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THE SPATIAL EVOLUTION OF QUEUES DURING THE MORNING COMMUTE IN A SINGLE CORRIDOR

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Abstract

This paper presents a qualitative description of the evolution of traffic congestion during the morning rush hour in a long freeway leading to a single destination. Traffic is generated at the freeway's many on-ramps during a short period of time and then is assumed to subside. Capacity limitations create queues on the ramps and the freeway, which is assumed to evolve according with the hydrodynamic theory **of** traffic flow. **A** special case that can be described with just a few parameters is analyzed in detail.

The simplicity of our scenario allows the results to be easily verified independently; they can then be used to check whether 'the existing traffic assignment models are consistent with the basic laws of traffic flow. We found that unreasonable results are obtained with "point queue'' models, currently a favored approach in the dynamic traffic assignment literature. **A** computer program, based on the cell transmission model (Daganzo, 1992), is put to the same test.

The note also discusses briefly the effect that a slower parallel arterial would have on the system's traffic. It is found that a route choice mechanism where drivers do not anticipate the system's evolution leads to unreasonable traffic patterns; i.e. patterns that would not be expected in reality. The anticipation phenomenon, thus, must be incorporated into any realistic model of dynamic network flows; unfortunately this increases the difficulty of developing detailed control strategies.

1 Introduction

This paper considers an important component of an urban region's transportation network: a freeway leading to its CBD and the parallel set of surface streets. It attempts to explain how traffic conditions and travel times evolve over time and space during the morning rush.

We assume that traffic behaves as predicted by the hydrodynamic theory of traffic flow without random incidents, and that **all** the traffic is headed for a unique destination at the end of the freeway. Instead of a numerical model based on detailed information, we seek general properties of the system and its behavior, which may lead to improved control schemes and future models. For example, if the demand is such that queues persist at all the ramps until they are dissipated, then our analysis shows that ramp metering cannot decrease the number of vehicle-hours spent in the system, although it can affect the distribution of delay across origins.

Our approach will use a simple idealized scenario that can be described with few input parameters so as to make the discussion transparent. The qualitative conclusions reached will extend beyond the scenario, and this will be justified as needed; the scenario and simple formulas that result are only used as vehicles for our discussion.

The following section describes the freeway, its demand and the traffic **flow** rules. The next two sections, the core of this note, describe the evolution of traffic on the freeway with the hydrodynamic model and with the "point queue" model. The last section introduces the arterial and examines the implications of route choice phenomena.

2 A Single Freeway

We consider here a freeway with a series of on-ramps leading to a major destination. Each origin is connected with the freeway by one on-ramp. The origins and on-ramps are consecutively numbered $i = 1, 2, 3, \ldots$, starting from the destination. The index **i** also refers to the freeway link directly upstream of ramp i. Note that off-ramps are ignored in our network since they have little effect on the scenario considered in this paper.

2.1 The demand

We assume that the freeway is empty and that the rush commences at time $t = 0$, when A_i vehicles suddenly leave each origin and form a queue at their respective on-ramp. **As** a result of our model we will determine, among other things, the approximate time d_i when the queue on ramp **i** dissipates.

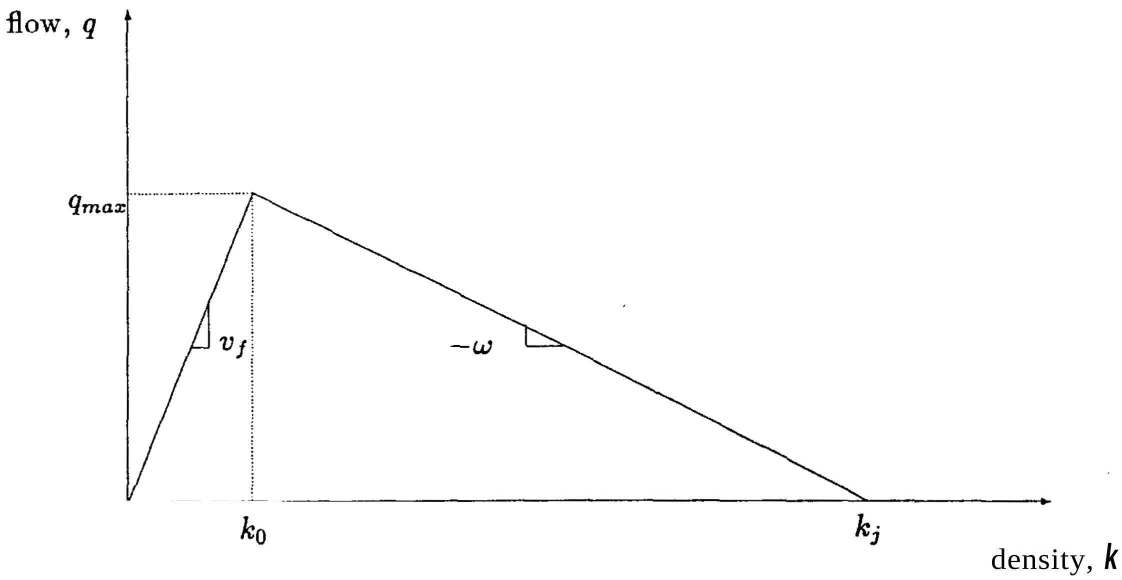

Figure 1: The piece-wise linear q-k relation

It should be clear that the behavior of traffic on the freeway would not change if some of the A_i vehicles on ramp i had left the origin after time $t = 0$, but not so late that they would avoid queuing at the ramp. As long as some vehicles are waiting to leave the ramp at all times in $(0, d_i)$, the actual size of the queue is irrelevant to the behavior of the freeway. (Although the focus of this report is freeway behavior, ramp delay is briefly considered in Secs. 3.3 and 4.) Thus our traffic generation assumption is not as restrictive as it may seem; it represents any arrival pattern where the queues at every ramp would build at about the same time and dissipate only once.

2.2 The traffic flow rules

Traffic on the freeway will be assumed to behave according to the hydrodynamic theory of traffic flow (Lighthill & Whitham, 1955, and Richards, 1956). Further, it will be assumed that the equation of state (the relationship between density k and flow q that holds at every point in space-time, $q(k)$) is triangular. A triangular relationship is defined by three constants (see Figure 1): the free flow speed, v_f, the optimum density, k_0 (or alternatively the maximum flow, $q_{max} = v_f k_0$), and the wave speed for $k > k_0$, $w > 0$. The expression is:

$$q = \begin{cases} v_f k & \text{if } k \leq k_0 \\ v_f k_0 - w(k - k_0) & \text{if } k_0 < k < k_0(1 + \frac{v_f}{\omega}) \end{cases}$$

The parameter w represents the unique speed at which flow disturbances propagate in the upstream direction within any (moving) queue. The ratio v_f/ω is approximately 6 for most freeways. The expression $k_0(1 + v_f/\omega)$ represents the jam density. Triangular $q(k)$ relations have been proposed by Newell (1991) for their simplicity of analysis and reasonable realism. Experiments to measure the accuracy of all these assumptions are being planned.

