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NON-COHERENCE OF ARITHMETIC HYPERBOLIC LATTICES

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ABSTRACT. We prove, under the assumption of the virtual fibration conjecture for arithmetic hyperbolic 3-manifolds, that all arithmetic lattices in $O(n, 1)$, $n \geq 4$, $n \neq 7$, are non-coherent. We also establish noncoherence of uniform arithmetic lattices of the simplest type in $SU(n, 1)$, $n \geq 2$, and of uniform lattices in $SU(2, 1)$ which have infinite abelianization.

1. INTRODUCTION

Recall that a group Γ is called *coherent* if every finitely-generated subgroup of Γ is finitely-presented. This paper is motivated by the following

Conjecture 1.1. *Let G be a semisimple Lie group which is not locally isomorphic to $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$. Then every lattice in G is noncoherent.*

In the case of lattices in $O(n, 1)$, this conjecture is due to Dani Wise. Conjecture 1.1 is true for all lattices containing direct product of two nonabelian free groups since the latter are incoherent. Therefore, it holds, for instance, for $SL(n, \mathbb{Z})$, $n \geq 4$. The case $n = 3$, to the best of my knowledge, is unknown (this problem is due to Serre, see the list of problems [47]).

Conjecture 1.1 is out of reach for non-arithmetic lattices in $O(n, 1)$ and $SU(n, 1)$, since we do not understand the structure of such lattices. However, all known constructions of nonarithmetic lattices lead to noncoherent groups: See [28] for the case of Gromov–Piatetsky-Shapiro construction; the same argument proves noncoherence of nonarithmetic reflection lattices (see e.g. [45]) and non-arithmetic lattices obtained via Agol’s [1] construction. In the case of lattices in $PU(n, 1)$, all known nonarithmetic groups are commensurable to the ones obtained via Deligne–Mostow construction [11]. Such lattices contain fundamental groups of complex-hyperbolic surfaces which fiber over hyperbolic Riemann surfaces. Noncoherence of such groups is proven in [25], see also section 9.

In this paper we will discuss the case of arithmetic subgroups of rank 1 Lie groups. Conjecture 1.1 was proven in [28] for non-uniform arithmetic lattices in $O(n, 1)$, $n \geq 6$ (namely, it was proven that the noncoherent examples from [27] embed in such lattices). The proof of Conjecture 1.1 in the case of all arithmetic lattices of the *simplest type* appears as a combination of [28] and [2]. In particular, it covers the case of all non-uniform arithmetic lattices ($n \geq 4$) and all arithmetic lattices in $O(n, 1)$ for n even, since they are of the simplest type. For odd $n \neq 3, 7$, there are also arithmetic lattices in $O(n, 1)$ of “quaternionic origin” (see Section 7 for the detailed definition), while for $n = 7$ there is one more family of arithmetic groups associated

with octonions. One of the keys to the proof of noncoherence above is virtual fibration theorem for various classes of hyperbolic 3-manifolds. Our main result (Theorem 1.3) will be conditional to the existence of such fibrations:

Assumption 1.2. *We will assume that every arithmetic hyperbolic 3-manifold M of “quaternionic origin” admits a virtual fibration, i.e., M has a finite cover which fibers over the circle.*

We discuss in Section 3 what is currently known about the existence of virtual fibrations on such arithmetic manifolds. In short, assuming that a recent paper on subgroup separability by Dani Wise is correct, they all do.

Our main result is

Theorem 1.3. *Under the assumption 1.2, Conjecture 1.1 holds for all arithmetic lattices of quaternionic type.*

In section 9 we will also provide some partial corroboration to Conjecture 1.1 for arithmetic subgroups of $SU(n, 1)$. More precisely, we will prove it for uniform arithmetic lattices of the simplest type (also called *type 1 arithmetic lattices*) in $SU(n, 1)$ and for all uniform lattices (arithmetic or not) in $SU(2, 1)$ with (virtually) positive first Betti number. We will also prove the conjecture for all lattices in the isometry groups of the quaternionic–hyperbolic spaces \mathbf{HH}^n and the octonionic–hyperbolic plane \mathbf{OH}^2 . These groups are noncoherent since they contain arithmetic lattices from $O(4, 1)$ or $O(8, 1)$.

Historical Remarks. The fact that the group $F_2 \times F_2$ is non-coherent was known for a very long time (at least since [16]). Moreover, it was proven by Baumslag and Roseblade [4] that “most” finitely generated subgroups of $F_2 \times F_2$ are not finitely-presented. It therefore follows that many higher rank lattices (e.g., $SL(n, \mathbb{Z})$, $n \geq 4$) are non-coherent. It was proven by P. Scott [41] that finitely generated 3-manifold groups are all finitely-presented. In particular, lattices in $SL(2, \mathbb{C})$ are coherent. The first examples of incoherent geometrically finite groups in $SO(4, 1)$ were constructed by the author and L. Potyagailo [27, 38, 39]. These examples were generalized by B. Bowditch and G. Mess [8] who constructed incoherent uniform arithmetic lattices in $SO(4, 1)$. (For instance, the reflection group in the faces of right-angled 120-cell in \mathbb{H}^4 is one of such lattices.) Their examples, of course, embed in all other rank 1 Lie groups (except for $SO(n, 1)$, $SU(n, 1)$, $n = 1, 2, 3$). Noncoherent arithmetic lattices in $SU(2, 1)$ (and, hence, $SU(3, 1)$) were constructed in [25]. All these constructions were ultimately based on either existence of hyperbolic 3-manifolds fibering over the circle (in the case of discrete subgroups in $SO(n, 1)$) or existence of complex-hyperbolic surfaces which admit singular holomorphic fibrations over hyperbolic complex curves. A totally new source of noncoherent geometrically finite groups comes from the recent work of D. Wise [51]: He proved that fundamental groups of many (in some sense, most) polygons of finite groups are noncoherent. On the other hand, according to [26], the fundamental group of every even-sided (with at least 6 sides) hyperbolic polygons of finite groups embeds as a discrete convex-cocompact subgroup in some $O(n, 1)$.

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with me the results [52], to Jonathan Hillmann for providing reference to Theorem 4.1, to Ben McReynolds and Andrei Rapinchuk for help with classification of arithmetic subgroups of $Isom(\mathbf{H}\mathbb{H}^n)$ and $Isom(\mathbf{O}\mathbb{H}^2)$ and to John Millson for providing me with references. I am also grateful to Leonid Potyagailo and Ernest Vinberg as this paper has originated in our joint work [28] on non-coherence of non-uniform arithmetic lattices.

2. RELATIVE SEPARABILITY FOR COLLECTIONS OF SUBGROUPS

Recall that a subgroup H in a group G is called *separable* if for every $g \in G \setminus H$ there exists a finite index subgroup $G' \subset G$ containing H but not g . For instance, if $H = \{1\}$ then separability of H amounts to residual finiteness of G . A group G is called LERF if every finitely-generated subgroup of G is separable. According to [42], LERF property is stable under group commensurability. Examples of LERF groups include: Free groups (M. Hall [20]), surface groups (P. Scott [42]), certain hyperbolic 3-manifold groups (R. Gitik [13], D. Wise [50]), groups commensurable to right-angled hyperbolic Coxeter groups (P. Scott [42], F. Haglund [17]), non-uniform arithmetical lattices of small dimension (I. Agol, D. Long and A. Reid [3] and M. Kapovich, L. Potyagailo, E. B. Vinberg [28]), and other classes of groups (F. Haglund and J. Świątkowski [18], D. Wise [49]). Note that in a number of these results one has to replace finite generation of subgroups with geometric finiteness/quasiconvexity. On the other hand, there are 3-dimensional graph-manifolds whose fundamental groups are not LERF, see [33]. For the purposes of this paper, we will need LERF property only for the surface groups.

Remark 2.1. Dani Wise has a recent preprint [52] which among other things proves that if M is a closed 3-dimensional hyperbolic manifold containing an incompressible surface whose fundamental group is quasi-fuchsian, then $\pi_1(M)$ is LERF. His proof is rather long and intricate; whether or not it is correct is unclear at this point. For the purposes of our paper, we note the following. Suppose that M is a closed 3-manifold, whose fundamental group is of quaternionic origin in the sense of section 7. Then, by applying [31] or [34], see also [30] and the last paragraph of the introduction to [31], one concludes that M admits a finite cover M' so that $b_1(M') \geq 2$ (actually, the first Betti number of a finite cover could be made arbitrarily large). It then follows from Thurston's results on Thurston's norm [44], that M' contains a nonseparating incompressible surface Σ which is not a fiber in a fibration over the circle. Therefore, $\pi_1(\Sigma) \subset \pi_1(M') \subset SL(2, \mathbb{C})$ is a quasi-fuchsian subgroup. Hence, Wise's theorem applies to this class of arithmetic hyperbolic 3-manifolds.

In this paper we will be using a *relative version* of subgroup separability, which deals with finite collections of subgroups of G . It is both stronger than subgroup separability (since it deals with collections of subgroups) and weaker than subgroup separability (since it does not require as much as separability in the case of a single subgroup). Actually, we will need this concept only in the case of pairs of subgroups, but we included the more general discussion for the sake of completeness.

Let G be a group, $H_1, H_2 \subset G$ be subgroups. We say that a double coset H_1gH_2 is *trivial* if it equals the double coset $H_1 \cdot 1 \cdot H_2 = H_1 \cdot H_2$. In other words, $g \in H_1 \cup H_2$.

Given a finitely-generated group G , we let Γ_G denote its Cayley graph (here we are abusing the notation by suppressing the choice of a finite generating set which will be irrelevant for our purposes). Recall also that a *geometric action* of a group G on a metric space X is an isometric properly discontinuous cocompact action.

Definition 2.2. Let G be a group, $\mathcal{H} = \{H_1, H_2, \dots, H_M\}$ be a collection of subgroups of G . We will say that \mathcal{H} is *relatively separable* in G if for every finite collection of nontrivial double cosets $H_i g_k H_j, i, j \in \{1, \dots, M\}, k = 1, \dots, K$, there exists a finite index subgroup $G' \subset G$ which is disjoint from the above double cosets.

In the case when $G, H_i, i = 1, \dots, M$, are finitely generated, separability can be reformulated as follows:

Given a number R , there exists a finite index subgroup $G' \subset G$ so that for each $g \in G'$ either $g \in H_i \cdot H_j$ for some i, j (and, hence, $d(\Gamma_{H_i}, g\Gamma_{H_j}) = d(\Gamma_{H_i}, \Gamma_{H_j})$) or

$$d(g\Gamma_{H_i}, \Gamma_{H_j}) \geq R$$

for all $i, j \in \{1, \dots, M\}$.

Equivalently, in the above property one can replace Γ_G with a space X on which G acts geometrically and Γ_{H_i} 's with H_i -invariant subsets $X_i \subset X$ with compact quotients $X_i/H_i, i = 1, \dots, M$.

In what follows we will use the following notation

Notation 2.3. Given a metric space X , a subset $Y \subset X$ and a real number $R \geq 0$, let $B_R(Y) := \{x \in X : d(x, Y) \leq R\}$, i.e., the R -neighborhood of Y . For instance for $x \in X$, $B_R(x)$ is the closed R -ball in X centered at x . We let $proj_Y : X \rightarrow Y$ denote the nearest-point projection of X to Y .

Below are several useful examples illustrating relative separability.

Example 2.4. Suppose that $M = 1$. Then \mathcal{H} is relatively separable provided that $H = H_1$ is separable in G .

