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The generalized continuum hypothesis can fail everywhere

By Matthew Foreman and W. Hugh Woodin

1. Introduction

In 1874, Cantor [C1] showed that every set has cardinality strictly smaller than the cardinality of its power set. Cantor asked [C2] whether for infinite sets X there is a set Y of cardinality strictly between cardinality X and cardinality 2^X. The special case of $X = \mathbf{Z}$ (the integers) was Hilbert's first problem in his famous list [Hi] (the continuum hypothesis).

In this paper we show that it is consistent with Zermelo-Frankel set theory with the full Axiom of Choice (modulo large cardinals) that for every set X there is a set Y such that the cardinality of Y lies strictly between the cardinality of X and the cardinality of the power set of X.

It was previously shown in Gödel [G] that the generalized continuum hypothesis (G.C.H) was consistent; i.e., for every infinite cardinal X the cardinal successor to X was 2^X. In 1963, Cohen [Co] showed that it was consistent that the continuum hypothesis failed. Easton [E] showed that subject to relatively mild restrictions (König's theorem) essentially arbitrary behavior of the power set operation could occur at regular cardinals.

Singular cardinals presented a significantly more difficult matter. The first work on them was done by Prikry and Silver [P] and [Si1] who showed that the G.C.H. can fail at singular strong limit cardinals. Magidor [M1] showed that it was consistent that it fail at the first singular strong limit and even that one could have the first failure of the G.C.H. at a singular strong limit.

After Magidor's work it was generally believed that arbitrary behavior was possible. Silver, however, showed [Si] that if the G.C.H. holds below a singular cardinal κ of uncountable cofinality then it holds at κ. Galvin and Hajnal [G-H] showed that under more general conditions the behavior of the power set below a singular cardinal κ of uncountable cofinality strongly affects the power set at κ.

These results are all "local" results. The consistent global behavior of the power set operation was not settled. We prove the following theorem.

Theorem. *Let κ be a supercompact cardinal with infinitely many inaccessible cardinals above κ. Then there is a partial ordering* $\mathbf{P}$ *such that in* $V^{\mathbf{P}}$, $V_\kappa \models \mathrm{ZFC} +$ *for all* λ, $2^\lambda > \lambda^+$.

In fact we can arrange, by our choice of partial orderings that $V^{\mathbf{P}} \models \kappa$ is $\beth_n(\kappa)$-supercompact. Solovay [So] has shown that if κ is supercompact then $2^{\beth_\omega(\kappa)} = \beth_\omega(\kappa)^+$; hence this is near best possible. Woodin extended this result to get:

THEOREM (Woodin). *If there is a supercompact cardinal then there is a model of* ZFC *in which* $2^\kappa = \kappa^{++}$ *for each cardinal* κ.

The general problem of the global behavior of the power set operation remains open although Shelah in [Sh1], [Sh2] has shown that significant restrictions hold even at cardinals of cofinality ω.

Mitchell [Mi] has extended Jensen's results showing that the failure of G.C.H. at a singular cardinal is a large cardinal property and Gittik [Gi] has obtained a good upper bound on the consistency strength of this property.

We now outline the argument. The idea of this proof is to marry the techniques developed by Magidor [M1] with a technique developed by Radin [R1] for adding a club set to a supercompact cardinal κ and keeping κ supercompact.

We shall begin with a model in which κ is supercompact and in which for every n, $\beth_n(\kappa)^{<\beth_n(\kappa)} = \beth_n(\kappa)$ and $\beth_n(\kappa)$ is the n^{th} weakly inaccessible cardinal past κ. Such a model is easily obtained by the technique of Silver in [Si1] from a model in which κ is supercompact and in which there are infinitely many strongly inaccessible cardinals above κ.

Our final model is obtained from this model which for the moment we call V_0, as follows. We generically add a closed unbounded subset, $C \sim \langle \kappa_\alpha \colon \alpha < \kappa \rangle$, of κ. For this we use Radin forcing and so the powerset function in V_0 at each κ_α behaves locally as it does at κ; i.e., for all n, $\beth_n(\kappa_\alpha)^{<\beth_n(\kappa_\alpha)} = \beth_n(\kappa_\alpha)$ and $\beth_n(k_\alpha)$ is the n^{th} weakly inaccessible cardinal past κ_α. Further, $V_0[C]$ is a mild enough extension of V_0 so that this remains true in $V_0[C]$.

For each α let $\lambda_\alpha = \beth_4(\kappa_\alpha)$. At the same time we generically add $\kappa_{\alpha+1}$ subsets to λ_α using the usual Cohen style conditions. Add $(\lambda_\alpha, \kappa_{\alpha+1})$ from V_0. Since $\lambda_\alpha^{<\lambda_\alpha} = \lambda_\alpha$ these conditions are $< \lambda_\alpha$ closed and satisfy the λ_α^+ chain condition. Hence cardinals are not collapsed.

The final generic may be regarded as a sequence $\langle \kappa_\alpha g_\alpha \colon \alpha < \kappa \rangle$ where for each $\alpha < \kappa$, $g_\alpha \subset \text{Add}(\lambda_\alpha, \kappa_{\alpha+1})$ is V_0-generic.

As is the case with Radin forcing, this forcing factors nicely in that for each limit ordinal $\alpha_0 < \kappa$ and for each n,

$$V_0[\langle \kappa_\alpha g_\alpha \colon \alpha < \kappa \rangle] = V_0[\langle \kappa_\alpha g_\alpha : \alpha < \alpha_0 \rangle][g_{\alpha_0} \times \cdots \times g_{\alpha_n}][\langle \kappa_\alpha g_\alpha \colon \alpha_n < \alpha < \kappa \rangle].$$

The first stage is a generic extension of V_0 for a version of the forcing defined at κ_{α_0}, $g_{\alpha_0} \times \cdots \times g_{\alpha_n}$ is $V_0[\langle \kappa_\alpha g_\alpha\colon \alpha < \alpha_0 \rangle]$-generic and the third stage is a generic extension of $V_0[\langle \kappa_\alpha g_\alpha\colon \alpha < \alpha_0 \rangle][g_{\alpha_0} \times \cdots \times g_{\alpha_n}]$ for the forcing defined in $V_0[\langle \kappa_\alpha g_\alpha\colon \alpha < \alpha_0 \rangle][g_{\alpha_0} \times \cdots \times g_{\alpha_n}]$ (except that $\mathrm{Add}(\lambda_\alpha, \kappa_{\alpha+1})$ is still computed in V_0). The forcing for the first stage is $\kappa^+_{\alpha_0}$-cc and all the new bounded subsets of λ_{α_0} in $V_0[\langle \kappa_\alpha g_\alpha\colon \alpha < \kappa \rangle]$ appear by the first stage. Thus it follows by induction that in $V_0[\langle \kappa_\alpha g_\alpha\colon \alpha < \kappa \rangle]$, the G.C.H. fails everywhere below κ. Finally the entire forcing preserves the regularity of κ; this follows by a master condition argument analogous to that for Radin forcing, and so $V_0[\langle \kappa_\alpha g_\alpha\colon \alpha < \kappa \rangle] \restriction \kappa$ is a model of ZFC in which the G.C.H. fails everywhere.

The forcing we construct is a generalization of Radin forcing in the spirit that the forcing constructed by Magidor is a generalization of Prikry forcing. In defining the conditions, we modify the definition of the Radin measures to incorporate constraints for the g_α's. As is the case in Magidor's construction we proceed by defining a 'supercompact' version of the forcing. The final notion of forcing is obtained by a projection of this forcing. Our situation differs from Magidor's in an essential fashion as we want to keep κ regular. It is for this that we need to define explicitly the projected forcing (to allow a master condition argument) rather than to pass to an inner model as Magidor does.

We give an outline by sections:

In Section 2 we prove abstract lemmas about preservation of cardinals, mostly relying on an abstract characterization of the "Prikry property." We define what a strong factorization is and discuss master conditions. We show that the existence of a master condition at a cardinal κ for an elementary embedding j with critical point κ and a partial ordering $\mathbf{P}$ implies that κ remains regular after forcing with $\mathbf{P}$.

In Section 3 we define the measure sequences used in the proof. These are analogues to the measure sequences used by Radin in his construction. Our measure sequences are further complicated by the need to compute the projected forcing. Hence, they concentrate on more than just the sequences of shorter length. We make a fairly complicated inductive definition, stating and verifying the induction hypothesis as we go along. We define a projection map from our measure sequences to "projected" measure sequences.

In Section 4 we define and prove some properties of the measure sequences that we will use. These are mostly normality properties of measures.

In Section 5 we define the forcing notions we will use. Roughly a forcing condition is a finite sequence of 5-tuples each giving

a) a point on the Radin sequence,

b) a measure sequence and some sets of measure one to guide further sequences,

c) a condition in a partial order for adding lots of subsets of $\beth_n(\gamma)$ where γ is the cardinal of the point on the Radin sequence,

d) a collection of restrictions on the addition of subsets of smaller cardinals.

If we do $\beth_n(\kappa)$-supercompact Radin forcing we automatically get that if λ is a point on the Radin sequence then for each $m \leq n$, $V \models \beth_m(\lambda) =$ the m^{th} weakly inaccessible cardinal above λ and if $\alpha = \beth_m(\lambda)$, $|\alpha^{<\alpha}| = \alpha$. If λ^* is the next point on the Radin sequence above λ then our partial ordering will add λ^* many Cohen subsets of $\beth_n(\lambda)$. In our final model the power set operation is as follows:
For $m < n$, $\beth_m(\lambda) = \beth_m(\lambda)^V$ and $\beth_{n+1}(\lambda) = (\lambda)^*$.

In Section 6 we define the projected forcing and show that the map from measure sequences to projected measure sequences induces a (Boolean algebra) projection between the original forcing and the projected forcing.

In Section 7 we show that there is a master condition for j and the projected forcing. From this, using the results of Section 2 we conclude that

$$V_\kappa \models \text{“ZFC + for all cardinals } \alpha, 2^\alpha > \alpha^+\text{.”}$$

In Section 8 we describe how to manipulate this proof to preserve the fact that κ is $\beth_n(\kappa)$-supercompact.

We have attempted to keep our notation within the standard notations. Some idiosyncrasies are as follows:

We use the word *measure* for a countably complete non-principal ultrafilter. We use the phrase *almost everywhere* in conjunction with a measure μ to mean “on a set in the filter μ.”

If p, q are forcing conditions, we use $p \leq q$ to mean that p gives more information than q. We use $p \| \varphi$ to mean $p \Vdash \varphi$ or $p \Vdash \neg \varphi$, and if p, q are forcing conditions in a partial ordering $\mathbf{P}$ and G is the canonical term for the generic object, we abbreviate $p \Vdash q \in G$ by $p \Vdash q$. We will use the abbreviation “club” for closed and unbounded.

Both Boolean algebra and partial ordering notation are used. If $\mathbf{P}$ is a partial ordering we write $\mathbf{P}/p$ for $\langle \{q \in \mathbf{P}\colon q \leq p\}, \leq_{\mathbf{P}} \rangle$, and similarly for B/p. Our partial orderings and Boolean algebras will be non-atomic. If $i\colon \mathbf{P} \to Q$ is order-preserving and one-to-one then i is a neat-embedding if whenever $A \subseteq \mathbf{P}$ is a maximal antichain i'', $A \subseteq Q$ is a maximal antichain in Q. We use the symbol $\vee$ for Boolean join and $\wedge$ for meet.

We use $\text{Add}(\kappa, \lambda)$ for the partial ordering for adding λ-many Cohen subsets of κ. Specifically, a condition in $\text{Add}(\kappa, \lambda)$ is a partial function $p\colon \kappa \times \lambda \to 2$ with domain having cardinality $< \kappa$. The ordering is reverse inclusion. We assume that the reader is familiar with basic properties of this partial ordering (see [J]).

This work was done in 1979 while both authors were students at the University of California at Berkeley.

2. Preliminaries

In this section we introduce some abstract notions we will use in our proof to preserve cardinals and cofinalities.

Definition (2.1). A partial ordering $\mathbf{P}$ is κ-Prikry if and only if $\mathbf{P} = \bigcup_{\alpha<\gamma} F_\alpha$ where:

i) Each F_α is κ-closed (i.e., if $\langle p_\beta: \beta < \delta < \kappa\rangle \subseteq F_\alpha$ is a decreasing sequence of conditions then there is a $q \in F_\alpha$, $q \leq p_\beta$, for all $\beta < \delta$).

ii) If $b \in \mathscr{B}(\mathbf{P})$, $p \in F_\alpha$, then there is a $q \leq p$ such that $q \| b$ and $q \in F_\alpha$.

Examples (2.2). Partial orderings with the κ-Prikry property are:

a) $\mathbf{P}$ = Prikry forcing through a measurable cardinal $\lambda \geq \kappa$ (see [P]). Here two conditions lie in the same F_α if and only if they have the same length.

b) Any κ-closed partial ordering.

If $\mathbf{P} = \bigcup_{\alpha<\gamma} F_\alpha$ is a Prikry then p *refines* q if and only if $p \leq q$ and for some $\alpha, p, q \in F_\alpha$. If $\langle p_\beta: \beta < \delta\rangle$ is a decreasing sequence of elements of $\mathbf{P}$ then $\langle p_\beta: \beta < \delta\rangle$ is a *tower of refinements* if and only if for some α, $\langle p_\beta: \beta < \delta\rangle \subseteq F_\alpha$.

The following lemma is easy to see:

LEMMA (2.3). *If κ is a regular and $\mathbf{P}$ is κ-Prikry then forcing with $\mathbf{P}$ adds no new bounded subsets of κ*; i.e., $([\kappa]^{<\kappa})^V = ([\kappa]^{<\kappa})^{V^{\mathbf{P}}}$.

To make the G.C.H. fail at many places one must do two things:

First, add many subsets of many cardinals. Second, one must preserve a lot of cardinals in order to avoid re-creating the G.C.H. at some cardinal. In [E] Easton developed a technique for doing this; namely his partial orderings were of the form (κ c.c.) $\times$ (κ-closed) for suitably many cardinals κ. We generalize his technique with the following notion:

We say that $\mathbf{P}$ *strongly factors* at κ below p if and only if there is a κ c.c. partial ordering Q and a κ-Prikry partial ordering R with witness $R = \bigcup_{\alpha<\gamma} F_\alpha$ such that:

a) $\mathscr{B}(\{p' \in \mathbf{P}: p' \leq p\})$ has a dense set isomorphic to $Q \times R$ (identify $Q \times R$ with this dense subset).

b) If $b \in \mathscr{B}(\{p' \in \mathbf{P}: p' \leq p\})$ and $(q, r) \in Q \times R$ with $r \in F_\alpha$, then there is an $r' \leq r$, $r' \in F_\alpha$, and a maximal antichain $A \subseteq Q/q$ such that for all $q' \in A, (q', r') \| b$.

Example (2.4). If $\mathbf{P}/p \simeq Q \times R$ where Q is κ-c.c. and R is κ-closed then $\mathbf{P}/p$ strongly factors at κ below p.

LEMMA (2.5). *Suppose κ is a regular cardinal and $\mathbf{P}$ strongly factors at κ below p; then $p \Vdash_{\mathbf{P}} \kappa$ is a cardinal. Further, if $\mathscr{B}(Q \times R) \simeq \mathscr{B}(\mathbf{P}/p)$ witnesses strong factoring at κ, then $V^Q \models R$ adds no new bounded subsets of κ.*

Proof. Since Q is κ-c.c., it is enough to show the latter statement. Let τ be a term for a subset of $\lambda < \kappa$. Let $(q, r) \in Q \times R$. By strong factorization we can build a tower of refinements of r, $\langle r_\beta: \beta < \lambda \rangle$ and maximal antichains $\langle A_\beta: \beta < \lambda \rangle$ in Q/q so that for all $q' \in A_\beta$, $(q', r_\beta) \|$"$\beta \in \tau$." Since $\langle r_\beta: \beta < \lambda \rangle$ is a tower of refinements there is an $r^* \leq r_\beta$ for all $\beta < \lambda$. Then

$$(q, r^*) \Vdash \tau = \{\beta: \text{there is a } q' \in A_\beta \cap G_Q, (q', r^*) \Vdash \beta \in \tau\}$$

where G_Q is the generic object for Q. Thus $(q, r^*) \Vdash \tau \in V^Q$. □

Strong factorization allows us to preserve cardinals but we must use another technique to keep a cardinal κ inaccessible. (This cardinal κ will be where we cut off the universe.)

Let $j: V \to M$ be an elementary embedding with critical point κ. Let $\mathbf{P}$ be a partial ordering and $i: \mathbf{P} \to j(\mathbf{P})$ be a neat embedding. If there is a condition $m \in j(\mathbf{P})$ such that for all $p \in \mathbf{P}$, $0 \neq i(p) \wedge m \leq j(p)$, then m is called a *master condition* for j, $\mathbf{P}$, and i.

Standard theory says that if we force with $j(\mathbf{P})/m$ to get a generic filter $H \subseteq j(\mathbf{P})$ and let G be the generic filter on $\mathbf{P}$ induced by i, then j can be extended to $\hat{j}: V[G] \to M[H]$. Consequently, if $m \wedge i(p) \neq 0$ for all $p \in \mathbf{P}$, $p \Vdash \kappa$ is a regular cardinal.

In our situation we will start with j a $\beth_{n+1}(\kappa)$-supercompact embedding. We will build a κ^+-c.c. partial ordering $\mathbf{P}$ and a master condition m such that there is a dense collection of $p \in i(\mathbf{P})/m$ such that $i(\mathbf{P})/p$ strongly factors at κ^+ with witness $\mathbf{P} \times \mathbf{R}$ and i the natural embedding of $\mathbf{P}$ into $\mathbf{P} \times \mathbf{R}$.

LEMMA (2.6). *Let $j, \mathbf{P}, m, \mathbf{P} \times \mathbf{R}$ be as above; if $\mathbf{R} = \bigcup_{\alpha < \gamma} F_\alpha$ witnesses strong factorization and each F_α is $\beth_{n+1}(\kappa)^+$-closed then $V^{\mathbf{P}} \models \kappa$ is $\beth_n(\kappa)$-supercompact.*

Proof. Enumerate the terms for subsets of $[\beth_n(\kappa)]^{<\kappa}$ in $V^{\mathbf{P}}$, $\langle \tau_\delta: \delta < \beth_{n+1}(\kappa) \rangle$, and terms for regressive functions from $[\beth_n(\kappa)]^{<\kappa}$ into $\beth_n(\kappa)$, $\langle f_> : \beta < \beth_{n+1}(\kappa) \rangle$.

Since m is a master condition,

$$m \Vdash_{j(\mathbf{P})} \text{“There is an elementary embedding } \hat{j}: V[G] \to M[H]\text{”}$$

where $H \subseteq j(\mathbf{P})/m$ is generic and $G \subseteq \mathbf{P}$ is the generic object induced by the canonical embedding of $\mathbf{P}$ into $j(\mathbf{P})/p = \mathbf{P} \times \mathbf{R}$ for some $p \in H$. Hence, for each δ and each condition $q \leq p$, we can find a maximal antichain of conditions deciding the value $\|j''\beth_n(\kappa) \in \hat{j}(\tau_\delta)\|$. Further, for each $\beta < \beth_{n+1}(\kappa)$ we can find a maximal antichain of conditions $A \subseteq j(\mathbf{P})$ below p such that for all $p' \in A$ there is a $\xi \in \beth_n(\kappa)$ such that $p' \Vdash \hat{j}(f_\beta)(j''\beth_n(\kappa)) = \hat{j}(\xi)$.

Thus we can build a tower of refinements in R, $\langle r_\delta \colon \delta < \beth_{n+1}(\kappa)\rangle$, such that for all δ, there is a maximal antichain $A_\delta \subseteq \mathbf{P}$ such that for all $p \in A_\delta$, $(p, r_\delta)\|$"$j''\beth_n(\kappa) \in j(\tau_\delta)$" and for all $p \in A_\delta$, there is a $\xi \in \beth_n(\kappa)$ such that $(p, r_\delta) \Vdash j(f_\delta)(j''\beth_n(\kappa)) = j(\xi)$.