In order to predict traffic behavior on the freeway with the hydrodynamic theory, the above is not sufficient. Boundary conditions must **also** be defined at the location of each non-empty ramp, to reflect the amount of ramp traffic that is allowed to enter, depending on conditions. If the freeway is congested at the ramp location, the ramp flow might be restricted. This phenomenon can be captured by means of mixing constants that indicate the fraction of ramp vehicles (a) and freeway vehicles $(1 - a)$ that will flow immediately downstream of the junction, assuming that the flow of ramp vehicles does not exceed the ramp capacity.

Although a could depend on the freeway density, it is reasonable to assume that it is constant for our purposes; perhaps close to $\frac{1}{2m}$, where m is the number of freeway lanes (Yagar, **1993).** If a is constant and the freeway is in a congested steady state during part of the rush hour, its link flow will decrease geometrically by a factor of $(1 - a)$ in the upstream direction. This means that downstream ramps will discharge more flow than their upstream counterparts, essentially giving the commuting advantage to downstream inhabitants.

This qualitative observation is also true if one assumes that $a = 1$ (and that the ramp flow is limited to a fixed amount). The main difference is that the freeway link flows would now decrease arithmetically instead of geometrically, and the ramp flows would vary more drastically.

The remainder of this paper assumes that $Q = 1$ because the same qualitative conclusions are reached, and the model with $\alpha = 1$ leads to a graphical presentation that only involves a few traffic states and exhibits more contrast; it is better suited for comparison with graphical computer output. The theoretical and numerical results for $a < 1$ are summarized in the Appendix.

In the following section the evolution of traffic on the freeway is examined. Subsection **3.1** provides a qualitative description of traffic behavior for general condtions regarding freeway geometry and demand. Subsection **3.2** provides quantitative formulae for a special case that can be described with just a few parameters. Subsection **3.3** explains the implications of the above results for control, considering the impact that a control scheme would have on ramp queueing delay.

3 The Traffic Evolution

Let us assume that the flow from every ramp is metered at a rate $q_r > 0$ (while there is a queue) and that $\alpha = 1$. Then, as long as on-ramp traffic is not blocked by downstream freeway congestion, on ramp queues will discharge at a rate q_r.

3.1 Qualitative description

Immediately after $t = 0$, as vehicles from upstream ramps travel downstream, flow on each freeway link will increase. The hydrodynamic model adopted here predicts that it should do *so* in steps. The time between steps is the time it takes a vehicle to travel between ramps; and the size of each step is q_r.

Flow will have reached capacity in a freeway section upstream of a ramp in the time that it takes a vehicle to travel q_{max}/q_r freeway links. Since the maximum freeway flow that can get past an active ramp is $q_{max} - q_r$, capacity flows will be precluded from advancing from then on. Our hydrodynamic model predicts that under these conditions queues of slow moving vehicles will form upstream of every ramp and that those queues will grow spatially at rate *w*. When the queues reach the next upstream ramp, (in the time that it takes for the shock wave to travel between ramps; e.g. **5** min) the freeway flow immediately downstream of ramps 2, **3, ...** will decrease to $q_{max} - q_r$ from **q,.. As** ramps 2, **3, ...** continue to emit vehicles at rate q_r, the freeway flow directly upstream of these ramps will be further restricted to $q_{max} - 2q_r$; another set of shockwaves will be generated.

Consideration shows that in the time that it takes a wave to travel the length of q_{max}/q_r freeway links (about **30** min) the system will have reached an equilibrium state as depicted in Figure 2, assuming of course that no ramp queues have dissipated. Link flows would be

$$q_i = \begin{cases} q_{max} \quad iq_r & \text{if } i \le q_{max}/q_r \\ 0 \quad \_ & \text{if } i > q_{max}/q_r \end{cases}$$

This occurs because priority downstream origins block upstream traffic. This stable pattern will persist until some of the downstream ramps dissipate their queues.

The stable pattern is characterized by certain link occupancies, *n;,* and link travel times,-ti. The freeway link occupancy is **a** linearly decreasing function of the link flow,

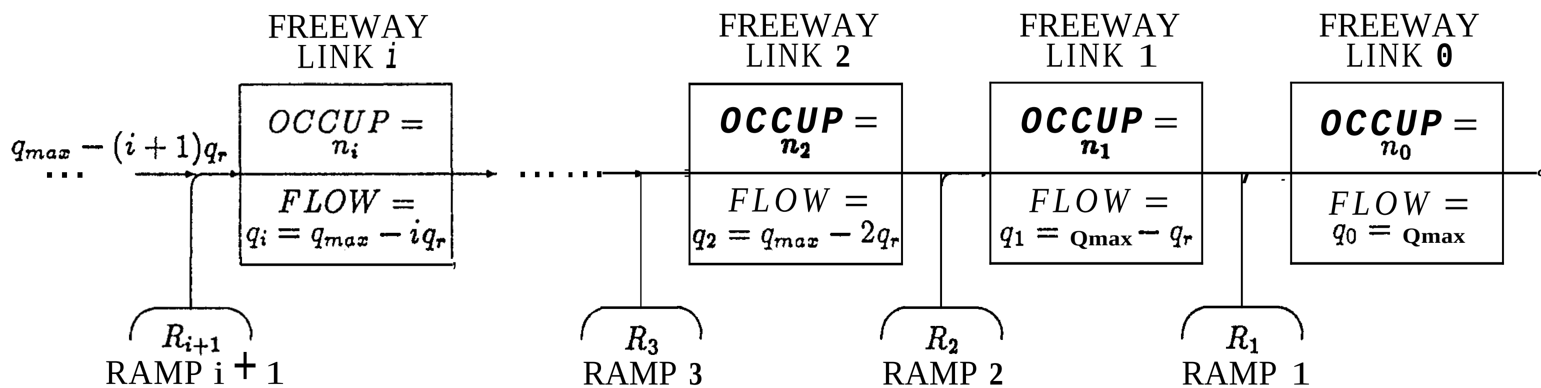

Figure 2: Schematic representation of the freeway system

given the equation of state (Figure 1). If I?; is the length of link i, the relationship is:

$$n_i = \left[\frac{v_f k_0 - q_i}{\omega} + k_0\right] l_i = \left[\frac{i q_r}{\omega} + k_0\right] l_i \ .$$

The link travel times for a vehicle traversing link **i** during the stable pattern is:

$$t_i = \frac{n_i}{q_i} = \frac{\left[\frac{i q_r}{\omega} + k_0\right] l_i}{q_{max} - i q_r}$$

The stable pattern will persist until one of the downstream ramp queues dissipates. Upstream of such a ramp freeway flow will then increase by q_r units and after a suitable delay the first ramp to have been blocked will commence to discharge. If no other ramps dissipate their queues during this delay, another stable pattern will then have been reached.