Proof. Suppose that H is separable in G . Given R , there are only finitely many distinct nontrivial double cosets

$$Hg_k H, k = 1, \dots, K,$$

so that

$$d(g\Gamma_H, \Gamma_H) < R$$

for $g \in Hg_k H$. Let $G' \subset G$ be a subgroup containing H but not g_1, \dots, g_K . Then G' is disjoint from

$$\bigcup_{k=1}^K Hg_k H$$

and the claim follows. \square

Although the converse to the above example is, probably, false, relative separability suffices for typical applications of subgroup separability. For instance, suppose that G is the fundamental group of a 3-manifold M and $\mathcal{H} = \{H\}$ is a relatively separable surface subgroup of G . Then a finite cover of M contains an incompressible surface (whose fundamental group is a finite index subgroup in H).

Example 2.5. Let G be an arithmetic lattice of the simplest type in $O(n, 1)$ and $H_1, \dots, H_M \subset G$ be the stabilizers of distinct “rational” hyperplanes L_1, \dots, L_M in \mathbb{H}^n , i.e., L_i/H_i has finite volume, $i = 1, \dots, M$. Then $\mathcal{H} = \{H_1, \dots, H_M\}$ is relatively separable in G . See [28].

Example 2.6. As a special case of the second example, suppose that G is a surface group and H_1, \dots, H_M are cyclic subgroups. Then $\mathcal{H} = \{H_1, \dots, H_M\}$ is relatively separable in G .

Recall that finitely generated subgroups of surface groups are separable. Therefore, one can generalize the last example as follows:

Proposition 2.7. *Suppose that G is a word-hyperbolic group which is separable with respect to its quasiconvex subgroups. Let H_1, \dots, H_M be residually finite quasiconvex subgroups with finite pairwise intersections. Then $\mathcal{H} = \{H_1, \dots, H_M\}$ is relatively separable in G .*

Proof. Let $H \subset G$ be a subgroup generated by sufficiently deep finite index torsion-free subgroups $H'_i \subset H_i$ ($i = 1, \dots, M$). Then, according to [14], H is isomorphic to the free product $H'_1 * H'_2 * \dots * H'_M$ and is quasiconvex. Let X denote the Cayley graph of G and $X_i, i = 1, \dots, M$ the Cayley graphs of H_1, \dots, H_M . Since the groups H_i are quasiconvex and have finite intersections, for every $R < \infty$, there exists r so that for $i \neq j$, the projection of $B_R(X_i)$ to X_j is contained in $B_r(1)$.

Let Y_j denote the preimage in X of $B_r(1)$ under the nearest-point projection $X \rightarrow X_j$. Thus, $B_R(X_j) \subset Y_j, i \neq j$. Moreover, if we choose r large enough then

$$Y_j^c \cap Y_i^c = \emptyset, \quad \forall i \neq j,$$

where Y_j^c denotes the complement of Y_j in X .

Since the group H_j is residually finite, there exists a finite index subgroup $H'_j \subset H_j$ so that Y_j is a *sub-fundamental domain* for the action $H'_j \curvearrowright X$, i.e.,

$$h(Y_j) \cap Y_j = \emptyset, \forall h \in H'_j \setminus \{1\}.$$

Therefore, one can apply the ping-pong arguments to the collection of sub-fundamental domains Y_1, \dots, Y_M as follows:

Every nontrivial element h is the product

$$h_{i_1} \circ h_{i_2} \circ \dots \circ h_{i_m}$$

where $h_{i_k} \in H_{i_k} \setminus \{1\}$ and $i_k \neq i_{k+1}$ for each $k = 1, \dots, m-1$. Then, arguing inductively on m , we see that for each j ,

$$h(Y_j) \subset Y_l^c,$$

where $l = i_1$.

We now claim that $d(X_i, h(X_j)) \geq R$ provided that $h \notin H_i \cdot H_j$. We write down h in the normal form as above with $l = i_1$.

1. Suppose first that $i \neq l$. Then $B_R(X_i) \subset Y_l$. On the other hand, by taking any $m \neq j$, we get

$$h(X_j) \subset h(Y_m) \subset Y_l^c.$$

Thus $B_R(X_i) \subset X_l$ has empty intersection with $h(X_j) \subset Y_l^c$ and the claim follows.

2. Suppose now that $i = l = i_1$. Then $s = i_2 \neq i$; set

$$g := h_{i_2} \circ \dots \circ h_{i_m}.$$

Then

$$d(X_i, h(X_j)) = d(X_i, h_i g(X_j)) = d(X_i, g(X_j)).$$

Now, by appealing to Case 1, we get

$$d(X_i, h(X_j)) = d(X_i, g(X_j)) \geq R.$$

We hence conclude that \mathcal{H} is relatively separable in the subgroup generated by H_1, \dots, H_M .

There are only finitely many nontrivial double coset classes $H_i g_k H_j$, $k = 1, \dots, K$, in $H_i \backslash G / H_j$ so that for the elements $g \in H_i g_k H_j \subset G$, we have

$$d(X_i, g(X_j)) < R.$$

Note that $g_k \notin H$ unless $g_k \in H_i \cdot H_j$ (in which case the corresponding double coset would be trivial). Since H is quasiconvex in G and does not contain g_k , $k = 1, \dots, K$, by the subgroup separability of G , there exists a finite index subgroup $G' \subset G$ containing H , so that $g_1, \dots, g_K \notin G'$. Therefore, G' has empty intersection with each of the double cosets $H_i g_k H_j$, $k = 1, \dots, K$ (for all $i, j \in \{1, \dots, M\}$). It therefore follows that for every $g \in G' \setminus H_i \cdot H_j$,

$$d(X_i, g(X_j)) \geq R.$$

Hence \mathcal{H} is relatively separable in G . □

In order to apply this proposition in section 8, we need the following simple

Lemma 2.8. *Let $N \triangleleft G$ be a normal finitely-generated subgroup and \mathcal{H} be a finite set of subgroups in N which is relatively separable in N . Assume that G/N is residually finite. Then \mathcal{H} is relatively separable in G .*

Proof. Let $H_i g_k H_j$, $g_k \in A \subset G$, be the double cosets that we want to avoid. We will construct the appropriate finite-index subgroup in G in two steps. First, consider those double cosets $H_i g_k H_j$, $g_k \in A_1$, which are contained in N . By separability of \mathcal{H} in N , there exists a finite-index subgroup $N' \subset N$ which is disjoint from these double cosets. This subgroup may not be normal in G , but it is standard that there exists a finite-index subgroup $G_1 \subset G$ containing N_1 (the subgroup G_1 is the normalizer of N_1 in G). Hence, G_1 is disjoint from the first set of double cosets. Now, consider double cosets $H_i g_k H_j$, $g_k \in A_2$, so that $g_k \notin N$. Then, since G/N is residually finite, there exists a finite-index subgroup $G_2 \subset G$ so that $g_k \notin G_2$, $g_k \in A_2$, but $N \subset G_2$. Clearly, G_2 is disjoint from the double cosets $H_i g_k H_j$, $g_k \in A_2$. Now, taking $G' = G_1 \cap G_2$ we obtain a finite-index subgroup in G which is disjoint from the double cosets $H_i g_k H_j$, $g_k \in A_1 \cup A_2$. □

3. VIRTUAL FIBRATION CONJECTURE

W. Thurston conjectured that every closed hyperbolic 3-manifold M is *virtually fibered*, i.e., it admits a finite cover M' which is fibered over the circle. This conjecture, known as *Thurston's Virtual Fibration Conjecture* was proven by I. Agol in [2] under the assumption that $\pi_1(M)$ is virtually RFRS. Moreover, under this assumption, M admits infinitely many non-isotopic virtual fibrations.

The latter condition holds provided that $\pi_1(M)$ contains a finite-index subgroup which embeds in a right-angled Coxeter (or Artin) group. This allowed Agol to prove [2] the virtual fibration conjecture for all arithmetic manifolds of the simplest type: Immersed closed totally-geodesic surfaces provide a way to virtually embed $\pi_1(M)$ in a right-angled Artin group using the results of Haglund and Wise [19]. Paper [52] of Dani Wise claims virtual RFRS condition for fundamental groups of all closed hyperbolic 3-manifolds containing an incompressible surface with quasi-fuchsian fundamental group. Recall that every closed arithmetic 3-manifold of quaternionic origin admits a finite cover containing such an incompressible surface, see Remark 2.1. Thus, assuming Dani Wise's results [52], if M is a closed arithmetic 3-manifold of quaternionic origin, then Virtual Fibration Conjecture holds for M .

We also note that if M is arithmetic, then existence of one virtual fibration implies existence of infinitely many virtual fibrations since the commensurator $Comm(\Gamma)$ of $\Gamma = \pi_1(M)$ is dense in $PSL(2, \mathbb{C})$: If $F \triangleleft \Gamma$ is a normal surface subgroup, there exists $\alpha \in Comm(\Gamma)$ so that $F' := \alpha F \alpha^{-1}$ is not commensurable to F . Therefore, $F' \cap \Gamma$ is a surface subgroup corresponding to a different fibration of the manifold $M' = \mathbb{H}^3/\Gamma'$, where $\Gamma' := \Gamma \cap \alpha \Gamma \alpha^{-1}$.

4. NORMAL SUBGROUPS OF POINCARÉ DUALITY GROUPS

Recall that a group Γ is called an n -dimensional *Poincaré Duality group* (over \mathbb{Z}), abbreviated $PD(n)$ group, if there exists $z \in H_n(\Gamma, D)$, so that

$$\cap z : H^i(\Gamma, M) \rightarrow H_{n-i}(\Gamma, \overline{M})$$

is an isomorphism for $i = 0, \dots, n$ and every $\mathbb{Z}\Gamma$ -module M . Here $\overline{M} = D \otimes M$, where $D \cong H^n(\Gamma, \mathbb{Z}\Gamma)$ is the *dualizing module*. For instance, if X is a closed n -manifold so that $X = K(\Gamma, 1)$, then Γ is a $PD(n)$ group. The converse holds for $n = 2$, while for $n \geq 3$ the converse is an important open problem (for groups which admit finite $K(\Gamma, 1)$).

Recall also that a group Γ is called FP_r (over \mathbb{Z}) if there exists a partial resolution

$$P_r \rightarrow P_{r-1} \rightarrow \dots \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$$

of finitely-generated projective $\mathbb{Z}\Gamma$ -modules. For instance, Γ is FP_1 iff it is finitely-generated, while every finitely-presented group is FP_2 (the converse is false, see [5]). We refer the reader to [6] for a comprehensive discussion of $PD(n)$ and FP_r groups. We will need the following theorem in the case $n = 4$, $r = 2$:

Theorem 4.1. (*J. Hillman [23, Theorem 1.19], see also [24].*) *Let π be a $PD(n)$ -group with an FP_r normal subgroup K such that $G = \pi/K$ is a $PD(n-r)$ group and $2r \geq n - 1$. Then K is a $PD(r)$ -group.*

5. BISECTORS

Given $p \neq q \in \mathbb{H}^n$ the *bisector* $Bis(p, q)$ is the set of points in \mathbb{H}^n equidistant from p and q . Define the closed half-spaces $Bis(p, q)^\pm$ bounded by $Bis(p, q)$ by requiring $Bis(p, q)^+$ to contain q and $Bis(p, q)^-$ to contain p .

