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Classification of a Family of Free Bogoliubov Actions of $\mathbb{R}$ on $L(\mathbb{F}_\infty)$ up to Cocycle Conjugacy

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Joshua Keneda

2019

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ABSTRACT OF THE DISSERTATION

Classification of a Family of Free Bogoliubov Actions of $\mathbb{R}$ on $L(\mathbb{F}_\infty)$ up to Cocycle Conjugacy

by

Joshua Keneda
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2019
Professor Dimitri Shlyakhtenko, Chair

Much of the research into operator algebras concerns the classification problem for von Neumann algebras, where one hopes to find useful invariants to categorize von Neumann algebras up to isomorphism. Murray and von Neumann initiated the study of these objects and reduced the classification problem to that of classifying the so-called factors. And while Type I factors are fully understood, the Type II and Type III cases are still active areas of research. Relatedly, one can attempt to classify group actions on von Neumann algebras up to a suitable notion of equivalence, for example unitary or cocycle conjugacy.

The aim of this dissertation is to classify a family of free Bogoliubov actions of $\mathbb{R}$ on the Type II_1 free group factor $L(\mathbb{F}_\infty)$ up to cocycle conjugacy. We consider a certain collection $\mathfrak{C}$ of measure classes on $\mathbb{R}$, corresponding to the spectral measure classes of the infinitesimal generators for orthogonal representations $\alpha : \mathbb{R} \to \mathcal{O}(H)$, with H separable and infinite-dimensional. Such representations give rise to a free Bogoliubov action σ^α of $\mathbb{R}$ on $L(\mathbb{F}_\infty)$ and an associated von Neumann algebra: the crossed product $L(\mathbb{F}_\infty) \rtimes_{\sigma^\alpha} \mathbb{R}$. Note that the cocycle conjugacy of two actions gives an isomorphism between their crossed product von Neumann algebras. We show that our family of free Bogoliubov actions are completely classified up to cocycle conjugacy by their associated spectral measure class $[\alpha] \in \mathfrak{C}$.

Our main technical tool for this result mirrors a recent result of Houdayer, Shlyakhtenko, and Vaes and relates the equality of the spectral measure classes $[\alpha]$, $[\beta]$ to the embeddability

(in the sense of Popa's intertwining-by-bimodules) of the group algebra $L_\alpha(\mathbb{R})$ into $L_\beta(\mathbb{R})$ inside of their shared crossed product $M \simeq L(\mathbb{F}_\infty) \rtimes_{\sigma^\alpha} \mathbb{R} \simeq L(\mathbb{F}_\infty) \rtimes_{\sigma^\beta} \mathbb{R}$.

By restricting to representations α of $\mathbb{R}$ which act trivially on a large subspace of H, we force $L_\alpha(\mathbb{R})$ to have large (non-amenable) relative commutant in $L(\mathbb{F}_\infty) \rtimes_{\sigma^\alpha} \mathbb{R}$. This allows us to use solidity arguments and a rigidity result of Houdayer and Ueda to "trap" $L_\alpha(\mathbb{R})$ inside of $L_\beta(\mathbb{R})$ within the crossed product (i.e. $L_\alpha(\mathbb{R}) \preceq_M L_\beta(\mathbb{R})$). Applying the technical tool of the previous paragraph, we obtain that $[\alpha] = [\beta]$, which establishes the desired cocycle conjugacy invariant.

The dissertation of Joshua Keneda is approved.

Rowan Killip

Tim Austin

Sorin Popa

Dimitri Shlyakhtenko, Committee Chair

University of California, Los Angeles

2019

To my family, whose love and kindness are exceeded only by their independent interest in the classification of group actions on von Neumann algebras

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CHAPTER 1

von Neumann Algebra Preliminaries and Constructions

1.1 Topologies on $B(H)$, von Neumann algebras, and the Bicommutant Theorem

Let H be a complex Hilbert space. We will usually be interested in the case where H is separable and infinite dimensional, in which case H has a countable orthonormal basis. For $T : H \to H$ a linear operator, we define the *operator norm* of T to be:

$$\|T\|_\infty = \sup_{\{\xi \in H : \|\xi\| = 1\}} \|T\xi\|.$$

We say that a linear operator $T : H \to H$ is *bounded* if the supremum in the above definition is finite and denote by $B(H)$ the set of bounded linear operators on H. One can check that $B(H)$ is precisely the set of continuous linear operators $T : H \to H$ when H is equipped with the norm induced by its inner product. Note that $B(H)$ forms an algebra over $\mathbb{C}$ whose multiplication operation is given by composition of operators: $TS := T \circ S$. We note that the operator norm satisfies $\|TS\| \leq \|T\|\|S\|$ for all $T, S \in B(H)$. (Remark: We will suppress the ∞ from the notation for the operator norm if there's no chance of confusion.) The algebra $B(H)$ comes equipped with a natural conjugate-linear involution $*$ given by the adjoint. Recall that the adjoint T^* of an operator $T \in B(H)$ is the unique bounded linear operator satisfying for all $\xi, \eta \in H$:

$$\langle \xi, T\eta \rangle = \langle T^*\xi, \eta \rangle.$$

We remark that the adjoint is an anti-homomorphism of rings (i.e. $(TS)^* = S^*T^*$) and that the operator norm satisfies $\|T\|^2 = \|T^*\|^2 = \|T^*T\|$ for all $T \in B(H)$.

There are several natural topologies we can consider on $B(H)$ which are weaker than the operator norm topology. For all of the following topologies and for Λ a directed set, we have that a net $\{T_\lambda \in B(H) : \lambda \in \Lambda\}$ converges to an operator T if and only if the net $\{T_\lambda - T : \lambda \in \Lambda\}$ converges to zero. So it suffices to describe which nets of operators go to zero.

