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A Tannakian Result for Profinite Groups

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ABSTRACT OF THE DISSERTATION

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The classical Tannaka reconstruction theorem [25] allows one to recover a compact group G (up to isomorphism) from the monoidal category of finite dimensional representations of G over $\mathbb{C}$, $\mathrm{Rep}_{\mathbb{C}}(G)$, as the tensor preserving automorphisms of the forgetful functor $\mathrm{Rep}_{\mathbb{C}}(G) \longrightarrow \mathrm{Vec}_{\mathbb{C}}$. Now let G be a profinite group, K a finite extension of $\mathbb{Q}_p$ and $\mathrm{Ban}_G(K)$ the category of K-Banach space representations (of G). $\mathrm{Ban}_G(K)$ can be equipped with a tensor product bifunctor $(-)\hat{\otimes}_K(-)$ and has a forgetful functor $\omega : \mathrm{Ban}_G(K) \longrightarrow \mathrm{Ban}(K)$. Then, using an anti-equivalence of categories ([21, Thm 2.3]) between $\mathrm{Ban}_G(K)$ and the category of Iwasawa G-modules, we prove that a profinite group G can be recovered from $\mathrm{Ban}_G(K)$ as the group of tensor preserving automorphisms of ω, in particular $G \cong \mathrm{Aut}^{\otimes}(\omega)$.

Chapter 1

Introduction

Fix primes $p \neq l$ and a finite extension K of $\mathbb{Q}_p$. Interest in representations of p-adic Lie groups (groups with the structure of a manifold over a p-adic field) such as $GL_n(K)$ on nonarchimedean Banach spaces has been motivated in large part by developments in the local p-adic Langlands program for GL_n (for a survey of known results see [3]). Loosely stated, the local (l-adic) Langlands correspondence relates (isomorphism classes of) n-dimensional continuous representations ρ of $\mathrm{Gal}(\overline{K}/K)$ over $\overline{\mathbb{Q}_l}$ with (isomorphism classes of) irreducible smooth representations $\pi(\rho)$ of $GL_n(K)$. Via a (reversible) completion process due to Vigneras ([26]) one relates smooth representations $\pi(\rho)$ with unitary K-Banach space representations $\widehat{\pi(\rho)}$ of $GL_n(K)$. The goal of the p-adic local Langlands programme is to extend the above to the case $l = p$.

Going from l-adic to p-adic representations on the Galois side greatly increases the complexity of the representation theory. In an attempt to enrich the representation theory on the Banach side, Schneider and Teitelbaum, in a joint effort, began a systematic study ([24], [22], [21], [23]) of the continuous linear action of a topological group G on various kinds of topological vector spaces. In particular focusing on the cases of a profinite G such as compact p-adic Lie group $\mathrm{GL}_n(O)$; and locally profinite G such as the locally compact p-adic Lie group $\mathrm{GL}_n(K)$, or more generally a locally L-analytic group (where L is finite over $\mathbb{Q}_p$).

Locally convex vector spaces from nonarchimedean functional analysis provides a good technical framework for studying more general topological vector spaces. Smooth representations of V correspond to an action map $G \times V \longrightarrow V$ with the discrete topology on V. Considering more complex topologies on V facilitates the study of more subtle representations.

The simplest locally convex spaces are Banach spaces (complete vector spaces whose topology is defined by a single norm) and when endowed with a continuous G-action are called Banach space representations (of G). In [21] Schneider and Teitelbaum (extending upon a result by Schikhof [18, Thm 4.6]) prove that the category of Banach space representations of G (for a profinite group G) is anti-equivalent to the category of Iwasawa G-modules (flat, compact, linearly topological O-modules with a continuous (left) action of the Iwasawa algebra $\Lambda(G)$).

This result is certainly useful in allowing one to relate questions about Banach space representations to Iwasawa G-modules (and indeed this anti-equivalence of categories is an essential ingredient of the main recovery theorem in this dissertation), but application to the Langlands program is limited by the pathological nature of general Banach space representations. For example, there exist non-isomorphic topologically irreducible Banach space representations E and F of G for which nonetheless there exists a nonzero G-equivariant continuous linear map $E \longrightarrow F$. And even the simplest profinite groups such as $\mathbb{Z}_p$ have infinitely many infinite dimensional, topologically irreducible Banach space representations (Diarra constructs explicit examples in [9]).

Motivated by the need to impose additional finiteness conditions on $\mathrm{Ban}_G(K)$ to develop a more manageable theory, Schneider and Teitelbaum introduced the concept of an *admissible* Banach space representation ([21, §3]). Admissible Banach space representations correspond (via the anti-equivalence of categories) to finitely generated $K[\![G]\!]$-modules (where $K[\![G]\!] := K \otimes O[\![G]\!]$).

One of the key results in Lazard's seminal work on p-adic Lie groups ([15]) is that the

Iwasawa algebra $\Lambda(G)$ for a compact p-adic Lie group G is Noetherian. In particular it would follow that $K[\![G]\!]$ is also Noetherian and the category of finitely generated $K[\![G]\!]$-modules is abelian. Thus the category of admissible Banach space representations is also abelian.

The proof of this result uses several important constructions due to Lazard (in the case $O = \mathbb{Z}_p$). A p-valuation ω on G is a map $\omega : G \setminus \{1\} \longrightarrow (0, \infty)$ satisfying some technical conditions (cf [20, §23]). Extending ω to a map $\tilde{\omega}$ on $\mathbb{Z}_p[\![G]\!]$ one can can equip $\mathbb{Z}_p[\![G]\!]$ with an exhaustive filtration indexed by $m \in \mathbb{N}$:

$$\mathbb{Z}_p[\![G]\!]_{m/N} := \{\lambda \in \mathbb{Z}_p[\![G]\!] : \tilde{\omega}(\lambda) \geq m/N\}$$

(here $N \in \mathbb{N}$ is a common denominator of values of $\tilde{\omega}$ on $\mathbb{Z}_p[\![G]\!]$). The associated graded ring is isomorphic to a polynomial ring in finitely variables over a Noetherian ring and is therefore Noetherian. Applying a lifting result ([2, Chap. III §2.9 Cor. 2]) we deduce that $\Lambda(G)$ is itself also Noetherian.

There is already some evidence that the category of admissible Banach space representations may be suitable to developing a Langlands correspondence, for instance Colmez ([5]) has constructed topologically irreducible admissible unitary Banach space representations of $GL_2(\mathbb{Q}_p)$ that correspond to 2-dimensional irreducible Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

Although we focus entirely on Banach space representations, it is important to mention that much of the effort by Schneider and Teitelbaum has been directed towards developing the theory of locally analytic representations ([24], [22]). These represenions are given by a continuous action of G on V such that the orbit maps $g \longmapsto gv$ are locally analytic functions on G.

The work of this dissertation began by asking the question of whether it is possible to recover a profinite group G from its category of Banach space representations $\mathrm{Ban}_G(K)$ by exploiting the anti-equivalence of categories

$$\mathrm{Ban}_G(K)^{\leq 1} \xrightarrow{\sim} \mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$$

due to Schneider and Teitelbaum mentioned above.

This kind of recovery theorem goes back to the work of Tannaka and Krein in the 1930s and 1940s, and is part of a (quite general) duality principle known as Tannaka-Krein duality.

1.1 Tannaka-Krein duality

As a natural precursor to Tannaka-Krein duality, the theory of Pontryagin duality allows one to associate to a locally compact abelian group G the dual group $\hat{G}$ consisting of continuous characters of G (these are just the 1-dimensional unitary representations of G). Endowing $\hat{G}$ with the topology of uniform convergence one proves that $\hat{G}$ is a locally compact topological group and the canonical homomorphism

$$\mathrm{ev} : G \longrightarrow \hat{\hat{G}}$$

given by evaluation at $g \in G$ is an isomorphism. In this way G can be recovered from $\hat{G}$.

The first results in what is now called Tannaka-Krein duality were Tannaka's reconstruction theorem ([25]) in 1939 and Krein's recognition theorem ([14]) in 1949. As a generalization of Pontryagin duality to nonabelian (compact) groups Tannaka proved that one can recover a compact group (up to isomorphism), from its category of finite dimensional complex representations (with some additional structure). Krein's theorem gives the complementary result; characterizing the categories that arise as the representation category of a compact group G.

I will briefly outline Tannaka's classical result (see [13, §1] for a more detailed account). To generalize Pontryagin duality to nonabelian compact groups Tannaka realized that it becomes necessary to consider *all* finite dimensional representations of G (over $\mathbb{C}$), not just the 1-dimensional unitary ones. These do not form a group (as in the abelian case) but rather a category. Let $\mathrm{Rep}_{\mathbb{C}}(G)$ denote the category of finite dimensional representations of G over $\mathbb{C}$, these are pairs (V, ρ_V) where V is a finite dimensional complex vector space and $\rho_V : G \longrightarrow GL(V)$ is a group homomorphism.

$\mathrm{Rep}_{\mathbb{C}}(G)$ is a monoidal category with respect to the usual tensor product over $\mathbb{C}$ and forgetting the G-action on objects V in $\mathrm{Rep}_{\mathbb{C}}(G)$ gives us a functor

$$\omega : \mathrm{Rep}_{\mathbb{C}}(G) \longrightarrow \mathrm{Vec}_{\mathbb{C}}$$

The set of endomorphisms of ω (Definition A.5) can be endowed with a topology (the coarsest topology with respect to which the projections $\mathrm{End}(\omega) \longrightarrow \mathrm{End}(V)$ are all continuous) and there is a continuous natural map $\pi : G \longrightarrow \mathrm{End}(\omega)$ defined by multiplication by $g \in G$

$$\pi(g)_V = \rho_V(g)$$

Denoting the tensor preserving self-conjugate endomorphisms of ω by $\mathcal{T}(G)$ (self-conjugate here means $\lambda \in \mathrm{End}(\omega)$ satisfies $\overline{\lambda} = \lambda$ where $\overline{\lambda}$ is the complex conjugate of λ), the key observations are that $\mathcal{T}(G)$ is a closed *subgroup* of $\mathrm{End}(\omega)$ and every $\pi(g)$ is in $\mathcal{T}(G)$. In fact, π is an isomorphism.

Theorem 1.1 (Tannaka)**.** *For any compact group G*

$$\pi : G \longrightarrow \mathcal{T}(G)$$

is an isomorphism of topological groups

So G may be recovered from $\mathrm{Rep}_{\mathbb{C}}(G)$, but what if we are given an abstract (monoidal) category $\mathcal{C}$, what are necessary and sufficient conditions to recover a compact group G from $\mathcal{C}$? This question is answered by Krein's theorem.

Theorem 1.2 (Krein)**.** *Let $\mathcal{C}$ be a $\mathbb{C}$-linear monoidal category, equipped with a $\mathbb{C}$-linear faithful monoidal functor $\omega : \mathcal{C} \longrightarrow \mathit{Vec}_{\mathbb{C}}$.*

There is an equivalence of categories $\mathcal{C} \longrightarrow \mathit{Rep}_{\mathbb{C}}(G)$ if and only if the following conditions hold

1. *$\mathcal{C}$ is semisimple (every object can be decomposed as a coproduct of simple objects)*

2. *$hom_{\mathcal{C}}(X,Y)$ is either 1-dimensional (when $X \cong Y$) or $hom_{\mathcal{C}}(X,Y) = 0$*

It is important to remark that obviously this is a modern reformulation of the classical results that utilizes the language of category theory. The original results were approached from the point of view of classical harmonic analysis ([12, §30]).

The next major development in Tannaka-Krein duality theory can be attributed to Saavedra (and Grothendieck) in the 1970s ([17]). Working as a student of Grothendieck's, Saavedra with some considerable effort vastly generalized Tannaka-Krein duality to recover a pro-algebraic group (a pro-algebraic group is an inverse limit of algebraic groups), from its category of representations. To this end Saavedra defined the notion of a *rigid* category; a technical condition that at a key step in the proof ensures that $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is represented by an affine *group* scheme, rather than a monoid (rigidity is carefully defined in [7]), and a *Tannakian* category (although his original definition misses a necessary non-degeneracy condition $k = \mathrm{End}(\mathbf{1})$).

Deligne and Milne ([7]) noticed the errors in Saavedra's original work, correctly redefined the notion of a Tannakian category and reproved the required results in the ”neutral” case ([7]).

Definition 1.1. A (neutral) Tannakian category over k is a rigid abelian tensor category $(\mathcal{C},\otimes)$ such that $k = \mathrm{End}(\mathbf{1})$ for which there exists an exact faithful k-linear tensor functor $\omega : \mathcal{C} \longrightarrow \mathrm{Vec}_k$. Any such functor is called a fiber functor for $\mathcal{C}$.

It is possible to recover an affine group scheme G from its category of finite dimensional representations.

Theorem 1.3 (Saavedra, Deligne, Milne)**.** *Let G be an affine group scheme over k and ω be the forgetful functor $\omega : Rep_k(G) \longrightarrow Vec_k$. The natural map $G \longrightarrow \underline{Aut}^{\otimes}(\omega)$ is an isomorphism of functors (of k-algebras).*

As before there is a recognition theorem that characterizes the categories that arise as the category of representations of an affine group scheme. These are precisely the (neutral) Tannakian categories.

Theorem 1.4 (Saavedra, Deligne, Milne)**.** *Let* $(\mathcal{C},\otimes)$ *be a (neutral) Tannakian category with fibre functor* $\omega : \mathcal{C} \longrightarrow Vec_k$*, then:*

1. *The functor* $\underline{Aut}^{\otimes}(\omega)$ *of k-algebras is represented by an affine group scheme G;*

2. *The functor* $\mathcal{C} \longrightarrow Rep_k(G)$ *defined by* ω *is an equivalence of tensor categories*

It took almost 10 years for the main results to be extended to the much more general non-neutral case by Deligne ([6]). Since then Tannaka-Krein duality has been proven to hold in a large number of cases, and the subject has enjoyed a resurgence in popularity.

Tannakian formalism has allowed complex algebraic objects to be constructed out of perhaps relatively simpler (Tannakian) categories. Applying this duality to the category of motives, one constructs Grothendieck's motivic Galois group. Tannaka-Krein duality has been successfully applied to the study and construction of quantum groups ([27]). And there is perhaps even some hope of constructing the (conjectural) Langlands groups with Tannaka-Krein duality, although this is very much still hopeful.

1.2 Results in this dissertation

Having made a case for why Banach space representations are interesting and important objects to study and provided some historical context, let us set up and state the main results of this dissertation.

Let G be a profinite group (compact, Hausdorff and totally disconnected topological group) and K a finite extension of $\mathbb{Q}_p$. Denoting the categories of K-Banach spaces and K-Banach space representations of G by $\mathrm{Ban}(K)$ and $\mathrm{Ban}_G(K)$ respectively, we have a forgetful functor (forgetting the G-action structure):

$$\omega : \mathrm{Ban}_G(K) \longrightarrow \mathrm{Ban}(K)$$

Given K-Banach spaces E and F it is possible to endow the abstract K-vector space $E \otimes_K F$ with a norm (cf section 3.3), denoted $\|\|_{E\otimes F}$. Taking the completion with respect to this norm allows one to define a completed tensor product $\hat{\otimes}_K$ in the category $\mathrm{Ban}(K)$ (with no G-action). By defining a suitable G-action on $E \otimes_K F$ this definition extends to $\mathrm{Ban}_G(K)$.

The goal of my research was to prove a Tannaka type reconstruction result and recover G from the data $(\mathrm{Ban}_G(K), \hat{\otimes}_K, \omega)$. The basic idea is straightforward; following the constructions in other cases, one would expect to recover G as the tensor preserving automorphisms of ω. The hope was we can prove there is an isomorphism of topological groups:

$$G \xrightarrow{\cong} \mathrm{Aut}^{\otimes}(\omega)$$

There did not seem to be a clear way to proceed in $\mathrm{Ban}_G(K)$ directly, but as mentioned in the introduction we can use an anti-equivalence of categories ([21, Thm 2.3]):

$$\mathrm{Ban}_G(K)^{\leq 1} \xrightarrow{\sim} \mathrm{Mod}^{fl}_{cpt}(\Lambda(G))$$

to transfer the problem of recovering G to $\mathrm{Mod}^{fl}_{cpt}(\Lambda(G))$, the category of Iwasawa G-modules (cf section 5.1), where it would perhaps be easier to solve.

We interpret the aforementioned anti-equivalence as factoring through an intermediate category $\mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$ introduced by Emerton ([10, §2.4]).

$$\mathrm{Ban}_G(K)^{\leq 1} \underset{(-)\otimes K}{\overset{(-)^{\circ}}{\rightleftarrows}} \mathrm{Mod}_G^{\varpi-cts}(O)^{fl} \underset{(-)^d}{\overset{(-)^d}{\rightleftarrows}} \mathrm{Mod}^{fl}_{cpt}(\Lambda(G))$$

To make progress towards the recovery theorem a tensor product is defined for each category in a way that is compatible with the equivalences between them. The real technical challenge here was with $\mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$. Making use of a completed tensor product construction due to Brumer ([4]) we first define the tensor product of objects M and N in the simpler category $\mathrm{Mod}_{cpt}^{fl}(O)$ by:

$$M\hat{\otimes}_O N := \varprojlim_{M',N'} M/M' \otimes_O N/N'$$

Each object in $\mathrm{Mod}_{cpt}^{fl}(O)$ comes equipped with a fundamental family of open neighborhoods of 0 by O-submodules. Here the inverse limit is taken over all the fundamental open O-submodules $M' \subseteq M$ and $N' \subseteq N$ respectively.

Since the open submodules M' and N' are not assumed to have a $\Lambda(G)$-module structure it difficult to define a $\Lambda(G)$-action on $M\hat{\otimes}_O N$ directly. This issue is tackled in section 5.2.3 by explicitly constructing G-stable submodules in every open O-submodule $M' \subseteq M$, then extending the resulting $O[G]$-action to a $\Lambda(G)$-action structure. Combined with a $\hat{\otimes}_O$-coalgebra structure we construct on $\Lambda(G)$

$$c : \Lambda(G) \longrightarrow \Lambda(G)\hat{\otimes}_O \Lambda(G)$$

we can define a $\Lambda(G)$-action on $M\hat{\otimes}_O N$, and thus $(-)\hat{\otimes}_O(-)$ is a completed tensor product (bifunctor) on $\mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$ as required. To make any use of the anti-equivalence of categories between $\mathrm{Ban}_G(K)^{\leq 1}$ and $\mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$ we prove two key compatibility results (cf section 5.3)

$$(E\hat{\otimes}_K F)^\circ \cong E^\circ \hat{\otimes}_O F^\circ \quad \text{and} \quad (M\hat{\otimes}_O N)^d \cong M^d \hat{\otimes}_O N^d$$

We relate the tensor preserving automorphisms of the forgetful functors ω_1 on $\mathrm{Ban}_G(K)^{\leq 1}$ and ω_3 on $\mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$ by proving an isomorphism (cf section 6.2.1)

$$\mathrm{Aut}^{\otimes}(\omega_1) \cong \mathrm{Aut}^{\otimes}(\omega_3)$$

which allows us to transfer the recovery problem to $\mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$. Seemingly unrelated, G coincides with the group-like elements of $\Lambda(G)$, denoted $\Lambda(G)^{gp}$

$$G \underset{\text{Lemma 5.5}}{=} \Lambda(G)^{gp}$$

The key lemma in chapter 6 relates (tensor preserving) automorphisms of ω_3 with group-like elements of $\Lambda(G)$.