Consider a configuration C of 3-dimensional subspaces H_1, H_2, H_3 in \mathbb{H}^n and geodesics $\gamma_1, \dots, \gamma_4$, so that:

1. $\gamma_i \subset H_i, i = 1, 2, 3, \gamma_4 \subset H_3$.
2. $H_1 \cap H_2 = \gamma_2, H_2 \cap H_3 = \gamma_3$ and the angles at these intersections are $\geq \alpha > 0$.
3. γ_i, γ_{i+1} ($i = 1, 2, 3$) are at least distance $R_0 > 0$ apart.

The following is elementary:

Lemma 5.1. *Given α, r and R_0 , there exists R_* so that if $d(\gamma_2, \gamma_3) \geq R_*$ then $d(H_1, H_3) \geq r$.*

Therefore, from now on, we will also fix a number $r > 0$ so that the configuration C satisfies:

4. H_1, H_3 are at least distance $r > 0$ apart.

Let $p_i, q_i \in \gamma_i$ denote the points closest to $\gamma_{i-1}, \gamma_{i+1}$ respectively. We let m_i denote the midpoint of $\overline{q_i p_{i+1}}, i = 1, 3$. Let n_i denote the midpoint of $\overline{p_i q_i}, i = 2, 3$. Our assumptions imply that $d(q_i, p_{i+1}) \geq R_0, i = 1, \dots, 3$.

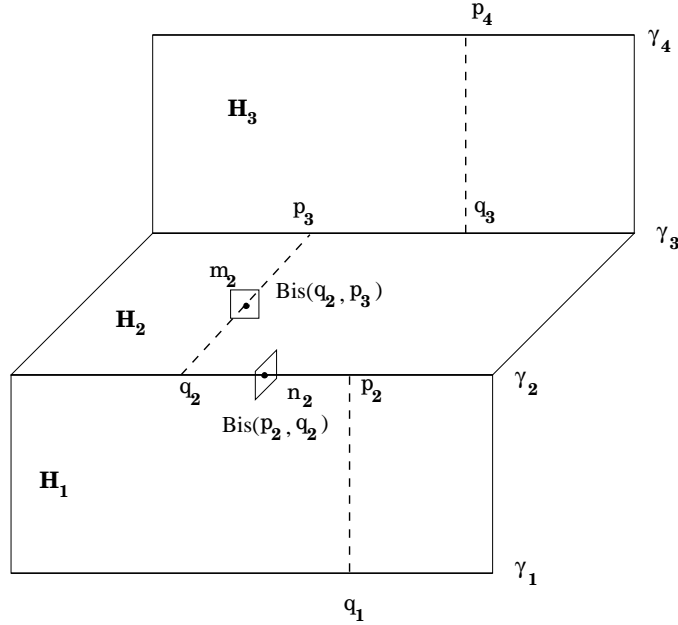


FIGURE 1. Configuration of planes and lines.

See Figure 1.

Lemma 5.2. *There exist $R_1 = R_1(\alpha, r, R_0), \rho_1 = \rho_1(\alpha, r, R_0) > 0$ so that if $d(q_2, p_2) \geq R_1$ then:*

1. *The bisectors $Bis(p_2, q_2)$ and $Bis(q_2, p_3)$ are within distance ρ_1 .*

2.

$$\gamma_1 \subset \text{Bis}(p_2, q_2)^-, H_3 \subset \text{Bis}(p_2, q_2)^+.$$

3. $d(\gamma_1, \text{Bis}(p_2, q_2)) \geq \rho_1$ and $d(H_3, \text{Bis}(p_2, q_2)) \geq \rho_1$. See Figure 1.

Proof. Observe that as $d = d(q_2, p_2)$ goes to infinity, the bisector $\text{Bis}(p_2, q_2)$ converges to an ideal point of the geodesic γ_2 . Since both $\gamma_2, \text{Bis}(q_2, p_3)$ are orthogonal to $\overline{q_2 p_3}$ and $d(p_3, q_2) \geq R_0 > 0$, it follows that $\text{Bis}(p_2, q_2), \text{Bis}(q_2, p_3)$ are disjoint for large d . Similarly, since γ_2 is within positive distance from H_3 , it follows that $H_3 \subset \text{Bis}(p_2, q_2)^+$ for large d . Similarly, $\gamma_1 \subset \text{Bis}(p_2, q_2)^-$ for large d (since $d(q_1, p_2) \geq R_0 > 0$). Alternatively, one compute R_1, ρ_1 directly using hyperbolic trigonometry. \square

The next two lemmas are proven by the same arguments as Lemma 5.2.

Lemma 5.3. *Assume that $d(q_2, p_3) = R_0$. Then there exist $R_2 = R_2(\alpha, r, R_0), \rho_2 = \rho_2(\alpha, r, R_0) > 0$ so that if $d(p_2, q_2) \geq R_1, d(p_3, q_3) < R_1$ (where R_1 is as in Lemma 5.2), and $d(q_3, p_4) \geq R_2$, then:*

1. *The bisectors $\text{Bis}(p_2, q_2)$ and $\text{Bis}(q_3, p_4)$ are within positive distance.*2. $H_1 \cup H_2 \subset \text{Bis}(q_3, p_4)^-$.3. $d(H_1 \cup H_2, \text{Bis}(q_3, p_4)) \geq \rho_2$.

Lemma 5.4. *Assume again that $d(q_2, p_3) = R_0$ and, in addition, $d(p_2, q_2) < R_1, d(p_3, q_3) < R_1$. Then there exist $R_3 = R_3(\alpha, r, R_0), \rho_3 = \rho_3(\alpha, r, R_0) > 0$ so that if*

$$d(q_1, p_2) \geq R_3, \quad d(q_3, p_4) \geq R_4,$$

then:

1. *The bisectors $\text{Bis}(q_1, p_2)$ and $\text{Bis}(q_3, p_4)$ are within distance.*2. $H_2 \cup H_3 \subset \text{Bis}(q_1, p_2)^+$ and $H_2 \cup H_1 \subset \text{Bis}(q_3, p_4)^-$.3. $d(H_2 \cup H_3, \text{Bis}(q_1, p_2)) \geq \rho_3, d(H_2 \cup H_1, \text{Bis}(q_3, p_4)) \geq \rho_3$.

We now set $R_4 := \max(R_2, R_3)$ and $\rho_4 := \min(\rho_1, \rho_2, \rho_3)$. Thus, $R_4 = R_4(\alpha, r, R_0)$ and $\rho_4 := \rho_4(\alpha, r, R_0)$.

We will say that H_i is *large* (relative to the configuration C) if $d(q_i, p_{i+1}) \geq R_4$ and γ_i is *large* if $d(p_i, q_i) \geq R_1$. If H_i or γ_i is not large, we will call it *small*.

6. A COMBINATION THEOREM FOR QUADRILATERALS OF GROUPS

Combination theorems in theory of Kleinian groups provide a tool for proving that a subgroup of $O(n, 1)$ generated by certain discrete subgroups is also discrete and, moreover, has prescribed algebraic structure. The earliest example of such theorem is ‘‘Schottky construction’’ (actually, due to F. Klein) producing free discrete subgroups of $O(n, 1)$. This was generalized by B. Maskit in the form of Klein-Maskit combination theorems where one constructs amalgamated free products and HNN extensions acting properly discontinuously on \mathbb{H}^n . More generally, the same line of arguments applies to graphs of groups. Complexes of groups are higher-dimensional generalizations of graphs of groups. A combination theorem for polygons of finite groups was proven in [26]. The goal of this section is to prove a combination theorem for certain quadrilaterals of infinite groups.

Let G_1 be a discrete subgroup in $Isom(L)$, $L \cong \mathbb{H}^3$. Pick two nonconjugate maximal cyclic subgroups $G_{\epsilon_1}, G_{\epsilon_2}$ in G_1 . For $i = 1, 2$, let $\gamma_i = L(\epsilon_i) \subset L$ denote the invariant geodesics of G_{ϵ_i} .

We will assume that:

Assumption 6.1. 1. The distance between these geodesics is $R_0 > 0$.

2. There exists $R > R_0$ such that for all distinct geodesics β, γ in the G_1 -orbits of γ_1, γ_2 , the distance $d(\beta, \gamma)$ is at least R unless there exists $g \in G_1$ which carries $\beta \cup \gamma$ to $\gamma_1 \cup \gamma_2$.

3. There exists $R_1 > 0$ so that for each $\gamma = \gamma_i, i = 1, 2$ and geodesics $\beta_1, \beta_2 \in G_0 \cdot (\gamma_1 \cup \gamma_2)$ so that $d(\beta_j, \gamma) = R_0$ ($j = 1, 2$), it follows that

$$d(\text{proj}_\gamma(\beta_1), \text{proj}_\gamma(\beta_2)) \geq R_1.$$

We will specify the choices of R and R_1 later on.

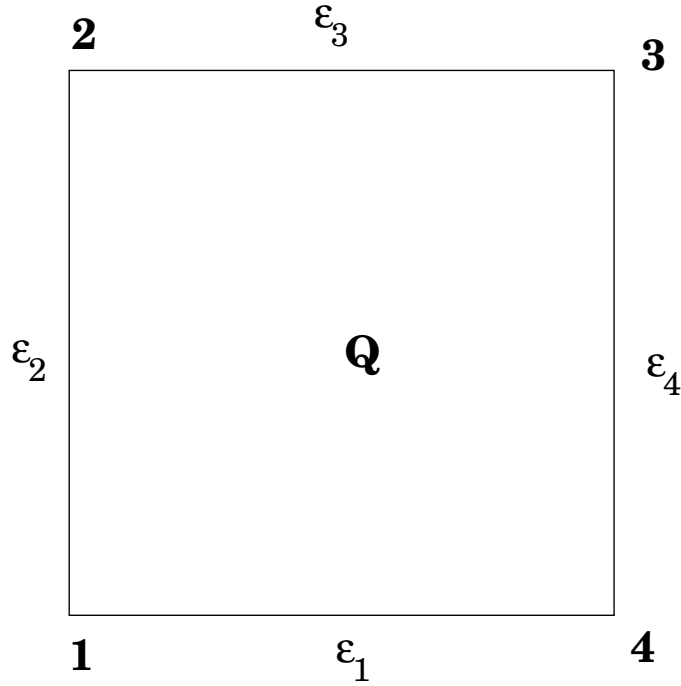


FIGURE 2. Square of groups.

We then define a quadrilateral Q of groups where the vertex groups G_1, \dots, G_4 are copies of G_1 and the edge groups $G_{\epsilon_1}, \dots, G_{\epsilon_4}$ are copies of $G_{\epsilon_1}, G_{\epsilon_2}$. In particular, Q has trivial face-group. We refer the reader to [9] for the precise definitions of complexes of groups and their fundamental groups, we note here only that for each vertex v of Q incident to an edge e , the structure of a quadrilateral of groups prescribes an embedding $G_e \hookrightarrow G_v$. The fundamental group $G = \pi_1(Q)$ of this quadrilateral of groups is the direct limit of the diagram of groups and homomorphisms given by Q .

More specifically, we require that Q admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action, generated by two involutions σ_1, σ_2 , so that

$$\sigma_2(G_1) = G_2, \sigma_1(G_1) = G_4,$$

and σ_i fixes G_{ϵ_i} , $i = 1, 2$. The isomorphisms

$$G_1 \rightarrow G_i, \quad i = 2, 3, 4$$

are induced by $\sigma_1, \sigma_1\sigma_2, \sigma_2$ respectively.

Let X denote the universal cover of the complex of groups Q . Then X is a square complex where the links of vertices are bipartite graphs. Since $G_{\epsilon_1} \cap G_{\epsilon_2} = \{1\}$, it follows that the links of X contain no bigons. Thus, X is a CAT(0) square complex. Let X^1 denote 1-skeleton of the complex X ; we metrize X^1 by declaring every edge to have unit lengths. Recall that for $k > 0$ a path η in X^1 is a k -local geodesic if every sub-path of length k in η is a geodesic in X^1 .