Definition 1.1.1 (Strong Operator Topology) We have $T_\lambda \to 0$ in the strong operator topology if and only if for all $\xi \in H$,

$$\|T_\lambda \xi\| \to 0.$$

In other words, the strong operator topology corresponds to pointwise convergence in norm. If a set is closed in the strong operator topology, we will sometimes call it strongly closed.

Definition 1.1.2 (Weak Operator Topology) We have $T_\lambda \to 0$ in the weak operator topology if and only if for all $\xi, \eta \in H$,

$$\langle T_\lambda \xi, \eta \rangle \to 0.$$

A closed set in this topology will sometimes be called weakly closed when there is no chance of confusion.

Definition 1.1.3 (Ultraweak Topology) We have $T_\lambda \to 0$ in the ultraweak (or σ-weak) topology if and only if for all $(\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}} \in l^2(\mathbb{N}) \otimes H$,

$$\sum_{n\in\mathbb{N}} \langle T_\lambda \xi_n, \eta_n \rangle \to 0.$$

We note that the ultraweak topology is also the weak-* topology induced by the Banach space predual of $B(H)$, which is usually represented as the set of trace-class operators on $B(H)$. See [Hou11] for a more operator topologies on $B(H)$.

From the above definitions, it's clear that the operator norm topology is stronger than both the strong operator and ultraweak topologies, which are both stronger than the weak operator topology. We will use the above topologies to distinguish two important types of subalgebras of $B(H)$.

Definition 1.1.4 (C*-algebra) A $*$-subalgebra A of $B(H)$ is called a *(concrete) C*-algebra* if it is closed under the operator norm topology.

Definition 1.1.5 (von Neumann Algebra) A $*$-subalgebra M of $B(H)$ is called a *(concrete) von Neumann algebra* if it is closed under the weak operator topology.

Clearly, every von Neumann algebra is also a C*-algebra. We remark that all von Neumbann algebras are in fact unital (in the sense of a unital ring). The unit 1_M of a von Neumann algebra M will usually be denoted by 1. In $B(H)$, for example, we write $\mathbb{C}1$ (or just $\mathbb{C}$) to refer to the scalar multiples of the identity operator, otherwise denoted id_H or id, on H. Mirroring the terminology used for operators in $B(H)$, we will say the following for elements a, p, v, u of a von Neumann algebra (or C*-algebra) M:

Definition 1.1.6 (C*-algebra/von Neumann Algebra Terminology)

1.) a is called *self-adjoint* if $a^* = a$.

2.) a is called *positive* (written $a \geq 0$) if $a = b^*b$ for some $b \in M$.

3.) p is called a *projection* if $p = p^* = p^2$.

4.) v is called a *partial isometry* if both vv^* and v^*v are projections.

5.) v is called an *isometry* if $v^*v = 1$.

6.) u is called a *unitary* if $u^*u = uu^* = 1$.

(Note: we may call v^*v the *initial/source projection* of v and vv^* the *final/range projection* of v. These correspond to the orthogonal projections in $B(H)$ onto the orthocomplement of the kernel of v and onto the image of v, respectively.)

These algebras were first studied by von Neumann and Murray in their series of papers "On Rings of Operators." von Neumann noticed that the above topological/analytic characterization of von Neumann algebras could be replaced with an equivalent algebraic condition. To state his theorem, we define the *commutant* of a collection of operators $\mathcal{S} \subset B(H)$ as follows:

$$\mathcal{S}' := \{T \in B(H) : TS = ST \quad \forall S \in \mathcal{S}\}.$$

Similarly, we can define the bicommutant $\mathcal{S}'' := (\mathcal{S}')'$. We note that clearly $\mathcal{S} \subset \mathcal{S}''$. Furthermore, as long as $\mathcal{S}$ is closed under taking adjoints, the following theorem guarantees that $\mathcal{S}''$ is the von Neumann algebra generated by $\mathcal{S}$, i.e. the weak operator closure of the $*$-algebra generated by $\mathcal{S}$.

Theorem 1.1.7 *(von Neumann's Bicommutant Theorem) For a $*$-subalgebra M of $B(H)$, the following are equivalent:*

1.) $M = M''$.

2.) M is strongly closed.

3.) M is ultraweakly closed.

4.) M is weakly closed.

We remark that an (abstract) von Neumann algebra could be defined as a $*$-algebra M which admits a faithful representation on some Hilbert space H and satisfies any of the above equivalent conditions. Although the weak and strong operator topologies on M may depend on the choice of representation, Sakai showed that the ultraweak topology is in fact independent of this choice (he showed that any von Neumann algebra, viewed as a Banach space, has a unique Banach space predual up to isomorphism), and combining his result with the above theorem guarantees that restricting our attention to concrete von Neumann algebras won't cause any loss of generality in practice. To conclude this section, we give a few examples of von Neumann algebras.

Example 1.1.8 (Abelian von Neumann Algebras) Let (X, μ) be a probability measure space. We can consider $L^\infty(X, \mu) \subset B(L^2(X, \mu))$ by letting $L^\infty(X, \mu)$ act by pointwise multiplication. I.e. if $f \in L^\infty(X, \mu)$ and $g \in L^2(X, \mu)$, we let

$$f(g) = f(x)g(x).$$

Under this embedding, we have $\|f\|_\infty = \|f\|$, i.e. the L^∞ norm corresponds to the operator norm on $B(L^2(X))$.

One can check that $L^\infty(X)' = L^\infty(X)$, (in other words, $L^\infty(X)$ is *maximal abelian* in $B(L^2(X))$), so, in particular, $L^\infty(X)'' = L^\infty(X)$, and $L^\infty(X)$ is a von Neumann algebra.