Let $G \subseteq \mathbf{P}$ be generic and let $r \leq \langle r_\delta \colon \delta < \beth_{n+1}(\kappa)\rangle$. It is easy to check that $V[G] \Vdash \{\tau_\delta^{V[G]}\colon$ there is a $p \in G, (p, r) \Vdash \hat{j}''\beth_n(\kappa) \in \hat{j}(\tau_\delta)\}$ is a κ-complete, normal, fine ultrafilter on $\mathscr{P}_\kappa(\beth_n(\kappa))$. □

We will be using the following notion of a projection map:

Definition (2.7). An order-preserving map $\pi\colon \mathbf{P} \to Q$ is called a *projection map* if and only if for all $p \in \mathbf{P}$ there is a $q \leq \pi(p)$ such that for all $q' \leq q$ there is a $p' \leq p$, $\pi(p') \leq q'$.

$$\begin{array}{ccc} \mathbf{P}|p & \geq & p' \\ \downarrow & & \downarrow \\ Q|\pi(p) \geq q \geq q' & \geq & \pi(p') \end{array}$$

The following proposition is standard:

Proposition (2.8). *If $\pi\colon \mathbf{P} \to Q$ is a projection map and $G \subset \mathbf{P}$ is generic, then $\pi''G \subseteq Q$ is generic.*

It is also easy to check that if $\pi\colon \mathbf{P} \to Q$ is a projection map and $b \in \mathscr{B}(Q)$ is a Boolean value and $p\|b \in \pi''G$ then $\pi(p)\|b$.

3. The measure sequences

We now build the measure sequences used in the proof. We start with an easy lemma:

Lemma (3.1). *Suppose $j\colon V \to M$ is an elementary embedding with $\{j(\alpha)\colon \alpha < \lambda\} \in M$. Suppose $a \in M$ and there is a function $H \in V$ such that $a \in j(\text{dom } H)$ and $j(H)(a) = \{j(\alpha)\colon \alpha < \lambda\}$. Let $N_a = \{j(f)(a)\colon f$ is a function in V and $a \in j(\text{dom } f)\}$; then:*

a) $N_a \prec M$ and $j\colon V \to N_a$ is an elementary embedding.

b) N_a is closed under λ-sequences.

Proof. a) is standard. We indicate the proof of b). Let $\{x_\alpha\colon \alpha < \lambda\} \subseteq N_a$. Then there is a sequence of functions $\langle f_\alpha\colon \alpha < \lambda\rangle \in V$ such that for all $\alpha < \lambda$,

$j(f_\alpha)(a) = x_\alpha$. Let

$$F\colon \bigcup\{\operatorname{dom} f_\alpha\colon \alpha < \lambda\} \cap \operatorname{dom} H \to V$$

be defined by $F(b) = \{f_\alpha(b)\colon \alpha \in H(b)\}$. Then $j(F)(a) = \{x_\alpha\colon \alpha < \lambda\}$. □

Let $j\colon V \to M$ be a $(2^\lambda)^+$-supercompact embedding with $\kappa = \operatorname{crit}(j)$ and $\lambda = \beth_\omega(\kappa)$. We will inductively define a sequence $\langle M_\alpha\colon \alpha < \lambda_1\rangle$ for some $\lambda_1 < (\beth_2(\kappa))^+$ such that $M_0 = j''\beth_3(\kappa)$ and each M_i, $i > 0$, will be a measure on:

$$\left(\mathscr{P}_\kappa(\beth_3(\kappa)) \times R(\kappa)^{<\kappa}\right) \times R(\kappa)^{<\kappa}$$

We will write $\lambda^* = \beth_3(\kappa)$ in what follows and call such a sequence a *measure sequence*.

We will simultaneously be defining a sequence of functions $\langle g_\alpha\colon 0 < \alpha < \lambda_1\rangle$ where for each β, $\operatorname{dom} g_\beta = A_\beta$ is a set of measure one for M_β and $g_\beta\colon A_\beta \to R_\kappa$.

Having arrived at α, we will have $\langle M_\beta\colon \beta < \alpha\rangle$ and $\langle g_\beta\colon 0 < \beta < \alpha\rangle$ defined. We define a measure M_α as follows:

$$M_\alpha(X) = 1 \text{ if and only if } \left(M_0, \langle M_\beta\colon 0 < \beta < \alpha\rangle, \left(g_\beta\colon 0 < \beta < \alpha\right)\right) \in j(X).$$

It is easy to check that, in M, the sequence $(M_0, \langle M_\beta\colon \beta < \alpha\rangle) \in \mathscr{P}_{j(\kappa)}(j\beth_3(\kappa)) \times R(j(\kappa))^{<j(\kappa)})$ and $\langle g_\beta\colon \beta < \alpha\rangle) \in R(j(\kappa))^{<j(\kappa)}$. Hence, M_α is a measure on $\mathscr{P}_\kappa\beth_3(\kappa) \times R(\kappa)^{<\kappa} \times R(\kappa)^{<\kappa}$.

We now proceed to make the definitions needed to define g_α. For notational simplicity we write $\vec{M}_{<\beta}$ for $\langle M_{\beta'}\colon \beta' < \beta\rangle$ and $\vec{g}_{<\beta}$ for $\langle g_{\beta'}\colon \beta' < \beta\rangle$. Similarly if $\langle A_{\beta'}\colon \beta' < \beta\rangle$ is a sequence of sets we write $\vec{A}_{<\beta}$. If $(\vec{u}, \vec{f})$ is a sequence of measures $\vec{u}_{<\beta} = \langle u_{\beta'}\colon \beta' < \beta\rangle$ and functions $\vec{f}_{<\beta} = \langle f_\delta\colon \delta < \beta\rangle$, then we say that the length of $(\vec{u}, \vec{f})$ is $\beta \overset{\text{def}}{=} l(\vec{u}) \overset{\text{def}}{=} l(\vec{f})$. If $\kappa' < \lambda'$ are cardinals we will abuse notation and write $\mathscr{P}_{\kappa'}(\lambda')$ for those elements x of $\mathscr{P}_{\kappa'}(\lambda')$ such that $x \cap \kappa' \in \kappa'$. This is a set of measure one for any supercompact measure on $\mathscr{P}_{\kappa'}(\lambda')$. If $x, y \in \mathscr{P}_{\kappa'}(\lambda')$ we write $x \prec y$ if $x \subseteq y$ and o.t. $x < y \cap \kappa'$. If $\vec{u}$ is a measure sequence with $u_0 \in \mathscr{P}_{\kappa'}(\lambda')$ then $\kappa_{\vec{u}} = u_0 \cap \kappa$ and $\lambda_{\vec{u}} =$ o.t. u_0. If $\vec{u}, \vec{v}$ are measure sequences and $\vec{u}_0 \prec \vec{v}_0$ we get a natural map $i_{u_0, v_0}\colon \lambda_{\vec{u}} \to \lambda_{\vec{v}}$. This naturally induces a map $i_{u_0, v_0}\colon \mathscr{P}_{\kappa_u}(\lambda_u) \to \mathscr{P}_{\kappa_v}(\lambda_v)$. Hence, there is a natural map

$$i_{u,v}\colon \left(\mathscr{P}_{\kappa_u}(\lambda_u) \times R(\kappa_u)^{<\kappa_u} \times R(\kappa_u)^{<\kappa_u}\right) \to \left(\mathscr{P}_{\kappa_v}(\lambda_v) \times R(\kappa_v)^{<\kappa_v} \times R(\kappa_v)^{<\kappa_v}\right)$$

when $i_{u,v}$ is the identity on R_{κ_u}. If $u_0 \prec v_0$, we can let $u_0^* = i''_{\vec{u}\vec{v}}\lambda_{\vec{u}}$ and $\vec{u}^*$ be $\vec{u}$ with u_0 replaced by u_0^*. This allows us to speak about $\vec{u}$ being in a set of measure one for $\vec{v}$. Clearly u_0^* depends on v_0 but in practice the appropriate $\vec{v}$ will be clear from context. Similarly if $u_0 \in \mathscr{P}_\kappa(\lambda^*)$ we get a function $i_{u_0}\colon \lambda_u \to \lambda^*$ and

$$i_u\colon \left(\mathscr{P}_{\kappa_u}(\lambda_u) \times R(\kappa_u)^{<\kappa_u} \times R(\kappa_u)^{<\kappa_u}\right) \to \left(\mathscr{P}_\kappa(\lambda^*) \times R(\kappa)^{<\kappa} \times R(\kappa^{<\kappa})\right).$$

If u, v are measure sequences we write $u \prec v$ if and only if $u_0 \prec v_0$ and $i_{u_0}(l(u_0)) < l(v_0)$.

Let $\lhd$ be a well-ordering of $H((2^\lambda)^+)$.

LEMMA 3.2. *Let $M_{<\alpha}$ be a sequence of measures defined as above. Then there is a sequence of sets of measure one $A_{<\alpha}$ for $M_{<\alpha}$ such that:*

a) *If $(\vec{u}, \vec{h}) \in A_\beta$ then $u_0 \in \mathscr{P}_\kappa(\lambda^*)$ and $\vec{u}$ is a measure sequence on $(\mathscr{P}_{\kappa_u}(\lambda_u) \times R(\kappa_u)^{<\kappa_u}) \times R(\kappa_u)^{<\kappa_u}$ and $\vec{h}$ is a sequence of functions defined on sets of $\vec{u}$ measure one.*

b) *If $(\vec{u}, \vec{h}) \in A_\beta$ then $l(\vec{u}) < \lambda_u$ and $i_u(l(\vec{u})) = \beta$.*

c) *If $\beta' \neq \beta$ then $A_{\beta'} \cap A_\beta = \varnothing$.*

d) *If $(\vec{u}, \vec{h}) \in A_\beta$ then $\langle H(\beth_4(\sup(u_0))), \in, (\vec{u}, \vec{h}), \lhd |H(\beth_4(\sup u_0))\rangle \equiv \langle H^M(\beth_4(\sup j''\lambda^*)), \in, (M_{<\beta}, g_{<\beta}), \lhd \rangle^M$ and $j(\vec{u}) \prec M_{<\beta}$.*

Proof. Note that b) $\Rightarrow$ c) and d) $\Rightarrow$ a). We prove d) as the proof of b) is easier and similar.

Let A_β be the collection of $(\vec{u}, \vec{h})$ with the desired properties. Then $\langle M_{<\beta}, g_{<\beta}\rangle \in j(A_\beta)$ since $\langle H(\beth_4(\sup j''\lambda^*)), \in, (M_{<\beta}, g_{<\infty}), \lhd \rangle \equiv \langle H(j(\beth_4(\sup j''\lambda^*)))^M, \in, j(M_{<\beta}, g_{<\beta}), j(\lhd)\rangle$. Hence, $M_\beta(A_\beta) = 1$. □

Note that by c) we can consider $g_{<\alpha}$ as a single function with domain $\bigcup A_{<\alpha}$. Also, d) implies that $(\vec{u}, \vec{h}) \in A_\beta$ is built by the same process as $\langle M_{<\beta}, g_{<\beta}\rangle$ was, since the portion of j we use is in $H(\beth_4(\sup j''\lambda^*))^M$.

Since $(\mathscr{P}(\beth_3(\kappa)) \times R(\kappa)^{<\kappa}) \times R(\kappa)^{<\kappa}$ can be canonically identified with a subset of $\mathscr{P}_\kappa(\beth_3(\kappa)) \times R(\kappa)$, we do so from now on.

The functions $\langle g_\alpha: \alpha < \lambda_1\rangle$ will take values in particular partial orderings. In our situation the function g_α on a sequence $(\vec{u}, \vec{h})$ will have a value in $\mathrm{Add}(\beth_4(\kappa_u), \kappa)$; however in other applications, g_α may take values in other partial orderings such as $\mathrm{Col}(\kappa_u^{+4}, \kappa)$ (see [F]).

Let $i: V \to M'$ be the ultrapower by the measure M_β. Then g_β represents an element of $\mathrm{Add}(\beth_4(\kappa), i(\kappa))$ in M'.

We note that M' is the transitive collapse of the model $N_{\langle M_{<\beta}, g_{<\beta}\rangle} \subseteq M$, as in Lemma 3.1. Hence M' is closed under λ^*-sequences since $M_0 = j''\lambda^*$ (i.e. the function H in Lemma 3.1 is $H((\vec{u}, \vec{h}) = u_0)$.

We put a partial ordering on the collection of functions h defined on M_α sets of measure one, A, such that for $(\vec{u}, \vec{k}) \in A$, $h(\vec{u}, \vec{k}) \in \mathrm{Add}(\beth_4(\kappa_u), \kappa)$. We let $h \leq_{\text{a.e.}} h'$ if and only if for M_α-measure one worth of $(\vec{u}, \vec{k}), h(\vec{u}, \vec{k}) \leq h'(\vec{u}, \vec{k})$.

LEMMA 3.3. *Suppose $\beth_4(\kappa)$ is regular and $\langle h_\delta: \delta < \gamma \leq \beth_3(\kappa)\rangle$ is a sequence of functions as above such that if $\delta < \delta'$ then $h_{\delta'} \leq_{\text{a.e.}} h_\delta$. Then there is a* *function h defined on a set A of measure one for M_α such that for all $(\vec{u}, \vec{k}) \in A$, $h(\vec{u}, \vec{k}) \in \mathrm{Add}(\beth_4(\kappa_u), \kappa)$ and for all $\delta < \gamma$, $h \leq_{\text{a.e.}} h_\delta$.*

Proof. We note that the functions h_δ represent a descending sequence of elements $p_\delta = [h_\delta] \in \mathrm{Add}(\beth_4(\kappa), i(\kappa))^{M'}$ that lies in M'. Since $\mathrm{Add}(\beth_4(\kappa), i(\kappa))$ is $\beth_4(\kappa)$-closed in M' there is a $p \in \mathrm{Add}(\beth_4(\kappa), i(\kappa))^{M'}$ such that for all δ, $p \leq p_\delta$. Let h be any function that represents p in M'. Then since $p \leq p_\delta$, $h \leq_{\text{a.e.}} h_\delta$. □

We restrict our attention to a special collection of measure sequences: Let

$$U_0^\alpha = \left\{\langle \vec{u}, \vec{h}\rangle : \langle \vec{u}, \vec{h}\rangle \text{ has properties a), b), d) of Lemma 3.2}\right\}.$$

Let

$$U_{i+1}^\alpha = \left\{(\vec{u}, \vec{h}) : \text{for all } 0 < \delta < l(\vec{u}) \{(v^*, f) : (\vec{v}, f) \in U_i^\alpha\} \text{ is of } u_\delta \text{ measure one}\right\}.$$

Then for all i, and all $\beta < \alpha$, $M_\beta(U_i^\alpha) = 1$. Let

$$U_\infty^\alpha = \left(\bigcap_{i \in \omega} U_i^\alpha\right) \cup \left\{\langle M_{<\beta}, g_{<\beta}\rangle : \beta < \alpha\right\}.$$

For $\beta < \alpha$, $M_\beta(U_\infty^\alpha) = 1$ and for all $(\vec{u}, \vec{h}) \in U_\infty^\alpha$, all $0 < \delta < l(\vec{u})$ $u_\delta(\{(\vec{v}^*, \vec{f}) : (\vec{v}, \vec{f}) \in U_\infty^\alpha\}) = 1$. From now on we will assume all of our measure sequences lie in U_∞^α.

We note that $U_\infty^\beta \subseteq U_\infty^{\beta'}$ for $\beta < \beta'$. In the sequel we neglect this dependence on α. Our final U_∞ will actually be $U_\infty^{\lambda_1}$.

By using Lemma 3.2, for each $a = (\vec{u}, \vec{h}) \in U_\infty$, we get a notion $U_\infty(a)$ by this same process. It is easy to check that $U_\infty(a) = \{(\vec{v}^*, \vec{f}) : (\vec{v}, \vec{f}) \in U^\infty$ and $v_0 \prec u_0\}$.

We need to define a projection map π from pairs $(\vec{u}, \vec{h})$ where $\vec{u}$ is a measure sequence on $\mathscr{P}_{\kappa_u}(\lambda_u) \times R(\kappa_u)$ and $\vec{h}$ has as domain sets $\vec{A}$ of $\vec{u}$ measure one to pairs $\langle \vec{w}, \vec{\mathscr{F}}\rangle$ where $\vec{w}$ is a measure sequence on R_{κ_u} and $\vec{\mathscr{F}}$ is a sequence of filters on some Boolean algebras $\langle Q(\vec{w}, \beta) : \beta < l(\vec{w})\rangle$.

If $l(\vec{u}) = 1$ then $\pi(\vec{u}) = \kappa_u$.

Suppose π has been defined for all pairs $(\vec{v}, \vec{g})$ with $\kappa_{\vec{v}} < \kappa_{\vec{u}}$. Let $U_\infty^\pi = \{\pi(\vec{v}, \vec{g}) : (\vec{v}, \vec{g}) \in U_\infty$ and $\kappa_{\vec{v}} < \kappa_{\vec{u}}\}$. We will let $\pi(\vec{u}, \vec{h}) = (\pi^*(\vec{u}), \pi^*(\vec{h}))$, where $\pi^*(\vec{u}) = \langle \pi^*(u_\beta) : \beta < l(\vec{u})\rangle$ and $\pi^*(\vec{h}) = \langle \pi^*(h_\beta) : \beta < l(\vec{u})\rangle$. Let $\pi^*(u_0) = \kappa_u$. If $l(\vec{u}) = \gamma$ and $0 < \beta < \gamma$, define the measure $\pi^*(u_\beta)$ on sets $X \subseteq U_\infty^\pi \restriction R_{\kappa_u}$ by $\pi^*(u_\beta)(X) = 1$ if and only if $u_\beta(\{(v, g) : \pi(v, g) \in X\}) = 1$ (i.e., $u_\beta(\pi^{-1}(X)) = 1$). By standard arguments each $\pi^*(u_\beta)$ is a measure. It is also easy to check that each $\pi^*(u_\beta)$ is a measure on $R_{\kappa_u}^{<\kappa_u}$.

For the measure sequence $\vec{u}_{<\gamma}$ and each $\beta < \gamma$, let $Q(\pi^*(\vec{u}), \beta) = \{e$: e is a function defined on a set of measure one B for $\pi^*(u_\beta)$ and for each $a \in B$, $e(a) \in \mathscr{B}(\mathrm{Add}(\beth_4(\kappa_a), (\kappa_u)))\}$ be the ultraproduct. We identify functions in $Q(\pi^*(\vec{u}), \beta)$ that are equal on sets of $\pi^*(u_\beta)$-measure one and define the Boolean operations in the natural way.

Let $u_{<\gamma}$ be a measure sequence defined on U_∞. For each sequence $A_{<\gamma}$ of sets of measure one for $u_{<\gamma}$ and functions $\vec{h}_{<\gamma}$ with $\operatorname{dom} h_\beta = A_\beta$ for $\beta < \gamma$ and $h(a) \in \mathscr{B}(\mathrm{Add}(\beth_4(\kappa_a)), (\kappa_u))$, and each $\beta < \gamma$, let

$$b\left(\vec{u}, \vec{h}, \vec{A}, \beta\right)(c) = \bigvee \left\{h_\beta(a) \colon a \in A_\beta \text{ and } \pi(a) = c\right\}.$$

Let

$$\mathscr{F}_{h_\beta} = \left\{b\left(\vec{u}, \vec{h}, \vec{A}, \beta\right) \colon A \text{ is a sequence of measure one sets for } \vec{u}\right\}.$$

Then $\mathscr{F}_{h_\beta}$ generates a filter on $Q(\pi^*(\vec{u}), \beta)$.

We let $\pi^*(h_\beta) = \mathscr{F}_{h_\beta}$ and $\pi^*(\vec{h}) = \langle \pi^*(h_\beta) \colon \beta < l(\vec{u})\rangle = \mathscr{F}_{\vec{h}}$.

CLAIM 3.4. *Let $f \in Q(\pi^*(M_{<\alpha+1}), \beta)$ and h be a function with domain B of M_β measure one such that for all $(\vec{v}, \vec{k}) \in B$, $h(\vec{v}, \vec{k}) \in \mathrm{Add}(\beth_4(\kappa_{\vec{v}}), \kappa)$. Then there is a function $g \leq_{\text{a.e.}} h$ with domain $A \subseteq B$, $M_\beta(A) = 1$, such that either*

1) *$b(M_{<\alpha+1}, g, A, \beta) \leq f$ in $Q(\pi^*(M_{<\alpha+1}), \beta)$, or*
2) *$b(M_{<\alpha+1}, g, A, \beta) \wedge f = 0$ in $Q(\pi^*(M_{<\alpha+1}), \beta)$.*

(We ignore coordinates except β.)