Note that if we choose edge groups which generate a free subgroup of G , then the links of the vertices of X are trees. Define the group $\tilde{G} = \langle G, \sigma_1, \sigma_2 \rangle$ generated by G, σ_1, σ_2 . Then \tilde{G} is a finite extension of G . The group \tilde{G} acts on X with exactly two orbit types of edges. For every edge e we define $Type(e) = \epsilon_i$ if $g(e)$ projects (under the map $X \rightarrow Q$) to ϵ_i for some $g \in \tilde{G}$.

We next describe a construction of representations $\phi : G \rightarrow O(5, 1)$ which are discrete and faithful provided that R is sufficiently large and R_1 is chosen appropriately. Let S_i denote 3-dimensional subspaces in \mathbb{H}^5 which intersect L along γ_i at the angles $\alpha_i \geq \alpha > 0$, $i = 1, 2$. We assume that $S_1 \cap S_2$ is a geodesic which is within distance $\geq \frac{r}{2} > 0$ from L and that S_1 is orthogonal to S_2 . Let $\sigma_i, i = 1, 2$ denote commuting isometric involutions in \mathbb{H}^5 with the fixed-point sets $S_i, i = 1, 2$ respectively.

Let $L_1 := L, L_2 := \sigma_2(L_1), L_3 := \sigma_1\sigma_2(L_1), L_4 := \sigma_1(L_1)$. Then

$$(1) \quad d(L_1, L_3) = d(L_2, L_4) \geq r.$$

We have the (identity) discrete embedding $\phi_1 : G_1 \rightarrow Isom(L_1) \subset Isom(\mathbb{H}^5)$. We will assume that $\phi_1(G_{\epsilon_i})$ stabilizes $\gamma_i, i = 1, 2$.

Given this data, we define a representation $\phi : G \rightarrow Isom(\mathbb{H}^5)$ so that the symmetries σ_1, σ_2 of the quadrilateral of groups Q correspond to the involutions $\sigma_1, \sigma_2 \in O(5, 1)$:

1. $\phi_1 = \phi|_{G_1}$.
2. $\phi_2 = \rho|_{G_2} = Ad_{\sigma_2} \circ \phi_1, \phi_4 = \phi|_{G_4} = Ad_{\sigma_1} \circ \phi_1$.
3. $\phi|_{G_3} = Ad_{\sigma_1\sigma_2} \circ \phi_1$.

Thus ϕ extends to a representation (also called ϕ) of the group $\tilde{G} = \langle G, \sigma_1, \sigma_2 \rangle$ generated by G, σ_1, σ_2 .

Our main result is

Theorem 6.2. *If in the above construction R, R_1 is sufficiently large (with fixed R_0, r, α), then ϕ is discrete and faithful.*

Proof. Our proof is analogous to the one in [26]. Every vertex x of X is associated with a 3-dimensional hyperbolic subspace $L(x) \subset \mathbb{H}^5$, namely, it is a subspace stabilized by the vertex group of x in G . If $x = g(x_1), g \in \tilde{G}$, where $x_1 \in Q$ is stabilized by G_1 , then $L(x) = g(L_1)$.

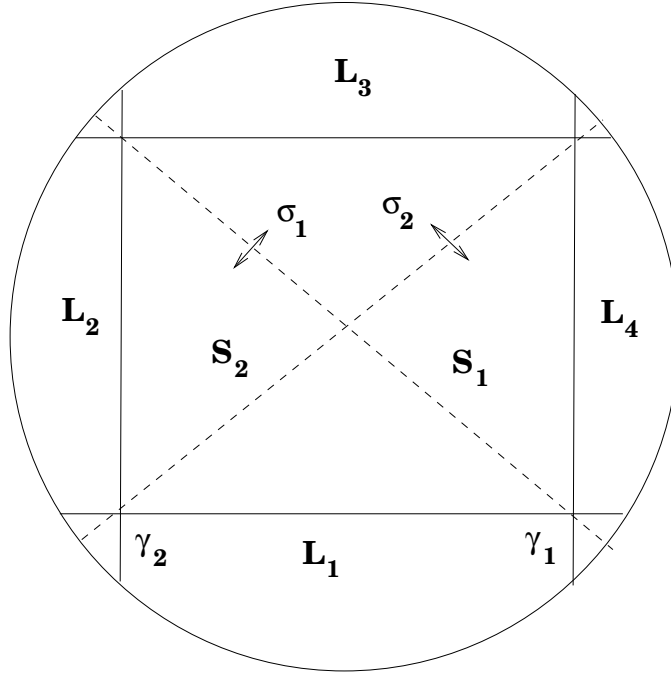


FIGURE 3. Constructing a representation ϕ : Projective model of \mathbb{H}^5 .

Similarly, every edge e of X is associated a geodesic $L(e) \subset \mathbb{H}^5$ stabilized by G_e . Hence, if e connects vertices x, y then

$$L(e) = L(x) \cap L(y).$$

The following properties of the subspaces $L(x), L(e)$ follow directly from the Assumption 6.1 and inequality (1):

If x, y are vertices of X which belong to a common 2-face and are within distance 2 in X^1 , the subspaces $L(x), L(y)$ are within positive distance r from each other. If e, f are distinct edges incident to a common vertex x of X then

a) either e, f belong to a common 2-face of X , in which case

$$d(L(e), L(f)) = R_0,$$

b) otherwise, $d(L(e), L(f)) \geq R$.

In order to prove Theorem 6.2, it suffices to show that for sufficiently large R , there exists $\delta > 0$ so that for all distinct edges f, f' in X ,

$$(2) \quad d(L(f), L(f')) \geq \delta.$$

Consider a 4-chain \mathcal{C} of edges (f_1, \dots, f_4) in X^1 so that $f_i \cap f_{i+1} = y_i, i = 1, 2, 3$, and the concatenation $f_1 \cup f_2 \cup f_3 \cup f_4$ is a local 3-geodesic in X^1 . Two chains (f_1, \dots, f_4) and (f'_1, \dots, f'_4) are said to be *combinatorially isomorphic* if:

1. $Type(f_i) = Type(f'_i), i = 1, \dots, 4$, and
2. f_i, f_{i+1} belong to a common 2-face iff f'_i, f'_{i+1} belong to a common 2-face for $i = 1, 2, 3$.

Clearly, there are only finitely many combinatorial isomorphism types of 4-chains in X^1 .

Each 4-chain $\mathcal{C} = (f_1, \dots, f_4)$ corresponds to a configuration C of three hyperbolic 3-dimensional subspaces $L(y_1), L(y_2), L(y_3)$, and four geodesics $L(f_1), \dots, L(f_4)$ as in section 5. The isometry class of C depends only on the combinatorial isomorphism class of \mathcal{C} .

We now describe the choices of R_1 and R :

1. We will assume that R_1 is such that for each combinatorial isomorphism class of 4-chains in X^1 , the corresponding configuration C of hyperbolic subspaces in \mathbb{H}^5 satisfies $R_1 \geq R_1 = R_1(C)$ where the function R_1 is defined in section 5.

2. We will assume that R is chosen so that for each combinatorial isomorphism class of 4-chains in X^1 , the corresponding configuration C of hyperbolic subspaces in \mathbb{H}^5 satisfies $R \geq R_4 = R_4(C)$; we let ρ be the minimum of the numbers $\rho_4(C)$, where the minimum is taken over all combinatorial isomorphism classes of 4-chains in X^1 . Here R_4 and ρ_4 are the functions introduced in section 5.

We claim that one can take $\delta = \min(2\rho, r, R_0)$ in (2).

If f, f' share a vertex then $d(L(f), L(f')) \geq R_0$. If e, f do not share a vertex and belong to a common 2-face then $d(L(f), L(f')) \geq r$. We therefore consider the generic case when none of the above occurs.

Consider a geodesic path η in X^1 connecting f to f' ; the path η is a concatenation of the edges

$$\eta = e_2 \cup \dots \cup e_{k-1}.$$

Then $k \geq 3$ and for $e_1 = f, e_k = f'$, the concatenation

$$\theta = e_1 \cup \eta \cup e_k$$

is a 3-local geodesic. Set $e_i = [x_i x_{i+1}]$, $i = 1, \dots, k$. Following section 5, for each geodesic $L(e_i)$ we define points p_i (closest to $L_{e_{i-1}}$) and q_i (closest to $L_{e_{i+1}}$).

We will say that an edge e in θ is *large* if the corresponding geodesic $L(e)$ is large in the sense of section 5. Similarly, a vertex x_i in θ is *large* if $L(x_i)$ is large in the sense of section 5, i.e., $d(q_i, p_{i+1}) \geq R$. Accordingly, x_i is *small* if $d(q_i, p_{i+1}) = R_0$ and e is *small* if it is not large.

The next lemma immediately follows from Assumption 6.1:

Lemma 6.3. *For every edge $e_i = [x_i, x_{i+1}]$, at least one of e_i, x_i, x_{i+1} is large.*

Remark 6.4. Note that we can have a vertex x_i so that all three x_i, e_i, e_{i-1} are small.

We also define a sequence of bisectors in \mathbb{H}^n corresponding to the path θ as follows:

For each large cell $c = e_i, c = x_i$ in θ we take the corresponding bisector $Bis(c) = Bis(p_i, q_i), Bis(c) = Bis(q_i, p_{i+1})$ defined as in section 5. Recall that we also have half-spaces $Bis(c)^\pm \subset \mathbb{H}^n$ bounded by the bisectors $Bis(c)$.

We define the natural total order $>$ on the cells in θ by requiring that $e_j > e_i, x_j > x_i$ if $j > i$ and that $e_i > x_i$.

Proposition 6.5. 1. If $c > c'$ then $Bis(c')^+ \subset Bis(c)^+$.

2. If $e_i \neq c$ then $d(L(e_i), Bis(c)) \geq \rho > 0$. Moreover, if $e_i < c$ then $L(e_i) \subset Bis(c)^-$, while if $e_i > c$ then $L(e_i) \subset Bis(c)^+$.

Proof. Part 1. If c, c' are consecutive (with respect to the order $>$) large cells in θ then they belong to a common 4-chain. Therefore, the assertion follows from Lemmas 5.2, 5.3, 5.4 in this case. The general case follows by induction since for three consecutive large cells $c_1 < c_2 < c_3$ in θ , we have

$$Bis(c_3)^+ \subset Bis(c_2)^+ \subset Bis(c_1)^+.$$

Part 2. (a) If e_i, c are not separated (with respect to the order $<$) by any large cells, then they belong to a common 4-chain and the assertion follows from Lemmas 5.2, 5.3, 5.4 and the definition of ρ . (b) In the general case, the assertion follows from (a) and Part 1. \square

Corollary 6.6. $d(L(e_1), L(e_k)) \geq 2\rho > 0$.

Proof. Note first that x_1, x_{k+1}, e_1, e_k cannot be large. Since the (combinatorial) length of θ is at least 3, by Lemma 6.3 there exists a large cell c in θ (every edge e_i , $1 < i < k$, will contain a large cell c). Then

$$e_1 < c < e_k$$

and, by part 3 of the above lemma,

$$L(e_1) \subset Bis(c)^-, L(e_k) \subset Bis(c)^+$$

Moreover

$$\min(d(L(e_1), Bis(c)), d(L(e_k), Bis(c))) \geq \rho.$$

Corollary follows. \square

This concludes the proof of Theorem 6.2. \square

Remark 6.7. By using the arguments of the proof of Theorem 6.2, one can also show that $\phi(G)$ is convex-cocompact.