Lemma (Key Lemma)**.**

$$Aut^{\otimes}(\omega_3) \cong \Lambda(G)^{gp} \tag{1.1}$$

is a continuous isomorphism of topological groups

Combining all this together we prove the main result:

Theorem 1.5 (Main Result)**.** *Let G be a profinite group, and $Ban_G(K)$ be the category of K-Banach space representations, with tensor product bifunctor $\hat{\otimes}_K$ and a forgetful functor ω (that forgets the G-action). There is a continuous isomorphism of topological groups:*

$$Aut^{\otimes}(\omega) \cong G$$

As a corollary of this result we give a mild classification result for Iwasawa algebras. By definition $\Lambda(G)$ is the completed group ring of G over the ring of integers O, defined (cf section 4.1) as the following inverse limit of group rings:

$$\Lambda(G) = O[\![G]\!] := \varprojlim_{N} O[G/N]$$

The O-algebra $\Lambda(G)$ plays a role analogous to the group ring of a finite group and can be seen as natural generalization that takes into account the topology of G. Indeed for finite groups (with the discrete topology) we have $\Lambda(G) = O[G]$.

The following remained a conjecture for a long time in the theory of finite groups.

Conjecture 1.1. *Let G and H be finite groups. If* $\mathbb{Z}[G] \cong \mathbb{Z}[H]$*, then* $G \cong H$

This problem had been studied extensively, and proven to hold in a large number of cases before a counterexample to the general statement was constructed in [11]. Being interested in profinite groups and Iwasawa algebras, a natural question one may ask is to what extent is a profinite group G determined by $\Lambda(G)$? More precisely, what conditions must we place on $\Lambda(G)$ and $\Lambda(H)$ to guarantee $G \cong H$?

Being isomorphic as algebras is not enough since this is not a sufficient condition even for finite G. Instead we prove:

Corollary 1.1. *Let G, H be two profinite groups such that there exists a topological isomorphism of O-algebras* $\varphi : \Lambda(G) \longrightarrow \Lambda(H)$ *that is compatible with the* $\hat{\otimes}_O$*-coalgebra structures of* $\Lambda(G)$ *and* $\Lambda(H)$*, denoted* c_1 *and* c_2 *respectively, such that the following diagram commutes:*

$$\begin{array}{ccc} \Lambda(G)\hat{\otimes}_O\Lambda(G) & \xrightarrow{\varphi\otimes\varphi} & \Lambda(H)\hat{\otimes}_O\Lambda(H) \\ \uparrow c_1 & & \uparrow c_2 \\ \Lambda(G) & \xrightarrow[\varphi]{} & \Lambda(H) \end{array}$$

Then $G \cong H$.

1.3 Summary of chapters

Chapter 2 gives is a quick summary of the basic definitions and constructions from nonarchimedean functional analysis we will need in this dissertation. Nonachimedean fields, normed spaces (in particular Banach spaces) are defined and some of their basic properties are given. We briefly discuss some duality theory from nonarchimedean functional analysis, in particular the lack of a reflexive property for (infinite dimensional) Banach spaces over spherically complete fields.

In chapter 3 and 4 we introduce the categories: $\mathrm{Ban}(K)$, $\mathrm{Ban}(K)^{\leq 1}$, $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$ and $\mathrm{Mod}^{fl}_{cpt}(O)$, defining a tensor product in each category and the functors between them. Important compatibility results are proven in these chapters (cf section 3.4 and 4.4).

In chapter 5 we extend results from previous chapters to new categories $\mathrm{Ban}_G(K)^{\leq 1}$, $\mathrm{Mod}^{\varpi-cts}_G(O)^{fl}$ and $\mathrm{Mod}^{fl}_{cpt}(\Lambda(G))$ whose objects are endowed with a continuous action by a profinite group G. Tensor products in each category are defined in section 5.2 and in section 5.3 the compatibility results of chapter 3 and 4 are generalized to these new categories.

Finally in chapter 6 we prove the main result Theorem 1.5. First, using the work from chapter 5 we translate the problem from $\mathrm{Ban}_G(K)^{\leq 1}$ to $\mathrm{Mod}^{fl}_{cpt}(\Lambda(G))$ (cf section 6.2.1). Then we define a map from the "group-like" elements in $\Lambda(G)$ to automorphisms of the forgetful functor ω_3 on $\mathrm{Mod}^{fl}_{cpt}(\Lambda(G))$ and prove it is an isomorphism (cf section 6.2.2). At the end of section 6.2 we put everything together and finish off the proof of Theorem 1.5. We finish the chapter with a corollary of the main recovery theorem and tentative suggestions for future research in this topic.

1.4 Notation

Throughout K will be a finite extension of $\mathbb{Q}_p$ for a fixed prime p. The ring of integers in K will be denoted O and will have ϖ as a distinguished uniformizer. In particular the natural profinite topology on O is ϖ-adic.

G will be a fixed profinite group and $O[[G]]$ is the Iwasawa algebra (or completed group ring) of G over O (cf section 4.1) which we will often denote by $\Lambda(G)$.

1.5 Category definitions

1.5.1 Categories without G

$\mathrm{Ban}(K)$, the category of K-Banach spaces with continuous linear maps as morphisms.

$\mathrm{Ban}(K)^{\leq 1}$, the category of K-Banach spaces $(E, \|\|)$ satisfying $\|E\| \subseteq |K|$, with continuous norm decreasing (satisfying $\|f\| \leq 1$ with respect to the operator norm) linear maps as morphisms.

$\mathrm{Mod}^{\varpi-cts}(O)^{fl}$, the category of flat ϖ-adically complete and separated O-modules, with O-linear maps as morphisms.

$\mathrm{Mod}^{fl}_{cpt}(O)$, the category of flat, compact and linearly topological O-modules, with continuous O-linear maps as morphisms.

1.5.2 Categories with G

(See section 5.1 for detailed definitions)

$\mathrm{Ban}_G(K)$, the category of K-Banach space representations of G, with G-equivariant continuous linear maps as morphisms.

$\mathrm{Ban}_G(K)^{\leq 1}$, the category of K-Banach space representations $(E, \|\|)$ satisfying $\|E\| \subseteq |K|$, with G-equivariant norm-decreasing linear maps as morphisms.

$\mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$, the category of flat ϖ-adically continuous G-representations, with G-equivariant O-linear maps as morphisms.

$\mathrm{Mod}^{fl}_{cpt}(\Lambda(G))$, the category of Iwasawa G-modules over O, with continuous G-equivariant O-linear maps as morphisms.

Chapter 2

Nonarchimedean functional analysis

In this chapter we begin with a brief summary of the important definitions, results, and constructions from nonarchimedean functional analysis that will prove useful later. The full details can be found in the excellent reference [19].

2.1 Nonarchimedean fields

Let K be any field.

Definition 2.1. A nonarchimedean absolute value (on K) is a function $|\ | : K \longrightarrow \mathbb{R}$ satisfying for any a, b in K

1. $|a| \geq 0$
2. $|a| = 0$ if and only if $a = 0$
3. $|ab| = |a| \cdot |b|$
4. $|a+b| \leq \max(|a|, |b|)$

To avoid pathological examples we will only consider *non-trivial* absolute values (those for which there exists some $x \in K$ such that $|x| \neq 0, 1$).

Example 2.1 (The p-adic absolute value on $\mathbb{Q}$)**.** Any rational x can be uniquely written as $x = p^{-n}\frac{a}{b}$ where $p \nmid ab$ and $n \in \mathbb{Z}$, then the absolute value defined by:

$$\left|p^{-n}\frac{a}{b}\right|_p := p^n$$

is nonarchimedean.

The only difference between the regular definition of an absolute value and Definition 2.1 is property 4 (the *strict* triangle inequality). It is this property in particular that is responsible for the terminology *nonarchimedean*. It is an easy consequence of the definition that $|n \cdot 1_K| \leq |1_K| = 1$ holds for all $n \in \mathbb{N}$, thus p-adic fields do not satisfy the Archimedean property:

$$x < y \implies \exists n \in \mathbb{N} \text{ such that } n \cdot x > y$$

The strange topological properties of p-adic fields and spaces over p-adic fields are all due to the strict triangle inequality.

Definition 2.2. A field K is called *nonarchimedean* if it is equipped with a nonarchimedean absolute value $|\ |$ such that the corresponding metric space is complete.

Examples.

1. The field of p-adic numbers $\mathbb{Q}_p$ with (the extension of) the p-adic absolute value is a nonarchimedean field. More generally, given K a finite extension of $\mathbb{Q}_p$ (say $[K : \mathbb{Q}_p] = n$), the absolute value $|\ |_p$ extends (uniquely) to K and is given by

$$|x|_K = \left(\left|\mathrm{Norm}_{K/\mathbb{Q}_p}(x)\right|_p\right)^{\frac{1}{n}}$$

   $(K, |\ |_K)$ is a nonarchimedean field

2. Taking the union over finite degree extensions of all degrees we can extend the p-adic absolute value uniquely to the algebraic completion $\overline{\mathbb{Q}_p}$, which will in general not be complete

(as a metric space). The completion $\widehat{\overline{\mathbb{Q}_p}}$ (sometimes denoted $\mathbb{C}_p$) is a nonarchimedean field that is not locally compact.

3. Let q be a power of p. The field of Laurent series $\mathbb{F}_q((T))$ over the finite field $\mathbb{F}_q$ with the absolute value given by
$$\left|\sum_{k=-m}^{\infty} a_k T^k\right| := q^m$$
is a nonarchimedean field with positive characteristic.

The metric $|x-y|$ induces a topology with respect to which addition and multiplication in K are continuous maps, thus K is a topological field. For any $a \in K$ and $\varepsilon > 0$ the subsets

$$B_\varepsilon(a) := \{x \in K : d(x,a) \leq \varepsilon\} \quad \text{and} \quad B_\varepsilon^-(a) := \{x \in K : d(x,a) < \varepsilon\}$$

are respectively called closed and open balls in K. As a consequence of the strict triangle inequality $B_\varepsilon(a)$ and $B_\varepsilon^-(a)$ are both open and closed. This means that for any fixed $a \in K$ both the open and closed balls will form a fundamental system of (open) neighborhoods of $a \in K$.

The (closed) unit ball $O := B_1(0)$ is called the *ring of integers* of K. This is a ring due to the strict triangle inequality, and has the (open) unit ball as an ideal $\mathfrak{m} := B_1^-(0)$. O is a principal ideal domain with unique maximal ideal $\mathfrak{m}$ for which we fix ϖ as a generator (called a *uniformizer* of K). The quotient $k = O/\mathfrak{m}$ is called the *residue field* of K (k is always finite in our case). O has a canonical topology given by powers of the ideal $\mathfrak{m} = \varpi O$ called the ϖ-adic topology. Topologically O is compact, Hausdorff and totally disconnected, in particular O is a profinite ring:
$$O \cong \varprojlim_n (O/\mathfrak{m}^n)$$

In general the topological properties of a nonarchimedean field K can be quite varied. Often a stronger completeness condition is required for results to hold. In particular the Hahn-Banach Theorem (Theorem 2.1) only holds for vector spaces over *spherically complete* fields.

Definition 2.3. A field K is called spherically complete if any decreasing sequence of balls $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ in K has non-empty intersection.

Our primary interest will be the case when K is a finite extension of $\mathbb{Q}_p$. In this case K is both locally compact and discretely valued (meaning $|K^\times|$ is a discrete subgroup of $(\mathbb{R}^{>0}, \times)$). Both of these conditions imply spherical completeness, whereas $\mathbb{C}_p$ is an example of a non spherically complete field.

2.2 Locally convex topologies and normed vector spaces

From now on K will always mean a nonarchimedean field with absolute value $|\ |$ as in the previous section.

Definition 2.4. A subset A of a vector space V is called convex if it is of the form $A = v + A_0$ where v is a vector in V and A_0 is an O-submodule of V.

Recall that we call a vector space *topological* if it is endowed with a topology with respect to which addition and scalar multiplication are continuous. As we will see shortly the topology of a locally convex vector space is defined by a family of special O-submodules called lattices (hence open sets are convex according to the above definition).

This is quite a general class of topological vector spaces and although we will not need the full flexibility that locally convex topologies allow for, many of the concepts and constructions are best stated in general terms (although simplifications will be made wherever possible).

Definition 2.5. A *lattice* L in a K-vector space V is an O-submodule with the property that for any $v \in V$ there exists $a \in K^\times$ such that $av \in L$.

A lattice $L \subseteq V$ can be characterized by the fact that the natural map $K \otimes_O L \longrightarrow V$ given by $a \otimes v \longmapsto av$ (which is injective for any O-submodule) is a bijection.

Definition 2.6. Let V be a K-vector space. A topology on V that is defined by a family of lattices $\{L_j\}_{j\in J}$ satisfying:

1. for any $j \in J$ and $a \in K^\times$ there exists a $k \in J$ such that $L_k \subseteq aL_j$

2. for any i, j in J there exists $k \in J$ such that $L_k \subseteq L_i \cap L_j$

is called the locally convex topology on V (defined by $\{L_j\}_{j\in J}$).

A topological vector space endowed with a locally convex topology is called a locally convex vector space, or sometimes just locally convex space.

One can show easily that addition and scalar multiplication are continuous with respect to any locally convex topology, so any locally convex vector space is topological. We are especially interested in locally convex topologies that arise from a single norm.

Definition 2.7. Let V be a vector space over a nonarchimedean field K . A (nonarchimedean) norm on V is a function $\|\| : V \longrightarrow \mathbb{R}$ satisfying for any v, w in V and a in K:

1. $\|av\| = |a|\|v\|$

2. $\|v+w\| \leq \max(\|v\|, \|w\|)$

3. $\|v\| = 0$ if and only if $v = 0$

A vector space endowed with a norm is called a normed vector space.

Remark 2.1. A function (typically denoted q) that only satisfying properties 1 and 2 is called a (nonarchimedean) *seminorm*. We will not need to discuss seminorms much but it is important to at least mention the natural correspondence between lattices and seminorms. Given a collection of seminorms $\{q_i\}_{i\in I}$ on V, for any finite subset $q_{i_1}, \cdots, q_{i_k}$ and $\varepsilon > 0$ the O-module

$$V(q_{i_1}, \cdots, q_{i_k}, \varepsilon) := \{v \in V : q_{i_1}(v), \cdots, q_{i_k}(v) \leq \varepsilon\}$$

is a lattice. On the other hand, given a lattice L we can define a seminorm called a gauge, associated to L (Definition 2.11). One can equivalently define a locally convex topology on V purely in terms of seminorms.

As with fields, we will always consider a normed space $(V, \|\|)$ to be a metric space with respect to $d(v,w) := \|v-w\|$. In this way $(V, \|\|)$ is a Hausdorff (by property 2) topological space. For any $v \in V$ and $\varepsilon > 0$

$$B_\varepsilon(v) := \{w \in V : \|v-w\| \leq \varepsilon\} \quad \text{and} \quad B_\varepsilon^-(v) := \{w \in V : \|v-w\| < \varepsilon\}$$

are respectively called closed and open balls in V, as before $B_\varepsilon(v)$ and $B_\varepsilon^-(v)$ are both open and closed. Both the open and closed balls will form a fundamental system of (open) neighborhoods of $v \in V$. It should be immediately clear that $B_\varepsilon(0)$ (and $B_\varepsilon^-(0)$) are lattices in V and one checks easily that they satisfy the requirements of a locally convex topology, thus any normed vector space is a locally convex vector space.

Definition 2.8. A normed K-vector space is called a K-Banach space if the corresponding metric space is complete.

Remark 2.2. We will not consider a Banach space E to be equipped with any particular norm defining its topology, when it is necessary to fix a norm we will write $(E, \|\|)$.

Examples. 1. K^n with the norm $\|(a_1, \cdots, a_n)\| = \max_i |a_i|$

2. Let X be any set and denote the set of bounded functions $f : X \longrightarrow K$ by $l^\infty(X)$. This is naturally a K-vector space with pointwise addition and scalar multiplication, but may be endowed with a norm:

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

$(l^\infty(X), \|\|_\infty)$ is a K-Banach space.

Certain closed subspaces of $l^\infty(X)$ give other important examples of Banach spaces:

(a) $c_0(X) := \{f \in l^\infty(X) : \forall \varepsilon > 0 \; |f(x)| < \varepsilon \text{ for almost all } x\}$

(b) Assuming X is compact, $C(X,K) := \{f \in l^\infty(X) : f \text{ is continuous}\}$

Definition 2.9. Let V be a vector space, two norms $\|\|_1$ and $\|\|_2$ on V are said to be *equivalent* if there exist constants $c_1, c_2 > 0$ such that

$$c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_1 \text{ for all } v \in V$$

Remark 2.3. Equivalent norms $\|\|_1$ and $\|\|_2$ on V define equivalent topologies.

The following definition generalizes the notion of bounded sets to locally convex vector spaces. It can be used to intrinsically characterize the locally convex vector spaces with a topology defined by a single norm.

Definition 2.10. A subset $B \subseteq V$ is called *bounded* if for any open lattice $L \subseteq V$ there exists $a \in K$ such that $B \subseteq aL$

It should be clear that if $(V, \|\|)$ is a normed space we recover the usual definition of a boundedness (i.e. $\sup_{x \in B} \|x\| < \infty$)

The notion of a gauge associated to a lattice will be important in later sections.

Definition 2.11. Let L be a lattice in V, then L induces a *gauge* seminorm on V:

$$\begin{aligned} p_L : V &\longrightarrow \mathbb{R} \\ v &\longmapsto \inf_{v \in aL} |a|_K \end{aligned} \tag{2.1}$$

Remark 2.4. If L is an open bounded lattice in a Banach space V, then p_L is a norm (since L is bounded) and so we denote it by $\|\|_L$. The topology on V induced by $\|\|_L$ is equivalent to the given topology of V because L is open. In fact the topology of a Haudorff locally convex vector space V can be defined by a single norm if and only if there exists a bounded open lattice $L \subseteq V$.

2.3 Duality theory

In this section we state some of the basic results in the rich duality theory of locally convex vector spaces. We will see that given a normed space V, the Hahn-Banach theorem guarantees that the dual space V' is nonzero and (by Prop. 2.2) is even a Banach space. Unfortunately Prop 2.3 dispels any hope of a naïve isomorphism $V \cong (V')'$ as for vector spaces.

Proposition 2.1. *Let $(V, \|\|_1)$, $(W, \|\|_2)$ be two normed vector spaces. A linear map $f : V \longrightarrow W$ is continuous if and only if there exists $c \geq 0$ such that $\|f(v)\|_2 \leq c\|v\|_1$ for any $v \in V$*

Proof. [19, Prop 3.1] □

Remark 2.5. A continuous f satisfying $\|f(v)\|_2 \leq \|v\|_1$ (i.e. $c \leq 1$) is called *norm decreasing*.

