, \tag{4.90}$$

then for any initial data $\left(\boldsymbol{u}^T(0,\cdot),\ \boldsymbol{v}^T(0,\cdot)\right)^T \in \boldsymbol{H}$, *there exists a unique (mild) solution*

$$\left(\boldsymbol{u}^T(t,\cdot),\boldsymbol{v}^T(t,\cdot)\right)^T \in C([0,\infty);\boldsymbol{H}). \tag{4.91}$$

to (4.90), *and it holds that*

$$\left\|\left(\boldsymbol{u}^T(t,\cdot),\boldsymbol{v}^T(t,\cdot)\right)^T\right\|_{(L^2(0,1))^{n+m}} \leq \sqrt{\frac{C_2}{C_1}}e^{-a/2t}\left\|\left(\boldsymbol{u}^T(0,\cdot),\boldsymbol{v}^T(0,\cdot)\right)^T\right\|_{(L^2(0,1))^{n+m}}, \tag{4.92}$$

where

$$a := \underline{\lambda} \min \left\{ \zeta_1(\nu),\ \frac{1}{2\max\left\{d_i; i\ = \overline{1,m}\right\}}\zeta_2(\nu) \right\} > 0. \tag{4.93}$$

Proof. Define the energy of the system (4.39)–(4.42) as

$$\begin{aligned} E(t) =& \frac{1}{2}\left\|(\boldsymbol{\gamma}^T(t,\cdot),\ \boldsymbol{\delta}^T(t,\cdot))^T\right\|_{\boldsymbol{H}}^2, \\ =& \frac{1}{2}\int_0^1 e^{-\nu x}\boldsymbol{\gamma}^T(t,x)[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\boldsymbol{\gamma}(t,x)\,\mathrm{d}x \\ &+ \frac{1}{2}\int_0^1 (1+x)\boldsymbol{\delta}^T(t,x)\boldsymbol{D}[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\boldsymbol{\delta}(t,x)\,\mathrm{d}x \end{aligned} \tag{4.94}$$

then its derivative

$$\begin{aligned} \dot{E}(t) =& \operatorname{Re}\left\{\int_0^1 e^{-\nu x}\boldsymbol{\gamma}^T(t,x)[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\overline{\boldsymbol{\gamma}_t(t,x)}\,\mathrm{d}x\right\} \\ &+ \operatorname{Re}\left\{\int_0^1 (1+x)\boldsymbol{\delta}^T(t,x)[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\overline{\boldsymbol{\delta}_t(t,x)}\,\mathrm{d}x\right\} \\ \le& -\frac{1}{2}\zeta_1(\nu)\int_0^1 e^{-\nu x}|\boldsymbol{\gamma}(t,x)|^2\,\mathrm{d}x - \frac{1}{2}\zeta_2(\nu)\int_0^1 |\boldsymbol{\delta}(t,x)|^2\,\mathrm{d}x \\ \le& -aE(t), \end{aligned} \tag{4.95}$$

where (4.85) has been used. The proof can then be completed with (4.53). □

Remark 4.1. Following the proof of [67, Lemma 3.1], it can be similarly derived that the solution to the system (4.90) is also finite time stable.

4.4 Notes and references

This chapter presents a preliminary stability analysis for the coupled systems of first-order hyperbolic PDEs. This, together with the study of exponential stability of the non-local cascaded hyperbolic system, can serve as a guidance for the controller design and other related control problems.

Chapter 4 contains reprints or adaptions of the following papers: 1. A. Diagne, S.-X. Tang*, M. Diagne and M. Krstic, "Stabilization of the Bilayer *Saint-Venant* Model by backstepping boundary control", *IEEE Transactions on Automatic Control*, under review. 2. S.-X. Tang, B.-Z. Guo and M. Krstic, "Control

designs for a coupled hyperbolic system with a matched boundary disturbance", in preparation. The dissertation author is one of the primary investigators and authors of these papers, and would like to thank Ababacar Diagne, Mamadou Diagne, Bao-Zhu Guo and Miroslav Krstic for their contributions.

Chapter 5

Backstepping Control of the Coupled Hyperbolic Systems with Spatially Dependent Coefficients

5.1 Introduction

5.1.1 Problem statement

The class of systems discussed in this chapter (see, Figure 5.1) are

$$\boldsymbol{u}_t(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{u}_x(t,x) = \boldsymbol{S}^{uu}(x)\boldsymbol{u}(t,x) + \boldsymbol{S}^{uv}(x)\boldsymbol{v}(t,x), \tag{5.1}$$

$$\boldsymbol{v}_t(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{v}_x(t,x) = \boldsymbol{S}^{vu}(x)\boldsymbol{u}(t,x) + \boldsymbol{S}^{vv}(x)\boldsymbol{v}(t,x), \tag{5.2}$$

$$\boldsymbol{u}(t,0) = \boldsymbol{Q}\boldsymbol{v}(t,0), \tag{5.3}$$

$$\boldsymbol{v}(t,1) = \boldsymbol{R}\boldsymbol{u}(t,1) + \boldsymbol{U}(t), \tag{5.4}$$

$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), \tag{5.5}$$

$$\boldsymbol{v}(0,x) = \boldsymbol{v}_0(x), \tag{5.6}$$

where

$$\boldsymbol{u}(t,x) = \Big[u_1(t,x),\, u_2(t,x),\, \ldots,\, u_n(t,x)\Big]^T, \tag{5.7}$$

$$\boldsymbol{v}(t,x) = \Big[v_1(t,x),\, v_2(t,x),\, \ldots,\, v_m(t,x)\Big]^T \tag{5.8}$$

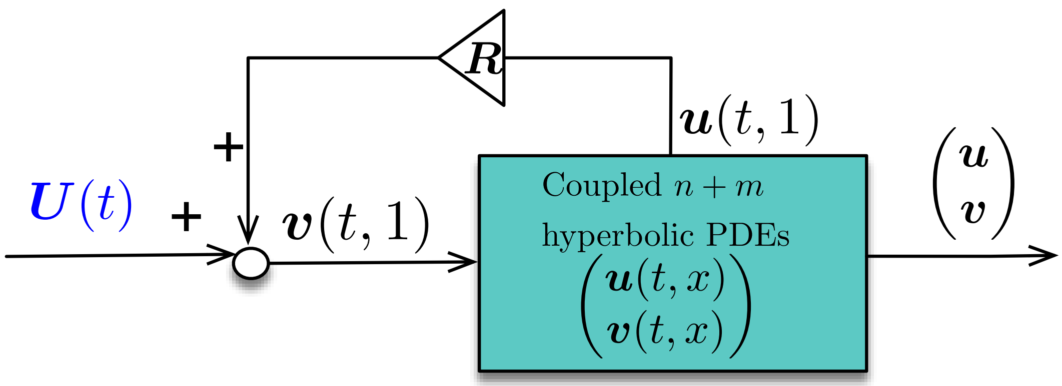

Figure 5.1: Block diagram of the coupled hyperbolic control system.

are the systems states with $t > 0,\ x \in [0,1]$. The following spatially varying matrices stand for the transport speeds:

$$\mathbf{\Lambda}^{\mathrm{r}}(x) = \mathrm{diag}\left[\lambda_1^{\mathrm{r}}(x)\,,\ \lambda_2^{\mathrm{r}}(x)\,,\ \cdots\,,\ \lambda_n^{\mathrm{r}}(x)\right] \in \mathcal{M}_n(\mathbb{R}), \tag{5.9}$$

$$\mathbf{\Lambda}^{\mathrm{l}}(x) = \mathrm{diag}\left[\lambda_1^{\mathrm{l}}(x)\,,\ \lambda_2^{\mathrm{l}}(x)\,,\ \cdots\,,\ \lambda_m^{\mathrm{l}}(x)\right] \in \mathcal{M}_m(\mathbb{R}), \tag{5.10}$$

where

$$0 < \lambda_1^{\mathrm{r}}(x) < \lambda_2^{\mathrm{r}}(x) < \cdots < \lambda_n^{\mathrm{r}}(x), \tag{5.11}$$

$$-\lambda_m^{\mathrm{l}}(x) < -\lambda_2^{\mathrm{l}}(x) \cdots < -\lambda_1^{\mathrm{l}}(x) < 0,\ \forall x \in [0,1]. \tag{5.12}$$

And the in-domain parameters are given as

$$\boldsymbol{S}^{uu}(x) = \left(S_{ij}^{uu}(x)\right)_{1\le i\le n, 1\le j\le n}, \tag{5.13}$$

$$\boldsymbol{S}^{uv}(x) = \left(S_{ij}^{uv}(x)\right)_{1\le i\le n, 1\le j\le m}, \tag{5.14}$$

$$\boldsymbol{S}^{vu}(x) = \left(S_{ij}^{vu}(x)\right)_{1\le i\le m, 1\le j\le n}, \tag{5.15}$$

$$\boldsymbol{S}^{vv}(x) = \left(S_{ij}^{vv}(x)\right)_{1\le i\le m, 1\le j\le m}. \tag{5.16}$$

The matrices $\boldsymbol{Q}$, $\boldsymbol{R}$ of boundary parameters are given as

$$\boldsymbol{Q} = \{q_{ij}\}_{1\le i\le n, 1\le j\le m}, \tag{5.17}$$

$$\boldsymbol{R} = \{r_{ij}\}_{1\le i\le m, 1\le j\le n}, \tag{5.18}$$

and

$$\boldsymbol{U}(t) = \left[U_1(t),\, U_2(t),\, \ldots,\, U_m(t)\right]^T \tag{5.19}$$

denotes the boundary control inputs. The control objective is to (exponentially) stabilize this class of coupled systems of n leftwards and m rightwards convecting transport partial differential equations (PDEs) with spatially varying coefficients.

All the investigation will be conducted under the following assumption, which is the same as Assumption 4.1.

Assumption 5.1.

1. $\mathbf{\Lambda^r}(x) \in C^1\left(0,1;\mathcal{M}_n(\mathbb{R})\right), \mathbf{\Lambda^l}(x) \in C^1\left(0,1;\mathcal{M}_m(\mathbb{R})\right)$, and we denote

$$\underline{\lambda} := \min\left\{\lambda_i^r(x), \lambda_j^l(x);\ x\in[0,1],\ i=\overline{1,n},\ j=\overline{1,m}\right\}, \tag{5.20}$$

$$\overline{\lambda} := \max\left\{\lambda_i^r(x), \lambda_j^l(x);\ x\in[0,1],\ i=\overline{1,n},\ j=\overline{1,m}\right\}. \tag{5.21}$$

2. $\boldsymbol{S}^{uu}(x) \in L^\infty\left(0,1;\mathcal{M}_n(\mathbb{R})\right), \boldsymbol{S}^{uv}(x) \in L^\infty\left(0,1;\mathcal{M}_{n,m}(\mathbb{R})\right)$,
$\boldsymbol{S}^{vu}(x) \in L^\infty\left(0,1;\mathcal{M}_{m,n}(\mathbb{R})\right)$, $\boldsymbol{S}^{vv}(x) \in L^\infty\left(0,1;\mathcal{M}_m(\mathbb{R})\right)$, and we denote

$$M_{\boldsymbol{S}} := \max_{x\in[0,1]}\{\|\boldsymbol{S}^{uu}(x)\|, \|\boldsymbol{S}^{uv}(x)\|\}, \tag{5.22}$$

3. $\boldsymbol{Q} \in \mathcal{M}_{n,m}(\mathbb{R})$, $\boldsymbol{R} \in \mathcal{M}_{m,n}(\mathbb{R})$, and we denote

$$\bar{q} := \|\boldsymbol{Q}^T\boldsymbol{Q}\|. \tag{5.23}$$

Here and in the sequel, $\|\cdot\|$ denotes the 2−norm of a matrix.

5.1.2 Literature review

Control problems of coupled hyperbolic systems have received a lot of attention from researchers. Indeed, extensive studies have been conducted towards the design of control methodologies for this class of systems, see, e.g., [68, 69, 70]. Specifically, several efforts have been made on the stabilization problems for the coupled systems of first-order linear hyperbolic PDEs, see, e.g., [71, 72, 73], in the past decades.

In this chapter, the PDE backstepping control method is employed for the controller designs of the considered systems. Through backstepping, we design a full state feedback controller, which stabilize the system while requiring the full state information. In order to unlock the limitation of requiring a full state estimation, we design an exponentially convergent Luenberger observer, which helps reconstruct the full system state over the domain by employing sensors located only at the boundary. Then, based on the full state feedback controller and the state observer, an (exponentially stabilizing) output feedback controller is constructed.

It is worth noting that the specific controller design strategy can be referred to [74], in which the stabilization problem for a general coupled heterodirectional (or, more strictly, bidirectional) system of hyperbolic PDEs with constant coefficients is solved. The present work can be treated as a generalization of the result obtained in [74], from the system with constant system coefficients to the one with time-varying coefficients. In our stability proofs, we employ a novel Lyapunov function in which the parameters need to be successively determined.

5.1.3 Organization

The outline of this chapter is as follows. A state feedback controller is designed in Section 5.2, in which the system of coupled $n+m$ transport PDEs is exponentially stabilized by m full state feedback controllers located at the boundary. In Section 5.3, a state observer is first designed in Subsection 5.3.1 for this coupled system, which recovers the full system state information with only m boundary measurements. Then, based on the results from Section 5.2 and Subsection 5.3.1, Subsection 5.3.2 constructs an output feedback controller, with which exponential stability is achieved for the closed-loop control system. Finally in Section 5.4, a conclusion is presented and some possible future work directions are discussed.

5.2 State feedback stabilization

We employ the PDE backstepping method. First, we construct a backstepping transformation to map the system (5.1)–(5.2) into a target system with desirable stability property, which follows from the one constructed in [74].

5.2.1 Target system and backstepping transformation

Consider the following target system (see, Figure 5.2):

$$\begin{aligned}\partial_t\boldsymbol{\alpha}(t,x)+\boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x\boldsymbol{\alpha}(t,x)&=\boldsymbol{S}^{uu}(x)\boldsymbol{\alpha}(t,x)+\boldsymbol{S}^{uv}(x)\boldsymbol{\beta}(t,x)\\&\quad+\int_0^x\boldsymbol{C}^{\mathrm{r}}(x,\xi)\boldsymbol{\alpha}(t,\xi)\,\mathrm{d}\xi+\int_0^x\boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{\beta}(t,\xi)\,\mathrm{d}\xi, \qquad (5.24)\end{aligned}$$

$$\partial_t\boldsymbol{\beta}(t,x)-\boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x\boldsymbol{\beta}(t,x)=\boldsymbol{\Delta}(x)\boldsymbol{\beta}(t,0), \qquad (5.25)$$

$$\boldsymbol{\alpha}(t,0)=\boldsymbol{Q}\boldsymbol{\beta}(t,0), \qquad (5.26)$$

$$\boldsymbol{\beta}(t,1) = \mathbf{0}, \tag{5.27}$$

where

$$\boldsymbol{\Delta}(x) = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \delta_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \delta_{m,1}(x) & \cdots & \delta_{m,m-1}(x) & 0 \end{bmatrix}, \tag{5.28}$$

and $\boldsymbol{C}^{\mathrm{r}}$, $\boldsymbol{C}^{\mathrm{l}}$ are matrices of functions defined on the triangular domain

$$\mathbb{T} = \Big\{(x,\xi) \in \mathbb{R}^2 |\ 0 \le \xi \le x \le 1\Big\}.$$

Here, $\boldsymbol{C}^{\mathrm{r}}$, $\boldsymbol{C}^{\mathrm{l}}$ and $\boldsymbol{\Delta}(x)$ are all to be determined from the backstepping transformation introduced immediately later.

In order to map the system (5.1)–(5.6) into the desired target system (5.24)–(5.27), we consider the following backstepping transformation:

$$\begin{pmatrix} \boldsymbol{\alpha}(t,x) \\ \boldsymbol{\beta}(t,x) \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}(t,x) \\ \boldsymbol{v}(t,x) \end{pmatrix} - \int_0^x \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{G}(x,\xi) & \boldsymbol{H}(x,\xi) \end{pmatrix} \begin{pmatrix} \boldsymbol{u}(t,\xi) \\ \boldsymbol{v}(t,\xi) \end{pmatrix} \mathrm{d}\xi. \tag{5.29}$$

Here, the to-be-determined kernels $\boldsymbol{G}$ and $\boldsymbol{H}$ are defined on the domain $\mathbb{T}$.

Comparing the system equations (5.1)–(5.6) and (5.24)–(5.27), we derive through some calculations and integration by parts that $\boldsymbol{G}$ and $\boldsymbol{H}$ need to satisfy the following system of equations:

$$\begin{aligned} &\boldsymbol{G}_\xi(x,\xi)\boldsymbol{\Lambda}^{\mathbf{r}}(\xi) - \boldsymbol{\Lambda}^{\mathbf{l}}(x)\boldsymbol{G}_x(x,\xi) \\ &\qquad = -\boldsymbol{G}(x,\xi)\left[\boldsymbol{\Lambda}^{\mathbf{r}}(\xi)\right]' - \boldsymbol{G}(x,\xi)\boldsymbol{S}^{uu}(\xi) - \boldsymbol{H}(x,\xi)\boldsymbol{S}^{vu}(\xi), \end{aligned} \tag{5.30}$$

$$\begin{aligned} &\boldsymbol{H}_\xi(x,\xi)\boldsymbol{\Lambda}^{\mathbf{l}}(\xi) + \boldsymbol{\Lambda}^{\mathbf{l}}(x)\boldsymbol{H}_x(x,\xi) \\ &\qquad = -\boldsymbol{H}(x,\xi)[\boldsymbol{\Lambda}^{\mathbf{r}}(\xi)]' + \boldsymbol{G}(x,\xi)\boldsymbol{S}^{uv}(\xi) + \boldsymbol{H}(x,\xi)\boldsymbol{S}^{vv}(\xi), \end{aligned} \tag{5.31}$$

$$\boldsymbol{G}(x,x)\boldsymbol{\Lambda}^{\mathbf{r}}(x) + \boldsymbol{\Lambda}^{\mathbf{l}}(x)\boldsymbol{G}(x,x) = -\boldsymbol{S}^{vu}(x), \tag{5.32}$$

$$\boldsymbol{H}(x,x)\boldsymbol{\Lambda}^{\mathbf{l}}(x) - \boldsymbol{\Lambda}^{\mathbf{l}}(x)\boldsymbol{H}(x,x) = \boldsymbol{S}^{vv}(x), \tag{5.33}$$

$$\boldsymbol{G}(x,0)\boldsymbol{\Lambda}^{\mathbf{r}}(0)\boldsymbol{Q} - \boldsymbol{H}(x,0)\boldsymbol{\Lambda}^{\mathbf{l}}(0) = -\boldsymbol{\Delta}(x). \tag{5.34}$$

The existence, uniqueness and regularity of the backstepping transformation (5.29) could be guaranteed similarly as [74], by adding some artificial boundary conditions, for which the proof is omitted here. Then, the continuity of the kernels

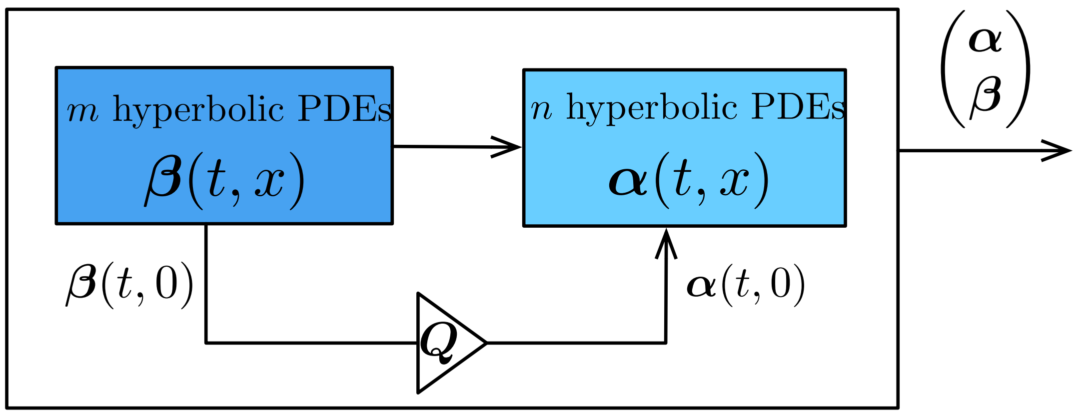

Figure 5.2: Block diagram of the target system.

guarantees the existence of a unique inverse transformation. We write the inverse transformation as

$$\begin{pmatrix} \boldsymbol{u}(t,x) \\ \boldsymbol{v}(t,x) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}(t,x) \\ \boldsymbol{\beta}(t,x) \end{pmatrix} - \int_0^x \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{\mathcal{G}}(x,\xi) & \boldsymbol{\mathcal{H}}(x,\xi) \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}(t,\xi) \\ \boldsymbol{\beta}(t,\xi) \end{pmatrix} \mathrm{d}\xi, \tag{5.35}$$

then we immediately have from (5.29) and (5.35) that the kernels $\boldsymbol{\mathcal{G}}(x,\xi), \boldsymbol{\mathcal{H}}(x,\xi)$ need to satisfy

$$0 = \boldsymbol{G}(x,\xi) + \boldsymbol{\mathcal{G}}(x,\xi) - \int_\xi^x \boldsymbol{\mathcal{H}}(x,\eta)\boldsymbol{G}(\eta,\xi)\ \mathrm{d}\eta, \tag{5.36}$$

$$0 = \boldsymbol{H}(x,\xi) + \boldsymbol{\mathcal{H}}(x,\xi) - \int_\xi^x \boldsymbol{\mathcal{H}}(x,\eta)\boldsymbol{H}(\eta,\xi)\ \mathrm{d}\eta. \tag{5.37}$$

With the continuity of $\boldsymbol{G}(x,\xi)$, $\boldsymbol{H}(x,\xi)$, well-posedness of the solution $\boldsymbol{\mathcal{G}}(x,\xi)$, $\boldsymbol{\mathcal{H}}(x,\xi)$ to the system of equations (5.36)–(5.37) can be derived. And the solution can be calculated through the method of successive approximations, see, [22, Section 4.4].

In the mean time, $\delta_{i,j}(x)$ for $i = \overline{2,\ m}$, $j = \overline{1,\ m-1}$ can be obtained, and the following equations are obtained for $\boldsymbol{C}^{\mathrm{r}}(x,\xi)$, $\boldsymbol{C}^{\mathrm{l}}(x,\xi)$:

$$\boldsymbol{C}^{\mathrm{r}}(x,\xi) = \boldsymbol{S}^{uv}(x)\boldsymbol{G}(x,\xi) + \int_\xi^x \boldsymbol{C}^{\mathrm{l}}(x,\eta)\boldsymbol{G}(\xi,\eta)\ \mathrm{d}\eta, \tag{5.38}$$

$$\boldsymbol{C}^{\mathrm{l}}(x,\xi) = \boldsymbol{S}^{uv}(x)\boldsymbol{H}(x,\xi) + \int_\xi^x \boldsymbol{C}^{\mathrm{l}}(x,\eta)\boldsymbol{H}(\xi,\eta)\ \mathrm{d}\eta. \tag{5.39}$$

By the method of successive approximation, it can be proved that the functions $\boldsymbol{C}^{\mathrm{r}}(x,\xi)$, $\boldsymbol{C}^{\mathrm{l}}(x,\xi)$ are continuous on $\mathbb{T}$. Moreover, we assume that

$$\int_0^1 [\boldsymbol{\Lambda}^{\mathrm{l}}(\xi)]^{-1}\boldsymbol{\Delta}(\xi)\,\mathrm{d}\xi - \boldsymbol{I}$$

is nonsingular,where $\boldsymbol{I}$ denotes the unit matrix.

Hence, the control law $\boldsymbol{U}(t)$ can be obtained by plugging the transformation (5.29) into (5.4). Indeed, (5.27) implies that

$$\boldsymbol{U}(t) = -\boldsymbol{R}\boldsymbol{u}(t,1) + \int_0^1 [\boldsymbol{G}(1,\xi)\boldsymbol{u}(t,\xi) + \boldsymbol{H}(1,\xi)\boldsymbol{v}(t,\xi)]\,\mathrm{d}\xi. \tag{5.40}$$

5.2.2 Stability of the target system

The exponential stability of the target system (5.24)–(5.27) can be referred to Lemma 4.4. Note that the novelty, compared with other results in this area, e.g., [74], lies in a newly proposed Lyapunov function. In particular, we employ in Subsection 4.3.2 a Lyapunov function where the parameters needs to be successively determined.

Lemma 5.1. *For any given initial data* $((\boldsymbol{\alpha}^0)^T,\ (\boldsymbol{\beta}^0)^T)^T = (\boldsymbol{\alpha}^T(0,\cdot),\ \boldsymbol{\beta}^T(0,\cdot))^T \in (\mathcal{L}^2([0,1]))^{n+m}$*, the equilibrium* $(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T)^T = (\mathbf{0}^T,\ \mathbf{0}^T)^T$ *of the target system* (5.24)–(5.27) *determined by* (5.28), (5.38) *and* (5.39) *is exponentially stable in the* $\mathcal{L}^2$*-norm:*

$$\|(\boldsymbol{\alpha}^T(t,\cdot),\boldsymbol{\beta}^T(t,\cdot))^T\|^2_{(\mathcal{L}^2([0,1]))^{n+m}} := \int_0^1 \boldsymbol{\alpha}^T(t,x)\overline{\boldsymbol{\alpha}(t,x)} + \boldsymbol{\beta}^T(t,x)\overline{\boldsymbol{\beta}(t,x)\,\mathrm{d}x}. \tag{5.41}$$

5.2.3 Stability of the closed-loop control system

With the exponential stability of the target system, and with existence, uniqueness, regularity and invertibility of the backstepping transformation, the exponential stability of the closed-loop control system with the designed state feedback controller (5.40), see, Figure 5.3, can then be derived.

Theorem 5.1. *For any given initial data* $((\boldsymbol{u}^0)^T,\ (\boldsymbol{v}^0)^T)^T = (\boldsymbol{u}^T(0,\cdot),\ \boldsymbol{v}^T(0,\cdot))^T \in (\mathcal{L}^2([0,1]))^{n+m}$*, the equilibrium* $(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T = (\mathbf{0}^T,\ \mathbf{0}^T)^T$ *of the closed-loop system* (5.1)–(5.6) *with the designed controller* (5.40) *is exponentially stable in the sense of* $\mathcal{L}^2$*-norm:*

$$\|(\boldsymbol{u}^T(t,\cdot),\boldsymbol{v}^T(t,\cdot))^T\|^2_{(\mathcal{L}^2([0,1]))^{n+m}} := \int_0^1 \boldsymbol{u}^T(t,x)\overline{\boldsymbol{u}(t,x)} + \boldsymbol{v}^T(t,x)\overline{\boldsymbol{v}(t,x)}\,\mathrm{d}x. \tag{5.42}$$

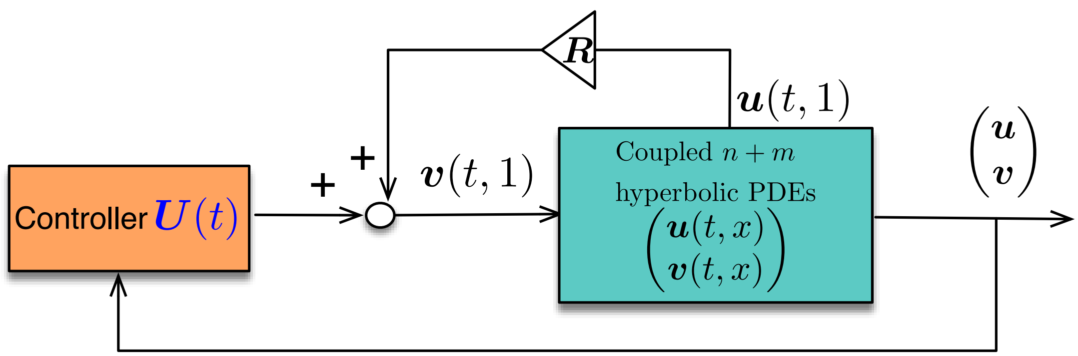

Figure 5.3: Block diagram of the closed-loop control system.

5.3 Output feedback stabilization

The backstepping controller (5.40) requires a full state measurement across the spatial domain. In this section we design a boundary state observer for estimating the distributed states of the system (5.1)–(5.6) over the whole spatial domain using the measured boundary output $\boldsymbol{y}(t) = \boldsymbol{v}(t, 0)$, which could help avoid the full state measurement in a to-be-designed output feedback controller.

5.3.1 State observer design

A. A Luenberger observer

Consider the following state observer (see, Figure 5.4), which consists of a copy of the plant (5.1)–(5.6) plus output injection terms:

$$\partial_t \hat{\boldsymbol{u}}(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x \hat{\boldsymbol{u}}(t,x)$$
$$= \boldsymbol{S}^{uu}(x)\hat{\boldsymbol{u}}(t,x) + \boldsymbol{S}^{uv}(x)\hat{\boldsymbol{v}}(t,x) + \boldsymbol{P}_1(x)[\boldsymbol{y}(t) - \hat{\boldsymbol{v}}(t,0)], \quad (5.43)$$

$$\partial_t \hat{\boldsymbol{v}}(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x \hat{\boldsymbol{v}}(t,x)$$
$$= \boldsymbol{S}^{vu}(x)\hat{\boldsymbol{u}}(t,x) + \boldsymbol{S}^{vv}(x)\hat{\boldsymbol{v}}(t,x) + \boldsymbol{P}_2(x)[\boldsymbol{y}(t) - \hat{\boldsymbol{v}}(t,0)], \quad (5.44)$$

$$\hat{\boldsymbol{u}}(t,0) = \boldsymbol{Q}\boldsymbol{y}(t), \quad (5.45)$$

$$\hat{\boldsymbol{v}}(t,1) = \boldsymbol{R}\hat{\boldsymbol{u}}(t,1) + \boldsymbol{U}(t). \quad (5.46)$$

Here, $\left(\hat{\boldsymbol{u}}^T, \hat{\boldsymbol{v}}^T\right)^T$ is the estimated state vector, and the output injection coefficients $\boldsymbol{P}_1(x)$ and $\boldsymbol{P}_2(x)$ are to be found so that the estimated state vector $\left(\hat{\boldsymbol{u}}^T,\ \hat{\boldsymbol{v}}^T\right)^T$ converges to the real state vector $(\boldsymbol{u}^T, \boldsymbol{v}^T)^T$.

Let $\begin{pmatrix} \tilde{\boldsymbol{u}}^T & \tilde{\boldsymbol{v}}^T \end{pmatrix}^T = \begin{pmatrix} \boldsymbol{u}^T - \hat{\boldsymbol{u}}^T & \boldsymbol{v}^T - \hat{\boldsymbol{v}}^T \end{pmatrix}^T$, then we obtain the following observer error system:

$$\partial_t \tilde{\boldsymbol{u}}(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x \tilde{\boldsymbol{u}}(t,x) = \boldsymbol{S}^{uu}(x)\tilde{\boldsymbol{u}}(t,x) + \boldsymbol{S}^{uv}(x)\tilde{\boldsymbol{v}}(t,x) - P_1(x)\tilde{\boldsymbol{v}}(t,0), \quad (5.47)$$

$$\partial_t \tilde{\boldsymbol{v}}(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x \tilde{\boldsymbol{v}}(t,x) = \boldsymbol{S}^{vu}(x)\tilde{\boldsymbol{u}}(t,x) + \boldsymbol{S}^{vv}(x)\tilde{\boldsymbol{v}}(t,x) - \boldsymbol{P}_2(x)\tilde{\boldsymbol{v}}(t,0), \quad (5.48)$$

$$\tilde{\boldsymbol{u}}(t,0) = \boldsymbol{0}, \quad (5.49)$$

$$\tilde{\boldsymbol{v}}(t,1) = \boldsymbol{R}\tilde{\boldsymbol{u}}(t,1). \quad (5.50)$$

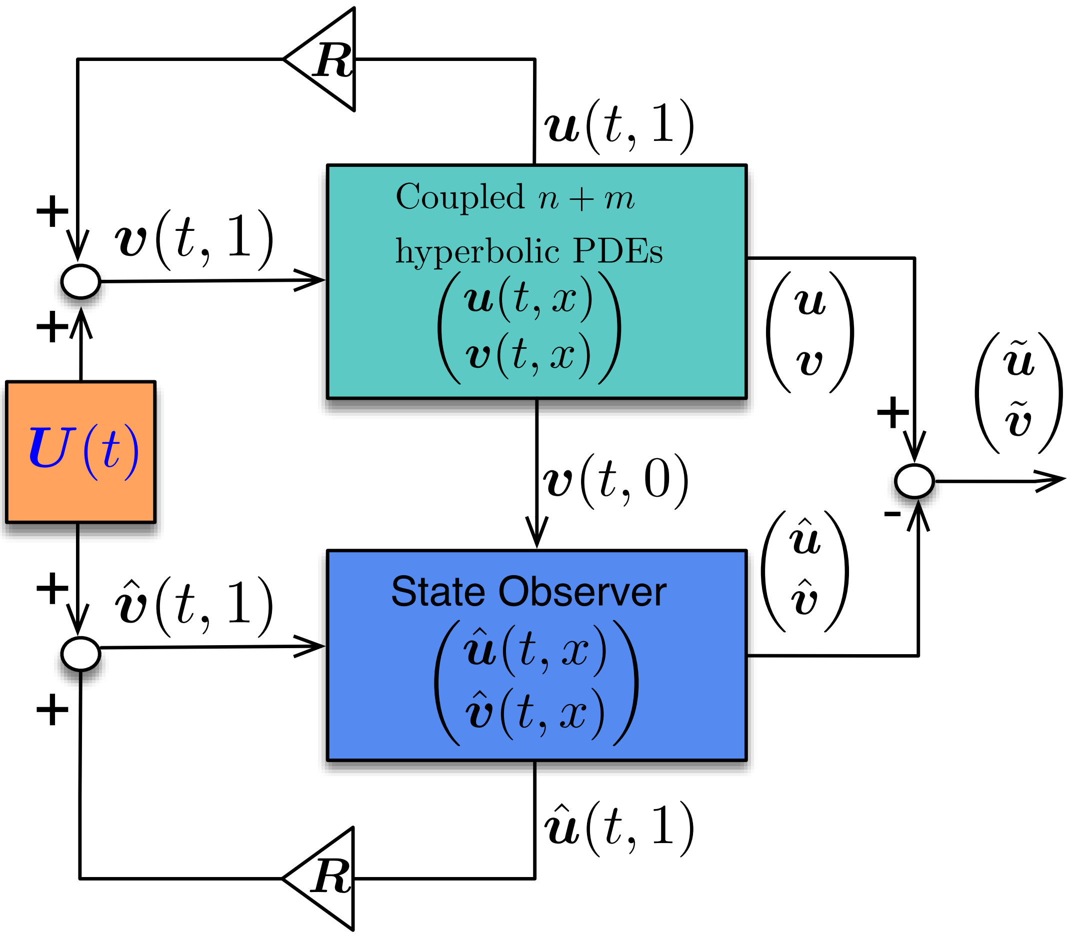

Figure 5.4: Block diagram of the state observer.

Thus, our objective now is to determine $\boldsymbol{P}_1(x)$ and $\boldsymbol{P}_2(x)$ so that the observer error $(\tilde{\boldsymbol{u}}^T, \tilde{\boldsymbol{v}}^T)^T$ converges to the origin in the sense of $\mathcal{L}^2$-norm:

$$\|(\boldsymbol{u}^T - \hat{\boldsymbol{u}}^T, \boldsymbol{v}^T - \hat{\boldsymbol{v}}^T)^T\|^2_{(\mathcal{L}^2([0,1]))^{n+m}} := \int_0^1 (\boldsymbol{u} - \hat{\boldsymbol{u}})^T\overline{(\boldsymbol{u} - \hat{\boldsymbol{u}})} + (\boldsymbol{v} - \hat{\boldsymbol{v}})^T\overline{(\boldsymbol{v} - \hat{\boldsymbol{v}})}\, \mathrm{d}x. \tag{5.51}$$

B. Backstepping transformation and the target system

We consider the following backstepping transformation:

$$\begin{pmatrix} \tilde{\boldsymbol{u}}(t,x) \\ \tilde{\boldsymbol{v}}(t,x) \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{\alpha}}(t,x) \\ \tilde{\boldsymbol{\beta}}(t,x) \end{pmatrix} + \int_0^x \begin{pmatrix} \boldsymbol{0} & \boldsymbol{M}(x,\xi) \\ \boldsymbol{0} & \boldsymbol{N}(x,\xi) \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{\alpha}}(t,\xi) \\ \tilde{\boldsymbol{\beta}}(t,\xi) \end{pmatrix} \mathrm{d}\xi, \tag{5.52}$$

where the to-be-determined kernels $\boldsymbol{M}$ and $\boldsymbol{N}$ are defined on the triangular domain $\mathbb{T}$, map the error system (5.47)–(5.50) into the following exponentially stable target system:

$$\partial_t\tilde{\boldsymbol{\alpha}}(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x\tilde{\boldsymbol{\alpha}}(t,x) = \boldsymbol{S}^{uu}(x)\tilde{\boldsymbol{\alpha}}(t,x) + \int_0^x \boldsymbol{D}^r(x,\xi)\tilde{\boldsymbol{\alpha}}(t,\xi)\ \mathrm{d}\xi, \tag{5.53}$$

$$\partial_t\tilde{\boldsymbol{\beta}}(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x\tilde{\boldsymbol{\beta}}(t,x) = \boldsymbol{S}^{vu}(x)\tilde{\boldsymbol{\alpha}}(t,x) + \int_0^x \boldsymbol{D}^l(x,\xi)\tilde{\boldsymbol{\alpha}}(t,\xi)\ \mathrm{d}\xi, \tag{5.54}$$

$$\tilde{\boldsymbol{\alpha}}(t,0) = \boldsymbol{0}, \tag{5.55}$$

$$\tilde{\boldsymbol{\beta}}(t,1) = \boldsymbol{R}\tilde{\boldsymbol{\alpha}}(t,1) - \int_0^1 \tilde{\boldsymbol{\Delta}}(\xi)\tilde{\boldsymbol{\beta}}(t,\xi)\,\mathrm{d}\xi. \tag{5.56}$$

Here, the functions $\boldsymbol{D}^r(x,\xi)$, $\boldsymbol{D}^l(x,\xi)$ and $\tilde{\boldsymbol{\Delta}}(\xi)$ are also to be determined later.

Through some lengthy calculations and integration by parts, the following PDEs are derived for the transformation kernels $\boldsymbol{M}(x,\xi)$ and $\boldsymbol{N}(x,\xi)$:

$$\begin{aligned} &-[\boldsymbol{M}_\xi(x,\xi)\boldsymbol{\Lambda}^{\mathrm{l}}(\xi) + \boldsymbol{M}(x,\xi)[\boldsymbol{\Lambda}^{\mathrm{l}}(\xi)]'] + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{M}_x(x,\xi) \\ &\qquad = \boldsymbol{S}^{uu}(x)\boldsymbol{M}(x,\xi) + \boldsymbol{S}^{uv}(x)\boldsymbol{N}(x,\xi), \end{aligned} \tag{5.57}$$

$$\begin{aligned} &-[\boldsymbol{N}_\xi(x,\xi)\boldsymbol{\Lambda}^{\mathrm{l}}(\xi) + \boldsymbol{N}(x,\xi)[\boldsymbol{\Lambda}^{\mathrm{l}}(\xi)]'] - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{N}_x(x,\xi) \\ &\qquad = \boldsymbol{S}^{vu}(x)\boldsymbol{M}(x,\xi) + \boldsymbol{S}^{vv}(x)\boldsymbol{N}(x,\xi), \end{aligned} \tag{5.58}$$

$$\boldsymbol{M}(x,x)\boldsymbol{\Lambda}^{\mathrm{l}}(x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{M}(x,x) = \boldsymbol{S}^{uv}(x), \tag{5.59}$$

$$\boldsymbol{N}(x,x)\boldsymbol{\Lambda}^{\mathrm{l}}(x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{N}(x,x) = \boldsymbol{S}^{vv}(x). \tag{5.60}$$

In the mean time, we derive that the observer gains can be defined by

$$\boldsymbol{P}_1(x) = -\boldsymbol{M}(x,0)\boldsymbol{\Lambda}^{\mathrm{l}}(0),\ \ \boldsymbol{P}_2(x) = -\boldsymbol{N}(x,0)\boldsymbol{\Lambda}^{\mathrm{l}}(0), \tag{5.61}$$

and the functions $\boldsymbol{D}^r(x,\xi), \boldsymbol{D}^l(x,\xi)$ and $\tilde{\boldsymbol{\Delta}}(\xi)$ are defined by the following equations:

$$\boldsymbol{D}^r(x,\xi) + \boldsymbol{M}(x,\xi)\boldsymbol{S}^{vu}(\xi) + \int_\xi^x \boldsymbol{M}(x,\eta)\boldsymbol{D}^l(\eta,\xi)\,\mathrm{d}\eta = 0, \tag{5.62}$$

$$\boldsymbol{D}^l(x,\xi) + \boldsymbol{N}(x,\xi)\boldsymbol{S}^{vu}(\xi) + \int_\xi^x \boldsymbol{N}(x,\eta)\boldsymbol{D}^l(\eta,\xi)\,\mathrm{d}\eta = 0, \tag{5.63}$$

$$\tilde{\boldsymbol{\Delta}}(\xi) = \boldsymbol{N}(1,\xi) - \boldsymbol{R}\boldsymbol{M}(1,\xi). \tag{5.64}$$

C. Inverse transformation

The existence, uniqueness and regularity of the transformation (5.52) can also be discussed by following [74], and then the continuity of the kernels guarantees the existence of a unique inverse transformation. We write the inverse transformation as

$$\begin{pmatrix} \tilde{\boldsymbol{\alpha}}(t,x) \\ \tilde{\boldsymbol{\beta}}(t,x) \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{u}}(t,x) \\ \tilde{\boldsymbol{v}}(t,x) \end{pmatrix} + \int_0^x \begin{pmatrix} \mathbf{0} & \boldsymbol{\mathcal{M}}(x,\xi) \\ \mathbf{0} & \boldsymbol{\mathcal{N}}(x,\xi) \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{u}}(t,x) \\ \tilde{\boldsymbol{v}}(t,x) \end{pmatrix} \mathrm{d}\xi, \tag{5.65}$$

then we immediately have from (5.52) and (5.65) that

$$\begin{aligned} \tilde{\boldsymbol{\alpha}}(t,x) =& \tilde{\boldsymbol{\alpha}}(t,x) + \int_0^x \boldsymbol{M}(x,\xi)\tilde{\boldsymbol{\beta}}(t,\xi)\ \mathrm{d}\xi \\ &+ \int_0^x \boldsymbol{\mathcal{M}}(x,\xi)\left[\tilde{\boldsymbol{\beta}}(t,\xi) + \int_0^\xi \boldsymbol{N}(\xi,\eta)\tilde{\boldsymbol{\beta}}(t,\eta)\ \mathrm{d}\eta\right]\mathrm{d}\xi, \end{aligned} \tag{5.66}$$

$$\begin{aligned} \tilde{\boldsymbol{\beta}}(t,x) =& \tilde{\boldsymbol{\beta}}(t,x) + \int_0^x \boldsymbol{N}(x,\xi)\tilde{\boldsymbol{\beta}}(t,\xi)\ \mathrm{d}\xi \\ &+ \int_0^x \boldsymbol{\mathcal{N}}(x,\xi)\left[\tilde{\boldsymbol{\beta}}(t,\xi) + \int_0^\xi \boldsymbol{N}(\xi,\eta)\tilde{\boldsymbol{\beta}}(t,\eta)\ \mathrm{d}\eta\right]\mathrm{d}\xi. \end{aligned} \tag{5.67}$$

Thus, the kernels $\boldsymbol{\mathcal{M}}(x,\xi), \boldsymbol{\mathcal{N}}(x,\xi)$ need to satisfy

$$0 = \boldsymbol{M}(x,\xi) + \boldsymbol{\mathcal{M}}(x,\xi) + \int_\xi^x \boldsymbol{\mathcal{M}}(x,\eta)\boldsymbol{N}(\eta,\xi)\,\mathrm{d}\eta, \tag{5.68}$$

$$0 = \boldsymbol{N}(x,\xi) + \boldsymbol{\mathcal{N}}(x,\xi) + \int_\xi^x \boldsymbol{\mathcal{N}}(x,\eta)\boldsymbol{N}(\eta,\xi)\,\mathrm{d}\eta. \tag{5.69}$$

In order to solve the system of equations (5.68)–(5.69), we could also use the method of successive approximations, see, [22, Section 4.4].