As time progresses other stable patterns may arise. They all have in common that the first q_{max}/q_r ramps with queues remain emitting vehicles and the rest are either empty or blocked. Flow on link 0 remains at capacity throughout – during the stable patterns and the transition periods. Note as well that ramps close to the destination clear before those far away. When the last q_{max}/q_r ramps are emitting vehicles freeway speeds will have returned to normal and the downstream portion of the freeway will still be at capacity. **As** the queues on these ramps dissipate, downstream freeway flows will decrease.

3.2 Quantitative results

Here we develop quantitative results for an idealized case that can be described with few parameters and is easy to analyze: we assume that the freeway ramps are evenly spaced l distance units apart and that q_{max}/q_r is an integer, m. If we let t_0 denote the sum of the times that it takes for a vehicle to travel one link (l/v_f) and for a wave to do the same (l/w), $t_0 = \frac{l}{v_f} + \frac{l}{\omega}$, the first stable pattern is reached at time $t_b = mt_0$. This is also the time when ramps such that $i > q_{max}/q_r$ become blocked:

$$t_b = mt_0 = \frac{q_{max}}{q_r} t_0 = \frac{q_{max}}{q_r} l \left(\frac{1}{v_f} + \frac{1}{\omega}\right).$$

The number of vehicles initially discharged by one of these ramps is:

$$A_b = q_{max} t_0 = l \left(\frac{q_{max}}{v_f} + \frac{q_{max}}{\omega}\right) = l k_{jam}$$

which is also the maximum number of vehicles that fit in a freeway link.

Assuming that $A; > A_b$ (the usual case if there is a congestion problem during the morning rush), the last $(A; - A_b)$ vehicles on the ramp will have to wait until $(i - q_{max}/q_r)$

downstream ramps have discharged and the clearing signal has had the time to reach ramp i. The specific times when ramp i begins to discharge again, t_{ri}, and ends to discharge, t_{di}, will depend on the downstream A_i's, but the formula is not particularly important for our purposes. The simplest case where all the $A;$ are equal, $A; = A$, will suffice to illustrate some issues.

The time-space diagram for this problem is displayed in Figure 3 for a case with $m = 4$ and $v_f/\omega = 2$,' and with only 12 on-ramps. The figure does not depict vehicle trajectories; the solid lines represent time-space interfaces between the traffic states prevailing in various regions of the time-space plane. These traffic states, labeled "O, A, B, C, D, C', B', A', O'", correspond to the points on the $q(k)$ curve also displayed in the Figure.[2] Since our $q(k)$ relation exhibits only two wave speeds (v_f and w), the hydrodynamic solution is easy to construct using the following two well known facts:

> (i) Because there is a change in freeway flow at the ramp's location whenever the ramp is emitting vehicles a stationary interface must be located there. The flow on the upstream side of the interface must be q_r flow units less than the downstream flows.
>
> (ii) Non-stationary interfaces must move with a speed equal to the ratio of the change in flows on both sides of the interface to the change in density. If the scales of representation are chosen appropriately (as we have done) interfaces are parallel to the $q - k$ diagram line that joins the two states.

In the figure, a horizontal interface corresponds to an active ramp; a blocked ramp is identified by an interruption in a horizontal line at the ramp's position. The figure also displays explicitly the times $t,;$ and t_{di} for ramp i = 8. The first and last set of 4 ramps behave a little differently from the 4 in between, since the former are affected by upstream boundary conditions. The expression for $t_{di} - t_{ri}$, obvious from the figure, applies to the middle ramps (excluding the first and last m in general cases):

$$t_{di} - t_{ri} = \frac{A - A_b}{\quad}, \quad \text{if } m < i < i_{max} - m. \tag{1}$$

It should also be clear from the geometry of the figure that:

$$t_{di} \;=\; t_b + \left(\frac{A}{q_r} - t_b\right)\left(\frac{i}{m}\right)^+ + \left(\frac{l}{\omega}\right)\left\{-1 + 2i - \left(\frac{i}{m}\right)^+ m\right\}, \tag{2a}$$

which can be approximated by a smooth function of i:

$$t_{di} \;\approx\; \quad + \left(\frac{A}{q_r} - t_b\right)\left(\quad\right) \tag{2b}$$

'A value close to 6 would be more reasonable, but the figure would be less clear.

[2]The reader is encouraged to verify this.