Let V, W be as above. The set of all continuous K-linear maps $f : V \longrightarrow W$ form a vector subspace of $Hom_K(V,W)$ and is denoted $\mathcal{L}(V,W)$.

Proposition 2.2. *$\mathcal{L}(V,W)$ is a normed K-vector space with respect to the operator norm*

$$\|f\| := \sup \left\{ \frac{\|f(v)\|_2}{\|v\|_1} : v \in V \setminus \{0\} \right\}$$

Additionally, if W is a Banach space then so is $\mathcal{L}(V,W)$.

Remark 2.6. If $(V, \|\|)$ is a K-Banach space such that $\|V\| \subseteq |K|$ then the operator norm on $\mathcal{L}(V,W)$ is given by

$$\|f\| = \sup_{\|v\|=1} \|f(v)\|_W$$

Given any K-normed space V it follows from the above discussion that the vector space $V' := \mathcal{L}(V,K)$ endowed with the operator norm is a Banach space (since K is complete and thus a Banach space with norm $|\ |$), called the dual Banach space to V. It is not obvious *a priori* that there exist any nonzero continuous linear forms on V. Fortunately over spherically complete fields one can use the Hahn-Banach theorem to construct nonzero continuous linear forms and so the dual of a (nonzero) Banach space is always nonzero.

Theorem 2.1 (Hahn-Banach)**.** *Let K be a spherically complete field, $(U, \|\|)$ a normed K-vector space and $U_0 \subseteq U$ a vector subspace. Given any linear form $l_0 : U_0 \longrightarrow K$ such that $|l_0(v)| \leq \|v\|$ for all $v \in U_0$, there exists a linear form $l : U \longrightarrow K$ such that $l|_{U_0} = l_0$ and $|l(v)| \leq \|v\|$ for all $v \in U$.*

For two locally convex vector spaces V and W there is a general procedure to endow the vector space $\mathcal{L}(V,W)$ with various locally convex topologies. Let $\mathcal{B}$ be a fixed family of bounded subsets of V. For a given $B \in \mathcal{B}$ and open lattice $M \subseteq W$ the set

$$\mathcal{L}(B,M) := \{f \in \mathcal{L}(V,W) : f(B) \subseteq W\}$$

is a lattice in $\mathcal{L}(V,W)$ and the family of lattices

$$\{\mathcal{L}(B,M) : B \in \mathcal{B},\ M \subseteq W \text{ open lattice}\}$$

defines a locally convex topology on $\mathcal{L}(V,W)$. We let $\mathcal{L}_{\mathcal{B}}(V,W)$ denote the associated locally convex vector space.

There are two common topologies that are put on V' corresponding to different choices for the set $\mathcal{B}$:

1. The locally convex topology induced by $\mathcal{B} = \{\text{all one point subsets of } V\}$ is called the *weak* topology. V' endowed with the weak topology is called the *weak dual* of V and is denoted V'_s.

2. The locally convex topology induced by $\mathcal{B} = \{\text{all bounded subsets of } V\}$ is called the *strong* topology. V' endowed with the strong topology is called the *strong dual* of V and is denoted V'_b.

Remark 2.7. The topology on V_b' is equivalent to one defined by the operator norm.

As a consequence of Remark 2.7 (at least for Banach spaces) the strong dual is the more interesting of the two and one may hope that the duality map

$$\begin{aligned} \delta : V &\longrightarrow (V_b')_b' \\ v &\longmapsto \delta_v(l) := l(v) \end{aligned} \tag{2.2}$$

extends to an endofunctor on some category of Banach spaces. The duality map δ is always a topological isomorphism onto its image. This motivates the following definition.

Definition 2.12. A Banach space is called *reflexive* if the duality map δ is a topological isomorphism

One could completely recover a reflexive Banach space V from its strong dual V_b'. Unfortunately the situation for infinite dimensional K-Banach spaces is a little more complicated.

Proposition 2.3. *Suppose that K is spherically complete and that V is a K-Banach space; then V is reflexive if and only if V is finite dimensional.*

Proof. [19, Prop 11.1] □

So over spherically complete fields K, infinite dimensional K-Banach spaces are never reflexive. As disappointing as this result is, there are two ways in which the situation can be salvaged.

Firstly we can drop the insistence on V being a Banach space and instead consider locally convex vector spaces *of compact type*. These are vector spaces V that are (locally convex) inductive limits of Banach spaces V_n

$$V = \varinjlim_n V_n$$

with transition maps $i_n : V_n \longrightarrow V_{n+1}$ that are both injective and *compact* (i_n is called compact if there exists an open lattice $U \subseteq V_n$ such that $i_n(U)$ is compact in V_{n+1}).

Despite the additional complexity inherent in their definition, vector spaces of compact type have a well behaved duality theory, in particular both V and its strong dual V_b' are reflexive. Moreover the strong dual V_b' is the (topological) projective limit of $(V_n)_b'$

$$V_b' = \varprojlim_n (V_n)_b'$$

A countable projective limit of Banach spaces is complete, but will not in general have a topology defined by a single norm, but rather a countable number of seminorms. Such locally convex vector spaces are called Fréchet spaces and the Fréchet spaces that occur as strong duals of vector spaces of compact type can be characterized by the property of being nuclear (a complicated finiteness condition [19, Ch. IV]). The following proposition summarizes the above as an equivalence of categories.

Proposition 2.4. *The functor $V \longmapsto V_b'$ is an anti-equivalence between the category of vector spaces of compact type and the category of nuclear Fréchet spaces.*

The second way one may get around the lack of reflexivity for infinite dimensional Banach spaces is to consider a different functor in place of the strong dual $(-)_b'$. In particular by restricting to the unit ball $((E)_b')^\circ \subseteq (E)_b'$ and endowing the resulting O-module with the topology of pointwise convergence we induce an anti-equivalence of categories

$$\begin{aligned} \mathrm{Ban}(K) &\longrightarrow \mathrm{Mod}_{cpt}^{fl}(O) \\ E &\longmapsto (E_b')^\circ \text{ with pointwise topology} \end{aligned}$$

between the categories of K-Banach spaces and flat, compact, and linearly topological modules over O (Schikhof [18]). Schneider and Teitelbaum extend this result to an anti-equivalence between K-Banach space representations of G and Iwasawa G-modules ([21]). These results are central in what follows and so we postpone their discussion until Section 4.3.

Chapter 3

K-Banach spaces and the unit ball functor

In this chapter we introduce an important category of K-Banach spaces E satisfying $\|E\| \subseteq |K|$, that have norm-decreasing continuous linear maps as morphisms. Denoting this category by $\mathrm{Ban}(K)^{\leq 1}$, we prove that the functor

$$(-)^{\circ} : (E, \|\|) \longmapsto \{e \in E : \|e\| \leq 1\} =: E^{\circ}$$

induces an equivalence of categories between $\mathrm{Ban}(K)^{\leq 1}$ and the category of ϖ-adically complete, separated, and torsion-free O-modules. Going further, we define a tensor product in each category in such a way that it is compatible with this equivalence.

3.1 Two categories of Banach spaces

Our natural starting point is $\mathrm{Ban}(K)$, the category of K-Banach spaces with continuous linear maps as morphisms. As we mentioned in Remark 2.2 we do not consider a specified norm to be part of the structure of a given Banach space in $\mathrm{Ban}(K)$, so we typically denote these objects by E.

Closely related to $\mathrm{Ban}(K)$ is the category $\mathrm{Ban}(K)^{\leq 1}$ of K-Banach spaces $(E, \|\|)$ that satisfy $\|E\| \subseteq |K|$, with norm decreasing linear maps as morphisms. The objects in this category do come equipped with a specified norm and have an important structural result ([19, 10.2]).

Proposition 3.1. *Every K-Banach space $(E, \|\|)$ such that $\|E\| \subseteq |K|$ is isometrically isomorphic to a K-Banach space $(c_0(X), \|\|_\infty)$ for some set X.*

Given an object $(E, \|\|)$ in $\mathrm{Ban}(K)^{\leq 1}$ we can simply forget the specified norm to define a functor

$$\begin{aligned} \text{forget} : \mathrm{Ban}(K)^{\leq 1} &\longrightarrow \mathrm{Ban}(K) \\ (E, \|\|) &\longmapsto E \end{aligned} \tag{3.1}$$

On the other hand let E be an object in $\mathrm{Ban}(K)$ and pick a $\|\|$ norm that defines its topology. We can construct another norm $\|\|'$ on E in the following way

$$\|e\|' := \inf\{r \in |K| : r \geq \|e\|\}$$

that satisfies $\|E\|' \subseteq |K|$ and is equivalent to $\|\|$ since $|\varpi| \leq \|v\|/\|v\|' \leq 1$.

The main difference between these two categories is in their morphism sets between objects. For any two Banach spaces E and F, the set of morphisms in $\mathrm{Ban}(K)$, which we denote by $\mathcal{L}(E,F)$, has the structure of a K-vector space. In fact, as we discussed in chapter 2, endowed with the operator norm $\|\|$ $\mathcal{L}(E,F)$ is itself a K-Banach space.

It is immediate from the definitions that a continuous linear map $f : E \longrightarrow F$ is norm decreasing if and only if $\|f\| \leq 1$. This means that the norm decreasing linear maps are in the unit ball $\{f \in \mathcal{L}(E,F) : \|f\| \leq 1\}$. Since $\|\lambda f\| = |\lambda| \cdot \|f\|$ this is an O-module, but if we tensor with $\mathbb{Q}$ we can recover $\mathcal{L}(E,F)$. To summarize, the functor (3.1) induces an equivalence of categories:

$$(\mathrm{Ban}(K)^{\leq 1})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Ban}(K)$$

between $\mathrm{Ban}(K)$ and $\mathrm{Ban}(K)^{\leq 1}$ localized at $\mathbb{Q}$ (cf. Definition A.2).

As a consequence we will always be able to recover the full category of K-Banach spaces whenever necessary and otherwise we will mostly work in the category $\mathrm{Ban}(K)^{\leq 1}$.

3.2 An equivalence between $\mathbf{Ban}(K)^{\leq 1}$ and $\mathbf{Mod}^{\varpi-cts}(O)^{fl}$

As mentioned in the introduction, one of the essential ingredients in the proof of the main recovery theorem is an anti-equivalence of categories

$$\mathrm{Ban}_G(K) \xrightarrow{\sim} \mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$$

between K-Banach space representations of G and Iwasawa G-modules (flat, compact, and linearly topological O-modules with a continuous action of the completed group ring $\Lambda(G)$, see section 5.1).

This can be interpreted as factoring through an intermediate category introduced by Emerton ([10]), denoted $\mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$, and consisting of flat, ϖ-adically continuous representations of G over O (also defined in section 5.1). The unit ball of any K-Banach space representation is an object in this category, in fact the unit ball functor induces an equivalence of categories

$$(-)^{\circ} : \mathrm{Ban}_G(K) \xrightarrow{\sim} \mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$$

In the absence of a G-action the corresponding anti-equivalence

$$(-)^{\circ} : \mathrm{Ban}(K) \xrightarrow{\sim} \mathrm{Mod}^{\varpi-cts}(O)^{fl}$$

is between K-Banach spaces and a category which we denote (in order to match Emerton's notation for $\mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$) by $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$.

Definition 3.1. $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$ is the category of flat ϖ-adically complete and separated O-

modules with O-linear maps as morphisms (automatically continuous).

We can interpret the objects V in $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$ as flat O-modules such that

$$V = \varprojlim_n V/\varpi^n V$$

In fact here flat just means torsion-free.

Lemma 3.1. *Over any principal ideal domain a module is flat if and only if it is torsion-free.*

Proof. [1, I §2.4 Prop. 3(ii)] □

We will prove shortly that the unit ball functor induces an equivalence of categories between $\mathrm{Ban}(K)^{\leq 1}$ and $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$, with quasi-inverse given by extension of scalars:

$$(-)_K : \mathrm{Mod}^{\varpi-cts}(O)^{fl} \xrightarrow{\sim} \mathrm{Ban}(K)$$
$$V \longmapsto V_K := V \otimes_O K$$

The following lemma is useful in relating morphisms between the two categories.

Lemma 3.2. *Given $(E, \|\|_E)$, $(F, \|\|_F)$ in Ban$(K)^{\leq 1}$, a continuous linear map $f : E \longrightarrow F$ is norm decreasing if and only if $f(E^\circ) \subseteq F^\circ$, where E° (resp. F°) is the unit ball in E (resp. F).*

Proof. The only if direction is obvious. For the converse implication suppose $f(E^\circ) \subseteq F^\circ$ but f is not norm decreasing. This means there exists $e_0 \in E$ such that $\|f(e_0)\|_F > \|e_0\|_E$. Clearly $\|e_0\|$ must be nonzero so we can find some $\lambda_{e_0} \in K$ such that $\|\lambda_{e_0} \cdot e_0\|_E = 1$ (here we use $\|E\| \subseteq |K|$). Then $f(\lambda_{e_0} \cdot e_0) \in F^\circ$ and in particular we have:

$$\begin{aligned}\|f(\lambda_{e_0} \cdot e_0)\|_F &= \|\lambda_{e_0} \cdot f(e_0)\|_F \\ &= |\lambda_{e_0}| \cdot \|f(e_0)\|_F \leq 1\end{aligned}$$

So:

$$\|e_0\|_E < \|f(e_0)\|_F \leq |\lambda_{e_0}|^{-1} \implies |\lambda_{e_0}| \cdot \|e_0\|_E = \|\lambda_{e_0} \cdot e_0\|_E < 1$$

□

Lemma 3.3. $(-)^\circ$ *and* $(-)_K$ *are quasi-inverse functors and induce an equivalence of categories:*

$$Ban(K)^{\leq 1} \xrightarrow{\sim} Mod^{\varpi-cts}(O)^{fl}$$

Proof. Given $(E, \|\|_E)$ in $\mathrm{Ban}(K)^{\leq 1}$ the unit ball $E^\circ = \{e \in E : \|e\|_E \leq 1\}$ is a torsion free O-module with a topology induced from E. Since $\|E\| \subseteq |K| = q^{\mathbb{Z}} \cup \{0\}$ we can see that $\{\varpi^n E^\circ\}_{n\geq 0}$ is a fundamental system of open neighborhoods of 0 in E°. Since E° is a closed subset of E (ϖ-adically complete and separated) it is also ϖ-adically complete and separated, and so is an object in $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$. By Lemma 3.2 the unit ball functor acts on morphisms by restricting its domain to the unit ball.

Given an object V in $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$ we can tensor with K to get the vector space $V_K := V \otimes_O K$. Since V is flat, $V \otimes_O O \longrightarrow V \otimes_O K$ is injective and we view $V = V \otimes_O O$ as sitting injectively in V_K as a lattice (in fact as the unit ball).

Since V is ϖ-adically separated, the induced gauge (cf. Definition 2.11) p_V is a norm on V_K with respect to which (V, p_V) is a Banach space. Rewriting p_V as $\|\|_V$, we explicitly verify that $\|V_K\|_V \subseteq |K|$. Given x in V_K

$$\|x\|_V := q^{-max\{n: x \in \varpi^n V\}} \in |K|$$

Regarding morphisms, if $f : V \longrightarrow W$ is a morphism in $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$, then we define f_K by:

$$f_K(v \otimes x) := f(v) \otimes x$$

Extending by linearity we get a morphism $f_K : V_K \longrightarrow V_K$ that maps $(V_K)^\circ$ into $(W_K)^\circ$ and thus by Lemma 3.2 must be norm decreasing.

Finally we note that for any V in $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$ and E in $\mathrm{Ban}(K)^{\leq 1}$ we have the following natural isomorphisms

$$V \cong (V_K)^\circ \quad \text{and} \quad E \cong (E^\circ)_K$$

Thus $(-)^\circ$ and $(-)_K$ are quasi-inverse functors that induce an equivalence of categories

$$Ban(K)^{\leq 1} \xrightarrow{\sim} Mod^{\varpi-cts}(O)^{fl}$$

□

3.3 Tensor products in $\mathrm{Ban}(K)^{\leq 1}$ and $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$

In this section we discuss how to endow the categories $\mathrm{Ban}(K)^{\leq 1}$ and $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$ with a tensor product bifunctor. Since taking a naïve tensor product (over K and O respectively) will not ensure completeness in general we will discuss how to complete the tensor product in each case.

In nonarchimedean functional analysis there are many ways to define a completed tensor product for two locally convex vector spaces E and F. The general idea is to take the abstract tensor product over K, endow the resulting K-vector space $E \otimes_K F$ with a locally convex topology (this is the step that has many different constructions), and take the (Hausdorff) completion.

Since we are interested in the case when E and F are K-Banach spaces (hence have a norm) it is possible to endow the K-vector space $E \otimes_K F$ with a norm $\|\|_{E\otimes F}$, then we can define $E\hat{\otimes}_K F$ as the completion of $E \otimes_K F$ with respect to $\|\|_{E\otimes F}$. The induced topology is called the projective tensor product topology, a locally convex topology with an important continuity property.

Definition 3.2. The *projective* tensor product topology on the (abstract) vector space $E \otimes_K F$ is the finest locally convex topology such that the canonical map

$$E \times F \longrightarrow E \otimes_K F \tag{3.2}$$

is jointly continuous (continuous as a map between topological spaces).

We mention another way to characterize the projective tensor product topology that has the advantage of being explicit about the open sets in $E \otimes_K F$. Suppose $\mathcal{E}$ (resp. $\mathcal{F}$) denote the families of open lattices in E (resp. F) and let $U \in \mathcal{E}$ and $V \in \mathcal{F}$. Applying Lemma 3.1 we can deduce that the O-modules U, V are flat O-modules, thus $U \otimes_O V \longrightarrow E \otimes_O F$ is injective. Since $E \otimes_O F = E \otimes_K F$, $U \otimes_O V$ is a lattice in $E \otimes_K F$. In particular the family of all such $U \otimes_O V$ generates the projective tensor product topology on $E \otimes_K F$.

By contrast, the *inductive* tensor product topology on $E \otimes_K F$ is the finest locally convex topology with respect to which (3.2) is only separably continuous, meaning for any $e \in E$ the map

$$F \longrightarrow E \otimes_K F$$
$$f \longmapsto e \otimes f$$

is continuous (similarly we can fix $f \in F$ to consider a map $E \longrightarrow E \otimes_K F$). For general locally convex spaces the inductive topology is finer than the projective one, but for K-Banach spaces, (in fact even for Frechét spaces) the two topologies coincide.

Definition 3.3. Let $(E, \|\|_E)$ and $(F, \|\|_F)$ be two normed vector spaces. The *tensor product norm* $\|\|_{E\otimes F}$ on $E \otimes_K F$ is defined as:

$$\|u\|_{E\otimes F} := \inf\left\{\max_{1\le i\le r} \|e_i\|_E \cdot \|f_i\|_F : u = \sum_{i=1}^{r} e_i \otimes f_i\right\} \tag{3.3}$$

Remark 3.1. The tensor product norm $\|\|_{E\otimes F}$ has the following properties

1. $\|\|_{E\otimes F}$ is a norm on $E \otimes_K F$ if and only if $\|\|_E$ and $\|\|_F$ are norms on E and F.