D. Stability of the target system and convergence of the designed observer

We could prove exponential stability of the target system (5.53)–(5.56), by following the idea in the proof for Lemma 4.4 and employing a Lyapunov function in which the parameters need to be successively determined.

Lemma 5.2. *For any given data* $\left((\tilde{\boldsymbol{\alpha}}^0)^T, (\tilde{\boldsymbol{\beta}}^0)^T\right)^T \in (\mathcal{L}^2([0,1]))^{n+m}$, *the system* (5.53)–(5.56), *with* (5.57)–(5.60), (5.62)–(5.64), *is exponentially stable in the* $\mathcal{L}^2$ *sense:*

$$\|(\tilde{\boldsymbol{\alpha}}^T(t,\cdot),\ \tilde{\boldsymbol{\beta}}^T(t,\cdot))^T\|^2_{(\mathcal{L}^2([0,1]))^{n+m}} := \int_0^1 \tilde{\boldsymbol{\alpha}}^T(t,x)\overline{\tilde{\boldsymbol{\alpha}}(t,x)} + \tilde{\boldsymbol{\beta}}^T(t,x)\overline{\tilde{\boldsymbol{\beta}}(t,x)}\,\mathrm{d}x. \tag{5.70}$$

Furthermore, for any given data $((\boldsymbol{u}^0)^T,\ (\boldsymbol{v}^0)^T,\ (\hat{\boldsymbol{u}}^0)^T,\ (\hat{\boldsymbol{v}}^0)^T)^T \in (\mathcal{L}^2([0,1]))^{2(n+m)}$, *the observer* (5.43)–(5.46) *exponentially converges to the system* (5.1)–(5.6) *in the* $\mathcal{L}^2$ *sense, see,* (5.51).

The proof is omitted here for simplicity.

5.3.2 Output feedback controller design

Based on the backstepping controller (5.40) designed in Section 5.2, which requires a full state measurement, and the observer (5.43)–(5.46) designed in Section 5.3.1, which reconstructs the state over the whole spatial domain through the boundary measurement $\boldsymbol{v}(t,0)$, we could design an observer-based output feedback controller (see, Figure 5.5):

$$\boldsymbol{U}(t) = -\boldsymbol{R}\hat{\boldsymbol{u}}(t,1) + \int_0^1 [\boldsymbol{G}(1,\xi)\hat{\boldsymbol{u}}(t,\xi) + \boldsymbol{H}(1,\xi)\hat{\boldsymbol{v}}(t,\xi)]\,\mathrm{d}\xi, \tag{5.71}$$

which works with the help of the observer (5.43)–(5.46).

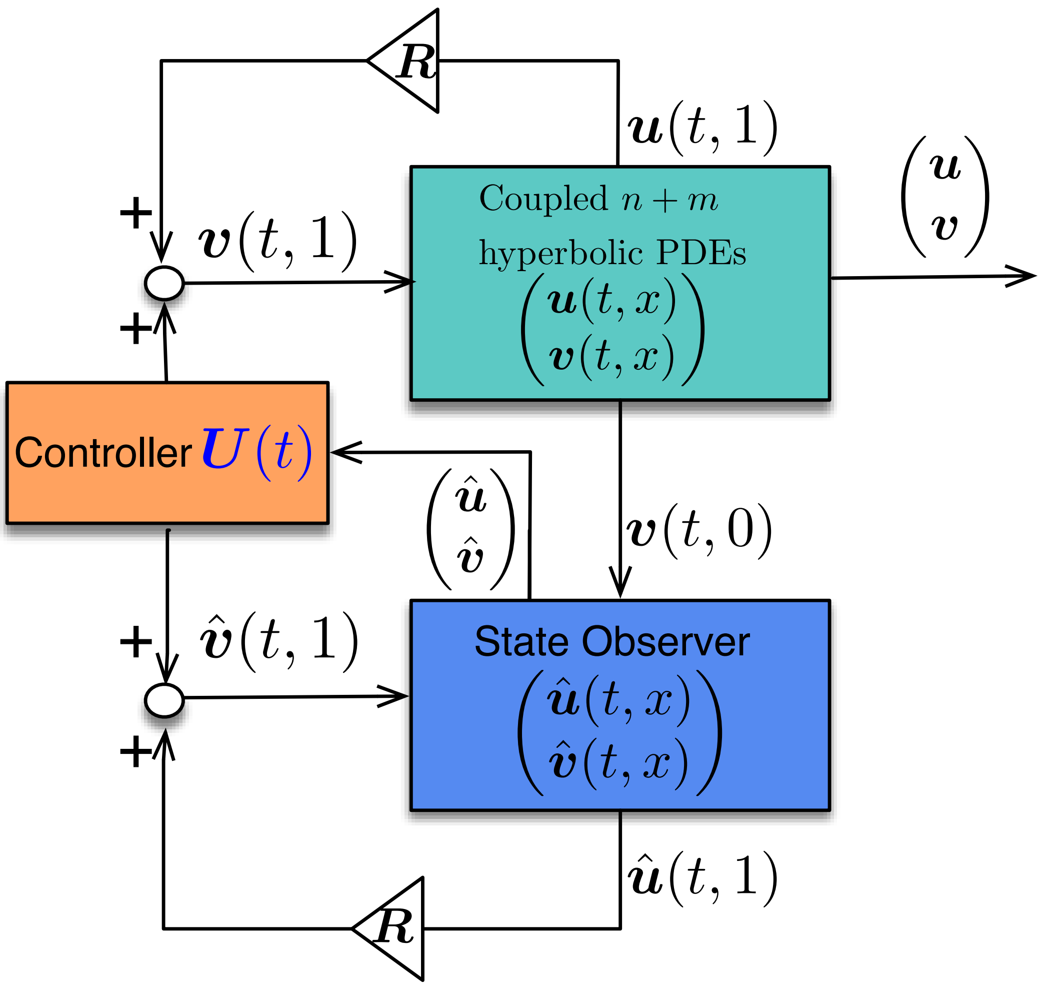

Figure 5.5: Block diagram of the closed-loop output control system.

Theorem 5.2. *For any initial data*

$$((\boldsymbol{u}^0)^T, (\boldsymbol{v}^0)^T, (\hat{\boldsymbol{u}}^0)^T, (\hat{\boldsymbol{v}}^0)^T)^T \in (\mathcal{L}^2([0,1]))^{2(n+m)},$$

the closed-loop $(\boldsymbol{u}^T_, \boldsymbol{v}^T, \ \hat{\boldsymbol{u}}^T, \ \hat{\boldsymbol{v}}^T)^T$*-system, consisting of the original system* (5.1)–(5.6)*, the observer* (5.43)–(5.46) *defined by* (5.57)–(5.61)*, and the controller* (5.71) *with the kernels* $\boldsymbol{G}$ *and* $\boldsymbol{H}$ *defined by* (5.30)–(5.34)*, is exponentially stable in the sense of the* $\mathcal{L}^2$*-norm:*

$$\begin{aligned}&\|(\boldsymbol{u}^T(t,\cdot), \boldsymbol{v}^T(t,\cdot), \hat{\boldsymbol{u}}^T(t,\cdot), \hat{\boldsymbol{v}}^T(t,\cdot))\|^2_{(\mathcal{L}^2([0,1]))^{n+m}} \\ &:= \int_0^1 \left[\boldsymbol{u}^T(t,x)\overline{\boldsymbol{u}(t,x)} + \boldsymbol{v}^T(t,x)\overline{\boldsymbol{v}(t,x)} + \hat{\boldsymbol{u}}^T(t,x)\overline{\hat{\boldsymbol{u}}(t,x)} + \hat{\boldsymbol{v}}^T(t,x)\overline{\hat{\boldsymbol{v}}(t,x)}\right] \mathrm{d}x.\end{aligned} \tag{5.72}$$

The proof is omitted here, for which a Lyapunov function can be constructed by also following the the idea in the proof for Lemma 4.4, in which a (weighted) Lyapunov function is employed where the parameters need to be successively determined, and the idea in [22, Section 5.2] as well, in which a weighted Lyapunov function is constructed.

5.4 Notes and references

By applying the PDE backstepping method, this chapter is devoted to the stabilization of a class of coupled transport systems. Both state feedback boundary controllers and output feedback boundary controllers by means of non-collocated observed state are designed, and the resulting closed-loop control systems are exponentially stable with arbitrarily prescribed decay rates.

Chapter 5 contains reprints or adaptions of the following papers: 1. A. Diagne, S.-X. Tang*, M. Diagne and M. Krstic, "Stabilization of the Bilayer *Saint-Venant* Model by backstepping boundary control", *IEEE Transactions on Automatic Control*, under review. 2. S.-X. Tang, B.-Z. Guo and M. Krstic, "Control designs for a coupled hyperbolic system with a matched boundary disturbance", in preparation. The dissertation author is one of the primary investigators and authors of these papers, and would like to thank Ababacar Diagne, Mamadou Diagne, Bao-Zhu Guo and Miroslav Krstic for their contributions.

Chapter 6

Control Designs for the Coupled Hyperbolic Systems with a Matched Boundary Disturbance

6.1 Introduction

6.1.1 Problem statement

In this chapter, the coupled first-order linear hyperbolic partial differential equation (PDE) systems are subjected to an external disturbance which is assumed to be matched with the Dirichlet control input. The main goal is to stabilize the system while rejecting or attenuating the disturbance.

More specifically, we are concerned with stabilizing the following coupled system of n leftwards and m rightwards transport PDEs with spatially varying coefficients (see, Figure 6.1):

$$\boldsymbol{u}_t(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{u}_x(t,x) = \boldsymbol{S}^{uu}(x)\boldsymbol{u}(t,x) + \boldsymbol{S}^{uv}(x)\boldsymbol{v}(t,x), \tag{6.1}$$

$$\boldsymbol{v}_t(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{v}_x(t,x) = \boldsymbol{S}^{vu}(x)\boldsymbol{u}(t,x) + \boldsymbol{S}^{vv}(x)\boldsymbol{v}(t,x), \tag{6.2}$$

$$\boldsymbol{u}(t,0) = \boldsymbol{Q}\boldsymbol{v}(t,0), \tag{6.3}$$

$$\boldsymbol{v}(t,1) = \boldsymbol{R}\boldsymbol{u}(t,1) + \boldsymbol{U}(t) + \boldsymbol{d}(t), \tag{6.4}$$

$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), \tag{6.5}$$

$$\boldsymbol{v}(0,x) = \boldsymbol{v}_0(x), \tag{6.6}$$

where

$$\boldsymbol{u}(t,x) = \Big[u_1(t,x),\, u_2(t,x),\, \ldots,\, u_n(t,x)\Big]^T, \tag{6.7}$$

$$\boldsymbol{v}(t,x) = \Big[v_1(t,x),\, v_2(t,x),\, \ldots,\, v_m(t,x)\Big]^T \tag{6.8}$$

are the systems states with $t > 0,\ x \in [0,1]$. The matrices

$$\boldsymbol{\Lambda}^{\mathbf{r}}(x) = \operatorname{diag}\Big[\lambda_1^{\mathrm{r}}(x)\,,\, \lambda_2^{\mathrm{r}}(x)\,,\, \cdots\,,\, \lambda_n^{\mathrm{r}}(x)\Big] \in \mathcal{M}_n(\mathbb{R}), \tag{6.9}$$

$$\boldsymbol{\Lambda}^{\mathbf{l}}(x) = \operatorname{diag}\Big[\lambda_1^{\mathrm{l}}(x)\,,\, \lambda_2^{\mathrm{l}}(x)\,,\, \cdots\,,\, \lambda_m^{\mathrm{l}}(x)\Big] \in \mathcal{M}_m(\mathbb{R}), \tag{6.10}$$

where

$$0 < \lambda_1^{\mathrm{r}}(x) < \lambda_2^{\mathrm{r}}(x) < \cdots < \lambda_n^{\mathrm{r}}(x), \tag{6.11}$$

$$-\lambda_m^{\mathrm{l}}(x) < -\lambda_2^{\mathrm{l}}(x) \cdots < -\lambda_1^{\mathrm{l}}(x) < 0,\ \forall x \in [0,1], \tag{6.12}$$

and the in-domain parameters are

$$\boldsymbol{S}^{uu}(x) = \left(S_{ij}^{uu}(x)\right)_{1\le i\le n, 1\le j\le n}, \tag{6.13}$$

$$\boldsymbol{S}^{uv}(x) = \left(S_{ij}^{uv}(x)\right)_{1\le i\le n, 1\le j\le m}, \tag{6.14}$$

$$\boldsymbol{S}^{vu}(x) = \left(S_{ij}^{vu}(x)\right)_{1\le i\le m, 1\le j\le n}, \tag{6.15}$$

$$\boldsymbol{S}^{vv}(x) = \left(S_{ij}^{vv}(x)\right)_{1\le i\le m, 1\le j\le m}. \tag{6.16}$$

The matrices $\boldsymbol{Q}$, $\boldsymbol{R}$ are given as

$$\boldsymbol{Q} = \{q_{ij}\}_{1\le i\le n, 1\le j\le m}, \tag{6.17}$$

$$\boldsymbol{R} = \{r_{ij}\}_{1\le i\le m, 1\le j\le n}, \tag{6.18}$$

and

$$\boldsymbol{U}(t) = \Big[U_1(t),\, U_2(t),\, \ldots,\, U_m(t)\Big]^T, \tag{6.19}$$

$$\boldsymbol{d}(t) = \Big[d_1(t),\, d_2(t),\, \ldots,\, d_m(t)\Big]^T \tag{6.20}$$

denote the boundary control inputs and the external disturbances entering at the input boundary, respectively.

All the investigation will be conducted under the following assumption.

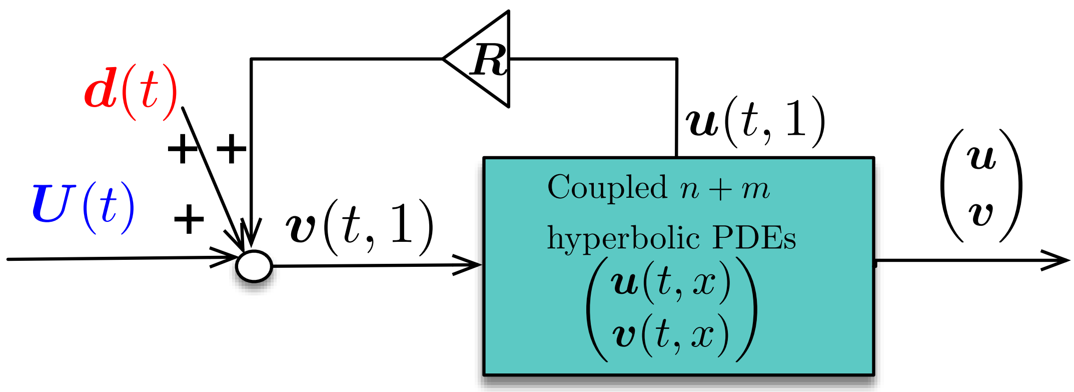

Figure 6.1: Block diagram of the control systems with input-matched disturbance.

Assumption 6.1.

1. $\mathbf{\Lambda^r}(x) \in C^1\left(0,1;\mathcal{M}_n(\mathbb{R})\right), \mathbf{\Lambda^l}(x) \in C^1\left(0,1;\mathcal{M}_m(\mathbb{R})\right)$, and we denote

$$\underline{\lambda} := \min\left\{\lambda_i^r(x), \lambda_j^l(x);\ x \in [0,1],\ i = \overline{1,n},\ j = \overline{1,m}\right\}, \tag{6.21}$$

$$\overline{\lambda} := \max\left\{\lambda_i^r(x), \lambda_j^l(x);\ x \in [0,1],\ i = \overline{1,n},\ j = \overline{1,m}\right\}. \tag{6.22}$$

2. $\boldsymbol{S}^{uu}(x) \in L^\infty\left(0,1;\mathcal{M}_n(\mathbb{R})\right),\ \boldsymbol{S}^{uv}(x) \in L^\infty\left(0,1;\mathcal{M}_{n,m}(\mathbb{R})\right),$
   $\boldsymbol{S}^{vu}(x) \in L^\infty\left(0,1;\mathcal{M}_{m,n}(\mathbb{R})\right)$, $\boldsymbol{S}^{vv}(x) \in L^\infty\left(0,1;\mathcal{M}_m(\mathbb{R})\right)$, and we denote

$$M_{\boldsymbol{S}} := \max_{x\in[0,1]}\{\|\boldsymbol{S}^{uu}(x)\|, \|\boldsymbol{S}^{uv}(x)\|\}, \tag{6.23}$$

3. $\boldsymbol{Q} \in \mathcal{M}_{n,m}(\mathbb{R})$, $\boldsymbol{R} \in \mathcal{M}_{m,n}(\mathbb{R})$, and we denote

$$\bar{q} := \|\boldsymbol{Q}^T\boldsymbol{Q}\|. \tag{6.24}$$

4. $\boldsymbol{d}(t) \in L^\infty(0,\infty) \cap C^1(0,\infty)$, and we denote

$$M_d := \operatorname*{ess\,sup}_{t\geq 0}\{|\boldsymbol{d}(t)|\}. \tag{6.25}$$

Here and in the sequel, $|\cdot|$ denotes the Euclidean norm and $\|\cdot\|$ denotes the 2−norm of a matrix.

6.1.2 Literature review

Two approaches are to be utilized in this chapter for dealing with the control matched disturbances: sliding mode control (SMC) and active disturbance rejection control (ADRC). Before applying these control algorithms to handle the disturbance, a backstepping transformation from Chapter 5 is applied to pretreat the coupled system. As a result, the transformed system is a cascaded one, i.e., with a simpler structure. In the meantime, the control problem becomes simplified. This technique could be referred to [75, 76].

Following the SMC design process, a sliding mode surface (SMS) is first found, on which the system is exponentially stable. Then, inspired by a derived “reaching condition”, the SMC is constructed. Disturbance rejection is achieved for the resulting closed-loop system with SMC. In particular, convergence of state trajectories to the chosen infinite-dimensional SMS takes place in a finite time. Then on the sliding surface, the system is exponentially stable with a decay rate depending on the spatially varying system coefficients. Since SMC deals with the worst case, it is enforced by high gain feedback. It is worth noting that the SMC configuration in [77, 75] comes across a problem of algebraic loops and is thus not realizable. This aforementioned problem is solved in this chapter by considering another SMS.

Before building the ADRC, an extended state observer (ESO) is first constructed, which serves as a disturbance estimator. Rather than the usual constant high gain, the designed ESO in this chapter has a time-varying high gain instead. Then, the designed ADRC works jointly and simultaneously with the ESO, achieving asymptotic attenuation of the disturbance.

6.1.3 Organization

The remaining parts of this chapter is organized as follows. In Section 6.2, the invertible Volterra integral transformation in Chapter 5 is applied to map the original coupled systems into cascaded ones, and some preliminary studies are made on the linear operators associated to these transformed systems. In Section 6.3, an infinite-dimensional sliding surface is first designed, on which the systems are exponentially stable. Then, a sliding mode boundary controller is designed and the finite time “reaching condition” is derived for achieving the disturbance rejection. Existence and uniqueness of solution to the resulting closed-loop systems are proved as well. Section 6.4 is devoted to disturbance attenuation by the ADRC approach. An extended state estimator with a time varying gain is designed to estimate the disturbance. Based on the idea of canceling the disturbance with the approximated values provided by the estimator, a feedback controller is subsequently constructed. With regards to the resulting closed-loop control system, well-posedness of its solution is proved. Moreover, the solution is shown to tend to any arbitrary vicinity of zero as the time goes to infinity. Finally, Section 6.5

gives some concluding remarks and possible future research directions.

6.2 Preliminaries

In this section, we first transform the coupled system into a cascaded one, then we conduct some preliminary analysis on the new system through the operator semigroup theory.

6.2.1 A backstepping transformation

Following Chapter 5, we introduce a backstepping transformation:

$$\begin{pmatrix} \boldsymbol{\alpha}(t,x) \\ \boldsymbol{\beta}(t,x) \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}(t,x) \\ \boldsymbol{v}(t,x) \end{pmatrix} - \int_0^x \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{G}(x,\xi) & \boldsymbol{H}(x,\xi) \end{pmatrix} \begin{pmatrix} \boldsymbol{u}(t,\xi) \\ \boldsymbol{v}(t,\xi) \end{pmatrix} \mathrm{d}\xi, \tag{6.26}$$

where

$$\boldsymbol{\alpha}(t,x) = \Big[\alpha_1(t,x),\, \alpha_2(t,x),\, \ldots,\, \alpha_n(t,x)\Big]^T, \tag{6.27}$$

$$\boldsymbol{\beta}(t,x) = \Big[\beta_1(t,x),\, \beta_2(t,x),\, \ldots,\, \beta_m(t,x)\Big]^T. \tag{6.28}$$

The matrices $\boldsymbol{G}(x,\xi)$, $\boldsymbol{H}(x,\xi)$ of kernel functions are defined on the triangular domain $\mathbb{T} = \Big\{(x,\xi) \in \mathbb{R}^2 |\ 0 \le \xi \le x \le 1\Big\}$ and satisfy the following system of PDEs:

$$\begin{aligned} &\boldsymbol{G}_\xi(x,\xi)\boldsymbol{\Lambda}^{\mathbf{r}}(\xi) - \boldsymbol{\Lambda}^{\mathbf{l}}(x)\boldsymbol{G}_x(x,\xi) \\ &\qquad = -\boldsymbol{G}(x,\xi)\,[\boldsymbol{\Lambda}^{\mathbf{r}}(\xi)]' - \boldsymbol{G}(x,\xi)\boldsymbol{S}^{uu}(\xi) - \boldsymbol{H}(x,\xi)\boldsymbol{S}^{vu}(\xi), \end{aligned} \tag{6.29}$$

$$\begin{aligned} &\boldsymbol{H}_\xi(x,\xi)\boldsymbol{\Lambda}^{\mathbf{l}}(\xi) + \boldsymbol{\Lambda}^{\mathbf{l}}(x)\boldsymbol{H}_x(x,\xi) \\ &\qquad = -\boldsymbol{H}(x,\xi)[\boldsymbol{\Lambda}^{\mathbf{r}}(\xi)]' + \boldsymbol{G}(x,\xi)\boldsymbol{S}^{uv}(\xi) + \boldsymbol{H}(x,\xi)\boldsymbol{S}^{vv}(\xi), \end{aligned} \tag{6.30}$$

$$\boldsymbol{G}(x,x)\boldsymbol{\Lambda}^{\mathbf{r}}(x) + \boldsymbol{\Lambda}^{\mathbf{l}}(x)\boldsymbol{G}(x,x) = -\boldsymbol{S}^{vu}(x), \tag{6.31}$$

$$\boldsymbol{H}(x,x)\boldsymbol{\Lambda}^{\mathbf{l}}(x) - \boldsymbol{\Lambda}^{\mathbf{l}}(x)\boldsymbol{H}(x,x) = \boldsymbol{S}^{vv}(x), \tag{6.32}$$

$$\boldsymbol{G}(x,0)\boldsymbol{\Lambda}^{\mathbf{r}}(0)\boldsymbol{Q} - \boldsymbol{H}(x,0)\boldsymbol{\Lambda}^{\mathbf{l}}(0) = -\boldsymbol{\Delta}(x). \tag{6.33}$$

Similarly as [67], one more artificial boundary condition is needed to ensure well-posedness of the kernel equations. Here,

$$\boldsymbol{\Delta}(x) = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \delta_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \delta_{m,1}(x) & \cdots & \delta_{m,m-1}(x) & 0 \end{bmatrix}, \tag{6.34}$$

where $\delta_{i,j}(x)$ for $i = \overline{2,\ m}$, $j = \overline{1,\ m-1}$ are decided from the transformation (6.26) and its inverse, and we assume that $\int_0^1 [\boldsymbol{\Lambda}^{\mathrm{l}}(\xi)]^{-1}\boldsymbol{\Delta}(\xi)\,\mathrm{d}\xi - \boldsymbol{I}$ is nonsingular, where $\boldsymbol{I}$ denotes the unit matrix.

As derived in Chapter 5, the transformation (6.26) is invertible and the inverse is:

$$\begin{pmatrix} \boldsymbol{u}(t,x) \\ \boldsymbol{v}(t,x) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}(t,x) \\ \boldsymbol{\beta}(t,x) \end{pmatrix} - \int_0^x \begin{pmatrix} 0 & 0 \\ \boldsymbol{\mathcal{G}}(x,\xi) & \boldsymbol{\mathcal{H}}(x,\xi) \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}(t,\xi) \\ \boldsymbol{\beta}(t,\xi) \end{pmatrix} \mathrm{d}\xi, \tag{6.35}$$

where the matrices of kernel functions $\boldsymbol{\mathcal{G}}(x,\xi), \boldsymbol{\mathcal{H}}(x,\xi)$ satisfy

$$0 = \boldsymbol{G}(x,\xi) + \boldsymbol{\mathcal{G}}(x,\xi) - \int_\xi^x \boldsymbol{\mathcal{H}}(x,\eta)\boldsymbol{G}(\eta,\xi)\ \mathrm{d}\eta, \tag{6.36}$$

$$0 = \boldsymbol{H}(x,\xi) + \boldsymbol{\mathcal{H}}(x,\xi) - \int_\xi^x \boldsymbol{\mathcal{H}}(x,\eta)\boldsymbol{H}(\eta,\xi)\ \mathrm{d}\eta. \tag{6.37}$$

The transformation (6.26) maps the the system (6.1)–(6.6) into the following system:

$$\begin{aligned} \partial_t\boldsymbol{\alpha}(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x\boldsymbol{\alpha}(t,x) &= \boldsymbol{S}^{uu}(x)\boldsymbol{\alpha}(t,x) + \boldsymbol{S}^{uv}(x)\boldsymbol{\beta}(t,x) \\ &\quad + \int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)\boldsymbol{\alpha}(t,\xi)\,\mathrm{d}\xi + \int_0^x \boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{\beta}(t,\xi)\,\mathrm{d}\xi, \end{aligned} \tag{6.38}$$

$$\partial_t\boldsymbol{\beta}(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x\boldsymbol{\beta}(t,x) = \boldsymbol{\Delta}(x)\boldsymbol{\beta}(t,0), \tag{6.39}$$

$$\boldsymbol{\alpha}(t,0) = \boldsymbol{Q}\boldsymbol{\beta}(t,0), \tag{6.40}$$

$$\begin{aligned} \boldsymbol{\beta}(t,1) &= \boldsymbol{U}(t) + \boldsymbol{d}(t) + \boldsymbol{R}\boldsymbol{\alpha}(t,1) \\ &\quad - \int_0^1 \left(\boldsymbol{G}(1,\xi) + \int_\xi^1 \boldsymbol{H}(1,\eta)\boldsymbol{\mathcal{G}}(\eta,\xi)\,\mathrm{d}\eta\right)\boldsymbol{\alpha}(t,\xi)\,\mathrm{d}\xi \\ &\quad - \int_0^1 \left(\boldsymbol{H}(1,\xi) + \int_\xi^1 \boldsymbol{H}(1,\eta)\boldsymbol{\mathcal{H}}(\eta,\xi)\,\mathrm{d}\eta\right)\boldsymbol{\beta}(t,\xi)\,\mathrm{d}\xi, \end{aligned} \tag{6.41}$$

where (6.38)–(6.39) is a "cascade" of the $\boldsymbol{\beta}$-system and the $\boldsymbol{\alpha}$-system, see, Figure 6.2. Here, $\boldsymbol{C}^{\mathrm{r}}(x,\xi)$, $\boldsymbol{C}^{\mathrm{l}}(x,\xi)$ are matrices of functions defined on the domain $\mathbb{T}$ and satisfy the following (cascaded) system of equations:

$$\boldsymbol{C}^{\mathrm{r}}(x,\xi) = \boldsymbol{S}^{\mathrm{l}}(x)\boldsymbol{G}(x,\xi) + \int_{\xi}^{x} \boldsymbol{C}^{\mathrm{l}}(x,\eta)\boldsymbol{G}(\xi,\eta)\ \mathrm{d}\eta, \tag{6.42}$$

$$\boldsymbol{C}^{\mathrm{l}}(x,\xi) = \boldsymbol{S}^{\mathrm{l}}(x)\boldsymbol{H}(x,\xi) + \int_{\xi}^{x} \boldsymbol{C}^{\mathrm{l}}(x,\eta)\boldsymbol{H}(\xi,\eta)\ \mathrm{d}\eta. \tag{6.43}$$

And we denote

$$M_{\boldsymbol{C}} := \max_{(x,\xi)\in\mathbb{T}} \{\|\boldsymbol{C}^{r}(x,\xi)\|, \|\boldsymbol{C}^{l}(x,\xi)\|\}. \tag{6.44}$$

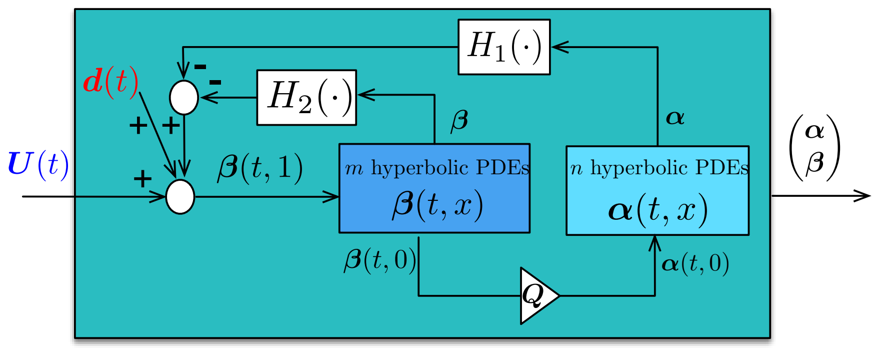

Figure 6.2: Block diagram of the transformed systems. H_1 and H_2 are operators, of which the meaning is clear from the equation (6.41).

6.2.2 The transformed system

Let

$$\boldsymbol{U}(t) = -\boldsymbol{R}\boldsymbol{\alpha}(t,1) + \int_0^1 \left(\boldsymbol{G}(1,\xi) + \int_\xi^1 \boldsymbol{H}(1,\eta)\boldsymbol{\mathcal{G}}(\eta,\xi)\,\mathrm{d}\eta\right)\boldsymbol{\alpha}(t,\xi)\,\mathrm{d}\xi$$
$$+ \int_0^1 \left(\boldsymbol{H}(1,\xi) + \int_\xi^1 \boldsymbol{H}(1,\eta)\boldsymbol{\mathcal{H}}(\eta,\xi)\,\mathrm{d}\eta\right)\boldsymbol{\beta}(t,\xi)\,\mathrm{d}\xi + \boldsymbol{U}_0(t), \tag{6.45}$$

where $\boldsymbol{U}_0(t)$ is a new controller to be designed, then (6.38)–(6.41) becomes

$$\partial_t\boldsymbol{\alpha}(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x\boldsymbol{\alpha}(t,x) = \boldsymbol{S}^{uu}(x)\boldsymbol{\alpha}(t,x) + \boldsymbol{S}^{uv}(x)\boldsymbol{\beta}(t,x)$$
$$+ \int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)\boldsymbol{\alpha}(t,\xi)\,\mathrm{d}\xi + \int_0^x \boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{\beta}(t,\xi)\,\mathrm{d}\xi, \tag{6.46}$$

$$\partial_t\boldsymbol{\beta}(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x\boldsymbol{\beta}(t,x) = \boldsymbol{\Delta}(x)\boldsymbol{\beta}(t,0), \tag{6.47}$$

$$\boldsymbol{\alpha}(t,0) = \boldsymbol{Q}\boldsymbol{\beta}(t,0), \tag{6.48}$$

$$\boldsymbol{\beta}(t,1) = \boldsymbol{U}_0(t) + \boldsymbol{d}(t). \tag{6.49}$$

A. Abstract form of the transformed system

Consider the state Hilbert space $\boldsymbol{H} = \left(L^2(0,1)\right)^{n+m}$ with the inner product given by

$$< (\boldsymbol{f}_1^T,\ \boldsymbol{g}_1^T)^T, (\boldsymbol{f}_2^T,\ \boldsymbol{g}_2^T)^T >_{\boldsymbol{H}}$$
$$= \int_0^1 \left(e^{-\nu x}\boldsymbol{f}_1(x)^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\overline{\boldsymbol{f}_2(x)} + (1+x)\boldsymbol{g}_1(x)^T\boldsymbol{D}[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\overline{\boldsymbol{g}_2(x)}\right)\mathrm{d}x,$$
$$\forall\ (\boldsymbol{f}_1^T,\ \boldsymbol{g}_1^T)^T, (\boldsymbol{f}_2^T,\ \boldsymbol{g}_2^T)^T \in \boldsymbol{H}, \tag{6.50}$$

where the parameter ν is chosen as a positive constant large enough to satisfy the following inequality:

$$\nu - \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 - \left(\frac{M_{\boldsymbol{C}}}{\underline{\lambda}}\right)^2\left(\frac{1}{\nu}+1\right) - 3 > 0, \tag{6.51}$$

and the weight matrix

$$\boldsymbol{D} := \mathrm{diag}\{d_1\,,\, d_2\,,\, \cdots\,,\, d_{m-1}\,,\, d_m\} > \boldsymbol{0}, \tag{6.52}$$

and the positive constants $d_1,\ d_2\ ,\ \cdots\ ,\ d_{m-1},\ d_m$ are chosen successively as follows:

$$d_m \geq \max\left\{\bar{q},\ \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + \frac{1}{\nu} + 2(m-1)\right\}, \tag{6.53}$$

$$d_{m-1} \geq \max\left\{\bar{q} + \int_0^1 (1+x) d_m^2 \frac{1}{\lambda_m^{\mathrm{l}}(x)^2}\delta_{m,m-1}^2(x)\,\mathrm{d}x,\ \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + \frac{1}{\nu} + 2(m-1)\right\}, \tag{6.54}$$

$$d_{m-2} \geq \max\left\{\bar{q} + \int_0^1 (1+x) \sum_{i=m-1}^{m} d_i^2 \frac{1}{\lambda_i^{\mathrm{l}}(x)^2}\delta_{i,m-2}^2(x)\,\mathrm{d}x,\right.$$
$$\left.\left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + \frac{1}{\nu} + 2(m-1)\right\}, \tag{6.55}$$

$$\vdots$$

$$d_1 \geq \max\left\{\bar{q} + \int_0^1 (1+x) \sum_{i=2}^{m} d_i^2 \frac{1}{\lambda_i^{\mathrm{l}}(x)^2}\delta_{i,1}^2(x)\,\mathrm{d}x,\ \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + \frac{1}{\nu} + 2(m-1)\right\}. \tag{6.56}$$

Denote $\|\cdot\|_{\boldsymbol{H}}$ as the norm induced by the inner product (6.104), then Section 4.3.2 tells that there exists two constants C_1, C_2 such that the following inequality holds:

$$\begin{aligned} C_1 &\|(\boldsymbol{f}^T(t,\cdot),\ \boldsymbol{g}^T(t,\cdot))^T\|^2_{(L^2(0,1))^{n+m}} \\ &\leq \left\|\left(\boldsymbol{f}^T(t,\cdot),\ \boldsymbol{g}^T(t,\cdot)\right)^T\right\|^2_{\boldsymbol{H}} \\ &\leq C_2 \|(\boldsymbol{f}^T(t,\cdot),\ \boldsymbol{g}^T(t,\cdot))^T\|^2_{(L^2(0,1))^{n+m}}. \end{aligned} \tag{6.57}$$

Define an operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A})(\subset \boldsymbol{H}) \to \boldsymbol{H}$ as follows:

$$\begin{aligned} \mathcal{A}(\boldsymbol{f}^T(x), \boldsymbol{g}^T(x))^T = &([-\boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{f}'(x) + \boldsymbol{S}^{uu}(x)\boldsymbol{f}(x) + \boldsymbol{S}^{uv}(x)\boldsymbol{g}(x) \\ &+ \int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)\boldsymbol{f}(\xi)\,\mathrm{d}\xi + \int_0^x \boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{g}(\xi)\,\mathrm{d}\xi\Big]^T, \\ &\left[\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{g}' + \boldsymbol{\Delta}(x)\boldsymbol{g}(0)\right]^T\Big)^T,\ \forall(\boldsymbol{f},\boldsymbol{g})^T \in \mathrm{Dom}(\mathcal{A}), \end{aligned} \tag{6.58}$$

$$\mathrm{Dom}(\mathcal{A}) = \{(\boldsymbol{f}^T, \boldsymbol{g}^T)^T \in \left(H^1(0,1)\right)^{n+m};\ \boldsymbol{f}(0) = \boldsymbol{Q}\boldsymbol{g}(0), \boldsymbol{g}(1) = \boldsymbol{0}\}, \tag{6.59}$$

for which the adjoint operator is

$$\mathcal{A}^*(\boldsymbol{\phi}^T(x), \boldsymbol{\psi}^T(x))^T$$

$$
\begin{aligned}
&= \Bigg(\bigg[\boldsymbol{\Lambda}^{\mathrm{r}}(x)\{-\nu\boldsymbol{\phi}(x) + \boldsymbol{\phi}'(x) + [\boldsymbol{S}^{uu}(x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\boldsymbol{\phi}(x)\} \\
&\qquad + e^{\nu x}\boldsymbol{\Lambda}^{\mathrm{r}}(x)\int_x^1 e^{-\nu\xi}[\boldsymbol{C}^{\mathrm{r}}(\xi,x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(\xi)]^{-1}\boldsymbol{\phi}(\xi)\,\mathrm{d}\xi\bigg]^T, \\
&\quad \bigg[\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{D}^{-1}\frac{e^{-\nu x}}{1+x}\{[\boldsymbol{S}^{uv}(x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\boldsymbol{\phi}(x)\} \\
&\quad + \frac{1}{1+x}\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{D}^{-1}\int_x^1 e^{-\nu\xi}[\boldsymbol{C}^{\mathrm{l}}(\xi,x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(\xi)]^{-1}\boldsymbol{\phi}(\xi)\,\mathrm{d}\xi \\
&\quad - \frac{1}{1+x}\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{\psi}(x) - \boldsymbol{\Lambda}^{\mathbf{l}}(x)\boldsymbol{\psi}'(x)\bigg]^T\Bigg)^T,\ \forall(\boldsymbol{\phi},\boldsymbol{\psi})^T\in \mathrm{Dom}(\mathcal{A}^*),
\end{aligned}
\tag{6.60}
$$

$$
\begin{aligned}
\mathrm{Dom}(\mathcal{A}^*) = \Big\{ &\left(\boldsymbol{\phi}^T,\boldsymbol{\psi}^T\right)^T \in \left(H^1(0,1)\right)^2; \\
&\boldsymbol{Q}^T\boldsymbol{\phi}(0) = \boldsymbol{D}\boldsymbol{\psi}(0) - \int_0^1 (1+x)\boldsymbol{\Delta}(x)^T\boldsymbol{D}[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\boldsymbol{\psi}(x)\,\mathrm{d}x,\ \boldsymbol{\phi}(1)=\mathbf{0}\Big\}.
\end{aligned}
\tag{6.61}
$$

Taking the inner product with $(\boldsymbol{\phi}^T,\boldsymbol{\psi}^T)^T \in \mathrm{Dom}(\mathcal{A}^*)$ on both sides of (6.46)–(6.49), one gets

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \begin{pmatrix}\boldsymbol{\alpha}(t,\cdot)\\ \boldsymbol{\beta}(t,\cdot)\end{pmatrix}, \begin{pmatrix}\boldsymbol{\phi}\\ \boldsymbol{\psi}\end{pmatrix}\right\rangle_{\boldsymbol{H}} = \left\langle \begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}, \mathcal{A}^*\begin{pmatrix}\boldsymbol{\phi}\\ \boldsymbol{\psi}\end{pmatrix}\right\rangle_{\boldsymbol{H}} \\
&+ 2\left\langle \begin{pmatrix}0\\ \frac{1}{x+1}\delta(x-1)\boldsymbol{\Lambda}^{\mathrm{l}}(x)\end{pmatrix}(\boldsymbol{U}_0(t)+\boldsymbol{d}(t)), \begin{pmatrix}\boldsymbol{\phi}\\ \boldsymbol{\psi}\end{pmatrix}\right\rangle_{\mathrm{Dom}(\mathcal{A}^*)'\times\mathrm{Dom}(\mathcal{A}^*)},
\end{aligned}
\tag{6.62}
$$

then the system (6.46)–(6.49) can be written into the following abstract form in $\boldsymbol{H}$:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} = \mathcal{A}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} + \mathcal{B}(\boldsymbol{U}_0(t)+\boldsymbol{d}(t)), \tag{6.63}
$$

where

$$
\mathcal{B} = \begin{pmatrix}\mathbf{0}\\ \frac{2}{x+1}\delta(x-1)\boldsymbol{\Lambda}^{\mathrm{l}}(x)\end{pmatrix}, \tag{6.64}
$$

and $\delta(\cdot)$ denotes the Dirac distribution.