If μ is localizable (e.g. if it is σ-finite), then $L^1(X, \mu)$ is the predual fo $L^\infty(X)$, and the corresponding weak-$*$ topology on $L^\infty(X)$ is the (unique, by Sakai) ultraweak topology on $L^\infty(X)$. It can be shown that every abelian von Neumann algebra is isomorphic to $L^\infty(X, \mu)$ for some (X, μ).

Example 1.1.9 (Matrices) Of course, $B(H)$ itself is a von Neumann algebra. In particular, when H is finite dimensional with dimension n, we see that $M_n(\mathbb{C})$ is a von Neumann algebra. We remark that the only operators which commute with all of $B(H)$ are constant multiples of the identity operator, so that $B(H) \cap B(H)' = \mathbb{C}1$. Thus, $B(H)$ (and therefore $M_n(\mathbb{C})$) is a factor in the terminology of the next section.

Example 1.1.10 (Opposite Algebra) Let $M \subset B(H)$ be a von Neumann algebra (or a C*-algebra). Define M^{op} to be the algebra with the same underlying set and addition as M but with multiplication reversed, i.e. $x^{op} = x$, and $x^{op}y^{op} = (yx)^{op} = yx$ for all $x, y \in M$. Then M^{op} is in fact a von Neumann algebra. To see this, it's easiest to first define $\overline{M}$. Given $x \in M$, we define $\overline{x}$ on the conjugate Hilbert space $\overline{H}$ via $\overline{x}\overline{\xi} = \overline{x\xi}$ and set $\overline{M} = \{\overline{x} : x \in M\}$. Clearly $\overline{M}$ is a von Neumann algebra, and the map $x^{op} \mapsto \overline{x^*}$ provides an isomorphism between M^{op} and $\overline{M}$.

Example 1.1.11 (Direct Sum) If $M \subset B(H)$ and $N \subset B(K)$ are von Neumann algebras, then $M \oplus N \subset B(H \oplus K)$ is again a von Neumann algebra.

1.2 Type Classification of Factors

Now that we've defined von Neumann algebras, we can begin the project of classifying them up to isomorphism. Define a *factor* to be a von Neumann algebra M with trivial center, i.e. $\mathcal{Z}(M) = M \cap M' = \mathbb{C}1$. In their original series of papers, Murray and von Neumann showed that every von Neumann algebra on a (separable) Hilbert space admits a decomposition into a direct integral of factors. We won't use or describe the direct integral decomposition here, but we note that, in some sense, they reduced the classification question to that of the classification of these so-called factors. These factors have been further classified into three 'types,' which we describe below.

First, we need some terminology. It can be shown that every von Neumann algebra is generated by its projections, so we will categorize the factors based on the behavior of their projections. For this, we first need to describe Murray and von Neumann's comparison theory for projections in M. Suppose that $p, q \in M$ are projections. We say that p is a sub-projection of q (written $p \leq q$) if $pq = qp = p$. In terms of an underlying Hilbert space, this corresponds to the situation where p is the projection onto a subspace $V \subset H$, q is the projection onto a subspace $W \subset H$, and we have $V \subset W$. In this way, we get a partial ordering on the set of projections in our von Neumann algebra. Unfortunately for our purposes, this partial ordering is too restrictive, so Murray and von Neumann introduced another way of comparing projections as follows.

We say that p and q are *(Murray-von Neumann) equivalent* (written $p \sim q$) if there exists a partial isometry $v \in M$ such that $p = v^*v$ and $q = vv^*$. We also write $p \lesssim q$ if there's a partial isometry $v \in M$ such that $p = v^*v$ and $vv^* \leq q$. We note that $\lesssim$ induces a partial ordering on the $\sim$-equivalence classes of projections of M, and this is the ordering that will prove useful for type classification of factors, being more flexible than the partial ordering that comes from subspace containment (above).

Definition 1.2.1 We say that a projection p is:

1.) ...*minimal* if for all projections $q \in M$,

$$q \lesssim p \implies q = p.$$

2.) ...*finite* if p is not equivalent to any strict sub-projection, i.e.

$$p \sim q, p \geq q \implies p = q.$$

3.) ...*infinite* if p is equivalent to some strict sub-projection.

4.) ...*purely infinite* if p has no finite sub-projections.

5.) ...*semifinite* if p is infinite and the supremum of an increasing family of finite sub-projections.

Note that a minimal projection is necessarily finite.

We will call a von Neumann algebra M *diffuse* if it has no minimal projections. We will also call M *finite* or *(purely) infinite* if its unit is finite or (purely) infinite, respectively. Our first step in trying to classify these factors will be to divide them into three 'types,' depending on the behavior of their projections.

Definition 1.2.2 (Type I) A factor M is of *Type I* if M has a minimal projection.

Type I factors are completely understood. It can be shown that such a factor is necessarily isomorphic to $B(H)$ for some Hilbert space H, so these factors are determined up to isomorphism by the cardinality of any orthonormal basis for H. If H is finite-dimensional with dimension n, then the corresponding Type I factor is isomorphic to $M_n(\mathbb{C})$, the $*$-algebra of $n \times n$ matrices over $\mathbb{C}$. This factor is sometimes called the Type I_n factor. This is in contrast to their infinite-dimensional counterparts, sometimes denoted Type I_∞ factors. We note that these finite-dimensional Type I factors possess a unique faithful normal tracial state (terminology defined in the next section) given by the usual (normalized) trace on matrices. This distinguishes these factors from the infinite-dimensional Type I factors, but it also hints at a tool we can use to further classify Type II factors, i.e. whether they support such a trace.