Proof. Let $f \in Q(\pi^*(M_{<\alpha+1}), \beta)$. Let f' have domain $\{(\vec{v}, \vec{k}) \colon \pi(\vec{v}, \vec{k}) \in \operatorname{dom} f\}$ by $f'(\vec{v}, \vec{k}) = f(\pi(\vec{v}, \vec{k}))$. Consider $T = \{a\colon f'(a) \wedge h(a) \neq 0$ in $\mathscr{B}(\mathrm{Add}(\beth_4(\kappa_a), (\kappa)))\}$. If $T \in M_\beta$, then for $a \in T$ let $g(a) \leq f'(a) \wedge h(a)$, $g(a) \in \mathrm{Add}(\beth_4(\kappa_a), (\kappa))$. Clearly $b(M_{<\alpha+1}, g, T, \beta) \leq f$ in $Q(\pi^*(M_{<\alpha+1}), \beta)$. If $T \notin M_\beta$ then $T^* = \pi^{-1}(\operatorname{dom} f) \cap \sim T \in M_\beta$ and for all $a \in T^*$, $f'(a) \wedge h(a) = 0$. Hence, $b(M_{<\alpha+1}, h, T^*, \beta) \wedge f = 0$ in $Q(\pi^*(M_{<\alpha+1}), \beta)$. □

COROLLARY 3.5. *For each function h defined on a set B of M_α-measure one such that for all $(\vec{v}, \vec{k}) \in B$, $h(\vec{v}, \vec{k}) \in \mathrm{Add}(\beth_4(\kappa_\nu)\kappa)$, there is a function $g_\alpha \leq_{\text{a.e.}} h$ with domain $A_\alpha \in M_\alpha$ such that $\mathscr{F}_{g_\alpha}$ is an ultrafilter on $Q(\pi^*(M_{<\alpha+1}), \alpha)$.*

Proof. Since $|U_\infty^\pi| = \kappa$, we know $|Q(\pi^*(M_{<\alpha+1}), \alpha)| = 2^\kappa$. By Lemmas 3.3. and 3.4, we can build a sequence of functions $\langle h_\delta \colon \delta < 2^\kappa\rangle$, each h_δ defined on a set of M_α-measure one such that:

1) For all $a \in \operatorname{dom} h_\delta$, $h_\delta(a) \in \mathrm{Add}(\beth_4(\kappa_a), \kappa)$.
2) If $\delta < \delta'$ then $h_{\delta'} \leq_{\text{a.e.}} h_\delta$.

3) If $f \in Q(\pi^*(M_{<\alpha+1}), \alpha)$, then for some $A \in M_\alpha$ and some δ, $b(M_{<\alpha+1}, h_\delta, A, \alpha) \leq f$ in $Q(M_{<\alpha+1}, \alpha)$ or $b(M_{<\alpha+1}, h_\delta, A, \alpha) \wedge f = 0$.
By Lemma 3.3, there is a function $g_\alpha \leq_{\text{a.e.}} h_\delta$ for all $\delta < 2^\kappa$. By 3), for all $f \in Q(\pi^*(M_{<\alpha+1}), \alpha)$, there is a set $A \in M_\alpha$ such that either $b(M_{<\alpha+1}, g_\alpha, A, \alpha) \leq f$ or $b(M_{<\alpha+1}, g_\alpha, A, \alpha) \wedge f = 0$. Hence, $\mathscr{F}_{g_\alpha}$ is an ultrafilter on $Q(\pi(M_{<\alpha+1}), \alpha)$. □

Letting g_α be as in Corollary 3.5, we complete the definition of (M_α, g_α) which allows us to build inductively the sequence $\langle M_{<\alpha, g_{<\alpha}}\rangle$ for all $\alpha < \lambda^*$.

We continue the definition until we find $\lambda_0 < \lambda_1$ such that $\pi(M_{\lambda_0}) = \pi(M_{\lambda_1})$. Since each measure $\pi(M_\alpha)$ is a measure on $R(\kappa)$, there is a pair (λ_0, λ_1) such that $\pi(M_{\lambda_0}) = \pi(M_{\lambda_1})$ and $\lambda_1 < (2^{2^\kappa})^+ < \lambda^*$.

Note that if $\langle b_\delta: \delta < \gamma\rangle$, $\gamma < \kappa$, is a descending sequence of elements of $\mathscr{F}_{g_\alpha}$ then there is a $b \in \mathscr{F}_{g_\alpha}$, $b \leq b_\delta$, in $Q(\pi(M_{<\alpha+1}), \alpha)$. This is because the measure M_α is κ-complete. For each δ we can find an A_δ of M_α-measure one so that $b(M_{<\alpha+1}, g_\alpha, A_\delta, \alpha) \leq b_\delta$. Letting $A = \bigcap_{\delta<\gamma} A_\delta$ we get $b(M_{<\alpha+1}, g_\alpha, A, \alpha) \leq b_\delta$ for all $\delta < \gamma$.

Finally, by the definition of $\pi(M_{<\alpha})$, if $\delta < \alpha$, $B \subseteq U_\infty^\pi$ and $A = \pi^{-1}(B)$, then the following are equivalent:

a) B has $\pi(M_\delta)$-measure one;
b) A has M_δ-measure one;
c) $\langle M_{<\delta}, g_{<\delta}\rangle \in j(A)$;
d) $\pi(M_{<\delta}, g_{<\delta}) \in j(B)$.

By 3.2, every $(\vec{u}, \vec{h}) \in U_\infty$ has this property when j is replaced by a supercompact embedding constructing $(\vec{u}, \vec{h})$ that agrees with $i_{\vec{u}}$ on the ordinals.

4. Some properties of the measure sequences

Suppose $\langle A_\beta: \beta < \alpha\rangle = A_{<\alpha}$ is a sequence of sets of $M_{<\alpha}$-measure one (i.e., $M_\alpha(A_\alpha) = 1$). Suppose $0 < \beta < \alpha$ and

$$\rho: \left(\mathscr{P}_\kappa(\lambda^*) \times R(\kappa)^{<\kappa}\right) \times R(\kappa)^{<\kappa} \to \mathscr{P}\left(\mathscr{P}_\kappa(\lambda^*) \times R(\kappa)^{<\kappa} \times R(\kappa)^{<\kappa}\right)$$

is a partial function such that for all (x, y, z), $M_\beta(\rho(x, y, z)) = 1$. We define the diagonal intersection $\Delta\rho$ to be $\{(w, \vec{v}, \vec{g})$: for all (x, y, z) such that $x \prec w$ and $y, z \in R(\kappa_w)$, if $\rho(x, y, z)$ is defined then $(w, \vec{v}, \vec{g}) \in \rho(x, y, z)\}$.

LEMMA (4.1). *Let ρ be as above. Then $M_\beta(\Delta\rho) = 1$.*

Proof. We must see that $(M_{<\beta}, g_{<\beta}) \in j(\Delta\rho)$. So we have to check that for all $(x, y, z) \in M$, if $M \models$ "$x \prec j''\lambda^*$ and $y, z \in R(\kappa)$" then $(M_{<\beta}, g_{<\beta}) \in j(\rho)(x, y, z)$.

Since $\kappa_x < \kappa$, $x = j(x')$ for some $x' \in \mathscr{P}_\kappa(\lambda^*)$. Hence, $j(\rho(x', y, z)) = j(\rho)(j(x'), j(y), j(z)) = j(\rho)(x, y, z)$. But $\rho(x', y, z)$ is of M_β-measure one; hence $(M_{<\beta}, g_{<\beta}) \in j(\rho(x', y, z))$. Thus $(M_{<\beta}, g_{<\beta}) \in j(\rho)(x, y, z)$. Hence, $(M_{<\beta}, g_{<\beta}) \in \Delta j(\rho) = j(\Delta p)$. □

We will often consider sequences $\vec{\rho} = \langle \rho_{\beta'}\colon \beta' < \beta \rangle$ where each $\rho_{\beta'}$ is a function as above. We let $\Delta\vec{\rho}$ be the sequence of sets $\langle \Delta\rho_{\beta'}\colon \beta' < \beta\rangle$. So $\Delta\vec{\rho}$ is a sequence of sets of measure one for $M_{<\beta}$.

We note that by Lemma 3.2, if $(\vec{u}, \vec{h})$ is in U_∞ then $\vec{u}$ is closed under diagonal intersections.

Finally, if $(x, \vec{y}, \vec{z}), (w, \vec{v}, \vec{g}) \in \mathscr{P}_\kappa(\lambda^*) \times R(\kappa)^{<\kappa} \times R(\kappa)^{<\kappa}$, we write $(x, \vec{y}, \vec{z}) \prec (w, \vec{v}, \vec{g})$ if $x \prec w$ and $\vec{y}, \vec{z} \in R(\kappa_w)$.

Recall that if $(\vec{u}, \vec{h}) \in U_\infty$ there is a map

$$i_u\colon \mathscr{P}_{\kappa_u}(\lambda_u) \times R(\kappa_u)^{<\kappa_u} \times R(\kappa_u)^{<\kappa_u} \to \mathscr{P}_\kappa(\lambda^*) \times R(\kappa)^{<\kappa} \times R(\kappa)^{<\kappa}.$$

LEMMA (4.2). *There are sets* $\langle C_\beta\colon \beta < \alpha\rangle = C_{<\alpha}$ *of* $M_{<\alpha}$*-measure one such that for all* $(\vec{u}, \vec{h}) \in C_\beta$ *there are sets* $\langle D_\delta\colon \delta < l(\vec{u})\rangle$ *of* $\vec{u}$*-measure one such that*:

i) For all $a \in D_\delta$, $h_\delta(a) \geq g_{i_u(\delta)}(i_u(a))$ in $\mathrm{Add}(\beth_4(\kappa_a), \kappa)$.

ii) *If* $(\vec{u}, \vec{h}) \in U^\infty$ *then* $\pi^*(\vec{h})$ *is a sequence of ultrafilters on* $\vec{Q}(\pi(\vec{u}), < l(\vec{u}))$.

Proof. Let $\langle A_\beta\colon \beta < \alpha\rangle$ be as in Lemma 3.2. By 3.2 d), ii) holds for all $(\vec{u}, \vec{h}) \in A_\beta$. If $(\vec{u}, \vec{h}) \in A_\beta$ and $\delta < l(\vec{u})$ let $D_\delta(\vec{u}, \vec{h})$ be the collection of all a such that $h_\delta(a) \geq g_{i_u(\delta)}(i_u(a))$. For $\delta < \beta$, $j(D)_\delta(M_{<\beta}, g_{<\beta}) = \{a\colon g_\delta(a) \geq j(g)_{j(\delta)}(j(a))\}$, since $i_{M_{<\beta}}$ computed in M is $j \restriction (\mathscr{P}_\kappa(\lambda^*) \times R(\kappa)^{<\kappa} \times R(\kappa)^{<\kappa})$. But for all $a \in A_\delta$, $g_\delta(a) \in \mathrm{Add}(\beth_4(\kappa_a), \kappa)$; hence $g_\delta(a) = j(g_\delta(a)) = j(g)_{j(\delta)}(j(a))$.

Thus for all $\delta < \beta$, $j(D)_\delta(M_{<\beta}, g_{<\beta})$ has M_δ-measure one. Hence $(M_{<\beta}, g_{<\beta}) \in j(\{(\vec{u}, \vec{h})\colon \text{for all } \delta < l(\vec{u}),\ D_\delta(\vec{u}, \vec{h}) \text{ has } u_\delta\text{-measure one}\})$. Taking $C_\beta = \{(\vec{u}, \vec{h})\colon \text{for all } \delta < l(\vec{u}),\ D_\delta(\vec{u}) \text{ has } u_\delta\text{-measure one}\}$ we have proved the claim. □

5. The forcing

We now define our first forcing notion.

As described in the introduction, we start with a model V with a $\beth_\omega(\kappa)$-supercompact embedding $j\colon V \to M$ with critical point κ such that for all $n \in \omega$, $\beth_n(\kappa)$ is weakly inaccessible.

As in the model for our argument, [M1], we will build a "big" partial ordering **P** that does considerable damage to V. In particular it makes κ singular.

We then pass to an inner model $V' \subseteq V^{\mathbf{P}}$ such that in V', κ is inaccessible and the G.C.H. fails everywhere below κ. Unlike [M1] we explicitly compute the regular subordering $\mathbf{P}^{\pi} \hookrightarrow \mathbf{P}$ such that $V' = V^{\mathbf{P}^{\pi}}$ We will use this knowledge about $\mathbf{P}^{\pi}$ to show that $\mathbf{P}^{\pi}$ has a master condition.

A *suitable* 5-*tuple* for $\langle M_{<\alpha}, g_{<\alpha}\rangle$ is of the form $\langle(\vec{u}, \vec{f})\vec{A}\vec{k}s\rangle$ where $(\vec{u}, \vec{f}) \in U_\infty$, $\vec{A}$ is a sequence of sets of measure one for $\vec{u}$. The sequence $\vec{k}$ is a sequence of functions such that for $\delta < l(\vec{u})$, k_δ: $A_\delta \to R(\kappa)$, and for each $b \in A_\delta$, $k_\delta(b) \in \mathrm{Add}(\beth_4(\kappa_b), \kappa_u)$, and $k_\delta(b) \leq f_\delta(b)$. Finally $s \in \mathrm{Add}(\beth_4(\kappa_{\vec{u}}), \kappa)$. We say that a suitable 5-tuple $((\vec{v}, \vec{d})\vec{B}\vec{k}s)$ is *addable* to the suitable 5-tuple $((\vec{u}, \vec{h})\vec{A}\vec{f}\, s^*)$ if and only if:

a) $\vec{v} \prec \vec{u}$ so that $i_{\vec{v}\vec{u}}$ is defined. As before we let $v_0^* = \text{image } i''_{\vec{v}\vec{u}}\lambda_v$ and $\vec{v}^*$ be $\vec{v}$ with v_0 replaced by v_0^*:

b) $(\vec{v}^*, \vec{d}) \in A_\gamma$ where $\gamma = i_{\vec{v}\vec{u}}(l(\vec{v}))$.

c) If $a \in B_\delta$ then $i_{\vec{v}\vec{u}}(a) \in A_\gamma$ where $\gamma = i_{\vec{v}\vec{u}}(\delta)$.

d) If $a \in B_\delta$ then $k_\delta(a) \leq f_\gamma(i_{\vec{v}\vec{u}}(a))$ in $\mathrm{Add}(\beth_4(\kappa_a), \kappa_v)$ where $\gamma = i_{\vec{v}\vec{u}}(\delta)$.

e) $s \leq f_\gamma(\vec{v}^*, \vec{d})$ where $\gamma = i_{\vec{v}\vec{u}}(l(\vec{v}))$.

We note that neither s^* nor $\vec{h}$ was mentioned in a)–e); hence we can define: $((\vec{v}, \vec{d})\vec{B}\vec{k}s)$ is *addable* to the 4-tuple $((\vec{u}, \vec{h})\vec{A}, \vec{f})$ if and only if a)–e) hold.

If $t_1 = ((\vec{u}_1, \vec{f}_1)\vec{A}_1\vec{k}_1 s_1)$ and $t_2 = ((\vec{u}_2, \vec{f}_2)\vec{A}_2\vec{k}_2 s_2)$ then t_2 *shrinks* t_1 if and only if $(\vec{u}_1, \vec{f}_1) = (\vec{u}_2, \vec{f}_2)$, for all $\delta < l(\vec{u}_1)$, $(A_2)_\delta \subseteq (A_1)\delta$, and $(\vec{k}_2)_\delta \leq (\vec{k}_1)_\delta$ everywhere in $\vec{A}_2$ and $s_2 \leq s_1$ in $\mathrm{Add}(\beth_4(\kappa_{u_1}, \kappa))$.

Using Lemma 3.2, for each $a \in U_\infty$, we get a natural definition for a 5-tuple suitable for a. Namely we require that $(\vec{u}, \vec{f}) \in U_\infty^a$ and $\vec{A}$ is a sequence of sets of measure one for $\vec{u}$, etc.

Similarly for suitable five tuples t_1 and t_2 for a, we get the notion that t_1 is addable to t_2.

We now note that if $t = ((\vec{u}, \vec{f})\vec{A}\vec{k}s)$ is a suitable 5-tuple, $(\vec{v}, \vec{h}) \in U_\infty$, $v_0 \prec u_0$ and $(\vec{v}^*, \vec{h}) \in A_\delta$, some $\delta < l(\vec{u})$, then there is a canonical candidate for a suitable 5-tuple expanding $(\vec{v}, \vec{h})$ that is addable to t. We assume that:

i) $A_\delta \cap A_{\delta'} = \emptyset$ for $\delta \neq \delta'$.

ii) $v_0 \prec u_0$ and $(\vec{v}^*, \vec{h}) \in A_\delta$ where $i_{\vec{v}, \vec{u}}(l(\vec{v})) = \delta$.

iii) There are sets of measure one $\langle D_\gamma\colon \gamma < l(\vec{v})\rangle$ for $\vec{v}$ such that $i''_{\vec{v}, \vec{u}}$, $D_\gamma \subseteq A_\delta$ and for all $a \in D_\gamma$, $h_\gamma(a) \geq k_\delta(i_{\vec{v}, \vec{u}}(a))$ where $\delta = i_{\vec{v}, \vec{u}}(\gamma)$.

We define the *canonical* (maximal) *expansion of* $(\vec{v}, \vec{h})$ *to a suitable* 5-*tuple addable to* t to be $s = \langle(\vec{v}, \vec{h})\vec{A}'k's'\rangle$ where for $\gamma < l(\vec{v})$,

$$A'_\gamma = \{a \in \operatorname{dom} i_{\vec{v}, \vec{u}}\colon i_{\vec{v}, \vec{u}}(a) \in A_\delta, \delta = i_{vu}(\gamma) \text{ and } h_\gamma(a) \geq k_\delta(i_{vu}(a))\}.$$

For $\gamma < l(\vec{v})$ and $a \in A'_\gamma$ we let $k'_\gamma(a) = k_\delta(i_{vu}(a))$, $\delta = i_{vu}(\gamma)$ and $s' = k_\delta((\vec{v}, \vec{h}))$ where $\delta = i_{vu}(l(\vec{v}))$.

LEMMA (5.1). *Let* $t = ((\vec{u}, \vec{f})\vec{A}\vec{k}s)$ *be a suitable* 5-*tuple. Then there is a sequence of sets* $\vec{A}^*$ *shrinking* $\vec{A}$ *such that for all* $(\vec{v}, \vec{h}) \in A^*_\delta$, $\delta < l(\vec{u})$, $(\vec{v}, \vec{h})$ *can be expanded to a suitable* 5-*tuple addable to* t. *Further we can shrink* t *to a* $t^{**} = \langle(\vec{u}, \vec{f}), \vec{A}^{**}\vec{k}s\rangle$ *so that any* $(\vec{v}, \vec{h}) \in \vec{A}^{**}_\delta$ *can be expanded to a* 5-*tuple addable to* t^{**}.

Proof. Lemma (3.2), we can shrink $\vec{A}$ to $\vec{A}'$ so that i) and ii) hold in the conditions for $(\vec{v}, \vec{h})$ to have a canonical expansion. By Lemma (4.2) we can shrink $\vec{A}'$ to $\vec{A}^*$ so that iii) holds. Then, if $(\vec{v}, \vec{h}) \in A^*_\delta$ the canonical expansion of $(\vec{v}, \vec{h})$ is defined and hence we can expand $(\vec{v}, h)$ to a suitable 5-tuple addable to t.

To build t^{**}, we construct an ω-sequence $\langle \vec{A}^i : i \in \omega \rangle$ so that for any $(\vec{v}, \vec{h})$ if $(\vec{v}^*, \vec{h}) \in \vec{A}^{i+1}$ then the canonical expansion of $(\vec{v}, \vec{h})$ is addable to $t_i = \langle(\vec{u}, \vec{f})\vec{A}^i\vec{k}s\rangle$. Let $\vec{A}^{**} = \bigcap_{i \cap \omega} \vec{A}^i$. Then for any $(\vec{v}, \vec{h})$, if $(\vec{v}^*, \vec{h}) \in \vec{A}^{**}$ then the canonical expansion of $(\vec{v}, \vec{h})$ is addable to $t^{**} = \langle(\vec{v}, \vec{h})\vec{A}^{**}\vec{k}s\rangle$. □

We also note that addability is a transitive relation since if $v \prec w \prec u$ then $i_{vu} = i_{wu} \circ i_{vw}$.

LEMMA (5.2). *Let* $t^* = \langle(\vec{v}, \vec{h})\vec{A}'\vec{k}'s'\rangle$ *be addable to* $t = \langle(\vec{u}, \vec{f})\vec{A}\vec{k}s)\rangle$. *Suppose* $i_{\vec{v}\vec{u}}(l(\vec{v})) = \delta$. *For* $\beta > \delta$ *and for* u_β-*almost all* $(\vec{w}, \vec{h}')$, $\vec{t}^*$ *is addable to the canonical expansion of* $(\vec{w}, \vec{h}')$.