B. Exponentially stability of $e^{\mathcal{A}t}$ and admissibility of $\mathcal{B}$

Lemma 6.1 (Lemma 4.4)**.** *The C_0-semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ is exponentially stable.*

Lemma 6.2. *$\mathcal{B}$ is admissible for the C_0-semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$.*

Proof. Consider the observation problem of the dual system of (6.63):

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{\alpha}^* \\ \boldsymbol{\beta}^*\end{pmatrix} = \mathcal{A}^*\begin{pmatrix}\boldsymbol{\alpha}^* \\ \boldsymbol{\beta}^*\end{pmatrix}, \tag{6.65}$$

$$y^* = \mathcal{B}^*\begin{pmatrix}\boldsymbol{\alpha}^* \\ \boldsymbol{\beta}^*\end{pmatrix}. \tag{6.66}$$

where

$$\mathcal{B}^* = \begin{pmatrix}0 & \langle 2\delta(x-1)\boldsymbol{D}, \cdot\rangle\end{pmatrix}. \tag{6.67}$$

(Part One) Differentiate the Lyapunov function

$$\begin{aligned}E_{(\boldsymbol{\alpha}^{*T},\ \boldsymbol{\beta}^{*T})^T}(t) = {} & \frac{1}{2}\int_0^1 e^{-\nu x}\boldsymbol{\alpha}^{*T}(t,x)[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\overline{\boldsymbol{\alpha}^*(t,x)}\,\mathrm{d}x \\ & + \frac{1}{2}\int_0^1 (1+x)\boldsymbol{\beta}^{*T}(t,x)\boldsymbol{D}[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\overline{\boldsymbol{\beta}^*(t,x)}\,\mathrm{d}x\end{aligned} \tag{6.68}$$

with respect to t along the solution to (6.65)–(6.66), then

$$\begin{aligned}\dot{E}_{(\boldsymbol{\alpha}^{*T},\ \boldsymbol{\beta}^{*T})^T}(t) \le {} & -\boldsymbol{\beta}^{*T}(t,1)\boldsymbol{D}\overline{\boldsymbol{\beta}^*(t,1)} - \frac{1}{2}\zeta_1(\nu)\int_0^1 e^{-\nu x}|\boldsymbol{\alpha}(t,x)|^2\,\mathrm{d}x \\ & - \frac{1}{2}\zeta_2(\nu)\int_0^1 |\boldsymbol{\beta}(t,x)|^2\,\mathrm{d}x \qquad (6.69)\\ \le {} & 0, \qquad (6.70)\end{aligned}$$

and hence

$$E_{(\boldsymbol{\alpha}^{*T},\ \boldsymbol{\beta}^{*T})^T}(T) \le E_{(\boldsymbol{\alpha}^{*T},\ \boldsymbol{\beta}^{*T})^T}(0), \quad \forall T > 0. \tag{6.71}$$

For any $T > 0$,

$$\begin{aligned}\int_0^T |\boldsymbol{y}^*(t)|^2\,\mathrm{d}t &= 4\int_0^T |\boldsymbol{D}\boldsymbol{\beta}^*(t,1)|^2\,\mathrm{d}t \\ &\le 4\max\left\{d_i; i = \overline{1,m}\right\}\int_0^T \boldsymbol{\beta}^{*T}(t,1)\boldsymbol{D}\overline{\boldsymbol{\beta}^*(t,1)}\,\mathrm{d}t.\end{aligned} \tag{6.72}$$

From (6.69),

$$\int_0^T \boldsymbol{\beta}^{*T}(t,1)\boldsymbol{D}\overline{\boldsymbol{\beta}^*(t,1)}\,\mathrm{d}t \le E_{(\boldsymbol{\alpha}^{*T},\ \boldsymbol{\beta}^{*T})^T}(0) - E_{(\boldsymbol{\alpha}^{*T},\ \boldsymbol{\beta}^{*T})^T}(t)$$

$$\leq E_{(\boldsymbol{\alpha}^{*T},\ \boldsymbol{\beta}^{*T})^T}(0), \tag{6.73}$$

and then

$$\int_0^T |\boldsymbol{y}^*(t)|^2 \,\mathrm{d}t \leq 4 \max\left\{d_i; i = \overline{1,m}\right\} E_{(\boldsymbol{\alpha}^{*T},\ \boldsymbol{\beta}^{*T})^T}(0). \tag{6.74}$$

(Part Two) For any given $(\boldsymbol{\phi}^T, \boldsymbol{\psi}^T)^T \in \boldsymbol{H}$, we consider the existence of $\left(\boldsymbol{\phi}_1^T, \boldsymbol{\psi}_1^T\right)^T \in \mathrm{Dom}(\mathcal{A}^*)$ such that

$$\mathcal{A}^* \left(\boldsymbol{\phi}_1^T, \boldsymbol{\psi}_1^T\right)^T = (\boldsymbol{\phi}^T, \boldsymbol{\psi}^T)^T, \tag{6.75}$$

which is equivalent to the following cascaded ordinary differential equation (ODE) system:

$$\begin{aligned}
&\boldsymbol{\phi}_1'(x) - \nu\boldsymbol{\phi}_1(x) + [\boldsymbol{S}^{\mathrm{r}}(x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\boldsymbol{\phi}_1(x) \\
&+ e^{\nu x}\int_x^1 e^{-\nu\xi}[\boldsymbol{C}^{\mathrm{r}}(\xi, x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(\xi)]^{-1}\boldsymbol{\phi}_1(\xi)\,\mathrm{d}\xi = [\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\boldsymbol{\phi}, \qquad (6.76)\\
&\boldsymbol{\psi}_1(x)' + \frac{1}{1+x}\boldsymbol{\psi}_1(x) \\
&= -[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\boldsymbol{\psi} + \boldsymbol{D}^{-1}\frac{e^{-\nu x}}{1+x}\{[\boldsymbol{S}^{\mathrm{l}}(x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\boldsymbol{\phi}(x)\} \\
&\quad + \frac{1}{1+x}[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{D}^{-1}\int_x^1 e^{-\nu\xi}[\boldsymbol{C}^{\mathrm{l}}(\xi, x)]^T[\boldsymbol{\Lambda}^{\mathrm{r}}(\xi)]^{-1}\boldsymbol{\phi}(\xi)\,\mathrm{d}\xi. \qquad (6.77)
\end{aligned}$$

Note that

$$\mathcal{B}^*\mathcal{A}^{*-1}(\boldsymbol{\phi}^T, \boldsymbol{\psi}^T)^T = 2\boldsymbol{D}\boldsymbol{\psi}_1(1). \tag{6.78}$$

Hence, by lengthy derivations, it can be proved that $\mathcal{B}^*\mathcal{A}^{*-1}$ is bounded on $\boldsymbol{H}$.

Results from Part One and Part Two show that $\mathcal{B}^*$ is admissible for the C_0-semigroup $e^{\mathcal{A}^*t}$ generated by $\mathcal{A}^*$, and so is $\mathcal{B}$ for $e^{\mathcal{A}t}$. □

Lemma 6.1 together with Lemma 6.2 give the following lemma.

Lemma 6.3. *For any initial data* $\left(\boldsymbol{\alpha}^T(0,\cdot),\ \boldsymbol{\beta}^T(0,\cdot)\right)^T \in \boldsymbol{H}, \boldsymbol{U} \in (L^2_{\mathrm{loc}}(0,\infty))^m$, *there exists a unique (mild) solution to* (6.46)–(6.49), *which satisfies* (6.63) *and thus also* (6.62).

6.3 Sliding mode control design

6.3.1 Design of a sliding mode surface

Choose the sliding surface surface as

$$\boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T} = \left\{(\boldsymbol{f},\boldsymbol{g})^T \in \boldsymbol{H};\ \int_0^1 x\boldsymbol{g}(x)\,\mathrm{d}x = \boldsymbol{0}\right\}, \tag{6.79}$$

which is a closed subspace of the state space $\boldsymbol{H}$. Then set the corresponding sliding mode function for the system (6.46)–(6.49) as

$$\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t) = \int_0^1 x\boldsymbol{\beta}(t,x)\,\mathrm{d}x. \tag{6.80}$$

On the sliding surface surface $\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t) \equiv 0, \forall t \geq 0$, the system (6.46)–(6.49) becomes

$$\begin{aligned}
&\partial_t\boldsymbol{\alpha}(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x\boldsymbol{\alpha}(t,x) = \boldsymbol{S}^{uu}(x)\boldsymbol{\alpha}(t,x) + \boldsymbol{S}^{uv}(x)\boldsymbol{\beta}(t,x)\\
&\qquad + \int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)\boldsymbol{\alpha}(\xi)\,\mathrm{d}\xi + \int_0^x \boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{\beta}(\xi)\,\mathrm{d}\xi, && (6.81)\\
&\partial_t\boldsymbol{\beta}(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x\boldsymbol{\beta}(t,x) = \boldsymbol{\Delta}(x)\boldsymbol{\beta}(t,0), && (6.82)\\
&\boldsymbol{\alpha}(t,0) = \boldsymbol{Q}_0\boldsymbol{\beta}(t,0), && (6.83)\\
&\int_0^1 x\boldsymbol{\beta}(t,x)\,\mathrm{d}x = \boldsymbol{0}. && (6.84)
\end{aligned}$$

Define an operator $\boldsymbol{\mathcal{A}} : \mathrm{Dom}(\boldsymbol{\mathcal{A}})(\subset \boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}) \to \boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}$ as follows:

$$\begin{aligned}
&\boldsymbol{\mathcal{A}}(\boldsymbol{f}^T(x),\boldsymbol{g}^T(x))^T\\
&= \Big(\Big[-\boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{f}'(x) + \boldsymbol{S}^{uu}(x)\boldsymbol{f}(x) + \boldsymbol{S}^{uv}(x)\boldsymbol{g}(x)\\
&\qquad + \int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)\boldsymbol{f}(\xi)\,\mathrm{d}\xi + \int_0^x \boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{g}(\xi)\,\mathrm{d}\xi\Big]^T,\\
&\quad \left[\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{g}' + \boldsymbol{\Delta}(x)\boldsymbol{g}(0)\right]^T\Big)^T,\ \forall(\boldsymbol{f},\boldsymbol{g})^T \in \mathrm{Dom}(\boldsymbol{\mathcal{A}}), && (6.85)\\
&\mathrm{Dom}(\boldsymbol{\mathcal{A}}) = \Big\{(\boldsymbol{f}^T,\boldsymbol{g}^T)^T \in \boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T} \cap \left(H^1(0,1)\right)^{n+m};\\
&\qquad \boldsymbol{f}(0) = \boldsymbol{Q}\boldsymbol{g}(0), \int_0^1 x[\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{g}'(x) + \boldsymbol{\Delta}(x)\boldsymbol{g}(0)]\,\mathrm{d}x = \boldsymbol{0}\Big\}\\
&= \Big\{(\boldsymbol{f}^T,\boldsymbol{g}^T)^T \in \boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T} \cap \left(H^1(0,1)\right)^{n+m}; \boldsymbol{f}(0) = \boldsymbol{Q}\boldsymbol{g}(0),
\end{aligned}$$

$$\boldsymbol{\Lambda}^{\mathrm{l}}(1)\boldsymbol{g}(1)+\int_0^1 x\boldsymbol{\Delta}(x)\,\mathrm{d}x\boldsymbol{g}(0)-\int_0^1[\boldsymbol{\Lambda}^{\mathrm{l}}(x)+x(\boldsymbol{\Lambda}^{\mathrm{l}}(x))']\boldsymbol{g}(x)\,\mathrm{d}x=\mathbf{0}\Big\},\quad (6.86)$$

then the system (6.81)–(6.84) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}=\boldsymbol{\mathcal{A}}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}. \tag{6.87}$$

Lemma 6.4. *Assume that*

$$\boldsymbol{\Lambda}^{\mathrm{l}}(0)+\int_0^1 s\boldsymbol{\Delta}(s)\,\mathrm{d}s-\int_0^1\boldsymbol{\Delta}(s)\,\mathrm{d}s \tag{6.88}$$

is nonsingular, and

$$C_{\boldsymbol{g}_1}:=\frac{1}{\underline{\lambda}^2}\int_0^1\left[\|\boldsymbol{\Psi}(x)\|+\|[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]'\|\right]^2\mathrm{d}x<1, \tag{6.89}$$

$$C_{\boldsymbol{f}_1}:=\int_0^1\int_0^x\left\|\int_s^x[\boldsymbol{\Lambda}^{\mathrm{r}}(\xi)]^{-1}\boldsymbol{C}^{\mathrm{r}}(\xi,s)\,\mathrm{d}\xi e^{\int_x^s[\boldsymbol{\Lambda}^{\mathrm{r}}(\eta)]^{-1}\boldsymbol{S}^{\mathrm{r}}(\eta)\,\mathrm{d}\eta}\right\|^2\mathrm{d}s\,\mathrm{d}x<1, \tag{6.90}$$

where

$$\begin{aligned}\boldsymbol{\Psi}(s)=&-\left[\int_0^1 s\boldsymbol{\Delta}(s)\,\mathrm{d}s+\int_1^x\boldsymbol{\Delta}(s)\,\mathrm{d}s\right]\left[\boldsymbol{\Lambda}^{\mathrm{l}}(0)+\int_0^1 s\boldsymbol{\Delta}(s)\,\mathrm{d}s-\int_0^1\boldsymbol{\Delta}(s)\,\mathrm{d}s\right]^{-1}\\ &\times[\boldsymbol{\Lambda}^{\mathrm{l}}(s)+(s-1)(\boldsymbol{\Lambda}^{\mathrm{l}}(s))']+\boldsymbol{\Lambda}^{\mathrm{l}}(s)+s[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]',\end{aligned} \tag{6.91}$$

then $\boldsymbol{\mathcal{A}}^{-1}$ *exists and is compact on* $\boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}$.

Proof. For any given $(\boldsymbol{f}^T,\boldsymbol{g}^T)^T\in\boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}$, consider the existence of $(\boldsymbol{f}_1^T,\boldsymbol{g}_1^T)^T\in$ Dom($\boldsymbol{\mathcal{A}}$) such that

$$\boldsymbol{\mathcal{A}}(\boldsymbol{f}_1^T,\boldsymbol{g}_1^T)^T=(\boldsymbol{f}^T,\boldsymbol{g}^T)^T. \tag{6.92}$$

First, the $\boldsymbol{g}_1(x)$-equation

$$\boldsymbol{\Lambda}^{\mathrm{l}}(x)\boldsymbol{g}_1'+\boldsymbol{\Delta}(x)\boldsymbol{g}_1(0)=\boldsymbol{g}, \tag{6.93}$$

$$\boldsymbol{\Lambda}^{\mathrm{l}}(1)\boldsymbol{g}_1(1)+\int_0^1 x\boldsymbol{\Delta}(x)\,\mathrm{d}x\boldsymbol{g}_1(0)-\int_0^1[\boldsymbol{\Lambda}^{\mathrm{l}}(x)+x(\boldsymbol{\Lambda}^{\mathrm{l}}(x))']\boldsymbol{g}_1(x)\,\mathrm{d}x=\mathbf{0} \tag{6.94}$$

gives the following integral equation:

$$\boldsymbol{g}_1(x)=[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\Big\{\int_1^x\boldsymbol{g}(s)\,\mathrm{d}s-\left[\int_0^1 s\boldsymbol{\Delta}(s)\,\mathrm{d}s+\int_1^x\boldsymbol{\Delta}(s)\,\mathrm{d}s\right]\boldsymbol{g}_1(0)$$

$$+\int_0^1 \left[\mathbf{\Lambda}^1(s)+s(\mathbf{\Lambda}^1(s))'\right]\boldsymbol{g}_1(s)\,\mathrm{d}s+\int_1^x [\mathbf{\Lambda}^1(s)]'\boldsymbol{g}_1(s)\,\mathrm{d}s\Big\}, \tag{6.95}$$

where

$$\boldsymbol{g}_1(0)=\left[\mathbf{\Lambda}^1(0)+\int_0^1 s\mathbf{\Delta}(s)\,\mathrm{d}s-\int_0^1 \mathbf{\Delta}(s)\,\mathrm{d}s\right]^{-1} \times\Big\{-\int_0^1 \boldsymbol{g}(s)\,\mathrm{d}s+\int_0^1[\mathbf{\Lambda}^1(s)+(s-1)(\mathbf{\Lambda}^1(s))']\boldsymbol{g}_1(s)\,\mathrm{d}s\Big\}. \tag{6.96}$$

That is,

$$\begin{aligned}\boldsymbol{g}_1(x)=[\mathbf{\Lambda}^1(x)]^{-1}\Big\{&(T_{\boldsymbol{g}_1}\boldsymbol{g}_1)(x)+\int_1^x \boldsymbol{g}(s)\,\mathrm{d}s\\ &+\left[\int_0^1 s\mathbf{\Delta}(s)\,\mathrm{d}s+\int_1^x \mathbf{\Delta}(s)\,\mathrm{d}s\right]\\ &\times\left[\mathbf{\Lambda}^1(0)+\int_0^1 s\mathbf{\Delta}(s)\,\mathrm{d}s-\int_0^1 \mathbf{\Delta}(s)\,\mathrm{d}s\right]^{-1}\int_0^1 \boldsymbol{g}(s)\,\mathrm{d}s\Big\},\end{aligned} \tag{6.97}$$

where

$$\begin{aligned}&(T_{\boldsymbol{g}_1}\boldsymbol{g}_1)(x)\\ &=-\left[\int_0^1 s\mathbf{\Delta}(s)\,\mathrm{d}s+\int_1^x \mathbf{\Delta}(s)\,\mathrm{d}s\right]\left[\mathbf{\Lambda}^1(0)+\int_0^1 s\mathbf{\Delta}(s)\,\mathrm{d}s-\int_0^1 \mathbf{\Delta}(s)\,\mathrm{d}s\right]^{-1}\\ &\quad\times\left[\int_0^1[\mathbf{\Lambda}^1(s)+(s-1)(\mathbf{\Lambda}^1(s))']g_1(s)\,\mathrm{d}s\right]\\ &\quad+\int_0^1[\mathbf{\Lambda}^1(s)+s(\mathbf{\Lambda}^1(s))']\boldsymbol{g}_1(s)\,\mathrm{d}s+\int_1^x[\mathbf{\Lambda}^1(s)]'\boldsymbol{g}_1(s)\,\mathrm{d}s.\end{aligned} \tag{6.98}$$

Consider the operator $P_{\boldsymbol{g}_1}:(L^2(0,1))^m\to(L^2(0,1))^m$ defined by

$$\begin{aligned}(P_{\boldsymbol{g}_1}\cdot)(x)=[\mathbf{\Lambda}^1(x)]^{-1}\Big\{&(T_{\boldsymbol{g}_1}\cdot)+\int_1^x \boldsymbol{g}(s)\,\mathrm{d}s\\ &+\left[\int_0^1 s\mathbf{\Delta}(s)\,\mathrm{d}s+\int_1^x \mathbf{\Delta}(s)\,\mathrm{d}s\right]\\ &\times\left[\mathbf{\Lambda}^1(0)+\int_0^1 s\mathbf{\Delta}(s)\,\mathrm{d}s-\int_0^1 \mathbf{\Delta}(s)\,\mathrm{d}s\right]^{-1}\int_0^1 \boldsymbol{g}(s)\,\mathrm{d}s\Big\},\end{aligned} \tag{6.99}$$

then

$$\left\|P_{\boldsymbol{g}_1}\boldsymbol{g}_1^1 - P_{\boldsymbol{g}_1}\boldsymbol{g}_1^2\right\|_{(L^2(0,1))^m}$$
$$\leq \frac{1}{\underline{\lambda}^2}\int_0^1 \left[\|\boldsymbol{\Psi}(s)\| + \|[\boldsymbol{\Lambda}^{\mathrm{l}}(s)]'\|\right]^2 \mathrm{d}x\, \|\boldsymbol{g}_1(x) - \boldsymbol{g}_2(x)\|^2_{(L^2(0,1))^m}. \tag{6.100}$$

It can then be derived that under the assumption (6.89), there exists a unique solution $\boldsymbol{g}_1(x)$.

Second, $\boldsymbol{f}_1(x)$ needs to satisfy the following equation:

$$-\boldsymbol{\Lambda}^{\mathrm{r}}(x)\boldsymbol{f}_1'(x) + \boldsymbol{S}^{uu}(x)\boldsymbol{f}_1(x) + \boldsymbol{S}^{uv}(x)\boldsymbol{g}_1(x)$$
$$+ \int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)\boldsymbol{f}_1(\xi)\,\mathrm{d}\xi + \int_0^x \boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{g}_1(\xi)\,\mathrm{d}\xi = \boldsymbol{f}, \tag{6.101}$$
$$\boldsymbol{f}_1(0) = \boldsymbol{Q}\boldsymbol{g}_1(0). \tag{6.102}$$

As proved in Lemma 4.2, under the assumption (6.90), there exists a unique solution $\boldsymbol{f}_1$ to the initial-value problem (6.101)–(6.102).

Hence, $\boldsymbol{\mathcal{A}}^{-1}$ exists and is compact on $\boldsymbol{S}^{(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T)^T}$ by the Sobolev embedding theorem. □

Assume that[1]

$$m_\lambda := \max_{2\leq i\leq m}\left\{\frac{1}{\lambda_1^{\mathrm{l}}(1)}\left[\max_{x\in[0,1]}|\lambda_1^{\mathrm{l}}(x) + x(\lambda_1^{\mathrm{l}}(x))'|\right],\right.$$
$$\left.\frac{\sqrt{2}}{\lambda_1^{\mathrm{l}}(1)}\left[\max_{x\in[0,1]}|\lambda_i^{\mathrm{l}}(x) + x(\lambda_i^{\mathrm{l}}(x))'|\right]\right\} < 1, \tag{6.103}$$

then we define a new inner product in $\boldsymbol{H}$ as follows:

$$< (\boldsymbol{f}_1^T,\ \boldsymbol{g}_1^T)^T, (\boldsymbol{f}_2^T,\ \boldsymbol{g}_2^T)^T >_{\boldsymbol{\mathcal{H}}}$$
$$= \int_0^1 \left(e^{-\upsilon x}\boldsymbol{f}_1(x)^T[\boldsymbol{\Lambda}^{\mathrm{r}}(x)]^{-1}\overline{\boldsymbol{f}_2(x)} + (1+x)\boldsymbol{g}_1(x)^T\boldsymbol{\mathcal{D}}[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\overline{\boldsymbol{g}_2(x)}\right)\mathrm{d}x,$$
$$\forall\ (\boldsymbol{f}_1^T,\ \boldsymbol{g}_1^T)^T, (\boldsymbol{f}_2^T,\ \boldsymbol{g}_2^T)^T \in \boldsymbol{H}, \tag{6.104}$$

where the parameter υ is chosen as a positive constant large enough to satisfy the following inequality:

$$\upsilon - \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 - \left(\frac{M_{\boldsymbol{C}}}{\underline{\lambda}}\right)^2\left(\frac{1}{\upsilon} + 1\right) - 3 > 0, \tag{6.105}$$

[1] For example, consider the case when $\boldsymbol{\Lambda}^{\mathrm{l}}(x) = \frac{1}{x}\boldsymbol{I}$.

the weight matrix

$$\boldsymbol{\mathcal{D}} := \operatorname{diag}\{d_1\,,\ d_2\,,\ \cdots\,,\ d_{m-1}\,,\ d_m\} > \mathbf{0}, \tag{6.106}$$

and the positive constants d_1 , d_2 , $\cdots$, d_{m-2}, d_{m-1} , d_m are chosen successively as follows:

$$d_m \geq \max\left\{\bar{q}, \frac{1}{1-m_\lambda}\left[\left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + \frac{1}{\upsilon} + 2(m-1)\right]\right\}, \tag{6.107}$$

$$d_{m-1} \geq \max\left\{\bar{q} + \int_0^1 (1+x) d_m^2 \frac{1}{\lambda_m^1(x)^2} \delta_{m,m-1}^2(x)\,\mathrm{d}x \right. \\ \left. +4d_m \frac{1}{\lambda_m^1(1)^2}\int_0^1 x^2(m-1)|\delta_{m,m-1}(x)|^2\,\mathrm{d}x, \frac{1}{1-m_\lambda}\left[\left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + \frac{1}{\upsilon} + 2(m-1)\right]\right\}, \tag{6.108}$$

$$d_{m-2} \geq \max\left\{\bar{q} + \int_0^1 (1+x)\sum_{i=m-1}^{m} d_i^2 \frac{1}{\lambda_i^1(x)^2}\delta_{i,m-2}^2(x)\,\mathrm{d}x \right. \\ + 4\sum_{j=m-1}^{m} d_j \frac{1}{\lambda_j^1(1)^2}\int_0^1 x^2(j-1)\left|\delta_{j,m-2}(x)\right|^2\mathrm{d}x, \\ \left. \frac{1}{1-m_\lambda}\left[\left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + \frac{1}{\upsilon} + 2(m-1)\right]\right\}, \tag{6.109}$$

$$\vdots$$

$$d_2 \geq \max\left\{\bar{q} + \int_0^1 (1+x)\sum_{i=3}^{m} d_i^2 \frac{1}{\lambda_i^1(x)^2}\,\mathrm{d}\delta_{i,2}^2(x)x \right. \\ \left. + 4\sum_{j=3}^{m} d_j \frac{1}{\lambda_j^1(1)^2}\int_0^1 x^2(j-1)\left|\delta_{j,2}(x)\right|^2\mathrm{d}x, \frac{1}{1-m_\lambda}\left[\left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + \frac{1}{\upsilon} + 2(m-1)\right]\right\}, \tag{6.110}$$

$$d_1 \geq \max\left\{\bar{q} + \int_0^1 (1+x)\sum_{i=2}^{m} d_i^2 \frac{1}{\lambda_i^1(x)^2}\delta_{i,1}^2(x)\,\mathrm{d}x \right. \\ \left. + 4\sum_{j=2}^{m} d_j \frac{1}{\lambda_j^1(1)^2}\int_0^1 x^2(j-1)\left|\delta_{j,1}(x)\right|^2\mathrm{d}x, \frac{1}{1-m_\lambda}\left[\left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 + \frac{1}{\upsilon} + 2(m-1)\right]\right\}. \tag{6.111}$$

Denote $\|\cdot\|_{\boldsymbol{\mathcal{H}}}$ as the norm induced by the inner product (6.104), then it is obvious that the norm $\|\cdot\|_{\boldsymbol{\mathcal{H}}}$ is also equivalent to the L^2-norm. Indeed, the following inequality holds:

$$C_3\|(\boldsymbol{f}^T(t,\cdot),\, \boldsymbol{g}^T(t,\cdot))^T\|^2_{(L^2(0,1))^{n+m}} \le \left\|\left(\boldsymbol{f}^T(t,\cdot),\ \boldsymbol{g}^T(t,\cdot)\right)^T\right\|^2_{\boldsymbol{\mathcal{H}}} \le C_4\|(\boldsymbol{f}^T(t,\cdot),\ \boldsymbol{g}^T(t,\cdot))^T\|^2_{(L^2(0,1))^{n+m}}, \tag{6.112}$$

where

$$C_3 := \frac{1}{\bar{\lambda}}\min\left\{e^{-\upsilon},\ d_i;\, i = \overline{1,m}\right\} > 0, \tag{6.113}$$

$$C_4 := \frac{1}{\underline{\lambda}}\max\left\{1,\ 2d_i;\, i = \overline{1,m}\right\} > 0. \tag{6.114}$$

Lemma 6.5. *$\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}}^*$ are dissipative in $\boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}$, and $\boldsymbol{\mathcal{A}}$ generates a C_0-semigroup $e^{\boldsymbol{\mathcal{A}}t}$ of contractions in $\boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}$.*

Proof. Let $(\boldsymbol{f}^T,\ \boldsymbol{g}^T)^T \in \mathrm{Dom}(\boldsymbol{\mathcal{A}})$, then

$$\begin{aligned}
&\mathrm{Re}\left\langle \boldsymbol{\mathcal{A}}(\boldsymbol{f}^T,\ \boldsymbol{g}^T)^T,\ \ (\boldsymbol{f}^T,\ \boldsymbol{g}^T)^T\right\rangle_{\boldsymbol{\mathcal{H}}}\\
&\quad\le -\frac{1}{2}e^{-\upsilon}|\boldsymbol{f}(1)|^2 - \frac{1}{2}f_1(\upsilon)\int_0^1 e^{-\upsilon x}|\boldsymbol{f}(x)|^2\,\mathrm{d}x\\
&\quad - \frac{1}{2}\mathrm{Re}\int_0^1 \boldsymbol{g}(x)^T\boldsymbol{D}\overline{\boldsymbol{g}(x)}\,\mathrm{d}x - \frac{1}{2}\left[-\left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 - \frac{1}{\upsilon} - 2(m-1)\right]\int_0^1 |\boldsymbol{g}(x)|^2\,\mathrm{d}x\\
&\quad + \left\{\int_0^1 \left[[\boldsymbol{\Lambda}^1(x) + x(\boldsymbol{\Lambda}^1(x))']\boldsymbol{g}(x) - x\boldsymbol{\Delta}(x)\boldsymbol{g}(0)\right]\mathrm{d}x\right\}^T [\boldsymbol{\Lambda}^1(1)]^{-1}\boldsymbol{D}[\boldsymbol{\Lambda}^1(1)]^{-1}\\
&\qquad \times \int_0^1 \left[[\boldsymbol{\Lambda}^1(x) + x(\boldsymbol{\Lambda}^1(x))']\overline{\boldsymbol{g}(x)} - x\boldsymbol{\Delta}(x)\overline{\boldsymbol{g}(0)}\right]\mathrm{d}x\\
&\quad + |g_1(0)|^2\left(\int_0^1 \frac{(1+x)}{2}\sum_{i=2}^m \delta_{i,1}^2(x)d_i^2\frac{1}{\lambda_i^1(x)^2}\,\mathrm{d}x - \frac{1}{2}(d_1 - \bar{q})\right)\\
&\quad + |g_2(0)|^2\left(\int_0^1 \frac{(1+x)}{2}\sum_{i=3}^m \delta_{i,2}^2(x)d_i^2\frac{1}{\lambda_i^1(x)^2}\,\mathrm{d}x - \frac{1}{2}(d_2 - \bar{q})\right)\\
&\quad + \cdots\\
&\quad + |g_{m-2}(0)|^2\left(\int_0^1 \frac{(1+x)}{2}\sum_{i=m-1}^m \delta_{i,m-2}^2(x)d_i^2\frac{1}{\lambda_i^1(x)^2}\,\mathrm{d}x - \frac{1}{2}(d_{m-2} - \bar{q})\right)\\
&\quad + |g_{m-1}(0)|^2\left(\int_0^1 \frac{(1+x)}{2}\delta_{m,m-1}^2(x)d_m^2\frac{1}{\lambda_m^1(x)^2}\,\mathrm{d}x - \frac{1}{2}(d_{m-1} - \bar{q})\right)
\end{aligned}$$

$$+ |g_m(0)|^2\left(-\frac{1}{2}(d_m - \bar{q})\right), \tag{6.115}$$

where the equalities in (6.85)–(6.86) have been used, and we have from (6.105) that

$$\zeta_1(\upsilon) := \upsilon - \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 - \left(\frac{M_{\boldsymbol{C}}}{\underline{\lambda}}\right)^2 \left(\frac{1}{\upsilon} + 1\right) - 3 > 0. \tag{6.116}$$

Through some length calculations, we obtain

$$\begin{aligned} &\mathrm{Re}\left\langle \boldsymbol{\mathcal{A}}(\boldsymbol{f}^T,\ \boldsymbol{g}^T)^T,\ \ (\boldsymbol{f}^T,\ \boldsymbol{g}^T)^T\right\rangle_{\boldsymbol{\mathcal{H}}} \\ &\quad \le -\frac{1}{2}e^{-\upsilon}|\boldsymbol{f}(1)|^2 - \frac{1}{2}\zeta_1(\upsilon)\int_0^1 e^{-\upsilon x}|\boldsymbol{f}(x)|^2\,\mathrm{d}x - \frac{1}{2}\zeta_3(\upsilon)\int_0^1 |\boldsymbol{g}(x)|^2\,\mathrm{d}x \\ &\quad \le 0, \end{aligned} \tag{6.117}$$

where (6.103), (6.107)–(6.111) have been used in the first inequality, and the last line uses (6.116) and the following inequality derived from (6.107)–(6.111):

$$\zeta_3(\upsilon) := (1 - m_\lambda)\min\{d_i; i = \overline{1, m}\} - \left(\frac{M_{\boldsymbol{S}}}{\underline{\lambda}}\right)^2 - \frac{1}{\upsilon} - 2(m-1) > 0. \tag{6.118}$$

Hence, $\boldsymbol{\mathcal{A}}$ is dissipative in $\boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}$. Similarly, it can be proved that $\boldsymbol{\mathcal{A}}^*$ is also dissipative in $\boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}$. According to the Lumer-Phillips theorem [53, Corollary 4.4], $\boldsymbol{\mathcal{A}}$ generates a C_0-semigroup $e^{\boldsymbol{\mathcal{A}}t}$ of contractions in $\boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}$. □

The following lemma can be proved by following a similar way as the proof of Lemma 4.4.

Lemma 6.6. *The C_0-semigroup $e^{\boldsymbol{\mathcal{A}}t}$ generated by $\boldsymbol{\mathcal{A}}$ is exponentially stable. Thus, for any initial data $\left(\boldsymbol{\alpha}^T(0,\cdot),\ \boldsymbol{\beta}^T(0,\cdot)\right)^T \in \boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}$, there exists a unique (mild) solution to (6.81)-(6.84) such that*

$$\left(\boldsymbol{\alpha}^T(t,\cdot),\ \boldsymbol{\beta}^T(t,\cdot)\right)^T \in C([0,\infty); \boldsymbol{H}). \tag{6.119}$$

Moreover, the system (6.81)–(6.84) is exponentially stable in $\boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}$.

Transforming $\boldsymbol{S}^{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}$ and $\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t)$ through (6.26) gives the following sliding surface surface $\boldsymbol{S}^{\left(\boldsymbol{u}^T,\ \boldsymbol{v}^T\right)^T}$ and the corresponding sliding mode function

$\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)$ for the original system (6.1)–(6.6) :

$$\boldsymbol{S}^{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T} = \Big\{(\boldsymbol{f}^T, \boldsymbol{g}^T)^T \in \boldsymbol{H};\\ \int_0^1 x\left[\boldsymbol{g}(x) - \int_0^x [\boldsymbol{G}(x,\xi)\boldsymbol{f}(\xi) + \boldsymbol{H}(x,\xi)\boldsymbol{g}(\xi)]\,\mathrm{d}\xi\right]\mathrm{d}x = 0\Big\} \tag{6.120}$$

and

$$\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t) = \int_0^1 x\left[\boldsymbol{v}(t,x) - \int_0^x [\boldsymbol{G}(x,\xi)\boldsymbol{u}(t,\xi) + \boldsymbol{H}(x,\xi)\boldsymbol{v}(t,\xi)]\,\mathrm{d}\xi\right]\mathrm{d}x. \tag{6.121}$$

On $\boldsymbol{S}^{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}$, the original system (6.1)–(6.6) becomes

$$\partial_t \boldsymbol{u}(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x \boldsymbol{u}(t,x) = \boldsymbol{S}^{uu}(x)\boldsymbol{u}(t,x) + \boldsymbol{S}^{uv}(x)\boldsymbol{v}(t,x), \tag{6.122}$$

$$\partial_t \boldsymbol{v}(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x \boldsymbol{v}(t,x) = \boldsymbol{S}^{vu}(x)\boldsymbol{u}(t,x) + \boldsymbol{S}^{vv}(x)\boldsymbol{v}(t,x), \tag{6.123}$$

$$\boldsymbol{u}(t,0) = \boldsymbol{Q}\boldsymbol{v}(t,0), \tag{6.124}$$

$$\int_0^1 x\left[\boldsymbol{v}(t,x) - \int_0^x [\boldsymbol{G}(x,\xi)\boldsymbol{u}(t,\xi) + \boldsymbol{H}(x,\xi)\boldsymbol{v}(t,\xi)]\,\mathrm{d}\xi\right]\mathrm{d}x = 0. \tag{6.125}$$

From Lemma 6.1 and the equivalence between the two systems (6.81)–(6.84) and (6.122)–(6.125), exponentially stability of the system (6.122)–(6.125), i.e., the system (6.1)–(6.6) on the sliding surface, is also obtained.

Lemma 6.7. *For any initial data* $\left(\boldsymbol{u}^T(0,\cdot), \boldsymbol{v}^T(0,\cdot)\right)^T \in \boldsymbol{S}^{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}$*, there exists a unique (mild) solution to* (6.1)–(6.6) *such that*

$$\left(\boldsymbol{u}^T(t,\cdot), \boldsymbol{v}^T(t,\cdot)\right)^T \in C([0,\infty); \boldsymbol{H}). \tag{6.126}$$

Moreover, the system (6.1)–(6.6) *is exponentially stable on* $\boldsymbol{S}^{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}$.

6.3.2 State feedback controller

To motivate the controller design, differentiating (6.80) with respect to t, we get

$$\dot{\boldsymbol{S}}_{(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T)^T}(t) = \int_0^1 x\boldsymbol{\beta}_t(t,x)\,\mathrm{d}x$$

$$= \int_0^1 x[\boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x\boldsymbol{\beta}(t,x) + \boldsymbol{\Delta}(x)\boldsymbol{\beta}(t,0)]\,\mathrm{d}x$$
$$= \boldsymbol{\Lambda}^{\mathrm{l}}(1)[\boldsymbol{U}_0(t) + \boldsymbol{d}(t)] - \int_0^1 [\boldsymbol{\Lambda}^{\mathrm{l}}(x) + x(\boldsymbol{\Lambda}^{\mathrm{l}}(x))']\boldsymbol{\beta}(t,x)\,\mathrm{d}x + \int_0^1 x\boldsymbol{\Delta}(x)\boldsymbol{\beta}(t,0)\,\mathrm{d}x. \tag{6.127}$$

Choose

$$\begin{aligned}\boldsymbol{U}_0(t) =& [\boldsymbol{\Lambda}^{\mathrm{l}}(1)]^{-1}\left\{\int_0^1 [\boldsymbol{\Lambda}^{\mathrm{l}}(x) + x(\boldsymbol{\Lambda}^{\mathrm{l}}(x))']\boldsymbol{\beta}(t,x)\,\mathrm{d}x - \int_0^1 x\boldsymbol{\Delta}(x)\boldsymbol{\beta}(t,0)\,\mathrm{d}x\right\}\\ & - K\frac{\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t)}{\left[\overline{\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t)}^T\boldsymbol{\Lambda}^{\mathrm{l}}(1)\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t)\right]^{1/2}},\end{aligned} \tag{6.128}$$

where the positive constant K satisfies

$$K > \frac{\max\{\lambda_i^{\mathrm{l}}(1);\ 1 \le i \le m\}M_d}{\sqrt{\min\{\lambda_i^{\mathrm{l}}(1);\ 1 \le i \le m\}}}, \tag{6.129}$$

then

$$\dot{\boldsymbol{S}}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t) = \boldsymbol{\Lambda}^{\mathrm{l}}(1)\left[\boldsymbol{d}(t) - K\frac{\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T(t)}}{\left[\overline{\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t)}^T\boldsymbol{\Lambda}^{\mathrm{l}}(1)\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t)\right]^{1/2}}\right]$$
$$\text{for}\quad \boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t) \neq \mathbf{0}, \tag{6.130}$$

that is,

$$\dot{\boldsymbol{S}}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t) = \boldsymbol{\Lambda}^{\mathrm{l}}(1)\left[\boldsymbol{d}(t) - K\frac{\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)}{\left[\overline{\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)}^T\boldsymbol{\Lambda}^{\mathrm{l}}(1)\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)\right]^{1/2}}\right]$$
$$\text{for}\quad \boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t) \neq \mathbf{0}. \tag{6.131}$$

Thus, the following holds:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left|\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)\right|^2\\ &= 2\mathrm{Re}\left[\overline{\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)}^T\dot{\boldsymbol{S}}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)\right]\\ &= 2\mathrm{Re}\left[\overline{\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)}^T\boldsymbol{\Lambda}^{\mathrm{l}}(1)\boldsymbol{d}(t)\right]\end{aligned}$$

$$
\begin{aligned}
&- 2K\left[\overline{\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)}^T\boldsymbol{\Lambda}^{\mathrm{l}}(1)\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)\right]^{1/2}\\
&\le -2c\left|\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)\right|,\ \forall t\ge 0 \text{ a.e.}, \qquad (6.132)
\end{aligned}
$$

where

$$
c=\sqrt{\min\{\lambda_i^{\mathrm{l}}(1);\ 1\le i\le m\}}K-\max\{\lambda_i^{\mathrm{l}}(1);\ 1\le i\le m\}M_d>0. \qquad (6.133)
$$

The inequality (6.132) is the finite time "reaching condition" for the system (6.1)–(6.6). Note that, different from dealing with ODE systems, at present we do not know yet whether $\dot{\boldsymbol{S}}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)$, i.e., $\dot{\boldsymbol{S}}_{(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T)^T}(t)$, always exists. This following lemma will give a rigorous proof.

Lemma 6.8. *There exists a unique, continuous, nonzero solution to* (6.130) *on some interval* $[0, T_{\max}]$.