We remark that if H is infinite-dimensional, then $B(H)$ is necessarily an infinite factor. For example, if H has orthonormal basis $\{\xi_1, \xi_2, ..., \xi_n, ... : n \in \mathbb{N}\}$, then the shift operator defined by

$$L(\xi_n) = \xi_{n+1}$$

is an isometry that witnesses the equivalence of the projections onto H and the closed span of $\{\xi_2, ..., \xi_n, ... : n \in \mathbb{N} \setminus \{1\}\}$. So 1 is equivalent to a strict sub-projection in this case. Conversely, if H is finite-dimensional, then any isometry is necessarily a unitary, and $B(H) \simeq M_n(\mathbb{C})$ is therefore a finite factor. So the finiteness of $B(H)$ corresponds precisely to the finiteness of the dimension of H.

We now sort the remaining (non-Type I) factors into two classes based on whether they have any finite projections.

Definition 1.2.3 (Type II) A factor M is of *Type II* if it has no minimal projection but does have finite projections.

Definition 1.2.4 (Type III) A factor M is of *Type III* if it has no finite projections (i.e. M is purely infinite).

Clearly, all factors belong to exactly one of these types. Much of the research into von Neumann algebras has been focused on obtaining a better understanding of Type II and Type III factors. We introduce some tools for distinguishing these factors in the following section.

1.3 States and the Gelfand-Naimark-Segal Construction

Throughout this section, let M be a von Neumann algebra, and let $\phi : M \to \mathbb{C}$ be a linear functional.

Definition 1.3.1 We say that ϕ is:

1.) ...*unital* if $\phi(1) = 1$.

2.) ...*positive* if $\phi(x^*x) \geq 0$ for all $x \in M$.

3.) ...a *state* if ϕ is positive and $\phi(1) = 1$.

4.) ...*faithful* if $\phi(x^*x) > 0$ for all $0 \neq x \in M$.

5.) ...*tracial* if $\phi(xy) = \phi(yx)$ for all $x, y \in M$.

6.) ...*normal* if ϕ is continuous for the ultraweak (i.e. weak-*) topology on M.

We note that positive linear functionals are necessarily bounded, with $\|\phi\| = \phi(1)$.

Given $M \subset B(H)$, there is a natural way to produce normal states on M by considering the so-called "vector states," given by taking a unit vector $\xi \in H$ and defining

$$\phi_\xi(x) = \langle \xi, x\xi \rangle.$$

(Remark: here, we assume that the inner product on H is complex linear in the second variable.)

The Gelfand-Naimark-Segal (GNS) construction will give us a converse to the above, allowing us to produce Hilbert spaces on which our algebra acts, given a normal state on M.

Suppose ϕ is a normal state on M. The GNS construction is as follows:

First, we equip M with the following semi-definite sesquilinear form:

$$\langle x, y\rangle_\phi := \phi(x^*y).$$

Quotienting by the ideal of zero-length vectors $I = \{x \in M : \phi(x^*x) = 0\}$ and then completing with respect to the pre-inner product $\langle \cdot, \cdot\rangle_\phi$ on M/I yields a Hilbert space H_ϕ. The left multiplication action of M on itself descends to a represenation π_ϕ of M on H_ϕ satisfying

$$\pi_\phi(x)\widehat{y} = \widehat{xy}$$

for all $x, y \in M$, where we define $\widehat{y} := y + I \in M/I \subset H_\phi$. Note that in the event that ϕ is faithful, we may identify M with its isomorphic image $\pi_\phi(M) \subset B(H_\phi)$, so we may suppress π_ϕ from our notation and write x instead of $\pi_\phi(x)$ when there is no chance of confusion.

In this way, we obtain a representation of M on H_ϕ where the original state ϕ can be realized, through this construction, as a vector state corresponding to the vector $\widehat{1} \in H_\phi$. That is,

$$\phi(x) = \langle \widehat{1}, x\widehat{1}\rangle_\phi.$$

So the GNS construction gives us a correspondence between normal states and cyclic vectors for cyclic representations of M.

Example 1.3.2 Consider $M = L^\infty(X, \mu)$ where μ is a probability measure. Integration against the probability measure μ gives us a faithful, normal trace state τ_μ on M. The GNS construction with respect to this state recovers the natural representation of $L^\infty(X)$ on $L^2(X)$ defined in 1.1.8, with cyclic vector 1_X.

Example 1.3.3 (Enveloping von Neumann Algebra) Let A be a C*-algebra, and let $S(A) \subset A^*$ denote the state space of A (a subset of the Banach space dual of A). The representation $\pi = \bigoplus_{\phi \in S(A)} \pi_\phi$ of A is called the *universal representation* of A. By the above, it contains every cyclic representation of A as a subrepresentation. Since arbitrary representations are direct sums of cyclic representations, every representation is a multiple of subrepresentations of π. We define the *enveloping von Neumann algebra* of A to be $\pi(A)''$, i.e. the von Neumann algebra generated by A in the universal representation. The Sherman-Takeda theorem gives a canonical identification between $\pi(A)''$ and A^{**}, the Banach space double dual of A, so we may simply denote the enveloping algebra by A^{**}.

Remark 1.3.4 More generally, the above notions of positivity and normality can be extended to maps $\varphi : M \to N$ between von Neumann algebras M and N. For a certain class of such maps, there is a construction analogous to the GNS construction called the Stinespring dilation. These maps are called *completely positive*. We'll defer the discussion of complete positivity and the Stinespring dilation to 3.1, but we can give the general definition of positivity and normality of φ below.

Definition 1.3.5 (Normality) We will call a map $\varphi : M \to N$ between von Neumann algebras M and N *normal* if it is continuous with respect to the ultraweak topologies on M and N.