Proof. By Lemma 3.2, d) it is enough to verify this in M when $t = \langle\langle M_{<\alpha}, g_{<\alpha}\rangle\vec{A}\vec{k}s\rangle$ for some $\vec{A}\vec{k}s$. For $\beta > \delta$, t^* is addable to the canonical expansion of M_β-almost all $(\vec{w}, \vec{h}')$ if and only if t^* is addable (in M) to the canonical expansion of $(M_{<\beta}, g_{<\beta})$ with respect to $j(t)$. Let $\langle(M_{<\beta}, g_{<\beta})\vec{B}\vec{l}\vec{t}\,\rangle$ be this canonical expansion. Then for all $\gamma < \beta$, $A_\gamma \subset B_\gamma$ and for all $a \in A_\gamma$, $\vec{k}_\gamma(a) \leq \vec{l}_\gamma(a)$. By hypothesis, t^* is addable to $\langle(M_{<\alpha} g_{<\alpha})\vec{A}\vec{k}s\rangle$, hence addable to $\langle(M_{<\beta}, g_{<\beta})\vec{A}_{<\beta}\vec{k}_{<\beta}s\rangle$ and thus addable to $\langle(M_{<\beta}, g_{<\beta})\vec{B}\vec{l}\vec{t}\rangle$. □

Putting 5.1 and 5.2 together, we get for each suitable 5-tuple $t = \langle(\vec{u}, \vec{f})\vec{A}\vec{k}s\rangle$, there is a sequence of $\vec{u}$ measure one sets $\vec{A}'$ such that if $(\vec{v}, \vec{h}) \in A'_\delta$ and $(\vec{w}, \vec{g}) \in A'_\beta$, $\delta < \beta$ and $v_0 \prec w_0$, then the canonical expansion of $(\vec{v}, \vec{h})$ is addable to the canonical expansion of $(\vec{w}, \vec{g})$. Hence for $(\vec{v}, \vec{h})$, $(\vec{w}, \vec{g}) \in \bigcup_{\beta < l(u)} A_\beta$, if $\vec{v} \prec \vec{w}$ then the canonical expansion of $(\vec{v}, \vec{h})$ is addable to the canonical expansion of $(\vec{w}, \vec{g})$.

For each $a \in U_\infty$ we define a partial ordering $\mathbf{P}_a$. An element of $\mathbf{P}_a$ will be a sequence of suitable five-tuples and a four-tuple.

If $\vec{a} = (\vec{w}, \vec{h})$ then $p \in \mathbf{P}_a$ if and only if there is a sequence of suitable 5-tuples for a, $t_1, \ldots, t_n$, with $t_i = \langle(\vec{u}_i, \vec{f}_i)\vec{A}_i\vec{k}_i s_i\rangle$ and $p =$

$\langle t_1 \cdots t_n, (\vec{w}, \vec{h})\vec{A}\vec{k}\rangle$ such that:

a) $(u_i)_0 \prec (u_{i+1})_0$, $(u_i)_0 \in \mathscr{P}_{\kappa_w}(\lambda_w)$;

b) $\vec{A}$ is a sequence of sets of measure one for $\vec{w}$;

c) $\vec{k}$ is a sequence of functions where for each $\delta < l(\vec{w})$ the domain of k_δ is A_δ and for all $b \in A_\delta$, $k_\delta(b) \in \mathrm{Add}(\beth_4(\kappa_b), \kappa_\omega)$;

d) $s_i \in \mathrm{Add}(*\beth*_4(\kappa_{\vec{u}_i}), \kappa_{\vec{u}_{i+1}})$.

We will write κ_i for $\kappa_{\vec{u}_i}$.

If $p, q \in \mathbf{P}_a$, $p = (t_1 \cdots t_u, (\vec{w}, \vec{h})\vec{A}\vec{k})$ and $q = (s_1 \cdots s_l, (\vec{w}, \vec{h})\vec{B}\vec{k}')$ then $p \leq q$ if and only if p can be derived from q by

1) adding 5-tuples t with $(s_i)_0 \prec (t)_0 \prec (s_{i+1})_0$ and t addable to s_{i+1},
2) adding t's addable to s_1,
3) adding t's with $(s_l)_0 \prec (t)_0$ with t addable to $\langle(\vec{w}, \vec{h})\vec{A}, \vec{k}\rangle$,
4) shrinking the 5-tuples s_i,
5) $\vec{A} \subseteq \vec{B}$ and for all $a \in \vec{A}$, $\delta < l(\vec{w})$, $k_\delta(a) \leq k'_\delta(a)$.

It is easy to verify that this is a partial ordering. In particular transitivity follows easily from the transitivity of addability.

This partial ordering is not separative as defined. We now describe canonical representatives of each equivalence class in the separative quotient. Let $p = \langle t_1 \cdots t_n(\vec{\omega}, \vec{h})\vec{A}\vec{k}\rangle$ be a condition where $t_i = \langle(\vec{u}_i, \vec{f}_i)\vec{A}_i\vec{k}_i s_i\rangle$. For $1 < i \leq n$, $\delta < l(u_i)$, let

$$(B_i)_\delta = \{(\vec{v}, \vec{g}) \colon (\vec{u}_{i-1})_0 < (\vec{v})_0 \text{ and } s_{i-1} \in \mathrm{Add}(\beth_4(\kappa_{i-1}), \kappa_{\vec{v}})\} \cap (A_i)_\delta.$$

We claim that $(B_i)_\delta$ is of $(u_i)_\delta$-measure one.

Consider $\alpha \leq \lambda_1$ and $M_{<\alpha}$. For all $\delta' < \alpha$, $M_{\delta'}$ is a κ-complete measure on sequences $(\vec{v}, \vec{g})$ that is fine with respect to the first coordinate $(\vec{v})_0$. Hence for all $x \in \mathscr{P}_\kappa(\lambda^*)$ and all $\gamma < \kappa$, $\{(\vec{v}, \vec{g}) \colon x \subseteq (\vec{v})_0$ and $\gamma < \kappa_{\vec{v}}\}$ is M_δ-measure one. Since $u_i \in U_\infty$, by Lemma 3.2, B_δ is of $(u_i)_\delta$-measure one. Let

$$B_\delta = \{(\vec{v}, \vec{g}) \in A_\delta(\vec{u}_n)_0 \prec (\vec{v})_0 \text{ and } s \in \mathrm{Add}(\beth_4(\kappa_n), \kappa_v)\}.$$

Then B_δ is ω_δ-measure one similarly.

Let $t'_i = \langle(\vec{u}_i, \vec{h}_i)\vec{B}_i\vec{k}_i s_i\rangle$. Let $p' = \langle t'_1 \cdots t'_n(\vec{\omega}, \vec{h})\vec{B}\vec{k}\rangle$. Then p and p' are in the same class in the separative quotient and p' is a canonical representative. In the rest of the proof we will frequently tacitly assume we are working with the canonical representatives of elements of $\mathbf{P}_a$, the p''s. It is easy to check that the canonical representative of an equivalence class is the minimal element of that class.

The *length* of a condition p, $l(p)$, is the number of 5-tuples occurring in p. We say that p *refines* q if and only if $p \leq q$ and $l(p) = l(q)$.

If $p = \langle t_1 \cdots t_n, (\vec{w}, \vec{h})\vec{A}\vec{k}\rangle$ then $(t_1, \ldots, t_n)$ is the *lower part* of p and $((\vec{w}, \vec{h})\vec{A}, \vec{k})$ is the *upper part* of p.

We will now start a sequence of lemmas showing that $\mathbf{P}_a$ strongly factors at many places.

LEMMA (5.3). *Suppose* $p \in \mathbf{P}_{(\vec{\omega},\vec{h})}$, $l(p) = n$, $1 \le i \le n$, *and* $l(\vec{u}_{i+1}) = 1$. *Then there is a condition* $p' \in \mathbf{P}_{(\vec{u}_i,\vec{f}_i)}$ *and a condition* $p'' \in \mathbf{P}_{(\vec{\omega},\vec{h})}$ *such that*

$$\mathbf{P}_{(\vec{\omega},\vec{h})}/p \simeq \mathbf{P}_{(\vec{u}_i,\vec{f}_i)}/p' \times \mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})/s_i \times \mathbf{P}_{(\vec{\omega},\vec{h})}/p''.$$

Proof. We may assume that p is a canonical representative of its class. For $j < i$ and $t_j = \langle (u_j, f_j)\vec{A}_j\vec{k}_j s_j\rangle$ we have $(\vec{u}_j)_0 \prec (\vec{u}_i)_0$. Hence $\vec{u}_j^*$ is defined with respect to $\vec{u}_i$. Let $t_j^* = \langle (u_j^*, f_j)\vec{A}_j\vec{k}_j s_j\rangle$ and let $p' = \langle t_1^* \cdots t_{i-1}^*(\vec{u}_i, \vec{f}_i)\vec{A}_i\vec{k}_i\rangle$. Let $p'' = \langle t_{i+1} \cdots t_n(\vec{\omega},\vec{h})\vec{A}\vec{k}\rangle$. We can define a map

$$\varphi\colon \mathbf{P}_{(\vec{u}_i,f_i)}/p' \times \mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})/s_i \times \mathbf{P}_{(\vec{\omega},\vec{h})}/p'' \hookrightarrow \mathbf{P}_{(\vec{\omega},\vec{h})}/p$$

that simply concentrates conditions and changes the first coordinates of the measure sequences. Explicitly: φ takes the triple

$$\left(\left\langle t_1^{\#} \cdots t_m^{\#}\left(\vec{u}_i, \vec{f}_i\right)\vec{A}_i^{\dagger}\vec{k}_i^{\dagger}\right\rangle, s^{\dagger}, \left\langle t_1^{\dagger\dagger} \cdots t_r^{\dagger\dagger}(\vec{\omega},\vec{h})\vec{B}^{\dagger\dagger}\vec{k}^{\dagger\dagger}\right\rangle\right)$$

to the condition

$$\left\langle t_1^{\dagger} \cdots t_m^{\dagger}\left\langle\left(\vec{u}_i, \vec{f}_i\right)\vec{A}_i^{\dagger}\vec{k}_i^{\dagger}s^{\dagger}\right\rangle t_1^{\dagger\dagger} \cdots t_r^{\dagger\dagger}(\vec{\omega},\vec{h})\vec{B}^{\dagger\dagger}\vec{k}^{\dagger\dagger}\right\rangle$$

where $t_l^{\dagger}$ is $t_l^{\#}$, but with $(u_l^{\#})_0$ changed to $i''_{u_i w}(u_l^{\#})_0$. Note that $t_1^{\dagger\dagger} = \langle(\vec{u}_{i+1}, \vec{f}_{i+1})\vec{A}_{i+1}^{\dagger\dagger}\vec{k}_{i+1}^{\dagger\dagger}\vec{s}_{i+1}^{\dagger\dagger}\rangle$ since $l(\vec{u}_{i+1}) = 1$ (so nothing can be added to t_{i+1}). It is easy to see that φ is order-preserving and maps onto $\mathbf{P}_{\langle\vec{\omega},\vec{h}\rangle}/p$. □1

We will frequently have maps similar to the φ in the previous proof. The maps are essentially concatenation with trivial changes to make the result a condition in the appropriate partial ordering. In the following we will use the notation $\langle\ \ \rangle$ for this map. So, for example if φ is as above we would write $\varphi(q, s, q^{\#})$ as $\langle qsq^{\#}\rangle$. If we have the similar map

$$\varphi\colon \left(\mathbf{P}_{(\vec{u}_i,\vec{f}_i)}/p' \times \mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})\right) \times \mathbf{P}_{\langle\vec{\omega},\vec{h}\rangle}/p'' \to \mathbf{P}_{\langle\vec{\omega},\vec{h}\rangle}/p$$

that has as its domain *pairs* of conditions (one in the product of $\mathbf{P}_{(\vec{u}_i,\vec{f}_i)}/p'$ and $\mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})$, the other in $\mathbf{P}_{\langle\vec{\omega},\vec{h}\rangle}/p''$), we would write $\varphi(q, q') = \langle q, q'\rangle$. Using Lemma 5.3 as justification we will tend not to distinguish carefully between measure sequences changed only in the first coordinate, as in $t^{\#}$ and $t^{\dagger}$ above, for example.

The following lemma is an approximation to the strong factorization property.

LEMMA (5.4). *Suppose* $b \in \mathscr{B}(\mathbf{P}_{\langle\vec{w},\vec{h}\rangle})$, p, p', s_i *are as in Lemma* 5.3 *with* $i = n$. *Then there are a maximal antichain* $A \subseteq \mathbf{P}_{(\vec{u}_n,\vec{h}_n)}/p' \times \mathrm{Add}(\beth_4(\kappa_n), \kappa_{\vec{w}})/s_n$

and a condition $p'' \in \mathbf{P}_{\langle\vec{w},\vec{h}\rangle}$ *with no lower part such that*

1) $\langle p' s_n p''\rangle$ *refines* p.
2) *For all* $q \in A$, $\langle qp''\rangle \parallel b$

(i.e. p'' *decides* b *"up to lower parts"*).

Proof. We first construct an upper part

$$\langle(\vec{w},\vec{h})\vec{A}^*\vec{k}^*\rangle \leq \langle(\vec{w},\vec{h})\vec{A}\vec{k}\rangle$$

such that for any lower part q^*, if $\langle q^*(\vec{w}\vec{h})\vec{A}^*\vec{k}^*\rangle$ is a condition in $\mathbf{P}_{\langle\vec{w},\vec{h}\rangle}$ and $\langle q^*(\vec{w}\vec{h})\vec{A}^*\vec{k}^*\rangle \leq p \wedge \langle(\vec{w},\vec{h})\vec{A}^*\vec{k}^*\rangle$, and there are $\vec{B}$, $\vec{g}$ such that $\langle q^*(\vec{w},\vec{h})\vec{B}\vec{g}\rangle \parallel b$ then $\langle q^*(\vec{w},\vec{h})\vec{A}^*\vec{k}^*\rangle \parallel b$.

To do this: Enumerate the lower parts $\langle q_\alpha : \alpha < \beth_3(\kappa_w)\rangle$. Build a sequence of sets of measure one $\langle \vec{A}_\alpha : \alpha < \beth_3(\kappa_w)\rangle$, $\vec{A}_\alpha$-measure one for $\vec{w}$, and an a.e.-decreasing sequence of functions $\langle \vec{k}_\alpha : \alpha < \beth_3(\kappa_w)\rangle$.

At stage β: Let $\vec{k}'_\beta$ be a sequence of functions $\langle(\vec{k}'_\beta)_\delta : \delta < l(w)\rangle$ defined on w_δ sets of measure one such that

a) $(\vec{k}'_\beta)_\delta((u,h)) \in \mathrm{Add}(\beth_4(\kappa_u), \kappa_w)$,

b) for all $\alpha < \beta$, $(\vec{k}'_\beta)_\delta \leq (\vec{k}_\alpha)_\delta$ a.e. in w_δ.

Such a $\vec{k}'_\beta$ exists by Lemma 3.3.

If $\langle q_\beta(\vec{w},\vec{h})\vec{A}\vec{k}'_\beta\rangle \leq p \wedge \langle(\vec{w},\vec{h})\vec{A}\vec{k}'_\beta\rangle$ and there are a sequence of sets $\vec{B}$ of $\vec{w}$-measure one and functions $\vec{g}$ defined on $\vec{w}$-measure one, $\vec{g} \leq \vec{k}'_\beta$ a.e., $(\vec{g})_\delta(a) \in \mathrm{Add}(\beth_4(\kappa_a), \kappa_w)$, $\delta < l(\vec{w})$, and $\langle q_\beta(\vec{w},\vec{h})\vec{B}\vec{g}\rangle \parallel b$, let $\vec{A}_\beta = \vec{B}$ and $\vec{k}_\beta = \vec{g}$. Otherwise, $\vec{A}_\beta = \vec{A}$ and $\vec{k}_\beta = \vec{k}'_\beta$.

It is easy to verify by induction that for all β, $a \in \mathrm{dom}(\vec{k}_\beta)_\delta$ we have $(\vec{k}_\beta)_\delta(a) \in \mathrm{Add}(\beth_4(\kappa_a), \kappa_w)$ and the $\vec{k}_\beta$'s are a descending sequence modulo sets of measure one in $\vec{w}$.

Let $\vec{k}^*$ be a function sequence such that for all $\delta < l(w)$, and all $\beta < \beth_3(\kappa_w)$, $\mathrm{dom}(\vec{k}^*)_\delta$ is of w_δ-measure one, $(\vec{k}^*_\delta)(a) \in \mathrm{Add}(\beth_4(\kappa_u), \kappa_w)$ and $\vec{k}^* \leq \vec{k}_\beta \pmod{\vec{w}}$.

Let $\vec{\rho}$ be defined by $\rho(q_\beta)_\delta = (\vec{A}_\beta)_\delta \cap \{a : (\vec{k}^*)_\delta(a) \leq (\vec{k}_\beta)_\delta(a)\}$. Let $\vec{A}^* = \Delta\vec{\rho}$. By Lemma 4.1, $\vec{A}^*$ is a sequence of sets of $\vec{w}$-measure one.

For all β, if $\langle q_\beta(\vec{w},\vec{h})\vec{A}^*\vec{k}^*\rangle \leq p \wedge \langle(\vec{w},\vec{h})\vec{A}^*\vec{k}^*\rangle$ then: $\langle q_\beta(\vec{w},\vec{h})\vec{A}^*\vec{k}^*\rangle \leq \langle q_\beta(\vec{w},\vec{h})\vec{A}_\beta\vec{k}_\beta\rangle$. Thus, if some $\langle q_\beta(\vec{w},\vec{h})\vec{B}\vec{g}\rangle \parallel b$ then $\langle q_\beta(\vec{w},\vec{h})\vec{A}_\beta\vec{k}_\beta\rangle \parallel b$ so that $\langle q_\beta(\vec{w},\vec{h})\vec{A}^*\vec{k}^*\rangle \parallel b$. Hence $\langle(w,h)\vec{A}^*\vec{k}^*\rangle$ is as desired.

Now build a $k^{**} \leq k^*$ such that for all $\beta < \beth_3(\kappa_w)$ and $\delta < l(\vec{w})$ and $b' = \pm b$, if $\{(\vec{u},\vec{f})$: for some $\vec{c}\vec{k}s$, $\langle(\vec{u},\vec{f})\vec{c}\vec{k}s\rangle$ is addable to $\langle(\vec{w},\vec{h})\vec{A}^*\vec{k}^{**}\rangle$ and $\langle q_\beta\langle(\vec{u},\vec{f})\vec{c}\vec{k}s\rangle(\vec{w},\vec{h})\vec{A}^*\vec{k}^{**}\rangle \Vdash b'\}$ is $(\vec{w})_\delta$-measure one then

$$\left\{(\vec{u},\vec{f}): \text{for some } \vec{c}\vec{k},\ \left\langle(\vec{u},\vec{f})\vec{c}\vec{k}k^{**}(\vec{u},\vec{f})\right\rangle \text{ is addable to } \left\langle(\vec{w},\vec{h})\vec{A}^*\vec{k}^{**}\right\rangle\right.$$

and

$$\left\langle q_\beta\left\langle\left(\vec{u},\vec{f}\right)\vec{c}\vec{k}k^{**}\left(\vec{u},\vec{f}\right)\right\rangle\left(\vec{w},\vec{h}\right)\vec{A}^{*}\vec{k}^{**}\right\rangle \Vdash b'\Big\}$$

is $(\vec{w})_\delta$-measure one. Do this by constructing a decreasing sequence of functions below k^*, $\langle g_\beta\colon \beta < \beth_3(\kappa_w)\rangle$. At stage β: let $\vec{g}$ be $\leq$ -a.e.-$\vec{g}_\alpha$ for all $\alpha < \beta$. For each $(\vec{u},\vec{f}) \in \operatorname{dom}(\vec{g})_\delta$ choose $(\vec{g}_\beta)_\delta(\vec{u},\vec{f}) \leq (\vec{g})_\delta(\vec{u},\vec{f})$ so that if $s \leq (\vec{g}_\beta)_\delta(\vec{u},\vec{f})$, $b' \neq b$ and there are $\vec{c},\vec{k}$ such that $\langle(\vec{u},\vec{f})\vec{c}\vec{k}s\rangle$ is addable to $\langle(\vec{w},\vec{h})\vec{A}^*\vec{g}\rangle$ and $\langle q_\beta\langle(\vec{u},\vec{f})\vec{c}\vec{k}s\rangle(\vec{w},\vec{h})\vec{A}^*\vec{g}\rangle \Vdash b'$, then

$$\langle q_\beta\langle\left(\vec{u},\vec{f}\right)\vec{c}\vec{k}\left(\vec{g}_\beta\right)_\delta\left(\vec{u},\vec{f}\right)\rangle\left(\vec{w},\vec{h}\right)\vec{A}^*\vec{g} \;\Vdash b'.$$

With $\vec{k}^{**} \leq$ -a.e. $\vec{q}_\beta$ every $\vec{g}_\beta$ works as desired.