Proof. Suppose that for some $T_0\ge 0$, $\boldsymbol{S}_{(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T)^T}(T_0)=\boldsymbol{S}_0\ne\boldsymbol{0}$, then (6.130) is equivalent to:

$$
\begin{aligned}
\boldsymbol{S}_{(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T)^T}(t) &= \boldsymbol{S}_0+\boldsymbol{\Lambda}^{\mathrm{l}}(1)\int_{T_0}^t \boldsymbol{d}(\tau)\,\mathrm{d}\tau\\
&- K\boldsymbol{\Lambda}^{\mathrm{l}}(1)\int_{T_0}^t \frac{\boldsymbol{S}_{(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T)^T(\tau)}}{\left[\overline{\boldsymbol{S}_{(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T)^T}(\tau)}^T\boldsymbol{\Lambda}^{\mathrm{l}}(1)\boldsymbol{S}_{(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T)^T}(\tau)\right]^{1/2}}\,\mathrm{d}\tau,\ \forall t\ge T_0.
\end{aligned}
\qquad (6.134)
$$

Define a closed subspace of $C\left[T_0, T_0+\frac{b_1}{b_2M_\Omega}|\boldsymbol{S}_0|\right]$ by

$$
\begin{aligned}
\Omega=\Big\{&\boldsymbol{S}\in C\left[T_0,T_0+\frac{b_1}{b_2M_\Omega}|\boldsymbol{S}_0|\right];\\
&\boldsymbol{S}(T_0)=\boldsymbol{S}_0,\ |\boldsymbol{S}(t)|\ge\frac{b_2-b_1}{b_2}|\boldsymbol{S}_0|,\ \forall\ t\in\left[T_0,T_0+\frac{b_1}{b_2M_\Omega}|\boldsymbol{S}_0|\right]\Big\}, \qquad (6.135)
\end{aligned}
$$

where

$$
M_\Omega:=\max\{\lambda_i^{\mathrm{l}}(1);\ 1\le i\le m\}\left[M_d+K\frac{1}{\sqrt{\min\{\lambda_i^{\mathrm{l}}(1);\ 1\le i\le m\}}}\right], \qquad (6.136)
$$

and $(b_1, b_2) \in (\mathbb{N}^*)^2$ satisfies

$$b_2 > b_1 \left(1 + K \frac{\max\{\lambda_i^{\mathrm{l}}(1);\ 1 \leq i \leq m\}}{\sqrt{\min\{\lambda_i^{\mathrm{l}}(1);\ 1 \leq i \leq m\}}} \left[1 + \frac{\max\{\lambda_i^{\mathrm{l}}(1);\ 1 \leq i \leq m\}}{\min\{\lambda_i^{\mathrm{l}}(1);\ 1 \leq i \leq m\}}\right] \frac{1}{M_\Omega}\right). \tag{6.137}$$

Consider a mapping F on Ω by

$$(F\boldsymbol{S})(t) = \boldsymbol{S}_0 + \boldsymbol{\Lambda}^{\mathrm{l}}(1) \int_{T_0}^{t} \boldsymbol{d}(\tau)\, \mathrm{d}\tau - K\boldsymbol{\Lambda}^{\mathrm{l}}(1) \int_{T_0}^{t} \frac{\boldsymbol{S}(\tau)}{\left[\overline{\boldsymbol{S}(\tau)}^T \boldsymbol{\Lambda}^{\mathrm{l}}(1)\boldsymbol{S}(\tau)\right]^{1/2}}\, \mathrm{d}\tau, \tag{6.138}$$

then

$$\begin{aligned}
|(F\boldsymbol{S})(t)| &\geq |\boldsymbol{S}_0| - \max\{\lambda_i^{\mathrm{l}}(1);\ 1 \leq i \leq m\} M_d(t - T_0) \\
&\quad - K \frac{\max\{\lambda_i^{\mathrm{l}}(1);\ 1 \leq i \leq m\}}{\sqrt{\min\{\lambda_i^{\mathrm{l}}(1);\ 1 \leq i \leq m\}}}(t - T_0) \\
&= |\boldsymbol{S}_0| - M_\Omega(t - T_0) \\
&\geq \frac{b_2 - b_1}{b_2} |\boldsymbol{S}_0|,\ \forall \boldsymbol{S} \in \Omega,
\end{aligned} \tag{6.139}$$

that is, $F\Omega \subset \Omega$. Denote

$$\Upsilon(\boldsymbol{S}) := \left[\overline{\boldsymbol{S}(\tau)}^T \boldsymbol{\Lambda}^{\mathrm{l}}(1)\boldsymbol{S}(\tau)\right]^{1/2}, \tag{6.140}$$

then

$$\begin{aligned}
&|(F\boldsymbol{S}_1)(t) - (F\boldsymbol{S}_2)(t)| \\
&\leq K\boldsymbol{\Lambda}^{\mathrm{l}}(1) \int_{T_0}^{t} \left| \frac{\boldsymbol{S}_1(\tau)}{\Upsilon(\boldsymbol{S}_1(\tau))} - \frac{\boldsymbol{S}_2(\tau)}{\Upsilon(\boldsymbol{S}_2(\tau))} \right| \mathrm{d}\tau \\
&= K\boldsymbol{\Lambda}^{\mathrm{l}}(1) \int_{T_0}^{t} \left| \frac{[\boldsymbol{S}_1(\tau) - \boldsymbol{S}_2(\tau)]\Upsilon(\boldsymbol{S}_2(\tau))}{\Upsilon(\boldsymbol{S}_1(\tau))\Upsilon(\boldsymbol{S}_2(\tau))} - \frac{\boldsymbol{S}_2(\tau)[\Upsilon(\boldsymbol{S}_1(\tau)) - \Upsilon(\boldsymbol{S}_2(\tau))]}{\Upsilon(\boldsymbol{S}_1(\tau))\Upsilon(\boldsymbol{S}_2(\tau))} \right| \mathrm{d}\tau \\
&\leq K\boldsymbol{\Lambda}^{\mathrm{l}}(1) \int_{T_0}^{t} \left[\left| \frac{\boldsymbol{S}_1(\tau) - \boldsymbol{S}_2(\tau)}{\Upsilon(\boldsymbol{S}_1(\tau))} \right| \right. \\
&\qquad \left. + \left| \frac{\boldsymbol{S}_2(\tau)[\Upsilon(\boldsymbol{S}_1(\tau) - \boldsymbol{S}_1(\tau))]^2}{\Upsilon(\boldsymbol{S}_1(\tau))\Upsilon(\boldsymbol{S}_2(\tau))[\Upsilon(\boldsymbol{S}_1(\tau)) + \Upsilon(\boldsymbol{S}_2(\tau))]} \right| \right] \mathrm{d}\tau \\
&\leq K\boldsymbol{\Lambda}^{\mathrm{l}}(1) \int_{T_0}^{t} \left[\left| \frac{\boldsymbol{S}_1(\tau) - \boldsymbol{S}_2(\tau)}{\Upsilon(\boldsymbol{S}_1(\tau))} \right| \right.
\end{aligned}$$

$$+ \frac{\max\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\}\ |\boldsymbol{S}_1(\tau) - \boldsymbol{S}_2(\tau)|^2}{|\Upsilon(\boldsymbol{S}_1(\tau))| \min\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\}[|\boldsymbol{S}_1(\tau)| + |\boldsymbol{S}_2(\tau)|]}\Bigg] \mathrm{d}\tau$$
$$\le K \max\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\} \left[1 + \frac{\max\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\}}{\min\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\}}\right]$$
$$\times \int_{T_0}^{t} \left|\frac{\boldsymbol{S}_1(\tau) - \boldsymbol{S}_2(\tau)}{\Upsilon(\boldsymbol{S}_1(\tau))}\right| \mathrm{d}\tau$$
$$\le K \max\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\} \left[1 + \frac{\max\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\}}{\min\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\}}\right] \frac{b_1}{b_2 M_\Omega} |\boldsymbol{S}_0|$$
$$\times \frac{1}{\sqrt{\min\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\}}} \frac{b_2}{(b_2 - b_1)|\boldsymbol{S}_0|} \|\boldsymbol{S}_1 - \boldsymbol{S}_2\|_\Omega$$
$$= M_F \|\boldsymbol{S}_1 - \boldsymbol{S}_2\|_\Omega, \tag{6.141}$$

where $\|\boldsymbol{S}\|_\Omega = \|\boldsymbol{S}\|_{C\left[T_0, T_0 + \frac{b_1}{b_2 M_\Omega}|\boldsymbol{S}_0|\right]}$, and we have from (6.137) that

$$M_F := K \frac{\max\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\}}{\sqrt{\min\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\}}} \left[1 + \frac{\max\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\}}{\min\{\lambda_i^{\rm l}(1);\ 1 \le i \le m\}}\right] \frac{b_1}{(b_2 - b_1) M_\Omega}$$
$$< 1. \tag{6.142}$$

Then, the mapping F is a contraction mapping on Ω. By the Banach fixed point theorem, the proof can be completed.

□

The corresponding closed-loop control system (6.46)–(6.49) becomes

$$\partial_t \boldsymbol{\alpha}(t,x) + \boldsymbol{\Lambda}^{\rm r}(x) \partial_x \boldsymbol{\alpha}(t,x) = \boldsymbol{S}^{uu}(x) \boldsymbol{\alpha}(t,x) + \boldsymbol{S}^{uv}(x) \boldsymbol{\beta}(t,x)$$
$$+ \int_0^x \boldsymbol{C}^{\rm r}(x,\xi) \boldsymbol{\alpha}(t,\xi)\, \mathrm{d}\xi + \int_0^x \boldsymbol{C}^{\rm l}(x,\xi) \boldsymbol{\beta}(t,\xi)\, \mathrm{d}\xi, \tag{6.143}$$
$$\partial_t \boldsymbol{\beta}(t,x) - \boldsymbol{\Lambda}^{\rm l}(x) \partial_x \boldsymbol{\beta}(t,x) = \boldsymbol{\Delta}(x) \boldsymbol{\beta}(t,0), \tag{6.144}$$
$$\boldsymbol{\alpha}(t,0) = \boldsymbol{Q} \boldsymbol{\beta}(t,0), \tag{6.145}$$
$$\boldsymbol{\beta}(t,1) = [\boldsymbol{\Lambda}^{\rm l}(1)]^{-1} \left\{ \int_0^1 [\boldsymbol{\Lambda}^{\rm l}(x) + x(\boldsymbol{\Lambda}^{\rm l}(x))'] \boldsymbol{\beta}(t,x)\, \mathrm{d}x - \int_0^1 x \boldsymbol{\Delta}(x) \boldsymbol{\beta}(t,0)\, \mathrm{d}x \right\}$$
$$- (K + c) \frac{\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t)}{\left[\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t)^T \boldsymbol{\Lambda}^{\rm l}(1) \overline{\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t)}\right]^{1/2}} + \boldsymbol{d}(t)$$
$$\triangleq [\boldsymbol{\Lambda}^{\rm l}(1)]^{-1} \left\{ \int_0^1 [\boldsymbol{\Lambda}^{\rm l}(x) + x(\boldsymbol{\Lambda}^{\rm l}(x))'] \boldsymbol{\beta}(t,x)\, \mathrm{d}x - \int_0^1 x \boldsymbol{\Delta}(x) \boldsymbol{\beta}(t,0)\, \mathrm{d}x \right\}$$

$$+\tilde{\boldsymbol{d}}(t) \quad \text{for} \quad \boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t) \neq \mathbf{0}, \tag{6.146}$$

where

$$\tilde{\boldsymbol{d}}(t) = \left[\tilde{d}_1(t),\, \tilde{d}_2(t),\, \ldots,\, \tilde{d}_m(t)\right]^T. \tag{6.147}$$

The system (6.143)–(6.146) can be written as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} = \mathcal{A}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}\\
&+\mathcal{B}\left\{[\mathbf{\Lambda}^1(1)]^{-1}\left[\int_0^1 [\mathbf{\Lambda}^1(x) + x(\mathbf{\Lambda}^1(x))']\boldsymbol{\beta}(t,x)\,\mathrm{d}x - \int_0^1 x\mathbf{\Delta}(x)\boldsymbol{\beta}(t,0)\,\mathrm{d}x\right] + \tilde{\boldsymbol{d}}(t)\right\},
\end{aligned} \tag{6.148}$$

where the operators $\mathcal{A}$ and $\mathcal{B}$ are defined in (6.58)–(6.59) and (6.64), respectively. From Lemma 6.1 and Lemma 6.2, the following lemma is obtained.

Lemma 6.9. *For any initial data* $\left(\boldsymbol{\alpha}^T(0,\cdot), \boldsymbol{\beta}^T(0,\cdot)\right)^T \in \boldsymbol{H}$, *there exists* $T_{\max} \geq 0$, *depending on initial data, such that the system* (6.143)–(6.146) *admits a unique solution*

$$\left(\boldsymbol{\alpha}^T(t,\cdot), \boldsymbol{\beta}^T(t,\cdot)\right)^T \in C([0, T_{\max}]; \boldsymbol{H}), \tag{6.149}$$

and

$$\int_0^1 x\boldsymbol{\beta}(t,x)\,\mathrm{d}x = \mathbf{0} \tag{6.150}$$

for all $t \geq T_{\max}$. *Moreover,*

$$\boldsymbol{S}_{\left(\boldsymbol{\alpha}^T,\ \boldsymbol{\beta}^T\right)^T}(t) = \int_0^1 x\boldsymbol{\beta}(t,x)\,\mathrm{d}x \tag{6.151}$$

is continuous and monotone in $[0, T_{\max}]$.

From (6.128), the sliding mode boundary controller for the system (6.1)–(6.6) is thus defined as follows:

$$\boldsymbol{U}(t) = -\boldsymbol{R}\boldsymbol{u}(t,1) + \int_0^1 [\boldsymbol{G}(1,\xi)\boldsymbol{u}(t,\xi) + \boldsymbol{H}(1,\xi)\boldsymbol{v}(t,\xi)]\,\mathrm{d}\xi$$

$$
\begin{aligned}
&+ [\boldsymbol{\Lambda}^{\mathrm{l}}(1)]^{-1}\Bigg\{ \int_0^1 [\boldsymbol{\Lambda}^{\mathrm{l}}(x) + x(\boldsymbol{\Lambda}^{\mathrm{l}}(x))']\\
&\qquad\qquad \times \left[\boldsymbol{v}(t,x) - \int_0^x [\boldsymbol{G}(x,\xi)\boldsymbol{u}(t,\xi) + \boldsymbol{H}(x,\xi)\boldsymbol{v}(t,\xi)]\,\mathrm{d}\xi\right]\mathrm{d}x\\
&\qquad - \int_0^1 x\boldsymbol{\Delta}(x)\boldsymbol{v}(t,0)\,\mathrm{d}x\Bigg\}\\
&- (K+c)\frac{\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)}{\left[\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)^T\boldsymbol{\Lambda}^{\mathrm{l}}(1)\overline{\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)}\right]^{1/2}}. \qquad (6.152)
\end{aligned}
$$

Then, the resulting closed-loop system (see, Figure 6.3) is

$$
\partial_t\boldsymbol{u}(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x\boldsymbol{u}(t,x) = \boldsymbol{S}^{uu}(x)\boldsymbol{u}(t,x) + \boldsymbol{S}^{uv}(x)\boldsymbol{v}(t,x), \qquad (6.153)
$$

$$
\partial_t\boldsymbol{v}(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x\boldsymbol{v}(t,x) = \boldsymbol{S}^{vu}(x)\boldsymbol{u}(t,x) + \boldsymbol{S}^{vv}(x)\boldsymbol{v}(t,x), \qquad (6.154)
$$

$$
\boldsymbol{u}(t,0) = \boldsymbol{Q}\boldsymbol{v}(t,0), \qquad (6.155)
$$

$$
\begin{aligned}
&\boldsymbol{v}(t,1) = \boldsymbol{d}(t) + \int_0^1 [\boldsymbol{G}(1,\xi)\boldsymbol{u}(t,\xi) + \boldsymbol{H}(1,\xi)\boldsymbol{v}(t,\xi)]\,\mathrm{d}\xi\\
&+ [\boldsymbol{\Lambda}^{\mathrm{l}}(1)]^{-1}\Bigg\{ \int_0^1 [\boldsymbol{\Lambda}^{\mathrm{l}}(x) + x(\boldsymbol{\Lambda}^{\mathrm{l}}(x))']\\
&\qquad\qquad \times \left[\boldsymbol{v}(t,x) - \int_0^x [\boldsymbol{G}(x,\xi)\boldsymbol{u}(t,\xi) + \boldsymbol{H}(x,\xi)\boldsymbol{v}(t,\xi)]\,\mathrm{d}\xi\right]\mathrm{d}x\\
&\qquad - \int_0^1 x\boldsymbol{\Delta}(x)\boldsymbol{v}(t,0)\,\mathrm{d}x\Bigg\}\\
&- (K+c)\frac{\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)}{\left[\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)^T\boldsymbol{\Lambda}^{\mathrm{l}}(1)\overline{\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)}\right]^{1/2}} \text{ for } \boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t) \neq \boldsymbol{0}. \qquad (6.156)
\end{aligned}
$$

By the equivalence between the system (6.143)–(6.146) and the system (6.153)–(6.156), the following theorem can be proved, which includes the result in Lemma 6.7.

Theorem 6.1. *For any initial data* $\left(\boldsymbol{u}^T(0,\cdot), \boldsymbol{v}^T(0,\cdot)\right)^T \in \boldsymbol{H}$, *there exists* $T_{\max} \geq 0$, *depending on initial data, such that the system* (6.153)–(6.156) *admit a unique solution*

$$
\left(\boldsymbol{u}^T(t,\cdot), \boldsymbol{v}^T(t,\cdot)\right)^T \in C([0,T_{\max}];\boldsymbol{H}) \qquad (6.157)
$$

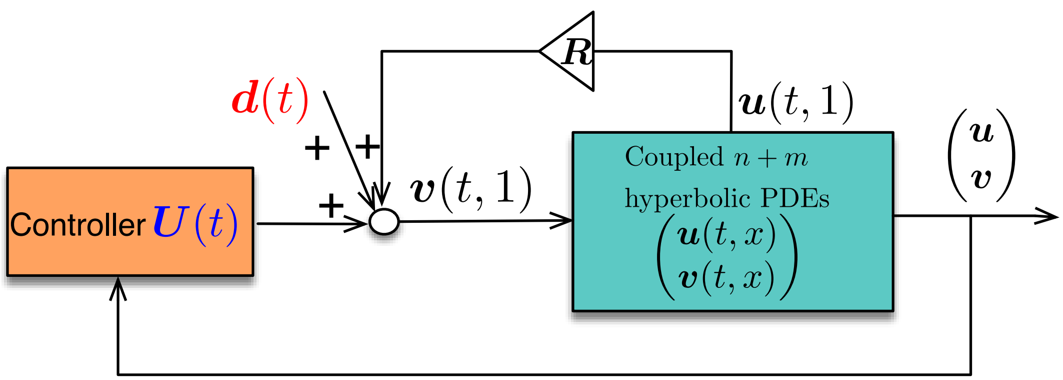

Figure 6.3: Block diagram of the closed-loop systems with SMC.

and

$$\boldsymbol{S}_{(\boldsymbol{u},\boldsymbol{v})^T}(t) = \int_0^1 x\left[\boldsymbol{v}(t,x) - \int_0^x [\boldsymbol{G}(x,\xi)\boldsymbol{u}(t,\xi) + \boldsymbol{H}(x,\xi)\boldsymbol{v}(t,\xi)]\,\mathrm{d}\xi\right]\mathrm{d}x = \mathbf{0} \tag{6.158}$$

for all $t \geq T_{\max}$. *Moreover,* $\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)$ *is continuous and monotone in* $[0, T_{\max}]$. *On the sliding mode surface* $\boldsymbol{S}_{(\boldsymbol{u}^T,\ \boldsymbol{v}^T)^T}(t)$, *the system* (6.153)–(6.156) *is exponentially stable.*

6.4 Active disturbance rejection control

An ESO with a time varying high gain is to be firstly presented, based on which the ADRC can be then designed.

6.4.1 Extended state observer with time varying gain

For any

$$\begin{pmatrix} 0 \\ \boldsymbol{\psi}(x) \end{pmatrix} \in \mathrm{Dom}(\mathcal{A}^*) \tag{6.159}$$

with

$$\boldsymbol{\psi}(x) := \Big[\psi_1(x),\, \psi_2(x),\, \ldots,\, \psi_m(x)\Big]^T, \tag{6.160}$$

it is required from (6.61) that the following equality is satisfied:

$$\boldsymbol{D}\boldsymbol{\psi}(0) = \int_0^1 (1+x)\boldsymbol{\Delta}(x)^T \boldsymbol{D}[\boldsymbol{\Lambda}^{\mathrm{l}}(x)]^{-1}\boldsymbol{\psi}(x)\,\mathrm{d}x, \tag{6.161}$$

which is equivalent to

$$\begin{cases} \psi_1(0) = d_1^{-1} \sum\limits_{i=2}^{m} \int_0^1 (1+x)\delta_{i,1}(x) d_i \left(\lambda_i^{\mathrm{l}}(x)\right)^{-1} \psi_i(x)\,\mathrm{d}x, \\ \psi_2(0) = d_2^{-1} \sum\limits_{i=3}^{m} \int_0^1 (1+x)\delta_{i,2}(x) d_i \left(\lambda_i^{\mathrm{l}}(x)\right)^{-1} \psi_i(x)\,\mathrm{d}x, \\ \qquad\qquad \vdots \\ \psi_{m-2}(0) = d_{m-2}^{-1} \sum\limits_{i=m-1}^{m} \int_0^1 (1+x)\delta_{i,m-2}(x) d_i \left(\lambda_i^{\mathrm{l}}(x)\right)^{-1} \psi_i(x)\,\mathrm{d}x, \\ \psi_{m-1}(0) = d_{m-1}^{-1} \int_0^1 (1+x)\delta_{m,m-1}(x) d_m \left(\lambda_m^{\mathrm{l}}(x)\right)^{-1} \psi_m(x)\,\mathrm{d}x, \\ \psi_m(0) = 0. \end{cases} \tag{6.162}$$

Note that it is always possible to choose a function $\varphi \in H^1(0,1)$ with a fixed value of $\varphi(0)$ to satisfy

$$\varphi(1) = 1. \tag{6.163}$$

Hence, following the equations (from the bottom line to the top line) in (6.162), one can successively choose $\psi_m(x)$, $\psi_{m-1}(x)$, $\cdots$, $\psi_2(x)$, $\psi_1(x)$ such that

$$\boldsymbol{\psi}(1) = \boldsymbol{e}_j, \forall j = 1, 2, \cdots, m. \tag{6.164}$$

Choose

$$\begin{pmatrix} 0 \\ \boldsymbol{\psi}_k \varpi(t) \end{pmatrix} \in \mathrm{Dom}(\mathcal{A}^*), \ \ k = \overline{1, m} \tag{6.165}$$

where

$$\boldsymbol{\psi}_k(1) = \boldsymbol{e}_k, \tag{6.166}$$

respectively, and the function $\varpi(t)$ is a positive C^1 function that satisfies

$$|\varpi(t)| \leq M_\varpi \text{ for all } t \geq 0 \text{ and some } M_\varpi > 0, \tag{6.167}$$

$$\lim_{t\to\infty} \left| \frac{\dot{\varpi}(t)}{\varpi(t)} \right| = 0. \tag{6.168}$$

Denote

$$\boldsymbol{\Psi}(x) := \left[\boldsymbol{\psi}_1(x),\, \boldsymbol{\psi}_2(x),\, \ldots,\, \boldsymbol{\psi}_m(x) \right], \tag{6.169}$$

then it is immediately derived from (6.166) and (6.169) that

$$\boldsymbol{\Psi}(1) = \boldsymbol{I}. \tag{6.170}$$

Let

$$\begin{aligned} y_k(t) &= \left\langle \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{\psi}_k(x)\varpi(t) \end{pmatrix} \right\rangle \\ &= \int_0^1 (1+x)\boldsymbol{\beta}(x)^T \boldsymbol{D}[\boldsymbol{\Lambda}^{\mathbf{1}}(x)]^{-1}\overline{\boldsymbol{\psi}_k(x)}\varpi(t)\,\mathrm{d}x, \end{aligned} \tag{6.171}$$

$$\begin{aligned} y_k^0(t) &= \left\langle \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \mathcal{A}^* \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{\psi}_k \varpi(t) \end{pmatrix} \right\rangle \\ &= -\int_0^1 \boldsymbol{\beta}(x)^T \boldsymbol{D} \left[\overline{\boldsymbol{\psi}_k(x)}\varpi(t) + (1+x)\overline{\boldsymbol{\psi}_k'(x)}\varpi(t) \right] \mathrm{d}x, \end{aligned} \tag{6.172}$$

then we can get from (6.62) that

$$\dot{\boldsymbol{y}}(t) = \boldsymbol{y}^0(t) + 2\boldsymbol{D}[\boldsymbol{U}_0(t) + \boldsymbol{d}(t)]\varpi(t), \tag{6.173}$$

where

$$\boldsymbol{y}(t) = \left[y_1(t),\, y_2(t),\, \ldots,\, y_m(t) \right]^T, \tag{6.174}$$

$$\boldsymbol{y}^0(t) = \left[y_1^0(t),\, y_2^0(t),\, \ldots,\, y_m^0(t)\right]^T. \tag{6.175}$$

Design an extended state estimator as follows:

$$\dot{\hat{\boldsymbol{y}}}(t) = 2\boldsymbol{D}\left[\boldsymbol{U}_0(t) + \hat{\boldsymbol{d}}(t)\right]\varpi(t) + \boldsymbol{y}^0(t) + r(t)(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)), \tag{6.176}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\hat{\boldsymbol{d}}(t)\varpi(t)] = r(t)^2\boldsymbol{D}^{-1}(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)), \tag{6.177}$$

where the time-varying gain function $r \in C(\mathbb{R}^+,\mathbb{R}^+)$ satisfies

$$\dot{r}(t) > 0, \tag{6.178}$$

$$\lim_{t\to\infty} r(t) = \infty, \tag{6.179}$$

$$\frac{\dot{r}(t)}{r(t)} \leq M_r, \forall t > 0, \text{ for some } M_r > 0, \tag{6.180}$$

$$\lim_{t\to\infty} \frac{|\dot{\boldsymbol{d}}(t)|}{r(t)} = \boldsymbol{0}. \tag{6.181}$$

In what follows, it will be clear that this estimator serves as an approximation of $\boldsymbol{d}(t)$. Besides the spatial dependency of some PDE coefficients, the analysis becomes more involved due to the time-varying high gain.

6.4.2 Active disturbance rejection control design

Design the following state feedback controller to (6.46)–(6.49):

$$\boldsymbol{U}_0(t) = -\hat{\boldsymbol{d}}(t), \tag{6.182}$$

then the closed-loop system becomes

$$\partial_t\boldsymbol{\alpha}(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x\boldsymbol{\alpha}(t,x) = \boldsymbol{S}^{uu}(x)\boldsymbol{\alpha}(t,x) + \boldsymbol{S}^{uv}(x)\boldsymbol{\beta}(t,x) + \int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)\boldsymbol{\alpha}(t,\xi)\,\mathrm{d}\xi + \int_0^x \boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{\beta}(t,\xi)\,\mathrm{d}\xi, \tag{6.183}$$

$$\partial_t\boldsymbol{\beta}(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x\boldsymbol{\beta}(t,x) = \boldsymbol{\Delta}(x)\boldsymbol{\beta}(t,0), \tag{6.184}$$

$$\boldsymbol{\alpha}(t,0) = \boldsymbol{Q}\boldsymbol{\beta}(t,0), \tag{6.185}$$

$$\boldsymbol{\beta}(t,1) = -\hat{\boldsymbol{d}}(t) + \boldsymbol{d}(t), \tag{6.186}$$

$$\dot{\hat{\boldsymbol{y}}}(t) = \boldsymbol{y}^0(t) + r(t)\left(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)\right), \tag{6.187}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{\boldsymbol{d}}(t)\varpi(t)\right] = r(t)^2\boldsymbol{D}^{-1}(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)). \tag{6.188}$$

6.4.3 Solution to the closed-loop systems

Let

$$\tilde{\boldsymbol{y}}(t) = r(t)\left(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)\right), \ \tilde{\boldsymbol{d}}(t) = \boldsymbol{d}(t) - \hat{\boldsymbol{d}}(t), \tag{6.189}$$

then

$$\dot{\tilde{\boldsymbol{y}}}(t) = \frac{\dot{r}(t)}{r(t)}\tilde{\boldsymbol{y}}(t) - r(t)\tilde{\boldsymbol{y}}(t) + 2r(t)\boldsymbol{D}\tilde{\boldsymbol{d}}(t)\varpi(t), \tag{6.190}$$

and the system (6.183)–(6.188) is equivalent to the following system:

$$\begin{aligned}
\partial_t\boldsymbol{\alpha}(t,x) + \boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x\boldsymbol{\alpha}(t,x) &= \boldsymbol{S}^{uu}(x)\boldsymbol{\alpha}(t,x) + \boldsymbol{S}^{uv}(x)\boldsymbol{\beta}(t,x) \\
&\quad + \int_0^x \boldsymbol{C}^{\mathrm{r}}(x,\xi)\boldsymbol{\alpha}(t,\xi)\,\mathrm{d}\xi + \int_0^x \boldsymbol{C}^{\mathrm{l}}(x,\xi)\boldsymbol{\beta}(t,\xi)\,\mathrm{d}\xi, && (6.191)\\
\partial_t\boldsymbol{\beta}(t,x) - \boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x\boldsymbol{\beta}(t,x) &= \boldsymbol{\Delta}(x)\boldsymbol{\beta}(t,0), && (6.192)\\
\boldsymbol{\alpha}(t,0) &= \boldsymbol{Q}\boldsymbol{\beta}(t,0), && (6.193)\\
\boldsymbol{\beta}(t,1) &= \tilde{\boldsymbol{d}}(t), && (6.194)\\
\dot{\tilde{\boldsymbol{y}}}(t) &= \frac{\dot{r}(t)}{r(t)}\tilde{\boldsymbol{y}}(t) - r(t)\tilde{\boldsymbol{y}}(t) + 2r(t)\boldsymbol{D}\tilde{\boldsymbol{d}}(t)\varpi(t), && (6.195)\\
\frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{\boldsymbol{d}}(t)\varpi(t)\right] &= -r(t)\boldsymbol{D}^{-1}\tilde{\boldsymbol{y}}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{d}(t)\varpi(t)\right]. && (6.196)
\end{aligned}$$

This is a cascaded ODE-PDE system. First, for the ODE subsystem (6.195)–(6.196), the following lemma can be proved.

Lemma 6.10. *The solution to the ODE subsystem* (6.195)*–*(6.196) *satisfies*

$$\left(\tilde{\boldsymbol{y}}(t)^T, \tilde{\boldsymbol{d}}(t)^T\right)^T \to \boldsymbol{0} \tag{6.197}$$

in the sense of the Euclidean norm.

Proof. The $\left(\tilde{\boldsymbol{y}}(t)^T, \tilde{\boldsymbol{d}}(t)^T\varpi(t)\right)$-subsystem (6.195)–(6.196) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \tilde{\boldsymbol{y}}(t) \\ \tilde{\boldsymbol{d}}(t)\varpi(t) \end{pmatrix} = \boldsymbol{F}r(t)\begin{pmatrix} \tilde{\boldsymbol{y}}(t) \\ \tilde{\boldsymbol{d}}(t)\varpi(t) \end{pmatrix} + \begin{pmatrix} \frac{\dot{r}(t)}{r(t)}\tilde{\boldsymbol{y}}(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{d}(t)\varpi(t)] \end{pmatrix}, \tag{6.198}$$

where the matrix

$$\boldsymbol{F} = \begin{pmatrix} -\boldsymbol{I} & 2\boldsymbol{D} \\ -\boldsymbol{D}^{-1} & \boldsymbol{0} \end{pmatrix}. \tag{6.199}$$

Since $\boldsymbol{F}$ is Hurwitz, then there exists a unique $2m \times 2m$ positive definite matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}\boldsymbol{F} + \boldsymbol{F}^T\boldsymbol{P} = -\boldsymbol{I}. \tag{6.200}$$

Consider the following Lyapunov function

$$V(t) = \begin{pmatrix} \tilde{\boldsymbol{y}}(t)^T & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix} \boldsymbol{P} \begin{pmatrix} \tilde{\boldsymbol{y}}(t)^T & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix}^T, \tag{6.201}$$

then

$$\begin{aligned} \lambda_{\min}(\boldsymbol{P}) & \left| \begin{pmatrix} \tilde{\boldsymbol{y}}(t)^T & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix}^T \right|^2 \\ & \leq V(t) \\ & \leq \lambda_{\max}(\boldsymbol{P}) \left| \begin{pmatrix} \tilde{\boldsymbol{y}}(t)^T & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix}^T \right|^2, \end{aligned} \tag{6.202}$$

where $\lambda_{\min}(\boldsymbol{P})$ and $\lambda_{\max}(\boldsymbol{P})$ denote the minimal and maximal eigenvalues of $\boldsymbol{P}$, respectively.

Taking the derivative of V along the solution to (6.198), we have

$$\begin{aligned} \dot{V}(t) = & \begin{pmatrix} \tilde{\boldsymbol{y}}(t)^T & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix} \boldsymbol{P} \begin{pmatrix} \dot{\tilde{\boldsymbol{y}}}(t)^T & \frac{\mathrm{d}}{\mathrm{dt}}\left[\tilde{\boldsymbol{d}}(t)^T\varpi(t)\right] \end{pmatrix}^T \\ & + \begin{pmatrix} \dot{\tilde{\boldsymbol{y}}}(t)^T & \frac{\mathrm{d}}{\mathrm{dt}}\left[\tilde{\boldsymbol{d}}(t)^T\varpi(t)\right] \end{pmatrix} \boldsymbol{P} \begin{pmatrix} \tilde{\boldsymbol{y}}(t)^T & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix}^T \\ = & \; r(t) \begin{pmatrix} \tilde{\boldsymbol{y}}(t)^T & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix} (\boldsymbol{P}\boldsymbol{F} + \boldsymbol{F}^T\boldsymbol{P}) \begin{pmatrix} \tilde{y}(t)^T & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix}^T \\ & + \begin{pmatrix} \tilde{\boldsymbol{y}}(t)^T & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix} \boldsymbol{P} \begin{pmatrix} \frac{\dot{r}(t)}{r(t)}\tilde{\boldsymbol{y}}(t)^T & \frac{\mathrm{d}}{\mathrm{dt}}\left[\boldsymbol{d}(t)^T\varpi(t)\right] \end{pmatrix}^T \\ & + \begin{pmatrix} \frac{\dot{r}(t)}{r(t)}\tilde{\boldsymbol{y}}(t)^T & \frac{\mathrm{d}}{\mathrm{dt}}\left[\boldsymbol{d}(t)^T\varpi(t)\right] \end{pmatrix} \boldsymbol{P} \begin{pmatrix} \tilde{\boldsymbol{y}}(t) & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix}^T \\ \leq & -r(t) \left| \begin{pmatrix} \tilde{\boldsymbol{y}}(t)^T & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix}^T \right|^2 + C_5 \left| \begin{pmatrix} \tilde{\boldsymbol{y}}(t)^T & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix}^T \right|^2 \\ & + C_6 \left| \frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{d}(t)^T\varpi(t)\right] \right| \left| \begin{pmatrix} \tilde{\boldsymbol{y}}(t)^T & \tilde{\boldsymbol{d}}(t)^T\varpi(t) \end{pmatrix}^T \right|, \end{aligned} \tag{6.203}$$

where C_5, C_6 are two positive constants and the last step used (6.180). Then, (6.202) and (6.203) give

$$\dot{V}(t) \leq -\frac{r(t)}{\lambda_{\max}(\boldsymbol{P})}V(t) + \frac{C_5}{\lambda_{\min}(\boldsymbol{P})}V(t) + \frac{C_6}{\sqrt{\lambda_{\min}(\boldsymbol{P})}} \left| \frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{d}(t)^T\varpi(t)\right] \right| \sqrt{V(t)}. \tag{6.204}$$

From (6.179), there exists $T > 0$ such that

$$r(t) \geq \frac{2\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})} C_5, \forall\, t \geq T. \tag{6.205}$$

By plugging (6.205) into (6.204), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{V(t)} \leq -\frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(t)\sqrt{V(t)} + \frac{C_6}{2\sqrt{\lambda_{\min}(\boldsymbol{P})}} \left| \frac{\mathrm{d}}{\mathrm{d}t} \left[\boldsymbol{d}(t)^T \varpi(t)\right] \right|, \ \forall\, t \geq T. \tag{6.206}$$

Solving

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{V(t)} = -\frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(t)\sqrt{V(t)} + \frac{C_6}{2\sqrt{\lambda_{\min}(\boldsymbol{P})}} \left| \frac{\mathrm{d}}{\mathrm{d}t} \left[\boldsymbol{d}(t)^T \varpi(t)\right] \right|, \ \forall\, t \geq T, \tag{6.207}$$

one obtains

$$\begin{aligned} \sqrt{V(t)} &= \sqrt{V(T)} e^{-\int_T^t \frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(\tau)\,\mathrm{d}\tau} \\ &\quad + \frac{C_6}{2\sqrt{\lambda_{\min}(\boldsymbol{P})}} \int_T^t e^{-\int_s^t \frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(\tau)\,\mathrm{d}\tau} \left| \frac{\mathrm{d}}{\mathrm{d}s} \left[\boldsymbol{d}(s)^T \varpi(s)\right] \right| \mathrm{d}s, \ \forall\, t \geq T, \end{aligned} \tag{6.208}$$

and

$$\begin{aligned} \lim_{t\to\infty} \sqrt{V(t)} &= \frac{C_6}{2\sqrt{\lambda_{\min}(\boldsymbol{P})}} \lim_{t\to\infty} \int_T^t e^{-\int_s^t \frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(\tau)\,\mathrm{d}\tau} \left| \frac{\mathrm{d}}{\mathrm{d}s} \left[\boldsymbol{d}(s)^T \varpi(s)\right] \right| \mathrm{d}s \\ &= \frac{C_6}{2\sqrt{\lambda_{\min}(\boldsymbol{P})}} \lim_{t\to\infty} \frac{\int_T^t e^{-\int_s^T \frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(\tau)\,\mathrm{d}\tau} \left| \frac{\mathrm{d}}{\mathrm{d}s} \left[\boldsymbol{d}(s)^T \varpi(s)\right] \right| \mathrm{d}s}{e^{-\int_t^T \frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(\tau)\,\mathrm{d}\tau}} \\ &= \frac{C_6}{2\sqrt{\lambda_{\min}(\boldsymbol{P})}} \lim_{t\to\infty} \frac{\left| \frac{\mathrm{d}}{\mathrm{d}t} \left[\boldsymbol{d}(t)^T \varpi(t)\right] \right|}{\frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(t)} \\ &= 0, \end{aligned} \tag{6.209}$$

where (6.179) is used in the first line, the L'Hôpital rule is used in the third line, and (6.167), (6.168), (6.179), (6.181) are used in the last line. Then, from the comparison principle [11, Lemma 3.4], it holds that

$$\lim_{t\to\infty} \sqrt{V(t)} = 0, \tag{6.210}$$

which, together with (6.202), further imply that

$$\lim_{t\to\infty} \tilde{\boldsymbol{y}}(t) = \mathbf{0}, \tag{6.211}$$

Thus,

$$\lim_{t\to\infty} (\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)) = \lim_{t\to\infty} \frac{\tilde{\boldsymbol{y}}(t)}{r(t)} = \mathbf{0}, \tag{6.212}$$

where (6.189) is used. Moreover, we have from (6.202) and (6.206) that

$$\begin{aligned} \left|\tilde{\boldsymbol{d}}(t)\right| \leq & \frac{\sqrt{V(T)} e^{-\int_T^t \frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(\tau)\,\mathrm{d}\tau}}{\sqrt{\lambda_{\min}(\boldsymbol{P})}\varpi(t)} \\ & + \frac{C_6}{2\lambda_{\min}(\boldsymbol{P})\varpi(t)} \frac{\int_T^t e^{-\int_s^T \frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(\tau)\,\mathrm{d}\tau} \left|\frac{\mathrm{d}}{\mathrm{d}s}\left[\boldsymbol{d}(s)^T \varpi(s)\right]\right| \mathrm{d}s}{e^{-\int_t^T \frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(\tau)\,\mathrm{d}\tau}}, \ \forall\, t \geq T. \end{aligned} \tag{6.213}$$

Since

$$\lim_{t\to\infty} \frac{\sqrt{V(T)} e^{-\int_T^t \frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(\tau)\,\mathrm{d}\tau}}{\sqrt{\lambda_{\min}(\boldsymbol{P})}\varpi(t)} = 0 \tag{6.214}$$

and

$$\begin{aligned} & \lim_{t\to\infty} \frac{\int_T^t e^{-\int_s^T \frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(\tau)\,\mathrm{d}\tau} \left|\frac{\mathrm{d}}{\mathrm{d}s}\left[\boldsymbol{d}(s)^T \varpi(s)\right]\right| \mathrm{d}s}{\varpi(t) e^{-\int_t^T \frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(\tau)\,\mathrm{d}\tau}} \\ & = \lim_{t\to\infty} \frac{\left|\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{d}(t)^T \varpi(t)\right]\right|}{\dot{\varpi}(t) - \varpi(t)\frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(t)} \\ & = \lim_{t\to\infty} \frac{\left|\dot{\boldsymbol{d}}(t)^T + \boldsymbol{d}(t)^T \frac{\dot{\varpi}(t)}{\varpi(t)}\right|}{\frac{\dot{\varpi}(t)}{\varpi(t)} - \frac{1}{4\lambda_{\max}(\boldsymbol{P})} r(t)} \\ & = 0, \end{aligned} \tag{6.215}$$

where (6.167), (6.168), (6.179) and (6.181) are used in deriving the two limits, then we obtain

$$\lim_{t\to\infty} \tilde{\boldsymbol{d}}(t) = \mathbf{0}. \tag{6.216}$$

The proof is therefore completed. □

Second, the convergence of the system (6.191)–(6.196) and also of the closed-loop system (6.183)–(6.188) can be proved.