Definition 1.3.6 (Positivity) A map $\varphi : M \to N$ is called *positive* if $\varphi(x^*x)$ is positive for all $x \in M$. In other words, φ takes positive elements of M to positive elements of N.

Definition 1.3.7 (Homomorphism) Let M, N be von Neumann algebras. A *homomorphism* $\varphi : M \to N$ is a normal, $\mathbb{C}$-linear ring $*$-homomorphism. That is, φ is a normal map satisfying

$$\varphi(\lambda x + y) = \lambda\varphi(x) + \varphi(y)$$

$$\varphi(x^*y) = \varphi(x)^*\varphi(y)$$

for all $\lambda \in \mathbb{C}, x, y \in M$.

The above definition also applies to maps between C*-algebras with the normality condition dropped. We remark that a homomorphism is automatically positive, since $\varphi(x^*x) =$

$\varphi(x)^*\varphi(x) \geq 0$. Additionally, one can check that a normal state ϕ on M gives rise to a normal representation $\pi_\phi : M \to B(H_\phi)$.

There are other natural choices of morphisms between von Neumann algebras, notably the normal, completely positive maps. We'll consider this choice of morphism in a later section, once we've seen the definitions of complete positivity and injectivity. For now, we move on to the discussion of factors of Type II.

1.4 Type II Factors

Type II factors possess non-trivial finite projections. We'll separate these factors into two further sub-classes depending on whether or not they are finite, i.e. whether their unit is a finite projection. We discuss the case where $1 \in M$ is finite in this section. We'll postpone the discussion of the case where 1 is infinite until we've covered semifiniteness (see 1.6).

1.4.1 Type II_1

Let M be a factor not of Type I. If $1 \in M$ is a finite projection, we will call M a II_1 factor. Murray and von Neumann showed that a II_1 factor has a unique tracial state, which we will usually denote by τ. (In fact, this unique tracial state characterizes Type II_1 factors among factors of Type II and III and is automatically faithful. Of course, finite Type I factors also have unique faithful tracial states.)

For these algebras, the GNS construction from the previous section applied to this canonical tracial state τ yields a Hilbert space $L^2(M, \tau)$, which we may sometimes denote by $L^2(M)$. We again write $\widehat{x}$ for the image of $x \in M$ after separation and completion in $L^2(M)$. There is a canonical anti-unitary operator $J : L^2(M) \to L^2(M)$ satisfying $J(\widehat{x}) = \widehat{x^*}$ and the following relations:

$$J^2 = 1$$

$$JMJ = M'$$

where, again, we use the faithfulness of τ to identify M with the corresponding left multiplication operators in $B(L^2(M))$. Given a tracial von Neumann algebra (M, τ) with faithful trace τ, we call this inclusion $M \subset B(L^2(M))$ the *standard representation* of M. The relation $JMJ = M'$ implies that M' is precisely the algebra of right multiplication operators on $B(L^2(M))$ defined by $\rho(x)\widehat{y} = \widehat{yx}$ for all $x \in M$, $\widehat{y} \in L^2(M)$. This is an easy consequence of the fact that

$$JxJ\widehat{y} = Jx\widehat{y^*} = J\widehat{xy^*} = \widehat{yx^*} = \rho(x^*)\widehat{y}.$$

Because ρ is an anti-isomorphism of M (i.e. it reverses the order of multiplication), we get that M' is naturally isomorphic to M^{op} in this case. In what follows, we will often simply write $\widehat{y}x$ instead of $\rho(x)\widehat{y}$. The left (respectively, right) multiplication operation of M on $L^2(M)$ makes $L^2(M)$ a left (resp. right) M-module, which we will define more carefully later. Note that $\xi = \widehat{1} \in L^2(M)$ is cyclic and separating for M (and therefore also for M').

Recall that for a convex cone $B \subset H$ in a Hilbert space H, we define the dual cone $B^\circ := \{\eta \in H : \langle \xi, \eta \rangle \geq 0 \text{ for all } \xi \in B\}$. A cone is called *self-dual* if $B = B^\circ$.

Remark 1.4.1 (Standard Form) Let $M \subset B(H)$ be a von Neumann algebra. We say that M is in *standard form* on H if there is an anti-unitary involution (i.e. an anti-linear isometry $J : H \to H$ satisfying $J^2 = 1$, typically called the *modular conjugation*) and a self-dual cone P in H satisfying the following relations:

1.) $JMJ = M'$

2.) $JxJ = x^*$ for all $x \in \mathcal{Z}(M)$

3.) $J\xi = \xi$ for all $\xi \in P$

4.) $xJxJP \subset P$ for all $x \in M$.

The following uniqueness theorem justifies the terminology "standard form":

Theorem 1.4.2 *(Thm IX.1.14, [Tak13]) Suppose that $\{M_1, H_1, J_1, P_1\}$ and $\{M_2, H_2, J_2, P_2\}$ are both standard forms and $\pi : M_1 \to M_2$ is an isomophism. Then there exists a unique unitary operator $u : H_1 \to H_2$ such that*

1.) $\pi(x) = uxu^*$ for all $x \in M_1$

2.) $J_2 = uJ_1u^*$

3.) $P_2 = uP_1$.

One can define a canonical positive cone in $L^2(M)$, and the above theorem guarantees that any standard form of M is unitarily equivalent to the standard representation. We remark that the above theorem was proven by Haagerup in the stated generality [Haa76], but a similar uniqueness statement was already known in the case that M admits a cyclic and separating vector. In particular, if M admits a cyclic and separating vector in H, then the representation of M on H is unitarily equivalent to the standard representation.

1.5 Constructions

The purpose of this section is to describe the construction of von Neumann algebras arising from tensor products, groups, and group actions. We start with the tensor product construction.