2. $\|\|_{E\otimes F}$ induces the projective tensor product topology on $E \otimes_K F$

3. $\|e \otimes f\|_{E\otimes F} = \|e\|_E \|f\|_F$

Proof. [19, §17] □

Lemma 3.4. *Given objects $(E, \|\|_E)$ and $(F, \|\|_F)$ in the category $Ban(K)^{\leq 1}$, the completion $E\hat{\otimes}_K F$ of $E \otimes_K F$ with respect to the norm $\|\|_{E\otimes F}$ is an object in $Ban(K)^{\leq 1}$.*

Proof. Let $\|\|_{E\otimes F}$ also denote the extension of $\|\|_{E\otimes F}$ to the K-Banach space $E\hat{\otimes}_K F$, it only remains to verify that $\|E\hat{\otimes}_K F\|_{E\otimes F} \subseteq |K|$. Since $u \neq 0$ and $|K|$ is discrete it will suffice to prove that $\|E \otimes_K F\|_{E\otimes F} \subseteq |K|$. Let u be in $E \otimes_K F$, by the discreteness of K we attain a minimal expression $u = \sum_{i=1}^{r} e_i \otimes f_i$ and hence

$$\|u\|_{E\otimes F} = \max_{1\leq i\leq r} \|e_i\|_E \cdot \|f_i\|_F \in |K^\times|$$

□

Defining a completed tensor product of two objects V, W in $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$ is a lot more straightforward. The tensor product over O may not be (ϖ-adically) complete so define the completed tensor product as the ϖ-adic completion

$$V\hat{\otimes}_O W := \varprojlim_n \left(\frac{V \otimes_O W}{\varpi^n (V \otimes_O W)} \right)$$

endowed with the induced topology this is an object in $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$

Lemma 3.5. *Given objects V, W in the category $Mod^{\varpi-cts}(O)^{fl}$, the ϖ-adic completion $V\hat{\otimes}_O W$ of $V \otimes_O W$ is an object in $Mod^{\varpi-cts}(O)^{fl}$.*

Proof. Since O is ϖ-adically complete the ϖ-adic completion $V\hat{\otimes}_O W$ is an O-module. By construction this is ϖ-adically complete and separated. It is easy to see that $V\hat{\otimes}_O W$ is O torsion-free and therefore by Lemma 3.1 $V\hat{\otimes}_O W$ is also flat.

□

3.4 Compatibility for $\mathbf{Ban}(K)^{\leq 1}$

We need to prove that the defined tensor products are compatible with the unit ball functor $(-)^\circ$, which by Lemma 3.3 induces an equivalence of categories.

Theorem 3.1. *For any objects $(E, \|\|_E)$, $(F, \|\|_F)$ in $Ban(K)^{\leq 1}$*

$$(E\hat{\otimes}_K F)^\circ \cong E^\circ \hat{\otimes}_O F^\circ$$

is a natural isomorphism in $Mod^{\varpi-cts}(O)^{fl}$.

First we establish some intermediate results

Lemma 3.6. *For any objects $(E, \|\|_E)$, $(F, \|\|_F)$ in $Ban(K)^{\leq 1}$*

$$E^\circ \otimes_O F^\circ \cong (E \otimes_K F)^\circ$$

is a (natural) topological isomorphism of O-modules.

Proof. Begin by observing that the map $E^\circ \otimes_O F^\circ \longrightarrow E \otimes_K F$ (induced by the identity map $E^\circ \otimes_O F^\circ \longrightarrow E \otimes_O F$) is injective and its image is in the unit ball of $E \otimes_K F$ with respect to the tensor product norm. This induces an injective O-module homomorphism:

$$\mathbf{Id} : E^\circ \otimes_O F^\circ \longrightarrow (E \otimes_K F)^\circ$$

To prove surjectivity pick a nonzero x in $(E \otimes_K F)^\circ$. Since $\|x\| \neq 0$ and $|K|$ is discrete, we can attain the infimum:

$$\|x\|_{E\otimes F} = \min\left\{ \max_{1\le i\le r} \|e_i\|_E \cdot \|f_i\|_F : x = \sum_{i=1}^{r} e_i \otimes_K f_i \right\}$$

Let $x = \sum_{i=1}^{r} e_i \otimes_K f_i$ be a minimal expression and rewrite it as:

$$x = \sum_{i=1}^{r} \varpi^{-m_i} e_i \otimes_K \varpi^{m_i} f_i$$

where the integers m_i are chosen such that $\|\varpi^{-m_i} e_i\|_E = |\varpi^{-m_i}|\|e_i\|_E = q^{m_i}\|e_i\|_E = 1$ for all i. Then:

$$\|x\|_{E\otimes F} = \max_{1\le i\le r} \|e_i\|_E \cdot \|f_i\|_F = \max_{1\le i\le r} q^{-m_i} \cdot \|f_i\|_F = \max_{1\le i\le r} \|\varpi^{m_i} f_i\|_F$$

Since $\|x\|_{E\otimes F} \le 1$ we deduce that $\|\varpi^{m_i} f_i\|_F \le 1$ for all i, therefore $\sum_i \varpi^{-m_i} e_i \otimes_O \varpi^{m_i} f_i$ is in $E^\circ \otimes_O F^\circ$ and

$$\mathbf{Id}\left(\sum_i \varpi^{-m_i} e_i \otimes_O \varpi^{m_i} f_i\right) = \sum_i \varpi^{-m_i} e_i \otimes_K \varpi^{m_i} f_i = x$$

Thus **Id** is an isomorphism.

To see that **Id** is continuous it suffices to note that any O-linear map between modules endowed with the ϖ-adic topology is continuous.

□

3.4.1 Proving Theorem 3.1

Proof. (Theorem 3.1)

Let **Id** be as in Lemma 3.6. We have the following commutative diagram:

$$\begin{array}{ccc} E^\circ\hat{\otimes}_O F^\circ & \overset{\mathbf{Id}^*}{\dashrightarrow} & (E\hat{\otimes}_K F)^\circ \\ \iota_1 \uparrow & & \uparrow \iota_2 \\ E^\circ \otimes_O F^\circ & \underset{\mathbf{Id}}{\longrightarrow} & (E \otimes_K F)^\circ \end{array}$$

where ι_1, ι_2 are injective and the bottom row O-modules are dense in the top row.

We need to first define a map $E^\circ\hat{\otimes}_O F^\circ \longrightarrow (E\hat{\otimes}_K F)^\circ$. One way to do so is to use the universal property of the completion $E^\circ\hat{\otimes}_O F^\circ$. Compose $\iota_2 \circ \mathbf{Id}$ to get a continuous, injective homomorphism $\mu : E^\circ \otimes_O F^\circ \longrightarrow (E\hat{\otimes}_K F)^\circ$ into the ϖ-adically complete $(E\hat{\otimes}_K F)^\circ$ which must factor through the ϖ-adic completion of $E^\circ \otimes_O F^\circ$. This defines $\mathbf{Id}^*$ as above.

Let x be in ker($\mathbf{Id}^*$), then (since ι_1 is dense) there is a sequence $\{x_n\}$ in $E^\circ \otimes_O F^\circ$ such that $\iota_1(x_n) \longrightarrow x$ in the completion $E^\circ\hat{\otimes}_O F^\circ$. $\{\mu(x_n)\}$ is a sequence in $(E\hat{\otimes}_K F)^\circ$ and must have a limit by completeness. By the continuity of μ we have $\lim_n \mu(x_n) = \mu(\lim_n x_n)$ and since the diagram commutes we must have $\mu(x) = 0$ hence (since μ is injective) $x = 0$.

Let y be in $(E\hat{\otimes}_K F)^\circ$, then there is a sequence $\{y_n\}$ in $(E \otimes_K F)^\circ$ such that $\iota_2(y_n) \longrightarrow y$ in $(E\hat{\otimes}_K F)^\circ$. Since $\mathbf{Id}$ is an isomorphism $x_n := \mathbf{Id}^{-1}(y_n)$ is a Cauchy sequence in $E^\circ \otimes_O F^\circ$, by completeness $\iota_1(x_n) \longrightarrow x \in E^\circ\hat{\otimes}_O F^\circ$. Since the diagram is commutative we have $\mathbf{Id}^*(x) = y$.

Since $\mathbf{Id}^*$ is a bijective O-linear map between two O-modules (with the ϖ-adic topology) it is an isomorphism in $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$ in this way we see that $(-)^\circ$ is indeed a tensor functor.

From this it is easy to see that so is $(-)_K$ is also a tensor functor. Indeed given V, W in $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$ we know that $V \cong (V_K)^\circ$ and $W \cong (W_K)^\circ$. We calculate:

$$(V\hat{\otimes}_O W)_K \cong ((V_K)^\circ\hat{\otimes}_O (W_K)^\circ)_K \underset{\text{Thm 3.1}}{\cong} ((V_K\hat{\otimes}_K W_K)^\circ)_K \cong V_K\hat{\otimes}_K W_K$$

□

Chapter 4

The Category $\mathrm{Mod}_{cpt}^{fl}(O)$

In this chapter we introduce a special category $\mathrm{Mod}_{cpt}^{fl}(O)$ of topological O-modules that is anti-equivalent to the category $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$ from Chapter 3. We define a tensor product in $\mathrm{Mod}_{cpt}^{fl}(O)$ and prove a compatibility result (as we did for $\mathrm{Ban}(K)^{\leq 1}$ with Theorem 3.1).

Definition 4.1. A topological O-module M is called *linear-topological* if it has a fundamental system of open neighborhoods of 0 by O-submodules.

$\mathrm{Mod}_{cpt}^{fl}(O)$ will denote the category of flat, compact and linearly topological O-modules, with continuous O-linear maps as morphisms.

Lemma 4.1. *A compact linear-topological O-module M is flat if and only if $M \cong \prod_{i\in I} O$ as topological O-modules for some set I*

Proof. [8, Exp. VII_B (0.3.8)] □

Remark 4.1. As a consequence of Lemma 4.1 we can think of an object M in $\mathrm{Mod}_{cpt}^{fl}(O)$ as a (topological) direct product $\prod_{i\in I} O$ over some set I, with each O endowed with the ϖ-adic topology.

4.1 The completed group algebra $\Lambda(G)$

A profinite group G can be defined in two equivalent ways. Firstly as an inverse limit of discrete finite groups $G = \varprojlim G_i$ endowed with its profinite topology (with respect to which the canonical projection maps $p_i : G \longrightarrow G_i$ are always continuous). In this way G becomes a compact, Hausdorff and totally disconnected topological group. On the other hand given a compact, Hausdorff and totally disconnected topological group we recover the inverse limit definition since

$$G \cong \varprojlim_{N \in \mathcal{N}(G)} G/N$$

where $\mathcal{N}(G)$ denotes the set of open normal subgroups of G.

The inverse system of finite quotient groups G/N as above extends to a inverse system of group rings $O[G/N]$. The inverse limit of group rings $O[G/N]$ is called the completed group ring of G (over O).

Definition 4.2. The *completed group ring* (or *Iwasawa algebra*) of G over O is the inverse limit:

$$O[[G]] := \varprojlim_{N \in \mathcal{N}(G)} O[G/N] \tag{4.1}$$

(we will mostly denote $O[[G]]$ by $\Lambda(G)$)

Completed group rings generalize the concept of group rings to profinite groups, in particular if G is finite we have $\Lambda(G) = O[G]$. For every open normal subgroup N the corresponding group ring $O[G/N]$ is finitely generated and free over the profinite ring O and so are themselves profinite in their induced ϖ-adic topology. Endowed with the inverse limit topology we see that $\Lambda(G)$ is itself a profinite ring and can be written as an inverse limit of finite rings:

$$\Lambda(G) = \varprojlim_{N} \varprojlim_{n} (O/\varpi^n O)[G/N] \tag{4.2}$$

The canonical map $O[G] \longrightarrow \varprojlim_N O[G/N]$ is injective so we view $O[G]$ as a subring of $\Lambda(G)$, moreover the natural profinite topology on $O[G]$ has a fundamental system of open neighborhoods of 0 (indexed by N and n):

$$\ker(O[G] \longrightarrow (O/\varpi^n O)[G/N])$$

It follows from (4.2) that $\Lambda(G)$ is the (Hausdorff) completion of $O[G]$, thus $O[G]$ is dense in $\Lambda(G)$. $\Lambda(G)$ is O torsion-free and linearly topological with the O-submodules

$$I_N := \ker(\Lambda(G) \longrightarrow O[G/N]) \tag{4.3}$$

forming a fundamental system of open neighborhoods of 0. In particular $\Lambda(G)$ is an object in $\mathrm{Mod}_{cpt}^{fl}(O)$.

Proposition 4.1. *Let M be any complete, Hausdorff and linearly topological O-module. Given any continuous map $f: G \longrightarrow M$ there is a unique continuous O-module homomorphism f_Λ : $\Lambda(G) \longrightarrow M$ such that $f_\Lambda|_G = f$*

Proof. [20, Prop 19.2] □

For all but the simplest examples the Iwasawa algebra $\Lambda(G)$ has a very complex structure.

Example 4.1. $\Lambda(\mathbb{Z}_p)$ is topologically isomorphic to a power series ring in one variable over O. More generally we have:

$$O[[X_1, \cdots, X_d]] \cong \Lambda(\mathbb{Z}_p^d)$$

$$X_i \longmapsto g_i - 1$$

where $g_i = (\cdots, 0, 1, 0, \cdots)$ are the standard topological generators for $\mathbb{Z}_p^d$.

We collect some of the known properties of $\Lambda(G)$

1. The map $g \longmapsto g^{-1}$ on G lets us identify $\Lambda(G)$ and its opposite ring $\Lambda(G)^{op}$ (the same ring with multiplication reversed). In practice this means we can assume that all $\Lambda(G)$-modules have a left action.

2. G is contained in $O[G]^{\times}$ and therefore in $\Lambda(G)^{\times}$. The inclusion map $G \hookrightarrow \Lambda(G)$ is a homeomorphism onto its image ([20, Lemma 19.1]).

3. When G is a compact p-adic Lie group $\Lambda(G)$ is Noetherian ([20, Thm 33.4]).

4. When G is pro-p group (an inverse limit of finite p-groups) $\Lambda(G)$ is a local ring with maximal ideal $\ker(\Lambda(G) \longrightarrow O/\varpi O)$ ([20, Prop 19.7]).

4.2 A tensor product in $\mathbf{Mod}_{cpt}^{fl}(O)$

The definition of a tensor product $\otimes$ in this category is motivated by a desire for an isomorphism of completed group rings

$$\Lambda(G \times H) \cong \Lambda(G) \otimes \Lambda(H) \tag{4.4}$$

The isomorphism (4.4), combined with an O-algebra map induced by the diagonal map on G will let us endow $\Lambda(G)$ with a $\otimes$-coalgebra structure $c : \Lambda(G) \longrightarrow \Lambda(G) \otimes \Lambda(G)$. This coalgebra structure is essential to the proof of the main recovery theorem.

The following was introduced by Brumer in [4] to define a tensor product for pseudocompact modules (over pseudocompact rings). We have no need for that level of generality but the construction is useful here.

Definition 4.3. Let M and N be linearly topological O-modules, the *completed tensor product* over O is the inverse limit

$$M \hat{\otimes}_O N := \varprojlim_{M', N'} M/M' \otimes_O N/N' \tag{4.5}$$

taken over the open submodules M' and N' of M and N respectively. We endow the finite O-modules $M/M' \otimes_O N/N'$ with the discrete topology and $M\hat{\otimes}_O N$ with the inverse limit topology.

Lemma 4.2. *Let G and H be profinite groups, there is a (topological) isomorphism of O-modules*

$$\Lambda(G \times H) \cong \Lambda(G)\hat{\otimes}_O \Lambda(H) \tag{4.6}$$

Proof. By definition

$$\Lambda(G \times H) := \varprojlim_N O[(G \times H)/N] = \varprojlim_{N_1,N_2} O[G/N_1 \times H/N_2]$$

The first inverse limit is taken over the open normal subgroups of $G \times H$, which we rewrite as $N_1 \times N_2$ where $N_1 \lhd G$ and $N_2 \lhd H$. Since G and H are profinite, the quotients G/N_1 and H/N_2 are finite and so we have a (continuous) isomorphism between the finitely generated O-modules (each with the induced ϖ-adic topology):

$$O[G/N_1 \times H/N_2] \cong O[G/N_1] \otimes_O O[H/N_2] \tag{4.7}$$

Passing to the projective limit gives the (continuous) isomorphism:

$$\Lambda(G \times H) \cong \varprojlim_{N_1,N_2} (O[G/N_1] \otimes_O O[H/N_2])$$

By construction (4.3) for any open subgroup $N \lhd G$ we have an open O-submodule I_N in $\Lambda(G)$ such that $O[G/N] \cong \Lambda(G)/I_N$. In particular

$$\Lambda(G \times G) \cong \varprojlim_{N_1,N_2} (\Lambda(G)/I_{N_1} \otimes \Lambda(G)/I_{N_2}) =: \Lambda(G)\hat{\otimes}_O \Lambda(G)$$

□

Lemma 4.3. *Given objects M, N in $Mod^{fl}_{cpt}(O)$, the completed tensor product $M\hat{\otimes}_O N$ is an object in $Mod^{fl}_{cpt}(O)$*

Proof. By Remark 4.1 we may suppose

$$M \cong \prod_I O \text{ and } N \cong \prod_J O$$

for some sets I and J. By the construction of the topology on $\prod_I O$ we know that the open submodules of M all have the following form:

$$\prod_{s\in S} \varpi^{n_s} O \times \prod_{I\setminus S} O$$

where S is a finite set and n_s is some non-negative integer. By taking $n = \max(n_s)$ we can see that

$$U_{n,S} := \prod_{s\in S} \varpi^{n} O \times \prod_{I\setminus S} O \subseteq \prod_{s\in S} \varpi^{n_s} O \times \prod_{I\setminus S} O$$

In particular open sets of the type $U_{n,S}$ form a cofinal subset of all open sets in M ordered by reverse inclusion, thus it will suffice to take the inverse limit over these simpler open subsets.

We calculate:

$$\begin{aligned} M\hat{\otimes}_O N &:= \varprojlim_{M',N'} M/M' \otimes_O N/N' \\ &= \varprojlim (O/\varpi^n O)^S \otimes_O (O/\varpi^m O)^T \\ &\cong \varprojlim (O/\varpi^n O \otimes_O O/\varpi^m O)^{S\times T} \end{aligned}$$

Assuming without loss of generality that $n \geq m$ we get

$$M\hat{\otimes}_O N \cong \varprojlim (O/\varpi^n O)^{S\times T} \cong O^{I\times J}$$

O is compact and thus by Tychonoff's Theorem, so is $O^{I\times J}$. It is easy to see that

$\prod \varpi^n O \times \prod O$ form a fundamental system of neighborhoods of 0 in $O^{I\times J}$ and finally, applying Remark 4.1 we see that $M\hat{\otimes}_O N$ is flat. □

4.3 Schikhof duality of topological O-modules

As we discussed at the end of chapter 2 one of the technical challenges in the theory of Banach spaces is the lack of a reflexive property that would allow us to move freely between a Banach space and its dual space. Nevertheless (as mentioned in section 2.3) there is a workable alternative.

