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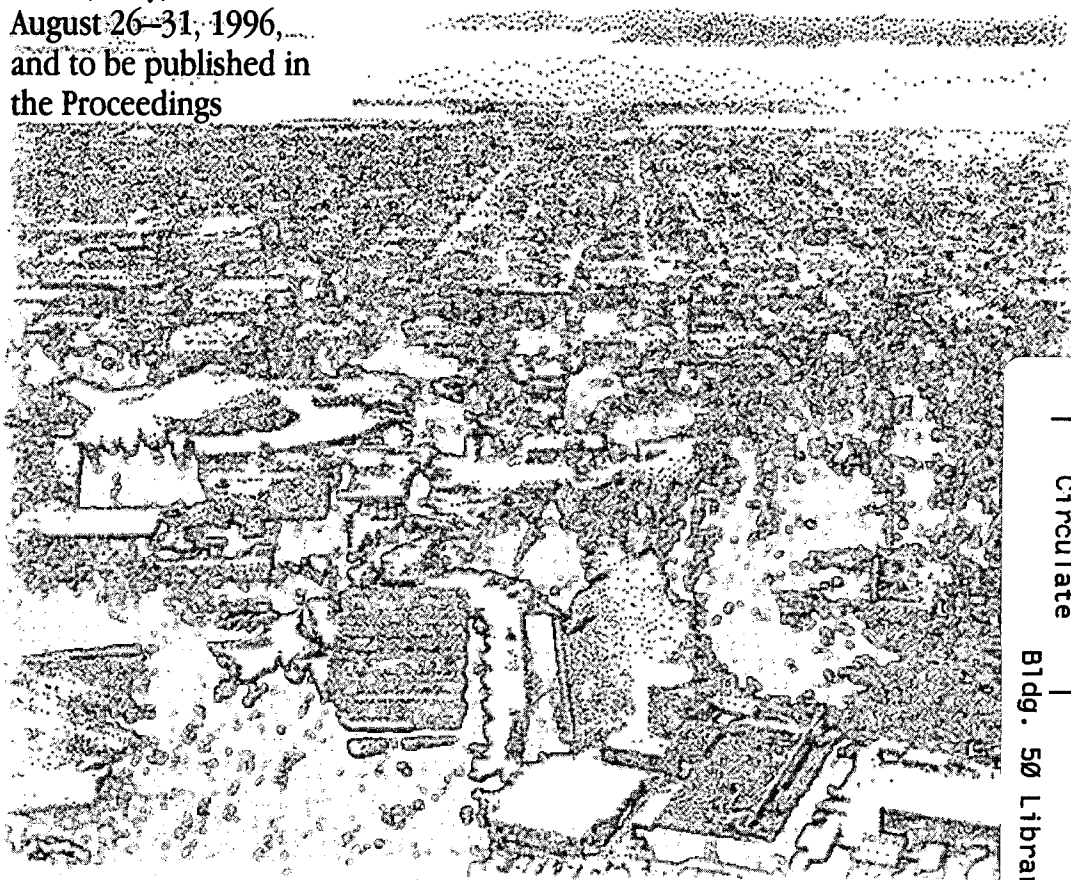
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Kim, K.J.

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Start-Up Noise in 3-D Self-Amplified Spontaneous Emission

K.-J. Kim
Accelerator and Fusion Research Division

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Start-Up Noise in 3-D Self-Amplified Spontaneous Emission[*]

Kwang-Je Kim

Ernest Orlando Lawrence Berkeley National Laboratory
Berkeley, CA 94720 USA

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1. Introduction

The Self-Amplified-Spontaneous Emission (SASE) is receiving a renewed interest recently in connection with the proposal to build x-ray FEL facilities [1,2]. A consistent theory of how the initial incoherent undulator radiation develops into an exponentially growing coherent signal was derived in 1-D case in references [3] and [4]. How the theory could be extended to a 3-D case was explained in reference [5]. In the case of the parallel electron beam, the problem was explicitly solved in references [6] and [7]. In particular, the equivalent noise power was identified as the undulator radiation in one gain length in reference [7]. In this paper, we study how the result should be modified when the electrons' angular divergence is taken into account. We find that the equivalent noise power is the portion of the undulator power within the coherent phase space area, a result that should have been expected.

2. Derivation

At the end of the first gain length, the radiation amplitude to a good approximation is the sum of the undulator radiation by individual electrons, and can be written as follows:

$$E_u(x) = \sum_{i=1}^{N_e} U(x - x_i)e^{-ik(x \cdot \phi_i)}e^{i\theta_i}, \tag{1}$$

where Ne is the total number of the electrons, x_i, ϕ_i, and θ_i are respectively the transverse position, transverse angle, and the longitudinal phase of the ith electron; $U(x)$ is the undulator radiation amplitude due to a reference electron, and $k=2\pi/\lambda$ wave number; and λ is the wavelength. We will approximate $U(x)$ by a Gaussian function:

$$U(x) = U_0 \frac{\exp\left(-\frac{x^2}{4\sigma_r^2}\right)}{\sqrt{2\pi\sigma_r^2}}, \tag{2}$$

where σ_r is the rms source size of the undulator radiation. Representing the undulator radiation as in the above has been useful in developing an understanding of its phase space properties [8,9].