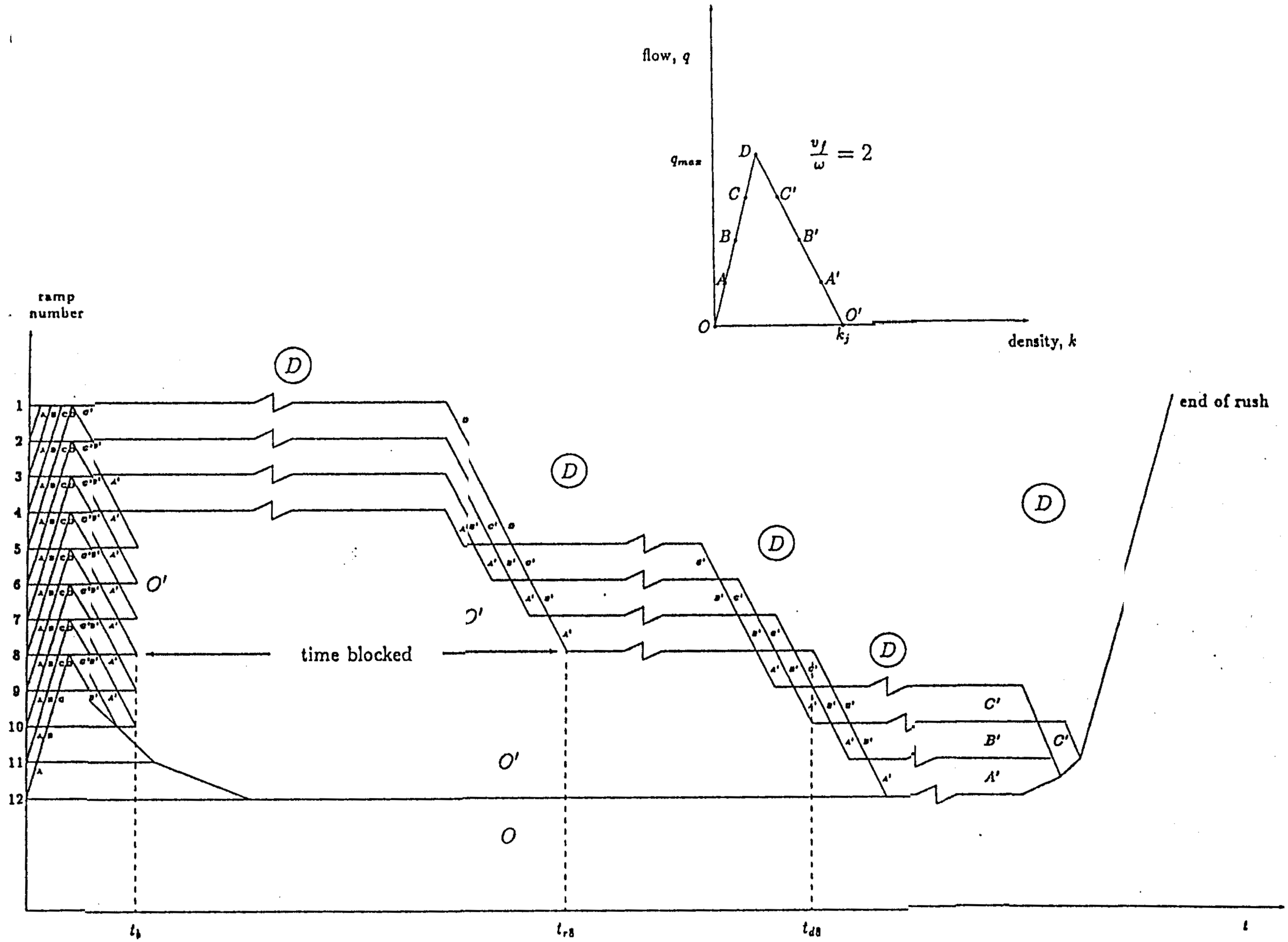

Figure 3: **Time-space solution** to the morning **traffic** problem

The first two terms of (2b) represent the time that it takes to discharge the queues blocking ramp i (including ramp i itself). The third term represents the time for the clearing signal to travel back to ramp i. Because $t_b = q_{max} t_0 / q_r$ and $m = q_{max}/q_r$ we can write:

$$t_{di} \approx \frac{A}{2q_r} + \left(\frac{A}{q_{max}} - t_0 + \frac{l}{\omega}\right) i \quad \text{if } m < i < i_{max} - m \tag{3}$$

which increases with i at rate: $(A/q_{max} - t_0 + l/\omega)$.

Because the difference between t_{di} and t_{ri} is constant, t_{ri} also increases at the same rate with i. Figure 4 displays graphically the relationship between i and the times when the ramp actually discharges vehicles, including $i \in (1, m)$.

The times t_{di} represent the times when an additional vehicle could conceivably enter the freeway at ramp i and travel at the free flow speed all the way to the destination. As such, it can be viewed as the time of return to normalcy for origin i.

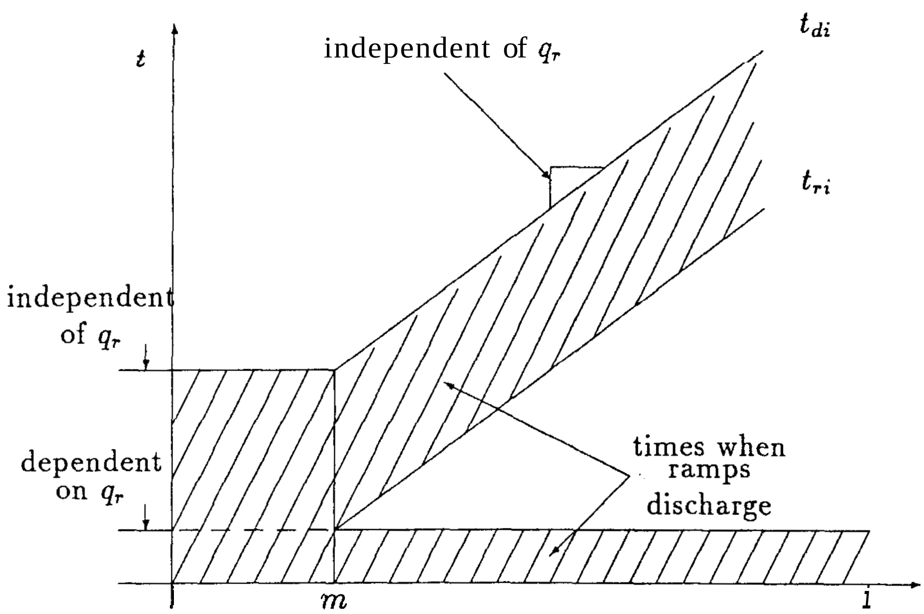

Figure 4: Periods of activity for the different ramps

3.3 Implications for control

Except for an initial time, m/v_f, the bottleneck remains at capacity throughout the rush hour. This observation is true independent of q_r and therefore independent of a ramp metering strategy if we assume (plausibly) that changes in q_r don't change q_{max}. (It is also true if $\alpha < 1$.) This indicates that the $i_{max}A$ vehicles that use the systems are removed at a maximum rate that is independent of q_r. It follows that the total number of vehicle-hours in the system cannot be reduced by ramp metering.

Ramp metering cannot even influence the number of vehicle-hours on the freeway (metering one ramp simply allows vehicles from other ramps to spend time on the freeway). To illustrate this we show that vehicles enter the freeway from **all** ramps at a combined rate which varies cyclically around a long term average that is independent of the metering rate.

At any given time, t', the number of ramps discharging (at rate q_r) into the freeway is given by the number of horizontal lines that would be intersected by the line $t = t'$ in Figure **3.** During any stable pattern, without moving interfaces, the number is $m = q_{max}/q_r$, but the number is less during a transition period. The number fluctuates periodically.

For a freeway with $i_{max} \to \infty$, the number of active ramps is on **average:**[3]

$$m \left\{ \left(\frac{A}{q_r} - mt_0 \right) / \left(\frac{A}{q_r} - mt_0 + \frac{ml}{w} \right) \right\} .$$

The average flow discharged into the freeway before the end of the rush also varies cyclically around an average which is the product of the above and q_r:

$$q_{max} \left\{ (A - q_{max}t_0) / \left(A - q_{max}t_0 + q_{max}\frac{l}{\omega} \right) \right\} . \tag{4}$$

This quantity is independent of q_r.

4 Traffic Evolution With Point Queues

The term "point queue" is used here to refer to the limiting case **of** the hydrodynamic model with $k_{jam} \to \infty$ (or $w \to 0$). This limiting case exhibits the feature that queues never grow to be so long that they restrict entry into a link.

This limiting model is becoming popular in the dynamic traffic assignment literature because it is somewhat tractable and (since it includes transient queuing phenomena) gives

[3]This can be easily seen from the figure if we imagine that there are m "on/off" moving switches that control the ramp flows; a ramp emits flow when it has an on switch. During any stable period there **is** one "on" switch on each of the m active ramps. Whenever a ramp queue dissipates, its switch is turned off and sent to the first upstream ramp that is blocked at the speed of the interface. The trip takes $\frac{ml}{\omega}$ time units; see Figure **3.** Upon arrival it is turned on until the queue dissipates; i.e. it stays on for a time $\frac{A}{q_r} - mt_0$. For a very long freeway, the average number of active ramps is equal to the average number of "on" switches, which in turn is equal to the product of m and the fraction of time that a switch is on. This is the formula that is presented.

the illusion of realism. Unfortunately, the assumption $w \to 0$ cannot properly capture the real life processes that can grind traffic to a halt; with point queues, the upstream end of every link is always ready to admit vehicles. As the material in this subsection shows, the consequences of this assumption can be drastic indeed.