In the 1990s Schikhof proved there is an anti-equivalence of categories between norm-polar Banach spaces (normed spaces E over discretely valued fields K are norm-polar if and only if $\|E\| \subseteq |K|$) and certain kinds of compactoid spaces (a generalization of compact sets to locally convex vector spaces). Specializing to our particular situation this the anti-equivalence between $\mathrm{Ban}(K)^{\leq 1}$ and $Mod_{cpt}^{fl}(O)$ in [21, Thm 1.2].

The anti-equivalence depends on a compactness result:

Theorem 4.1 (Alaoglu's Theorem)**.** *Let E be a K-Banach space, the unit ball of the dual space E' is compact when endowed with the weak topology.*

Let $(E, \|\|)$ be a Banach space in $\mathrm{Ban}(K)^{\leq 1}$ and in keeping with notation from [21] for the time being let E^d denote the unit ball of the dual space E' endowed with the topology of pointwise convergence. This is flat, compact (by Theorem 4.1) and linearly topological O-module. Moreover E^d can be written more suggestively as:

$$E^d = Hom_O(E^\circ, O)$$

Here E° is the unit ball in $(E, \|\|)$. This defines a functor $E \longmapsto E^d$ between $\mathrm{Ban}(K)^{\leq 1}$ and

$\text{Mod}_{cpt}^{fl}(O)$ that factors through the category $\text{Mod}^{\varpi-cts}(O)^{fl}$:

$$(E, \|\|) \longmapsto E^{\circ} \longmapsto Hom_O(E^{\circ}, O)$$

On the other hand to any flat, compact and linearly topological O-module M we can associate the Banach space:

$$M^d := Hom_O^{cts}(M, K)$$

endowed with the norm $\|l\| := \max_{m \in M} |l(m)|$. One can check that maps in the image of this functor are naturally norm-decreasing (in fact $f^d : N^d \longrightarrow M^d$ is an isometry if and only if $f : M \longrightarrow N$ is surjective), so $M \longmapsto M^d$ defines a functor between $\text{Mod}_{cpt}^{fl}(O)$ and $\text{Ban}(K)^{\leq 1}$. The unit ball in M^d with respect to its given norm is the O-module $Hom_O^{cts}(M, O)$, this again is an element of $\text{Mod}^{\varpi-cts}(O)^{fl}$ and we can recover $Hom_O^{cts}(M, K)$ by tensoring with K (cf. section 3.2). In this way we view the functor $M \longmapsto M^d$ also naturally factoring through $\text{Mod}^{\varpi-cts}(O)^{fl}$ as:

$$M \longmapsto Hom_O^{cts}(M, O) \longmapsto Hom_O^{cts}(M, O) \otimes_O K$$

We will refer to the contravariant functor $\text{Hom}_O^{cts}(-, O)$ as the Schikhof dual functor and we will denote it by $(-)^d$. In light of the equivalence between $\text{Ban}(K)^{\leq 1}$ and $\text{Mod}^{\varpi-cts}(O)^{fl}$ the real content of [21, Thm 1.2] is the anti-equivalence of categories induced by $(-)^d$ (in our notation):

Theorem 4.2 (Schikhof)**.** *The functors $V \longmapsto V^d := Hom_O(V, O)$ (equipped with the topology of pointwise convergence) and $M \longmapsto M^d := Hom_O^{cts}(M, O)$ are quasi-inverse and induce an anti-equivalence of categories between $Mod^{\varpi-cts}(O)^{fl}$ and $Mod_{cpt}^{fl}(O)$.*

4.4 Compatibility for $\mathbf{Mod}_{cpt}^{fl}(\mathcal{O})$

In this section we prove that the tensor product we have defined in $\mathrm{Mod}_{cpt}^{fl}(\mathcal{O})$ is compatible with the Schikhof duality functor.

Theorem 4.3. *For any objects M, N in $Mod_{cpt}^{fl}(\mathcal{O})$*

$$(M\hat{\otimes}_{\mathcal{O}}N)^d \cong M^d\hat{\otimes}_{\mathcal{O}}N^d$$

is a natural isomorphism in $Mod^{\varpi-cts}(\mathcal{O})^{fl}$.

First we establish some intermediate results.

Lemma 4.4. *There is a topological isomorphism of $\mathcal{O}$-modules*

$$\Phi : Hom_{\mathcal{O}}^{cts}(M\hat{\otimes}_{\mathcal{O}}N, \mathcal{O}) \longrightarrow \varprojlim_n Hom_{\mathcal{O}}^{cts}(M\hat{\otimes}_{\mathcal{O}}N, \mathcal{O}/\varpi^n\mathcal{O})$$

with the ϖ-adic topology on $Hom_{\mathcal{O}}^{cts}(M\hat{\otimes}_{\mathcal{O}}N, \mathcal{O})$, discrete topology on $Hom_{\mathcal{O}}^{cts}(M\hat{\otimes}_{\mathcal{O}}N, \mathcal{O}/\varpi^n\mathcal{O})$.

Proof. The isomorphism Φ is induced by the universal property of inverse limits. First we define

$$\Phi_n : \mathrm{Hom}_{\mathcal{O}}^{cts}(M\hat{\otimes}_{\mathcal{O}}N, \mathcal{O}) \longrightarrow \mathrm{Hom}_{\mathcal{O}}^{cts}(M\hat{\otimes}_{\mathcal{O}}N, \mathcal{O}/\varpi^n\mathcal{O})$$

Let $\lambda \in \mathrm{Hom}_{\mathcal{O}}^{cts}(M\hat{\otimes}_{\mathcal{O}}N, \mathcal{O})$, then $\Phi_n(\lambda) = \lambda_n$ is defined as the composition of λ with the canonical projection map $\mathcal{O} \longrightarrow \mathcal{O}/\varpi^n\mathcal{O}$. For each n notice that the kernel $\ker(\Phi_n) = \varpi^n\mathrm{Hom}_{\mathcal{O}}^{cts}(M\hat{\otimes}_{\mathcal{O}}N, \mathcal{O})$ is open, thus each Φ_n is continuous and the maps Φ_n induce a continuous homomorphism

$$\Phi : \mathrm{Hom}_{\mathcal{O}}^{cts}(M\hat{\otimes}_{\mathcal{O}}N, \mathcal{O}) \longrightarrow \varprojlim_n \mathrm{Hom}_{\mathcal{O}}^{cts}(M\hat{\otimes}_{\mathcal{O}}N, \mathcal{O}/\varpi^n\mathcal{O})$$
$$\lambda \longrightarrow (\lambda_n)$$

For injectivity, suppose $\Phi(\lambda) = 0$. Then we must have $\lambda_n = 0$ for all n, hence $\lambda = 0$. On the other hand, given $(\lambda_n) \in \varprojlim \mathrm{Hom}_O^{cts}(M\hat{\otimes}_O N, O/\varpi^n O)$ we can define $\lambda = \varprojlim \lambda_n \in \mathrm{Hom}_O^{cts}(M\hat{\otimes}_O N, O)$ by:

$$\left(\varprojlim \lambda_n\right)(x) = (\lambda_n(x)_n) \in \varprojlim O/\omega^n O = O$$

Then $\Phi(\lambda) = (\lambda_n)$. □

Lemma 4.5. *There is a topological isomorphism of O-modules*

$$\Psi : \varinjlim_{M',N'} Hom_O\left(\frac{M}{M'} \otimes \frac{N}{N'}, O/\varpi^n O\right) \longrightarrow Hom_O^{cts}(M\hat{\otimes}_O N, O/\varpi^n O)$$

with the discrete topology on $Hom_O^{cts}(M\hat{\otimes}_O N, O/\varpi^n O)$ and $Hom_O\left(\frac{M}{M'} \otimes \frac{N}{N'}, O/\varpi^n O\right)$

Proof. The isomorphism Ψ is induced by the universal property of direct limits. First we define:

$$\Psi_{M'N'} : Hom_O\left(\frac{M}{M'} \otimes \frac{N}{N'}, O/\varpi^n O\right) \longrightarrow Hom_O^{cts}(M\hat{\otimes}_O N, O/\varpi^n O)$$

Let $\mu \in Hom_O\left(\frac{M}{M'} \otimes \frac{N}{N'}, O/\varpi^n O\right)$ and define

$$\Psi_{M'N'}(\mu) := \mu_{M'N'} = \mu \circ pr_{M'N'} \in Hom_O^{cts}(M\hat{\otimes}_O N, O/\varpi^n O)$$

where $p_{M'N'}$ are the (continuous) projections $p_{M'N'} : M\hat{\otimes}_O N \longrightarrow \frac{M}{M'} \otimes \frac{N}{N'}$. The O-modules $Hom_O\left(\frac{M}{M'} \otimes \frac{N}{N'}, O/\varpi^n O\right)$ are all discrete, thus each $\Psi_{M'N'}$ is continuous and the maps $\Psi_{M'N'}$ induce the continuous O-module homomorphism:

$$\Psi : \varinjlim_{M',N'} Hom_O\left(\frac{M}{M'} \otimes \frac{N}{N'}, O/\varpi^n O\right) \longrightarrow Hom_O^{cts}(M\hat{\otimes}_O N, O/\varpi^n O)$$

For injectivity suppose $\Psi(\mu) = 0$. By the definition of direct limits there exists M', N' such that

$\mu \in Hom_O\left(\frac{M}{M'} \otimes \frac{N}{N'}, O/\varpi^n O\right)$ with $\Psi_{M'N'}(\mu) = pr_{M'N'} \circ \mu = 0$, so we must have $\text{im}(pr_{M'N'}) \subseteq \ker(\mu)$. Since each $pr_{M'N'}$ is surjective $\ker(\mu) = \frac{M}{M'} \otimes \frac{N}{N'}$, hence $\mu = 0$.

To prove surjectivity let $\nu \in Hom_O^{cts}(M\hat{\otimes}_O N, O/\varpi^n O)$. Since $O/\varpi^n O$ is a discrete O-module and ν is continuous, $\nu^{-1}(0) = \ker(\nu)$ is open in $M\hat{\otimes}_O N$. By construction $M\hat{\otimes}_O N$ has a fundamental system of open neighborhoods of 0 by:

$$\ker\left(M\hat{\otimes}_O N \longrightarrow \frac{M}{M'} \otimes \frac{N}{N'}\right)$$

Thus there is a pair M', N' such that $\ker(pr_{M'N'}) \subseteq \ker(\nu)$. In particular ν must factor through $\frac{M}{M'} \otimes \frac{N}{N'}$ which defines a map $\mu_{M'N'} : \frac{M}{M'} \otimes \frac{N}{N'} \longrightarrow O/\varpi^n O$

$$\begin{array}{ccc} \frac{M}{M'} \otimes \frac{N}{N'} & \overset{\mu_{M'N'}}{\dashrightarrow} & O/\varpi^n O \\ {\scriptstyle pr_{M'N'}}\uparrow & \nearrow{\scriptstyle \nu} & \\ M\hat{\otimes}_O N & & \end{array}$$

Then $\Psi(\mu_{M'N'}) = \nu$ □

4.4.1 Proving Theorem 4.3

Proof. Our aim is to prove

$$(M\hat{\otimes}_O N)^d \cong M^d \hat{\otimes}_O N^d$$

is an isomorphism in $\text{Mod}^{\varpi-cts}(O)^{fl}$. We remind the reader that the tensor product in $(M\hat{\otimes}_O N)^d$ is taken in $Mod_{cpt}^{fl}(O)$ and so is completed over the open O-modules of M and N, whereas the tensor product in $M^d \hat{\otimes}_O N^d$ is taken in $Mod^{\varpi-cts}(O)^{fl}$ and is completed ϖ-adically. It should be clear from the context which tensor product is being used.

$(M\hat{\otimes}_O N)^d$ is by definition the O-module $\text{Hom}_O^{cts}(M\hat{\otimes}_O N, O)$. We will use a sequence of isomorphisms to reduce this to a finite problem which can be easily solved. We begin by exploiting the ϖ-adic completeness of $\mathcal{O}$ and using Lemma 4.4:

$$\mathrm{Hom}^{cts}_{\mathcal{O}}(M\hat{\otimes}_{\mathcal{O}}N,\mathcal{O}) \cong \varprojlim_{n} Hom^{cts}_{\mathcal{O}}(M\hat{\otimes}_{\mathcal{O}}N,\mathcal{O}/\varpi^n\mathcal{O}) \tag{4.8}$$

Next we want to take advantage of the finite quotients $\frac{M}{M'}\otimes\frac{N}{N'}$, so we use Lemma 4.5 to get:

$$Hom^{cts}_{\mathcal{O}}(M\hat{\otimes}_{\mathcal{O}}N,\mathcal{O}/\varpi^n\mathcal{O}) \cong \varinjlim_{M',N'} Hom_{\mathcal{O}}\left(\frac{M}{M'}\otimes\frac{N}{N'},\mathcal{O}/\varpi^n\mathcal{O}\right) \tag{4.9}$$

Since $\frac{M}{M'}\otimes\frac{N}{N'}$ is discrete, homomorphisms are automatially continuous. Fix $M'\cong(\varpi^s\mathcal{O})^S\times\mathcal{O}^{I\setminus S}$ and $N'\cong(\varpi^t\mathcal{O})^T\times\mathcal{O}^{J\setminus T}$ and assume $s\geq t$, then:

$$\begin{aligned} Hom_{\mathcal{O}}\left(\frac{M}{M'}\otimes\frac{N}{N'},\mathcal{O}/\varpi^n\mathcal{O}\right) &\cong \bigoplus_{S\times T} Hom_{\mathcal{O}}(\mathcal{O}/\varpi^s\mathcal{O},\mathcal{O}/\varpi^n\mathcal{O}) \\ &\cong Hom_{\mathcal{O}}\left(\frac{M}{M'},\mathcal{O}/\varpi^n\mathcal{O}\right)\otimes_{\mathcal{O}} Hom_{\mathcal{O}}\left(\frac{N}{N'},\mathcal{O}/\varpi^n\mathcal{O}\right) \end{aligned}$$

Direct products commute with tensor products, thus we can split the (double) direct product in (4.9) as a tensor product of direct limits:

$$\begin{aligned} &\varinjlim_{M',N'} Hom_{\mathcal{O}}\left(\frac{M}{M'}\otimes\frac{N}{N'},\mathcal{O}/\varpi^n\mathcal{O}\right) \\ &\cong \varinjlim_{M',N'}\left(Hom_{\mathcal{O}}\left(\frac{M}{M'},\mathcal{O}/\varpi^n\mathcal{O}\right)\otimes_{\mathcal{O}} Hom_{\mathcal{O}}\left(\frac{N}{N'},\mathcal{O}/\varpi^n\mathcal{O}\right)\right) \\ &\cong \varinjlim_{M'} Hom_{\mathcal{O}}\left(\frac{M}{M'},\mathcal{O}/\varpi^n\mathcal{O}\right)\otimes_{\mathcal{O}} \varinjlim_{N'} Hom_{\mathcal{O}}\left(\frac{N}{N'},\mathcal{O}/\varpi^n\mathcal{O}\right) \end{aligned}$$

Reapplying Lemma 4.5 we get rid of the direct limits above:

$$\cong Hom^{cts}_{\mathcal{O}}(M,\mathcal{O}/\varpi^n\mathcal{O})\otimes_{\mathcal{O}} Hom^{cts}_{\mathcal{O}}(N,\mathcal{O}/\varpi^n\mathcal{O})$$

We can rewrite this as

$$\left(Hom_{\mathcal{O}}^{cts}(M,\mathcal{O}) \otimes_{\mathcal{O}} Hom_{\mathcal{O}}^{cts}(N,\mathcal{O})\right) \otimes_{\mathcal{O}} (\mathcal{O}/\varpi^n \mathcal{O})$$

Simplifying the notation and reintroducing the inverse limit from (4.8) finishes the proof:

$$\begin{aligned}(M\hat{\otimes}_{\mathcal{O}}N)^d &\cong \varprojlim_n \left(M^d \otimes_{\mathcal{O}} N^d\right) \otimes_{\mathcal{O}} (\mathcal{O}/\varpi^n \mathcal{O}) \\ &\cong \varprojlim_n \frac{\left(M^d \otimes_{\mathcal{O}} N^d\right)}{\varpi^n \left(M^d \otimes_{\mathcal{O}} N^d\right)} \\ &=: M^d \hat{\otimes}_{\mathcal{O}} N^d\end{aligned}$$

□

Chapter 5

Extending results to new categories

In this chapter we extend the results of Chapters 3 and 4 to K-Banach spaces and topological O-modules that carry an action of a profinite group G. These will be the categories of Banach space representations $\mathrm{Ban}_G(K)$ and $\mathrm{Ban}_G(K)^{\leq 1}$, the category of flat ϖ-adically continuous G representations $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$ (over O) and the category of Iwasawa G-modules $\mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$ (over O).

We will begin by defining the new categories and tensor products. The goal will be to prove $\{Ban_G(K)^{\leq 1}, \hat{\otimes}_K\}$, $\{Mod_G^{\varpi-cts}(O)^{fl}, \hat{\otimes}_O\}$ and $\left\{Mod_{cpt}^{fl}(\Lambda(G)), \hat{\otimes}_O\right\}$ are tensor categories with tensor functors:

$$(-)^\circ : \{Ban_G(K)^{\leq 1}, \hat{\otimes}_K\} \longrightarrow \left\{Mod_G^{\varpi-cts}(O)^{fl}, \hat{\otimes}_O\right\}$$
$$(-)^d : \left\{Mod_{cpt}^{fl}(\Lambda(G)), \hat{\otimes}_O\right\} \longrightarrow \left\{Mod_G^{\varpi-cts}(O)^{fl}, \hat{\otimes}_O\right\}$$

The main result of this chapter is the generalized compatibility result:

Theorem 5.1. *For any $(E, \|\|_E)$, $(F, \|\|_F)$ in $Ban_G(K)^{\leq 1}$, and any M, N in $Mod_{cpt}^{fl}(\Lambda(G))$, we have natural isomorphisms:*

$$(E\hat{\otimes}_K F)^\circ \cong E^\circ \hat{\otimes}_O F^\circ$$

$$(M\hat{\otimes}_O N)^d \cong M^d \hat{\otimes}_O N^d$$

in $Mod_G^{\varpi-cts}(O)^{fl}$

5.1 The new categories

Definition 5.1. A K-Banach space representation of G is a K-Banach space E together with a G-action by continuous linear automorphisms such that the map $G\times E\longrightarrow E$ describing the action is continuous.

$\mathrm{Ban}_G(K)$ will denote the category of K-Banach space representations of G, with G-equivariant continuous linear maps as morphisms. Let $\mathrm{Ban}_G(K)^{\leq 1}$ denote the category of K-Banach space representations $(E,\|\|)$ satisfying $\|E\|\subseteq|K|$, with G-equivariant norm-decreasing linear maps as morphisms.

As in section 3.1 the forgetful functor (that forgets the specified norm for objects $(E,\|\|)$ in $\mathrm{Ban}_G(K)^{\leq 1}$) induces an equivalence of categories:

$$(\mathrm{Ban}_G(K)^{\leq 1})_{\mathbb{Q}}\xrightarrow{\sim}\mathrm{Ban}_G(K)$$

Breuil introduced the concept of a unitary Banach space representation

Definition 5.2. A Banach space representation E of G is called unitary if its topology can be defined by a G-invariant norm $\|\|$, meaning $\|g\cdot x\|=\|x\|$ for any g in G and x in E.