Lemma 6.11. *For any initial data* $\left(\boldsymbol{\alpha}(0,\cdot)^T, \boldsymbol{\beta}(0,\cdot)^T\right)^T \in \boldsymbol{H}$, *there exists a unique solution*

$$\left(\boldsymbol{\alpha}(t,\cdot)^T, \boldsymbol{\beta}(t,\cdot)^T\right)^T \in C([0,\infty);\boldsymbol{H}) \tag{6.217}$$

to the closed-loop systems (6.183)*–*(6.188)*. Moreover, the solution tend to arbitrary vicinity of zero in the following sense:*

$$\lim_{t\to\infty}\left[\left\|\begin{pmatrix} \boldsymbol{\alpha}(\cdot,t)^T & \boldsymbol{\beta}(\cdot,t)^T \end{pmatrix}^T\right\|_{\boldsymbol{H}} + |\hat{\boldsymbol{y}}(t)| + |\boldsymbol{d}(t) - \hat{\boldsymbol{d}}(t)|\right] = 0. \tag{6.218}$$

Proof. From (6.197), for any given $\epsilon > 0$, there exist $t_0 > 0$ such that $\left|\tilde{\boldsymbol{d}}(t)\right| < \epsilon$ for all $t > t_0$. The $\left(\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T\right)^T$-subsystems (6.191)–(6.194) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \mathcal{A}\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} + \mathcal{B}\tilde{\boldsymbol{d}}(t). \tag{6.219}$$

From Lemma 6.3, there exists a unique solution to the system (6.219):

$$\begin{aligned}\begin{pmatrix} \boldsymbol{\alpha}(\cdot,t) \\ \boldsymbol{\beta}(\cdot,t) \end{pmatrix} =& e^{\mathcal{A}t}\begin{pmatrix} \boldsymbol{\alpha}(\cdot,0) \\ \boldsymbol{\beta}(\cdot,0) \end{pmatrix} + \int_0^t e^{\mathcal{A}(t-\tau)}\mathcal{B}\tilde{\boldsymbol{d}}(\tau)\,\mathrm{d}\tau \\ =& e^{\mathcal{A}t}\begin{pmatrix} \boldsymbol{\alpha}(\cdot,0) \\ \boldsymbol{\beta}(\cdot,0) \end{pmatrix} + \int_{t_0}^t e^{\mathcal{A}(t-\tau)}\mathcal{B}\tilde{\boldsymbol{d}}(\tau)\,\mathrm{d}\tau \\ &+ e^{\mathcal{A}(t-t_0)}\int_0^{t_0} e^{\mathcal{A}(t_0-\tau)}\mathcal{B}\tilde{\boldsymbol{d}}(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{6.220}$$

With admissibility of $\mathcal{B}$, it can be derived that

$$\left\|\int_0^{t_0} e^{\mathcal{A}(t_0-\tau)}\mathcal{B}\tilde{\boldsymbol{d}}(\tau)\,\mathrm{d}\tau\right\|_{\boldsymbol{H}}^2 \le C_{t_0}\left\|\tilde{\boldsymbol{d}}\right\|^2_{\left(L^2_{\mathrm{loc}}(0,t_0)\right)^m} \le C_{t_0}t_0^2\left\|\tilde{\boldsymbol{d}}\right\|^2_{(L^\infty(0,t_0))^m}, \forall t_0 \ge 0, \tag{6.221}$$

where C_{t_0} is a constant that is independent of $\tilde{\boldsymbol{d}}$. With exponential stability of $e^{\mathcal{A}t}$, it can be derived that

$$\left\|\int_{t_0}^t e^{\mathcal{A}(t-\tau)}\mathcal{B}\tilde{\boldsymbol{d}}(\tau)\,\mathrm{d}\tau\right\|_{\boldsymbol{H}} = \left\|\int_0^t e^{\mathcal{A}(t-\tau)}\mathcal{B}(\boldsymbol{0} \underset{t_0}{\diamond} \tilde{\boldsymbol{d}})(\tau)\,\mathrm{d}\tau\right\|_{\boldsymbol{H}}$$

$$\leq N \left\| \tilde{\boldsymbol{d}} \right\|_{(L^\infty(0,\infty))^m} \leq N\epsilon, \tag{6.222}$$

where N is a constant independent of $\tilde{\boldsymbol{d}}$, and $\boldsymbol{d}_1 \underset{s}{\diamond} \boldsymbol{d}_2$ denotes the s-concatenation of $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$:

$$(\boldsymbol{d}_1 \underset{s}{\diamond} \boldsymbol{d}_2)(\tau) = \begin{cases} \boldsymbol{d}_1(\tau), & 0 \leq \tau \leq s, \\ \boldsymbol{d}_2(\tau - s), & \tau > s. \end{cases} \tag{6.223}$$

Since from Lemma 4.4, we know that there exist three positive constants C_1, C_2, a such that

$$\|e^{\mathcal{A}t}\| \leq \sqrt{\frac{C_2}{C_1}} e^{-a/2t}, \tag{6.224}$$

then from (6.220), (6.221) and (6.222), we obtain

$$\left\| \begin{pmatrix} \boldsymbol{\alpha}(\cdot,t) \\ \boldsymbol{\beta}(\cdot,t) \end{pmatrix} \right\|_{\boldsymbol{H}} \leq e^{-a/2t} \left\| \begin{pmatrix} \boldsymbol{\alpha}(\cdot,0) \\ \boldsymbol{\beta}(\cdot,0) \end{pmatrix} \right\|_{\boldsymbol{H}} + C_{t_0} e^{-a/2(t-t_0)} \|\tilde{\boldsymbol{d}}\|_{(L^\infty(0,t_0))^m} + N\epsilon. \tag{6.225}$$

With the arbitrariness of ϵ, the proof is completed. □

By the equivalence between the transformations (6.26) and (6.35), exponentially stability of the closed-loop $\begin{pmatrix} \boldsymbol{u}(\cdot,t)^T & \boldsymbol{v}(\cdot,t)^T \end{pmatrix}^T$-system (see, Figure 6.4) is obtained. The closed-loop construction and the result are summarized in the following main theorem.

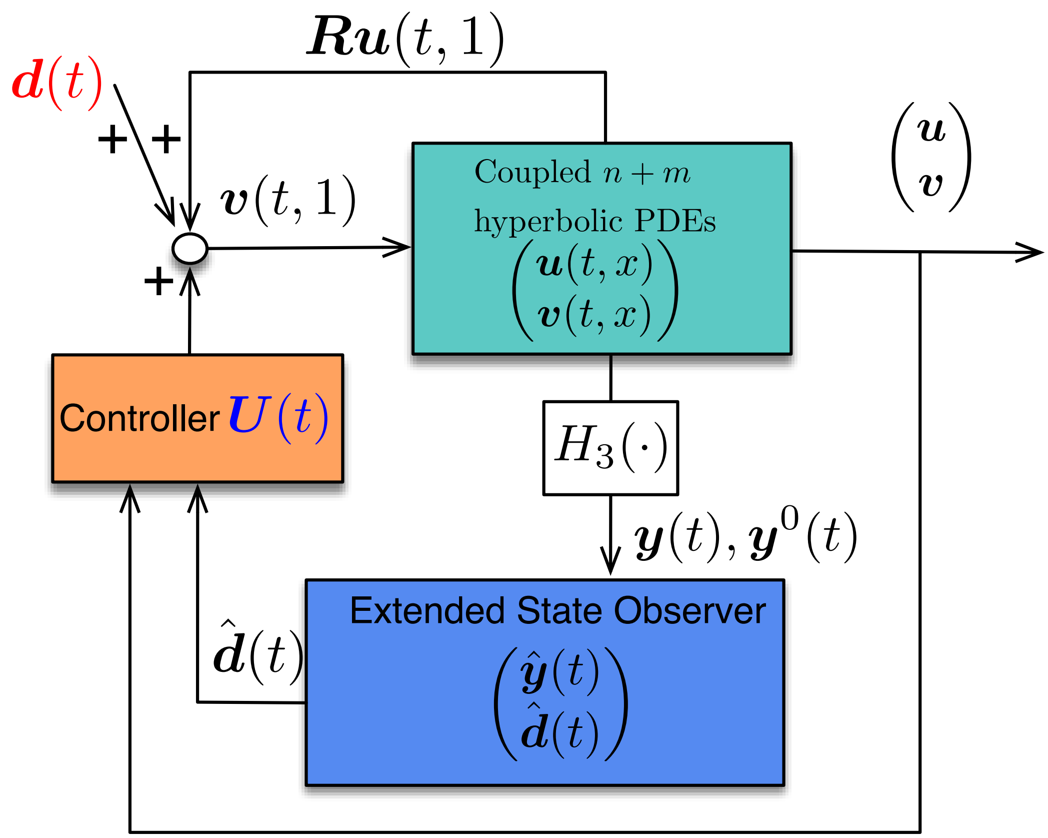

Figure 6.4: Block diagram of the closed-loop systems with ADRC. H_3 is an operator of which the meaning is clear from the equations (6.171)–(6.175).

Theorem 6.2. *For any initial data* $\left(\boldsymbol{u}^T(0,\cdot),\boldsymbol{v}^T(0,\cdot)\right)^T\in\boldsymbol{H}$, *there exists a unique (mild) solution*

$$\left(\boldsymbol{u}^T(t,\cdot),\boldsymbol{v}^T(t,\cdot)\right)^T\in C([0,\infty);\boldsymbol{H}) \tag{6.226}$$

to the following closed-loop system:

$$\partial_t\boldsymbol{u}(t,x)+\boldsymbol{\Lambda}^{\mathrm{r}}(x)\partial_x\boldsymbol{u}(t,x)=\boldsymbol{S}^{uu}(x)\boldsymbol{u}(t,x)+\boldsymbol{S}^{uv}(x)\boldsymbol{v}(t,x), \tag{6.227}$$

$$\partial_t\boldsymbol{v}(t,x)-\boldsymbol{\Lambda}^{\mathrm{l}}(x)\partial_x\boldsymbol{v}(t,x)=\boldsymbol{S}^{vu}(x)\boldsymbol{u}(t,x)+\boldsymbol{S}^{vv}(x)\boldsymbol{v}(t,x), \tag{6.228}$$

$$\boldsymbol{u}(t,0)=\boldsymbol{Q}\boldsymbol{v}(t,0), \tag{6.229}$$

$$\boldsymbol{v}(t,1)=\boldsymbol{R}\boldsymbol{u}(t,1)+\boldsymbol{U}(t)+\boldsymbol{d}(t), \tag{6.230}$$

$$\dot{\hat{\boldsymbol{y}}}(t)=2\boldsymbol{D}\left[\boldsymbol{U}_0(t)+\hat{\boldsymbol{d}}(t)\right]+\boldsymbol{y}^0(t)+r(t)(\boldsymbol{y}(t)-\hat{\boldsymbol{y}}(t)), \tag{6.231}$$

$$\dot{\hat{\boldsymbol{d}}}(t)=r(t)^2\boldsymbol{D}^{-1}(\boldsymbol{y}(t)-\hat{\boldsymbol{y}}(t)), \tag{6.232}$$

where the boundary controller is

$$\boldsymbol{U}(t)=-\boldsymbol{R}\boldsymbol{u}(t,1)+\int_0^1\left[\boldsymbol{G}(1,\xi)\boldsymbol{u}(t,\xi)+\boldsymbol{H}(1,\xi)\boldsymbol{v}(t,\xi)\right]\mathrm{d}\xi-\hat{\boldsymbol{d}}(t), \tag{6.233}$$

and

$$\boldsymbol{y}(t)=\left[y_1(t),\,y_2(t),\,\ldots,\,y_m(t)\right]^T, \tag{6.234}$$

$$\boldsymbol{y}^0(t)=\left[y_1^0(t),\,y_2^0(t),\,\ldots,\,y_m^0(t)\right]^T, \tag{6.235}$$

with

$$y_k(t)=\int_0^1(1+x)\boldsymbol{\beta}(x)^T\boldsymbol{D}[\boldsymbol{\Lambda}^{\mathbf{l}}(x)]^{-1}\overline{\boldsymbol{\psi}_k(x)}\,\mathrm{d}x, \tag{6.236}$$

$$y_k^0(t)=-\int_0^1\boldsymbol{\beta}(x)^T\boldsymbol{D}\left[\overline{\boldsymbol{\psi}_k(x)}+(1+x)\overline{\boldsymbol{\psi}_k'(x)}\right]\mathrm{d}x. \tag{6.237}$$

$\boldsymbol{\psi}_k(x),k=\overline{1,m}$ *is chosen as in Section 6.4.1.*

Moreover, the solution tend to arbitrary vicinity of zero as $t\to\infty$*:*

$$\lim_{t\to\infty}\left[\left\|\begin{pmatrix}\boldsymbol{u}(\cdot,t)^T & \boldsymbol{v}(\cdot,t)^T\end{pmatrix}^T\right\|_{\boldsymbol{H}}+|\hat{\boldsymbol{y}}(t)|+|\boldsymbol{d}(t)-\hat{\boldsymbol{d}}(t)|\right]=0. \tag{6.238}$$

6.5 Notes and references

In this chapter, a coupled system of first-order linear hyperbolic PDEs subject to boundary input disturbance is stabilized by the approaches of SMC and ADRC, respectively. Disturbance rejection and finite time stability is achieved for the resulting closed-loop control system with SMC, while disturbance attenuation and asymptotic stability are achieved for the resulting closed-loop system with ADRC.

One possible future work would be stabilization of linear coupled hyperbolic systems with non-matched boundary output disturbances.

It is also of great interest and importance to consider the application of this result into systems for which the models are amenable to this framework, e.g., the *Saint-Venant Exner* model and the bilayer *Saint-Venant* model. Also, coupled systems of first-order quasilinear hyperbolic PDEs have been topics of active research [70], and it is interesting to consider this class of systems with disturbance.

Chapter 6 contains reprints or adaptions of the following papers: 1. S.-X. Tang and M. Krstic, "Sliding mode control to stabilization of linear 2×2 hyperbolic systems with boundary input disturbance", in *Proceedings of the American Control Conference*, pp. 1027-1032, Portland, OR, USA, June 4-6, 2014. 2. S.-X. Tang, B.-Z. Guo and M. Krstic, "Active disturbance rejection control for 2×2 hyperbolic systems with input disturbance", in *Proceedings of the IFAC World Congress*, Vol. 19, No. 1, pp. 11385-11390, Cape Town, South Africa, August 24-29, 2014. 3. S.-X. Tang, B.-Z. Guo and M. Krstic, "Control designs for a coupled hyperbolic system with a matched boundary disturbance", in preparation. The dissertation author is the primary investigator and author of these papers, and would like to thank Bao-Zhu Guo and Miroslav Krstic for their contributions.

Part III

Parabolic Systems with Time-Dependent Coefficients

Chapter 7

State-of-Charge Estimation from a Thermal-Electrochemical Model of Lithium-Ion Batteries

7.1 Introduction

7.1.1 Motivation

Due to its high power and energy storage density, its lack of memory effect and low self discharge, lithium-ion technology is a common choice among the rechargeable battery family [78, 79]. Besides its wide employment in portable electronics, lithium-ion batteries are now being adopted in electrified transportation [80]: Electric Vehicles (EVs), Plug-in Hybrid Electric Vehicles (PHEVs) and Hybrid Electric Vehicles (HEVs). Lithium-ion technology is being considered for grid energy storage as well.

The key indicator for the amount of electrical energy available in batteries is the State-of-Charge (SoC) which, simply put, is the ratio of instantaneous battery capacity to its maximum capacity [81]. Thus, in order to predict the available power and energy in the battery during operation, online estimation of the SoC serves as an important factor for regulating both charging and discharging. Besides, it is generally required that the SoC remains within appropriate bounds all the time for safety reasons. Hence, a reliable and accurate estimation of the SoC is required for proper battery management.

7.1.2 Literature review

Accuracy of the SoC estimation depends highly on the quality of the selected model. Thus, one is encouraged to compare the different models available for describing the battery dynamics. Models for lithium-ion batteries can be divided into two classes. The first class consists of empirical models, in which the most frequently used ones are Equivalent Circuit Models (ECMs) [82]. ECMs use electric circuit elements such as voltage sources, resistances and RC networks to approximate the dynamics of the battery. Currently, most Battery Management Systems (BMS) employ ECMs for various tasks: power and energy estimation, cell balancing, thermal management, State-of-Health (SoH) estimation and charge/discharge control. SoC estimation methods with ECMs have been studied extensively. To name a few, Extended Kalman Filter (EKF) [83], Unscented Kalman Filters (UKF) [84], particle filters [85], linear parameter varying methods [86], polynomial chaos theory [87, 88], geometric observer [89] and sliding mode observer [90].

The second class of models are based on first principles. These electrochemical models [91] account for the main underlying physics in the battery, more precisely, they offer an explicit description of the battery dynamics in terms of the main electrochemical parameters and variables. The need for accurate SoC estimation as well as visibility of important electrochemical states and parameters, specially in high power and high energy applications, motivates the study of estimation based on electrochemical models. The widely employed Doyle-Fuller-Newman (DFN) model has been shown to accurately model the main phenomena in lithium-ion batteries [81]. However, the complexity of the model is too high for online estimation application or to conduct an observability analysis [92, 93]. Among the various approximations and reductions to the DFN model, the Single Particle Model (SPM) [94, 95] has been a regular option for online SoC estimation. The SPM is formulated by idealizing the lithium diffusion in the solid phase of each electrode as diffusion in a single spherical particle and assuming the electrolyte concentration to be uniform in both space and time. Thus, SPM is simple enough for the observer design [96, 97] and observability analysis.

Still, the SPM has several limitations, for example, being accurate only for low current [81]. Another limitation is that SPM is restricted to the cases with small variation in internal temperature, which comes from the fact that SPM ignores the dependence of the battery parameters on temperature. Indeed, lithium-ion batteries meet issues such as an increase in internal resistance and decrease of capacity, as functions of battery internal average temperature [98]. By preserving the temperature dynamics this limitation can be overcome.

Incorporating the temperature dynamics into the SPM results in a coupled Ordinary Differential Equation (ODE)-Partial Differential Equation (PDE) model, and the PDE backstepping approach [18] is employed to design an observer for this model. This method has been used for stabilization of unstable PDE systems, see [22], in which backstepping boundary controllers and observers are designed for some unstable parabolic, hyperbolic PDEs, etc. It has also been applied for stabilizing some coupled PDE-ODE systems [99, 100].

7.1.3 Organization

This work can be considered as a continuation and completion of an initial effort for SoC estimation from a thermal-electrochemical model of lithium-ion batteries in [101]. The rest of this chapter is organized as follows. In Section 7.2, a temperature-compensated SPM model is presented, and the corresponding SoC estimation problem is formulated in Section 7.3. In Section 7.4, a full state Luenberger observer is developed for estimation of the electrode lithium concentration through boundary state measurement, via the backstepping technique. The observer error system is proved to be exponentially stable with an arbitrarily designated decay rate, for which the well-posedness is derived by applying the abstract evolution equation theory. It is worth noting that solving the kernel function system for the backstepping transform is not trivial because of its dependence on the temperature [102, 103]. Under some more relaxed assumptions and simplifications than [101], the existence and regularity of the solution to the system is proved in this section. The SoC estimation accuracy is verified by the numerical simulation results presented in Section 7.5. Finally, a conclusion and some possible future works are given in Section 7.6.

7.2 The SPM-T model

In this section, the working mechanism of lithium-ion batteries is briefly introduced first, followed by an overview of the DFN model. Then an SPM-T model [101] is developed for the purpose of SoC estimation, which is a simplification of the DFN model and can be treated as a temperature-compensated SPM.

7.2.1 Working principles of lithium-ion batteries

A lithium-ion battery consists of three main regions: negative electrode, separator and positive electrode. Each electrode includes active materials, conductive fillers, a current collector and a binder. A porous configuration is fundamental in the lithium-ion batteries. The porous structure in the electrodes provides a large surface area and allows small distances between lithium ions and active material surfaces for reactions to occur, while the porous materials in the separator forbid the flow of electrons between the negative and positive electrode but allows the movement of lithium ions dissolved in the electrolyte. The active materials, intercalated in the lattices of the corresponding electrode, are insertion compounds, i.e. these are host structures in which lithium can be reversibly inserted or extracted. Electrolyte fills all remaining parts of the battery.

During discharging, lithium ions move from the interstitial site of the lattices in the negative electrode, through the electrolyte and the separator, to the one in the positive electrode.

As a result, the battery gives out power. During charging, lithium ions move back from the positive electrode to the negative one. At the same time, the battery absorbs power and energy from external power sources. In both cases, electrons flow in the opposite direction to the ions, through the outer circuit to which the battery is connected [104].

7.2.2 The DFN model

The DFN model is derived based on the porous structure all through the lithium-ion batteries [98, 81]. In the DFN model, each electrode is viewed as superposition of active materials, inert filler and electrolyte; justified from the porus configuration. As shown in Fig. 7.1, the battery is formulated as a Pseudo Two-Dimensional (P2D) model. One dimension represents the path along the spatial direction x from the anode through the electrolyte to the cathode, and the other dimension represents the path along the radial direction r_{s} of the ions into and out of the active material, by considering that spherical diffusion is symmetric.

Table 7.1: Nomenclature.

Variables	
c_{s}	Lithium concentration in the solid electrode
c_{ss}	Lithium concentration at the surface of the particle
c_{e}	Lithium concentration in the electrolyte
$\bar{c}_{\mathrm{s}}$	Volume averaged lithium concentration in the solid phase
j	Molar flux of lithium at the surface of the particle
ϕ_{s}	Electric potential in the solid electrode
ϕ_{e}	Electric potential in the electrolyte
η	Reaction overpotential
U	Open-circuit potential
i_0	Exchange current densities
i_{e}	Electrolyte current density normalized by cross-sectional area
T	Internal average temperature
T_{amb}	Ambient temperature
I	External current density normalized by cross-sectional area
V	Terminal voltage
$\bar{q}_{\mathrm{s}}$	Volume averaged flux

Table 7.1: Nomenclature, continued.

Parameters	
L	Length
D_{s}	Diffusion coefficient of lithium in the solid electrode
D_{e}	Diffusion coefficient of lithium in the electrolyte
$c_{\mathrm{s}}^{\max}$	Maximum lithium concentration in the solid electrode
R_{s}	Radius of the particle
α_{a}	Anodic transfer coefficient
α_{c}	Cathodic transfer coefficient
r_{eff}	Effective reaction rate in the solid electrode
R_{f}	Film resistance of the solid-electrolyte interphase (SEI)
R_{c}	Contact resistance between the electrode and current collector
ϵ_{s}	Volume fraction of the active material
ϵ_{e}	Volume fraction of the electrolyte
a_{s}	Interfacial surface area
F	Faraday's constant
R	Universal gas constant
N	(Nominal) total lithium ions in the battery
σ	Electronic conductivity in the solid electrode
κ	Ionic conductivity in the electrolyte
t_{c}^0	Transference number of the ions w.r.t. the solvent velocity
$f_{\mathrm{c/a}}$	Mean molar activity coefficient in the electrolyte
ρ^{avg}	Average density
c_{P}	Heat capacity
h_{cell}	Heat transfer coefficient
E	Activation energy coefficient

Super- and subscripts	
$+$	Positive electrode
$-$	Negative electrode
sep	Separator
s	Solid electrode
e	Electrolyte

This DFN model consists of several PDEs and ODEs, describing the mathematical relations among the input current, output voltage, SoC, temperature and other factors that influence the battery performance.

Diffusion dynamics of lithium ions in the solid phase of each electrode comes from Fick's law and is described as follows:

$$\frac{\partial c_{\mathrm{s}}^{\pm}}{\partial t}(t,x,r_s) = \frac{1}{r_{\mathrm{s}}^2}\frac{\partial}{\partial r_{\mathrm{s}}}\left[D_{\mathrm{s}}^{\pm}(T(t))r_{\mathrm{s}}^2\frac{\partial c_{\mathrm{s}}^{\pm}}{\partial r_{\mathrm{s}}}(t,x,r_{\mathrm{s}})\right],$$
$$t>0,\ x\in\left(0^{\pm},L^{\pm}\right),\ r_{\mathrm{s}}\in\left(0,R_{\mathrm{s}}^{\pm}\right), \tag{7.1}$$

$$\frac{\partial c_{\mathrm{s}}^{\pm}}{\partial r_{\mathrm{s}}}(t,x,0) = 0,\ t>0,\ x\in\left(0^{\pm},L^{\pm}\right), \tag{7.2}$$

$$\frac{\partial c_{\mathrm{s}}^{\pm}}{\partial r_{\mathrm{s}}}(t,x,R_{\mathrm{s}}^{\pm}) = -\frac{1}{D_{\mathrm{s}}^{\pm}(T(t))}j^{\pm}(t,x),\ t>0, x\in\left(0^{\pm},L^{\pm}\right), \tag{7.3}$$

$$c_{\mathrm{s}}^{\pm}(0,x,r_{\mathrm{s}}) = c_{\mathrm{s0}}^{\pm}(x,r_{\mathrm{s}}),\ x\in\left[0^{\pm},L^{\pm}\right], r_{\mathrm{s}}\in\left[0,R_{\mathrm{s}}^{\pm}\right], \tag{7.4}$$

where the temporal variable is t, the spatial variables are x and r_{s}. The states of the PDE model (7.1)–(7.4) are $c_{\mathrm{s}}^{\pm}(t,x,r_{\mathrm{s}})\in\mathbb{R}$, which denote the solid phase lithium-ion concentration. Diffusion coefficients are considered to be of functions with an Arrhenius-like dependency on the battery cell internal average temperature $T(t)$ [92]:

$$D_{\mathrm{s}}^{\pm}(T(t)) = D_{\mathrm{s}}^{\pm}(T(0))e^{E_{D_{\mathrm{s}}^{\pm}}\frac{T(t)-T(0)}{T(t)T(0)}}. \tag{7.5}$$

Pore wall molar ion fluxes at the surface of the particles are denoted by $j^{\pm}(t,x)$ and are related to the overpotential $\eta(t,x)$ by the following Butler-Volmer equation:

$$j^{\pm}(t,x) = \frac{i_0^{\pm}(t,x)}{F}\left[e^{\frac{\alpha_{\mathrm{a}}F}{RT(t)}\eta(t,x)} - e^{-\frac{\alpha_{\mathrm{c}}F}{RT(t)}\eta(t,x)}\right], \tag{7.6}$$

where

$$i_0^{\pm}(t,x) = r_{\mathrm{eff}}^{\pm}(T(t))\left[c_{\mathrm{ss}}^{\pm}(t,x)\right]^{\alpha_{\mathrm{c}}}\left[c_{\mathrm{e}}(t,x)\left(c_{\mathrm{s}}^{\pm,\max} - c_{\mathrm{ss}}^{\pm}(t,x)\right)\right]^{\alpha_{\mathrm{a}}}, \tag{7.7}$$

$$c_{\mathrm{ss}}^{\pm}(t,x) \triangleq c_{\mathrm{s}}^{\pm}(t,x,R_s^{\pm}), \tag{7.8}$$

and the overpotential equation $\eta(t,x)$ for the ion intercalation reaction is

$$\eta(t,x) = \phi_{\mathrm{s}}(t,x) - \phi_{\mathrm{e}}(t,x) - U^{\pm}(c_{\mathrm{ss}}^{\pm}(t,x),T(t)) - R_{\mathrm{f}}^{\pm}(T(t))Fj^{\pm}(t,x). \tag{7.9}$$

Moreover, the functions $r_{\text{eff}}^{\pm}(T(t))$ and $R_{\text{f}}^{\pm}(T(t))$ are also considered to have an Arrhenius-like dependency on the battery cell internal average temperature $T(t)$:

$$r_{\text{eff}}^{\pm}(T(t)) = r_{\text{eff}}^{\pm}(T(0))e^{E_{r_{\text{eff}}^{\pm}}\frac{T(t)-T(0)}{T(t)T(0)}}, \tag{7.10}$$

$$R_{\text{f}}^{\pm}(T(t)) = R_{\text{f}}^{\pm}(T(0))e^{E_{R_{\text{f}}^{\pm}}\frac{T(t)-T(0)}{T(t)T(0)}}. \tag{7.11}$$

The following internal average temperature equation can be referred to [98, Section 12.3.7] and [105]:

$$\begin{aligned}\rho^{\text{avg}} c_P \frac{\mathrm{d}T}{\mathrm{d}t}(t) =& h_{\text{cell}}\left(T_{\text{amb}}(t) - T(t)\right) - I(t)V(t) \\ &+ I(t)\Big[U^+(\bar{c}_{\text{s}}^+(t,x), T(t)) - U^-(\bar{c}_{\text{s}}^-(t,x), T(t)) \\ &- T(t)\frac{\partial\left[U^+(\bar{c}_{\text{s}}^+(t,x), T(t)) - U^-(\bar{c}_{\text{s}}^-(t,x), T(t))\right]}{\partial T}\Big] \\ &+ R_c I(t)^2,\ t > 0, \end{aligned} \tag{7.12}$$

$$T(0) = T_{\text{amb}}(0), \tag{7.13}$$

where

$$\bar{c}_{\text{s}}^{\pm}(t,x) = \frac{3}{(R_{\text{s}}^{\pm})^3}\int_0^{R_{\text{s}}^{\pm}} r_{\text{s}}^2 c_{\text{s}}^{\pm}(t,x,r_{\text{s}})\,\mathrm{d}r_{\text{s}}. \tag{7.14}$$

Charge conservation in the electrodes gives the following equations for the ionic current:

$$\frac{\partial i_{\text{e}}^+}{\partial x}(t,x) = -a_{\text{s}}^+ F j^+(t,x), \tag{7.15}$$

$$\frac{\partial i_{\text{e}}^-}{\partial x}(t,x) = a_{\text{s}}^- F j^-(t,x), \tag{7.16}$$

where $a_{\text{s}}^{\pm} = 3\epsilon_{\text{s}}^{\pm}/R_{\text{s}}^{\pm}$ are the specific interfacial areas. Boundary conditions for the equations (7.15)–(7.16) are given in [81].

Equations for lithium concentration $c_{\text{e}}(t,x)$ in the electrolyte, solid electric potential $\phi_{\text{s}}(t,x)$ and electrolyte electric potential $\phi_{\text{e}}(t,x)$ are:

$$\frac{\partial c_{\text{e}}}{\partial t}(t,x) = \frac{\partial}{\partial x}\left[D_{\text{e}}\frac{\partial c_{\text{e}}}{\partial x}(t,x) + \frac{1-t_{\text{c}}^0}{\varepsilon_{\text{e}} F} i_{\text{e}}(t,x)\right], \tag{7.17}$$

$$\frac{\partial \phi_{\text{s}}}{\partial x}(t,x) = \frac{I(t) - i_{\text{e}}(t,x)}{\sigma}, \tag{7.18}$$

$$\frac{\partial \phi_{\rm e}}{\partial x}(t,x) = \frac{i_{\rm e}(t,x)}{\kappa} + \frac{2RT(t)}{F}(1-t_{\rm c}^0) \times \left(1 + \frac{{\rm d} \ln f_{\rm c/a}}{{\rm d} \ln c_{\rm e}}(t,x)\right) \frac{\partial \ln c_{\rm e}}{\partial x}(t,x), \quad (7.19)$$

where the Kirchoff's law and Ohm's law have been used in (7.18). The readers could also refer to [81] for more details of the equations (7.17)–(7.19), such as the corresponding initial and boundary conditions. Note that here positive current corresponds to battery discharging and negative current corresponds to battery charge.

7.2.3 The SPM-T model

Despite of the fact that full DFN model accurately models many aspects of the lithium-ion cells working mechanism, it is too complex to design a full state observer for online estimation. Thus, it is necessary to consider simplified models for the sake of implementation.

The SPM-T model is derived after making the following assumptions and simplifications:

- the electrolyte concentration is uniform in space and time,
- the lithium-ion diffusion is uniform in the x-direction.

As a result, the battery is formulated as a one-dimensional model, considering only the radial direction of the ions into and out of the active material. In the coupled SPM-T model, the SPM subsystem is postulated as follows [106]:

$$\frac{\partial c_{\rm s}^{\pm}}{\partial t}(t,r_s) = \frac{1}{r_{\rm s}^2}\frac{\partial}{\partial r_{\rm s}}\left[D_{\rm s}^{\pm}(T(t)) r_{\rm s}^2 \frac{\partial c_{\rm s}^{\pm}}{\partial r_{\rm s}}(t,r_{\rm s})\right], \; t>0, r_{\rm s} \in (0, R_{\rm s}^{\pm}), \quad (7.20)$$

$$\frac{\partial c_{\rm s}^{\pm}}{\partial r_{\rm s}}(t,0) = 0, \; t>0, \quad (7.21)$$

$$\frac{\partial c_{\rm s}^{\pm}}{\partial r_{\rm s}}(t,R_{\rm s}^{\pm}) = -\frac{1}{D_{\rm s}^{\pm}(T(t))} j^{\pm}(t), \; t>0, \quad (7.22)$$

$$c_{\rm s}^{\pm}(0,r_{\rm s}) = c_{\rm s,0}^{\pm}(r_{\rm s}), \; r_{\rm s} \in [0, R_{\rm s}^{\pm}]. \quad (7.23)$$

The states of the system (7.20)–(7.23) are $c_{\rm s}^{\pm}(t,r_{\rm s}) \in \mathbb{R}$, with the temporal variable t and the spatial variables $r_{\rm s}$. The diffusion coefficients $D_{\rm s}^{\pm}(T(t))$ satisfy (7.5). The

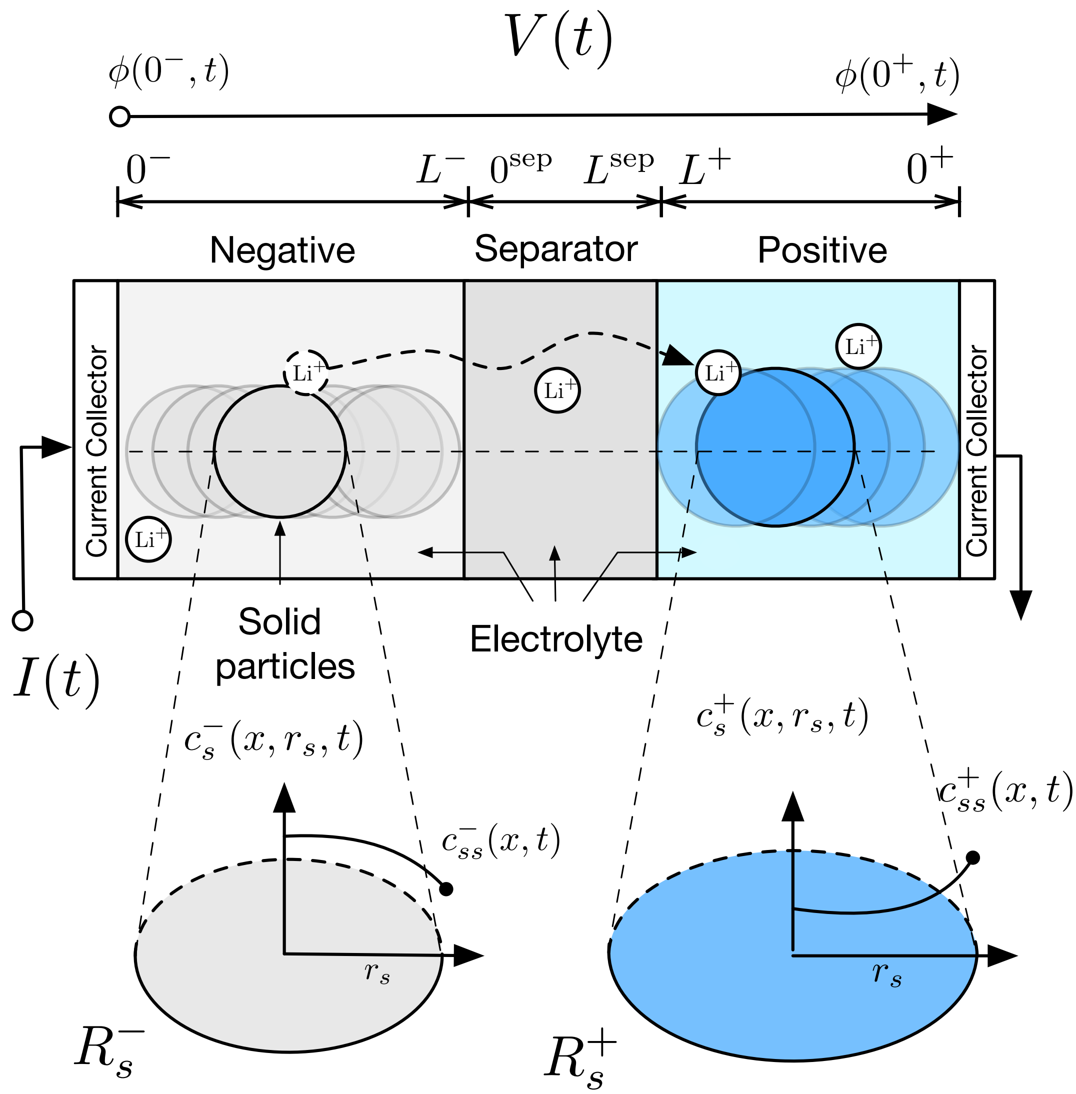

Figure 7.1: DFN Schematic. Lithium ions move from the negative electrode to the positive electrode during discharge and in the opposite direction during charge. In the DFN model, the three-dimensional problem is reduced to a Pseudo-Two Dimensional (P2D) one, and all intercalation particles are assumed to be spheres with a uniform, averaged radius.

average concentration is

$$\bar{c}_{\mathrm{s}}^{\pm}(t) = \frac{3}{(R_{\mathrm{s}}^{\pm})^3} \int_0^{R_{\mathrm{s}}^{\pm}} r_{\mathrm{s}}^2 c_{\mathrm{s}}^{\pm}(t, r_{\mathrm{s}}) \, \mathrm{d}r_{\mathrm{s}}. \tag{7.24}$$

The Butler-Volmer equation relating the the pore wall fluxes $j^{\pm}$ at the surface of the particles to the overpotential $\eta(t)$ becomes

$$j^{\pm}(t) = \frac{i_0^{\pm}(t)}{F} \left[e^{\frac{\alpha_a F}{RT(t)}\eta^{\pm}(t)} - e^{-\frac{\alpha_c F}{RT(t)}\eta^{\pm}(t)} \right], \tag{7.25}$$

where

$$i_0^{\pm}(t) = r_{\mathrm{eff}}^{\pm}(T(t)) \left[c_{\mathrm{ss}}^{\pm}(t) \right]^{\alpha_{\mathrm{c}}} \left[c_{\mathrm{e},0} \left(c_{\mathrm{s}}^{\pm,\max} - c_{\mathrm{ss}}^{\pm}(t) \right) \right]^{\alpha_{\mathrm{a}}}, \tag{7.26}$$

$$c_{\mathrm{ss}}^{\pm}(t) \triangleq c_{\mathrm{s}}^{\pm}(t, R_{\mathrm{s}}^{\pm}), \tag{7.27}$$

and

$$\eta^{\pm}(t) = \phi_{\mathrm{s}}^{\pm}(t) - U^{\pm}(c_{\mathrm{ss}}^{\pm}(t), T(t)) - FR_{\mathrm{f}}^{\pm}(T(t)) j^{\pm}(t), \tag{7.28}$$

with $r_{\mathrm{eff}}^{\pm}(T(t)), R_{\mathrm{f}}^{\pm}(T(t))$ satisfying (7.10) and (7.11), respectively. Here, $c_{\mathrm{e},0}$ denotes the electrolyte concentration at equilibrium. Assuming $\alpha_a = \alpha_c = \alpha$, then

$$\eta^{\pm}(t) = \frac{RT(t)}{\alpha F} \sinh^{-1}\left(\frac{F}{2 i_0^{\pm}(t)} j^{\pm}(t) \right). \tag{7.29}$$

Moreover, the relations between the pore wall fluxes $j^{\pm}$ at the surface of the particles and the current $I(t)$ become linear:

$$j^{+}(t) = -\frac{I(t)}{a_{\mathrm{s}}^{+} F L^{+}}, \; j^{-}(t) = \frac{I(t)}{a_{\mathrm{s}}^{-} F L^{-}}. \tag{7.30}$$

The T-equation becomes

$$\begin{aligned} \rho^{\mathrm{avg}} c_P \frac{\mathrm{d}T}{\mathrm{d}t}(t) =& h_{\mathrm{cell}} \left(T_{\mathrm{amb}}(t) - T(t) \right) + I(t) V(t) \\ &+ I(t) \Bigg[U^{+}(\bar{c}_{\mathrm{s}}^{+}(t), T(t)) - U^{-}(\bar{c}_{\mathrm{s}}^{-}(t), T(t)) \\ &- T(t) \frac{\partial [U^{+}(\bar{c}_{\mathrm{s}}^{+}(t), T(t)) - U^{-}(\bar{c}_{\mathrm{s}}^{-}(t), T(t))]}{\partial T} \Bigg] \\ &+ R_c I(t)^2, \; t > 0, \end{aligned} \tag{7.31}$$

$$T(0) = T_{\mathrm{amb}}(0). \tag{7.32}$$

The system states are the solid phase lithium-ion concentrations $c_{\mathrm{s}}^{\pm}(t, r_{\mathrm{s}}) \in \mathbb{R}$ in the PDE (7.20)–(7.23) and the internal average temperature $T(t)$ in the ODE (7.31)–(7.32).

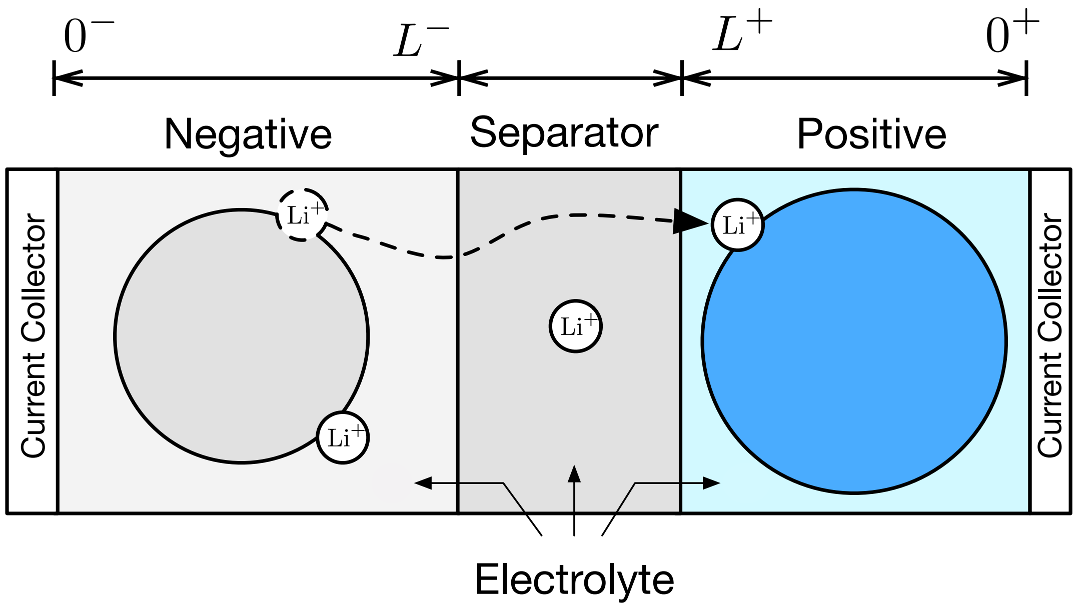

Figure 7.2: SPM Schematic. In this simplification, each electrode is modeled as a single spherical particle; representative of all particles in the electrode. Furthermore, lithium concentration in the electrolyte is assumed to be uniform in both time and space.