Let $\vec{A}_0^{**} = \vec{A}^*$. Define $\langle \vec{A}_i^{**}\colon i \in w\rangle$ by induction. At stage $i+1$: for each $\beta < \beth_3(\kappa_w)$ and $\delta < l(\vec{w})$ choose a $(\vec{w})_\delta$-measure one set $\rho(q_\beta)_\delta \subseteq A_\delta^*$ consisting of $(\vec{u},\vec{f})$ for which there are $\vec{c},\vec{k}$ such that $\langle(\vec{u},\vec{f})\vec{c}\vec{k}k^{**}(\vec{u},\vec{f})\rangle$ is addable to $\langle(\vec{w},\vec{h})A_i^{**}k^{**}\rangle$, $\vec{k}^{**}(\vec{u},\vec{f}) \leq \vec{g}_\beta(\vec{u},\vec{f})$, and such that if $b' = \pm b$ and there is *some* $(\vec{u},\vec{f}) \in \rho(q_\beta)_\delta$ with $\vec{c},\vec{k}$ such that $\langle(\vec{u},\vec{f})\vec{c}\vec{k}\vec{k}^{**}(\vec{u},\vec{f})\rangle$ is addable to $\langle(\vec{w},\vec{h})\vec{A}_i^{**}\vec{k}^{**}\rangle$ and $\langle\vec{q}_\beta\langle(\vec{u},\vec{f})\vec{c}\vec{k}k^{**}(\vec{u},\vec{f})\rangle(\vec{w},\vec{h})\vec{A}_i^{**}\vec{k}^{**}\rangle \Vdash b'$, then *every* $(\vec{u},\vec{f}) \in \rho(q_\beta)$ has this property. Let $\vec{A}_{i+1}^{**} = \vec{A}_i^{**} \cap \Delta\rho$. Let $\vec{A}^{**} = \bigcap_{i\in w}\vec{A}_i^{**}$.

If $b' = \pm b$ and $(\vec{u},\vec{f}) \in (\vec{A}^{**})_\delta$ and there are $\vec{c},\vec{k}$ and s such that $\langle(\vec{u},\vec{f})\vec{c}\vec{k}s\rangle$ are addable to $\langle(\vec{w},\vec{h})\vec{A}^{**}\vec{k}^{**}\rangle$ with $\langle q_\beta\langle(\vec{u},\vec{f})\vec{c}\vec{k}s\rangle(\vec{w},\vec{h})\vec{A}^{**}\vec{k}^{**}\rangle \Vdash b'$, then for every $(\vec{u},\vec{f}) \in (\vec{A}^{**})_\delta$ there are $\vec{c},\vec{k}$ such that $\langle(\vec{u},\vec{f})\vec{c}\vec{k}k^{**}(\vec{u},\vec{f})\rangle$ is addable to $\langle(\vec{w},\vec{h})\vec{A}^{**}\vec{k}^{**}\rangle$ and

$$\left\langle\vec{q}_\beta\left\langle\left(\vec{u},\vec{f}\right)\vec{c}\vec{k}k^{**}\left(\vec{u},\vec{f}\right)\right\rangle\left(\vec{w},\vec{h}\right)\vec{A}^{**}\vec{k}^{**}\right\rangle \Vdash b'.$$

We claim that $\langle p's_n(\vec{w},\vec{h})\vec{A}^{**}\vec{k}^{**}\rangle$ suffices for the lemma (i.e., $p'' = \langle(\vec{w},\vec{h})\vec{A}^{**}\vec{k}^{**}\rangle$).

Otherwise there is a $p^* \leq (p', s_n)$ for all $q' \leq p^*$ if $\langle q'q''(\vec{w},\vec{h})\vec{A}^{**}k^{**}\rangle$ decides b, with $q' \in \mathbf{P}_{(\vec{u}_n,\vec{h}_n)}/p' \times \operatorname{Add}(\beth_4(\kappa_n),\kappa_w)/s_n$. Then q'' has non-zero length. Take $q'q''$ of minimal length with $\langle q'q''(\vec{w},\vec{h})\vec{A}^{**}\vec{k}^{**}\rangle \Vdash b'$ for $b' = \pm b$. For some β, $q'q'' = \langle q_\beta(\vec{u},\vec{f})\vec{B}\vec{g}s\rangle$. By choice of $\vec{k}^{**}$ and $\vec{A}^{**}$ we may assume that $s = (\vec{k}^{**})_\delta(\vec{u},\vec{f})$ where $(\vec{u},\vec{f}) \in (\vec{A}^{**})_\delta$ for some δ, and for all $(\vec{u},\vec{f}) \in (\vec{A}^{**})_\delta$, there are $\vec{c},\vec{k}$ such that

$$\left\langle q_\beta\left\langle\left(\vec{u},\vec{f}\right)\vec{c}\vec{k}k^{**}\left(\vec{u},\vec{f}\right)\right\rangle\left(\vec{w},\vec{h}\right)\vec{A}^{**}\vec{k}^{**}\right\rangle \Vdash b'.$$

Define $\mathscr{K}(\vec{u},\vec{f}) =$ the $\vec{k}$ that works and $\mathscr{C}(\vec{u},\vec{f}) =$ the $\vec{c}$ that works.

By Lemma 3.2, $(\vec{w})_0 = i''_w\lambda_w$ comes from a $\beth_3(\kappa_w)$-supercompact embedding $i\colon V \to N$ that constructs $\vec{w}$. For $\alpha < \delta$, let $B_\alpha = \{(\vec{v},\vec{f}) \in A_\alpha^{**}\colon$ the canonical expansion of $(\vec{v}, \vec{f})$ is addable to

$$\left\langle\left(\vec{w} \upharpoonright \delta, \vec{h} \upharpoonright \delta\right) i(\mathscr{C})\left(\vec{w} \upharpoonright \delta, \vec{h} \upharpoonright \delta\right) i(\mathscr{K})\left(\vec{w} \upharpoonright \delta, \vec{h} \upharpoonright \delta\right)\right\rangle.$$

By Lemma 5.1, B_α has $(\vec{w})_\alpha$-measure one. Note that $i(\mathscr{C})(\vec{w} \upharpoonright \delta, \vec{h} \upharpoonright \delta) \subseteq \vec{A}^{**} \upharpoonright \delta$ coordinatewise and $i(\mathscr{K})(\vec{w} \upharpoonright \delta, \vec{h} \upharpoonright \delta) \leq \vec{k}^{**} \upharpoonright \delta$ pointwise, since

$$\left\langle\left(\vec{w} \upharpoonright \delta, \vec{h} \upharpoonright \delta\right) i(\mathscr{C})\left(\vec{w} \upharpoonright \delta, \vec{h} \upharpoonright \delta\right) i(\mathscr{K})\left(\vec{w} \upharpoonright \delta, \vec{h} \upharpoonright \delta\right) i(k^{**})\left(\vec{w} \upharpoonright \delta, \vec{h} \upharpoonright \delta\right)\right\rangle$$

is addable to $i(\langle(\vec{w}, \vec{h})\vec{A}^{**}\vec{k}^{**}\rangle)$. Thus, for $a \in i(\mathscr{C})(\vec{w}, \vec{h})_\alpha$, $i(a) \in i(\vec{A}^{**})_{i(\alpha)}$ so that $a \in (\vec{A}^{**})_\alpha$, and further, $i(\mathscr{K})(w \upharpoonright \delta, h \upharpoonright \delta)(a) \leq i(\vec{k}^{**})(i(a)) = \vec{k}^{**}(a)$.

For each $(\vec{v}, \vec{l}) \in B_\alpha$,

$$H(\vec{v}, \vec{l}\,) = \left\{\left(\vec{u}, \vec{f}\,\right)\text{: the canonical expansion of } \left(\vec{v}, \vec{l}\right) \text{ is addable to } \left\langle\left(\vec{u}, \vec{f}\,\right)\mathscr{C}\left(\vec{u}, \vec{f}\,\right)\mathscr{K}\left(\vec{u}, \vec{f}\,\right)\right\rangle\right\}$$

has $(\vec{w})_\delta$ measure one. Let $B_\delta = \Delta H \cap (A^{**})_\delta$. Then for $\alpha < \delta$, if $(\vec{v}, \vec{l}) \in B_\alpha$, $(\vec{u}, \vec{f}) \in B_\delta$ and $\vec{v} \prec \vec{u}$, then the canonical expansion of $(\vec{v}, \vec{l})$ is addable to $\langle(\vec{u}, \vec{f})\mathscr{C}(\vec{u}, \vec{f})\mathscr{K}(\vec{u}, \vec{f})\rangle$. For $\alpha > \delta$, let $B_\alpha = \{(\vec{v}, \vec{l})\colon \delta \in (\vec{v})_0\} \cap A^{**}_\alpha$.

Let $\vec{k}' \upharpoonright \delta = i(\mathscr{K})(w \upharpoonright \delta)$ and for $\alpha \geq \delta(\vec{k}')_\alpha, = (k^{**})_\alpha$.

By shrinking $\vec{B}$ further we may assume that for all $(\vec{u}, \vec{h}) \in B_\delta$, $\gamma < l(\vec{u})$, and all $(v, l) \in \mathscr{C}(\vec{u}, \vec{h})_\gamma$, $\mathscr{K}(\vec{u}, \vec{h})_\gamma(v, l) \geq (\vec{k}')_{i_{uw}(\gamma)}(i_{uw}(v, l))$. Then $\langle q_\beta(\vec{w}, \vec{h})\vec{B}\vec{k}'\rangle \leq \langle q_\beta(\vec{w}, \vec{h})\vec{A}^{**}\vec{k}^{**}\rangle$. So $\langle q_\beta(\vec{w}, \vec{h})\vec{B}\vec{k}'\rangle \nVdash b$. Hence there are $q', q'', q' \leq q_\beta$ and $\langle q'q''(\vec{w}, \vec{h})\vec{B}\vec{k}'\rangle \Vdash \neg b'$. Let q', q'' be arbitrary, satisfying these conditions. Suppose $q'' = \langle t'_l \cdots t'_r\rangle$, $t'_i = \langle(u'_i, h'_i)A'_i k'_i s'_i\rangle$.

Let i be minimal with $(u'_i, h'_i) \in B_\alpha$, $\alpha \geq \delta$. If no such i exists let $i = r + 1$.

Case 1. If $(u'_i, h'_i) \in B_\delta$ then

$$(**) \quad \left\langle q'q''(\vec{w}\vec{h})\vec{B}\vec{k}'\right\rangle \leq \left\langle q_\beta\left\langle(u'_i, h'_i)\mathscr{C}(u'_i, h'_i)\mathscr{K}(u'_i, h'_i)\vec{k}'(u'_i h'_i)\right\rangle(\vec{w}\vec{h})\vec{B}\vec{k}'\right\rangle,$$

since the canonical expansion of each (u'_m, h'_m), $m < i$, is addable to $\langle(u'_i, h'_i)\mathscr{C}(u'_i h'_i)\mathscr{K}(u'_i h'_i)\rangle$ and t'_m refines this canonical expansion.

But the inequality $(**)$ is a contradiction since the left-hand side forces $\neg b'$ and, because $(u'_i, h'_i) \in B_\delta$, the right-hand side forces b'.

Case 2. Assume otherwise; we treat the case $i < r$. If $i = r + 1$, it is similar and easier. Consider $\langle(\vec{u}'_i \vec{h}'_i)\vec{A}'_i \vec{k}'_i s'_i\rangle$. Since $\delta \in (\vec{u}'_i)_0$ for some $\beta < l(\vec{u}'_i)$, $i_{u'_i, w}(\beta) = \delta$.

Let $(u^\#, h^\#) \in (\vec{A}'_i)_\beta$ with $(u^*, h^*) = i_{u', w}(u^\#, h^\#)$ and $(u_{i-1})_0 \prec (u^*)_0$ and such that the canonical expansion t of $(u^\#, h^\#)$ is addable to t'_i. Then

$(u^*, h^*) \in B_\delta$ and t is addable to $\langle(\vec{w}\vec{h})\vec{B}\vec{k}'\rangle$. Let q^+ be the result of adding t to $\langle q'q''(\vec{w}, \vec{h})\vec{B}\vec{k}'\rangle$. Then for q^+ we are back in case one, a contradiction. Thus $p'' = \langle(\vec{w}, \vec{h})\vec{A}^{**}, \vec{k}^{**}\rangle$ satisfies the lemma. □

LEMMA (5.5). *Let* $p = \langle t_1 \cdots t_m(\vec{u}, \vec{h})\vec{A}\vec{k}\rangle \in \mathbf{P}_{(\vec{u}, \vec{h})}$, $t_i = \langle(\vec{u}_i, \vec{h}_i)\vec{A}_i\vec{k}_i s_i\rangle$. *Let* $b \in \mathscr{B}(\mathbf{P}_{(\vec{u}, \vec{h})})$. *Let* $p^* = \langle t_1, \ldots, t_{i-1}(\vec{u}_i, \vec{h}_i)\vec{A}_i\vec{k}_i\rangle \in \mathbf{P}_{(\vec{u}_i, \vec{h}_i)}$. *Then there are refinements* $t'_{i+1}, \ldots, t'_n$, *and* $\vec{A}' \subseteq \vec{A}$, $\vec{k}' \leq_{\text{a.e.}} \vec{k}$ *and a maximal antichain* $T \subseteq \mathbf{P}_{(\vec{u}_i, \vec{h}_i)}/p^* \times \mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})/s_i$ *such that for all* $(q, s) \in T$, $\langle qst'_{i+1} \cdots t'_n(\vec{u}, \vec{h})\vec{A}'\vec{k}'\rangle \parallel b$ (i.e., $t_{i+1} \cdots t_n, \vec{A}, \vec{k}$ *can be refined so that* b *can be decided up to the part below* t_i *and* s_i).

Proof. By induction on $0 \leq j \leq n - 1$, we prove it for $i = n - j$. For $j = 0$ this is Lemma 5.3. Assume true for $n - j = i + 1$. We prove it for $n - (j + 1) = i$.

By the induction hypothesis we can find $t'_{i+2}, \ldots t'_n, \vec{A}', \vec{k}'$ such that there is a maximal antichain

$$T' \subseteq \mathbf{P}_{\langle(\vec{u}_{i+1}, \vec{h}_{i+1})\rangle}\Big/\Big\langle t_1 \cdots t_i\Big(\vec{u}_{i+1}, \vec{h}_{i+1}\Big)\vec{A}_{i+1}k_{i+1}\Big\rangle \times \mathrm{Add}\big(\beth_4(\kappa_{i+1}), \kappa_{i+2}\big)$$

such that for all $(q, t) \in T'$, $\langle qtt'_{i+2}, \ldots t'_n(\vec{u}, \vec{h})\vec{A}'\vec{k}'\rangle \parallel b$.

Since $|\mathbf{P}_{(\vec{u}_{i+1}, \vec{h}_{i+1})}| = \beth_3(\kappa_{i+1})$ and $\mathrm{Add}(\beth_4(\kappa_{i+1}, \kappa_{i+2}))$ is $\beth_4(\kappa_{i+1})$-closed we can find an s^* and a maximal antichain

$$T^* \subseteq \mathbf{P}_{(\vec{u}_{i+1}, \vec{h}_{i+1})}\Big/\Big\langle t_i \cdots t_i\Big(\vec{u}_{i+1}, \vec{h}_{i+1}\Big)\vec{A}_{i+1}\vec{k}_{i+1}\Big\rangle$$

such that for all $q \in T^*$, $\langle qs^*t'_{i+2} \cdots t'_n(\vec{u}, \vec{h})\vec{A}', \vec{k}'\rangle \parallel b$.

In $\mathscr{B}(\mathbf{P}_{\langle(\vec{u}_{i+1}, \vec{h}_{i+1})\rangle})$, let

$$b^* = \bigvee \Big\{q \in T^*\colon \Big\langle qs^*t'_{i+2} \cdots t'_n(\vec{u}, \vec{h})\vec{A}', \vec{k}'\Big\rangle \leq b\Big\}.$$

Then $b^* \leq \langle t_1 \cdots t_i(\vec{u}_{i+1}, \vec{h}_{i+1})\vec{A}_{i+1}, \vec{k}_{i+1}\rangle$. By Lemma 5.3 we can find a maximal antichain $T \subseteq \mathbf{P}(\vec{u}_i, \vec{h}_i)/\langle t_1 \cdots t_{i-1}(\vec{u}_i, \vec{h}_i)\vec{A}_i, \vec{k}_i\rangle \times \mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})/s_i$ and $\vec{A}'_{i+1} \subseteq \vec{A}_{i+1}$, $\vec{k}'_{i+1} \leq_{\text{a.e.}} \vec{k}_{i+1}$ such that for all $(q, s) \in T$, $\langle qs(\vec{u}_{i+1}, \vec{h}_{i+1})\vec{A}_{i+1}, \vec{k}_{i+1}\rangle \parallel b^*$. Let $t'_{i+1} = \langle(\vec{u}_{i+1}, \vec{h}_{i+1})\vec{A}'_{i+1}\vec{k}'_{i+1}s^*\rangle$. Then t'_{i+1} refines t_{i+1}. Let $(q, s) \in T$. Then $\langle qs(\vec{u}_{i+1}, \vec{h}_{i+1})\vec{A}'_{i+1}, \vec{k}'_{i+1}\rangle \parallel b^*$ so that $\langle qs\langle(\vec{u}_{i+1}, \vec{h}_{i+1})\vec{A}'_{i+1}, \vec{k}'_{i+1}s^*\rangle t'_{i+2} \cdots t'_n(\vec{u}, \vec{h})\vec{A}'\vec{k}'\rangle \parallel b$. Hence $\langle t'_{i+1} \cdots t'_n(\vec{u}, \vec{h})\vec{A}', \vec{k}'\rangle$ works for the lemma. □

PROPOSITION (5.6). *Let* $p = \langle t_1 \cdots t_n(\vec{u}\vec{h})\vec{A}\vec{k}\rangle \in \mathbf{P}_{(\vec{u}, \vec{h})}$ *be as above.*

a) *Suppose* $l(u_{i+1}) = 1$. *Let* $\lambda \in [\beth_3(\kappa_i)^+, \kappa_{i+1})$. *Then* $\mathbf{P}_{(\vec{u}, \vec{h})}/p$ *strongly factors at* λ.

b) *Suppose that* $\lambda \in [\beth_3(\kappa_i)^+, \kappa_{i+1})$. *Then there is a* $p' \leq p$ *such that* $\mathbf{P}_{(\vec{u}, \vec{h})}/p'$ *factors at* λ.

Proof. a) By Lemma 5.3,

$$\mathbf{P}_{(\vec{u},\vec{h})}/p \simeq \mathbf{P}_{(\vec{u}_i,\vec{h}_i)}/\left\langle t_1 \cdots t_{i-1}\left(\vec{u}_i\vec{h}_i\right)\vec{A}_i\vec{k}_i\right\rangle$$
$$\times \operatorname{Add}(\beth_4(\kappa_i), \kappa_{i+1})/s_i \times \mathbf{P}_{(\vec{u},\vec{h})}/\left\langle t_{i+1} \cdots t_n\left(\vec{u}\vec{h}\right)\vec{A}\vec{k}\right\rangle.$$

Since $\beth_4(\kappa_i)$ is a regular limit λ with $|\lambda^{<\lambda}| = \lambda$, $\operatorname{Add}(\beth_4(\kappa_i), \kappa_{i+1})$ is $\beth_4(\kappa_i)^+$-c.c.

For $\lambda \in [\beth_3(\kappa_i)^+, \beth_4(\kappa_i)]$, let

$$\mathscr{F}_m \subseteq \operatorname{Add}(\beth_4(\kappa_i), \kappa_{i+1}) \times \mathbf{P}_{(\vec{u},\vec{h})}/\left\langle t_{i+1} \cdots t_n\left(\vec{u}\vec{h}\right)\vec{A}\vec{k}\right\rangle \stackrel{\text{def}}{=} R$$

be the collection of conditions of length m.