7. ARITHMETIC SUBGROUPS OF $O(n, 1)$

The goal of this section is to describe a “quaternionic” construction of arithmetic subgroups in $O(n, 1)$. For $n \neq 3, 7$, this construction covers all arithmetic subgroups. Our discussion follows [46], we refer the reader to [31] for the detailed proofs.

We begin by reviewing quaternion algebras over number fields and “hermitian vector spaces” over such algebras.

Let K be a field, D be a central quaternion algebra over K . In other words, the algebra $D = D(a, b)$ has the basis $\{1, i, j, k\}$, subject to the relations:

$$i^2 = a \in K, \quad j^2 = b \in K, \quad ij = -ji = k$$

and so that 1 generates the center of D . For instance, for $a = b = -1$ and $K = \mathbb{R}$, we get the algebra of Hamilton’s quaternions \mathbf{H} . Similarly, if $a = b = 1$ then D is naturally isomorphic to the algebra of 2×2 matrices $End(\mathbb{R}^2)$. One uses the notation

$$\lambda = x1 + yi + zj + wk = x + yi + zj + wk$$

for the elements of D , with $x, y, z, w \in K$. An element of D is *imaginary* if $x = 0$. We will identify K with the center of D :

$$K = K \cdot 1 \subset D.$$

One defines the *conjugation* on D by

$$\lambda = x + yi + zj + wk \mapsto \bar{\lambda} = x - yi - zj - wk.$$

Then $Tr(\lambda) = \lambda + \bar{\lambda}$, $N(\lambda) = \lambda\bar{\lambda}$ are the *trace* and the *norm* on D . Clearly, both trace and the norm are elements of K . In the case when $D \cong End(\mathbb{R}^2)$, the trace is twice the matrix trace and the norm is the matrix determinant. Suppose that K is a subfield of \mathbb{R} . We say that $\lambda \in D$ is *positive* (resp. *negative*) if it has positive (resp. negative) norm.

In what follows, we will assume that K is a totally real number field and D is a division algebra.

We will consider finite-dimensional “vector spaces” V over D , i.e., finite-dimensional right D -modules where we use the notation

$$v\lambda = v \cdot \lambda \in V$$

for $v \in V, \lambda \in D$. Such a module is isomorphic to D^n for some $n < \infty$, where n is the dimension of V as a D -module. Given $v \in V$ define $\langle v \rangle$ as the submodule in V generated by v .

A *skew-hermitian form* on V is a function $F(u, v) = \langle u, v \rangle \in D$, $u, v \in V$, so that:

$$\begin{aligned} F(u_1 + u_2, v) &= F(u_1, v) + F(u_2, v), \\ F(u\lambda, v\mu) &= \bar{\lambda}F(u, v)\mu, \quad F(u, v) = -\overline{F(v, u)}. \end{aligned}$$

Similarly, F is hermitian if

$$F(u\lambda, v\mu) = \bar{\lambda}F(u, v)\mu, \quad F(u, v) = \overline{F(v, u)}.$$

The form F is *nondegenerate* if

$$F(v, u) = 0 \quad \forall u \in V \Rightarrow v = 0.$$

In coordinates:

$$F(x, y) = \sum_{l, m} \bar{x}_l a_{lm} y_m, \quad a_{lm} = -\overline{a_{ml}}.$$

In particular, the diagonal entries of the Gramm matrix of F are imaginary. A *null-vector* is a vector with $F(v, v) = 0$, equivalently, $F(v, v)$ is not imaginary. We say that a vector v is *regular* if it is not null. From now on, we fix F .

Define $U(V, F)$, the group of *unitary* automorphisms of V , i.e., invertible endomorphisms which preserve F .

For a regular vector v , define the submodule v^\perp in V :

$$v^\perp = \{u \in V : \langle u, v \rangle = 0\}.$$

We will see below that

$$V = \langle v \rangle \oplus v^\perp$$

and the restriction of the form F to v^\perp is again nondegenerate.

Orthogonal projection. Suppose $\langle v, v \rangle = a \neq 0$. Define

$$Proj_v : V \rightarrow \langle v \rangle, Proj_v(u) = v \cdot a^{-1} \langle v, u \rangle.$$

Then $Proj_v \in End(V)$, $Proj_v|_{\langle v \rangle} = Id$, $Ker(Proj_v) = v^\perp$. In particular, for a vector $u \in V$,

$$u' = u - Proj_v(u) \in v^\perp.$$

Hence, $u = u' + u''$, with $u'' = Proj_v(u)$. It is now immediate that

$$V = \langle v \rangle \oplus v^\perp.$$

Since F is nondegenerate, the restriction $F|_{v^\perp}$ is also nondegenerate.

The existence of projections allows us to define Gram-Schmidt orthogonalization in V . In particular, V has an orthogonal basis in which F is diagonal.

Reflections. Given a regular vector $v \in V$, define the *reflection* $\sigma_v \in End(V)$ by

$$\sigma_v(u) := u - 2Proj_v(u).$$

It is immediate that $\sigma_v|_{\langle v \rangle} = -Id$ and $\sigma_v|_{v^\perp} = Id$. In particular, σ_v is an involution in $U(V, F)$.

Observe that reflections σ_u, σ_v commute iff either $\langle u \rangle = \langle v \rangle$ or $\langle u, v \rangle = 0$.

Proof of noncoherence of arithmetic lattices will use the following technical result:

Lemma 7.1. *Let V be 3-dimensional, p_1, p_2 be regular vectors which span a 2-dimensional submodule P in V so that the restriction of F to P is nondegenerate. Then there exist $u_1, u_2 \in V$ so that:*

1. $\langle p_m, u_m \rangle = 0, m = 1, 2$.
2. $\langle u_1, u_2 \rangle = 0$.

Proof. Since $F|_P$ is nondegenerate, it follows from the Gram-Schmidt orthogonalization that there exists $v \in V$ so that $P = v^\perp$ and $V = P \oplus \langle v \rangle$. In particular, v is a regular vector.

Orthogonalization implies that there exist vectors $u'_1, u'_2 \in P$ orthogonal to p_1, p_2 respectively. We will find vectors u_1, u_2 in the form

$$u_m = u'_m + v \cdot \lambda_m, \quad \lambda_m \in D, m = 1, 2,$$

It is immediate that $\langle p_m, u_m \rangle = 0, m = 1, 2$. We have

$$\langle u_1, u_2 \rangle = \langle u'_1, u'_2 \rangle + \bar{\lambda}_1 \langle v, u'_2 \rangle + \langle u'_1, v \rangle \lambda_2 + \bar{\lambda}_1 \langle v, v \rangle \lambda_2.$$

Set $\alpha := \langle v, v \rangle$, $\nu_1 := \langle u'_1, v \rangle$, $\nu_2 := \langle v, u'_2 \rangle$, $\mu := \langle u'_1, u'_2 \rangle$. By scaling u'_1 if necessary, we get

$$\nu_1 + \alpha \neq 0.$$

We now set $\lambda_1 = 1, \lambda_2 = \lambda$ (the unknown). Then the equation $\langle u_1, u_2 \rangle = 0$ has the solution

$$\lambda = -(\mu + \nu_2)(\nu_1 + \alpha)^{-1} \in D. \quad \square$$

Next, we now relate (V, F) to hyperbolic geometry. Regarding K as a subfield of \mathbb{R} , we define the completions $D_{\mathbb{R}}$ of D and $V_{\mathbb{R}}$ of V . We require D to be such that $D_{\mathbb{R}} \cong SL(2, \mathbb{R})$, i.e., to have zero divisors. Hence, at least one of the generators i, j, k

of D is negative, i.e., has negative norm. We assume that this is i ; thus $i^2 > 0$. By interchanging j and k if necessary, we obtain that j is also negative. By scaling i, j by appropriate real numbers we get $i^2 = j^2 = 1$. By abusing the terminology, we retain the notation i, j for these real multiples of the original generators of D .

Since $i^2 = 1$, the right multiplication by i ,

$$I(v) = v \cdot i, v \in V,$$

determines an involutive linear transformation of $V_{\mathbb{R}}$ (regarded as the real vector space). We obtain the eigenspace decomposition

$$V_{\mathbb{R}} = V_+ \oplus V_-,$$

where $I|_{V_{\pm}} = \pm Id$. (Note that $V_{\pm} \cap V = 0$.) Let $J : V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$ be given by the right multiplication by j : This is again a linear automorphism. For $v \in V_+$ we have

$$vji = -vij = -vj.$$

Therefore, J determines an isomorphism $V_+ \rightarrow V_-$.

Our next goal is to analyze the subspace V_+ . For $u, v \in V_+$ we have

$$c = \langle u, v \rangle = -i\langle u, v \rangle = \langle ui, v \rangle = \langle u, vi \rangle = \langle u, v \rangle i.$$

Hence,

$$-ic = c = ci.$$

Such c necessarily has the form $c = t(k - j), t \in \mathbb{R}$. Set $\alpha := k - j$. Then $F|_{V_+}$ takes values in $\mathbb{R}\alpha$. In particular, $F|_{V_+} = \alpha\varphi$, where φ is a real bilinear form.

Define $\beta := 1 + i$. Then

$$\begin{aligned} \alpha i = \alpha, i\beta = \beta i = \beta, \alpha^2 = 0, \beta\bar{\beta} = 0, \alpha\beta = \bar{\beta}\alpha = 2\alpha, \\ k\beta = \alpha, k\alpha = -\beta. \end{aligned}$$

We now consider the case when V is 1-dimensional, i.e., $V = D$. Then the form F is given by

$$F(x, y) = \bar{x}ay, a \in D, a = -\bar{a}.$$

Therefore,

$$D_+ = \{x\beta + y\alpha : x, y \in \mathbb{R}\}.$$

We now compute the form φ so that $F|_{D_+} = \alpha\varphi$. Note that the group

$$SL_1(D) = \{g : N(g) = g\bar{g} = 1\}$$

acts on the space of traceless matrices

$$D_0 = \{\lambda \in D_{\mathbb{R}} : Tr(\lambda) = \lambda + \bar{\lambda} = 0\}$$

by

$$Ad_g(\lambda) = \bar{g}\lambda g = g^{-1}\lambda g.$$

This action preserves the nondegenerate indefinite quadratic form $\lambda\bar{\lambda}$. Hence, this is the orthogonal action on $\mathbb{R}^{2,1}$ which has three nonzero orbit types. The relevant ones are positive ($N(\lambda) > 0$) and negative ($N(\lambda) < 0$) vectors in D_0 . They are represented by $\lambda = k$ (in which case $N(k) = 1$) and $\lambda = i$ (in which case $N(i) = -1$).

By changing the generator in $D_{\mathbb{R}}$, we replace a (in the definition of F) with $\bar{g}ag$. Hence, our analysis of the form φ reduces to two cases: $a = i, a = k$.

Case 1: $a = i$, $N(a) < 0$. Then for $v = a\beta + y\alpha \in D_+$, we have

$$\begin{aligned} F(v, v) &= \langle x\beta + y\alpha, x\beta + y\alpha \rangle = (x\bar{\beta} + y\bar{\alpha})i(x\beta + y\alpha) = \\ &= (x\bar{\beta} - y\alpha)(x\beta - y\alpha) = -2xy\alpha. \end{aligned}$$

Hence, $\varphi(v, v) = -2xy$, an indefinite form.