7.3 Problem statement

7.3.1 Estimation objective

The input to the SPM-T system is the applied current density $I(t)$, and the model output is the terminal voltage $V(t)$, i.e. the potential difference measured across the current collectors at the external boundaries of the electrodes:

$$\begin{aligned} V(t) =& \phi_{\rm s}^{+}(t) - \phi_{\rm s}^{-}(t) \\ =& \eta^{+}(t) - \eta^{-}(t) - \left(\frac{R_{\rm f}^{+}(T(t))}{a^{+}L^{+}} + \frac{R_{\rm f}^{-}(T(t))}{a^{-}L^{-}} \right) I(t) \\ &+ U^{+}(c_{\rm ss}^{+}(t), T(t)) - U^{-}(c_{\rm ss}^{-}(t), T(t)), \end{aligned} \tag{7.33}$$

where the equality (7.28) is used. Thus, the following system output function is derived from (7.29) and (7.33):

$$\begin{aligned} V(t) =& -\frac{RT(t)}{\alpha F}\left[\sinh^{-1}\left(\frac{1}{2i_0^{+}(t)} \frac{I(t)}{a^{+}L^{+}} \right) + \sinh^{-1}\left(\frac{1}{2i_0^{-}(t)} \frac{I(t)}{a^{-}L^{-}} \right) \right] \\ & - \left(\frac{R_{\rm f}^{+}(T(t))}{a^{+}L^{+}} + \frac{R_{\rm f}^{-}(T(t))}{a^{-}L^{-}} \right) I(t) + U^{+}(c_{\rm ss}^{+}(t), T(t)) - U^{-}(c_{\rm ss}^{-}(t), T(t)) \\ & \triangleq h_1(T(t), c_{\rm ss}^{+}(t), c_{\rm ss}^{-}(t), I(t)). \end{aligned} \tag{7.34}$$

The objective is to estimate the SoC in each electrode of the battery, which is defined by:

$$SoC^{\pm}(t) = \frac{3}{(R_{\rm s}^{\pm})^3} \int_0^{R_{\rm s}^{\pm}} r^2 \frac{c_{\rm s}^{\pm}(t,r)}{c_{\rm s,max}^{\pm}} \,{\rm d}r. \tag{7.35}$$

7.3.2 Output function inversion

A Luenberger observer is to be constructed for the positive electrode which uses the corresponding boundary state measurement. Note however that boundary values $c_{\rm ss}^{\pm}(t)$ appear in the output function $V(t)$ in a nonlinear fashion. Therefore, in order to proceed the observability analysis and design the observer, one needs to recover the surface concentration in one of the electrodes. For this purpose, an inversion of the output function with respect to this boundary value is required. Note that the leading term in the output voltage function is the open circuit potential. For the cases when the open circuit potential functions are invertible (with respect to the surface concentration) in both electrodes, the decision can be made based on which mean value of the partial derivative is bigger. In what follows, we consider the output function inversion with respect to the surface concentration in the positive electrode.

Write $V(t)$ as a function of $c_{\rm ss}^{\pm}(t)$, $\bar{c}_{\rm s}^{\pm}(t)$ and $I(t)$

Substituting (7.34) into (7.12) gives

$$\begin{aligned}
\rho^{\rm avg} c_P \frac{{\rm d}T}{{\rm d}t}(t) =& h_{\rm cell}\left(T_{\rm amb}(t) - T(t)\right) \\
&- I(t)\frac{RT(t)}{\alpha F}\left[\sinh^{-1}\left(\frac{1}{2i_0^+(t)}\frac{I(t)}{a^+L^+}\right) + \sinh^{-1}\left(\frac{1}{2i_0^-(t)}\frac{I(t)}{a^-L^-}\right)\right] \\
&- \left(\frac{R_{\rm f}^+(T(t))}{a^+L^+} + \frac{R_{\rm f}^-(T(t))}{a^-L^-} - R_c\right) I(t)^2 \\
&+ I(t)\left[U^+(c_{\rm ss}^+(t), T(t)) - U^-(c_{\rm ss}^-(t), T(t))\right] \\
&+ I(t)\Big\{U^+(\bar{c}_{\rm s}^+(t), T(t)) - U^-(\bar{c}_{\rm s}^-(t), T(t)) \\
&- T(t)\frac{\partial\left[U^+(\bar{c}_{\rm s}^+(t), T(t)) - U^-(\bar{c}_{\rm s}^-(t), T(t))\right]}{\partial T}\Big\}. \qquad (7.36)
\end{aligned}$$

In order to derive an expression for $T(t)$ in terms of $c_{\rm ss}^{\pm}(t)$, $\bar{c}_{\rm s}^{\pm}(t)$ and $I(t)$, the following approximations and assumptions are made. First,

$R_f^{\pm}(T(t)), r_{\rm eff}^{\pm}(T(t))$ are approximated by $R_f^{\pm}(T_{\rm amb}(t)), r_{\rm eff}^{\pm}(T_{\rm amb}(t))$, respectively, and all terms $U^{\pm}(\cdot, T(t))$ are approximated by $U^{\pm}(\cdot, T_{\rm amb}(t))$. Then, $\frac{\partial\left[U^+(\bar{c}_{\rm s}^+(t), T_{\rm amb}(t)) - U^-(\bar{c}_{\rm s}^-(t), T_{\rm amb}(t))\right]}{\partial T}$ is assumed to be a state-invariant, possibly time-varying, function, although in general it is dependent on the state $\bar{c}_{\rm s}^{\pm}(t)$. Note that the approximations and assumptions are made here only for the ease of analysis.

Rewrite the equation (7.36) as

$$\rho^{\rm avg} c_{\rm P} \frac{{\rm d}T}{{\rm d}t}(t) = \chi(t)T(t) + \omega(t), \qquad (7.37)$$

where

$$\chi(t) = -h_{\rm cell} - \frac{R}{\alpha F} I(t)\left[\sinh^{-1}\left(\frac{1}{2i_0^+(t)}\frac{I(t)}{a^+L^+}\right) + \sinh^{-1}\left(\frac{1}{2i_0^-(t)}\frac{I(t)}{a^-L^-}\right)\right]$$

$$
\begin{aligned}
& - I(t)\frac{\partial\left[U^{+}(\bar{c}_{\mathrm{s}}^{+}(t),T_{\mathrm{amb}}(t))-U^{+}(\bar{c}_{\mathrm{s}}^{+}(t),T_{\mathrm{amb}}(t))\right]}{\partial T}, && (7.38)\\
\omega(t) = & h_{\mathrm{cell}}T_{\mathrm{amb}}(t)-\left(\frac{R_{\mathrm{f}}^{+}(T_{\mathrm{amb}}(t))}{a^{+}L^{+}}+\frac{R_{\mathrm{f}}^{-}(T_{\mathrm{amb}}(t))}{a^{-}L^{-}}-R_{c}\right)I(t)^{2}\\
& + I(t)\left[U^{+}(c_{\mathrm{ss}}^{+}(t),T_{\mathrm{amb}}(t))-U^{-}(c_{\mathrm{ss}}^{-}(t),T_{\mathrm{amb}}(t))\right]\\
& + I(t)\left[U^{+}(\bar{c}_{\mathrm{s}}^{+}(t),T_{\mathrm{amb}}(t))-U^{-}(\bar{c}_{\mathrm{s}}^{-}(t),T_{\mathrm{amb}}(t))\right], && (7.39)
\end{aligned}
$$

then for the temperature, it holds that

$$
\begin{aligned}
T(t) = & T(0)e^{\frac{1}{\rho^{\mathrm{avg}}c_{\mathrm{P}}}\int_0^t \chi(\tau)\,\mathrm{d}\tau}+\frac{1}{\rho^{\mathrm{avg}}c_{\mathrm{P}}}\int_0^t e^{\frac{1}{\rho^{\mathrm{avg}}c_{\mathrm{P}}}\int_0^{t-\tau}\chi(\sigma)\,\mathrm{d}\sigma}\omega(\tau)\,\mathrm{d}\tau\\
\triangleq & g(c_{\mathrm{ss}}^{\pm}(t),\bar{c}_{\mathrm{s}}^{\pm}(t),I(t),T_{\mathrm{amb}}(t)). && (7.40)
\end{aligned}
$$

Substituting (7.40) into (7.34) yields the following simplified output function:

$$
\begin{aligned}
V(t) = & -\frac{R}{\alpha F}g(c_{\mathrm{ss}}^{\pm}(t),\bar{c}_{\mathrm{s}}^{\pm}(t),I(t),T_{\mathrm{amb}}(t))\\
& \times\left[\sinh^{-1}\left(\frac{1}{2i_0^{+}(t)}\frac{I(t)}{a^{+}L^{+}}\right)+\sinh^{-1}\left(\frac{1}{2i_0^{-}(t)}\frac{I(t)}{a^{-}L^{-}}\right)\right]\\
& -\left(\frac{R_{\mathrm{f}}^{+}(T_{\mathrm{amb}}(t))}{a^{+}L^{+}}+\frac{R_{\mathrm{f}}^{-}(T_{\mathrm{amb}}(t))}{a^{-}L^{-}}\right)I(t)\\
& + U^{+}(c_{\mathrm{ss}}^{+}(t),g(c_{\mathrm{ss}}^{\pm}(t),\bar{c}_{\mathrm{s}}^{\pm}(t),I(t),T_{\mathrm{amb}}(t)))\\
& - U^{-}(c_{\mathrm{ss}}^{-}(t),g(c_{\mathrm{ss}}^{\pm}(t),\bar{c}_{\mathrm{s}}^{\pm}(t),I(t),T_{\mathrm{amb}}(t)))\\
\triangleq & h_2(c_{\mathrm{ss}}^{\pm}(t),\bar{c}_{\mathrm{s}}^{\pm}(t),I(t)). && (7.41)
\end{aligned}
$$

Write $V(t)$ as a function of $c_{\mathrm{ss}}^{+}(t)$ and $I(t)$

In order to further simplify the output function, we establish relations between $c_{\mathrm{ss}}^{+}(t)$ and each one of $c_{\mathrm{ss}}^{-}(t),\ \bar{c}_{\mathrm{s}}^{+}(t),\ \bar{c}_{\mathrm{s}}^{-}(t)$. Consider the following approximate polynomial solution profiles of the electrode diffusion dynamics [107] [1]

$$
\bar{c}_{\mathrm{s}}^{\pm}(t)=c_{\mathrm{ss}}^{\pm}(t)-\frac{8R_s^{\pm}}{35}\bar{q}_{\mathrm{s}}^{\pm}(t)+\frac{R_{\mathrm{s}}^{\pm}}{35D_{\mathrm{s}}^{\pm}(T(t))}j^{\pm}(t), \qquad (7.42)
$$

[1] Note that (7.42) is obtained by assuming the following polynomial solution profile:

$$
\begin{aligned}
c_{\mathrm{s}}^{\pm}(t,r)= & \frac{39}{4}c_{\mathrm{ss}}^{\pm}(t)-3\bar{q}_{\mathrm{s}}^{\pm}(t)R_{\mathrm{s}}^{\pm}-\frac{35}{4}\bar{c}_s^{\pm}(t)+\left(-35c_{\mathrm{ss}}^{\pm}(t)+10\bar{q}_{\mathrm{s}}^{\pm}(t)R_{\mathrm{s}}^{\pm}+35\bar{c}_{\mathrm{s}}^{\pm}(t)\right)\frac{(r_{\mathrm{s}}^{\pm})^2}{(R_{\mathrm{s}}^{\pm})^2}\\
& +\left(\frac{105}{4}c_{\mathrm{ss}}^{\pm}(t)-7\bar{q}_{\mathrm{s}}^{\pm}(t)R_{\mathrm{s}}^{\pm}-\frac{105}{4}\bar{c}_{\mathrm{s}}^{\pm}(t)\right)\frac{r_{\mathrm{s}}^4}{(R_{\mathrm{s}}^{\pm})^4}.
\end{aligned}
$$

where the volume averaged fluxes $\bar{q}_{\mathrm{s}}^{\pm}(t)$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{q}_{\mathrm{s}}^{\pm}(t) = -\frac{30D_{\mathrm{s}}^{\pm}(T(t))}{(R_{\mathrm{s}}^{\pm})^2}\bar{q}_{\mathrm{s}}^{\pm}(t) - \frac{45}{2(R_{\mathrm{s}}^{\pm})^2}j^{\pm}(t), \tag{7.43}$$

and $T(t)$ is calculated from the following further simplified model:

$$\rho^{\mathrm{avg}}c_{\mathrm{P}}\frac{\mathrm{d}T}{\mathrm{d}t}(t) = h_{\mathrm{cell}}\left(T_{\mathrm{amb}}(t) - T(t)\right) + I(t)V(t) + R_c I(t)^2, \tag{7.44}$$

$$T(0) = T_{\mathrm{amb}}(0). \tag{7.45}$$

Note that this simplification is also only required for analysis and the complete thermal equation (7.31) can be used for simulations.

Now, since the total amount of lithium ions

$$N = \epsilon^+ L^+ \bar{c}_{\mathrm{s}}^+(t) + \epsilon^- L^- \bar{c}_{\mathrm{s}}^-(t) \tag{7.46}$$

is generally assumed to be conserved, then the average negative electrode concentration can be related to the average positive electrode concentration as follows:

$$\bar{c}_{\mathrm{s}}^-(t) = \alpha\bar{c}_{\mathrm{s}}^+(t) + \beta, \tag{7.47}$$

where

$$\alpha = -\frac{\epsilon^+ L^+}{\epsilon^- L^-},\ \beta = \frac{N}{\epsilon^- L^-}. \tag{7.48}$$

Here, the value of N could be calculated from the model initial data.

From (7.42) and (7.46), it could be obtained that

$$\bar{c}_{\mathrm{s}}^-(t) = \alpha\left(c_{\mathrm{ss}}^+(t) - \frac{8R_s^+}{35}\bar{q}_{\mathrm{s}}^+(t) + \frac{R_{\mathrm{s}}^+}{35D_{\mathrm{s}}^+(T(t))}j^+(t)\right) + \beta, \tag{7.49}$$

and

$$c_{\mathrm{ss}}^-(t) = \bar{c}_{\mathrm{s}}^-(t) + \frac{8R_{\mathrm{s}}^-}{35}\bar{q}_{\mathrm{s}}^-(t) - \frac{R_{\mathrm{s}}^-}{35D_{\mathrm{s}}^-(T(t))}j^-(t)$$

Moreover, the following terms are neglected from the temperature equation (7.31)–(7.32):

$$I(t)\left[U^+(\bar{c}_{\mathrm{s}}^+(t), T(t)) - U^-(\bar{c}_{\mathrm{s}}^-(t), T(t)) - T(t)\frac{\partial[U^+(\bar{c}_{\mathrm{s}}^+(t), T(t)) - U^-(\bar{c}_{\mathrm{s}}^-(t), T(t))]}{\partial T}\right].$$

$$=\alpha\left(c_{\rm ss}^{+}(t)-\frac{8R_s^{+}}{35}\bar{q}_{\rm s}^{+}(t)+\frac{R_{\rm s}^{+}}{35D_{\rm s}^{+}(T(t))}j^{+}(t)\right)+\beta$$
$$+\frac{8R_{\rm s}^{-}}{35}\bar{q}_{\rm s}^{-}(t)-\frac{R_{\rm s}^{-}}{35D_{\rm s}^{-}(T(t))}j^{-}(t). \tag{7.50}$$

Therefore, from (7.41), (7.42), (7.49) and (7.50), we could obtain a further simplified version of the output function:

$$V(t)=h_3(c_{\rm ss}^{+}(t),I(t)). \tag{7.51}$$

Inversion of the function h_3

As long as the function (7.51) is a one-to-one correspondence w.r.t. $c_{\rm ss}^{+}(t)$, uniformly in $I(t)$, one could invert it to derive the boundary concentration in the negative electrode[2] as

$$c_{\rm ss}^{+}(t)=h_0(V(t),I(t)). \tag{7.52}$$

7.3.3 Normalization and state transformation

We perform normalization and state transformation to simplify the system and also the observer structure. Let $r=r_{\rm s}/R_{\rm s}^{+}$ for normalization and proceed the state transformation $c(t,r)=r_{\rm s}c_{\rm s}^{+}(t,r_{\rm s})$, then the PDE subsystem (7.20)–(7.23) is transformed into [3]

$$\frac{\partial c}{\partial t}(t,r)=\frac{D_{\rm s}^{+}(T(t))}{(R_{\rm s}^{+})^2}\frac{\partial^2 c}{\partial r^2}(t,r),\ \ t>0,\ \ r\in(0,1) \tag{7.53}$$
$$c(t,0)=0,\ \ t>0 \tag{7.54}$$

[2]Note that the state observer can be designed for either the positive or the negative electrode. One could make the decision based on the ease of invertibility of the output function with respect to the boundary value.

[3]The normalization transformation

$$t'=\frac{D_s^{+}(T(t))}{(R_s^{+})^2}t$$

employed in [97] is not considered for use in this paper. The reason is: $T(t)$ is not known a priori in this case, and needs to be measured/derived at each time step. Thus, the corresponding inverse transformation could not be trivially obtained, and it would be difficult to transform the normalized system into the original coordinates.

$$\frac{\partial c}{\partial r}(t,1) - c(t,1) = -\frac{R_s^+}{D_s^+(T(t))}\frac{I(t)}{a^+FL^+},\ t > 0 \tag{7.55}$$

$$c(0,r) = c_0(r) = R_s^+ r c_s^+(0, R_s^+ r),\ r \in [0,1]. \tag{7.56}$$

Our estimation objective is now to design an observer for this normalized and transformed PDE system.

7.4 State observer design

With inversion of the output function in Section 7.3.2, the boundary concentration in the positive electrode is available for an observer design. Meanwhile, $T(t)$ is calculated from the simplified equation (7.44)–(7.45), and thus the function $D_s^+(T(t))$ could be treated as known. Furthermore, assume that $I(t), U^{\pm}(\cdot, T(t))$ and $V(t)$ are piecewise (real) analytic. In what follows, we only consider the proof piecewisely so that both $I(t)$ and $V(t)$ are analytic in each separate time interval. Then, from (7.12) and an assumption that $\frac{\partial U^{\pm}}{\partial T}$ are also analytic in each corresponding time interval, we could prove by induction that the n-th order derivative of $T(t)$ is differentiable for any nonnegative integer n, and that $T(t)$ is analytic as well in each time interval. Without loss of generality, consider $t \in [0, t_{\max}]$ where $t_{\max}$ is an appropriate finite time.

A Luenberger-type observer for the normalized and transformed PDE system (7.53)–(7.56) can be designed:

$$\frac{\partial \hat{c}}{\partial t}(t,r) = \frac{D_s^+(T(t))}{(R_s^+)^2}\frac{\partial^2 \hat{c}}{\partial r^2}(t,r) + p_1(t,r)(c(t,1) - \hat{c}(t,1)), \tag{7.57}$$

$$\hat{c}(t,0) = 0, \tag{7.58}$$

$$\frac{\partial \hat{c}}{\partial r}(t,1) - \hat{c}(t,1) = -\frac{R_s^+}{D_s^+(T(t))}\frac{I(t)}{a^+FL^+} + p_{10}(t)(c(t,1) - \hat{c}(t,1)), \tag{7.59}$$

$$\hat{c}(0,r) = \hat{c}_0(r), \tag{7.60}$$

where $\hat{c}_0(r)$ denotes the initial condition of the observer, and the boundary state error injection functions $p_1(t,r)$ and $p_{10}(t)$ are to be determined to guarantee the stability of the following observer error system:

$$\frac{\partial \tilde{c}}{\partial t}(t,r) = \frac{D_s^+(T(t))}{(R_s^+)^2}\frac{\partial^2 \tilde{c}}{\partial r^2}(t,r) - p_1(t,r)\tilde{c}(t,1), \tag{7.61}$$

$$\tilde{c}(t,0) = 0, \tag{7.62}$$

$$\frac{\partial \tilde{c}}{\partial r}(t,1) - \tilde{c}(t,1) = -p_{10}(t)\tilde{c}(t,1), \tag{7.63}$$

$$\tilde{c}(0,r) = c_0(r) - \hat{c}_0(r) \triangleq \tilde{c}_0(r), \tag{7.64}$$

with

$$\tilde{c}(t,r) = c(t,r) - \hat{c}(t,r). \tag{7.65}$$

In order to find the output injection gains, the PDE backstepping method [22] is employed. We would like to find an invertible transformation

$$\tilde{c}(t,r) = \tilde{w}(t,r) - \int_r^1 p(t,r,\iota)\tilde{w}(t,\iota)\,\mathrm{d}\iota \tag{7.66}$$

so that $\tilde{w}$ satisfies the following exponentially stable target system

$$\frac{\partial \tilde{w}}{\partial t}(t,r) = \frac{D_{\mathrm{s}}^{+}(T(t))}{(R_{\mathrm{s}}^{+})^2}\frac{\partial^2 \tilde{w}}{\partial r^2}(t,r) + \lambda \tilde{w}(t,r), \tag{7.67}$$

$$\tilde{w}(t,0) = 0, \tag{7.68}$$

$$\frac{\partial \tilde{w}}{\partial r}(t,1) = -\frac{1}{2}\tilde{w}(t,1), \tag{7.69}$$

$$\tilde{w}(0,r) = \tilde{w}_0(r), \tag{7.70}$$

where $\tilde{w}_0(r)$ denotes the initial condition to be determined for the target system, and $\lambda < \min_{t\geq 0}\{D_{\mathrm{s}}^{-}(T(t))\}/(4(R_{\mathrm{s}}^{+})^2)$ is a free parameter to be chosen, which determines the convergence rate of the observer (7.61)–(7.64) to the system (7.53)–(7.56).

The following lemma states the exponential stability of the $\tilde{w}$-system (7.67)–(7.70).

Lemma 7.1. *If*

$$\lambda < \frac{1}{4(R_{\mathrm{s}}^{-})^2}\min_{t\geq 0}\{D_{\mathrm{s}}^{+}(T(t))\}, \tag{7.71}$$

then for any initial value $\tilde{w}_0(\cdot) \in L^2(0,1)$, the $\tilde{w}$-system (7.67)–(7.70) has a (mild) solution $\tilde{w}(t,\cdot) \in L^2(0,1)$ and is exponentially stable at $\tilde{w} \equiv 0$. Moreover, if the boundary compatibility condition is satisfied, the solution is classical.

Proof. Consider the state Hilbert space $\boldsymbol{H} = L^2(0,1)$ with the usual inner product. For every $t \in [0, t_{\max}]$, define a linear operator $\mathcal{A}(t) : \mathrm{Dom}(\mathcal{A}(t)) \subset \boldsymbol{H} \to \boldsymbol{H}$ as follows:

$$\mathcal{A}(t) = \frac{D_{\mathrm{s}}^{+}(T(t))}{(R_{\mathrm{s}}^{+})^2}\psi_{rr}(t) + \lambda\psi, \quad \forall\psi \in \mathrm{Dom}(\mathcal{A}(t)), \tag{7.72}$$

$$\mathrm{Dom}(\mathcal{A}(t)) = \{\psi(t) \mid \psi(0,t) = 0,\ \psi_r\,(t,1) = -\frac{1}{2}\psi\,(t,1)\}. \tag{7.73}$$

Then the system (7.67)–(7.70) can be written into the following abstract equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \mathcal{A}(t)\psi(t),\ 0 \le t \le t_{\max}, \tag{7.74}$$

$$\psi(0) = \tilde{w}_0. \tag{7.75}$$

Note that $\mathrm{Dom}(\mathcal{A}(t))$ is dense in $\boldsymbol{H}$ and independent of t, and it can be proved that $\mathcal{A}(t)$ is for each t the infinitesimal generator of an exponential stable semigroup. Indeed, all the assumptions (P1)–(P3) in [53, Section 5.6] are satisfied. Thus, from [53, Section 5.6, Theorem 6.1], there exists a unique evolution system corresponding to (7.74)–(7.75) and (7.67)–(7.70) as well.

Furthermore, if considering the Lyapunov function

$$V(t) = \frac{1}{2}\int_0^1 \tilde{w}^2(t,r)\,\mathrm{d}r \tag{7.76}$$

then we could get

$$\dot{V}(t) \le -2\left(\frac{D_s^{+}(T(t))}{4(R_s^{+})^2} - \lambda\right)V(t) \tag{7.77}$$

where Poincaré inequality is employed. Thus, from (7.71), exponential stability of the $\tilde{w}$-system (7.67)–(7.70) is proved. □

By straightforward but lengthy calculations, we derive that the kernel function $p(r,\iota,t)$ needs to satisfy the following PDE system:

$$p_t(t,r,\iota) = \frac{D_{\mathrm{s}}^{+}(T(t))}{(R_{\mathrm{s}}^{+})^2}\left[(p_{rr}(t,r,\iota) - p_{\iota\iota}(t,r,\iota)\right] - \lambda p(t,r,\iota), \tag{7.78}$$

$$p(t,0,\iota) = 0, \tag{7.79}$$

$$p(t,r,r) = \frac{(R_{\mathrm{s}}^{+})^2}{2D_{\mathrm{s}}^{+}(T(t))}\lambda r, \tag{7.80}$$

$$p(0,r,\iota) = p_0(r,\iota), \tag{7.81}$$

for which the domain is $\mathcal{T} = \{(t,r,\iota) \mid 0 \le t \le t_{\max},\ 0 \le \iota \le r \le 1\}$, and $p_0(r,\iota)$ denotes the initial condition for the kernel system. Also, we derive that the output injection gains need to satisfy

$$p_1(t,r) = -\frac{D_{\rm s}^+(T(t))}{(R_{\rm s}^+)^2}\left(p_\iota(t,r,1) + \frac{1}{2}p(t,r,1)\right), \tag{7.82}$$

$$p_{10}(t) = \frac{3}{2} - \frac{(R_{\rm s}^+)^2}{2D_{\rm s}^+(T(t))}\lambda. \tag{7.83}$$

7.4.1 Well-posedness of the kernel function $p(t,r,\iota)$

Lemma 7.2. *The initial data* $p_0(\cdot,\cdot)$ *is an analytic function in* $\mathbb{T} = \{(r,\iota) \mid 0 \le \iota \le r \le 1\}$*, and the system* (A.55)–(A.56) *admits a unique analytic solution* $p(t,\cdot,\cdot)$ *in* $\mathcal{T}$*.*

Proof. We first transform the system (A.55)–(A.56) into an equivalent integral equation, and then apply the method of successive approximation.

Let

$$\xi = r + \iota,\ \eta = r - \iota \tag{7.84}$$

and

$$q(t,\xi,\eta) = p(t,r,\iota), \tag{7.85}$$

then we have from (A.55)–(A.56) that

$$q_t(t,\xi,\eta) = 4\frac{D_{\rm s}^+(T(t))}{(R_{\rm s}^+)^2}q_{\xi\eta}(t,\xi,\eta) - \lambda q(t,\xi,\eta), \tag{7.86}$$

$$q(t,\xi,-\xi) = 0, \tag{7.87}$$

$$q(t,\xi,0) = \frac{(R_{\rm s}^+)^2}{4D_{\rm s}^+(T(t))}\lambda\xi, \tag{7.88}$$

$$q(0,\xi,\eta) = p\left(0,\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right). \tag{7.89}$$

The equation (7.86) can be rewritten as

$$q_{\xi\eta}(t,\xi,\eta) = \frac{(R_{\rm s}^+)^2}{4D_{\rm s}^+(T(t))}\left(q_t(t,\xi,\eta) + \lambda q(t,\xi,\eta)\right). \tag{7.90}$$

Integrating (7.90) with respect to η from 0 to η and using boundary condition (7.89), we have

$$q_\xi(t,\xi,\eta) = \frac{(R_\mathrm{s}^+)^2}{4D_\mathrm{s}^+(T(t))}\lambda + \frac{(R_\mathrm{s}^+)^2}{4D_\mathrm{s}^+(T(t))}\int_0^\eta \left(q_t(t,\xi,\beta) + \lambda q(t,\xi,\beta)\right) \mathrm{d}\beta. \quad (7.91)$$

Integrating (7.91) with respect to ξ from $-\eta$ to ξ gives the following integro-differential equation (IDE):

$$q(t,\xi,\eta) = \frac{(R_\mathrm{s}^+)^2}{4D_\mathrm{s}^+(T(t))}\lambda(\xi+\eta) + \frac{(R_\mathrm{s}^+)^2}{4D_\mathrm{s}^+(T(t))}\int_{-\eta}^{\xi}\int_0^\eta \left(q_t(t,\alpha,\beta) + \lambda q(t,\alpha,\beta)\right) \mathrm{d}\beta\,\mathrm{d}\alpha, \quad (7.92)$$

where (7.87) is used. Let

$$C = \frac{(R_\mathrm{s}^+)^2}{4D_\mathrm{s}^+(T(t_0))e^{E_{D_\mathrm{s}^+}\times\frac{1}{T(0)}}} \quad (7.93)$$

and

$$f(t) = e^{E_{D_\mathrm{s}^+}\times\frac{1}{T(t)}}, \quad (7.94)$$

then we look for a solution of (7.92) in the form of

$$q(t,\xi,\eta) = \sum_{n=0}^{\infty} q^n(t,\xi,\eta), \quad (7.95)$$

where

$$q^0(t,\xi,\eta) = \lambda C(\xi+\eta)f(t), \quad (7.96)$$

and

$$q^{n+1}(t,\xi,\eta) = Cf(t)\int_{-\eta}^{\xi}\int_0^\eta \left[q_t^n(t,\alpha,\beta) + \lambda q^n(t,\alpha,\beta)\right] \mathrm{d}\beta\,\mathrm{d}\alpha. \quad (7.97)$$

Recall that $T(t)$ is analytic, and since it is physically impossible for the temperature to reach zero Kelvin, i.e., $T(t) \neq 0$, then it is reasonable to assume that $\frac{1}{T(t)}$ is an analytic function in $t \in [0, t_{\max}]$, and thus there exists a constant C_f such that for every nonnegative integer k, the following bound holds:

$$\left|f^{(k)}(t)\right| := \left|\frac{\mathrm{d}^k}{\mathrm{d}t^k}f(t)\right| \leq C_f^{k+1}k!. \quad (7.98)$$

Since the composition of analytic functions is analytic, then $q^0(t,\xi,\eta)$ is an analytic function in $t \in [0, t_{\max}]$ and it can be derived from (7.96) and (7.98) that

$$|\partial_t^i q^0(t,\xi,\eta)| \leq \lambda C C_f^{i+1} i!(\xi+\eta), \ \forall i \in \mathbb{N}, \tag{7.99}$$

with respect to (ξ,η), uniformly for $t \in [0, t_{\max}]$.

Assume that for any integer $n \geq 0$,

$$|\partial_t^m q^n(t,\xi,\eta)| \leq \lambda C^{n+1} C_f^{m+n+1} (C_f+\lambda)^n \frac{(m+2n)!}{2^n n!} \frac{\xi^n \eta^n (\xi+\eta)}{n!(n+1)!}, \tag{7.100}$$

then, for any integer $m \geq 0$, from (7.97), we derive

$$\begin{aligned}
&\left|\partial_t^m \tilde{q}^{n+1}(t,\xi,\eta)\right| \\
&= \left|\partial_t^m \left[C f(t) \int_{-\eta}^{\xi} \int_0^{\eta} [q_t^n(t,\alpha,\beta) + \lambda q^n(t,\alpha,\beta)] \,\mathrm{d}\beta \,\mathrm{d}\alpha \right]\right| \\
&= C \left| \sum_{i=0}^{m} \left[\binom{m}{i} \partial_t^{m-i} f(t) \int_{-\eta}^{\xi} \int_0^{\eta} \left[\partial_t^{i+1} q^n(t,\alpha,\beta) + \lambda \partial_t^i q^n(t,\alpha,\beta)\right] \mathrm{d}\beta \,\mathrm{d}\alpha \right] \right| \\
&\leq C \sum_{i=0}^{m} \left\{ \binom{m}{i} C_f^{m+n+2} (m-i)! \lambda C^{n+1} \left[C_f + \frac{\lambda}{i+2n+1} \right] (C_f+\lambda)^n \right. \\
&\qquad \left. \times \frac{(i+2n+1)!}{2^n n!} \right\} \frac{\xi^{n+1}\eta^{n+1}(\xi+\eta)}{(n+1)!(n+2)!} \\
&\leq \lambda C^{n+2} C_f^{m+n+2} (C_f+\lambda)^{n+1} \sum_{i=0}^{m} \left[\binom{m}{i} (m-i)! \frac{(i+2n+1)!}{2^n n!} \right] \frac{\xi^{n+1}\eta^{n+1}(\xi+\eta)}{(n+1)!(n+2)!} \\
&= \lambda C^{n+2} C_f^{m+n+2} (C_f+\lambda)^{n+1} \frac{(m+2(n+1))!}{2^{n+1}(n+1)!} \frac{\xi^{n+1}\eta^{n+1}(\xi+\eta)}{(n+1)!(n+2)!},
\end{aligned} \tag{7.101}$$

where the following equalities have been used:

$$\int_{-\eta}^{\xi} \int_0^{\eta} \frac{\alpha^n \beta^n (\alpha+\beta)}{n!(n+1)!} \,\mathrm{d}\beta \,\mathrm{d}\alpha = \frac{\xi^{n+1}\eta^{n+1}(\xi+\eta)}{(n+1)!(n+2)!}, \tag{7.102}$$

$$\sum_{i=0}^{m} \binom{m}{i} (m-i)!(i+j)! = \frac{(m+j+1)!}{j+1}. \tag{7.103}$$

By induction, (7.100) is proved for any integer $n \geq 0$.

Fixing $m = 0$ in (7.100) gives

$$|q^n(t,\xi,\eta)| \leq \lambda C^{n+1} C_f^{n+1} (C_f+\lambda)^n \frac{(2n)!}{2^n n!} \frac{\xi^n \eta^n (\xi+\eta)}{n!(n+1)!}. \tag{7.104}$$

Then the series $\sum_{n=0}^{\infty} q^n(t,\xi,\eta)$ could be proved to be absolutely and uniformly convergent. Indeed, the following bound holds:

$$\begin{aligned} |q(t,\xi,\eta)| &\le \sum_{n=0}^{\infty} |q^n(t,\xi,\eta)| \\ &\le \sum_{n=0}^{\infty} \lambda C^{n+1} C_f^{n+1} (C_f+\lambda)^n \frac{(2n)!}{2^n n!} \frac{\xi^n \eta^n (\xi+\eta)}{n!(n+1)!} \\ &= \lambda C C_f (\xi+\eta) \sum_{n=0}^{\infty} \phi_1(\xi,\eta;n), \end{aligned} \tag{7.105}$$

where

$$\phi_1(\xi,\eta;n) = [CC_f(C_f+\lambda)\xi\eta]^n \frac{(2n)!}{2^n n!} \frac{1}{n! \cdot (n+1)!}. \tag{7.106}$$

Since

$$\begin{aligned} \lim_{n\to\infty} \frac{\phi_1(\xi,\eta;n+1)}{\phi_1(\xi,\eta;n)} &= \lim_{n\to\infty} [CC_f(C_f+\lambda)\xi\eta] \frac{(2n+1)}{(n+1)(n+2)} \\ &= 0 < 1, \end{aligned} \tag{7.107}$$

then from D'Alembert's (ratio test) criterion, the series $\sum_{n=0}^{\infty} \phi_1(\xi,\eta;n)$ is convergent. Then, the existence and uniqueness of $q(t,\xi,\eta)$ and $p(t,r,\iota)$ is established, which are piecewise smooth. Moreover, the following bound holds for $p(t,r,\iota)$:

$$|p(t,r,\iota)| \le 2\lambda C C_f r \sum_{n=0}^{\infty} \phi_2(r,\iota;n), \tag{7.108}$$

where

$$\phi_2(r,\iota;n) = \phi_1(\xi,\eta;n). \tag{7.109}$$

□

7.4.2 Invertibility of the transformation (7.66)

Indeed, the continuity of the kernel $p(t,r,\iota)$ guarantees the existence of a unique inverse transformation. We write the inverse transformation as

$$\tilde{w}(t,r) = \tilde{c}(t,r) + \int_r^1 \rho(t,r,\iota)\tilde{c}(t,\iota)\,\mathrm{d}\iota, \tag{7.110}$$

then we could derive from (7.66) and (7.110) that the kernel $\rho(t, r, \iota)$ need to satisfy

$$\rho(t, r, \iota) = p(t, r, \iota) + \int_r^\iota p(t, r, \sigma)\rho(t, \sigma, \iota)\,\mathrm{d}\sigma. \tag{7.111}$$

In order to solve the equation (7.111), we could follow a similar (successive approximation) procedure as in Subsection 7.4.1, see also, [22, Section 4.4]. Similar well-posedness result of its inverse could also be obtained, and these derivations are omitted here.

Note also that the initial condition $\tilde{w}_0(r)$ for the target $\tilde{w}$-system (7.67)–(7.70) is determined by $\hat{c}_0(r)$ and $\rho_0(r, \iota) = \rho(0, r, \iota)$. Indeed, from (7.64) and (7.110), $\tilde{w}_0(r)$ can be calculated as

$$\tilde{w}_0(r) = c_0(r) - \hat{c}_0(r) + \int_r^1 \rho_0(r, \iota)[c_0(\iota) - \hat{c}_0(\iota)]\,\mathrm{d}\iota. \tag{7.112}$$

7.4.3 Exponential convergence of the designed observer

To conclude, some simplifications and assumptions are made for the ease of analysis. While deriving the equation for $V(t)$, it is assumed that

1. the functions $R_f^{\pm}(T(t))$, $r_{\text{eff}}^{\pm}(T(t))$ are approximated by $R_f^{\pm}(T_{\text{amb}}(t))$, $r_{\text{eff}}^{\pm}(T_{\text{amb}}(t))$, respectively, and all terms $U^{\pm}(\cdot, T(t))$ are approximated by $U^{\pm}(\cdot, T_{\text{amb}}(t))$;
2. the term $\frac{\partial\left[U^+(\bar{c}_s^+(t), T_{\text{amb}}(t)) - U^-(\bar{c}_s^-(t), T_{\text{amb}}(t))\right]}{\partial T}$ is assumed to be a state-invariant, possibly time-varying, function;
3. lithium diffusion in the electrodes is simplified by using a polynomial solution profile (7.42)–(7.43), along with a simplified temperature equation (7.44)–(7.45).

In the observer design, the simplified temperature equation (7.44)–(7.45) is employed. Furthermore, $I(t)$, $U^{\pm}(\cdot, T(t))$ and $V(t)$ are assumed to be piecewise analytic, and $\frac{\partial U^{\pm}}{\partial T}$ are also assumed to be analytic (in the corresponding time intervals).

Consider an appropriate time interval $[0, t_{\max}]$. With the well-posedness of the kernel function for the transformation (7.66) and also from the invertibility of the transformation, the following main theorem of this paper could be proved.

Theorem 7.1. *Under the above simplifications and assumptions, if*

$$\lambda < \frac{1}{4(R_s^+)^2} \min_{t \geq 0}\{D_s^+(T(t))\}, \tag{7.113}$$

then for any initial value $\hat{c}(0,\cdot) \in L^2(0,1)$*, the observer error* $\tilde{c}$*-system* (7.61)–(7.64) *is exponentially stable at* $\tilde{c} \equiv 0$ *in* L^2 *norm, which means the designed observer* (7.57)–(7.60) *is exponentially convergent to the system* (7.53)–(7.56)*.*

7.5 Simulation results

All the model parameters used in the simulations are listed in Appendix C.1, and both Open-Circuit Potential (OCP) functions for the electrodes are presented in Appendix C.2. Note that the OCP functions depend on the internal average temperature, and here a linear approximation is employed:

$$U^{\pm}(\theta^{\pm}, T) = U^{\pm}(\theta^{\pm}, T_{\text{amb}}) + \frac{\partial U^{\pm}(\theta^{\pm}, T_{\text{amb}})}{\partial T}(T - T_{\text{amb}}), \tag{7.114}$$

where

$$\theta^{\pm}(t) = \frac{c_{\text{ss}}^{\pm}(t)}{c_{\text{s}}^{\pm,\max}}, \tag{7.115}$$

and the ambient temperature is assumed to be constant:

$$T_{\text{amb}} = 298\ [K] = 24.85\ [^{\circ}C]. \tag{7.116}$$

The magnitude of the input current is characterized by the C-rate (per unit area) of the battery and is computed as follows:

$$\text{C} - \text{rate} = F\frac{\min\left\{\epsilon_{\text{s}}^{+} L^{+} c_{\text{s}}^{+,\max}, \epsilon_{\text{s}}^{-} L^{-} c_{\text{s}}^{-,\max}\right\}}{3600}. \tag{7.117}$$

Although the observer is derived under all previous assumptions and simplifications, it is shown in this section that by using the original equation for $T(t)$ and keeping the state dependency in all parameters and functions, the observer is still effective.

7.5.1 Simulation tests

Two simulation tests are performed to evaluate the effectiveness of the proposed observer. For both simulations, the lithium concentration in the positive electrode is initialized as 50% of the maximum value, which corresponds to 4.06V for the output voltage. And the tuning parameter λ in the designed observer is set as -1 in both tests.