Lemma 5.1. *For a compact group G and discretely valued field K the topology of a K-Banach space representation E of G can be defined by a G-invariant norm $\|\|$ satisfying $\|E\|\subseteq|K|$*

Proof. Any open O-submodule L in E contains a G-invariant open O-submodule $\bigcap_{g\in G}gL$.

It is clear that $\bigcap_{g\in G}gL$ is G-invariant and contained in L, so it remains to verify that it is open. By the continuity of the G-action on E we find an open normal subgroup $N\lhd G$ and open O-submodule in $L'\subseteq E$ such that $NL'\subseteq L$. Define $L'':=\bigcap_{n\in N}nL$

$NL' \subseteq L$ implies that $L' \subseteq nL$ for every $n \in N$, so L'' contains L' and therefore L'' itself is open. This means we have:

$$\bigcap_{g\in G} gL = \bigcap_{r\in G/N} \bigcap_{n\in N} rnL = \bigcap_{r\in G/N} rL''$$

G is compact so the quotient group G/N is both compact and discrete, therefore finite. It follows that $\bigcap_{g\in G} gL$ is open and G-invariant in E. By combining with Remark 2.4 we can see that given any open bounded lattice $L \subseteq E$ that defines the topology of E, we can construct an open (bounded) sublattice L'' that is also G-invariant. The induced gauge norm $\|\|_{L''}$ is then G-invariant, satisfies $\|E\| \subseteq |K|$ (by construction) and defines the same topology on E. $\square$

Example 5.1. Recall that $(C(X,K), \|\|_\infty)$ was one of the examples of K-Banach spaces given in Section 2.2 (provided X is compact). If we let $X = G$, $C(G,K)$ is the Banach space of bounded continuous functions $f : G \longrightarrow K$, with norm $\|f\|_\infty := \sup_{g\in G} |f(g)|$. G acts (continuously) on $C(G,K)$ by left translation

$$(\lambda \cdot f)(g) := f(\lambda^{-1} g)$$

Thus $C(G,K)$ is a K-Banach space representation.

Emerton introduced and studied the first properties of ϖ-adically continuous G-representations in [10, §2.4], originally defined over a wider class of rings he denoted $\mathrm{Comp}(O)$ consisting of complete local Noetherian O-algebras with finite residue fields. We make use of his definitions (with mild simplifications) here.

Definition 5.3. A ϖ-adically continuous G-representation over O is a ϖ-adically complete and separated $O[G]$-module V such that the G-action map $G \times V \longrightarrow V$ is continuous (with respect to the ϖ-adic topology on V).

$\mathrm{Mod}_G^{\varpi-cts}(O)$ will denote the category of ϖ-adically continuous G-representations, with G-equivariant O-linear maps as morphisms. The full subcategory of flat (torsion-free by Lemma

3.1) ϖ-adically continuous G-representations will be denoted by $\mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$

Example 5.2. $C(G,O)$, the O-module of continuous functions $f : G \longrightarrow O$ with a G-action by left translation is a (flat) ϖ-adically continuous representation of G over O. This is the unit ball in $(C(G,K), \|\|_\infty)$.

Following the definition in [21] we define Iwasawa G-modules over O.

Definition 5.4. An Iwasawa G-module over O is an O-module M in $\mathrm{Mod}_{cpt}^{fl}(O)$ together with a continuous action $\Lambda(G) \times M \longrightarrow M$ on M such that the induced O-action on M is the given O-module structure.

$\mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$ will denote the category of Iwasawa G-modules over O, with continuous G-equivariant O-linear maps as morphisms (any such morphism is automatically $\Lambda(G)$-linear)

Example 5.3. The compact O-algebra $\Lambda(G)$ is naturally a $\Lambda(G)$-module with its multiplication structure. In fact one can interpret $\Lambda(G)$ as the ring of O-valued continuous distributions on G, i.e.

$$\Lambda(G) = Hom_O(C(G,O),O)$$

5.2 Tensor products in the new categories

We need to endow each of the new categories with a tensor product structure. For the categories $\mathrm{Ban}_G(K)^{\leq 1}$ and $\mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$ defining a G-action that is compatible with the previous definitions can be proven to work with little difficulty. Defining a G-action in $\mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$ will be a little more complicated.

5.2.1 A tensor product in $\mathrm{Ban}_G(K)^{\leq 1}$

Let $(E, \|\|_E)$ and $(F, \|\|_F)$ be objects in $\mathrm{Ban}_G(K)^{\leq 1}$. The completed tensor product $E\hat{\otimes}_K F$, defined in section 3.3, is a K-Banach space (by Lemma 3.4), but does not have a G-action. We can use the continuous G-action maps on E and F to define a (continuous) G-action on $E\hat{\otimes}_K F$, by having G acts diagonally in each component.

Since the G-action is continuous on E and F we have

$$\|g\cdot x\|_E \leq A\cdot\|x\|_E \quad \text{and} \quad \|g\cdot y\|_F \leq B\cdot\|y\|_F$$

Then:

$$\begin{aligned}\|g\cdot(u\otimes v)\|_{E\otimes F} &= \|(g\cdot u)\otimes(g\cdot v)\|_{E\otimes F}\\ &= \|g\cdot u\|_E\|g\cdot v\|_F\\ &\leq AB\|u\|_E\|v\|_F\\ &= AB\|u\otimes v\|_{E\otimes F}\end{aligned}$$

There is a constant $C>0$ such that $\|g\cdot(u\otimes v)\|_{E\otimes F}\leq C\|u\otimes v\|_{E\otimes F}$, hence the G-action on $E\hat{\otimes}F$ is continuous.

5.2.2 A tensor product in $\mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$

Let V, W be objects in $\mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$, the completed tensor product $V\hat{\otimes}_O W$, defined in section 3.3 is in $\mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$ (by Lemma 3.5), but does not have a G-action.

Since $V\hat{\otimes}_O W := \varprojlim (V\otimes W)/\varpi^n(V\otimes W)$, an element $x\in V\hat{\otimes}_O W$ is a sequence $\{x_n\}\in \prod_n (V\otimes W)/\varpi^n(V\otimes W)$ satisfying $x_n\equiv x_{n+k} \pmod{\varpi^n}$ for all $k\in\mathbb{N}$.

Since G acts by continuous linear automorphisms, the G-action commutes with the O-module structure and hence

$$x_n\equiv x_{n+k} \pmod{\varpi^n} \Longleftrightarrow g\cdot x_n\equiv g\cdot x_{n+k} \pmod{\varpi^n}$$

So the G-action is continuous, acting diagonally in each component of the inverse limit $V\hat{\otimes}_O W$.

5.2.3 A tensor product in $\mathbf{Mod}_{cpt}^{fl}(\Lambda(G))$

As we have already seen, the O-module $M\hat{\otimes}_O N$ is flat, compact and linear topological, this is the content of Theorem 4.3. It remains to show that we can endow $M\hat{\otimes}_O N$ with a continuous $\Lambda(G)$-action in way that is compatible with its O-module structure.

The issue is that the open submodules of M and N do not come equipped with a $\Lambda(G)$-module structure (rather they are only assumed to be O-modules), so we cannot naïvely endow even the finite quotients M/M', N/N' with a (compatible) $\Lambda(G)$-action, let alone $M\hat{\otimes}_O N$. Nonetheless, the following result holds.

Theorem 5.2. *Given objects M, N in $Mod_{cpt}^{fl}(\Lambda(G))$, the completed tensor product $M\hat{\otimes}_O N$ is an object in $Mod_{cpt}^{fl}(\Lambda(G))$.*

Remark 5.1. It may be useful at times to consider the G-action on objects M in $\mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$, by this we will mean the continuous action induced by the (continuous) inclusion (cf Section 4.1)

$$\iota : G \hookrightarrow \Lambda(G)$$

By abuse of notation we will sometimes write g for the element $i(g) \in \Lambda(G)$, so $g \cdot m$ should be understood to mean $\iota(g) \cdot m$

The proof of Theorem 5.2 will occupy most of this section.

Step 1

As a first step we prove that for each fundamental open O-module M' in M we can construct a open submodule $\overline{M'} \subseteq M'$ that is G-stable.

Lemma 5.2. *Let M be an object in $Mod_{cpt}^{fl}(\Lambda(G))$. Given an open O-submodule $M' \subseteq M$, there exists $N \lhd G$ open such that $xM' = M'$ for all $x \in N$*

Proof. Let M' be an open O-submodule of M. By the continuity of the G-action we find an open normal N in G and an open O-submodule M'' of M' such that $NM'' \subseteq M'$.

Since $M'/M'' \subseteq M/M''$ and M/M'' is finite (M is profinite as an O-module). This lets us write M' as a (disjoint) union:

$$M' = \bigcup_{i=1}^{k} (m_i + M'') \tag{5.1}$$

Fix an index i, by the continuity of the map $G \times M \longrightarrow M$ at (e, m_i) we get a subgroup $N_i \subseteq G$ open (without loss of generality normal and contained in N) such that $N_i\{m_i\} \subseteq m_i + M''$. Repeat for all i and define:

$$N' := \bigcap_{i=1}^{k} N_i$$

Clearly N' is open and normal in G, we claim that $N'M' = M'$.

Let $m' \in M'$ and $n \in N'$. By (1.2) we can write $m' = m_i + m''$, so $nm' = nm_i + nm''$. By construction $n \in N_i$ for all i thus $nm_i \in m_i + M'' \subseteq M'$ and $N' \subseteq N$ so we have $nm'' \in NM'' \subseteq M'$. Thus $N'M' \subseteq M'$, the reverse inclusion is obvious. □

Lemma 5.3. *Let M be an object in $Mod_{cpt}^{fl}(\Lambda(G))$. Given an open O-submodule $M' \subseteq M$, there exists an open G-stable O-submodule in M'.*

Proof. Fix an open O-submodule $M' \subseteq M$. By Lemma 5.2 we find an open normal submodule N such that $NM' = M'$. Consider the following O-submodule of M':

$$\overline{M'} := \bigcap_{g \in G} gM'$$

This is obviously G-stable and to see that it is open we can write:

$$\bigcap_{g \in G} gM' = \bigcap_{r \in G/N} \bigcap_{n \in N} rnM' = \bigcap_{r \in G/N} \bigcap_{n \in N} rM' = \bigcap_{r \in G/N} rM'$$

Using $NM' = M'$ in the second equality.

Since G is profinite G/N is finite and so $\overline{M'}$ is a finite intersection of open sets, hence open. $\square$

The continuous $\Lambda(G)$-action on M induces a continuous G-action on $\overline{M'}$. Extending O-linearly we get an $O[G]$-action on $\overline{M'}$ defined by:

$$\left(\sum_g c_g g\right) \cdot m := \sum_g c_g (g \cdot m)$$

Since $O[G]$ is dense in $\Lambda(G)$ (cf section 4.1) the continuous $O[G]$-action on $\overline{M'}$ extends (uniquely) to a continuous $\Lambda(G)$-action. This way we can construct a fundamental system of open neighborhoods of 0 by $\Lambda(G)$-submodules, we summarize this as a lemma.

Lemma 5.4. *Any object M in $Mod_{cpt}^{fl}(\Lambda(G))$ has a fundamental system of open neighborhoods of* 0 *by $\Lambda(G)$-submodules.*

Step 2

Now we can use the $\Lambda(G)$-action on M' to define a continuous map $\Lambda(G) \times M/M' \longrightarrow M/M$ (M/M' is finite with the discrete topology). Writing out $M/M' = \{m_1, \cdots, m_N\}$ for each index i we can find an open ideal $I_i \subseteq \Lambda(G)$ that annihilates m_i, $I_i\{m_i\} = \{0\}$. This lets us define an open ideal

$$I := \bigcap_{i=1}^{k} I_i$$

By consttruction I annihilates M/M'. Repeating the above argument N we get an open ideal $J \subseteq \Lambda(G)$ that annihilates N/N'. Thus we can define a continuous action map:

$$\Lambda/I \otimes_O \Lambda/J \times M/M' \otimes_O N/N' \longrightarrow M/M' \otimes_O N/N'$$
$$(\lambda \otimes \mu, x \otimes y) \longmapsto \lambda x \otimes \mu y$$

Passing to the projective limit defines a $\Lambda(G)\hat{\otimes}_O\Lambda(G)$-action on $M\hat{\otimes}_O N$. Explicitly this means $\Lambda(G)\hat{\otimes}_O\Lambda(G)$ acts on each component $M/M' \otimes N/N'$ via the appropriate projection

$$p_{IJ} : \Lambda(G)\hat{\otimes}_O\Lambda(G) \longrightarrow \Lambda/I \otimes_O \Lambda/J$$

Step 3

Finally we define a $\hat{\otimes}_O$-coalgebra map $c : \Lambda(G) \longrightarrow \Lambda(G)\hat{\otimes}_O\Lambda(G)$ and use it to endow $M\hat{\otimes}_O N$ with a $\Lambda(G)$-action.

The diagonal map $\Delta : G \longrightarrow G \times G$ and the canonical injections $\iota_1, \iota_2 : G \longrightarrow G \times G$ given by $\iota_1(g) = (g, 1)$ and $\iota_2(g) = (1, g)$ are all continuous group homomorphisms that induce continuous (injective) O-algebra morphisms $\Delta_*, \iota_{1*}, \iota_{2*} : \Lambda(G) \longrightarrow \Lambda(G \times G)$. We can use ι_{1*} and ι_{2*} to make the isomorphism (4.6) from Lemma 4.2, which we now call ψ, explicit:

$$\begin{aligned} \psi : \Lambda(G)\hat{\otimes}_O\Lambda(G) &\xrightarrow{\cong} \Lambda(G \times G) \\ \lambda \otimes \mu &\longmapsto \iota_{1*}(\lambda) \cdot \iota_{2*}(\mu) \end{aligned}$$

Define $c : \Lambda(G) \longrightarrow \Lambda(G)\hat{\otimes}_O\Lambda(G)$ as a composition of Δ_* and ψ^{-1}:

$$\underbrace{\Lambda(G) \xrightarrow{\Delta_*} \Lambda(G \times G) \xrightarrow{\psi^{-1}} \Lambda(G)\hat{\otimes}_O\Lambda(G)}_{c}$$

Combined with the work of Step 2 we get a continuous $\Lambda(G)$-action on $M\hat{\otimes}_O N$ which finishes the proof of Theorem 5.2.

The $\hat{\otimes}_O$-coalgebra structure on $\Lambda(G)$ allow us to define "group-like" elements in $\Lambda(G)$. We say that $\lambda \in \Lambda(G)$ is a group-like element if $c(\lambda) = \lambda \otimes \lambda$. Denote the set of group-like elements $\{\lambda \in \Lambda(G) \setminus \{0\} \;:\; c(\lambda) = \lambda \otimes \lambda\}$ by $\Lambda(G)^{gp}$.

Lemma 5.5. $G = \Lambda(G)^{gp}$

Proof. In [20, §30] Schneider notes that the following simple equality holds:

$$G = \{\lambda \in \Lambda(G) \setminus \{0\} : \Delta_*(\lambda) = \iota_{1*}(\lambda) \cdot \iota_{2*}(\lambda)\}$$

By a projective limit argument it will suffice to prove this for a finite group G, in which case $\Lambda(G) = O[G]$ and therefore elements λ can be expressed as finite sums

$$\lambda = \sum_{g \in G} c_g g$$

A simple calculation then shows that $\Delta_*(\lambda) = \iota_{1*}(\lambda) \cdot \iota_{2*}(\lambda)$ implies:

$$c_g = c_g^2 \quad \text{and} \quad c_g c_h = 0 \quad \text{for} \quad g \neq h \tag{5.2}$$

Since λ is assumed to be nonzero, it has at least one nonzero coefficient c_{g_0}. Conditions (5.2) imply $c_{g_0} = 1$ and $c_h = 0$ for $h \neq g_0$. In particular $\lambda = g_0 \in G$.

It follows immediately from the definition of c that for each nonzero $\lambda \in \Lambda(G)$

$$\Delta_*(\lambda) = \iota_{1*}(\lambda) \cdot \iota_{2*}(\lambda) \quad \text{holds if and only if} \quad c(\lambda) = \lambda \otimes \lambda \quad \text{holds}$$

Thus $G = \{\lambda \in \Lambda(G) \setminus \{0\} : c(\lambda) = \lambda \otimes \lambda\}$ □

We will use Lemma 5.5 in the proof of Theorem 1.5.

5.3 Extending compatibility of structures

We can now prove that Theorem 3.1 and 4.3 generalize to the categories.

Theorem 5.3. *Suppose* $(E, \|\|_E)$, $(F, \|\|_F)$ *are objects in* $Ban_G(K)^{\leq 1}$*, and M, N are objects in*

$Mod^{fl}_{cpt}(\Lambda(G))$. *Then:*

$$(E\hat{\otimes}_K F)^{\circ} \cong E^{\circ}\hat{\otimes}_O F^{\circ}$$

$$(M\hat{\otimes}_O N)^d \cong M^d\hat{\otimes}_O N^d$$

are natural isomorphisms in $Mod_G^{\varpi-cts}(O)^{fl}$.

Remark 5.2. We know that these (topological) isomorphisms hold in $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$, when we can consider the G-action to be trivial. It is enough to prove that the isomorphisms in Theorem 3.1 and 4.3 are in fact G-equivariant.

Proof. **Equivariance for Theorem 3.1**

Recall that the proof of Theorem 3.1 made use of the following commutative diagram:

$$\begin{array}{ccc} E^{\circ}\hat{\otimes}_O F^{\circ} & \overset{\mathbf{Id}^*}{\dashrightarrow} & (E\hat{\otimes}_K F)^{\circ} \\ \iota_1 \uparrow & \overset{\mu}{\nearrow} & \uparrow \iota_2 \\ E^{\circ}\otimes_O F^{\circ} & \underset{\mathbf{Id}}{\longrightarrow} & (E\otimes_K F)^{\circ} \end{array}$$

The map **Id** induced by the identity and ι_1 and ι_2 are easily seen to be G-equivariant. In particular μ and hence $\mathbf{Id}^*(x)$ must be G-equivariant.