Case 2: $a = k$, $N(a) > 0$. Then for $v = a\beta + y\alpha \in D_+$, we have

$$\begin{aligned} F(v, v) &= \langle x\beta + y\alpha, x\beta + y\alpha \rangle = (x\bar{\beta} + y\bar{\alpha})k(x\beta + y\alpha) = \\ &= (x\bar{\beta} + y\bar{\alpha})(x\alpha - y\beta) = 2(x^2 + y^2)\alpha. \end{aligned}$$

Therefore, in this case the form φ is positive-definite.

To summarize, if $N(a) < 0$ then φ is indefinite, while if $N(a) > 0$ then the form φ is positive-definite; in both cases, φ is nondegenerate.

We now consider the general case. Without loss of generality, we may assume that F is diagonal, $\{v_1, \dots, v_n\}$ is an orthogonal basis in V . Clearly,

$$V_{\pm} = \bigoplus_{l=1}^n \langle v_l \rangle_{\pm},$$

where $\langle v_l \rangle_{\pm} = v_l \cdot D_{\pm}$. We say that a vector $v \in V$ is *positive* (resp. *negative*) if $N(\langle v, v \rangle) > 0$ (resp. $N(\langle v, v \rangle) < 0$).

Thus we obtain

Proposition 7.2. *The form φ is always nondegenerate. It has signature (p, q) iff F has “signature” (p, q) , i.e., some (equivalently, every) orthogonal basis $\{v_1, \dots, v_n\}$ of V contains exactly q negative vectors.*

In particular, φ has signature $(p, 1)$ iff one of the vectors of one (every) orthogonal basis of V is negative and the rest of the vectors are positive.

We assume from now on that F has the signature $(p, 1)$. Then (V, F) determines the hyperbolic $2p + 1$ -space $H(V)$ associated with the Lorentzian space (V_+, φ) . A vector $v \in V$ determines a geodesic $H(\langle v \rangle) \subset H(V)$ iff v is a negative vector. Similarly, $F|_{v^{\perp}}$ has hyperbolic signature iff v is a positive vector; thus $H(v^{\perp}) \subset H(V)$ is a hyperbolic hyperplane.

For a subspace $W_+ \subset V_+$ we define W_+^- as the closed light-cone

$$\{w \in W_+ : \varphi(w) \leq 0\}.$$

We obtain an embedding $U(V, F) \hookrightarrow \text{Isom}(H(V))$ given by the action of $U(V, F)$ on the Lorentzian space $(V_+, \varphi) \cong \mathbb{R}^{p,1}$.

Lemma 7.3. *Suppose that u, v are positive vectors in V . Then the hyperbolic hyperplanes $H(u^{\perp}), H(v^{\perp})$ are orthogonal iff u is orthogonal to v .*

Proof. Lemma immediately follows from the fact that the following are equivalent:

1. $H(u^{\perp}), H(v^{\perp})$ are orthogonal or equal.
2. σ_u commutes with σ_v .
3. u and v are orthogonal or generate the same submodule in V . □

Given the hyperbolic space $H(V)$, we say that a hyperplane $L \subset H(V)$ is K -rational if $L = H(u^\perp)$ for some positive vector $u \in V$. A subspace in $H(V)$ is K -rational if it appears as intersection of K -rational hyperplanes. One observes (using orthogonalization) that a geodesic $\gamma \subset H(V)$ is K -rational iff $\gamma = H(\langle v \rangle)$, where $v \in V$ is a negative vector.

We therefore obtain

Corollary 7.4. *Suppose that γ_1, γ_2 are distinct K -rational geodesics in the hyperbolic 5-space $\mathbb{H}^5 = H(V)$, $\dim(V) = 3$. Then there exist K -rational hyperplanes H_1, H_2 containing γ_1, γ_2 , so that H_1 is orthogonal to H_2 .*

Proof. Let $p_l \in V$ be such that $\gamma_l = H(\langle p_l \rangle)$, $l = 1, 2$. Then, according to Lemma 7.1, there exist $u_1, u_2 \in V$ so that

$$\langle p_l \rangle \subset u_l^\perp, l = 1, 2, \quad \langle u_1, u_2 \rangle = 0.$$

Since $F|_{\langle p_l \rangle}$ is indefinite, it follows that $F|_{u_l^\perp}$ is indefinite as well ($l = 1, 2$). Hence, u_l^\perp determines a K -rational hyperbolic hyperplane $H_l = H(u_l^\perp) \subset \mathbb{H}^3$. Clearly, $\gamma_l \subset H_l$, $l = 1, 2$. In view of the previous lemma, H_1 and H_2 are orthogonal provided that they actually intersect in \mathbb{H}^5 .

Pick a generator w of $W = u_1^\perp \cap u_2^\perp$. Since u_1, u_2 are positive and u_1, u_2, w form an orthonormal basis in V , it follows that w is negative. Therefore, $\gamma = H(W) = H(\langle w \rangle)$ is a geodesic in \mathbb{H}^5 . Hence, H_1, H_2 intersect along the geodesic γ at the right angle. \square

Let $\sigma_l := \sigma_{u_l}$, $l = 1, 2$ denote the reflections in the subspaces $U_l = u_l^\perp$. Since σ_1, σ_2 commute, their product is the involution τ whose fixed-point set is the 1-dimensional subspace $W = U_1 \cap U_2$. We now assume that the geodesics γ_1, γ_2 are within positive distance from each other. Set $\gamma := H(W)$ (a K -rational geodesic in \mathbb{H}^5) and set $L := H(P)$ (a 3-dimensional K -rational hyperbolic subspace in \mathbb{H}^5).

Lemma 7.5. *H and γ , L and $\tau(L)$ are within positive distance from each other.*

Proof. Observe that $U_l \neq P$, $l = 1, 2$ (since u_l is not a multiple of v). In particular, $U_l \cap P = \langle p_l \rangle$. Hence,

$$U_{l+}^- \cap P_+^- = \langle p_l \rangle_+^-.$$

If $w \in W_+^- \cap P_+^-$ then $w \in U_{l+}^- \cap P_+^- = \langle p_l \rangle_+^-$, $l = 1, 2$. However, $\langle p_1 \rangle_+^- \cap \langle p_2 \rangle_+^- = 0$ since we assumed that γ_1, γ_2 are within positive distance from each other. Thus $w = 0$. In particular, L and γ are within positive distance from each other.

Let $\rho \subset \mathbb{H}^5$ denote the geodesic segment with the end-points in L, γ , which is orthogonal to both. Then $\rho \cup \tau(\rho)$ is a geodesic segment connecting L and $\tau(L)$ and orthogonal to both subspaces. Hence, L and $\tau(L)$ are within positive distance from each other. \square

Suppose that F is a skew-hermitian form on V . Given an embedding $\tau : K \rightarrow \mathbb{R}$ we define the signature $sig_\tau(F)$ with respect to the subfield $\tau(F) \subset \mathbb{R}$: Note that the notion of positivity and negativity in V (and, hence, the signature) depends on the embedding τ .

At last, we are ready to define the class of “quaternionic” arithmetic lattices $G \subset O(n, 1)$. Let K be totally real, D be a quaternion algebra over K and V be an $n + 1$ -dimensional module over D . We assume that F is a nondegenerate hermitian form on V satisfying the following:

1. F has the signature $(n, 1)$.
2. For every nontrivial embedding $\tau : K \rightarrow \mathbb{R}$, the signature $\text{sig}_\tau(F)$ is $(n + 1, 0)$ (or $(0, n + 1)$).

Next, we need a notion of an “integer” automorphism of V . Let $O \subset D$ be an *order*, i.e., a lattice in D regarded as a vector space over K . An example of such order is given by $A^4 \subset D$, where A is the ring of integers of K .

The order O also determines the lattice $O^{n+1} \subset V = D^{n+1}$. We let $GL(V, O)$ denote the group of automorphisms of the D -module V which preserve the lattice O^{n+1} . If we regard automorphisms of V as “matrices” with coefficients in D , then the elements of $GL(V, O)$ are “matrices” with coefficients in O which admit inverses with the same property.

We now fix an order $O \subset D$. Then every subgroup Γ of $U(V, F)$ commensurable to the intersection $U(V, F) \cap GL(V, O)$ is an *arithmetic group of quaternionic type*. The embedding $U(V, F) \hookrightarrow O(n, 1)$ (induced by the identity embedding $K \hookrightarrow \mathbb{R}$) realizes Γ as a lattice in $O(n, 1)$.

Theorem 7.6. (See [31].) *Except for $n = 3, 7$, every arithmetic subgroup of $O(n, 1)$ appears as one of the groups Γ as above. In the case $n = 7$, there is an extra class of arithmetic groups associated with octaves rather than quaternions. For $n = 3$, there is yet another construction, also of quaternionic origin, which covers all arithmetic groups in this dimension, see [35] for the detailed description.*

8. PROOF OF NONCOHERENCE OF ARITHMETIC GROUPS OF QUATERNIONIC ORIGIN

Let $G_0 \subset O(3, 1)$ be an arithmetic lattice. According to our assumptions, there exists a finite index (torsion-free) subgroup $G'_0 \subset G_0$ so that the manifold $M^3 = \mathbb{H}^3/G'_0$ fibers over the circle. Let $F_0 \triangleleft G'_0$ denote the normal surface subgroup corresponding to the fundamental group of the surface fiber in this fibration. Pick two nonconjugate maximal cyclic subgroups G'_{e_1}, G'_{e_2} in G'_0 so that $G'_{e_i} \cap F_0 = \{1\}$.

Lemma 8.1. *We can choose G'_{e_1}, G'_{e_2} so that the pair $\{G'_{e_1}, G'_{e_2}\}$ is relatively separable in G_0 .*

Proof. According to discussion in section 1.2, there are at least two distinct virtual fibrations of the manifold M^3 . Let $J_0 \triangleleft G'_0$ be a normal surface subgroup corresponding to the second fibration. Then $J_0 \cap F_0$ has infinite index in both F_0, J_0 . Therefore, take two nontrivial elements $t_1, t_2 \in J_0$ so that:

$$\langle t_1 \rangle \neq \langle t_2 \rangle, \quad \langle t_1 \rangle \cap F_0 = \langle t_2 \rangle \cap F_0 = \{1\}.$$

After passing to a finite-index subgroup in G'_0 if necessary, we can assume that the subgroups $G'_{e_i} := \langle t_i \rangle, i = 1, 2$ are maximal cyclic subgroups of G'_0 and that they are not conjugate in G'_0 . The set $\{\langle t_1, t_2 \rangle\}$ is relatively separable in J_0 by Proposition 2.7

since, being a surface group, J_0 is LERF [42]. The subgroup J_0 is normal in G'_0 with $G'_0/J_0 \cong \mathbb{Z}$. Therefore, relative separability of $\{(t_1, t_2)\}$ in G'_0 follows from Lemma 2.8. \square

We let $\gamma_i, i = 1, 2$ denote the invariant geodesic of G'_{e_i} . Let $R_0 := d(\gamma_1, \gamma_2)$. We will assume, as in the beginning of Section 6.2 that R_0 is the distance between the projections of γ_1, γ_2 to the manifold M^3 .