Test One

Figures 7.3-7.7 correspond to the first simulation test, which uses a square wave current input, as depicted in Figure 7.3. The current consists of repeated cycles of: one hour of 1 C-rate constant discharging, followed by 90 minutes of zero input, then 1 C-rate constant charging and ending with 90 minutes of zero input. Only the first 250 minutes of the simulation results are shown in the figures.

Figure 7.4 presents the estimate of lithium concentration in the positive electrode, initialized by 66% of the maximum value, which corresponds to 3.8V for the output voltage. Then, both the true and the estimated SoC are shown in Figure 7.5, where the initial error is due to incorrect initialization of lithium concentration in the positive electrode. As one would expect, convergence of the output voltage coincides with convergence of the SoC, and this is shown in Figure 7.6. By setting $\lambda = -1$, convergence is achieved in less than 50 minutes. A faster convergence is anticipated by choosing a smaller value of λ.

Figure 7.7 compares the values of internal average temperature in both the SPM-T model and the observer. Note that, since the internal average temperature is monitored in an open-loop fashion, one needs to correctly initialize its value. This requirement can be achieved. Indeed, even if the initial value is not available, as long as the simulation starts from an equilibrium, the internal average temperature of the battery coincides with the ambient temperature after some settling time.

Test Two

Figures 7.8-7.12 correspond to a simulation test using a current input obtained from the Urban Dynamometer Driving Schedule (UDDS) and scaled to a current profile within the range of ± 4 C-rate of the battery. This current profile, as shown in Figure 7.8, is representative of current demands in automotive applications.

Figure 7.9 presents the estimate of lithium concentration in the positive electrode, also initialized by 66% of the maximum value, which corresponds to 3.8V as well for the output voltage. Effective estimation result of the SoC is shown in Figure 7.10. As seen from Figure 7.11, convergence of the output voltage also coincides with convergence of the SoC. In this case, convergence is achieved after 60 minutes. Finally, Figure 7.12 compares the values of internal average temperature in both the SPM-T model and the observer.

7.5.2 Numerical implementation

Numerical implementation of the SPM-T model follows the equations presented in Subsection 7.2.3. A finite-volume method is first used for the spatial discretization of the advection-diffusion equation (7.20)–(7.23), and then for time discretization of the resulting system of ODEs, the Euler-backward method is used. The observer is implemented using the same discretization procedure. Note that in the numerical implementation of the observer, lithium concentration in the negative electrode is approximated by the polynomial profile presented in [107], as described briefly in Subsection 7.3.2.

For the numerical implementation of the kernel function $p(r, s, t)$ and the computation of the observer gain, a trapezoidal approximation of the IDE (7.92) is used to obtain an ODE, which is then discretized in time with the Euler-backward method. As mentioned in Section 7.4, time normalization $t' = D_s^+(T(t))/(R_s^+)^2 t$ by the temperature-dependent function is not preferable; however, a normalization by some constant can always be done. In the above simulations, normalization is performed by

$$t' = \frac{D_s^+(T_{\mathrm{amb}})}{(R_s^+)^2} t, \tag{7.118}$$

where the temperature profile in the previous normalization function is replaced by the constant ambient temperature.

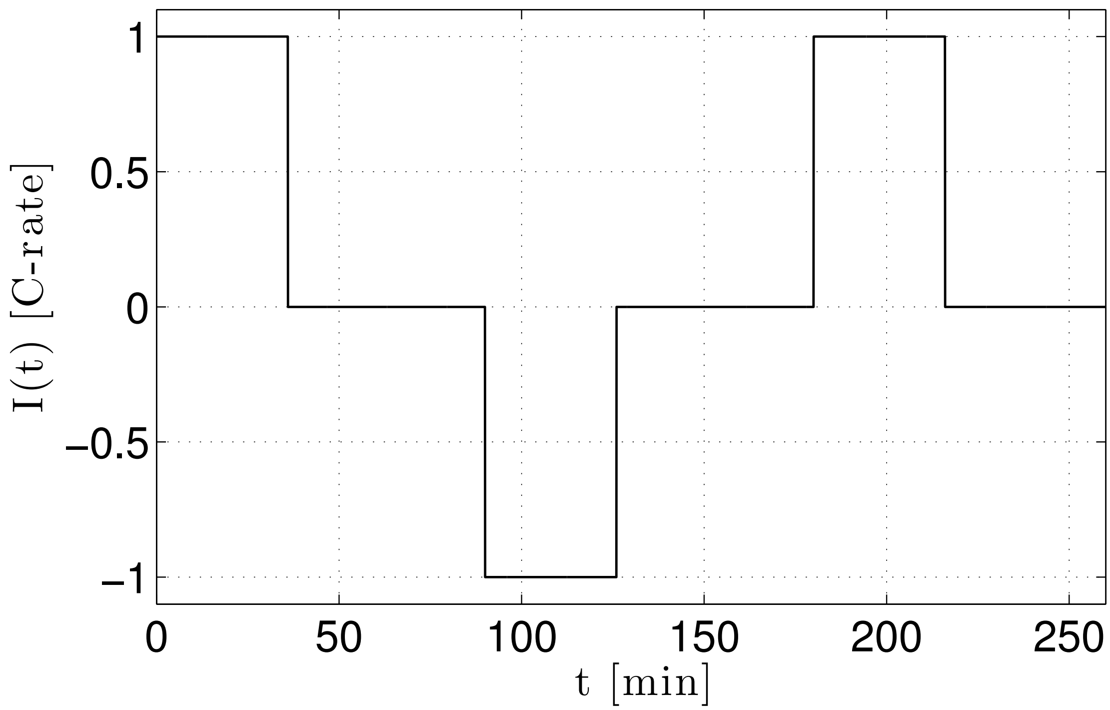

Figure 7.3: Current profile. Repeated cycles of 1C-rate constant discharging, resting and 1 C-rate charging.

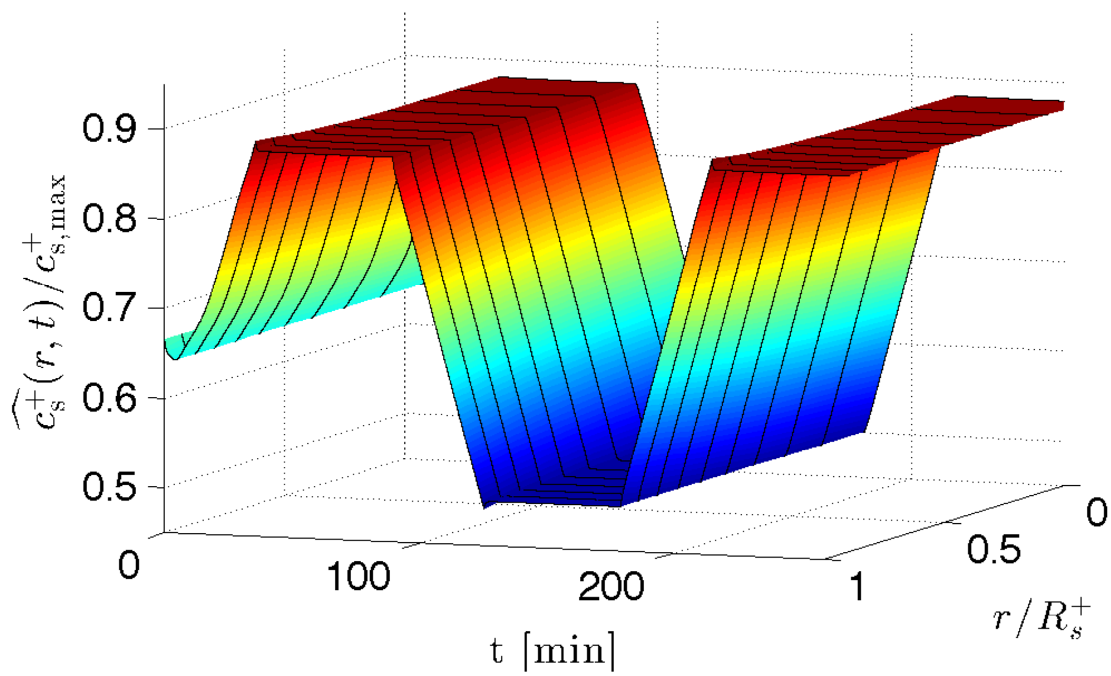

Figure 7.4: Estimate of lithium concentration in positive electrode.

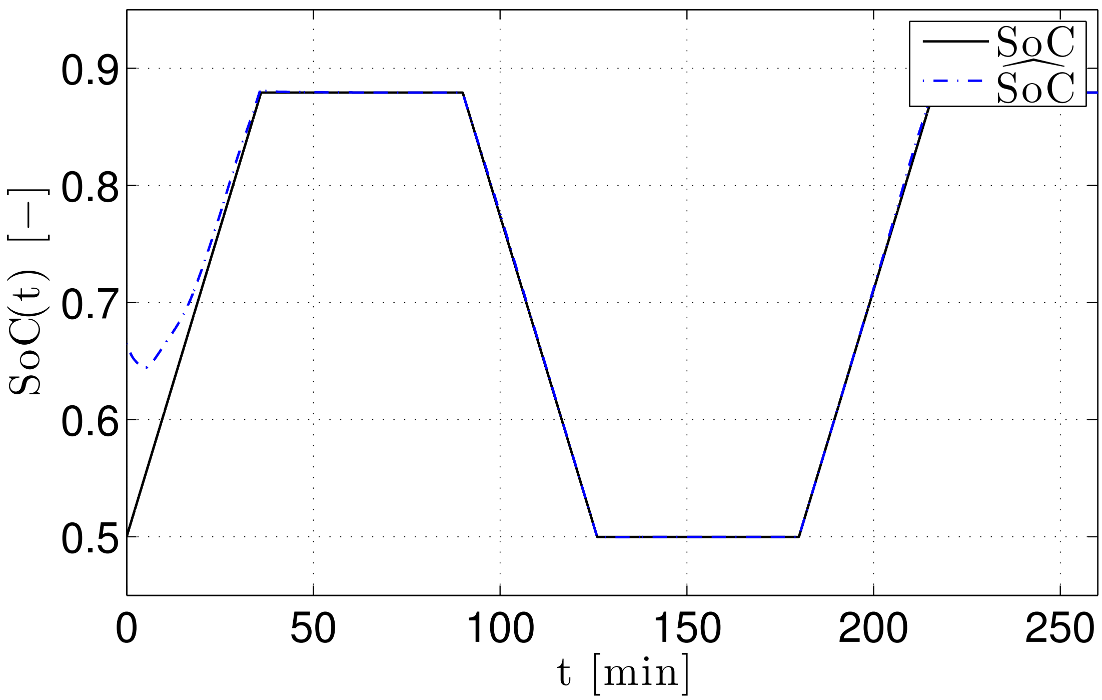

Figure 7.5: True and estimated SoC. Initial error in SoC estimation is due to incorrect initialization of lithium concentration in the positive electrode.

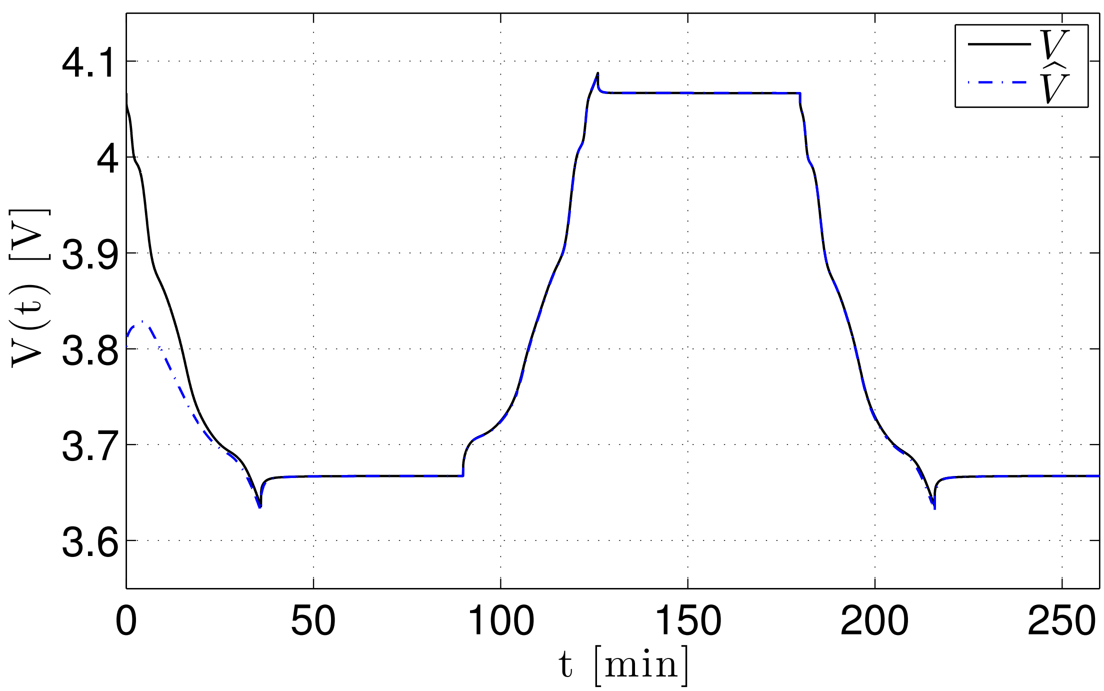

Figure 7.6: True and estimated voltage.

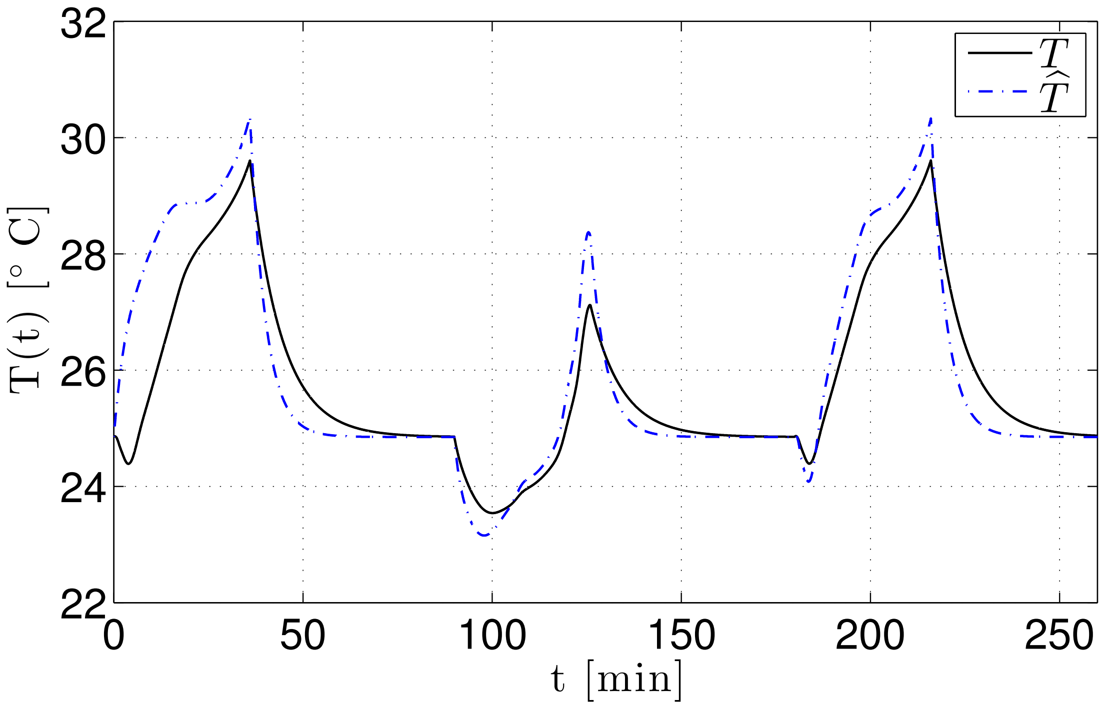

Figure 7.7: True and estimated internal average temperature.

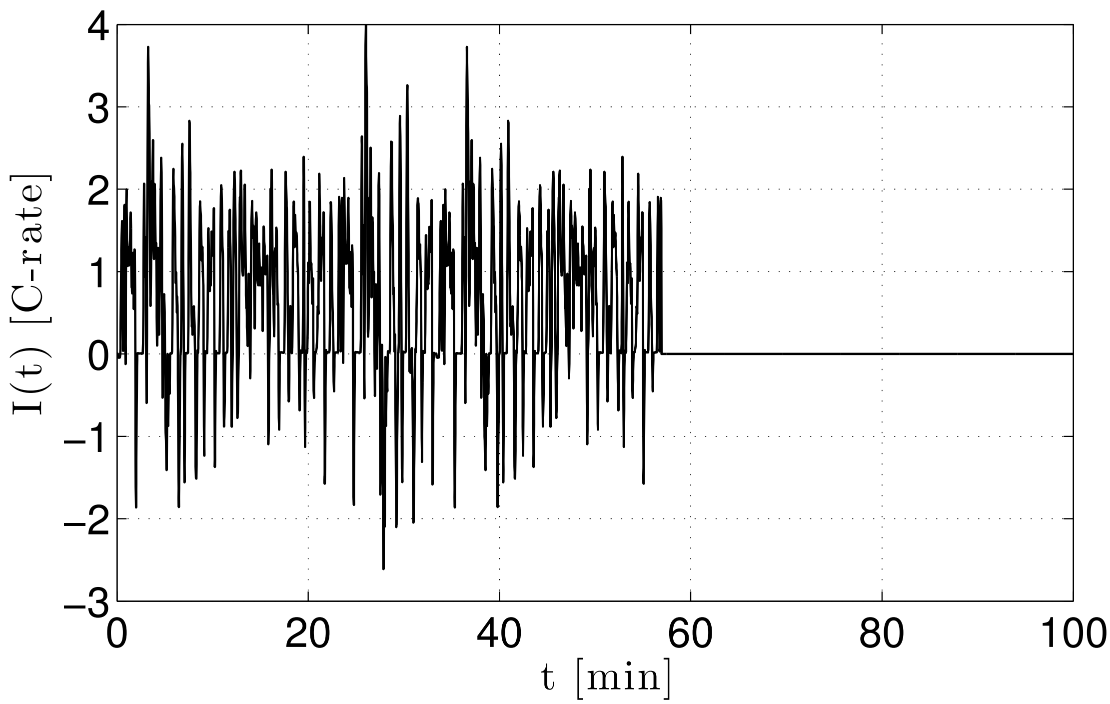

Figure 7.8: Current profile. The current density profile is obtained from th Urban Dynamometer Driving Schedule (UDDS) and scaled to input values no larger that ± 4C-rate.

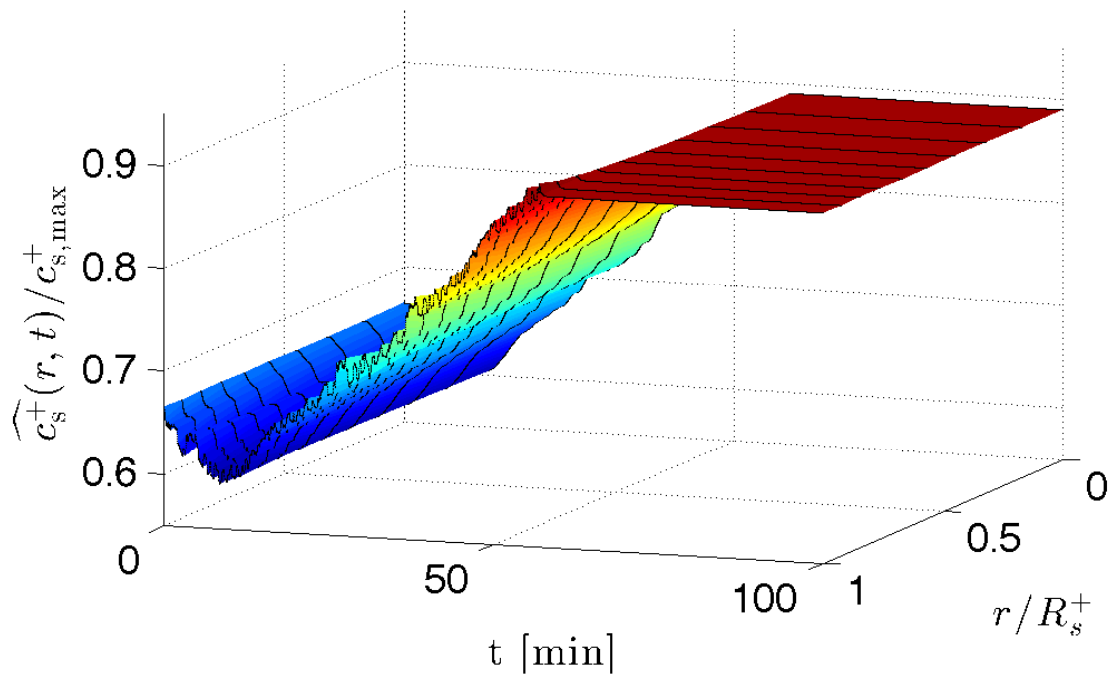

Figure 7.9: Estimate of lithium concentration in positive electrode.

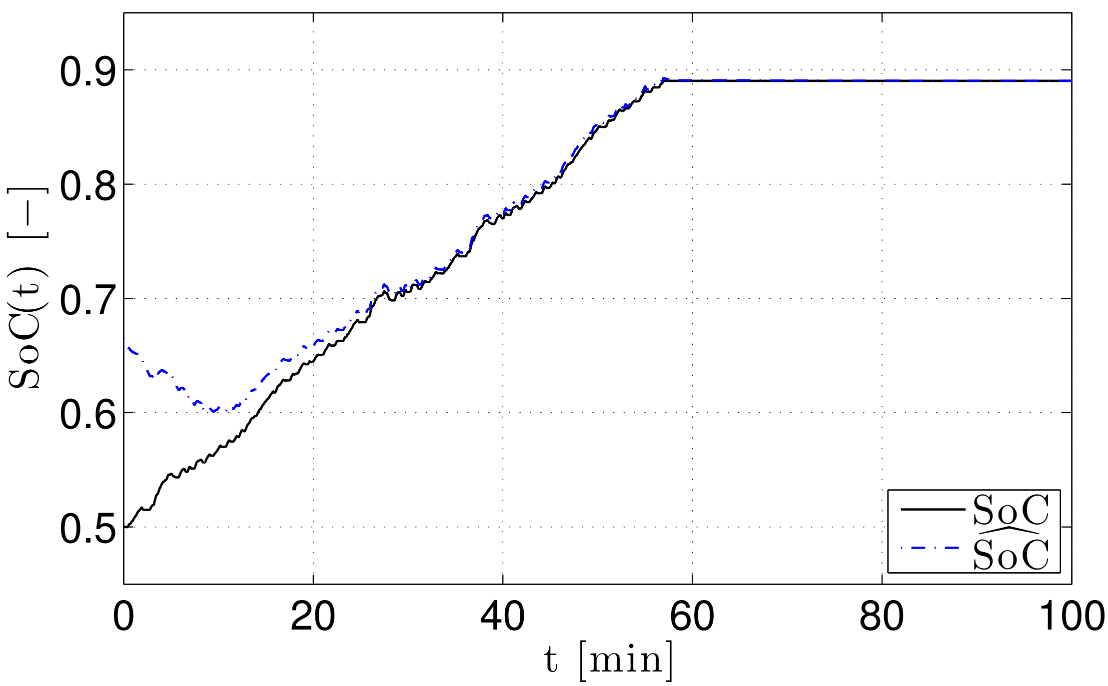

Figure 7.10: True and estimated SoC. Initial error in SoC is due to incorrect initialization of lithium concentration in the positive electrode.

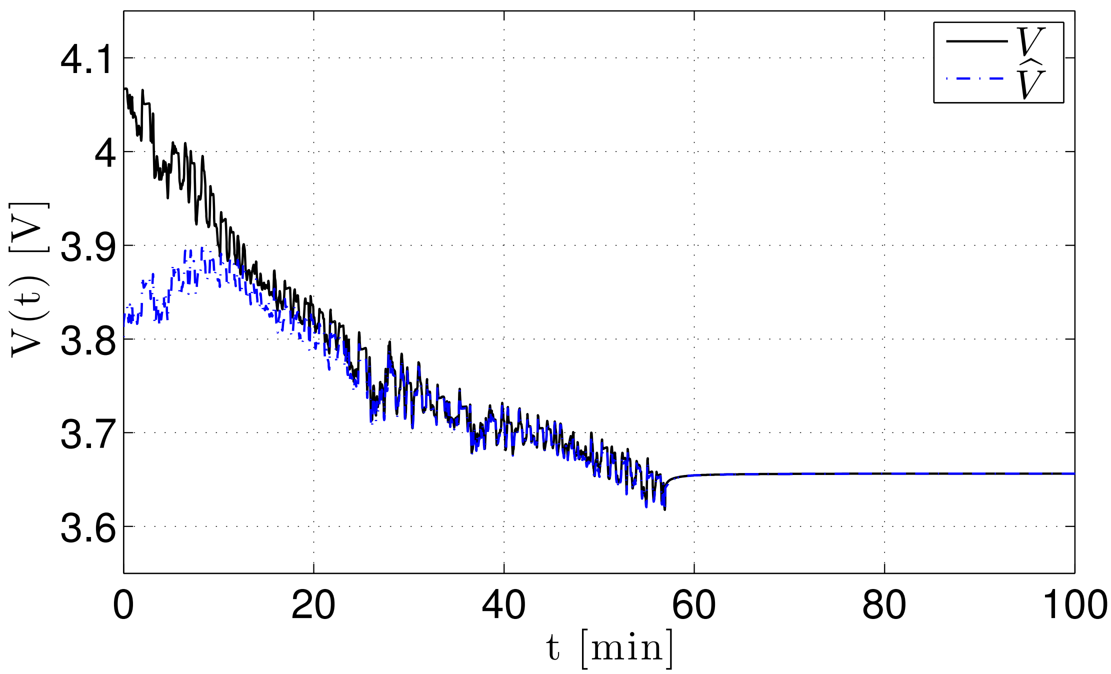

Figure 7.11: True and estimated voltage.

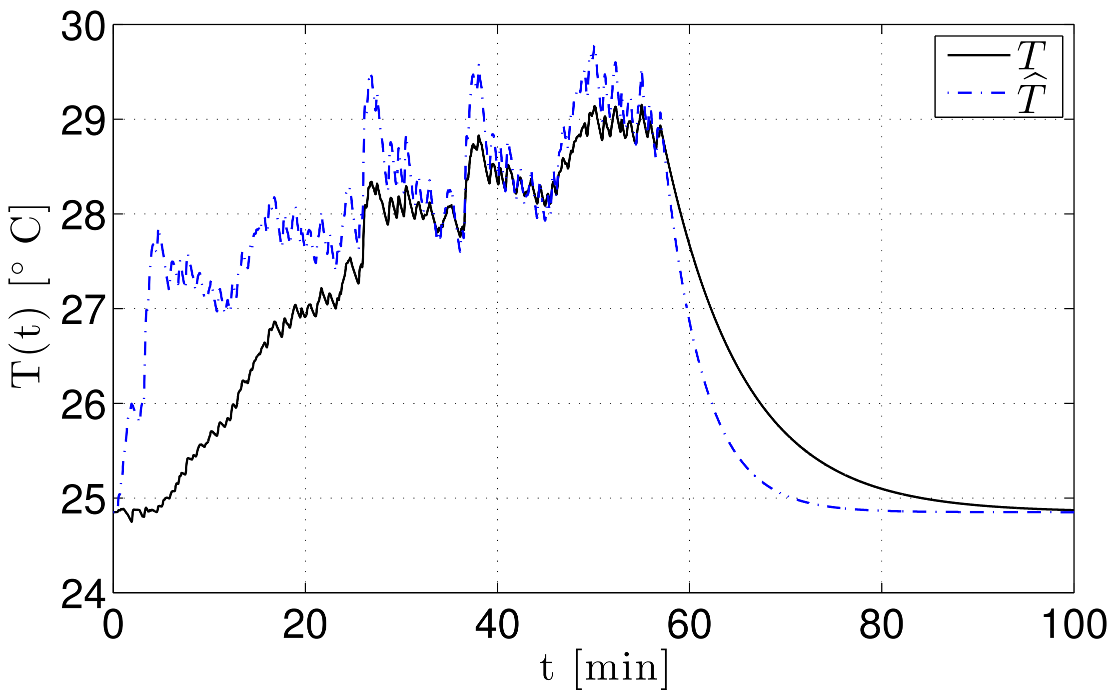

Figure 7.12: True and estimated interval average temperature.

7.6 Notes and references

This paper discusses the problem of SoC estimation for the lithium-ion batteries based on a thermal-electrochemical model. In this regard, an infinite-dimensional Luenberger observer is proposed. For the transformation between the observer error system and the exponentially stable target system, well-posedness of the time-varying PDE backstepping kernel functions are rigorously proved. Then, exponential stability of the observer error system is established, which proves effectiveness of the designed observer. We consider this result as an additional step in the effort to design estimation (and control) algorithms for lithium-ion batteries from electrochemical models, without relying on the discretization of the PDEs in these models.

Compared with estimation based on the SPM alone, the proposed observer has the advantage of taking into account the temperature dependence of model parameters. Simultaneously, the internal average temperature can be monitored in an open-loop fashion. The observer has also the advantage of requiring only one design/tuning parameter as compared with the possibly large number of tuning parameters required in estimation methods based on finite dimensional approximations of the battery model, such as EKF, UKF and particle filters.

Some simplifications are made in this paper, and their relaxation could be considered as a future research direction. Note that some system coefficients are actually state-dependent, and the relaxation would result in a challenging problem. Observer design for the battery internal temperature or the distributed temperature is also a problem worth investigating; this is an estimation problem for the full state of a coupled PDE-ODE systems. Another possible extension is to retain the concentration dynamics in the negative electrode and design one observer for each electrode [108]. One could even consider multiple active materials in the electrodes [109] or adding models for degradations (e.g. capacity fade) to the observer [110].

Chapter 7 contains reprints or adaptions of the following papers: 1. S.-X. Tang, Y.-B. Wang, Z. Sahinoglu, T. Wada, S. Hara and M. Krstic, "State-of-charge estimation for lithium-ion batteries via a coupled thermal-electrochemical model", in *Proceedings of the American Control Conference*, pp. 5871-5877, Chicago,

IL, USA, July 1-3, 2015. 2. S.-X. Tang, L. Camacho-Solorio, Y.-B. Wang and M. Krstic, “State-of-charge estimation from a thermal-electrochemical model of lithium-ion batteries”, *Automatica*, under review. The dissertation author is the primary investigator and author of these papers, and would like to acknowledge Leobardo Camacho-Solorio, Satoshi Hara, Miroslav Krstic, Zafer Sahinoglu, Toshihiro Wada and Yebin Wang for their contributions.

Chapter 8

Conclusion

This dissertation presents studies of three classes of partial differential equation (PDE) systems: Korteweg-de Vries (KdV) systems, coupled systems of first-order hyperbolic PDEs and parabolic systems with time-varying coefficients, which comes from the application of a real-world issue, i.e., state-of-charge estimation of Li-ion batteries.

Investigation in this dissertation centers on the problems of stability analysis, controller design and disturbance rejection, for which some useful control principles and tools are used. In detail, we have used the following three methods for the stability analysis: the center manifold method, the Lyapunov approach and the operator semigroup theory. Backstepping technique is employed all through the paper for the controller and observer designs. We have also modified the standard version of sliding mode control (SMC) algorithm and generalize the active disturbance control (ADRC) paradigm to deal with the control matched disturbance.

Looking into the future, the following are a few potential research directions, listed according to each of the three main parts in this chapter.

As seen from Part I, asymptotical stability analysis for the other cases when the nonlinear KdV equation has a second order or even higher center manifold is an ongoing research subject. And it remains an interesting topic to deal with the control matched disturbance within the linear Korteweg-de Vries equations with anti-diffusion (LKdVAD) system, hopefully via the SMC or the ADRC.

One possible extension from Part II would be to study the control problems related to the nonlinear coupled transport PDE systems. Some applications of the theoretical results can also be considered, such as the *Saint-Venant Exner* model and the bilayer *Saint-Venant* model, which are both nonlinear.

Inspired from the results in Part III, the stability analysis, control design and disturbance rejection problems can all be considered for the parabolic PDEs with time-varying coefficients.

Moreover, other types of PDEs such as the Euler-Bernoulli beam system can be discussed as well.

Appendix A

Appendices for Chapter 2

A.1 On the Solution a, b and c to Equations (2.75), (2.76) and (2.77)

Set

$$f_+(x) := a(x) + c(x),\ f_-(x) := a(x) - c(x), \tag{A.1}$$

and

$$\begin{cases} g_+(x) := \varphi_1(x)\varphi_1'(x) + \varphi_2(x)\varphi_2'(x) + \sqrt{3}c_1\varphi_1(x) - c_1\varphi_2(x), \\ g_-(x) := \varphi_1(x)\varphi_1'(x) - \varphi_2(x)\varphi_2'(x) - \sqrt{3}c_1\varphi_1(x) - c_1\varphi_2(x), \\ g(x) := \varphi_1(x)\varphi_2'(x) + \varphi_1'(x)\varphi_2(x) + c_1\varphi_1(x) - \sqrt{3}c_1\varphi_2(x). \end{cases} \tag{A.2}$$

First, adding each equation of (2.77) to the corresponding equation of (2.75), we have the following ODE for $f_+(x)$:

$$\begin{cases} f_+'''(x) + f_+'(x) + g_+(x) = 0, \\ f_+(0) = f_+(L) = 0,\ f_+'(L) = 0. \end{cases} \tag{A.3}$$

Second, subtracting each equation of (2.77) from the corresponding equation of (2.75), we obtain

$$\begin{cases} 2qb(x) + f_-'(x) + f_-'''(x) + g_-(x) = 0, \\ f_-(0) = f_-(L) = 0,\ f_-'(L) = 0, \end{cases} \tag{A.4}$$

which gives

$$b(x) = -\frac{1}{2q}(f'_-(x) + f'''_-(x) + g_-(x)). \tag{A.5}$$

Substitute (A.5) into (2.76), then the following ODE for $f_-(x)$ is obtained:

$$\begin{cases} f^{(6)}_-(x) + 2f^{(4)}_-(x) + f''_-(x) + 4q^2 f_-(x) + g'_-(x) + g'''_-(x) - 2qg(x) = 0, \\ f_-(0) = f_-(L) = \ f'_-(L) = f'''_-(L) = 0, \\ f'_-(0) + f'''_-(0) = 0, \ f''_-(L) + f^{(4)}_-(L) = 0, \end{cases} \tag{A.6}$$

where the second and third lines are borrowed from (A.4), and the last three lines are obtained from (A.5) and the boundary conditions of (2.26), (2.27), (2.76), (A.2), and (A.4).

Employing the method of undetermined coefficients, we get the following (unique) solution to the nonhomogeneous ODE (A.3) is

$$\begin{aligned} f_+(x) = \sum_{l=1}^{3} C_{+l} f_{+l}(x) + c_{+11}\cos\left(\frac{1}{\sqrt{21}}x\right) + c_{+12}\sin\left(\frac{1}{\sqrt{21}}x\right) + c_{+21}\cos\left(\frac{3}{\sqrt{21}}x\right) \\ + c_{+31}\cos\left(\frac{4}{\sqrt{21}}x\right) + c_{+32}\sin\left(\frac{4}{\sqrt{21}}x\right) + c_{+41}\cos\left(\frac{5}{\sqrt{21}}x\right) \\ + c_{+42}\sin\left(\frac{5}{\sqrt{21}}x\right) + c_{+51}\cos\left(\frac{6}{\sqrt{21}}x\right) + c_{+61}\cos\left(\frac{9}{\sqrt{21}}x\right), \end{aligned} \tag{A.7}$$

where the fundamental solutions $f_{+l}(x), \ l = 1, 2, 3$ are

$$f_{+1}(x) = 1, \ f_{+2}(x) = \cos(x), \ f_{+3}(x) = \sin(x), \tag{A.8}$$

and the constants are

$$c_{+11} = \frac{3c_1\Theta}{(\frac{1}{\sqrt{21}}) - (\frac{1}{\sqrt{21}})^3}, \quad c_{+12} = \frac{-3\sqrt{3}c_1\Theta}{-(\frac{1}{\sqrt{21}}) + (\frac{1}{\sqrt{21}})^3}, \ d_{21} = \frac{\Theta^2 \frac{18}{\sqrt{21}}}{(\frac{1}{\sqrt{21}}) - (\frac{1}{\sqrt{21}})^3}, \tag{A.9}$$

$$c_{+31} = \frac{-2c_1\Theta}{(\frac{1}{\sqrt{21}}) - (\frac{1}{\sqrt{21}})^3}, \ d_{32} = \frac{2\sqrt{3}c_1\Theta}{-(\frac{1}{\sqrt{21}}) + (\frac{1}{\sqrt{21}})^3}, \ d_{41} = \frac{c_1\Theta}{(\frac{1}{\sqrt{21}}) - (\frac{1}{\sqrt{21}})^3}, \tag{A.10}$$

$$c_{+42} = \frac{\sqrt{3}c_1\Theta}{-(\frac{1}{\sqrt{21}}) + (\frac{1}{\sqrt{21}})^3}, \ d_{51} = \frac{\Theta^2 \frac{18}{\sqrt{21}}}{(\frac{1}{\sqrt{21}}) - (\frac{1}{\sqrt{21}})^3}, \ d_{61} = \frac{\Theta^2 \frac{-18}{\sqrt{21}}}{(\frac{1}{\sqrt{21}}) - (\frac{1}{\sqrt{21}})^3}, \tag{A.11}$$

and

$$C_{+l} = \frac{\det(A_{+l})}{\det(A_+)}, \ l = 1, 2, 3. \tag{A.12}$$

Here,

$$A_+ = \begin{pmatrix} f_{+1}(0) & f_{+2}(0) & f_{+3}(0) \\ f_{+1}(L) & f_{+2}(L) & f_{+3}(L) \\ f'_{+1}(L) & f'_{+2}(L) & f'_{+3}(L) \end{pmatrix}, \tag{A.13}$$

and each A_{+l} is the matrix formed by replacing the l-th column of A_+ with a column vector $-b_+$, where

$$b_+ = \begin{pmatrix} b_{+1} & b_{+2} & b_{+3} \end{pmatrix}^{\mathrm{T}}, \tag{A.14}$$

and

$$b_{+1} = c_{+11} + c_{+21} + c_{+31} + c_{+41} + c_{+51} + c_{+61}, \tag{A.15}$$

$$\begin{aligned} b_{+2} = {} & c_{+11}\cos\left(\frac{1}{\sqrt{21}}L\right) + c_{+12}\sin\left(\frac{1}{\sqrt{21}}L\right) + c_{+21}\cos\left(\frac{3}{\sqrt{21}}L\right) \\ & + c_{+31}\cos\left(\frac{4}{\sqrt{21}}L\right) + c_{+32}\sin\left(\frac{4}{\sqrt{21}}L\right) + c_{+41}\cos\left(\frac{5}{\sqrt{21}}L\right) \\ & + c_{+42}\sin\left(\frac{5}{\sqrt{21}}L\right) + c_{+51}\cos\left(\frac{6}{\sqrt{21}}L\right) + c_{+61}\cos\left(\frac{9}{\sqrt{21}}L\right), \end{aligned} \tag{A.16}$$

$$\begin{aligned} b_{+3} = {} & -\frac{1}{\sqrt{21}}c_{+11}\sin\left(\frac{1}{\sqrt{21}}L\right) + \frac{1}{\sqrt{21}}c_{+12}\cos\left(\frac{1}{\sqrt{21}}L\right) - \frac{3}{\sqrt{21}}c_{+21}\sin\left(\frac{3}{\sqrt{21}}L\right) \\ & - \frac{4}{\sqrt{21}}c_{+31}\sin\left(\frac{4}{\sqrt{21}}L\right) + \frac{4}{\sqrt{21}}c_{+32}\cos\left(\frac{4}{\sqrt{21}}L\right) - \frac{5}{\sqrt{21}}c_{+41}\sin\left(\frac{5}{\sqrt{21}}L\right) \\ & + \frac{5}{\sqrt{21}}c_{+42}\cos\left(\frac{5}{\sqrt{21}}L\right) - \frac{6}{\sqrt{21}}c_{+51}\sin\left(\frac{6}{\sqrt{21}}L\right) - \frac{9}{\sqrt{21}}c_{+61}\sin\left(\frac{9}{\sqrt{21}}L\right). \end{aligned} \tag{A.17}$$