First we note that with the point queue model all the ramp queues will have dissipated by time $max_i\{A_i/q_r\}$. If the A_i are not too different, this is roughly the same time at which the queues on ramps 1 through m clear with the hydrodynamic model. The two times coincide for the idealized case with $A_i = A$. With point queues, the freeway does not return to normalcy at that time, however. Large queues will have formed on every freeway link that is away from the upstream system boundary.[4]

Unfortunately these queues don't dissipate quickly. Without any ramp flow, all the queues will receive the same number of vehicles as they emit in a unit time (q_{max}) and therefore will remain fixed in size. All the queues, that is, except the last non-zero upstream queue which will decline in size at rate q_{max}.

The specific times at which the individual queues dissipate can be found graphically with a construction as in Figure 3 (with $\omega \to 0$), or (more easily) with time versus cumulative count diagrams at every freeway link. This is not done here, however, because the qualitative properties of the result for $A_i = A$ are rather obvious as we now explain.

As explained in the prior footnote, the m links at the upstream end of the freeway do not develop queues; queues only grow on the remaining links after a delay of $(m+1)l/v_f$ time units. The build up will take place at the same rate (q_r) for all links, since links can emit a flow q_{max} but receive the sum of a flow q_{max}, emitted by the upstream freeway link, and the ramp flow q_r. If A is so large that the initial delay ($(m+1)l/v_f$) is negligible compared with A/q_r, then the maximum freeway queues will be reached approximately at time A/q_r and will be of size A on each link for the ideal case. (Essentially, the ramp queues will have been transferred to the freeway). From then on the freeway occupancy will decline.

The last upstream queue will dissipate first as it receives no traffic, but discharges it at rate q_{max}; its dissipation will reduce arrivals to the downstream queue, which will dissipate next, etc, ... Interestingly, the result of the point queue assumption is that freeway links return to normalcy from the upstream end to the downstream end. That is, in the reverse order as would occur in reality!

Figure 5 displays the predictions of the point queue and the hydrodynamic model for

[4] If the A_i/q_r are large relative to the time it takes to travel m links, which is the case of interest as explained earlier, then queues must begin to grow after time $(m+1)l/v$ on any freeway link that is more than m ramps away from the upstream system boundary. These queues continue to grow until the ramp flows are exhausted. If the A_i are large and the freeway has many ramps, the queues will be very large.

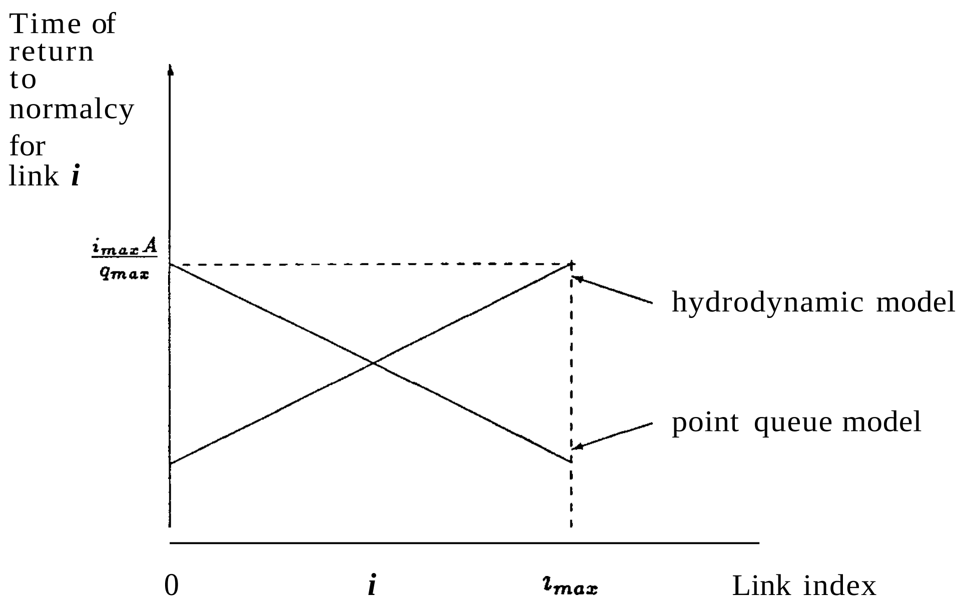

Figure 5: Comparison of point queue model and hydrodynamic model

a limiting case with $A/q_r, i_{max} \to \infty$. Note that both models predict the same clearing time for the system as a whole: $i_{max}A/q_{max}$. This should not be surprising, since both models keep link 0 saturated throughout the rush.

Constant saturation flow through link 0 **also** implies that the number of vehicle-hours in the system is the same as for the hydrodynamic model. However, the number of freeway vehicle-hours is overestimated, since vehicles are assumed to enter the freeway at a rate $q_r i_{max}$, which is much larger than (4). (In the limiting case with i_{max} and $A \to \infty$, the fractional overestimation error is unbounded.) This overestimation is a direct consequence of the absence of freeway link-to-link interactions (blocking) in the point queue model.

5 A Freeway And A Parallel Arterial

With the goal of assessing the reasonableness of a simple model of driver route choice behavior, we now allow people to travel on a slower one-way arterial road and choose the most convenient ramp. We do not allow them, however, to postpone or cancel their trips; e.g. as per the use of an alternative mode of transportation.

The arterial/freeway problem is easy to analyze if we assume that people evaluate routes continuously as they travel without anticipating future changes in link travel times. That is, the route travel time used in their decision at time t is the sum of the "current" link travel times, defined to be the projected link travel times under current conditions for a vehicle just entering each link.[5] It will be assumed that speed on the arterial is independent of flow; this is reasonable since arterial flows should be modest and in real life there may be a system of parallel streets ready to handle the flow. As part of our solutions, we seek the queue lengths on each ramp as a function of time, $R_i(t)$.

With our route choice model, a driver on the arterial at time t will compare two active adjoining ramps by considering the projected ramp queuing times, $R_{i+1}(t)/q_r$ and $R_i(t)/q_r$, and the current travel time on the intervening freeway link, $t_i(t) = n_i(t)/q_i(t)$. The travel time difference between the two choices (using ramp i and ramp $i+1$) is then:

$$\Delta time = \frac{R_{i+1}(t) - R_i(t)}{q_r} + t_i(t). \tag{5}$$

Note that an inactive ramp (one with $q_r = 0$) would never be selected.