Since G'_{e_1}, G'_{e_2} are separable in G'_0 , given a number R_1 , there exists a finite-index subgroup $G''_0 \subset G'_0$ so that:

For each $\gamma = \gamma_i, i = 1, 2$ and geodesics $\beta_1, \beta_2 \in G''_0 \cdot (\gamma_1 \cup \gamma_2)$ so that $d(\beta_j, \gamma) = R_0$ ($j = 1, 2$), it follows that

$$d(\text{proj}_\gamma(\beta_1), \text{proj}_\gamma(\beta_2)) \geq R_1.$$

Set $G''_{e_i} := G'_{e_i} \cap G''_0, i = 1, 2$. Without loss of generality, we may assume that G''_{e_1}, G''_{e_2} generate a free subgroup H of $F''_0 := F_0 \cap G''_0$. The subgroup H is separable in F''_0 ; since $G''_0/F''_0 \cong \mathbb{Z}$, the subgroup H is also separable in G''_0 . Therefore, in view of Lemma 2.8, the pair $\{G''_{e_1}, G''_{e_2}\}$ is weakly separable in G''_0 . Therefore, given a number R , one can find a finite-index subgroup $G'''_0 \subset G''_0$ so that:

For all distinct geodesics β, γ in the G'''_0 -orbit of γ_1, γ_2 , the distance $d(\beta, \gamma)$ is at least R unless there exists $g \in G'''_0$ which carries $\beta \cup \gamma$ to $\gamma_1 \cup \gamma_2$.

Therefore, the Assumption 6.1 (from section 6.2) is satisfied by the group $G_1 := G'''_0$ and its subgroups $G_{e_i} := G'_{e_i} \cap G_1, i = 1, 2$, with respect to the numbers R_1 and R .

Our next goal is to define R_1 and R_4 . We let $S_i, i = 1, 2$, be orthogonal 3-dimensional K -rational subspaces in \mathbb{H}^5 intersecting L_1 along the geodesics γ_1, γ_2 ; let $L_2 := \sigma_s(L_1), L_4 := \sigma_1(L_1)$. Since S_1, S_2 are K -rational, the involutions σ_1, σ_2 belong to the commensurator of the lattice Γ . Since the group generated by σ_1, σ_2 is finite, without loss of generality we may assume that these involutions normalize Γ (otherwise, we first pass to a finite-index subgroup in Γ).

Then $r > 0$ is the distance $d(L_2, L_4)$. Let α_1, α_2 denote the angles $\angle(L_1, L_4), \angle(L_1, L_2)$ and $\alpha := \min(\alpha_1, \alpha_2)$.

In view of Lemma 5.1, we will use $R \geq R_*$ so that for every $g \in G_1$,

$$d(g(L_i), L_j) \geq r, i, j \in \{2, 4\}, g(L_i) \neq L_j.$$

Lastly, we set $R_1 := R_1(\alpha, R_0, r)$ and $R_4 := R_4(\alpha, R_0, r)$. We then will use $R := R_4$.

As in section 6, we define a quadrilateral Q of groups with vertex groups isomorphic to G_1 , so that these isomorphisms send G_{e_1}, G_{e_4} to edge groups. Let $G := \pi_1(Q)$. Using the involutions σ_1, σ_2 as in section 6, we construct a discrete and faithful representation $\phi : G \rightarrow O(n, 1)$. Since $G_1 \subset \Gamma$ and σ_1, σ_2 normalize Γ , the image of this representation is contained in Γ . (This provides yet another proof of discreteness of $\phi(G)$, however we still have to use Combination Theorem 6.2 in order to conclude that ϕ is faithful.

In order to show incoherence of Γ it suffices to prove:

Lemma 8.2. *The group G is noncoherent.*

Proof. Let $\epsilon_1, \dots, \epsilon_4$ denote the edges of Q and $G_{\epsilon_i}, i = 1, \dots, 4$ denote the corresponding edge groups. Recall that $F_1 \triangleleft G_1$ is a normal surface subgroup. Let F_i denote the normal surface subgroups of $G_i, i = 2, 3, 4$, which are the images of F_1 under the isomorphisms $G_1 \rightarrow G_i$. Then

$$F_i \cap G_{\epsilon_{i-1}} = F_i \cap G_{\epsilon_i} = \{1\},$$

here and in what follows i is taken mod 4. Let F denote the subgroup of G generated by F_1, \dots, F_4 . Clearly, this group is finitely-generated. We will show that F is not finitely presented by proving that $F \cong F_+ *_N F_-$, where F_{\pm} are finitely-generated (actually, finitely-presented) and N is a free group of infinite rank.

We first describe G as an amalgamated free product: We cut the quadrilateral Q in half so that one half contains the vertices 1, 2, while the other half contains the vertices 3, 4. Accordingly, set

$$\begin{aligned} G_- &:= \langle G_1, G_2 \rangle \cong G_1 *_{G_{\epsilon_2}} G_2, & G_+ &:= \langle G_3, G_4 \rangle \cong G_3 *_{G_{\epsilon_4}} G_4, \\ E &:= \langle G_{\epsilon_1}, G_{\epsilon_3} \rangle \cong G_{\epsilon_1} * G_{\epsilon_3} \cong \mathbb{Z} * \mathbb{Z}. \end{aligned}$$

Then

$$G \cong G_- *_E G_+.$$

Similarly, we set

$$F_+ := F \cap G_+ \cong F_1 * F_2; \quad F_- := F \cap G_- \cong F_3 * F_4.$$

Since G_1 is generated by t_2 and F_1 , the group G_2 is generated by t_2 and F_2 . Hence, F_- is normal in G_- and $G_-/F_- \cong \mathbb{Z}$. Moreover, F_- has trivial intersection with G_{ϵ_1} (since F_1 does). It is immediate that $N := F_- \cap E$ is an infinite index nontrivial subgroup of E . Since E is free of rank 2, it follows that N is a free group of infinite rank. Clearly, $N = F_+ \cap E$ and we obtain

$$N \cong F_+ *_N F_-.$$

Therefore, F is finitely generated and infinitely presented since (by considering the Meyer-Vietoris sequence associated with the amalgam $N \cong F_+ *_N F_-$) $H_2(F, \mathbb{Z})$ has infinite rank. Thus, G is noncoherent. \square

9. COMPLEX-HYPERBOLIC AND QUATERNIONIC LATTICES

It is an important open problem in theory of lattices in rank 1 Lie groups $O(n, 1)$ and $SU(n, 1)$ if a lattice has positive virtual first Betti number, i.e., contains a finite-index subgroup with infinite abelianization. In this section we relate this problem to noncoherence in the case of $SU(n, 1)$. It was proven by D. Kazhdan [29] (see also [48]) that arithmetic lattices of the *simplest type* (or, first type) in $SU(n, 1)$ admit finite index (congruence) subgroups with infinite abelianization. Certain classes of non-arithmetic lattices in $SU(2, 1)$ (the ones violating integrality condition for arithmetic groups) are proven to have positive virtual first Betti number by S.-K. Yeung [53].

On the other hand J. Rogawski [40] proved that for arithmetic lattices Γ in $SU(2, 1)$ of *second type* (associated with division algebras), every congruence-subgroup $\Gamma' \subset \Gamma$ has finite abelianization. It is unknown if non-congruence subgroups in such lattices (if they exist at all!) can have infinite abelianization.

Below is the description of arithmetic lattices of the simplest type in $SU(n, 1)$ following [36] and [43]. Let K be a totally real number field; take a totally imaginary quadratic extension L/K and let \mathcal{O}_L be the ring of integers of L . Let $\sigma_1, \sigma_2, \dots, \sigma_k : L \rightarrow \mathbb{C}$ be the embeddings. Next, take a hermitian quadratic form in $n + 1$ variables

$$\varphi(z, \bar{z}) = \sum_{p,q=1}^{n+1} a_{pq} z_p \bar{z}_q$$

with coefficients in L . We require $\varphi^{\sigma_1}, \varphi^{\bar{\sigma}_1}$ to have signature $(n, 1)$ and require the forms φ^{σ_j} to have signature $(n + 1, 0)$ for the rest of the embeddings σ_j . Let $SU(\varphi)$ denote the group of special unitary automorphisms of the form φ on L^{n+1} . The embedding σ_1 defines a homomorphism $SU(\varphi) \rightarrow SU(n, 1)$ with relatively compact kernel. We will identify L with $\sigma_1(L)$, so $\sigma_1 = id$.

Definition 9.1. A subgroup Γ of $SU(n, 1)$ is said to be an arithmetic lattice of the *simplest type* if it is commensurable to $SU(\varphi, \mathcal{O}_L) = SU(\varphi) \cap SL(n + 1, \mathcal{O}_L)$.

By diagonalizing the form φ , we see that L^{n+1} contains a 3-dimensional subspace L^3 so that the restriction of φ to L^3 is a form of the signature $(2, 1)$. Therefore, an arithmetic lattice $\Gamma \subset SU(n, 1)$ of the simplest type intersects $SU(\varphi|_{L^3})$ along a lattice Γ' of the simplest type (regarded as a subgroup of $SU(2, 1)$). If Γ' is non-coherent, so is Γ . Therefore, we restrict our discussion to the case of isometries of the complex-hyperbolic plane $\mathbb{C}\mathbb{H}^2$.

Suppose that $\Gamma \subset SU(2, 1)$ is a torsion-free uniform lattice with infinite abelianization. Therefore, $b_1(M) > 0$, where $M = \mathbb{C}\mathbb{H}^2/\Gamma$. Since M is Kähler, its Betti numbers are even; therefore, there exists an epimorphism $\psi : \Gamma \rightarrow \mathbb{Z}^2$. There are two cases to consider:

Case 1. $Ker(\psi)$ is not finitely generated. Then, according to [12], there exists a holomorphic fibration $M \rightarrow R$ with connected fibers, where R is a hyperbolic Riemann surface-orbifold. It was proven in [25] that the kernel K of the homomorphism $\Gamma \rightarrow \pi_1(R)$ is finitely-generated but not finitely presented. Hence, Γ is non-coherent in this case.

Remark 9.2. Jonathan Hillman [21] suggested an alternative proof that K is not finitely presented. Namely, if K is of type FP_2 (e.g., is finitely presented) then it is a $PD(2)$ -group (see Theorem 4.1) and, hence, a surface group. It was proven by Hillman in [22] that the holomorphic fibration $M \rightarrow R$ has no singular fibers. Such fibrations cannot exist due to a result of K. Liu [32]. Thus, K is not finitely presented.

Case 2. $F = Ker(\psi)$ is finitely generated. If Γ were coherent, F would be also finitely-presented. It is proven by Jonathan Hillman (Theorem 4.1) that F has to be a surface group. We obtain the associated homomorphism $\eta : \mathbb{Z}^2 \rightarrow Out(F)$ (the mapping class group of a surface). Since Γ contains no rank 2 abelian subgroups, η is injective. Rank 2 abelian subgroups of the mapping class group have to contain nontrivial reducible elements [7]. Let $\gamma \in \mathbb{Z}^2 \setminus \{1\}$ be such that $\eta(\gamma)$ is a reducible element of the mapping class group. Hence, $\eta(\gamma)$ fixes a conjugacy class of some $\alpha \in F \setminus \{1\}$. It follows that Γ contains \mathbb{Z}^2 (generated by a lift of γ to Γ and by α). Contradiction.

We thus obtain:

Theorem 9.3. *Suppose that $\Gamma \subset SU(2,1)$ is a cocompact arithmetic group with infinite abelianization. Then Γ is non-coherent.*

Corollary 9.4. *Suppose that $\Gamma \subset SU(n,1)$ is a cocompact arithmetic group of the simplest type, where $n \geq 2$. Then Γ is non-coherent.*

We now consider quaternionic-hyperbolic lattices. Recall that all lattices in $\mathbf{H}\mathbb{H}^n$, $n \geq 2$, are arithmetic according to [10, 15]. On the other hand, $Isom(\mathbf{H}\mathbb{H}^1) \cong Isom(\mathbb{H}^4)$ and, hence, this group contains nonarithmetic lattices as well.