1.5.1 Tensor Products of von Neumann Algebras

We begin by describing finite tensor products, since the construction is elementary. Let M and N be von Neumann algebras on Hilbert spaces H and K respectively. The algebraic tensor product $M \otimes N$ acts on $H \otimes K$ by

$$(x \otimes y)(\xi \otimes \eta) = x\xi \otimes y\eta$$

for all $x \in M, y \in N, \xi \in H, \eta \in K$. We define the von Neumann tensor product of M and N, denoted $M \overline{\otimes} N$, to be $(M \otimes N)'' \subset B(H \otimes K)$.

We remark that we can identify M with $M \overline{\otimes} 1_N$ and N with $1_M \overline{\otimes} N$ in $M \overline{\otimes} N$. We also note that $M \overline{\otimes} N$ does not depend on the underlying Hilbert spaces H and K in the sense that if M and N were also faithfully represented on Hilbert spaces H' and K' respectively, then the von Neumann algebra $M \overline{\otimes} N$ on $H' \otimes K'$ will be isomorphic to the algebra constructed above. So we may safely omit the Hilbert spaces H and K from our notation.

If M and N were equipped with distinguished states ϕ and ψ respectively, then there is a tensor product state, denoted $\phi \otimes \psi$ on $M \overline{\otimes} N$ satisfying

$$(\phi \otimes \psi)(x \otimes y) = \phi(x)\psi(y)$$

for all $x \in M, y \in N$. The tensor product state is tracial if both ϕ and ψ are tracial.

Though special cases were already known, Tomita gave the first proof in 1967 of the following commutation relation for von Neumann tensor products:

Theorem 1.5.1 *(Tensor Product Commutants) Let M and N be as above, with $M \overline{\otimes} N \subset B(H \otimes K)$. Then*

$$(M \overline{\otimes} N)' = M' \overline{\otimes} N'.$$

Example 1.5.2 Given probability measure spaces (X, μ) and (Y, ν), we can consider the pointwise multiplication operators $L^\infty(X, \mu) \subset B(L^2(X, \mu))$ and $L^\infty(Y, \nu) \subset B(L^2(Y, \nu))$. The tensor product $L^\infty(X) \overline{\otimes} L^\infty(Y) \subset B(L^2(X) \otimes L^2(Y))$ is naturally (spatially) isomorphic to $L^\infty(X \times Y, \mu \times \nu) \subset B(L^2(X \times Y))$. In this case, the tensor product of the states given by integration against μ and ν corresponds to integration against the product measure $\mu \times \nu$.

Example 1.5.3 (Tensoring with Matrices) Let M be a von Neumann algebra and let $n \in \mathbb{N}$. The tensor product $M \overline{\otimes} M_n(\mathbb{C})$ of M with $n \times n$ matrices over $\mathbb{C}$ is isomorphic to $M \otimes M_n(\mathbb{C})$ (i.e. the algebraic tensor product), which in turn can be realized as $M_n(M)$, the ring of $n \times n$ matrices with entries in M with the natural matrix multiplication and adjoint operations. If τ is a (tracial) state on M, then the tensor product state of τ with the normalized trace tr on matrices corresponds to the (tracial) state given by $\frac{1}{n} \sum \tau(m_{ii})$ on $M_n(M)$. The algebra $M_n(M)$ is sometimes called an *amplification* of M and may sometimes be denoted M^n.

We now briefly describe the typical construction for infinite tensor products. Given a family $(M_i, H_i)_{i \in \mathbb{N}}$ of von Neumann algebras M_i on Hilbert spaces H_i, we could repeat the construction above to yield a von Neumann algebra on the Hilbert space tensor product $\otimes_{\mathbb{N}} H_i$. But the latter Hilbert space is rarely separable, so it is more natural to work with a construction that gives a smaller Hilbert space. For this purpose, we restrict our attention to families $(M_i, \phi_i)_{i \in \mathbb{N}}$ with ϕ_i a faithful state on M_i.

Let $N_j = \overline{\otimes}_{i=1}^{j} M_i$ (using the tensor product defined above), which comes equipped with the tensor product state $\otimes_1^j \phi_i$. Note that $N_j \subset N_k$ in a natural way if $j \leq k$. If we let $N = \cup_{j=1}^{\infty} N_j$, then N has a state ϕ satisfying $\phi(x) = \otimes_1^j \phi_i(x)$ if $x \in N_j$. We apply the GNS construction to ϕ and define the infinite tensor product via $\overline{\otimes}_{\mathbb{N}} M_i := N'' \subset B(H_\phi)$. The state ϕ is faithful for $\overline{\otimes}_{\mathbb{N}} M_i$ and is tracial if ϕ_i is tracial for all i.

Remark 1.5.4 For simplicity and to avoid discussion of cardinality, we will often restrict our attention (as we did above) to von Neumann algebras which admit faithful representations on separable Hilbert spaces. We will call such a von Neumann algebra M *separably acting* or simply *separable*. We remark that this is equivalent to the separability of the predual of M as a Banach space.