The source size for an undulator of length L is given by $\sigma_r = \sqrt{2\lambda L} / 4\pi$ [8]. In the present case, the length L is the amplitude gain length, which is twice the power gain length L_G. Thus,

$$\sigma_r = \frac{1}{2\pi}\sqrt{\lambda L_G} .$$ (3)

The power spectrum corresponding to eq. (1) is

$$\left(\frac{dP}{d\omega}\right)_{L_G}^{spont} = \left\langle \int |E_u(x)|^2 d^2x \right\rangle = N_e |U_0|^2 .$$ (4)

Here $\langle ... \rangle$ means taking the ensemble average. In deriving eq. (4), we have used the fact that the beam is initially not bunched, so that θ_i is a random variable and is independent from θ_j, when $i \neq j$.

Beyond the first gain length, the exponential gain must be taken into account. The field amplitude may then be represented by a sum of guided modes:

$$E(x,z) = \sum_n C_n \varepsilon_n(x) e^{\lambda_n z} ,$$ (5)

where $\varepsilon_n(x)$ is the field profile of the nth eigenmode, and λ_n is the corresponding eigenvalue. For the fundamental mode n=1, we have:

$$2 \operatorname{Re} \lambda_1 = 1 / L_G .$$ (6)

We assume that the mode functions satisfy the orthogonality condition:

$$\int \varepsilon_n(x) \varepsilon_m^*(x) d^2x = \delta_{nm} .$$ (7)

It should be stressed that eq. (7) is a simplifying assumption: the mode functions in general do not satisfy the simple orthogonality as in the above, but a more complicated one à la van Kampen [5,10].

The expansion coefficient C_n is then determined by equating $E_u(x)$ with $E(x,z)$ evaluated at $z = L_G$ and using eq. (7). Thus,

$$C_n e^{2\lambda_n L_G} = \int E_u(x) \varepsilon_n^*(x) d^2 x. \tag{8}$$

The power spectrum corresponding to eq. (5) is $dP/d\omega = \sum |C_n|^2 \exp(2(\mathrm{Re}\,\lambda_n)z)$. When there is no degeneracy, the power for the case $z \gg L_G$ is dominated by the fundamental mode. Thus we write:

$$\frac{dP}{d\omega} = \frac{1}{9}\left(\frac{dP_1}{d\omega}\right)_{noise} e^{z/L_G}, \tag{9}$$

where:

$$\left(\frac{dP_1}{d\omega}\right)_{noise} = 9\langle |C_1| \rangle^2. \tag{10}$$

Using eq. (8), eq. (10) becomes:

$$\left(\frac{dP_1}{d\omega}\right)_{noise} = 9e^{-2} \int \varepsilon_1(y) \langle E_u^*(y) E_u(x) \rangle \varepsilon_1(x) d^2 x\, d^2 y \tag{11}$$

Also from eq. (1),

$$\langle E_u^*(y) E_u(x) \rangle = N_e \int U(x - x_e) U^*(y - x_e) e^{-ik\phi_e \cdot (x-y)} f(x_e, \phi_e) dx_e\, d\phi_e \tag{12}$$

Here $f(x_e, \phi_e)$ is the electrons' phase space distribution function. We assume that it is of the following Gaussian form:

$$f(x_e, \phi_e) = \frac{1}{(2\pi)^2 \sigma_{xe}^2 \sigma_{\phi e}^2} \exp\left[-\frac{1}{2}\left(\frac{x_e^2}{\sigma_{xe}^2} + \frac{\phi_e^2}{\sigma_{\phi e}^2}\right)\right]. \tag{13}$$

The fundamental guided mode will also be approximated by a Gaussian profile similar to eq. (2):

$$\varepsilon_1(x) = \frac{1}{\sqrt{2\pi\sigma_G^2}} \exp\left(-\frac{x^2}{4\sigma_G^2}\right). \tag{14}$$

It is then straightforward to show that the effective noise spectrum is given by:

$$\left(\frac{dP_1}{d\omega}\right)_{noise} = 1.22 \left(\frac{dP}{d\omega}\right)_{L_G}^{spont} \frac{(\lambda/2)^2 \cdot 4}{(2\pi)^2 \left(\sigma_G^2 + \sigma_r^2 + \sigma_{xe}^2\right)\left(\sigma_{G'}^2 + \sigma_{r'}^2 + \sigma_{\phi e}^2\right)} . \tag{15}$$

In the above, we have introduced the diffraction limited angular divergences $\sigma_{r'}$ and $\sigma_{G'}$, corresponding respectively to σ_r and σ_G via [8]:

$$\sigma_r \sigma_{r'} = \lambda/4\pi, \quad \sigma_G \sigma_G' = \lambda/4\pi. \tag{16}$$