Because the flow from the first m ramps eventually blocks all flow from the remaining ramps as we have discussed in Sec. 3, it follows that the demand from origins $i = m+1, m+2, \ldots$, will eventually reject these ramps, moving on the arterial to one of the active ramps $(i \leq m)$. Thus, a pattern of ramp queues will develop where $R_i(t) = 0$ if $i > m$, and $R_i(t) > 0$, otherwise. The positive $R_i(t)$ will decrease in size with time, but as long as $R_i(t) > 0$ for all $i \leq m$, the freeway will behave as if there was no arterial and the stable pattern of Figure 2 prevailed; as explained in Sec. *3.1:*

$$t_i(t) = \frac{\left[\frac{iq_r}{\omega} + k_0\right] l_i}{q_{max} - iq_r}; \quad 0 \leq i \leq m.$$

On substituting this relation for $t_i(t)$ in Eq. (5), for the ideal case with $l_i = 1$, we obtain:

$$Atime = \frac{R_{i+1}(t) - R_i(t)}{q_r} + \frac{\left(\frac{iq_r}{\omega} + k_0\right) l}{Q_{max} - iq_r}, \quad 1 \leq i < m, \tag{6}$$

which gives the difference in current times between ramps, as a function of the queue sizes, $R_i(t)$. These, of course, are yet to be determined.

If we let τ denote the travel time on the arterial between two ramps, our route choice mechanism implies that[6]

$$Atime \leq \tau$$

which yields the following condition:

---

[5] An even more conservative model of route choice would assume that current travel times correspond to vehicles exiting each link.

[6] A pure equality would imply that people are indifferent between the two ramps. The inequality implies that the downstream ramp is favored; this can happen because drivers cannot backtrack.

$$R_{i+1}(t) - R_i(t) \leq q_r \left\{ \tau - \frac{l(iq_r/\omega + k_0)}{q_{max} - iq_r} \right\}$$

or

$$R_{i+1}(t) - R_i(t) \leq q_r \left\{ \tau - \frac{l}{v_f} \left[\frac{v_f i/\omega + m}{m - i} \right] \right\}. \tag{7}$$

The right side of Eq. (7) is positive for $i = 1$ (if $\tau \gg l/v_f$ as one would expect) and negative for $i = m - 1$ (since $\frac{ml}{v_f}$ should be greater than τ in most cases). We now argue that Eq. (7) should be a pure equality.

Initially, one would expect Eq. (7) to be a pure equality for $i \leq m$ because the (many) drivers from upstream blocked origins can choose among all i $(i \leq m)$ without traveling the wrong way; i.e. if one ramp was favored (with a small R_i) some drivers would travel to it and increase its R_i. As long as there are queues on all $i \leq m$, the $R_i(t)$ will decrease at the same rate and people will not have an incentive to jockey among ramps. Thus, one would expect Eq. (7) to remain as an equality until one of the queues dissipates. It follows that (as long as they are all positive) the $R_i(t)$ can be easily obtained for any given total number of ramp occupants

$$R(t) = \sum_{i=1}^{m} R_i(t) \tag{8}$$

with the following procedure:

> Starting with a given R_1, obtain R_2, R_3, ... ,R_m with Eq. (7) interpreted as an equality. Then add $(\frac{1}{m})(R(t) - \sum_{i=1}^{m} R_i)$ to each of the values.

The resulting values satisfy Eqs. (7) and (8); they are valid if none are negative.

Note that the ramp with the longest queue has an i where the RHS of Eq. (7) is close to zero. This means that it is located where the freeway travel time roughly matches the arterial travel time. Note as well that the minimum queue length arises at one of the extremes (either $i = 1$ or $i = m$). Consequently, since all the $R_i(t)$ decline at the same rate (q_r), one of the extreme ramps will eventually become idle. Consideration shows that no vehicles would move to the new empty ramp. This is obvious for ramp m, since vehicles are not allowed to backtrack. It is also true if ramp 1 clears first because, after dissipation, the queuing time of ramp 1 remains fixed at zero while the remaining queuing times decline. Drivers wouldn't want to move to ramp 1. It should be clear then that as soon as one of the ramps clears, the combined input flow to the freeway drops. As soon as this happens, freeway traffic must quickly return to an uncongested state with positive flows and rapidly moving vehicles (see Figure 6).

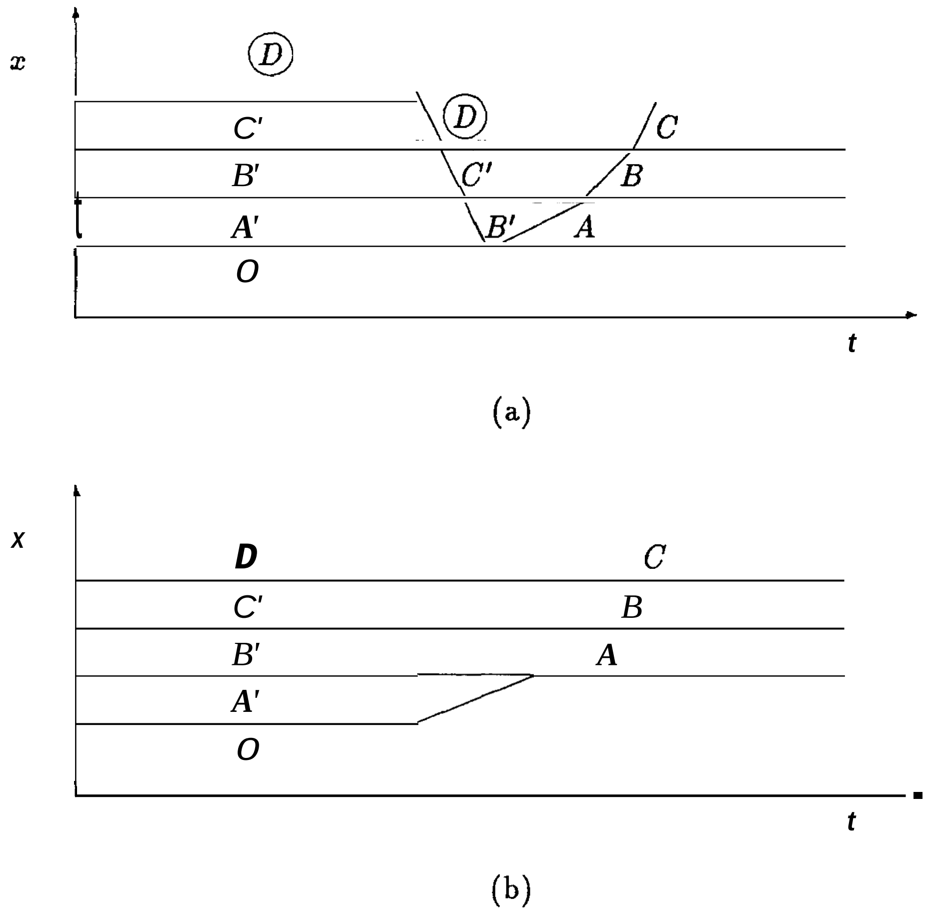

Figure 6: Time-space diagram of the system's return to normalcy when: (a) ramp $i = 1$ clears first, and (b) ramp $i = m$ clears first. Figure depicts the case with $m = 4$.