1.5.2 Group von Neumann Algebras and Crossed Product Constructions

1.5.2.1 Group von Neumann Algebras - Discrete Case

Consider a discrete group G. We first describe the group von Neumann algebra associated to G, which we will denote $L(G)$ (sometimes also denoted $\Gamma(G)$ in the literature). Let $l^2(G)$ denote the Hilbert space with orthonormal basis given by $\{g : g \in G\}$. Note that G acts on $l^2(G)$ by left translation. That is, there is a unique homomorphism $\lambda : G \to \mathcal{U}(B(l^2(G)))$ obtained by linearly extending the following relation for all $g, h \in G$:

$$\lambda_g(h) = gh.$$

Each λ_g is clearly a unitary, since it acts by permuting the orthonormal basis of $l^2(G)$. Similarly, there is a right action $\rho : G \to \mathcal{U}(B(l^2(G)))$ satisfying

$$\rho_g(h) = hg^{-1}.$$

We can now consider the group algebra $\mathbb{C}[G] := \text{span}\{\lambda_g : g \in G\} \subset B(l^2(G))$, noting that $\lambda_g \lambda_h = \lambda_{gh}$ for all $g, h \in G$. The adjoint endows $\mathbb{C}[G]$ with a $*$-algebra structure with $\lambda_g^* = \lambda_{g^{-1}}$. In particular, $\mathbb{C}[G]$ is closed under taking adjoints, and we may define $L(G) := \mathbb{C}[G]''$. Equivalently, $L(G)$ is the weak operator closure of, or the von Neumann algebra generated by, $\mathbb{C}[G] \subset B(l^2(G))$.

We remark that $L(G)$ comes equipped with a faithful, normal trace state given by expectation onto $\mathbb{C}\lambda_e = \mathbb{C}1$, where e denotes the identity element of G (so that $\lambda_e = \mathrm{id}$). In other words, the trace is given on linear combinations of λ_g with $\alpha_g \in \mathbb{C}$ by:

$$\tau(\sum \alpha_g \lambda_g) = \alpha_e.$$

The GNS representation corresponding to this trace is precisely $\lambda : G \to \mathcal{U}(B(l^2(G)))$. So, in order to obtain the group von Neumann algebra, we could have equivalently started with the initial data $(\mathbb{C}[G], \tau)$ and obtained $L(G)$ through the GNS representation.

It can be shown that the commutant of the left G-action is the weak operator closure of the right G-action, i.e. $\lambda(G)' = L(G)' = \rho(G)''$, and vice versa.

One can check that $L(G)$ is a factor iff G has all conjugacy classes infinite. In this case, G is called an ICC (infinite conjugacy class) group. If G had a finite conjugacy class $\{g_1, ..., g_n\}$, then the element of $L(G)$ given by $\sum \lambda_{g_i}$ would necessarily be central, since for any $g \in G$ we have

$$\lambda_g(\sum \lambda_{g_i}) = \sum \lambda_{gg_i} = \sum \lambda_{g_i g} = (\sum \lambda_{g_i})\lambda_g,$$

which follows from the fact that conjugation by g merely permutes our conjugacy class, so $\{gg_i : i \in [n]\} = \{g_i g : i \in [n]\}$. Note that if G is ICC, then $L(G)$ is a II_1 factor, because we've already seen that it comes equipped with a canonical faithful, normal trace state. Examples of ICC groups include $\mathrm{PSL}(n, \mathbb{Z})$, $\mathbb{F}_n$ (assuming in both cases that $2 \leq n \in \mathbb{N}$), and S_∞, defined below.

Example 1.5.5 Consider the group $G = \mathbb{Z}$ together with the canonical group trace τ defined above. Note that τ satisfies $\tau(\lambda_n) = \delta_{n=0}$.

Now let $\mathbb{S}^1 := \{e^{i\theta} : \theta \in [0, 2\pi)\}$ denote the circle group with normalized Haar measure given by $d\theta/2\pi$. If $z : \mathbb{S}^1 \to \mathbb{S}^1$ denotes the identity function, we can consider the group isomorphism given by $n \mapsto z^n$ between $\mathbb{Z}$ and $\{z^n : n \in \mathbb{Z}\} \subset C(\mathbb{S}^1)$.

Note that $\int z^n d\theta/2\pi = \int e^{in\theta} d\theta/2\pi = \delta_{n=0}$, so this identification takes the canonical group trace onto integration against the Haar measure of $\mathbb{S}^1$. Clearly, the GNS construction on the latter will yield $L^2(\mathbb{S}^1, d\theta/2\pi)$, and the definition of the weak operator topology applied to $L^\infty(\mathbb{S}^1) \subset B(L^2(\mathbb{S}^1)$ is easily seen to correspond to weak-* convergence in $L^\infty(\mathbb{S}^1)$. Since the span of $\{z^n : n \in \mathbb{Z}\}$ is weak-* dense in $L^\infty(\mathbb{S}^1)$, we must have that $\{z^n : n \in \mathbb{Z}\}'' = L^\infty(\mathbb{S}^1)$. By faithfulness of the traces involved, we can identify $L(\mathbb{Z}) \simeq L^\infty(\mathbb{S}^1)$ so that the canonical trace on the former corresponds to integration against the probability Haar measure on the latter. For this reason, if a unitary u has distribution satisfying $\phi(u^n) = \delta_{n=0}$ with respect to a state ϕ, we may call u a *Haar unitary* and can make the identification $(\{u, u^*\}'', \phi) \simeq (L(Z), \tau)$. Note that the spectrum of such a u is given by $\{z \in \mathbb{C} : |z| = 1\}$.

Example 1.5.6 Let $\mathbb{F}_n$ (resp. $\mathbb{F}_\infty$) denote the free group on $n \in \mathbb{N}$ (resp. countably many) generators. Clearly, $L(\mathbb{F}_1) = L(\mathbb{Z})$ is abelian. For all other $m \neq n \in \mathbb{N} \cup \{\infty\}$, it is a major open problem whether $L(\mathbb{F}_m) \simeq L(\mathbb{F}_n)$. Voiculescu developed machinery for studying these algebras and attempting to distinguish them. We'll detail some of this machinery in the section on Voiculescu's Free Gaussian Functor.