3. Interpretation

Equation (15) is the main result of this paper, and gives the effective start-up noise power of SASE evolution for an electron beam with arbitrary size and angular divergence. It generalizes the result of ref. [7] for the case of vanishing angular divergence. The noise power is maximized when $\sigma_G = \sigma_r$, and thus $\sigma_{G'} = \sigma_{r'}$. Equation (15) then becomes:

$$\left(\frac{dP_1}{d\omega}\right)_{noise} = 1.22 \left(\frac{dP}{d\omega}\right)_{L_G}^{spont} \frac{(\lambda/4\pi)^2}{\left(\sigma_G^2 + \frac{1}{2}\sigma_{xe}^2\right)\left(\sigma_{G'}^2 + \frac{1}{2}\sigma_{\phi e}^2\right)} . \tag{17}$$

This is in the form of the coherent fraction of the spontaneous emission [8], except for the replacement of $\sigma_{xe} \rightarrow \sigma_{xe}/\sqrt{2}$ and $\sigma_{\phi e} \rightarrow \sigma_{\phi e}/\sqrt{2}$ for the electron beam phase space contribution. When the electron beam phase space is negligible, then:

$$\left(\frac{dP_1}{d\omega}\right)_{noise} = 1.22 \left(\frac{dP}{d\omega}\right)_{L_G}^{spont} . \tag{18}$$

Apart from the factor 1.22, which is due to the particular approximation employed, eq. (18) states that all of the spontaneous power goes into the excitation of the fundamental mode in this case.

On the other hand, when the electron beam phase space is dominant $\left(\sigma_{xe} \gg \sigma_G, \sigma_{\phi e} \gg \sigma_{G'}\right)$, it becomes:

$$\left(\frac{dP_1}{d\omega}\right)_{noise} = 1.22 \left(\frac{dP}{d\omega}\right)_{LG}^{spont} \frac{4 \cdot (\lambda / 4\pi)^2}{\sigma_{xe}^2 \sigma_{\phi e}^2}. \tag{19}$$

The fraction of the undulator radiation exciting the fundamental mode is $1.22 \times 4 \approx 5$ times larger than the coherent fraction, i.e., the ratio of the coherent phase space area to the electron beam phase space area. To "understand" this result, note that, as the electron beam phase space becomes larger, higher order modes will become degenerate (have the same growth rate) with the fundamental mode. If n_d is the number of the degenerate modes, we expect:

$$\left(\frac{dP}{d\omega}\right)_{LG}^{spont} \approx n_d \left(\frac{dP_1}{d\omega}\right)_{noise}. \tag{20}$$

This leads to:

$$n_d \approx \frac{1}{5} \left(\frac{2\pi\sigma_{xe}\sigma_{\phi e}}{\lambda / 2}\right)^2. \tag{21}$$

That is, the number of the degenerate modes is smaller by a factor of 5 compared to that given by the ratio of the electron beam phase space area to the coherent phase space area. This appears reasonable since a higher order mode may be regarded as an off-axis fundamental mode, and the mode growth away from the center would be suppressed due to a smaller current density.

4. Comparison with Previous Results

When electrons are parallel, $\sigma_{\phi e} = 0$, and eq. (17) becomes

$$\left(\frac{dP_1}{d\omega}\right)_{noise} = 1.22 \left(\frac{dP}{d\omega}\right)_{LG}^{spont} \frac{\sigma_G^2}{\left(\sigma_G^2 + \frac{1}{2}\sigma_{xe}^2\right)}. \tag{22}$$

Using

$$\left(\frac{dP}{d\omega}\right)_{LG}^{spont} = \left(\frac{dP}{d\omega d\Omega}\right)_{LG}^{spont} 2\pi\sigma_{G'}^2, \tag{23}$$

where $dP/d\omega d\Omega$ is the angular density of the spectral power, eq. (23) becomes

$$\left(\frac{dP}{d\omega}\right)_{L_G}^{spont} = 1.22 \left(\frac{dP}{d\omega d\Omega}\right)_{L_G}^{spont} 2\pi\sigma_\theta^2, \tag{24}$$

with

$$\frac{1}{\sigma_\theta^2} = \frac{1}{\sigma_{G'}^2} + \frac{2\lambda^2}{\left(4\pi\sigma_{xe}\right)^2}. \tag{25}$$

Equation (24) is equivalent to eq. (12) of ref. [7] for the Gaussian electron distribution.

For a large electron beam, $\sigma_{xe} \gg \sqrt{2}\sigma_G$, eq. (22) becomes:

$$\left(\frac{dP}{d\omega}\right)_{noise} = 1.22 \left(\frac{dP}{d\omega}\right)_{L_G}^{spont} \frac{2\sigma_G^2}{\sigma_{xe}^2}. \tag{26}$$

Inserting the expression for the spontaneous radiation, one finds:

$$\left(\frac{dP_1}{d\omega}\right)_{noise} = 0.81\frac{\rho E_e}{2\pi}, \tag{27}$$

where ρ is the FEL scaling parameter [11] and E_e is the electron energy. Equation (27) closely

reproduces the result of ref. [5].

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