Similarly, the method of undetermined coefficients gives the following (unique) solution to the nonhomogeneous ODE system (A.6) is

$$\begin{aligned} f_-(x) = {} & \sum_{l=1}^{6} C_{-l} f_{-l}(x) + c_{-11}\cos\left(\frac{1}{\sqrt{21}}x\right) + c_{-12}\sin\left(\frac{1}{\sqrt{21}}x\right) + c_{-21}\cos\left(\frac{2}{\sqrt{21}}x\right) \\ & + c_{-31}\cos\left(\frac{4}{\sqrt{21}}x\right) + c_{-32}\sin\left(\frac{4}{\sqrt{21}}x\right) + c_{-41}\cos\left(\frac{5}{\sqrt{21}}x\right) \\ & + c_{-42}\sin\left(\frac{5}{\sqrt{21}}x\right) + c_{-51}\cos\left(\frac{8}{\sqrt{21}}x\right) + c_{-61}\cos\left(\frac{10}{\sqrt{21}}x\right), \end{aligned} \tag{A.18}$$

where the fundamental solutions $f_{-l}(x), l = 1, \cdots, , 6$ are

$$\begin{cases} f_{-1}(x) = e^{\alpha_1 x}\cos(\beta_1 x), & f_{-2}(x) = e^{\alpha_1 x}\sin(\beta_1 x), \\ f_{-3}(x) = e^{-\alpha_1 x}\cos(\beta_1 x), & f_{-4}(x) = e^{-\alpha_1 x}\sin(\beta_1 x), \\ f_{-5}(x) = \cos(\beta_2 x), & f_{-6}(x) = \sin(\beta_2 x), \end{cases} \tag{A.19}$$

with

$$\alpha_1 = \frac{\left(20+\sqrt{57}\right)^{\frac{1}{3}} - 7\left(20+\sqrt{57}\right)^{-\frac{1}{3}}}{2\sqrt{7}}, \tag{A.20}$$

$$\beta_1 = \frac{\left(20+\sqrt{57}\right)^{\frac{1}{3}} + 7\left(20+\sqrt{57}\right)^{-\frac{1}{3}}}{2\sqrt{21}}, \tag{A.21}$$

$$\beta_2 = \frac{\left(20+\sqrt{57}\right)^{\frac{1}{3}} + 7\left(20+\sqrt{57}\right)^{-\frac{1}{3}}}{\sqrt{21}}, \tag{A.22}$$

and the constants are

$$c_{-11} = \frac{-3\Theta^2\frac{40}{21^2} + 4q\Theta^2\frac{2}{\sqrt{21}} + 9qc_1\Theta}{(\frac{1}{\sqrt{21}})^6 - 2(\frac{1}{\sqrt{21}})^4 + (\frac{1}{\sqrt{21}})^2 - 4q^2}, \tag{A.23}$$

$$c_{-12} = \frac{-9\sqrt{3}qc_1\Theta}{(\frac{1}{\sqrt{21}})^6 - 2(\frac{1}{\sqrt{21}})^4 + (\frac{1}{\sqrt{21}})^2 - 4q^2}, \tag{A.24}$$

$$c_{-21} = \frac{3\Theta^2\frac{18}{21^2} - 4q\Theta^2\frac{3^2}{\sqrt{21}}}{(\frac{2}{\sqrt{21}})^6 - 2(\frac{2}{\sqrt{21}})^4 + (\frac{2}{\sqrt{21}})^2 - 4q^2}, \tag{A.25}$$

$$c_{-31} = \frac{3\Theta^2\frac{240}{21^2} - 4q\Theta^2\frac{12}{\sqrt{21}} - 6qc_1\Theta}{(\frac{4}{\sqrt{21}})^6 - 2(\frac{4}{\sqrt{21}})^4 + (\frac{4}{\sqrt{21}})^2 - 4q^2}, \tag{A.26}$$

$$c_{-32} = \frac{6\sqrt{3}qc_1\Theta}{(\frac{4}{\sqrt{21}})^6 - 2(\frac{4}{\sqrt{21}})^4 + (\frac{4}{\sqrt{21}})^2 - 4q^2}, \tag{A.27}$$

$$c_{-41} = \frac{-3\Theta^2\frac{600}{21^2} + 4q\Theta^2\frac{30}{\sqrt{21}} - 3qc_1\Theta}{(\frac{5}{\sqrt{21}})^6 - 2(\frac{5}{\sqrt{21}})^4 + (\frac{5}{\sqrt{21}})^2 - 4q^2}, \tag{A.28}$$

$$c_{-42} = \frac{-3\sqrt{3}qc_1\Theta}{(\frac{5}{\sqrt{21}})^6 - 2(\frac{5}{\sqrt{21}})^4 + (\frac{5}{\sqrt{21}})^2 - 4q^2}, \tag{A.29}$$

$$c_{-51} = \frac{3\Theta^2\frac{2048}{21^2} - 4q\Theta^2\frac{16}{\sqrt{21}}}{(\frac{8}{\sqrt{21}})^6 - 2(\frac{8}{\sqrt{21}})^4 + (\frac{8}{\sqrt{21}})^2 - 4q^2}, \tag{A.30}$$

$$c_{-61} = \frac{3\Theta^2\frac{1250}{21^2} + 4q\Theta^2\frac{5}{\sqrt{21}}}{(\frac{10}{\sqrt{21}})^6 - 2(\frac{10}{\sqrt{21}})^4 + (\frac{10}{\sqrt{21}})^2 - 4q^2}, \tag{A.31}$$

and

$$C_{-l} = \frac{\det(A_{-l})}{\det(A_{-})}, \ l = 1, \cdots, 6. \tag{A.32}$$

Here, the matrix A_{-} is defined by

$$A_{-} = \begin{pmatrix} \alpha_{-1} & \alpha_{-2} & \alpha_{-3} & \alpha_{-4} & \alpha_{-5} & \alpha_{-6} \end{pmatrix} \tag{A.33}$$

with

$$\alpha_{-l} = \begin{pmatrix} f_{-l}(0) \\ f_{-l}(L) \\ f'_{-l}(L) \\ f'_{-l}(0) + f'''_{-l}(0) \\ f'''_{-l}(L) \\ f''_{-l}(L) + f^{(4)}_{-l}(L) \end{pmatrix}, \ l = 1, \cdots, 6. \tag{A.34}$$

Each A_{-l} is the matrix formed by replacing the l-th column of A_{-} with a column vector $-b_{-}$, where

$$b_{-} = \begin{pmatrix} b_{-1} & b_{-2} & b_{-3} & b_{-4} & b_{-5} & b_{-6} \end{pmatrix}^{\mathrm{T}} \tag{A.35}$$

and

$$b_{-1} = c_{-11} + c_{-21} + c_{-31} + c_{-41} + c_{-51} + c_{-61}, \tag{A.36}$$

$$\begin{aligned} b_{-2} = {} & c_{-11}\cos\left(\frac{1}{\sqrt{21}}L\right) + c_{-12}\sin\left(\frac{1}{\sqrt{21}}L\right) + c_{-21}\cos\left(\frac{2}{\sqrt{21}}L\right) \\ & + c_{-31}\cos\left(\frac{4}{\sqrt{21}}L\right) + c_{-32}\sin\left(\frac{4}{\sqrt{21}}L\right) + c_{-41}\cos\left(\frac{5}{\sqrt{21}}L\right) \\ & + c_{-42}\sin\left(\frac{5}{\sqrt{21}}L\right) + c_{-51}\cos\left(\frac{8}{\sqrt{21}}L\right) + c_{-61}\cos\left(\frac{10}{\sqrt{21}}L\right), \end{aligned} \tag{A.37}$$

$$\begin{aligned} b_{-3} = & -\frac{1}{\sqrt{21}}c_{-11}\sin\left(\frac{1}{\sqrt{21}}L\right) + \frac{1}{\sqrt{21}}c_{-12}\cos\left(\frac{1}{\sqrt{21}}L\right) \\ & -\frac{2}{\sqrt{21}}c_{-21}\sin\left(\frac{2}{\sqrt{21}}L\right) \end{aligned}$$

$$
\begin{aligned}
&- \frac{4}{\sqrt{21}} c_{-31} \sin\left(\frac{4}{\sqrt{21}} L\right) + \frac{4}{\sqrt{21}} c_{-32} \cos\left(\frac{4}{\sqrt{21}} L\right) \\
&- \frac{5}{\sqrt{21}} c_{-41} \sin\left(\frac{5}{\sqrt{21}} L\right) + \frac{5}{\sqrt{21}} c_{-42} \cos\left(\frac{5}{\sqrt{21}} L\right) \\
&- \frac{8}{\sqrt{21}} c_{-51} \sin\left(\frac{8}{\sqrt{21}} L\right) - \frac{10}{\sqrt{21}} c_{-61} \sin\left(\frac{10}{\sqrt{21}} L\right), \qquad \text{(A.38)}
\end{aligned}
$$

$$
\begin{aligned}
b_{-4} =& \frac{20}{21\sqrt{21}} c_{-12} \cos\left(\frac{1}{\sqrt{21}} L\right) + \frac{20}{21\sqrt{21}} c_{-32} \cos\left(\frac{4}{\sqrt{21}} L\right) \\
&- \frac{20}{\sqrt{21}} c_{-42} \cos\left(\frac{5}{\sqrt{21}} L\right), \qquad \text{(A.39)}
\end{aligned}
$$

$$
\begin{aligned}
b_{-5} =& - \frac{20}{21\sqrt{21}} c_{-11} \sin\left(\frac{1}{\sqrt{21}} L\right) + \frac{20}{21\sqrt{21}} c_{-12} \cos\left(\frac{1}{\sqrt{21}} L\right) \\
&- \frac{34}{21\sqrt{21}} c_{-21} \sin\left(\frac{2}{\sqrt{21}} L\right) \\
&- \frac{20}{21\sqrt{21}} c_{-31} \sin\left(\frac{4}{\sqrt{21}} L\right) + \frac{20}{21\sqrt{21}} c_{-32} \cos\left(\frac{4}{\sqrt{21}} L\right) \\
&+ \frac{20}{21\sqrt{21}} c_{-41} \sin\left(\frac{5}{\sqrt{21}} L\right) - \frac{20}{21\sqrt{21}} c_{-42} \cos\left(\frac{5}{\sqrt{21}} L\right) \\
&+ \frac{344}{21\sqrt{21}} c_{-51} \sin\left(\frac{8}{\sqrt{21}} L\right) + \frac{790}{21\sqrt{21}} c_{-61} \sin\left(\frac{10}{\sqrt{21}} L\right), \qquad \text{(A.40)}
\end{aligned}
$$

$$
\begin{aligned}
b_{-6} =& - \frac{20}{21^2} c_{-11} \cos\left(\frac{1}{\sqrt{21}} L\right) - \frac{20}{21^2} c_{-12} \sin\left(\frac{1}{\sqrt{21}} L\right) \\
&- \frac{68}{21^2} c_{-21} \cos\left(\frac{2}{\sqrt{21}} L\right) \\
&- \frac{80}{21^2} c_{-31} \cos\left(\frac{4}{\sqrt{21}} L\right) - \frac{80}{21^2} c_{-32} \sin\left(\frac{4}{\sqrt{21}} L\right) \\
&+ \frac{100}{21^2} c_{-41} \cos\left(\frac{5}{\sqrt{21}} L\right) + \frac{100}{21^2} c_{-42} \sin\left(\frac{5}{\sqrt{21}} L\right) \\
&+ \frac{2752}{21^2} c_{-51} \cos\left(\frac{8}{\sqrt{21}} L\right) + \frac{7900}{21^2} c_{-61} \cos\left(\frac{10}{\sqrt{21}} L\right). \qquad \text{(A.41)}
\end{aligned}
$$

Therefore, we derive from (A.1) that

$$\begin{aligned}
a(x) &= \frac{1}{2}(f_+(x) + f_-(x)) \\
&= \frac{1}{2}\Bigg[\sum_{l=1}^{3} C_{+l} f_{+l}(x) + \sum_{l=1}^{6} C_{-l} f_{-l}(x) \\
&\quad +(c_{+11} + c_{-11})\cos\left(\frac{1}{\sqrt{21}}x\right) + (c_{+12} + c_{-12})\sin\left(\frac{1}{\sqrt{21}}x\right) \\
&\quad +c_{-21}\cos\left(\frac{2}{\sqrt{21}}x\right) + c_{+21}\cos\left(\frac{3}{\sqrt{21}}x\right) \\
&\quad +(c_{+31} + c_{-31})\cos\left(\frac{4}{\sqrt{21}}x\right) + (c_{+32} + c_{-32})\sin\left(\frac{4}{\sqrt{21}}x\right) \\
&\quad +(c_{+41} + c_{-41})\cos\left(\frac{5}{\sqrt{21}}x\right) + (c_{+42} + c_{-42})\sin\left(\frac{5}{\sqrt{21}}x\right) \\
&\quad +c_{+51}\cos\left(\frac{6}{\sqrt{21}}x\right) + c_{-51}\cos\left(\frac{8}{\sqrt{21}}x\right) \\
&\quad +c_{+61}\cos\left(\frac{9}{\sqrt{21}}x\right) + c_{-61}\cos\left(\frac{10}{\sqrt{21}}x\right)\Bigg],
\end{aligned} \tag{A.42}$$

and

$$\begin{aligned}
c(x) &= \frac{1}{2}(f_+(x) - f_-(x)) \\
&= \frac{1}{2}\Bigg[\sum_{l=1}^{3} C_{+l} f_{+l}(x) - \sum_{l=1}^{6} C_{-l} f_{-l}(x) \\
&\quad +(c_{+11} - c_{-11})\cos\left(\frac{1}{\sqrt{21}}x\right) + (c_{+12} - c_{-12})\sin\left(\frac{1}{\sqrt{21}}x\right) \\
&\quad -c_{-21}\cos\left(\frac{2}{\sqrt{21}}x\right) + c_{+21}\cos\left(\frac{3}{\sqrt{21}}x\right) \\
&\quad +(c_{+31} - c_{-31})\cos\left(\frac{4}{\sqrt{21}}x\right) + (c_{+32} - c_{-32})\sin\left(\frac{4}{\sqrt{21}}x\right) \\
&\quad +(c_{+41} - c_{-41})\cos\left(\frac{5}{\sqrt{21}}x\right) + (c_{+42} - c_{-42})\sin\left(\frac{5}{\sqrt{21}}x\right) \\
&\quad +c_{+51}\cos\left(\frac{6}{\sqrt{21}}x\right) - c_{-51}\cos\left(\frac{8}{\sqrt{21}}x\right) \\
&\quad +c_{+61}\cos\left(\frac{9}{\sqrt{21}}x\right) - c_{-61}\cos\left(\frac{10}{\sqrt{21}}x\right)\Bigg].
\end{aligned} \tag{A.43}$$

From (A.5), we obtain

$$
\begin{aligned}
b(x) &= -\frac{1}{2q}(f'_{-}(x) + f'''_{-}(x) + g_{-}(x)) \\
&= -\frac{1}{2q}\Bigg[\sum_{l=1}^{6} C_{-l}f'_{-l}(x) + \sum_{l=1}^{6} C_{-l}f'''_{-l}(x) \\
&\quad -\left(\frac{20}{21\sqrt{21}}c_{-11} + \frac{2}{\sqrt{21}}\Theta^2 + 3c_1\Theta\right)\sin\left(\frac{1}{\sqrt{21}}x\right) \\
&\quad +\left(\frac{20}{21\sqrt{21}}c_{-12} + 3\sqrt{3}c_1\Theta\right)\cos\left(\frac{1}{\sqrt{21}}x\right) \\
&\quad -\left(\frac{34}{21\sqrt{21}}c_{-21} + \frac{9}{\sqrt{21}}\Theta^2\right)\sin\left(\frac{2}{\sqrt{21}}x\right) \\
&\quad -\left(\frac{20}{21\sqrt{21}}c_{-31} + 2c_1\Theta\right)\sin\left(\frac{4}{\sqrt{21}}x\right) \\
&\quad +\left(\frac{20}{21\sqrt{21}}c_{-32} - 2\sqrt{3}c_1\Theta\right)\cos\left(\frac{4}{\sqrt{21}}x\right) \\
&\quad +\left(\frac{20}{21\sqrt{21}}c_{-41} + \frac{30}{\sqrt{21}}\Theta^2 + c_1\Theta\right)\sin\left(\frac{5}{\sqrt{21}}x\right) \\
&\quad -\left(\frac{20}{21\sqrt{21}}c_{-42} + \sqrt{3}c_1\Theta\right)\cos\left(\frac{5}{\sqrt{21}}x\right) \\
&\quad -\frac{12}{\sqrt{21}}\Theta^2\sin\left(\frac{6}{\sqrt{21}}x\right) + \left(\frac{8\times 43}{21\sqrt{21}}c_{-51} - \frac{16}{\sqrt{21}}\Theta^2\right)\sin\left(\frac{8}{\sqrt{21}}x\right) \\
&\quad +\left(\frac{790}{\sqrt{21}}c_{-61} - \frac{5}{\sqrt{21}}\Theta^2\right)\sin\left(\frac{10}{\sqrt{21}}x\right)\Bigg]. \qquad \text{(A.44)}
\end{aligned}
$$

A.2 Some discussions on the decay rate estimation ρ_u

If choosing $q_1 = q_2 = 0$, then for the class of LKdVAD equations (2.149)–(2.150) with

$$\lambda_2 \geqslant 0,\ \lambda_0 \leqslant \frac{1}{4L^2}\lambda_2, \qquad \text{(A.45)}$$

only stability ($\rho_u = 0$) can be derived by following the proofs in Section 2.3. However, from the following lemma, asymptotical stability also holds for the target systems.

Lemma A.1. *For each $\lambda \in \sigma(\mathbf{A})$, $Re\lambda < 0$. Moreover, $\mathbf{A}$ generates an asymptot-*

ically stable C_0-semigroup on $\mathcal{H}$.

Proof. Following the proof of Lemma 2.4, we can get that for each $\lambda \in \sigma(\mathbf{A})$, $Re\lambda \leqslant 0$. Let $\lambda \in \sigma(\mathbf{A})$ be on the imaginary axis and $f \in D(\mathbf{A})$ be its associated eigenfunction of $\mathbf{A}$, then we have

$$Re < \mathbf{A}f, f >= 0, \tag{A.46}$$

hence,

$$f'(L) = 0, \ \lambda_0 = \lambda_1 = \lambda_2 = 0. \tag{A.47}$$

That is, there exist $y(x) \in \mathcal{A}^3(0, L)\backslash\{0\}$ and λ on the imaginary axis such that

$$y''' - \lambda y = 0, x \in (0, L) \tag{A.48}$$

$$y'(0) = y''(0) = y(L) = y'(L) = 0. \tag{A.49}$$

Denote by $z \in H^3(\mathbb{R})$ its prolongation by 0, then

$$z''' - \lambda z = y(0)\delta_0'' - y''(L)\delta_L \ in \ \mathcal{D}'(\mathbb{R}), \tag{A.50}$$

where δ_{x_0} denotes the Dirac measure at x_0. This is equivalent to the existence of complex numbers ϕ, ψ, λ (with $\phi \neq 0, \ \psi \neq 0$) and a function $z \in H^3(\mathbb{R})$ with compact support in $[-L, L]$ such that

$$z''' - \lambda z = \phi\delta_0'' - \psi\delta_L \ in \ \mathcal{D}'(\mathbb{R}). \tag{A.51}$$

Take Fourier transformation, then

$$\left(\lambda_1 + \lambda_2 i\xi + (i\xi)^2\right) - \psi e^{-iL\xi} \text{ in } \mathcal{D}'(\mathbb{R}) \left((i\xi)^3 - \lambda\right) \hat{z}(\xi) = \phi(i\xi)^2 - \psi e^{-iL\xi} \text{ in } \mathcal{D}'(\mathbb{R}), \tag{A.52}$$

and (setting $\lambda = -ip^3$)

$$\hat{z}(\xi) = -i\frac{\phi\xi^2 + \psi e^{-iL\xi}}{\xi^3 - p^3}. \tag{A.53}$$

Thus, there exist $p \in \mathbb{C}$ and $(\phi, \psi) \in \mathbb{C}^2\backslash\{(x, y)|x \neq 0, y \neq 0\}$ such that

$$f(\xi) := \frac{\phi\xi^2 + \psi e^{-iL\xi}}{\xi^3 - p^3} \tag{A.54}$$

is an entire function in $\mathbb{C}$. Because the roots of $\xi^3 - p^3$ are $p, \omega p, \omega^2 p$, this holds only if they are all also roots of $\phi\xi^2 + \psi e^{-iL\xi}$. Then we have

$$e^{-iLp} = -\frac{\phi}{\psi}p^2 \tag{A.55}$$

$$e^{-iL\omega p} = -\frac{\phi}{\psi}\omega^2 p^2 \tag{A.56}$$

$$e^{-iL\omega^2 p} = -\frac{\phi}{\psi}\omega^4 p^2. \tag{A.57}$$

Substitute $(A.55)$ into $(A.56)$ and $(A.57)$, then

$$\left(-\frac{\phi}{\psi}p^2\right)^{\omega} = -\frac{\phi}{\psi}\omega^2 p^2 \tag{A.58}$$

$$\left(-\frac{\phi}{\psi}p^2\right)^{\omega^2} = -\frac{\phi}{\psi}\omega^4 p^2 \tag{A.59}$$

Multiply both sides of the above two equations, then we can get

$$p^2 = -\frac{\psi}{\phi}, -\omega\frac{\psi}{\phi} \ or \ -\omega^2\frac{\psi}{\phi}. \tag{A.60}$$

However, by substituting $(A.60)$ into $(A.58)$, we get contradictions for all three cases, which proves that for each $\lambda \in \sigma(\mathbf{A})$, $Re\lambda < 0$. Moreover, from (2.181),

$$V(t) \leqslant V(0), \tag{A.61}$$

then $\mathbf{A}$ generates an asymptotically stable C_0-semigroup on $\mathcal{H}$ by the Arendt-Batty-Lyubich-Phong theorem. □

Remark A.1. *If furtherly choosing* $\lambda_2 = \lambda_1 = \lambda_0 = 0$*, then the class of LKdVAD equations* (2.149)–(2.150) *has been proved to be exponentially stable in [111]. Thus, from Remark 2.5, we derive exponential stability of the target systems with* $\lambda_2 = \lambda_1 = q_1 = q_2 = 0,\ \lambda_0 \leq 0$.

Appendix B

More Discussions for Chapter 3

B.1 An example

Consider the state feedback stabilizing problem of the following subclass of LKdVAD control systems (3.1)–(3.2) (with $\lambda_2 = \lambda_1 = q_1 = q_2 = 0,\ L = 1$) as an example:

$$u_t(t,x) = u_{xxx}(t,x) + \lambda_0 u(t,x) \tag{B.1}$$

$$u_x(t,0) = 0,\ u_{xx}(t,0) = 0,\ u(t,1) = U(t). \tag{B.2}$$

Set the control parameters as $\mu_2 = \mu_1 = r_1 = r_2 = 0$, and thus the target systems (3.92)–(3.93) are as follows:

$$w_t(t,x) = w_{xxx}(t,x) + \mu_0 w(t,x) \tag{B.3}$$

$$w_x(t,0) = 0,\ w_{xx}(t,0) = 0,\ w(t,1) = 0. \tag{B.4}$$

Through spectrum analysis, we get that, for

$$\lambda_0 > 6.3297 \ \text{(an approximate value)}, \tag{B.5}$$

the open-loop systems $(B.1)$–$(B.2)$ (with $U(t) = 0$) have eigenvalues on RHS of the complex plane and thus are unstable. However, by choosing

$$\mu_0 < 6.3297, \tag{B.6}$$

all the eigenvalues of target systems are on LHS of the complex plane (see, e.g., TABLE 1) and thus the equivalent closed-loop control systems with controller (3.3)–(3.9) are asymptotically stable. What's more, for $\mu_0 \leq 0$, the closed-loop systems can be proved to be exponentially stable, as shown in Remark A.1, and the exponential decay rate can be arbitrarily large by choosing μ_0 to be small enough.

Remark B.1. The eigenvalues of $(B.1)$–$(B.2)$ (with $U(t) = 0$) and $(B.3)$–$(B.4)$ are $(ln\theta)^3 + \lambda_0$ and $(ln\theta)^3 + \mu_0$ respectively, where θ are roots of the following equation:

$$\theta + \theta^{\frac{-1+\sqrt{3}i}{2}} + \theta^{\frac{-1-\sqrt{3}i}{2}} = 0. \tag{B.7}$$

That is, the eigenvalues of target systems are open-loop eigenvalues shifted to the left along the real axis in the complex plane by the same distance $\lambda_0 - \mu_0$, which is consistent with statement in Remark 2.5.

To derive the control kernel function, from (3.46)–(3.48), first we have

$$G_0(\xi, \eta) = \frac{\mu_0 - \lambda_0}{6}\eta(\xi + \eta). \tag{B.8}$$

Then, we can get the following formula:

$$\begin{aligned} G^k(\xi, \eta) =& \sum_{i=0}^{[\frac{k}{3}]} \left(a_{i,0,k}\eta^{3k+2-3i} + \sum_{j=1}^{k+1-3i} a_{i,j,k}\eta^{3k+2-j-3i} \left(\xi^j - \eta^j\right) \right) \\ =& \sum_{i=0}^{[\frac{k}{3}]} \eta^{3k+2-3i} \left(\sum_{j=0}^{k+1-3i} b_{i,j,k} \left(\frac{\xi}{\eta}\right)^j \right) \end{aligned} \tag{B.9}$$

for $k \geqslant 1$, where all coefficients $a_{i,0,k}, a_{i,j,k}, b_{i,j,k}$ are constants and $[x]$ denotes the integer not larger than x. The corresponding series expression for the control kernel function $\kappa(x, y)$ can thus also be obtained.

B.2 Critical cases

Two critical cases regarding the left-end boundary conditions are considered in this section, one for $q_2 = \infty, q_1 \neq \infty$; and the other for $q_1 = \infty, q_2 \neq \infty$. Only

Table B.1: Real parts of first seven eigenvalues and closed-loop system.

Real parts of first 7 eig.	uncontrolled system with $\lambda_0 = 100$	closed-loop system with $\mu_0 = -100$
1st eig.	93.6703	-106.3297
2nd eig.	-61.1000	-261.1000
3rd eig.	-645.9000	-845.9000
4th eig.	-1.9467×10^3	-2.1467×10^3
5th eig.	-4.2501×10^3	-4.4501×10^3
6th eig.	-7.8423×10^3	-8.0423×10^3
7th eig.	-1.3010×10^4	-1.3210×10^4

state feedback stabilizing results are covered here, however, extension to output feedback stabilizing problems is also achievable.

Case 1

Consider the following class of LKdVAD:

$$u_t(x,t) = u_{xxx}(x,t) + \lambda_2 u_{xx}(x,t) + \lambda_1 u_x(x,t) + \lambda_0 u(x,t), x \in (0,L) \tag{B.10}$$

$$u(0,t) = 0,\ u_x(0,t) = 0,\ u(L,t) = U(t), \tag{B.11}$$

which is a critical case with $q_2 = \infty, q_1 \neq \infty$.

Remark B.2. Note that when $\lambda_2 = \lambda_0 = 0, \lambda_1 = 1$, the open-loop system of (B.10)–(B.11) is the linearized version of the KdV equation (2.7), under a coordinate change $x \mapsto L - x$.

Choose an exponentially stable target systems as

$$w_t(x,t) = w_{xxx}(x,t) + \lambda_2 w_{xx}(x,t) + \lambda_1 w_x(x,t) + \lambda_0 w(x,t), x \in (0,L) \tag{B.12}$$

$$w(0,t) = 0,\ w_x(0,t) = 0,\ w(L,t) = 0, \tag{B.13}$$

where

$$\lambda_2 \geqslant 0,\ \lambda_1 = \frac{\lambda_2^2 - \lambda_2^2}{3} + \lambda_1, \lambda_0 < \frac{1}{4L^2}\lambda_2. \tag{B.14}$$

Remark B.3.

1. Exponential stability of system $(B.12)$–$(B.13)$ with $(B.14)$ can be proved by following a similar way as Lemma 2.5 in Subsection 2.3.3. With a decay rate estimation

$$\rho_w^{cr} = \frac{1}{4L^2}\lambda_2 - \lambda_0 > 0, \tag{B.15}$$

the exponential decay rate can be arbitrarily large by choosing λ_2 large enough.

2. For a special case $(\lambda_2 = 0, \lambda_1 = 1)$ of the target system $(B.12)$–$(B.13)$, we know that the decay rate estimation $(B.15)$ is only optimal for some values of L [38], [60], for example, if

$$L \in \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}} \;\middle|\; (k,l) \in \mathbb{N}^2 \right\}. \tag{B.16}$$

In order to obtain a state feedback controller, we use the proposed transformation $u \mapsto w$:

$$w(x,t) = u(x,t)e^{\frac{\lambda_2 - \lambda_2}{3}x} - \int_0^x k(x,y)u(y,t)e^{\frac{\lambda_2 - \lambda_2}{3}y}dy, \tag{B.17}$$

with the kernel function $k(x,y) \in \mathbb{R}$ satisfying:

$$k_{xxx}(x,y) + k_{yyy}(x,y) + \lambda_2 \left(k_{xx}(x,y) - k_{yy}(x,y)\right) + \lambda_1 \left(k_x(x,y) + k_y(x,y)\right)$$
$$= \left[\frac{\lambda_2 - \lambda_2}{3}\left(\frac{(\lambda_2 - \lambda_2)(2\lambda_2 + \lambda_2)}{9} - \lambda_1\right) + \lambda_0 - \lambda_0\right]k(x,y) \tag{B.18}$$
$$k(x,x) = 0 \tag{B.19}$$
$$k_x(x,x) = \left(\frac{\lambda_0 - \lambda_0}{3} + \lambda_1\frac{\lambda_2 - \lambda_2}{9} - \frac{\lambda_2}{3}\left(\frac{\lambda_2 - \lambda_2}{3}\right)^2\right)x \tag{B.20}$$
$$k(x,0) = 0, \tag{B.21}$$

By following a similar way as the proofs in Subsection 3.2.2. A, the following lemma can be proved.

Lemma B.1. *There exists a unique C^∞ solution to the kernel function $k(x,y)$-system $(B.18)$–$(B.21)$. For the inverse transformation $w \mapsto u$:*

$$u(x,t) = w(x,t)e^{-\frac{\lambda_2 - \lambda_2}{3}x} - \int_0^x l(x,y)w(y,t)dye^{-\frac{\lambda_2 - \lambda_2}{3}x}, \tag{B.22}$$

kernel function $l(x,y)$-system:

$$\iota_{xxx}(x,y) + \iota_{yyy}(x,y) + \lambda_2 \left(\iota_{xx}(x,y) - \iota_{yy}(x,y)\right) + \lambda_1 \left(\iota_x(x,y) + \iota_y(x,y)\right)$$
$$= \left[\frac{\lambda_2 - \lambda_2}{3}\left(\lambda_1 - \frac{(\lambda_2 - \lambda_2)(2\lambda_2 + \lambda_2)}{9}\right) - \lambda_0 + \lambda_0\right]\iota(x,y) \tag{B.23}$$
$$\iota(x,x) = 0 \tag{B.24}$$
$$\iota_x(x,x) = \left[\frac{\lambda_0 - \lambda_0}{3} - \lambda_1\frac{\lambda_2 - \lambda_2}{9} + \left(\frac{\lambda_2 - \lambda_2}{3}\right)^2\frac{2\lambda_2 + \lambda_2}{9}\right]x \tag{B.25}$$
$$\iota(x,0) = 0 \tag{B.26}$$

also has a unique C^∞ solution.

From invertibility and continuity of transformations $(B.17), (B.22)$ and exponential stability of system $(B.12)$–$(B.13)$, the following theorem holds.

Theorem B.1. *For any initial value $u(\cdot,0) \in \mathcal{H}$, there exists a unique (mild) solution*

$$u(\cdot,t) \in C\left([0,\infty);\mathcal{H}\right) \tag{B.27}$$

to the closed-loop system $(B.10)$–$(B.11)$ with the following controller:

$$U(t) = \int_0^L k(L,y)u(y,t)e^{\frac{\lambda_2-\lambda_2}{3}(y-L)}dy, \tag{B.28}$$

in which the kernel function $k(x,y)$ is determined from $(B.18)$–$(B.21)$ and $(B.14)$. And there exists positive constants M_{u1}, ρ_{u1} such that

$$\|u(\cdot,t)\| \leqslant M_{u1}e^{-\rho_{u1}t}\|u(\cdot,0)\|. \tag{B.29}$$

Moreover, if $u(\cdot,0)$ satisfies boundary compatibility condition, then

$$u(\cdot,t) \in C^1\left([0,\infty);\mathcal{H}\right) \tag{B.30}$$

is the classical solution.

Case 2

Consider the following critical case (with $q_1 = \infty, q_2 \neq \infty$) of LKdVAD:

$$u_t(x,t) = u_{xxx}(x,t) + \lambda_2 u_{xx}(x,t) + \lambda_1 u_x(x,t) + \lambda_0 u(x,t), x \in (0,L) \tag{B.31}$$

$$u(0,t) = 0,\ u_{xx}(0,t) = 0,\ u(L,t) = U(t). \tag{B.32}$$

We firstly choose the parameter

$$\lambda_2 > 0 \tag{B.33}$$

and denote

$$\lambda_1 = \frac{\lambda_2^2 - \lambda_2^2}{3} + \lambda_1. \tag{B.34}$$

Then, for any positive constant $a > 0$, choose the parameter

$$\lambda_0 < \min\left\{0,\ a\lambda_1 - \frac{|a+\lambda_1|}{4\lambda_2}\right\}. \tag{B.35}$$

Thus, the following target systems are exponentially stable in the state energy space $H^1(0,L)$:

$$w_t(x,t) = w_{xxx}(x,t) + \lambda_2 w_{xx}(x,t) + \lambda_1 w_x(x,t) + \lambda_0 w(x,t), x \in (0,L) \tag{B.36}$$

$$w(0,t) = 0,\ w_{xx}(0,t) = 0,\ w(L,t) = 0. \tag{B.37}$$

Remark B.4.

1. We can not get exponential stability of the target system in the state energy space $L^2(0,L)$ here, but only get the stability result in $H^1(0,L)$, which is in a weaker sense.
2. In the proof for exponential stability of the target system, several inequalities are used, such as Young's inequality, Poincare inequality and the following inequality:

$$\left| |w_x(L,t)|^2 - |w_x(0,t)|^2 \right| \leq 2\|w_x\|\|w_{xx}\|, \tag{B.38}$$

which comes from Agmon's inequality.

In order to obtain the state feedback controller, we use the proposed transformation $u \mapsto w$:

$$w(x,t) = u(x,t)e^{\frac{\lambda_2-\lambda_2}{3}x} - \int_0^x k(x,y)u(y,t)e^{\frac{\lambda_2-\lambda_2}{3}y}dy, \tag{B.39}$$

with the kernel function $k(x,y) \in \mathbb{R}$ satisfying:

$$k_{xxx}(x,y) + k_{yyy}(x,y) + \lambda_2\left(k_{xx}(x,y) - k_{yy}(x,y)\right) + \lambda_1\left(k_x(x,y) + k_y(x,y)\right)$$
$$= \left[\frac{\lambda_2-\lambda_2}{3}\left(\frac{(\lambda_2-\lambda_2)(2\lambda_2+\lambda_2)}{9} - \lambda_1\right) + \lambda_0 - \lambda_0\right]k(x,y) \tag{B.40}$$

$$k(x,x) = 2\frac{\lambda_2-\lambda_2}{3} \tag{B.41}$$

$$k_x(x,x) = \left[\frac{\lambda_0-\lambda_0}{3} + \lambda_1\frac{\lambda_2-\lambda_2}{9} - \left(\frac{\lambda_2-\lambda_2}{3}\right)^2\frac{2\lambda_2+\lambda_2}{9}\right]x$$
$$- 2\frac{(\lambda_2-\lambda_2)(2\lambda_2+\lambda_2)}{9} \tag{B.42}$$

$$k_y(x,0) = \frac{2\lambda_2+\lambda_2}{3}k(x,0). \tag{B.43}$$

By the method of successive approximation, the following lemma can be proved.

Lemma B.2. *System of equations* $(B.40)$–$(B.43)$ *has a unique* C^∞ *solution, and system of equations for kernel function* $l(x,y)$ *of the inverse transformation* $w \mapsto u$*:*

$$u(x,t) = w(x,t)e^{-\frac{\lambda_2-\lambda_2}{3}x} - \int_0^x l(x,y)w(y,t)dye^{-\frac{\lambda_2-\lambda_2}{3}x}, \tag{B.44}$$

with the kernel function satisfying:

$$\iota_{xxx}(x,y) + \iota_{yyy}(x,y) + \lambda_2\left(\iota_{xx}(x,y) - \iota_{yy}(x,y)\right) + \lambda_1\left(\iota_x(x,y) + \iota_y(x,y)\right)$$
$$= \left[\frac{\lambda_2-\lambda_2}{3}\left(\lambda_1 - \frac{(\lambda_2-\lambda_2)(2\lambda_2+\lambda_2)}{9}\right) - \lambda_0 + \lambda_0\right]\iota(x,y) \tag{B.45}$$
$$\iota(x,x) = 0 \tag{B.46}$$
$$\iota_x(x,x) = \left[\frac{\lambda_0-\lambda_0}{3} - \lambda_1\frac{\lambda_2-\lambda_2}{9} + \left(\frac{\lambda_2-\lambda_2}{3}\right)^2\frac{2\lambda_2+\lambda_2}{9}\right]x \tag{B.47}$$
$$\iota(x,0) = 0 \tag{B.48}$$

also has a unique C^∞ *solution.*

From continuity of transformations $(B.39)$ and its inverse $(B.44)$, and exponential stability of system $(B.36)$–$(B.37)$, the following lemma holds.

Theorem B.2. *For any initial data* $u(\cdot,0) \in L^2(0,L)$*, there exists a unique (mild) solution to the closed-loop system* $(B.31)$–$(B.32)$ *with* $\lambda_2 \geq 0$ *and the controller*

$$U(t) = \int_0^L k(L,y)u(y,t)dy, \tag{B.49}$$

in which the kernel function $k(x,y)$ *is determined from* $(B.40)$–$(B.43)$ *and* $(B.33)$–$(B.35)$*, such that*

$$u(\cdot,t) \in C\left([0,\infty);\mathcal{A}\right), \tag{B.50}$$

and there exists positive constant M_{c2}, ρ_{c2} *such that*

$$\|u(\cdot,t)\| \leqslant M_{c2}e^{-\rho_{c2}t}\|u(\cdot,0)\|. \tag{B.51}$$

Moreover, if $u(\cdot,0)$ *satisfies boundary compatibility condition, then*

$$u(\cdot,t) \in C^1\left([0,\infty);\mathcal{A}\right) \tag{B.52}$$

is the classical solution.

Appendix C

Parameter Values and OCP Functions in Chapter 7

C.1 Parameter values

In the simulations, we use parameters of a lithium-ion battery with negative electrode LiC_6 and positive electrode $LiCoO_2$. All parameter values are from [112] and the references within, and are listed in Table C.1.

Table C.1: Physical parameters.

Electrochemical Parameters	Negative Electrode	Positive Electrode
L [m]	96×10^{-6}	60×10^{-6}
$D_{\mathrm{s}}(T_{\mathrm{amb}})$ [m^2/s]	3.5×10^{-14}	3.0×10^{-14}
$c_{\mathrm{s}}^{\max}$ [mol/m^3]	28600	51000
R_{s} [m]	8×10^{-6}	5×10^{-6}
$r_{\mathrm{eff}}(T_{\mathrm{amb}})$ [A $\cdot$ m$^{2.5}$/mol$^{1.5}$]	3.85948×10^{-4}	3.85948×10^{-4}
$R_{\mathrm{f}}(T_{\mathrm{amb}})$ [$\Omega \cdot$ m^2]	0	3.5×10^{-3}
ϵ_{s} [-]	0.4	0.6
a_{s} [1/m]	1.5×10^5	3.6×10^5
α_a [-]	0.5	
α_c [-]	0.5	
$c_{\mathrm{e},0}$ [mol/m^3]	1000	
N [mol/m^2]	1.796	
Thermal Parameters	–	
E_{D_s} [K]	3000	3000
$E_{r_{\mathrm{eff}}}$ [K]	24000	24000
$E_{R_{\mathrm{f}}}$ [K]	8000	1800
T_{amb} [K]	298 (24.85 [°C])	
R_{c} [$\Omega \cdot$ m^2]	3×10^{-2}	
ρ^{avg} [kg/m^2]	0.4983	
c_{P} [J/(kg $\cdot$ K)]	1000	
h_{cell} [W/(m$^2 \cdot K$)]	2	
Physical Constants	–	
F [s $\cdot$ A/mol)]	96485	
R [J/(mol $\cdot$ K))]	8.314472	

C.2 OCP functions

The OCP functions $U^{\pm}$ used in the simulations are borrowed from [112], where the authors fitted the data from [113, 114, 115].

For the positive electrode $LiCoO_2$,

$$\begin{aligned} U^{+}(\theta^{+}, T_{\text{amb}}) =& 2.16216 + 0.07645 \tanh(30.834 - 54.4806\theta^{+}) \\ &+ 2.1581 \tanh(52.294 - 50.294\theta^{+}) \\ &- 0.14169 \tanh(11.0923 - 19.8543\theta^{+}) \\ &+ 0.2051 \tanh(1.4684 - 5.4888\theta^{+}) \\ &+ 0.2531 \tanh\left(\frac{-\theta^{+} + 0.56478}{0.1316}\right) \\ &- 0.02167 \tanh\left(\frac{\theta^{+} - 0.525}{0.006}\right), \end{aligned} \tag{C.1}$$

and

$$\begin{aligned} \frac{\partial U^{+}}{\partial T}(\theta^{+}, T_{\text{amb}}) =& \left[-0.19952 + 0.928370\theta^{+} - 1.36455\left(\theta^{+}\right)^{2} + 0.61154\left(\theta^{+}\right)^{3}\right] \\ & / \left[1 - 5.661480\theta^{+} + 11.47636\left(\theta^{+}\right)^{2} - 9.82431\left(\theta^{+}\right)^{3} + 3.04876\left(\theta^{+}\right)^{4}\right]. \end{aligned} \tag{C.2}$$

For the negative electrode LiC_6 ,

$$\begin{aligned} U^{-}(\theta^{-}, T_{\text{amb}}) =& 0.194 + 1.5e^{-120.0\theta^{-}} + 0.0351 \tanh\left(\frac{\theta^{-} - 0.286}{0.083}\right) \\ &- 0.0045 \tanh\left(\frac{\theta^{-} - 0.849}{0.119}\right) - 0.035 \tanh\left(\frac{\theta^{-} - 0.9233}{0.05}\right) \\ &- 0.0147 \tanh\left(\frac{\theta^{-} - 0.5}{0.034}\right) - 0.102 \tanh\left(\frac{\theta^{-} - 0.194}{0.142}\right) \\ &- 0.022 \tanh\left(\frac{\theta^{-} - 0.9}{0.0164}\right) - 0.011 \tanh\left(\frac{\theta^{-} - 0.124}{0.0226}\right) \\ &+ 0.0155 \tanh\left(\frac{\theta^{-} - 0.105}{0.029}\right), \end{aligned} \tag{C.3}$$

and

$$\frac{\partial U^-}{\partial T}(\theta^-, T_{\text{amb}}) = \Big[0.00527 + 3.29927\theta^- - 91.79326\left(\theta^-\right)^2 + 1004.91101\left(\theta^-\right)^3$$
$$-5812.27813\left(\theta^-\right)^4 + 19329.75490\left(\theta^-\right)^5 - 37147.89470\left(\theta^-\right)^6$$
$$+38379.18127\left(\theta^-\right)^7 - 16515.05308\left(\theta^-\right)^8\Big]$$
$$/\Big[1 - 48.09287\theta^- + 1017.23480\left(\theta^-\right)^2 - 10481.80419\left(\theta^-\right)^3$$
$$+59431.30001\left(\theta^-\right)^4 - 195881.64880\left(\theta^-\right)^5 + 374577.31520\left(\theta^-\right)^6$$
$$-385821.16070\left(\theta^-\right)^7 + 165705.85970\left(\theta^-\right)^8\Big]. \tag{C.4}$$

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