Equivariance for Theorem 4.3

The isomorphism from Theorem 4.3 is a complicated composition of isomorphisms. It will suffice to show each isomorphism in the proof is G-equivariant. We verify the G-equivariance of Lemma 4.4 and 4.5

The isomorphism from Lemma 4.4 is given by:

$$\Phi : \mathrm{Hom}_O^{cts}(M\hat{\otimes}_O N, O) \longrightarrow \varprojlim_n \mathrm{Hom}_O^{cts}(M\hat{\otimes}_O N, O/\varpi^n O)$$

$$\lambda \longrightarrow (\lambda_n) = p_n \circ \lambda$$

where p_n are the canonical projection maps $p_n : O \longrightarrow O/\varpi^n O$. Let $g \in G$ and $x \in M\hat{\otimes}_O N$, then:

$$(g \cdot (p_n \circ \lambda))(x) = p_n(\lambda(g^{-1}x))$$
$$(p_n \circ (g\lambda))(x) = p_n(\lambda(g^{-1}x)$$

Thus $\Phi(g \cdot \lambda) = g \cdot \Phi(\lambda)$

The isomorphism from Lemma 4.5 is given by:

$$\Psi : \varinjlim_{M',N'} Hom_O\left(\frac{M}{M'} \otimes \frac{N}{N'}, O/\varpi^n O\right) \longrightarrow Hom_O^{cts}(M\hat{\otimes}_O N, O/\varpi^n O)$$
$$\mu \longmapsto \mu \circ p_{M'N'}$$

where $p_{M'N'}$ are the canonical projection maps as before, and μ is a given map:

$$\mu : \frac{M}{M'} \otimes \frac{N}{N'} \longrightarrow O/\varpi^n O$$

Let $g \in G$ and $x \in M\hat{\otimes}_O N$. We calculate:

$$(g \cdot (\mu \circ p_{M'N'}))(x) = (\mu \circ p_{M'N'})(g^{-1}x) = \mu(p_{M'N'}(g^{-1}x))$$
$$((g\mu) \circ p_{M'N'})(x) = (g\mu)(p_{M'N'}(x)) = \mu(g^{-1}p_{M'N'}(x))$$

Since $p_{M'N'}$ is G-equivariant, we get equality. Thus $\Psi(g \cdot \mu) = g \cdot \Psi(\mu)$. □

Chapter 6

The Recovery Theorem

In this section we combine the work from previous chapters to prove Theorem 1.5.

6.1 The recovery theorem

Theorem. *Let G be a profinite group, and $Ban_G(K)$ be the category of K-Banach space representations, with tensor product bifunctor $\hat{\otimes}_K$ and a forgetful functor ω (that forgets the G-action). There is a continuous isomorphism of topological groups:*

$$Aut^{\otimes}(\omega) \cong G$$

6.2 Proof of the recovery theorem

The proof can be broken up into 3 main steps:

Step 1 Prove

$$\mathrm{Aut}^{\otimes}(\omega_1) \underset{\text{Lemma 6.1}}{\cong} \mathrm{Aut}^{\otimes}(\omega_3)$$

to transfer the problem to $\mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$.

Step 2 Prove

$$\mathrm{Aut}^{\otimes}(\omega_3) \underset{\text{Lemma 6.2}}{\cong} \Lambda(G)^{gp}$$

to relate the (tensor preserving) automorphisms of ω_3 with the group-like elements of $\Lambda(G)$ (cf section 5.2.3).

Step 3 Apply Lemma 5.5

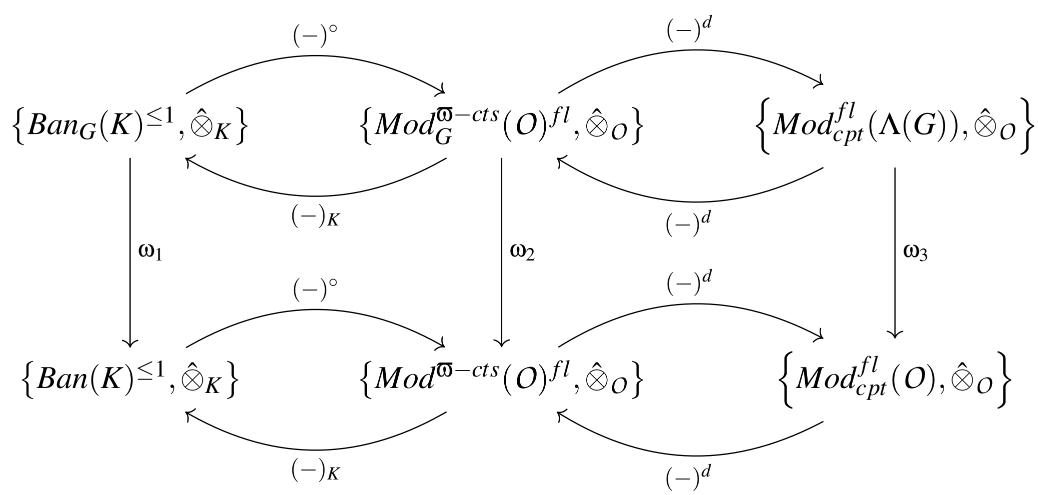
$$G = \Lambda(G)^{gp}$$

to finish the proof.

We have already discussed in Section 5.1 that we recover $\mathrm{Ban}_G(K)$ by localizing $\mathrm{Ban}_G(K)^{\leq 1}$ at $\mathbb{Q}$ and so we begin by transferring this problem to $\mathrm{Ban}_G(K)^{\leq 1}$ and renaming the forgetful functor ω as

$$\omega_1 : \mathrm{Ban}_G(K)^{\leq 1} \longrightarrow \mathrm{Ban}(K)^{\leq 1}$$

Each of the categories introduced in section 5.1 has a forgetful functor of its own that forgets the G-action. Naming these ω_1, ω_2, ω_3 we get the following diagram of categories and functors:



6.2.1 Proving Lemma 6.1

Recall that a monoid is set endowed with a binary operation that is associative and has an identity element. The set of all (tensor preserving) automorphisms of a functor naturally has the structure of a monoid with vertical composition (cf section A.2.1) as the binary operation.

Lemma 6.1. *With* ω_1, ω_2, ω_3 *as above, we have isomorphisms of monoids:*

$$Aut^{\otimes}(\omega_1) \cong Aut^{\otimes}(\omega_2) \cong Aut^{\otimes}(\omega_3) \tag{6.1}$$

Proof. The functors ω_1 and ω_2 are related by the following diagram:

$$\begin{array}{ccc} Ban_G(K)^{\leq 1} & \xrightarrow{(-)^{\circ}} & Mod_G^{\varpi-cts}(O)^{fl} \\ \Big\downarrow \omega_1 & & \Big\downarrow \omega_2 \\ Ban(K)^{\leq 1} & \xrightarrow[(-)^{\circ}]{} & Mod^{\varpi-cts}(O)^{fl} \end{array}$$

We can restrict to the unit ball then forget the G-action or forget the G-action and then restrict to the unit ball to get the exact same element of $\mathrm{Mod}^{\varpi-cts}(O)^{fl}$. This means we have an equality of functors $\omega_2 \circ (-)^{\circ} = (-)^{\circ} \circ \omega_1$ in the sense that, given any $(E, \|\|)$ in $\mathrm{Ban}_G(K)^{\leq 1}$ we have:

$$\omega_2(E^{\circ}) = (\omega_1(E))^{\circ} \tag{6.2}$$

Since the functors $(-)^{\circ}$ and $(-)_K$ are quasi-inverse, there is a natural isomorphism of functors

$$(-)^{\circ} \circ (-)_K \cong \mathbf{1} \tag{6.3}$$

where $\mathbf{1}$ is the identity functor on $\mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$. This means for any V in $\mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$ we have a natural isomorphism $V \cong (V_K)^\circ$, combining with (6.2) we get

$$\omega_2(V) \cong \omega_2((V_K)^\circ) = (\omega_1(V_K))^\circ$$

This is a component (at V) of the natural isomorphism $\varphi : \omega_2 \longrightarrow (-)^\circ \circ \omega_1 \circ (-)_K$ induced by (6.2) and (6.3) in the following way

$$\omega_2 \cong \omega_2 \circ \mathbf{1} \cong \omega_2 \circ (-)^\circ \circ (-)_K = (-)^\circ \circ \omega_1 \circ (-)_K \tag{6.4}$$

We can use the natural transformation φ and the functors $(-)^\circ$, $(-)_K$ to define:

$$\begin{aligned} \Phi : \mathrm{Aut}^{\otimes}(\omega_1) &\longrightarrow \mathrm{Aut}^{\otimes}(\omega_2) \\ \lambda &\longmapsto \varphi^{-1} \circ ((-)^\circ \cdot \lambda \cdot (-)_K) \circ \varphi \end{aligned} \tag{6.5}$$

where the natural transformation $(-)^\circ \cdot \lambda \cdot (-)_K$ above is an automorphism of $(-)^\circ \circ \omega_1 \circ (-)_K$ defined by:

$$((-)^\circ \cdot \lambda \cdot (-)_K)_V := (\lambda_{V_K})^\circ \tag{6.6}$$

(We are applying (A.3) with $\mathcal{F} = (-)_K$ and $\mathcal{K} = (-)^\circ$). Unraveling definitions we see that $\Phi(\lambda)_V = \varphi_V^{-1} \circ (\lambda_{V_K})^\circ \circ \varphi_V$

$$\begin{array}{ccc} \omega_2(V) & \overset{\Phi(\lambda)_V}{\dashrightarrow} & \omega_2(V) \\ \Big\downarrow \varphi_V & & \Big\uparrow \varphi_V^{-1} \\ \omega_1(V_K)^\circ & \underset{(\lambda_{V_K})^\circ}{\longrightarrow} & \omega_1(V_K)^\circ \end{array}$$

In particular, for any $v \in V$

$$\Phi(\lambda)_V(v) := \varphi_V^{-1}((\lambda_{V_K})^\circ(\varphi_V(v))) \tag{6.7}$$

By construction it should be clear that $\Phi(\lambda)$ is an automorphism of ω_2. To see that $\Phi(\lambda)$ is tensor preserving we note that by Lemma A.1 it suffices to prove that φ and $(-)^\circ \cdot \lambda \cdot (-)_K$ are tensor preserving.

Let $\omega_1^* = (-)^\circ \circ \omega_1 \circ (-)_K = (-)^\circ \circ (-)_K \circ \omega_2$, φ is shown to be tensor preserving by considering the following diagram

$$\begin{array}{ccc} \omega_2(V \otimes W) & \xrightarrow{\varphi_{V\otimes W}} & \omega_1^*(V \otimes W) \\ \| & & \downarrow \cong \\ \omega_2(V) \otimes \omega_2(W) & \xrightarrow{\varphi_V \otimes \varphi_W} & \omega_1^*(V) \otimes \omega_1^*(W) \end{array}$$

Noting that $\omega_1^*(V \otimes W) = (\omega_2(V \otimes W)_K)^\circ = ((\omega_2(V) \otimes \omega_2(W))_K)^\circ$. Finally $(-)^\circ \cdot \lambda \cdot (-)_K$ can be shown to be tensor preserving by applying $\lambda_{V_K \otimes W_K} = \lambda_{V_K} \otimes \lambda_{W_K}$ (since λ is tensor preserving).

This proves that Φ is a map $\mathrm{Aut}^\otimes(\omega_1) \longrightarrow \mathrm{Aut}^\otimes(\omega_2)$, we claim that this is a monoid isomorphism. First we show that Φ respects composition.

Let λ_1, λ_2 be in $\mathrm{Aut}^\otimes(\omega_1)$ and calculate:

$$\Phi(\lambda_2 \circ \lambda_1) := \varphi^{-1} \circ (-)^\circ(\lambda_2 \circ \lambda_1)(-)_K \circ \varphi$$

Since $((\lambda_2)_{V_K} \circ (\lambda_1)_{V_K})^\circ = ((\lambda_2)_{V_K})^\circ \circ ((\lambda_1)_{V_K})^\circ$ we get:

$$(-)^\circ(\lambda_2 \circ \lambda_1)(-)_K = (-)^\circ \lambda_2 (-)_K \circ (-)^\circ \lambda_1 (-)_K$$

Simplifying the above:

$$\begin{aligned}\Phi(\lambda_2 \circ \lambda_1) &= \varphi^{-1} \circ (-)^{\circ} \lambda_2 (-)_K \circ (-)^{\circ} \lambda_1 (-)_K \circ \varphi \\ &= (\varphi^{-1} \circ ((-)^{\circ} \lambda_2 (-)_K) \circ \varphi) \circ (\varphi^{-1} \circ ((-)^{\circ} \lambda_1 (-)_K) \circ \varphi) \\ &= \Phi(\lambda_2) \circ \Phi(\lambda_1)\end{aligned}$$

Finally we construct an inverse map Φ^{-1} in the obvious way. Now let φ' be the natural isomorphism $\omega_1 \longrightarrow (-)_K \circ \omega_2 \circ (-)^{\circ}$ and define:

$$\begin{aligned}\Phi^{-1} : \mathrm{Aut}^{\otimes}(\omega_2) &\longrightarrow \mathrm{Aut}^{\otimes}(\omega_1) \\ \mu &\longmapsto (\varphi')^{-1} \circ ((-)_K \cdot \mu \cdot (-)^{\circ}) \circ \varphi'\end{aligned}$$

$$\begin{aligned}\Phi^{-1}\Phi(\lambda) &= \Phi^{-1}\left(\varphi^{-1} \circ \lambda^* \circ \varphi\right) \\ &= (\varphi')^{-1} \circ (\varphi^{-1} \circ \lambda^* \circ \varphi)^* \circ \varphi' \\ &= (\varphi')^{-1} \circ \left[(-)_K \cdot (\varphi^{-1} \circ \lambda^* \circ \varphi) \cdot (-)^{\circ}\right] \circ \varphi' \\ &= (\varphi')^{-1} \circ \left[(-)_K \cdot (\varphi^{-1} \circ (-)^{\circ} \cdot \lambda \cdot (-)_K \circ \varphi) \cdot (-)^{\circ}\right] \circ \varphi'\end{aligned}$$

Given an object $V \in \mathrm{Mod}_G^{\varpi-cts}(O)^{fl}$, one can explicitly verify $\Phi^{-1}\Phi(\lambda)_V = \lambda_V$ holds for all V and hence $\Phi^{-1}\Phi(\lambda) = \lambda$. In particular $\mathrm{Aut}^{\otimes}(\omega_1) \cong \mathrm{Aut}^{\otimes}(\omega_2)$

Formally the argument for $\mathrm{Aut}^{\otimes}(\omega_2) \cong \mathrm{Aut}^{\otimes}(\omega_3)$ is the same. The difference in this case is that the Schikhof dual functors are contravariant and so a little care is taken with direction of arrows.

□

6.2.2 Proving Lemma 6.2

Recall that c is the $\hat{\otimes}_O$-coalgebra map defined in Section 5.2.3 as the composition:

$$c : \Lambda(G) \xrightarrow{\Delta_*} \Lambda(G \times G) \xrightarrow{\psi^{-1}} \Lambda(G)\hat{\otimes}_O\Lambda(G)$$

and we call the elements $\lambda \in \Lambda(G)$ satisfying $c(\lambda) = \lambda \otimes \lambda$ the group-like elements of $\Lambda(G)$

$$\Lambda(G)^{gp} := \{\lambda \in \Lambda(G) \setminus \{0\} \ : \ c(\lambda) = \lambda \otimes \lambda\}$$

Lemma 6.2. *With notation as above, there is a continuous isomorphism of groups*

$$Aut^{\otimes}(\omega_3) \cong \Lambda(G)^{gp} \tag{6.8}$$

Proof. Given a fixed $\lambda \in \Lambda(G)^{gp}$ and M in $Mod^{fl}_{cpt}(\Lambda(G))$ we construct a map:

$$\begin{aligned} \lambda_M : M &\longrightarrow M \\ m &\longmapsto \lambda \cdot m \end{aligned}$$

which we view as a morphism on the underlying O-modules, i.e. as a morphism in $Mod^{fl}_{cpt}(O)$. We claim that $\lambda = (\lambda_M)$ is in $\mathrm{Aut}^{\otimes}(\omega_3)$. To check the commutivity property let $f : M \longrightarrow N$ be a morphism in $Mod^{fl}_{cpt}(\Lambda(G))$, then $\omega_3(f)$ is the underlying O-module morphism in $Mod^{fl}_{cpt}(O)$ and we have a diagram:

$$\begin{array}{ccc} \omega_3(M) & \xrightarrow{\lambda_M} & \omega_3(M) \\ {\scriptstyle \omega_3(f)}\downarrow & & \downarrow{\scriptstyle \omega_3(f)} \\ \omega_3(N) & \xrightarrow{\lambda_N} & \omega_3(N) \end{array}$$

To show that this is commutative is a straightforward calculation:

$$(\omega_3(f)\circ\lambda_M)(m) = \omega_3(f)(\lambda m) = \lambda f(m)$$

$$(\lambda_N\circ\omega_3(f))(m) = \lambda_N(f(m)) = \lambda f(m)$$

The equality $\omega_3(f)(\lambda m) = \lambda f(m)$ follows from the fact that $\omega_3(f)(m) := f(m)$ for all $m\in M$. Since f is $\Lambda(G)$-linear $f(\lambda m) = \lambda f(m)$ and therefore $\omega_3(f)(\lambda m) = \lambda f(m)$.

To check that $\lambda = (\lambda_M)$ is tensor preserving we verify that the following diagram is commutative:

$$\begin{array}{ccc}
\omega_3(M\hat{\otimes}_O N) & \xrightarrow{\lambda_{M\otimes N}} & \omega_3(M\hat{\otimes}_O N) \\
\| & & \| \\
\omega_3(M)\hat{\otimes}_O\omega_3(N) & \xrightarrow{\lambda_M\otimes\lambda_M} & \omega_3(M)\hat{\otimes}_O\omega_3(N)
\end{array}$$

$$\lambda_{M\otimes N}(m\otimes n) = \lambda\cdot(m\otimes n) := c(\lambda)(m\otimes n) \tag{6.9}$$

The last equality follows from the construction of the $\Lambda(G)$-action on $M\hat{\otimes}_O N$. Since $\lambda\in\Lambda(G)^{gp}$

$$c(\lambda)(m\otimes n) = (\lambda\otimes\lambda)(m\otimes n) = \lambda m\otimes\lambda n \tag{6.10}$$

On the other hand

$$(\lambda_M\otimes\lambda_N)(m\otimes n) := \lambda_M(m)\otimes\lambda_N(n) = \lambda m\otimes\lambda n \tag{6.11}$$

Any $\lambda\in\Lambda(G)^{gp}$ defines an element (λ_M) in $\mathrm{Aut}^{\otimes}(\omega_3)$ and so we have a map:

$$\begin{aligned}\Lambda(G)^{gp} &\longrightarrow \mathrm{Aut}^{\otimes}(\omega_3)\\ \lambda &\longmapsto (\lambda_M)\end{aligned} \tag{6.12}$$

Moreover multiplication by the identity $1\in\Lambda(G)^{gp}$ is clearly the identity automorphism $(1_M$

is the identity map on M for every M) and (left) multiplication by a product $\mu\lambda$ is the same as multiplication by λ then μ ($\mu \cdot \lambda \longmapsto (\mu\lambda)_M = \mu_M \circ \lambda_M$). Together this means that at the very least (6.12) is an injective homomorphism of monoids. Next we prove surjectivity.