What is disturbing from this outcome is that the flow through the bottleneck will have declined below capacity, while ramp queues remain. Vehicles trapped in these queues would have been better served by waiting at ramp' $i = m + 1$ in anticipation of the freeway clearing time. Because they didn't, their delay 'and that of the overall system are increased. It is doubtful that such a phenomenon would be observed in real life. (If it did, closing the arterial would result in less delay.) It seems reasonable to conclude that drivers do anticipate the evolution based on past experience. Logical as this may be, the conclusion is unfortunate because it complicates the modeling of dynamic network flows; especially with regard to the development of detailed dynamic traffic-responsive control strategies.

Similar conclusions are reached if a two-way travel is allowed on the arterial. It is not difficult to see that the two-way system would still evolve in the same way until the dissipation time of the first ramp. After that time, people at ramp m and/or $m - 1$ may find it attractive to backtrack so as to enter ramp $m+1$,.without significant queuing or delay. This may or may not happen depending on the specific values of the parameters τ, l, v_f, etc. ..., appearing in Eq. (5). But if it does it would imply that some motorists from upstream origins would travel to a ramp, queue, and later decide that a previously bypassed upstream ramp was a better alternative. Commuters are unlikely to behave so naively.

A more realistic model of commuter behavior should be based on past experiences. We would expect experienced commuters always to maintain the first m ramps busy. If this is true, it would mean that the bottleneck is never starved for traffic and that the overall system delay is as low as it can be, given that there is a bottleneck. Ramp metering strategies could not reduce the overall system delay.

6 Conclusion and Summary

The note has developed the solution to the hydrodynamic model for a freeway carrying morning commuters to a single destination. Although one may take issue with the realism of the hydrodynamic model for a microscopic description of traffic (over short distances and small time intervals), its behavior on a macroscopic scale would seem satisfactory. It has been extensively studied, and is widely used by practicing engineers.[7] It is certainly more realistic than the point queue model, as it recognizes that as the freeway fills with vehicles the flow from upstream links must be restricted. The hydrodynamic model predicts a very different congestion dissipation pattern. It predicts that freeway links reach freeflow speed in the upstream direction, starting from the bottleneck, while the point queue model predicts the opposite; this highlights the latter's inability to represent real life situations where blocking arises.

A by-product of our analysis is a set of expressions that can be used to test numerical implementations of the hydrodynamic model. This is done in Figure 7, which displays the output of a prototype freeway network flow program currently being developed (Lin and Daganzo, 1993). The shading intensity in this figure, which should be compared to Figure 3, corresponds to the magnitude of the predicted traffic density. Figure 8 depicts the result for $\alpha = 1/4$. It should be compared with the appendix's results.

The introduction of a parallel arterial revealed that if drivers were to choose routes without anticipating the future evolution of the system, the queuing patterns that would develop would become unrealistic. This observation has important implications for the

[7] The value of purported improvements such as microscopic car-following simulations can be questioned, since the macroscopic implications of all the details have not been examined in sufficient detail. As is well known, e.g. from chaos theory, non-linear systems can be very sensitive to details.

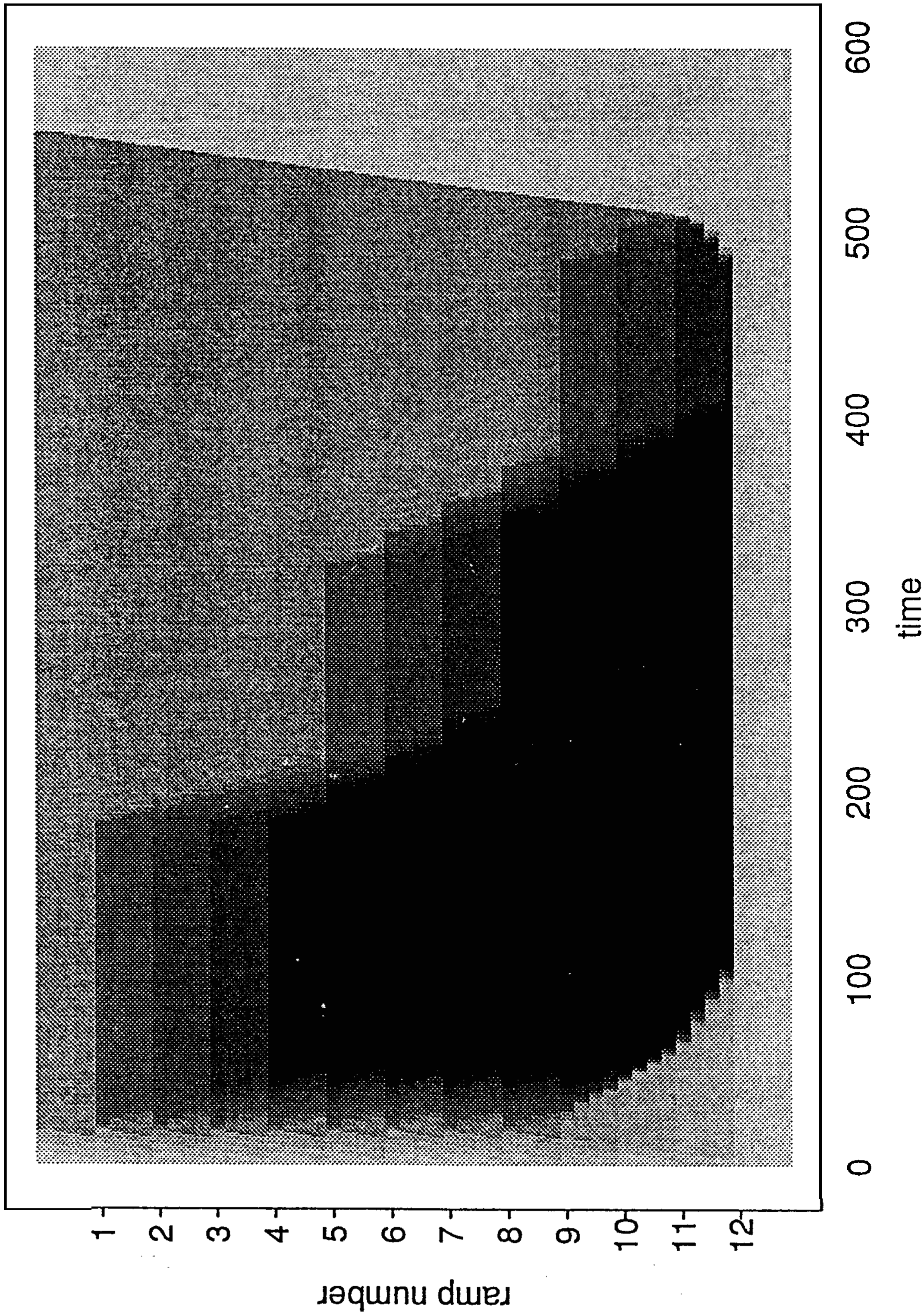

Figure 7: Traffic intensity plot to the morning traffic problem ($a = 1$)