Then $\langle \mathscr{F}_m \colon m \in \omega\rangle$ is a witness to the λ-Prikry property of R, since each $\mathscr{F}_m$ is $\beth_4(\kappa_i)$-closed and by Lemma 5.4 decides all Boolean values.

Since $|\mathbf{P}_{(\vec{u}_i,\vec{h}_i)}| = \beth_3(\kappa_i)$, $Q \stackrel{\text{def}}{=} \mathbf{P}_{(\vec{u}_i,\vec{h}_i)}$ is $\beth_3^+(\kappa_i)$-c.c. Then Lemma 5.5 implies that $Q \times R$ is a witness to strong factoring at λ.

If $\lambda \in [\beth_4(\kappa_i)^+, \kappa_{i+1})$ then

$$Q \stackrel{\text{def}}{=} \mathbf{P}_{(\vec{u}_i,\vec{h}_i)}/\left\langle t_1 \cdots t_{i-1}\left(\vec{u}_i\vec{h}_i\vec{A}_i\vec{k}_i\right)\right\rangle \times \operatorname{Add}(\beth_4(\kappa_i), \kappa_{i+1})$$

is λ-c.c. and $\mathbf{R} = \mathbf{P}_{(\vec{u},\vec{h})}/\langle t_{i+1}, \ldots t_n(\vec{u}\vec{h})\vec{A}\vec{k}\rangle$ is λ-Prikry by the analogous $\langle \mathscr{F}_m \colon m \in \omega\rangle$ ($q \in \mathscr{F}_m$ if and only if the length of q is m). Now Lemma 5.5 says this is a strong factoring.

b) If $l(u_{i+1}) = 1$ we are done by a). Otherwise, we can get $p' \leq p$ by adding a suitable 5-tuple t such that $\lambda < \kappa_t < \kappa_{i+1}$ and $l(u_t) = 1$. Then p' works by a). □

Definition (5.7). Let $G \subseteq \mathbf{P}_{(\vec{u},\vec{h})}$ be generic. Then κ is on the *Radin sequence* for G if and only if for some $p \in G$, $p = \langle t_1 \cdots t_n(\vec{u},\vec{h})\vec{A}\vec{k}\rangle$, $\kappa = \kappa_{u_i}$. It is easy to check that the Radin sequence for G is closed and unbounded in κ_u.

PROPOSITION (5.8). a) *κ is a limit point on the Radin sequence for $G \subseteq \mathbf{P}_{(\vec{u},\vec{h})}$ if and only if there is a $\in G$, $p = \langle t_1 \cdots t_n(\vec{u},\vec{h})\vec{A}\vec{k}\rangle$, $\kappa = \kappa_{u_i}$ and $l(u_i) > 1$.*

b) *If $\tau \in V^{\mathbf{P}_{(\vec{u},\vec{h})}}$ is a term for a subset of α, $p = \langle t_1 \cdots t_n(\vec{u},\vec{h})\vec{A}\vec{k}\rangle$ and $\alpha < \kappa_{u_i}$. Then $\tau \in V^{\mathbf{P}_{(\vec{u},\vec{h})}}/\langle t_1 \cdots t_{i-1}(\vec{u}_i, \vec{h}_i)\vec{A}_i\vec{k}_i\rangle$.*

c) *$\mathbf{P}_{(\vec{u},\vec{h})}$ does not collapse cardinals in the intervals $[\beth_3(\kappa_i)^+, \kappa_{i+1})$ where κ_i, κ_{i+1} are successive points in the Radin sequence.*

d) *If κ is a limit point of the Radin sequence then in $V^{\mathbf{P}_{(\vec{u},\vec{h})}}$, κ is a strong limit cardinal.*

Proof. a) If $l(u_i) = 1$ and $p = \langle t_1 \cdots t_{i-1}t_i \cdots t_n(\vec{u},\vec{h})\vec{A}\vec{k}\rangle$ then there is no 5-tuple addable to t_i; hence there is no element of the Radin sequence in (κ_{i-1}, κ_i).

If $l(u_i) > 1$ then for all $p = \langle t_1 \cdots t_{i-1}t_i \cdots t_n(\vec{u}, \vec{h})\vec{A}\vec{k}\rangle$ there is a suitable 5-tuple t such that $p' = \langle t_1 \cdots t_{i-1}tt_i \cdots t_n(\vec{u}, \vec{h})\vec{A}\vec{k}\rangle$ is a condition and $p' \leq p$. Hence κ_i is a limit point of the Radin sequence.

b) Without loss of generality, by Proposition 5.6 we can strongly factor $\mathbf{P}_{\langle \vec{u}, \vec{h}\rangle}/p$ at $\lambda \in (\alpha, \kappa_{u_i})$. Hence by Lemma 2.5,

$$\tau \in \mathbf{P}_{(\vec{u}, \vec{h})}\Big/\Big\langle t_1 \cdots t_{i-1}, \left(\vec{u}_i, \vec{h}_i\right)\vec{A}_i\vec{k}_i\Big\rangle.$$

c) This follows immediately from Proposition 5.6 and a).

d) follows immediately from a) and b) when we note that $|\mathbf{P}_{(\vec{u}, \vec{h})}| = \beth_3(\kappa_i)$.

Now 5.8 tells us that we have the G.C.H. failing in all intervals $(\beth_3^+(\kappa_i), \kappa_{i+1})$ where κ_i and κ_{i+1} are successive points on the Radin sequence. Unfortunately, this forcing collapses all cardinals in the interval $(\kappa, *\beth_3(\kappa))$ to κ if κ is a limit point in the Radin sequence. This recreates the G.C.H. at κ. Another problem is that we have not ensured that there is a regular limit point on the Radin sequence.

To perform both of these tasks it is necessary to look at a smaller "projected" forcing. This is treated in the next section.

Finally, we note that in other applications, we may want to collapse successor points on the Radin sequence (see e.g., [F]). Proposition 5.8 still guarantees that limit points are preserved.

In the situation in this paper it is easy to argue that successor points are preserved.

By Lemma 5.8 a), b), if κ is a successor point on the Radin sequence and $\tau \subseteq \alpha < \kappa$, then $\tau \in V^Q$ where Q is the forcing below κ. But $Q \simeq \mathbf{P}_{(\vec{v}, \vec{l})}/q \times \mathrm{Add}(\beth_4(\kappa^-), \kappa)/s$ where κ^- is the predecessor of κ in the Radin sequence. Since Q does not collapse cardinals, κ remains a cardinal.

6. The projected forcing

At this point, we have constructed partial orders $\mathbf{P}_{(u_{<\alpha}, h_{<\alpha})}$ that produce (among other things):

1) A closed unbounded set in $\mathscr{P}_{\kappa_u}(\beth_3(\kappa_u))$, $C = \{(\vec{v})_0 : \vec{v}$ is in some 5-tuple in the generic object$\}$ such that if $x, y \in C$, $\kappa_x < \kappa_y$, then $x \prec y$.

2) A bunch of subsets of $\beth_4(\kappa_v)$ for v's in some 5-tuple of the generic object.

By 1), at each κ_v, a limit point on the Radin sequence, we have collapsed $\beth_3(\kappa_v)$, but by 2) we have made the G.C.H. fail in the interval between $\beth_4(\kappa_v)$ and κ_{v*} where κ_{v*} is the next point on the Radin sequence.

Pictorially:

in V		in $V^{\mathbf{P}_{u<\alpha}, h<\alpha}$
κ_u	—	
κ_{v^*}	—	
		$2^{\beth_4(\kappa_v)} = \kappa_{v^*}$
$\beth_4(\kappa_v)$	—	
$\beth_3(\kappa_v)$	—	
$\beth_2(\kappa_v)$	—	cardinality κ_v
$\beth_1(\kappa_v)$	—	
κ_v	—	

If we could arrange not to collapse cardinals in the interval $[\kappa_v, \beth_4(\kappa_v)]$ we would have the G.C.H. failing in the interval (κ_w, κ_u) where κ_w is the first point on the Radin sequence. This is because the G.C.H. fails between κ_v and $\beth_4(\kappa_v)$ in the ground model. Adding κ_w many Cohen reals would then make the G.C.H. fail everywhere below κ_u.

To prevent this collapsing we want to "throw away" the sequence of elements in $\mathscr{P}_{\kappa_u}(\beth_3(\kappa_u))$ but keep the Radin sequence and the sequence of sets added by the various $\mathrm{Add}(\beth_4(\kappa_v), \kappa_{v^*})$'s.

With this in mind we now define the projected forcing.

Recall that in Section 3 we define a map $\pi\colon U_\infty \to U_\infty^\pi$. A suitable 5-tuple t for the projected forcing is of the form:

$$t = \left\langle (\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}s \right\rangle$$

where

i) $(\vec{w}, \vec{\mathscr{F}}) \in U_\infty^\pi$.

ii) $\vec{B}$ is a sequence of sets of measure one for $\vec{w}$.

iii) $(\vec{b})_\gamma\colon (\vec{B})_\gamma \to V$,

$$(\vec{b})_\gamma((w^*, \mathscr{F}^*)) \in \mathscr{B}(\mathrm{Add}(\beth_4(\kappa_{w^*}), \kappa_w)); \text{ hence } (\vec{b})_\gamma \in Q(\vec{w}, \gamma).$$

iv) $(\vec{b})_\gamma \in \mathscr{F}_\gamma$.

v) $s \in \mathrm{Add}(\beth_4(\kappa_w), \kappa)$.

We note that there is a canonical map

$$\pi\colon\{\text{suitable 5-tuples for } \mathbf{P}_{\langle \vec{u}, \vec{a}\rangle}\} \to \{\text{suitable 5-tuples for the projected forcing}\}$$

given by: $\pi((\vec{v}, \vec{h})\vec{A}\vec{k}s) = \langle \pi(\vec{v}, \vec{h})\pi''\vec{A}\vec{b}s\rangle$ where $(\vec{b})_\gamma = b(\vec{v}, \vec{k}, \vec{A}, \gamma) \in \mathscr{B}(Q(\pi^*(v), \gamma))$. If $t_1 = \langle(\vec{w}_1, \vec{\mathscr{F}}_1)\vec{A}\vec{b}_1 s_1\rangle$ and $t_2 = \langle(\vec{w}_2, \vec{\mathscr{F}}_2)\vec{B}\vec{b}_2 s_2\rangle$ then t_1

is addable to t_2 if and only if for some $\gamma < l(\vec{w}_2)$,

1) $(\vec{w}_1, \vec{\mathscr{F}}_1) \in (\vec{B})\gamma$,

2) $s_1 \leq (\vec{b}_2)_\gamma(\vec{w}_1, \vec{\mathscr{F}}_1)$.

3) There is an increasing function e: $l(\vec{w}_1) \to l(\vec{w}_2)$ such that for all $\xi < l(\vec{w}_1)$, $A_\xi \subseteq B_{e(\xi)}$ and for all $a \in A_\xi$, $(\vec{b}_1)_\xi(a) \leq (\vec{b}_2)_{e(\xi)}(a)$ in $\mathrm{Add}(\beth_4(\kappa_a), \kappa_{w_1})$.

Analogously we can define when a 5-tuple is addable to $\langle(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle$.

It is an easy consequence of the definitions that if t_1 and t_2 are suitable 5-tuples for $\mathbf{P}_{(\vec{u},\vec{h})}$ and t_1 is addable to t_2 then $\pi(t_1)$ is addable to $\pi(t_2)$.

Consequently, from Lemma 5.1:

LEMMA (6.1). *Let $t = \langle(w, \mathscr{F})\vec{B}\vec{b}s\rangle$ be a suitable 5-tuple for the projected forcing. Then there is a sequence of sets $\vec{B}^{**}$ of $\vec{w}$-measure one such that for all $(\vec{w}', \vec{\mathscr{F}}') \in (\vec{B}^{**})_\gamma$, $\gamma < l(\vec{w})$, there is a suitable 5-tuple $\langle(\vec{w}', \vec{\mathscr{F}}')\vec{B}'\vec{b}'s'\rangle$ addable to $\langle(w, \mathscr{F})\vec{B}^{**}\vec{b}s\rangle$.*

And from Lemma 5.2:

LEMMA (6.2). *Let $t = \langle(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}s\rangle$ be suitable for the projected forcing. Then there is a sequence of sets of measure one $\vec{B}^{**}$ such that if $\alpha < \beta < l(\vec{w})$ and $(w', \mathscr{F}') \in (\vec{B}^{**})_\alpha$, there are a set of measure one for $(\vec{w})_\beta, (\vec{B}^*)_\beta$ and an expansion to a 5-tuple $t' = \langle(w', \mathscr{F}')\vec{B}'\vec{b}', s'\rangle$ such that for all $(w^\#, \mathscr{F}^\#) \in (B^*)_\beta$ there is an expansion $t^\# = \langle(w^\#, \mathscr{F}^\#)\vec{B}^\#\vec{b}^\#, s^\#\rangle$ such that t' is addable to $t^\#$ and $t^\#$ is addable to t.*

Proof. Without loss of generality, $t = \pi((\vec{u}, \vec{h})\vec{A}\vec{k}s)$. Let $\vec{B}^{**} = \pi(\vec{A}^{**})$ as in Lemma 5.1. Let $(\vec{w}', \vec{\mathscr{F}}') \in (\vec{B}^{**})_\alpha$. Then there is $(\vec{u}', \vec{h}') \in (\vec{A}^{**})_\alpha$, $\pi(\vec{u}', \vec{h}') = (\vec{w}', \vec{\mathscr{F}}')$. Let $t' = \pi$ (the canonical expansion of $(\vec{u}', \vec{h}')$). Then by Lemma 6.2, $(\vec{u})_\beta$-almost all $(\vec{v}, \vec{k})$, the canonical expansion of $(\vec{u}', \vec{h}')$ is addable to the canonical expansion of $(\vec{v}, \vec{k})$. Hence for $(\vec{w})_\beta$ almost all $(w^\#, \mathscr{F}^\#)$, there is an expansion of $(w^\#, \mathscr{F}^\#)$, $t^\#$ so that t' is addable to $t^\#$. □

If $t = ((\vec{w}, \vec{h})\vec{B}\vec{b}s)$ and $t' = \langle(\vec{w}, \vec{h})\vec{B}'\vec{b}'s'\rangle$ we say t' shrinks t if and only if $\vec{B}'_\gamma \subseteq \vec{B}_\gamma$, $(\vec{b}')_\gamma \leq_{\text{a.e.}} (\vec{b})_\gamma$ for all $\gamma < l(\vec{w})$ and $s' \leq s$.

Let $(w, \mathscr{F}) \in U_\infty^\pi$. We define the *projected forcing* $\mathbf{P}^\pi_{(w, \mathscr{F})}$.

A condition $p \in \mathbf{P}^\pi_{(w, \mathscr{F})}$ is a finite sequence of suitable 5-tuples and a 4-tuple. Now $p = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle$, $t_i = \langle(\vec{w}_i, \vec{\mathscr{F}}_i)\vec{B}_i\vec{b}_i s_i\rangle$, where

a) $\kappa_{t_i} < \kappa_{t_j}$, $i < j$,

b) $s_i \in \mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})$,

c) $\vec{B}$ is a sequence of sets of measure one for $\vec{w}$,

d) $(\vec{b})_\gamma \in Q(\vec{w}, \gamma)$ and $(\vec{b})_\gamma \in \vec{\mathscr{F}}_\gamma$.

If $p, q \in \mathbf{P}^{\pi}_{(\vec{w}, \vec{\mathscr{F}})}$, $p = (t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b})$, $q = (t'_1 \cdots t'_l(\vec{w}, \vec{\mathscr{F}})\vec{B}'\vec{b}')$ then $p \le q$ if and only if p can be obtained from q by

1) adding suitable 5-tuples t such that $\kappa_{t'_i} < \kappa_t < \kappa_{t'_{i+1}}$ with t addable to t'_{i+1},

2) adding suitable 5-tuples t with $\kappa_t < \kappa_{t_1}$ with t addable to t_1,

3) adding suitable 5-tuples t with $\kappa_{t'_l} < \kappa_t$ with t addable to $((\vec{w}, \vec{\mathscr{F}})\vec{B}'\vec{b}')$,

4) shrinking some t'_i or $\vec{B}'$ or $\vec{b}'$.

Again we get a non-separative partial ordering. We choose representatives of the equivalence classes in the obvious way.

Again p *refines* q if and only if $p \le q$ and $l(p) = l(q)$, and if $p = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}'\vec{b}'\rangle$ then the *lower part* of p is $t_1, \ldots, t_n$. The *upper part* of p is $((\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b})$.

Remark (6.3). Any two conditions $p, q \in \mathbf{P}^{\pi}_{(\vec{w}, \vec{\mathscr{F}})}$ with the same lower part are compatible. Hence, $\mathbf{P}^{\pi}_{(\vec{w}, \vec{\mathscr{F}})}$ has the κ^+_w-chain condition.

This is true since if $p = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}_1\vec{b}_1\rangle$, $q = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}_2\vec{b}_2\rangle$. Then $(\vec{b}_1)_\gamma$ and $(\vec{b}_2)_\gamma$ both lie in the same filter on $Q(\vec{w}, \gamma)$, hence are $(\vec{w})_\gamma$-almost everywhere compatible. Choose $\vec{c}$ so that $(\vec{c})_\gamma \le (\vec{b}_1)_\gamma \wedge (\vec{b}_2)_\gamma$ and $\vec{C}$ so that $(\vec{C})_\gamma \subseteq (\vec{B}_1)_\gamma \cap (\vec{B}_2)_\gamma$. Then $r = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{C}\vec{c}\rangle \le p \wedge q$.

For $(\vec{u}, h) \in U_\infty$ we define a map π: $\mathbf{P}_{(\vec{u}, \vec{h})} \to \mathbf{P}^{\pi}_{\pi(\vec{u}, \vec{h})}$ by

$$\pi\big(\langle t_1 \cdots t_n(\vec{u}, \vec{h})\vec{A}\vec{k}\rangle\big) = \langle \pi(t_1) \cdots \pi(t_n)\pi(\vec{u}, \vec{h})\pi(\vec{A})\pi(\vec{k})\rangle.$$

Then π is clearly order-preserving since whenever t is addable to t', $\pi(t)$ is addable to $\pi(t')$.

We now show the main lemma of this section.

Lemma (6.3) (Projection Lemma). *Let $p \in \mathbf{P}_{(\vec{u}, \vec{h})}$, $(\vec{u}, \vec{h}) \in U_\infty$. Then there is a $q \le \pi(p)$ such that for all $q' \le q$ there is a $p' \le p$, $\pi(p') \le q'$. (Hence π is a projection map.) Further, q is a refinement of $\pi(p)$ and $\pi(p')$ refines q'.*

Proof. We first remark that it is enough to show this for conditions p with no lower part. To see this, let $p = \langle t_1 \cdots t_n(\vec{u}, \vec{h})\vec{A}\vec{k}\rangle)$ be a canonical representative of its class in the separative quotient, and $t_i = \langle(\vec{u}_i, \vec{h}_i)\vec{A}_i\vec{k}_i s_i\rangle$. Assume the lemma for conditions with no lower part. Then for all $(\vec{v}, \vec{g}) \in (\vec{A}_i)_\gamma$, $(u_{i-1})_0 \prec \vec{v}$ and $s_{i-1} \in \mathrm{Add}(\beth_4(\kappa_{i-1}, \kappa_{\vec{v}}))$.

Let $q_i = ((\pi(\vec{u}_i), \pi(\vec{h}_i))\vec{B}\vec{b})$ enforce the lemma for $\mathbf{P}^{\pi}_{\pi(\vec{u}_i, \vec{h}_i)}$ and the condition $\langle(\vec{u}_i, \vec{h}_i), \vec{A}_i, \vec{k}_i\rangle$, and let q^* work for $\mathbf{P}_{(\vec{u}, \vec{h})}$ below $\pi\langle(\vec{u}, \vec{h})\vec{A}\vec{k}\rangle$. Then it is easy to see that the condition $q = \langle q_1 s_1 q_2 s_2 \cdots q_n s_n q^*\rangle$ works for $\mathbf{P}_{(\vec{u}, \vec{h})}$ below p. Hence we assume $p = \langle(\vec{u}, \vec{h})\vec{A}\vec{k}\rangle$.

We build sequences of sets $\langle A^n_{<l(\vec{u})}: n \in w\rangle$, $A^n_{<l(\vec{u})}$ a sequence of $\vec{u}$-measure one sets.