Let η be in $\mathrm{Aut}^{\otimes}(\omega_3)$, we need to show that η is given by multiplication with a group like element of $\Lambda(G)$. Let M be in $\mathrm{Mod}^{fl}_{cpt}(\Lambda(G))$ and fix an element $m_0 \in M$. Consider the following morphism in $\mathrm{Mod}^{fl}_{cpt}(\Lambda(G))$:

$$\begin{aligned} f : \Lambda(G) &\longrightarrow M \\ \lambda &\longmapsto \lambda m_0 \end{aligned} \tag{6.13}$$

By functoriality the following diagram commutes:

$$\begin{array}{ccc} \omega_3(\Lambda(G)) & \xrightarrow{\eta_{\Lambda(G)}} & \omega_3(\Lambda(G)) \\ \omega_3(f) \downarrow & & \downarrow \omega_3(f) \\ \omega_3(M) & \xrightarrow{\eta_M} & \omega(M) \end{array}$$

and so we have

$$f \circ \eta_{\Lambda(G)} = \eta_M \circ f$$

Evaluating at $\lambda = 1 \in \omega_3(\Lambda(G))$:

$$\begin{aligned} (\eta_M \circ f)(1) = \eta_M(f(1)) = \eta_M(m_0) \\ (f \circ \eta_{\Lambda(G)})(1) = f(\eta_{\Lambda(G)}(1)) = \eta_{\Lambda(G)}(1) \cdot m_0 \end{aligned} \tag{6.14}$$

Now if we vary $m_0 \in M$ we see that

$$\eta_M(m) = \eta_{\Lambda(G)}(1) \cdot m \tag{6.15}$$

Since η is an automorphism of ω_3, we must necessarily have $\lambda_0 := \eta_{\Lambda(G)}(1) \neq 0$ and η is assumed to be tensor preserving thus $\eta_{M\otimes N} = \eta_M \otimes \eta_N$. A simple check gives us

$$\eta_{M\otimes N}(m\otimes n) = \lambda_0 \cdot (m\otimes n) := c(\lambda_0)(m\otimes n) \tag{6.16}$$

While on the other hand

$$(\eta_M \otimes \eta_N)(m\otimes n) = \eta_M(m)\otimes\eta_N(n) = \lambda_0\cdot m\otimes\lambda_0\cdot n =: (\lambda_0\otimes\lambda_0)\cdot m\otimes n \tag{6.17}$$

In particular $\lambda_0 \in \Lambda(G)^{gp}$, and so every tensor preserving automorphism of ω_3 is given by multiplication by some group-like element in $\Lambda(G)$.

The above work means $\mathrm{Aut}^{\otimes}(\omega_3)$ is isomorphic to $\Lambda(G)^{gp}$ as a monoid and therefore also as a group. By endowing $\mathrm{Aut}^{\otimes}(\omega_3)$ with the coarsest topology so that for each object M in $\mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$ the projections

$$\mathrm{Aut}^{\otimes}(\omega_3) \longrightarrow \mathrm{End}(\omega_3(M))$$

are continuous, $\mathrm{Aut}^{\otimes}(\omega_3)$ is even a topological group.

The natural topology to impose on $\mathrm{End}(\omega_3(M))$ is the compact-open topology with a basis of open sets indexed by K (compact in M) and E (open in M)

$$U_{K,E} := \{\varphi : M \longrightarrow M : \varphi(K) \subseteq E\}$$

To see that $\Lambda(G)^{gp} \longrightarrow \mathrm{Aut}^{\otimes}(\omega_3)$ is continuous it will suffice to prove $\Lambda(G)^{gp} \longrightarrow \mathrm{End}(\omega_3(M))$ is continuous for any M. This follows from the definition of the map (a group element g is sent to the the endomorphism given by multiplication by g) and the continuity of the G-action on each M. □

6.2.3 Proof of Theorem 1.5

Proof. First apply Lemma 6.1

$$\mathrm{Aut}^{\otimes}(\omega) = \mathrm{Aut}^{\otimes}(\omega_1) \cong \mathrm{Aut}^{\otimes}(\omega_3)$$

By Lemma 6.2 there is a continuous isomorphism of topological groups

$$\mathrm{Aut}^{\otimes}(\omega_3) \cong \Lambda(G)^{gp}$$

In particular since $\Lambda(G)^{gp} = G$ (Lemma 5.5)

$$G \cong \mathrm{Aut}^{\otimes}(\omega)$$

is a continuous isomorphism of topological groups □

6.3 A classification result for Iwasawa algebras

Corollary 6.1. *Let G, H be two profinite groups for which there exists a topological isomorphism of $\mathcal{O}$-algebras $\varphi : \Lambda(G) \longrightarrow \Lambda(H)$ that is compatible with the $\hat{\otimes}_{\mathcal{O}}$-coalgebra structures of $\Lambda(G)$ and $\Lambda(H)$, denoted c_1 and c_2 respectively, such that the following diagram commutes:*

$$\begin{array}{ccc} \Lambda(G)\hat{\otimes}_{\mathcal{O}}\Lambda(G) & \xrightarrow{\varphi\otimes\varphi} & \Lambda(H)\hat{\otimes}_{\mathcal{O}}\Lambda(H) \\ \uparrow{\scriptstyle c_1} & & \uparrow{\scriptstyle c_2} \\ \Lambda(G) & \xrightarrow[\varphi]{} & \Lambda(H) \end{array}$$

Then $G \cong H$.

Proof. The isomorphism φ induces a functor $\mathcal{F}_\varphi : \mathrm{Mod}_{cpt}^{fl}(\Lambda(H)) \longrightarrow \mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$ by restriction of scalars. Given an object M in $\mathrm{Mod}_{cpt}^{fl}(\Lambda(H))$, the $\Lambda(G)$-module structure on $\mathcal{F}_\varphi(M)$ is defined by:

$$\lambda \cdot m := \varphi(\lambda) \cdot m$$

This functor maps $\Lambda(H)$-equivariant morphisms $f : M \longrightarrow N$ in $\mathrm{Mod}_{cpt}^{fl}(\Lambda(H))$ to $\Lambda(G)$-equivariant morphisms $\mathcal{F}_\varphi(f) : \mathcal{F}_\varphi(M) \longrightarrow \mathcal{F}_\varphi(N)$ in $\mathrm{Mod}_{cpt}^{fl}(\Lambda(G))$ since for any $\lambda \in \Lambda(G)$ and M, N in $\mathrm{Mod}_{cpt}^{fl}(\Lambda(H))$ we have

$$\mathcal{F}_\varphi(f)(\lambda \cdot m) = \mathcal{F}_\varphi(f)(\varphi(\lambda) \cdot m) = \varphi(\lambda) \cdot \mathcal{F}_\varphi(f)(m) = \lambda \cdot \mathcal{F}_\varphi(f)(m)$$

In fact $\mathcal{F}_\varphi$ defines an equivalence of *tensor* categories as an immediate consequence of the commutative diagram above. Recall that the $\Lambda(G)$-action on $M\hat{\otimes}_O N$ is defined using the $\hat{\otimes}_O$-coalgebra map c_1. The commutivity of the diagram ensures the $\Lambda(G)$-action on $\mathcal{F}_\varphi(M\hat{\otimes}_O N)$ and $\mathcal{F}_\varphi(M)\hat{\otimes}_O \mathcal{F}_\varphi(N)$ coincide.

Applying the anti-equivalence of categories between Banach space representations of G and Iwasawa G-modules we deduce that we have an equivalence of categories

$$\mathrm{Ban}_G(K)^{\leq 1} \xrightarrow{\sim} \mathrm{Ban}_H(K)^{\leq 1}$$

Finally, Theorem 1.5 implies $G \cong H$

$\square$

6.4 Further research topics

Naturally the more learnt about a subject, the more questions seem to arise, and indeed there is much still left to learn about Banach representations of p-adic Lie groups. We expect continued interest in this field to drive developments and applications in the local p-adic Langlands program.

A natural first extension to the recovery theorem in this dissertation is to prove a Krein-type recognition theorem. Such a result would answer the following question: given an abstract category $\mathcal{C}$ (endowed with a tensor product $\otimes$ and fiber functor ω) what are the necessary and sufficient conditions we can place on $\{\mathcal{C}, \otimes, \omega\}$ to ensure an equivalence of categories

$$C \xrightarrow{\sim} \mathrm{Ban}_G(K)$$

(and therefore that we recover a profinite group $\mathrm{Aut}^{\otimes}(\omega)$ from $\mathcal{C}$).

Additionally, since in typical applications (in the local p-adic Langlands program especially) G is often only *locally* profinite (meaning *locally* compact, Hausdorff and totally disconnected), proving a generalized recovery theorem that allows for locally profinite G could be useful result.

The situation is a lot more complicated in this case however. One of the main issues is that we can no longer use the regular definition for $\Lambda(G)$ (which relied on G being compact), instead one can define $\Lambda(G)$ as a tensor product of $\Lambda(G_0)$ and $\mathcal{O}[G]$ where G_0 is a compact open subgroup in G. Of course $\Lambda(G)$ defined in this way is no longer necessarily Noetherian.

Appendix A

Definitions from Category Theory

A.1 Basic definitions

Definition A.1. A category $\mathcal{C}$ consists of the following data:

- A class of objects $\mathrm{ob}(\mathcal{C})$.
- A class of morphisms $\hom_{\mathcal{C}}(X,Y)$ between any two objects X, Y
- For each object X, a distinguished morphism $1_X \in \hom_{\mathcal{C}}(X,X)$ (called the identity morphism).
- A binary (composition) operation

$$\begin{aligned} \hom_{\mathcal{C}}(X,Y) \times \hom_{\mathcal{C}}(Y,Z) &\longrightarrow \hom_{\mathcal{C}}(X,Z) \\ (f,g) &\longmapsto gf \end{aligned} \tag{A.1}$$

Additionally morphisms must satisfy the following axioms:

(Associativity) For any morphisms $f \in \hom_{\mathcal{C}}(X,Y)$, $g \in \hom_{\mathcal{C}}(Y,Z)$, $h \in \hom_{\mathcal{C}}(Z,W)$

$$(hg)f = h(gf)$$

(Identity) For any morphisms $f \in \hom_{\mathcal{C}}(X, A)$ and $g \in \hom_{\mathcal{C}}(B, Y)$

$$f 1_X = f \quad \text{and} \quad 1_Y g = g$$

Definition A.2. Given an additive category $\mathcal{C}$, its localization at $\mathbb{Q}$, denoted $\mathcal{C}_{\mathbb{Q}}$, is the (additive) category with:

$$\mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\mathcal{C}_{\mathbb{Q}})$$

$$\hom_{\mathcal{C}_{\mathbb{Q}}}(E, F) = \hom_{\mathcal{C}}(E, F) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Remark A.1. Every module category over a ring is additive.

Definition A.3. Let $\mathcal{C}$, $\mathcal{D}$ be two categories. A (covariant) functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is a mapping that:

- Associates to every object X in $\mathcal{C}$ an object $\mathcal{F}(X)$ in $\mathcal{D}$.
- Associates to every morphism f in $\hom_{\mathcal{C}}(X, Y)$ a morphism $\mathcal{F}(f)$ in $\hom_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ such that $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$ and $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$

A contravariant functor is one that reverses the direction of arrows, so that given a morphism f in $\hom_{\mathcal{C}}(X, Y)$ $\mathcal{F}(f)$ is a morphism in $\hom_{\mathcal{D}}(\mathcal{F}(Y), \mathcal{F}(X))$

Definition A.4. Let $\mathcal{C}$, $\mathcal{D}$ be two categories. An equivalence of categories consists of two functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$, $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ and natural isomorphisms

$$\mathcal{G}\mathcal{F} \cong \mathbf{1}_{\mathcal{C}} \quad \text{and} \quad \mathcal{F}\mathcal{G} \cong \mathbf{1}_{\mathcal{D}}$$

where $\mathbf{1}_{\mathcal{C}}$ (resp. $\mathbf{1}_{\mathcal{D}}$) is the identity functor on $\mathcal{C}$ (resp. $\mathcal{D}$). Functors $\mathcal{F}$ and $\mathcal{G}$ as above are called quasi-inverse.

We say that the above is an equivalence of tensor categories if $\mathcal{F}$, $\mathcal{G}$ are tensor functors between tensor categories $\mathcal{C}$, $\mathcal{D}$.

Remark A.2. A functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ induces an equivalence of categories between $\mathcal{C}$ and $\mathcal{D}$ if and only if $\mathcal{F}$ is essentially surjective (each object in $\mathcal{D}$ is naturally isomorphic to $\mathcal{F}(X)$ for some X in $\mathcal{C}$) and faithfully full ($\mathcal{F}$ induces a bijection between $\hom_{\mathcal{C}}(X,Y)$ and $\hom_{\mathcal{D}}(\mathcal{F}(X),\mathcal{F}(Y))$).

A.2 Natural transformations

Definition A.5. Let $\mathcal{F},\ \mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ be functors between categories $\mathcal{C}$ and $\mathcal{D}$. A natural transformation $\eta : \mathcal{F} \longrightarrow \mathcal{G}$ is a collection of morphisms $\eta_X : \mathcal{G}(X) \longrightarrow \mathcal{F}(X)$ indexed by objects X in $\mathcal{C}$, that are compatible in the sense that for any morphism f in $\hom_{\mathcal{C}}(X,Y)$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\eta_X} & \mathcal{G}(X) \\ {\scriptstyle \mathcal{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}(Y) & \xrightarrow{\eta_Y} & \mathcal{G}(Y) \end{array}$$

An endomorphism of a functor $\mathcal{F}$ is a natural transformation $\eta : \mathcal{F} \longrightarrow \mathcal{F}$, the collection of all endomorphisms of $\mathcal{F}$ is denoted $\mathrm{End}(\mathcal{F})$.

If a endomorphism of $\mathcal{F}$ has a quasi-inverse it is called an automorphism of $\mathcal{F}$, the collection of all automorphisms of $\mathcal{F}$ is denoted $\mathrm{Aut}(\mathcal{F})$.

A.2.1 Operations on natural transformations:

1. ”Vertical” composition of natural transformations.

   Let $\mathcal{C}$, $\mathcal{D}$ be categories with functors $\mathcal{F},\ \mathcal{G},\mathcal{H} : \mathcal{C} \longrightarrow \mathcal{D}$ and natural transformations $\lambda : \mathcal{F} \longrightarrow \mathcal{G}, \mu : \mathcal{G} \longrightarrow \mathcal{H}$. Define the natural transformation $\mu \circ \lambda : \mathcal{F} \longrightarrow \mathcal{H}$ by

$$(\mu \circ \lambda)_X = \mu_X \circ \lambda_X \qquad \text{(A.2)}$$

for any object $X \in \mathcal{C}$

2. Natural transformations and functors.

   Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be categories with functors

$$\begin{aligned} \mathcal{F} : \mathcal{A} &\longrightarrow \mathcal{B} \\ \mathcal{G}, \mathcal{H} : \mathcal{B} &\longrightarrow \mathcal{C} \\ \mathcal{K} : \mathcal{C} &\longrightarrow \mathcal{D} \end{aligned}$$

   and let $\lambda : \mathcal{G} \longrightarrow \mathcal{H}$ be a natural transformation. We can form the following natural transformations:

$$\begin{aligned} \lambda\mathcal{F} : \mathcal{G}\mathcal{F} \longrightarrow \mathcal{H}\mathcal{F} \quad &\text{defined by:} \quad (\lambda\mathcal{F})_A := \lambda_{\mathcal{F}(A)} \\ \mathcal{K}\lambda : \mathcal{K}\mathcal{G} \longrightarrow \mathcal{K}\mathcal{H} \quad &\text{defined by:} \quad (\mathcal{K}\lambda)_B := \mathcal{K}\lambda_B \end{aligned} \tag{A.3}$$

A.3 Monoidal categories

Definition A.6. A (non-strict) tensor, or monoidal category is a category $\mathcal{C}$ equipped with a tensor product bifunctor $(-) \otimes (-) : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ and a distinguished object $\mathbf{1}$ such that

$$\mathbf{1} \otimes X \cong X \cong X \otimes \mathbf{1} \quad \text{and} \quad X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$$

are natural isomorphisms for any objects X, Y, Z in $\mathcal{C}$. There are also additional coherence conditions imposed to ensure morphisms constructed out of the isomorphisms above coincide, these are called the pentagon axiom ([16, pg 162]) and the hexagon axiom ([16, pg 184]). These conditions do not play a role in our proofs.

Definition A.7. Let $\mathcal{C}$, $\mathcal{D}$ be monoidal categories with tensor product bifunctors $(-) \otimes_1 (-)$ and $(-) \otimes_2 (-)$ respectively. A functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is called a tensor functor (or monoidal functor) if $\mathcal{F}(X \otimes_1 Y) \cong \mathcal{F}(X) \otimes_2 \mathcal{F}(Y)$ is a functorial isomorphism.

A tensor functor $\mathcal{F}$ is called strict if $\mathcal{F}(X \otimes_1 Y) = \mathcal{F}(X) \otimes_2 \mathcal{F}(Y)$

Definition A.8. Let $\mathcal{C}$, $\mathcal{D}$ be tensor categories with tensor functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$. A natural transformation $\eta : \mathcal{F} \longrightarrow \mathcal{G}$ is called tensor preserving (or monoidal) if for all objects X, Y in $\mathcal{C}$ the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{F}(X \otimes Y) & \xrightarrow{\eta_{X \otimes Y}} & \mathcal{G}(X \otimes Y) \\
\cong \downarrow & & \downarrow \cong \\
\mathcal{F}(X) \otimes \mathcal{F}(Y) & \xrightarrow{\eta_X \otimes \eta_Y} & \mathcal{G}(X) \otimes \mathcal{G}(Y)
\end{array}
$$

If $\mathcal{F}$ and $\mathcal{G}$ are strict tensor functors, then $\eta_{X \otimes Y} = \eta_X \otimes \eta_Y$ in the sense that, for any x in X and y in Y

$$\eta_{X \otimes Y}(x \otimes y) = \eta_X(x) \otimes \eta_Y(y)$$

We will denote the tensor preserving automorphisms of $\mathcal{F}$ by $\mathrm{Aut}^{\otimes}(\mathcal{F})$.

Lemma A.1. *Let $\mathcal{C}$, $\mathcal{D}$ be tensor categories with tensor functors $\mathcal{F}, \mathcal{G}, \mathcal{H} : \mathcal{C} \longrightarrow \mathcal{D}$. If the natural transformations $\lambda : \mathcal{F} \longrightarrow \mathcal{G}$ and $\mu : \mathcal{G} \longrightarrow \mathcal{H}$ are tensor preserving, then $\mu \circ \lambda$ is tensor preserving.*

Proof. This is almost immediate from the commutative diagram:

$$
\begin{array}{ccccc}
\mathcal{F}(X \otimes Y) & \xrightarrow{\lambda_{X \otimes Y}} & \mathcal{G}(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} & \mathcal{H}(X \otimes Y) \\
\cong \downarrow & & \downarrow \cong & & \downarrow \cong \\
\mathcal{F}(X) \otimes \mathcal{F}(Y) & \xrightarrow{\lambda_X \otimes \lambda_Y} & \mathcal{G}(X) \otimes \mathcal{G}(Y) & \xrightarrow{\mu_X \otimes \mu_Y} & \mathcal{H}(X) \otimes \mathcal{H}(Y)
\end{array}
$$

The top row is $(\mu \circ \lambda)_{X \otimes Y}$ and the bottom row is $(\mu \circ \lambda)_X \otimes (\mu \circ \lambda)_Y$. The left and right squares commute (by assumption), thus the entire diagram commutes. □

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