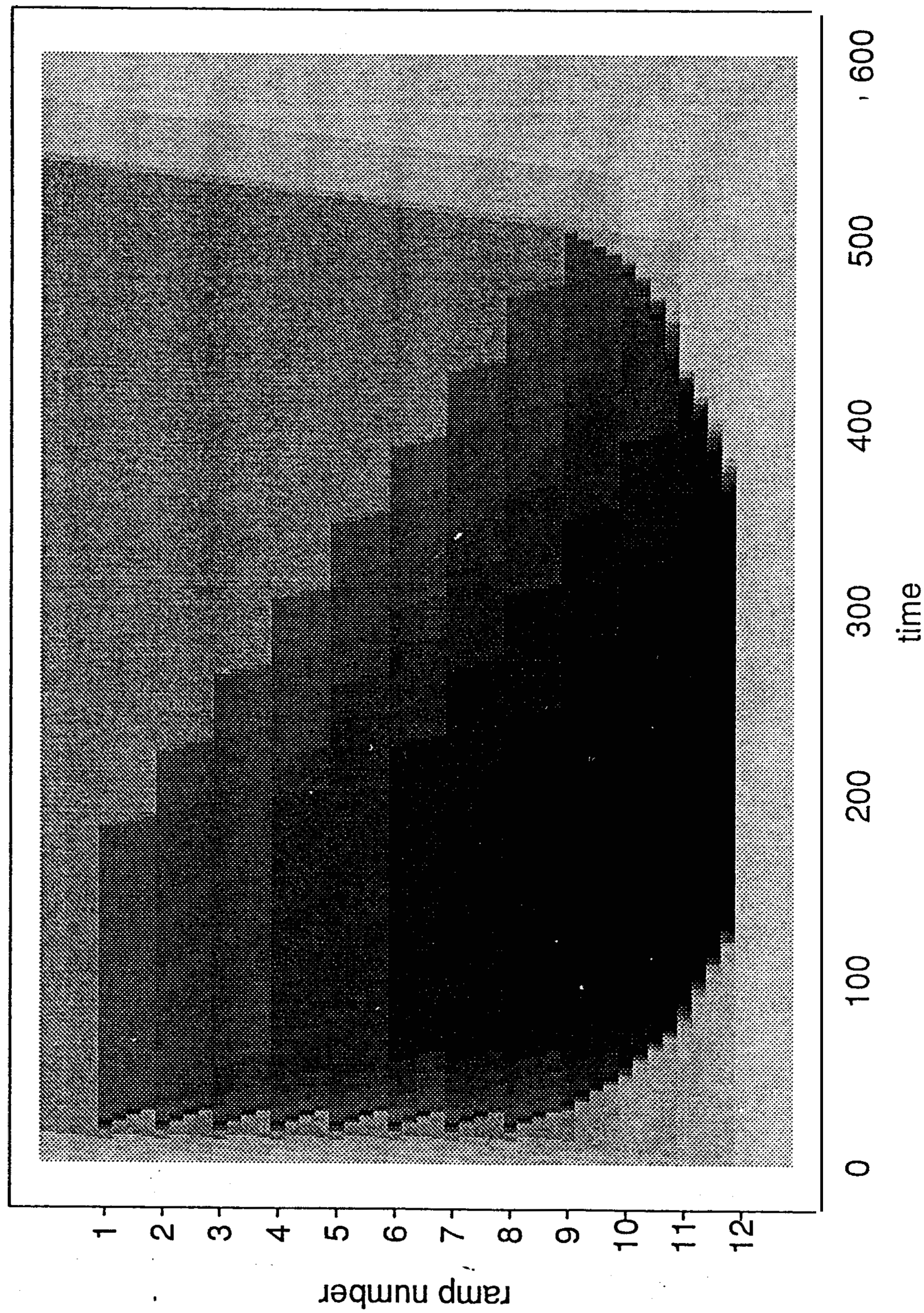

Figure 8: Traffic intensity plot to the morning **traffic** problem ***(a***=1/4)

direction of dynamic traffic network research; it seems to suggest that commuters choose routes based on expectations formed on prior travel days and that anticipation should be built into the models.

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Appendix

This appendix presents the expressions obtained by repeating the process of Sec. **3** for the more realistic case where α is small and the freeway is very long.[1] **As** was mentioned in the text, the results are qualitatively similar.

We consider the case where $\alpha \leq \frac{q_r}{q_{max}} = \frac{1}{2m}$ $(m = 1, 2, \ldots,)$, and recall that after an initial delay flows reach a stable pattern. Consideration shows that the stable flow on link i will be:

$$q_i = q_{max}(1 - \alpha)^i \quad i = 0, 1, 2, \ldots. \tag{1}$$

This state is reached at time:

$$t_i = \frac{l q_{max}}{v_f q_r} + \left(\frac{l}{\omega}\right) i. \tag{2}$$

The stable pattern is disturbed every time one of the downstream ramps clears and a wave propagates upstream. **As** the wave passes every link, the link flow will then increase by a factor of $1 - \alpha$. Link flows increase in steps until they return to capacity. The time when queues on ramp i are discharged, t_{di}, can be obtained recursively:

$$t_{di} = t_{d,i-1} + \frac{l}{\omega} + t_i' \quad i = 2, 3, \ldots. \tag{3}$$

The first term, $t_{d,i-1}$, is the time ramp $i - 1$ clears. The second term is the time it takes for the wave, generated by ramp $i - 1$ when it stops discharging vehicles, to travel from ramp $i - 1$ to ramp i. The last term, t_i', is the difference between the time link i receives the wave and the time queues on ramp i are cleared. It can be shown by induction that $t_2' = t_3' = \ldots = t_i'$ and that:

$$t_i' = \frac{A}{q_{max}} - \left(\frac{l}{v_f} + \frac{l}{\omega}\right) \quad i = 2, 3, \ldots. \tag{4}$$

Note that t_i' is independent of α.

Consideration shows that

$$t_{d1} = \frac{l q_{max}}{v_f q_r} + \frac{A - q_{max}\frac{l}{v_f}}{\alpha q_{max}}, \tag{5}$$

which together with (3) and (4) determines the t_{di}'s.

As occurred with $\alpha = 1$, the system returns to normal from the bottleneck up. Figure 8 depicts the computer output for $\alpha = 1/4$, which matches the predictions with (3) - (5). **A** diagram similar to Figure **3** would also match closely Figure 8.

[1]The expressions about to be mentioned do not apply to the ramps close to the upstream end of the freeway, as their behavior is influenced by upstream boundary conditions.