Let $A^0_{<l(\vec{u})} = \vec{A}$. By 5.2, we may assume by shrinking if necessary that if $(\vec{v}, \vec{g}) \in \bigcup A^0_{<l(\vec{u})}$ then the canonical expansion of $(\vec{v}, \vec{g})$ is addable to $((\vec{u}, \vec{h})\vec{A}^0\vec{k})$. Suppose we have $A^n_{<l(\vec{u})}$. For each $(\vec{w}, \vec{\mathscr{F}}) \in \bigcup_{\gamma<l(\vec{u})}\pi(A^n_\gamma)$, choose a set $T_{(\vec{w}, \vec{\mathscr{F}})} \subseteq \bigcup_{\gamma<l(\vec{u})}\vec{A}^n_\gamma$ with $|T_{(\vec{w}, \vec{\mathscr{F}})}| < \kappa_u$ such that:

a) For all $(\vec{v}, \vec{g}) \in T_{(\vec{w}, \vec{\mathscr{F}})}$, $\pi(\vec{v}, \vec{g}) = (\vec{w}, \vec{\mathscr{F}})$.

b) For all $(\vec{v}, \vec{g}) \in \bigcup_{\gamma<l(\vec{u})}A^n_\gamma$ if $\pi(\vec{v}, \vec{g}) = (\vec{w}, \vec{\mathscr{F}})$, then $k_{<l(\vec{u})}(\vec{v}, \vec{g}) \leq \bigvee\{k_{<l(\vec{u})}(\vec{v}^*, \vec{g}^*): (\vec{v}^*, \vec{g}^*) \in T_{(\vec{w}, \vec{\mathscr{F}})}\}$ in $\mathscr{B}(\mathrm{Add}(\beth_4(\kappa_v), \kappa_u))$. (Here $k_{<l(\vec{u})}(\vec{v}, \vec{g}) = k_\gamma(\vec{v}, \vec{g})$ where γ is the unique ordinal $(\vec{v}, \vec{g}) \in A_\gamma$.)

We note we can find such a $T_{(\vec{w}, \vec{\mathscr{F}})}$ by the κ_u-chain condition of $\mathrm{Add}(\beth_4(\kappa_w), \kappa_u)$.

Now choose a sequence of $\vec{u}$-measure one sets $\vec{A}^{n+1}_{<l(\vec{u})}$, such that for all $(\vec{w}, \mathscr{F}) \in \pi(A^n_{<l(\vec{u})})$, and all $(\vec{v}, \vec{g}) \in \bigcup A^{n+1}_{<l(\vec{u})}$, if $\kappa_{\vec{w}} < \kappa_{\vec{v}}$ then for all $\langle\vec{v}', \vec{g}'\rangle \in T_{(\vec{w}, \vec{\mathscr{F}})}$, $(\vec{v}')_0 \prec (\vec{v})_0$. (Such a set is guaranteed to exist by Lemma 4.1.)

Now let $A^*_{<l(\vec{u})} = \bigcap A^n_{<l(\vec{u})}$ (coordinatewise intersection).

Let $q = \pi((\vec{u}, \vec{h})\vec{A}^*\vec{k})$. We claim q is as desired.

Let $q' = \langle t_1' \cdots t_n'\pi(\vec{u}, \vec{h})\vec{B}'\vec{b}'\rangle$, $q' \leq q$, $t_i' = \langle(\vec{w}_i, \vec{\mathscr{F}}_i)B_ib_is_i\rangle$.

We find a sequence $(\vec{v}_i, \vec{g}_i)$ such that:

a) $(\vec{v}_i)_0 \prec (\vec{v}_{i+1})_0$,

b) $\pi(\vec{v}_i, \vec{g}_i) = (\vec{w}_i, \vec{\mathscr{F}}_i)$,

c) $k_{<l(\vec{u})}(\vec{v}_i, \vec{g}_i) \wedge s_i \neq 0$.

By induction on j, $0 \leq j \leq n-1$, we choose $\langle\vec{v}_{n-j}, \vec{g}_{n-j}\rangle$.

For $j = 0$, we arbitrarily choose $\langle\vec{v}_n, \vec{g}_n\rangle \in A^*_{<l(\vec{u})}$ such that $\pi(\vec{v}_n, \vec{g}_n) = (\vec{w}_n, \vec{\mathscr{F}}_n)$ and $\vec{k}(\vec{v}_n, \vec{g}_n) \wedge s_n \neq 0$. Such a choice is possible since $s_n \leq b(\vec{u}, \vec{k}, \vec{A}^*, \beta)$ for some β, $(w, \mathscr{F}) \in \pi(\vec{A}^*)_\beta$. Since $(\vec{v}_n, \vec{g}_n) \in A^*_{<l(\vec{u})}$, $(\vec{v}_n, \vec{g}_n) \in A^n_{<l(\vec{u})}$.

At stage $n-(j+1)$ we assume we have chosen $(\vec{v}_{n-j}, \vec{g}_{n-j}) \in \bigcup A^{n-j}_{<l(\vec{u})}$, $\pi(\vec{v}_{n-j}, \vec{g}_{n-j}) = (\vec{w}_{n-j}, \vec{\mathscr{F}}_{n-j})$. Since $(\vec{w}_{n-(j+1)}, \vec{\mathscr{F}}_{n-(j+1)}) \in \pi(A^*_{<l(\vec{u})})$, $(\vec{w}_{n-(j+1)}, \vec{\mathscr{F}}_{n-(j+1)}) \in \pi(A^{n-(j+1)}_{<l(\vec{u})})$. Hence, for all $(v, g) \in T_{(w_{n-(j+1)}, \mathscr{F}_{n-(j+1)})}$, $(v)_0 \prec (v_{n-j})_0$.

By the transitivity of addability, there is an expansion of $(\vec{w}_{n-(j+1)}, \vec{\mathscr{F}}_{n-(j+1)})$, $t^* = \langle(\vec{w}_{n-(j+1)}, \mathscr{F}_{n-(j+1)})\vec{B}^*\vec{b}^*s^*\rangle$ is addable to $\langle\pi(\vec{u}, \vec{h})\pi\vec{A}^*\pi\vec{k}\rangle$ such that $t'_{n-(j+1)}$ refines t^*. Hence for some $\gamma < l(\vec{u})$, $(\vec{w}_{n-(j+1)}, \vec{\mathscr{F}}_{n-(j+1)}) \in \pi(\vec{A}^*)_\gamma$ and $\pi(\vec{k})_\gamma(w_{n-(j+1)}, \vec{\mathscr{F}}_{n-(j+1)}) = b(\vec{u}, \vec{k}, \vec{A}^*, \gamma)(\vec{w}_{n-j+1}, \vec{\mathscr{F}}_{n-j+1}) \geq s^* \geq s_{n-(j+1)}$.

But $b(\vec{u}, \vec{k}, \vec{A}^*, \gamma)(\vec{w}_{n-(j+1)}, \mathscr{F}_{n-(j+1)}) = \bigvee\{k_\gamma(\vec{v}, \vec{g}): (\vec{v}, \vec{g}) \in (\vec{A}^*)_\gamma$ and $\pi(\vec{v}, \vec{g}) = (\vec{w}_{n-(j+1)}, \vec{\mathscr{F}}_{n-(j+1)})\}$. Hence $b(\vec{u}, \vec{k}, \vec{A}^*, \gamma)(\vec{w}_{n-(j+1)}, \mathscr{F}_{n-(j+1)}) \leq \bigvee\{\vec{k}(\vec{v}, \vec{g}): (\vec{v}, \vec{g}) \in T_{(w_{n-(j+1)}, \mathscr{F}_{n-(j+1)})}\}$.

Hence $s_{n-(j+1)} \leq \bigvee\{\vec{k}(\vec{v}, \vec{g}): (\vec{v}, \vec{g}) \in T_{(\vec{w}_{n-(j+1)}, \vec{\mathscr{F}}_{n-(j+1)})}\}$.

Thus we can find a $(\vec{v}_{n-(j+1)}, \vec{g}_{n-(j+1)}) \in T_{(\vec{w}_{n-(j+1)}, \vec{\mathscr{F}}_{n-(j+1)})}$ such that $\vec{k}(\vec{v}_{n-(j+1)}, \vec{g}_{n-(j+1)}) \wedge s_{n-(j+1)} \neq 0$. This completes the inductive choice of $(\vec{v}_i, \vec{g}_i)$, $l \leq i \leq n$.

By our assumption of $\vec{A}^0$, the canonical expansion of $(\vec{v}_i, \vec{g}_i)$ to a suitable 5-tuple is addable to $\langle(\vec{u}, \vec{h})\vec{A}^0\vec{k}\rangle$. Let $t_i^\#$ be the canonical expansion of $(\vec{v}_i, \vec{g}_i)$. Let $p^\# = \langle t_1^\# t_2^\# \cdots t_n^\#(\vec{u}, \vec{h})\vec{A}\vec{k}\rangle$. Then $p^\# \leq p$. It suffices for the lemma to show that we can refine $p^\#$ to p' such that $\pi(p') \leq q'$.

First note that if $t_i^\# = \langle(v_i, g_i)\vec{A}_i\vec{k}_i s_i^\#\rangle$ then $s_i^\# = \vec{k}(v_i, g_i)$. Let $s_i' \leq s_i \wedge s_i^\#$, $s_i' \in \mathrm{Add}(\beth_4(\kappa_{v_i}), \kappa_{v_{i+1}})$. Since $\pi(\vec{v}_i, \vec{g}_i) = (\vec{w}_i, \vec{\mathscr{F}}_i)$ and $\vec{\mathscr{F}}_i$ is a sequence of ultrafilters on $\langle Q(\vec{w}_i, \beta)\colon \beta < l(\vec{w}_i)\rangle$ and $\vec{k}_i \leq_{\text{a.e.}} \vec{g}_i$, $\vec{b}(\vec{v}_i, \vec{k}_i, \vec{A}_i) \in \vec{\mathscr{F}}_i$.

Hence for each $\gamma < l(w_i)$, $\{a\colon b(\vec{v}_i, \vec{k}_i, \vec{A}_i, \gamma)(a) \wedge (\vec{b}_i)_\gamma(a) \neq 0\}$ is $(\vec{w}_i)_\gamma$-measure one. Again, since $\vec{k}_i \leq_{\text{a.e.}} \vec{g}_i$ and $\mathscr{F}_{\vec{g}_i} = \vec{\mathscr{F}}_i$ is a sequence of ultrafilters, we can find a sequence of $\vec{v}_i$-measure one $\vec{A}_i^\#$ so that for all $\gamma < l(\vec{u})$ and all $a \in \pi(\vec{A}_i^\#)_\gamma$,

$$b\left(\vec{v}_i, \vec{k}_i, \vec{A}_i^\#, \gamma\right)(a) \leq b\left(\vec{v}_i, \vec{k}_i, \vec{A}_i, \gamma\right)(a) \wedge \left(\vec{b}_i\right)_\gamma(a).$$

Let $\vec{A}_i' \subseteq \vec{A}_i^\#$ be such that $\pi(\vec{A}_i') \leq \vec{B}_i$. Let $t_i^* = \langle(\vec{v}_i, \vec{g}_i)\vec{A}_i'\vec{k}_i's_i'\rangle$. Then by construction $\pi(t_i^*)$ shrinks t_i'.

Let $\vec{A}' \subseteq \vec{A}$ be such that $\pi(\vec{A}') \subseteq B'$ and $\vec{b}(\vec{u}, \vec{k}, \vec{A}') \leq \vec{b}'$. Let $t_i^* = \langle(v_i, g_i)\vec{A}_i', s_i'\rangle$. Then t_i^* refines $t_i^\#$; hence $p' = \langle t_1^* \cdots t_n^*(\vec{u}, \vec{h})\vec{A}'\vec{k}\rangle \leq p$. Since $\pi(t_i^*)$ shrinks t_i', $\pi(p') \leq q'$ as desired. □

Hence, forcing with $\mathbf{P}_{(\vec{u}, \vec{h})}$ induces, via π, a generic object on $\mathbf{P}^\pi_{\pi(\vec{u}, \vec{h})}$. Since for every condition q in $\mathbf{P}^\pi_{\pi(\vec{u}, \vec{h})}$ there is a $p \in \mathbf{P}_{(\vec{u}, \vec{h})}$ such that $\pi(p) \leq q$, we have:

Corollary (6.4). *When $G \subseteq \mathbf{P}^\pi_{\pi(\vec{u}, \vec{h})}$ is any generic ultrafilter, there is a generic ultrafilter $H \subseteq \mathbf{P}_{(\vec{u}, \vec{h})}$ such that $\pi''H = G$.*

We now argue that $\mathbf{P}^\pi_{(\vec{u}, \vec{h})}$ does not collapse any cardinals. We do this by showing that $\mathbf{P}^\pi_{(\vec{w}, \vec{\mathscr{F}})}$ strongly factors at every κ not in the Radin sequence. The following lemma and its proof are exactly as in Lemma 5.3.

Proposition (6.5). *Let $q \in \mathbf{P}^\pi_{(w, \mathscr{F})}$,*

$$q = \left\langle t_1 \cdots t_{i-1}\left\langle(w_i, \mathscr{F}_i), \vec{B}_i\vec{b}_i s_i\right\rangle t_{i+1} \cdots t_n(w, \mathscr{F})Bb\right\rangle \text{ and } l\left(\vec{w}_{i+1}\right) = 1.$$

Then

$$\mathbf{P}^\pi_{(w, \mathscr{F})}/q \simeq \mathbf{P}^\pi_{(w_i, \mathscr{F}_i)}/\left\langle t_1 \cdots t_{i-1}\left\langle(w_i, \mathscr{F}_i), \vec{B}_i\vec{b}_i\right\rangle\right. \times \mathrm{Add}\left(\beth_4(\kappa_i), \kappa_{i+1}\right)/s_i \times \mathbf{P}^\pi_{(\vec{w}, \vec{\mathscr{F}})}/q^*$$

where q^* is a refinement of $\langle t_{i+1} \cdots t_n(w, \mathscr{F})Bb\rangle$.

We now show that this is a witness to strong-factorization.

PROPOSITION (6.6). *Let*

$$q^{\#} = \left\langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\right\rangle \in \mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})},\ l(\vec{w}_{i+1}) = 1,\ \lambda \in [\kappa_i^+, \kappa_{i+1}).$$

Then $\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}/q^{\#}$ *strongly factors at* λ.

Proof. Let $q \le q^{\#}$. Let c be a boolean value in $\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}$. Choose $(\vec{u}, \vec{h}) \in U_\infty$ such that $\pi(\vec{u}, \vec{h}) = (\vec{w}, \vec{\mathscr{F}})$. Then $q \ge \pi(p)$ for some $p \in \mathbf{P}_{(\vec{u},\vec{h})}$ and $\pi(p)$ refines q. Say $p = \langle t_1' \cdots t_k'(\vec{u}, \vec{h})\vec{A}k\rangle$, $t_i' = \langle(\vec{u}_i, \vec{h}_i)\vec{A}_i\vec{k}_i s_i\rangle$, $\pi(t_i') = t_i$. By Lemma 5.3,

$$\mathbf{P}_{(\vec{u},\vec{h})}/p \simeq \mathbf{P}_{(\vec{u}_i,\vec{h}_i)}/\left\langle t_1' \cdots t_{i-1}'\left(\vec{u}_i, \vec{h}_i\right)\vec{A}_i k_i\right\rangle$$
$$\times \operatorname{Add}(\beth_4(\kappa_i), \kappa_{i+1})/s_i \times \mathbf{P}_{(\vec{u},\vec{h})}/p^*$$

where p^* refines $\langle t_{i+1}' \cdots t_n'(\vec{u}, \vec{h})\vec{A}k\rangle$. Let b be the Boolean value in $\mathscr{B}(\mathbf{P}_{(\vec{u},\vec{h})})$, $b = \|c \in \pi''G\|$ where $G \subseteq \mathbf{P}_{(\vec{u},\vec{h})}$ is a term for the generic ultrafilter and $\pi\colon \mathbf{P}_{(\vec{u},\vec{h})} \to \mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}$ is the projection map.

By Lemma 5.5, there are a refinement $p' \le p^*$, $p' \in \mathbf{P}_{(\vec{u},\vec{h})}$, and a dense set $T \subseteq \mathbf{P}_{(\vec{u}_i,\vec{h}_i)}/\langle t_i' \cdots t_{i-1}'(\vec{u}_i, \vec{h}_i)\vec{A}_i k_i\rangle \times \operatorname{Add}(\beth_4(\kappa_i), \kappa_{i+1})/s_i$ such that for all $(p'', s) \in T$, $\langle p'' s p'\rangle \| b$.

For each $p'' \in \mathbf{P}_{(\vec{u}_i,\vec{h}_i)}/\langle t_1' \cdots t_i'(\vec{u}_i, \vec{h}_i)\vec{A}_i k_i\rangle$, let $\pi^*(p'') \le \pi(p'')$ be such that for all $q^* \le \pi^*(p'')$ there is a $p^{\#} \le p''$, $\pi(p^{\#}) \le q^*$.

Let

$$T' \subset \mathbf{P}^{\pi}_{\pi(\vec{u}_j,\vec{h}_j)}/\pi\left(\left\langle t_1' \cdots t_{j-1}'\left(\vec{u}_j, \vec{h}_j\right)\vec{A}_j k_j\right\rangle\right) \times \operatorname{Add}(\beth_4(\kappa_i), \kappa_{i+1})/s_i$$

be $T' = \{\langle \pi^*(p''), s\rangle : \langle p'', s\rangle \in T\}$. Since π is a projection map, T' is a dense set and so contains a maximal antichain. Let $q' = \pi(p')$. Then for all $\langle q'', s\rangle \in T'$, $\langle q'' s q'\rangle \| c$.

Hence we have shown that given any $q \le q^{\#}$ we can refine the part of q above t_i, say by q', and find a maximal antichain T' of parts below t_{i+1} such that for all $\langle q'', s\rangle \in T'$, $\langle q'' s q'\rangle \| c$ (this is analogous to Lemma 5.5 for the projected forcing).

Case 1. $\lambda \in [\beth_4(\kappa_i)^+, \kappa_{i+1}]$. Then

$$\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}/q^{\#} \simeq \left[\mathbf{P}^{\pi}_{(\vec{w}_i,\vec{\mathscr{F}}_i)}/\left\langle t_1 \cdots t_{i-1}\left(\vec{w}_i, \vec{\mathscr{F}}_i\right)\vec{B}_i\vec{b}_i\right\rangle \times \operatorname{Add}(\beth_4(\kappa_i), \kappa_{i+1})\right]$$
$$\times \mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}/\langle t_{i+1} \cdots t_n(w, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle$$

and

$$\mathbf{P}^{\pi}_{(\vec{w}_i,\vec{\mathscr{F}}_i)}/\left\langle t_1 \cdots t_{i-1}\left(\vec{w}_i, \vec{\mathscr{F}}_i\right)\vec{B}_i b_i\right\rangle \times \operatorname{Add}(\beth_4(\kappa_i), \kappa_{i+1})$$

is λ-c.c. When $\mathscr{F}_l = \{q' \in \mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}/\langle t_{i+1} \cdots t_n(w, \mathscr{F})\vec{B}b\rangle: l(q') = l\}$ each $\mathscr{F}_l$ is closed under descending λ-sequences. Hence $\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}$ strongly factors at λ.

Case 2. $\lambda \in (\kappa_i, \beth_4(\kappa_i)]$. We associate the product forcing the other way:

$$\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}/q^{\#} \simeq \mathbf{P}^{\pi}_{(\vec{w}_i,\vec{\mathscr{F}}_i)}/\left\langle t_1 \cdots t_{i-1}\left(\vec{w}_i, \vec{\mathscr{F}}_i\right)\vec{B}_i b_i\right\rangle$$
$$\times\left[\mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})/s_i \times \mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}/\left\langle t_{i+1} \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\right\rangle\right].$$

Then $\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}}_i)}$ is λ-c.c. Let $\mathscr{F}_l = \{(s, r): l(r) = l\}$. We claim that for any Boolean value c and any $(s, r) \in \mathscr{F}_l$, there are $(s', r') \leq (s, r)$, $(s', r') \in \mathscr{F}_l$ and a maximal antichain $A \subseteq \mathbf{P}^{\pi}_{(\vec{w}_i,\vec{\mathscr{F}}_i)}/\langle t_1 \cdots t_{i-1}(\vec{w}_i, \vec{\mathscr{F}}_i)\vec{B}_i b_i\rangle$ such that for all $q \in A$, $\langle qs'r'\rangle \parallel c$.