Proposition 9.5. *Every arithmetic lattice in $\mathbf{H}\mathbb{H}^n$ is non-coherent.*

Proof. According to [37], all arithmetic lattices in $Isom(\mathbf{H}\mathbb{H}^n) \cong Sp(n,1)$ have the following form.

Let $K \subset \mathbb{R}$ be a totally-real number field, D be a central quaternion algebra over K , V be an $n+1$ -dimensional right D -module, F be a hermitian bilinear form on V (see section 7). Choose a basis where F is diagonal:

$$F(x, y) = \sum_{m=1}^{n+1} \bar{x}_m a_m y_m,$$

$a_m = \bar{a}_m$. Then the signature of F is (p, q) if (after permuting the coordinates) $a_m > 0, m = 1, \dots, p$ and $a_m < 0, m = p+1, \dots, n+1 = p+q$. Let $U(V, F)$ be the group of unitary transformations of (V, F) .

Given an embedding $\sigma : K \rightarrow \mathbb{R}$, we define a new form F^σ by applying σ to the coefficients of F . We now require F, D and K to be such that:

1. F has signature $(n, 1)$ and F^σ is definite for all embeddings σ different from the identity.
2. The completions of D with respect to all the embeddings $\sigma : K \rightarrow \mathbb{R}$ are isomorphic to Hamilton's quaternions \mathbf{H} (i.e., are division algebras).

In particular, the embedding $D \rightarrow \mathbf{H}$, induced by the identity embedding $K \hookrightarrow \mathbb{R}$, gives rise to a homomorphism $\eta : U(V, F) \rightarrow Sp(n, 1) = Isom(\mathbf{H}\mathbb{H}^n)$.

Let O be an order in D and set $\Gamma_{V,O} := U(V, F) \cap SL(V, O)$. Lastly, a group commensurable to $\eta(\Gamma_{V,O}) \subset Sp(n, 1)$ is called an *arithmetic* lattice in $Sp(n, 1)$. Note that the kernel of the homomorphism

$$\eta : \Gamma_{V,O} \rightarrow Sp(n, 1)$$

is finite. Hence, every arithmetic lattice in $Sp(n, 1)$ is abstractly commensurable to $\Gamma_{V,O}$ for some choice of K, D and O .

By restricting the form F to the 2-dimensional submodule W in V spanned by the first and last basis vectors, we obtain a hermitian form of signature $(1, 1)$. Therefore, every arithmetic lattice Γ in $Sp(n, 1)$ will contain a subgroup commensurable to $\eta(\Gamma_{W,O})$. The latter is an arithmetic lattice in \mathbb{H}^4 and, hence, is noncoherent according to [28] and [2]. Thus, Γ is incoherent as well. \square

The same argument applies to lattices Γ in the isometry group of the hyperbolic plane over Cayley octaves $Isom(\mathbf{O}\mathbb{H}^2)$, as every such lattice is arithmetic and contains

an arithmetic sublattice $\Gamma' \subset \text{Isom}(\mathbf{O}\mathbb{H}^1) \cong \text{Isom}(\mathbb{H}^8)$. (I owe this remark to Andrei Rapinchuk.) Since Γ' is noncoherent, so is Γ .

REFERENCES

- [1] I. AGOL, *Systoles of hyperbolic 4-manifolds*. Preprint, math.GT/0612290, 2006.
- [2] ———, *Criteria for virtual fibering*, J. Topol., 1 (2008), pp. 269–284.
- [3] I. AGOL, D. D. LONG, AND A. W. REID, *The Bianchi groups are separable on geometrically finite subgroups*, Ann. of Math. (2), 153 (2001), pp. 599–621.
- [4] G. BAUMSLAG AND J. E. ROSEBLADE, *Subgroups of direct products of free groups*, J. London Math. Soc., 30 (1984), pp. 44–52.
- [5] M. BESTVINA AND N. BRADY, *Morse theory and finiteness properties of groups*, Inventiones Math., 129 (1997), pp. 445–470.
- [6] R. BIERI, *Homological dimension of discrete groups*, Mathematical Notes, Queen Mary College, 1976.
- [7] J. BIRMAN, A. LUBOTZKY, AND J. MCCARTHY, *Abelian and solvable subgroups of the mapping class groups*, Duke Math. J., 50 (1983), pp. 1107–1120.
- [8] B. H. BOWDITCH AND G. MESS, *A 4-dimensional Kleinian group*, Transactions of AMS, 14 (1994), pp. 391–405.
- [9] M. BRIDSON AND A. HAEFLIGER, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.
- [10] K. CORLETTE, *Archimedean superrigidity and hyperbolic geometry*, Ann. of Math. (2), 135 (1992), pp. 165–182.
- [11] P. DELIGNE AND G. D. MOSTOW, *Commensurabilities among lattices in $\text{PU}(1, n)$* , vol. 132 of Annals of Mathematics Studies, Princeton University Press, 1993.
- [12] T. DELZANT, *L'invariant de Bieri Neumann Strebel des groupes fondamentaux des varietes Kaehleriennes*. math.DG/0603038, 2006.
- [13] R. GITIK, *Doubles of groups and hyperbolic LERF 3-manifolds*, Ann. of Math. (2), 150 (1999), pp. 775–806.
- [14] ———, *Ping-pong on negatively curved groups*, J. Algebra, 217 (1999), pp. 65–72.
- [15] M. GROMOV AND R. SCHOEN, *Harmonic maps into singular spaces and p -adic superrigidity for lattices in groups of rank one*, Publ. Math. of IHES, 76 (1992), pp. 165–246.
- [16] F. J. GRUNEWALD, *On some groups which cannot be finitely presented*, J. London Math. Soc. (2), 17 (1978), pp. 427–436.
- [17] F. HAGLUND, *Finite index subgroups of graph products*, Geom. Dedicata, 135 (2008), pp. 167–209.
- [18] F. HAGLUND AND J. ŚWIĄTKOWSKI, *Separating quasi-convex subgroups in 7-systolic groups*, Groups Geom. Dyn., 2 (2008), pp. 223–244.
- [19] F. HAGLUND AND D. WISE, *Special cube complexes*, Geom. Funct. Anal., 17 (2008), pp. 1551–1620.
- [20] M. HALL, JR., *Coset representations in free groups*, Trans. Amer. Math. Soc., 67 (1949), pp. 421–432.
- [21] J. A. HILLMAN, *Private communication*. 1998.
- [22] ———, *Complex surfaces which are fibre bundles*, Topology and its Applications, 100 (2000), pp. 187–191.
- [23] ———, *Four-manifolds, geometries and knots*, vol. 5 of Geometry & Topology Monographs, Geometry & Topology Publications, Coventry, 2002.
- [24] J. A. HILLMAN AND D. H. KOCHLOUKOVA, *Finiteness conditions and PD_r -group covers of PD_n -complexes*, Math. Z., 256 (2007), pp. 45–56.
- [25] M. KAPOVICH, *On normal subgroups in the fundamental groups of complex surfaces*. Preprint, math.GT/9808085, 1998.
- [26] ———, *Representations of polygons of finite groups*, Geom. Topol., 9 (2005), pp. 1915–1951.
- [27] M. KAPOVICH AND L. POTYAGAILO, *On absence of Ahlfors' finiteness theorem for Kleinian groups in dimension 3*, Topology and its Applications, 40 (1991), pp. 83–91.

- [28] M. KAPOVICH, L. POTYAGAILO, AND E. B. VINBERG, *Non-coherence of some non-uniform lattices in $\text{Isom}(\mathbb{H}^n)$* , Geometry and Topology Monographs, 14 (2008), pp. 335–351.
- [29] D. KAZHDAN, *Some applications of the Weil representation*, J. Analyse Mat., 32 (1977), pp. 235–248.
- [30] J.-P. LABESSE AND J. SCHWERMER, *On liftings and cusp cohomology of arithmetic groups*, Inventiones Math., 83 (1986), pp. 383–401.
- [31] J.-S. LI AND J. J. MILLSON, *On the first Betti number of a hyperbolic manifold with an arithmetic fundamental group*, Duke Math. J., 71 (1993), pp. 365–401.
- [32] K. LIU, *Geometric height inequalities*, Math. Res. Lett., 3 (1996), pp. 693–702.
- [33] D. D. LONG AND G. A. NIBLO, *Subgroup separability and 3-manifold groups*, Math. Z., 207 (1991), pp. 209–215.
- [34] A. LUBOTZKY, *Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem*, Ann. of Math., 144 (1996), pp. 441–452.
- [35] C. MACLACHLAN AND A. REID, *The arithmetic of hyperbolic 3-manifolds*, vol. 219 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2003.
- [36] D. B. MCREYNOLDS, *Arithmetic lattices in $SU(n, 1)$* . Preprint, <http://www.ma.utexas.edu/dmcreyn/downloads.html>, 2007.
- [37] V. PLATONOV AND A. RAPINCHUK, *Algebraic groups and number theory*, vol. 139 of Pure and Applied Mathematics, Academic Press Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.
- [38] L. POTYAGAILO, *The problem of finiteness for Kleinian groups in 3-space*, in “Knots 90” (Osaka, 1990), de Gruyter, Berlin, 1992, pp. 619–623.
- [39] ———, *Finitely generated Kleinian groups in 3-space and 3-manifolds of infinite homotopy type*, Transactions of the AMS, 344 (1994), pp. 57–77.
- [40] J. ROGAWSKI, *Automorphic representations of unitary groups in three variables*, vol. 123 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1990.
- [41] P. SCOTT, *Compact submanifolds of 3-manifolds*, Journ. of the LMS, 6 (1973), pp. 437–448.
- [42] ———, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. (2), 17 (1978), pp. 555–565.
- [43] M. STOVER, *Property (FA) and lattices in $SU(2, 1)$* , Internat. J. Algebra Comput., 17 (2007), pp. 1335–1347.
- [44] W. THURSTON, *A norm for the homology of 3-manifolds*, Memoirs of AMS, 339 (1985), pp. 100–130.
- [45] È. B. VINBERG, *Discrete groups generated by reflections in Lobačevskii spaces*, Mat. Sb. (N.S.), 72 (114) (1967), pp. 471–488; correction, *ibid.* 73 (115) (1967), 303.
- [46] È. B. VINBERG AND O. V. SHVARTSMAN, *Discrete groups of motions of spaces of constant curvature*, in Geometry, II, vol. 29 of Encyclopaedia Math. Sci., Springer, 1993, pp. 139–248.
- [47] C. WALL, *List of problems*, in Homological group theory (Proc. Sympos., Durham, 1977), C. T. C. Wall, ed., Cambridge Univ. Press, Cambridge, 1979, pp. 369–394.
- [48] N. WALLACH, *Square integrable automorphic forms and cohomology of arithmetic quotients of $SU(p, q)$* , Math. Ann., 266 (1984), pp. 261–278.
- [49] D. WISE, *Subgroup separability of graphs of free groups with cyclic edge groups*, Q. J. Math., 51 (2000), pp. 107–129.
- [50] ———, *Subgroup separability of the figure 8 knot group*, Topology, 45 (2006), pp. 421–463.
- [51] ———, *Morse theory, random subgraphs and incoherent groups*. Preprint, 2008.
- [52] ———, *The structure of groups with a quasiconvex hierarchy*. Preprint, 2009.
- [53] S.-K. YEUNG, *Virtual first Betti number and integrality of compact complex two-ball quotients*, Int. Math. Res. Not., 38 (2004), pp. 1967–1988.

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