By the discussion above we can find $r' \leq r$ in $\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}$ and a maximal antichain T',

$$T' \subseteq \mathbf{P}^{\pi}_{(\vec{w}_i,\vec{\mathscr{F}}_i)}/\left\langle t_1 \cdots t_{i-1}\left(\vec{w}_i, \vec{\mathscr{F}}_i\vec{B}_i\vec{b}_i\right)\right\rangle \times \mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})/s$$

such that for all $(q, s') \in T'$, $(q, s') \parallel c$. Since $|\mathbf{P}^{\pi}_{(\vec{w}_i,\vec{\mathscr{F}}_i)}| < \beth_4(\kappa_i)$ and $\mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})$ is $\beth_4(\kappa_i)$-closed there are an $s' \in \mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})$ and a maximal antichain $A \subset \mathbf{P}^{\pi}_{(\vec{w}_i,\vec{\mathscr{F}}_i)}/\langle t_1 \cdots t_{i-1}(\vec{w}_i, \vec{\mathscr{F}}_i)\vec{B}_i\vec{b}_i\rangle$ such that for all $q \in A$, (q, s') is below some element of T'. Then (s', r') works for the claim. Hence $\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}$ strongly factors at λ. □

Let $G \subseteq \mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}$ be generic. We define the Radin sequence for G to be $\{\kappa$: there are a $p = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle \in G$ and an $i \leq n$, $\kappa = \kappa_i\}$. Then as before this is a closed unbounded subset of κ_w.

The following is analogous to 5.8.

Proposition (6.7). *Let* $(\vec{w}, \vec{\mathscr{F}}) \in U^{\pi}_{\infty}$, $l(\vec{w}) > 1$, *and let* $p = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle \in \mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}$.

a) *If* $\lambda \in (\kappa_i, \kappa_{i+1})$ *then there is a* $p' \leq p$ *such that* $\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}/p'$ *strongly factors at* λ. *If* $l(\vec{w}_{i+1}) = 1$ *then* $\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}/p$ *strongly factors at all* $\lambda \in (\kappa_i, \kappa_{i+1})$.

b) *If* $\tau \in V^{\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}}$ *is a term for a subset of* α, $p = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{A}\vec{k}\rangle$ *and* $\alpha < \kappa_i$, *then* $\tau \in V^{Q}$ *where* $Q = \mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}/\langle t_1 \cdots t_{i-1}(\vec{w}_i, \vec{\mathscr{F}}_i)\vec{B}_i\vec{b}_i\rangle$.

c) *If* κ *is a limit point on the Radins sequence then in* $V^{\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}}$, κ *is a strong limit cardinal. Further*, κ_w *is a strong limit cardinal.*

d) $\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}$ *does not collapse cardinals.*

Proof. a) By 6.1 we can choose $p' \leq p$ such that if $t' = \langle \vec{w}', \vec{\mathscr{F}}'\vec{B}'\vec{b}', s'\rangle$ is the first 5-tuple in p' above t_i then $\kappa_{w'} > \lambda$ and $l(\vec{w}') = 1$. Then $\mathbf{P}^{\pi}_{(\vec{w},\vec{\mathscr{F}})}/p'$ strongly factors at λ by Proposition 6.6.

b) and c) follow immediately from Proposition 5.8 and the projection lemma.

d) Let λ be a cardinal in V. If $\lambda < \kappa_w$, choose a $p \in \mathbf{P}^{\pi}_{(\vec{w}, \vec{\mathscr{F}})}$, $p = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle$ such that $\lambda \leq \kappa_n$. Then, if $\lambda = \kappa_i$, $l(w_i) > 1$, λ is a strong limit cardinal by c). Otherwise, there is a $p' \leq p$ such that $\mathbf{P}^{\pi}_{(\vec{w}, \vec{\mathscr{F}})}/p'$ strongly factors at λ. Hence λ is preserved.

If $\lambda > \kappa_w$, we know that $\mathbf{P}_{(\vec{w}, \vec{\mathscr{F}})}$ is κ_w^+-c.c.; hence, λ is preserved.

We now have:

THEOREM (6.8). *Let* $(\vec{w}, \vec{\mathscr{F}}) \in U^{\pi}_{\infty}$. *Let* $G \times H \subseteq \mathbf{P}^{\pi}_{(\vec{w}, \vec{\mathscr{F}})} \times \mathrm{Add}(\omega, \kappa_1)$ *be generic where* κ_1 *is the first element of the Radin sequence. Then* $V[G \times H] \models \kappa_w$ *is a strong limit and the* G.C.H *fails up to* κ_w.

Proof. We can view the forcing as first forcing with $\mathbf{P}^{\pi}_{(\vec{w}, \vec{\mathscr{F}})}$ followed by adding κ_1 Cohen reals. Hence, if λ is a cardinal in V, λ remains a cardinal in $V[G \times H]$.

We must show that $2^{\lambda} > \lambda^+$.

Case 1. $\lambda \in [\kappa, \beth_4(\kappa))$ where κ is a point on the Radin sequence. Then $V \models 2^{\lambda} > \lambda^+$ and hence $V[G * H] \models 2^{\lambda} > \lambda^+$.

Case 2. $\lambda \in [\beth_4(\kappa), \kappa')$ where κ is a point on the Radin sequence and κ' is the next point on the Radin sequence.

Let $p = \langle t_1 \cdots t_i\ t_{i+1} \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle \in G$ where $\kappa_i < \lambda$ and $\kappa_{i+1} = \kappa'$. Then, by Proposition 6.5, $\mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})/s_i$ is a regular subordering of $\mathbf{P}^{\pi}_{(\vec{w}, \vec{\mathscr{F}})}/p$. Since $\mathbf{P}^{\pi}_{(\vec{w}, \vec{\mathscr{F}})}$ does not collapse cardinals, $2^{\lambda} \geq \kappa_{i+1} > \lambda^+$.

Case 3. $\lambda < \kappa_1$, and κ_1 is the first point on the Radin sequence. Then, since H adds κ_1 many Cohen reals, $2^{\lambda} \geq \kappa_1 > \lambda^+$. □

Hence at this stage we can kill the G.C.H. at many places. It only remains to see that for appropriately chosen $(\vec{w}, \vec{\mathscr{F}})$, $\mathbf{P}^{\pi}_{(\vec{w}, \vec{\mathscr{F}})}$ leaves $\kappa_{\vec{w}}$ regular.

7. The master condition

Let j be our original embedding from Section 3. Let $\kappa = \mathrm{crit}(j)$ and let $(M_{<\lambda_1}, g_{<\lambda_1})$ be the measure sequence defined there.

Our final forcing notion will be $\mathbf{P}^{\pi}_{\pi(M_{<\lambda_1}, g_{<\lambda_1})}$. From Section 6, to get a model $V' \models$ "ZFC + G.C.H. fails everywhere" it suffices to show that $\mathbf{P}^{\pi}_{\pi(M_{<\lambda_1}, g_{<\lambda_1})}$ leaves κ regular. Then, if $G \times H \subseteq \mathbf{P}^{\pi}_{\pi(M_{<\lambda_1}, g_{<\lambda_1})} \times \mathrm{Add}(\omega, \kappa_1)$ is generic, $V[G] \models$ "κ is inaccessible" and hence: $V_{\kappa}[G \times H] \models$ ZFC + the G.C.H. fails everywhere.

By the results of Section 2, it suffices to show that $\mathbf{P}^{\pi}_{\pi(M_{<\lambda_1}, g_{<\lambda_1})}$ has a master condition m. Let $\mathbf{P} = \mathbf{P}^{\pi}_{\pi(M_{<\lambda_1}, g_{<\lambda_1})}$.

We first note what j does to conditions $p \in \mathbf{P}$. If $p \in \mathbf{P}$ then $p = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle$ where $(\vec{w}, \vec{\mathscr{F}}) = \pi(M_{<\lambda_1}, g_{<\lambda_1})$. Each $t_i \in R(\kappa)$ so that $j(p) = \langle t_1 \cdots t_n \; j(\vec{w}, \vec{\mathscr{F}})j(\vec{B})j(\vec{b})\rangle \in j(\mathbf{P})$.

Since the measures $j(\vec{w})$ are $j(\kappa)$-complete and j is $\beth_\omega(\kappa)$-supercompact we can find a sequence of sets $\vec{B}^*_{<j(\lambda_1)}$ of $j(\vec{w})$-measure one so that for all $\vec{B}$ of $\vec{w}$-measure one, $j(\vec{B}) \supseteq \vec{B}^*$ coordinate-wise. Further, since $(j\vec{\mathscr{F}})_\gamma$ is closed under descending $< j(\kappa)$ sequences for all $\gamma < j(\lambda_1)$, there is a $\vec{b}^* \in j(\vec{\mathscr{F}})$ such that for all $\vec{b} \in \vec{\mathscr{F}}$, $j(\vec{b}) \geq \vec{b}^*$. Thus for all $p = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle\rangle$, $j(p) \geq \langle t_1 \cdots t_n \; j(\vec{w}, \mathscr{F})\vec{B}^*\vec{b}^*)\rangle$.

By the remarks at the end of Section 3 for all B of w_{λ_0}-measure one, $(w_{<\lambda_1}, \mathscr{F}_{<\lambda_1}) \in j(B)$. Hence for all $b \in \mathscr{F}_{\lambda_0}$ defined on a w_{λ_0}-set of measure one $(w_{<\lambda_1}, \mathscr{F}_{<\lambda_1}) \in \operatorname{dom} j(b)$. Since $\operatorname{Add}(\beth_4(\kappa), j(\lambda))$ is $\beth_4(\kappa)$-directed closed and there are at most 2^κ many $b \in \mathscr{F}_{\lambda_0}$ and $\mathscr{F}_{\lambda_0}$ is a filter, there is an $s^* \in \operatorname{Add}(\beth_4(\kappa), j(\kappa))$ such that $s^* \leq j(b)(\vec{w}, \vec{\mathscr{F}})$ for all $b \in \mathscr{F}_{\lambda_0}$.

LEMMA (7.1). *Let* $\langle(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle \in \mathbf{P}$ *be a condition with no lower part. Then* $\langle(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}s^*\rangle$ *is addable to* $j((\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b})$.

Proof. Since $j((\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}) = \langle j(\vec{w}, \mathscr{F})j\vec{B}\; j\vec{b}\rangle$, $j((\vec{B})_\gamma) \subseteq j(\vec{B})_{j(\gamma)}$. Further, for all $a \in (\vec{B})_\gamma$, $(\vec{b})_\gamma(a) \in R(\kappa)$, so that $(b)_\gamma(a) = j(\vec{b}_\gamma(a)) = j(\vec{b})_{j(\gamma)}(a)$. Hence j is the witness that clause 3) of the definition of addability holds.

Clause 1) holds because $(w_{<\lambda_1}, \mathscr{F}_{<\lambda_1}) \in j(B)_{j(\lambda_0)}$ and 2) holds since $s^* \leq j(\vec{b})_{j(\lambda_0)}(\vec{w}, \vec{\mathscr{F}})$. □

Let $\vec{B}, \vec{b}$ be such that $\langle(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle$ satisfies the conditions of Lemmas 6.1 and 6.2. We force below $r = \langle(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle \in \mathbf{P}$. Let $m = (\langle(\vec{w}, \mathscr{F})\vec{B}\vec{b}s^*\rangle j(\vec{w}, \vec{\mathscr{F}})\vec{B}^*\vec{b}^*)$. Then $m \leq j(r)$ by Lemma 7.1.

LEMMA (7.2). *Let* $p \in \mathbf{P}/r$. *Then* $j(p) \wedge m \neq 0$.

Proof. Let $p = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}'\vec{b}'\rangle$. Then $\vec{B}' \subseteq \vec{B}$ and $\vec{b}' \leq \vec{b}$ a.e. and, by Lemma 7.1, $\langle(\vec{w}, \vec{\mathscr{F}})\vec{B}'\vec{b}'s^*\rangle$ is addable to $\langle j(\vec{w}, \vec{\mathscr{F}})j(\vec{B}')j(\vec{b}')\rangle$. Hence

$$p' = \langle t_1 \cdots t_n\langle(\vec{w}, \mathscr{F})\vec{B}'\vec{b}'s^*\rangle j(\vec{w}, \vec{\mathscr{F}})j(\vec{B}')j(\vec{b}')\rangle \leq j(p).$$

By shrinking $j(\vec{B}')$ to $\vec{B}^*$ and $j(\vec{b}')$ to $\vec{b}^*$ we get a condition below both m and $j(p)$ since $p \leq \langle(\vec{w}, \mathscr{F})\vec{B}\vec{b}\rangle$. □

Let

$$i\colon \mathbf{P}/r \to j(\mathbf{P}) \text{ by } i\Big(\langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}'\vec{b}'\rangle\Big)$$

$$= \langle t_1 \cdots t_n(\vec{w}, \mathscr{F})\vec{B}'\vec{b}'s^* j(\vec{w}, \mathscr{F})B^*b^*\rangle.$$

Then i is clearly order-preserving, and sends maximal antichains to maximal antichains.

LEMMA (7.3). *m is a master condition for j, $\mathbf{P}/r$ and i.*

Proof. Let $p \leq r$ in $\mathbf{P}$, $p = \langle t_1 \cdots t_n(\vec{w}, \vec{\mathscr{F}})\vec{B}'\vec{b}'\rangle$. Then there are suitable 5-tuples $t'_1 \cdots t'_n$ such that t_i refines t'_i and each t'_i is addable to $\langle(\vec{w}, \vec{\mathscr{F}})\vec{B}\vec{b}\rangle$. Hence $\langle t'_1 \cdots t'_n\langle(\vec{w}, \mathscr{F})\vec{B}'\vec{b}'s^*\rangle j(\vec{w}, \vec{\mathscr{F}})\vec{B}^*\vec{b}^*\rangle \leq m$. So

$$\Big\langle t_1 \cdots t_n\Big\langle(\vec{w}, \mathscr{F})\vec{B}'\vec{b}'s^*\Big\rangle j(\vec{w}, \vec{\mathscr{F}})\vec{B}^*\vec{b}^*\Big\rangle = i(p) \leq m.$$

Hence i: $\mathbf{P}/r \to j(\mathbf{P})/m$ and $i(p) \wedge m = i(p)$.

It suffices to see that $i(p) \leq j(p)$. By Lemma 7.1, $\langle(\vec{w}, \vec{\mathscr{F}})\vec{B}'\vec{b}'s^*\rangle$ is addable to $\langle j(\vec{w}, \vec{\mathscr{F}})j(\vec{B}')j(\vec{b}')\rangle$. Hence

$$q = \langle t_1 \cdots t_n\langle(\vec{w}, \vec{\mathscr{F}})\vec{B}'\vec{b}'s^*\rangle j(\vec{w}, \vec{\mathscr{F}}), j(\vec{B}')j(\vec{b}')\rangle \leq j(p)$$

and $i(p)$ refines q. □

We have shown our main theorem:

THEOREM. *Suppose "ZFC + there is a supercompact cardinal with infinitely many strongly inaccessible cardinals as above" is consistent. Then so is "ZFC + for all κ, $2^\kappa > \kappa^+$".*

8. Miscellaneous remarks

In [So], Solovay showed that if κ is λ-supercompact where $\lambda > \kappa$ is a singular strong limit cardinal, then $2^\lambda = \lambda^+$. Hence, supercompact cardinals imply that there is a proper class of instances of the G.C.H. We now explore how much supercompactness we can preserve.

We first note that if $i, \mathbf{P}, m$, are as in Section 7 then by Proposition 6.6 and 6.7 for all $\lambda < \beth_4(\kappa)$ there is a dense set of $p \leq m$ in $j(\mathbf{P})$ such that $j(\mathbf{P})/p = \mathbf{P} \times \mathbf{R}$ and $\mathbf{P} \times \mathbf{R}$ is a strong factorization at λ. Further, the embedding i sends $\mathbf{P}$ to the first coordinate of $\mathbf{P} \times \mathbf{R}$. ($\mathbf{R} = \mathrm{Add}(\beth_4(\kappa), \kappa^*)/s^* \times j(\mathbf{P})/p^*$ for some κ^*, p^*.)

Hence, by Lemma (2.6), κ is $\beth_3(\kappa)$-supercompact.

The limitation on further supercompactness is the closure of $\mathrm{Add}(\beth_4(\kappa), \kappa^*)$. We can maintain $\beth_n(\kappa)$-supercompactness by making our basic forcing notion $\mathrm{Add}(\beth_{n+1}(\kappa_u), \kappa)$ instead of $\mathrm{Add}(\beth_4(\kappa_u), \kappa)$. Then all proofs work as before, and densely often in $j(\mathbf{P})/m$, and for each $\lambda < \beth_{n+1}(\kappa)$ there is a strong factorization at λ. Hence by 2.6 we get $\lambda_n(\kappa)$-supercompactness.

A by-product of this proof is the development of the "super-compact" Radin forcing. This forcing adds a new closed unbounded subset of $\mathscr{P}_\kappa(\lambda)$. We briefly sketch its development.

Let $j: V \to M$ be a $2^{2^{\lambda^+}}$-supercompact embedding with critical point κ. We define a sequence of measures as follows.

Let $M_0 = j''\lambda$. Let M_α concentrate on $\langle M_\beta : \beta < \alpha \rangle = M_{<\alpha}$. Then each measure is a measure on $\mathscr{P}_\kappa(\lambda) \times R(\kappa)$.

Since there are only 2^{2^λ} such measures we get $\lambda_0 < \lambda_1$ such that $M_{\lambda_0} = M_{\lambda_1}$. Such a λ_1 is a *repeat point*.

We get a set like U_∞ in an analogous way; i.e., there is a set U such that:

1) For each $\vec{m} \in U$, $m_\delta(U) = 1$ for all $\delta < l(m)$.

2) For all $\vec{m} \in U$ there is a $\delta < \lambda_1$ such that $\langle H(2^{2^{\lambda^+}}), \varepsilon, j, M_{<\delta} \rangle \equiv \langle H(2^{2^{\lambda^{+m}}}), \varepsilon, m_0, \vec{m} \rangle$.

3) $M_\delta(U) = 1$ for all $\delta < \lambda_1$.

A suitable pair is a pair $\langle \vec{m}, \vec{A} \rangle$ such that $\vec{A}$ is a sequence of sets of measure one for $\vec{m}$.

A pair $\langle \vec{m}, \vec{A} \rangle$ is *addable* to $\langle \vec{n}, \vec{B} \rangle$ if and only if

1) $\vec{m}_0 \prec \vec{n}_0$ so i_{mn} is defined.

2) $\vec{m} \in B_\gamma$ for some $\gamma < l(\vec{n})$.

3) There is a function $e: l(\vec{m}) \to l(\vec{n})$ such that $i_{mn}[A_\gamma] \subseteq B_{l(\gamma)}$.

An element p of the forcing $\mathbf{P}$ is a finite sequence $t_i \cdots t_n$ of suitable pairs and $(M_{<\lambda_1}, A_{<\lambda_1})$ where $A_{<\lambda_1}$ is a sequence of sets of measure one for $M_{<\lambda_1}$, $p = \langle t_1 \cdots t_n \; M_{<\lambda_1} A_{<\lambda_1} \rangle$ so that if $t_i = \langle \vec{m}_i \; \vec{A}_i \rangle$, $(\vec{m}_i)_0 \prec (\vec{m}_{i+1})_0$.

For $p, q \in \mathbf{P}$, $q \subseteq p$ if q can be obtained from p by shrinking sets of measure one and adding new pairs.

One shows that the "Prikry Property" for $\mathbf{P}$ is exactly analogous to Lemmas 5.4 and 5.5, except there are no s's, g's or k's to worry about. Then strong factorization at γ's between $\lambda(\alpha)$ and α' where α and α' are successive points on the Radin sequence is even easier as one does not have to worry about the closure of the $\mathrm{Add}(\delta, \gamma)$ partial orderings.

Hence, if κ started supercompact then forcing with $\mathbf{P}$ keeps κ supercompact, although it does collapse λ to have cardinality κ.

The second author has simplified the construction in this manuscript so that we do not need to build the g_α's as we did in Section 3 but can simply define

$g_\alpha(\vec{u}, \vec{f})$ as a function of u_0. We present the more complicated construction in this paper because it is more general; in particular it works in the context of [F].

The second author has also reduced the consistency strength of "ZFC + $\forall\kappa$, $2^\kappa > \kappa^+$" and "ZFC + $\forall\kappa$, $2^\kappa = \kappa^{++}$" to that of a $\mathscr{P}^2(\kappa)$-hypermeasurable.

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