To conclude this section, we mention the most important classification result concerning II_1 factors. While it is known that there are uncountably many non-isomorphic II_1 factors, Connes proved the following uniqueness result. For its statement, we first make the following definition:

Definition 1.5.7 (Hyperfiniteness) A von Neumann algebra M is called *hyperfinite* if it is generated by the countable union $\cup A_i$ of an increasing sequence of finite-dimensional subalgebras $A_1 \subset A_2 \subset ...$

Theorem 1.5.8 *(Uniqueness of the Hyperfinite II_1 Factor R) There is, up to isomorphism, only one hyperfinite II_1 factor.*

We won't prove Connes' uniqueness theorem here, but we can use it to produce two realizations of the hyperfinite II_1 factor (usually denoted R) by considering the constructions we've just defined.

Example 1.5.9 ($R \simeq \overline{\otimes}_{\mathbb{N}} M_2(\mathbb{C})$) Let $M_i = M_2(\mathbb{C})$ for all $i \in \mathbb{N}$ with the usual normalized trace. Applying the tensor product construction to $(M_i, tr)_{i\in\mathbb{N}}$ yields $R := \overline{\otimes}_{\mathbb{N}} M_2(\mathbb{C})$ with faithful tracial state τ. Note that, by construction, R is generated by $\cup_{j\in\mathbb{N}}\overline{\otimes}_1^j M_2(\mathbb{C})$, and $\overline{\otimes}_1^j M_2(\mathbb{C}) \simeq M_{2^j}(\mathbb{C})$. So R is generated by an ascending sequence of finite-dimensional subalgebras, i.e. it is hyperfinite.

Example 1.5.10 ($R \simeq L(S_\infty)$) Let S_∞ denote the group of finitely supported permutations on $\mathbb{N}$. Clearly, S_∞ is an ICC group, so $L(S_\infty)$ is a II_1 factor. Because S_∞ is the direct limit of the finite permutation groups (in the obvious way), $L(S_\infty)$ is generated by the union of the finite-dimensional von Neumann algebras given by $L(S_n)$. So $L(S_\infty)$ is a hyperfinite II_1 factor, and therefore $L(S_\infty) \simeq R$.

Remark 1.5.11 We will see in Theorem 3.2.5 that hyperfiniteness of $L(G)$ is equivalent to the amenability of G, so we can more generally realize R as $L(G)$ for any countable amenable ICC group G.

1.5.2.2 Crossed Product Construction

Now that we've defined the group von Neumann algebra $L(G)$ and seen some examples, we turn our attention to developing the crossed product in the von Neumann algebraic setting. As one might expect, the crossed product construction takes as input some action α of G on a von Neumann algebra M and produces as output a new von Neumann algebra, denoted $M \rtimes_\alpha G$.

We'll restrict our discussion to tracial von Neumann algebras (M, τ) for simplicity. By *tracial von Neumann algebra*, we mean that τ is a faithful, normal tracial state. We write $\mathrm{Aut}(M)$ for the group under composition of automorphisms (normal self-*-isomorphisms) of M. We first define a group action in our context. For more details on the following constructions, see [Tak13] Chapter X.

Definition 1.5.12 Let G be a discrete group and (M, τ) a tracial von Neumann algebra. An *action* of G on M is a group homomorphism $\alpha : G \to \mathrm{Aut}(M)$. We will occasionally write $\alpha : G \curvearrowright (M, \tau)$ (if α preserves the trace τ), $\alpha : G \curvearrowright M$, or $G \curvearrowright_\alpha M$ for such an action. If α is understood, we may also suppress it from the notation and write $G \curvearrowright M$.

Remark 1.5.13 If (M, τ) is tracial, we will assume that actions on M are trace-preserving unless otherwise stated. We may abuse notation slightly and write $\mathrm{Aut}(M)$ for $\mathrm{Aut}(M, \tau)$ if the trace-preserving condition is clear from context.

Example 1.5.14 Let $\widehat{\alpha} : G \to \mathcal{U}(B(H))$ be a unitary representation of G. If $M \subset B(H)$ is globally invariant under conjugation by $\widehat{\alpha}_g$, i.e.

$$\widehat{\alpha}_g M \widehat{\alpha}_g^* = M$$

for all $g \in G$, then $\alpha_g := \mathrm{Ad}\ (\widehat{\alpha}_g)$ defines a group action of G on M if $\mathrm{Ad}\ (\widehat{\alpha}_g)$ is τ-preserving for all g.

Given an action $\alpha : G \to \mathrm{Aut}(M)$, we can define the crossed product of $M \subset B(H)$ by α as follows. Let $l^2(G, H)$ denote the Hilbert space of square-summable H-valued functions on G, i.e. $l^2(G, H) = \{\xi = (\xi_g)_{g \in G} : \sum_g \|\xi_g\|^2 < \infty\}$ with the inner product $\langle (\xi_g), (\eta_g) \rangle = \sum_g \langle \xi_g, \eta_g \rangle$. We can define faithful representations $\pi_\alpha : M \to B(l^2(G, H))$ and $\lambda : G \to B(l^2(G, H))$ via the following relations:

$$\pi_\alpha(x)\xi_g = \alpha_g^{-1}(x)\xi_g$$
$$\lambda_g \xi_h = \xi_{g^{-1}h},$$

for all $x \in M$, $g, h \in G$, and $(\xi_g)_{g \in G} \in l^2(G, H)$.

We remark that, in the case where $(M, \tau) = (\mathbb{C}, \mathrm{id})$ and α is trivial, the above construction coincides with that of $L(G)$ at the beginning of the section. Indeed, if we identify $h \in l^2(G)$ with $\delta_{g=h} \in l^2(G, \mathbb{C})$ (i.e. the characteristic function of $\{h\}$), we see that the λ_g here corresponds precisely to the λ_g defined before. For general (M, τ), it is easy to see that we still have $L(G) = \lambda(G)''$ in the context of this construction.

Definition 1.5.15 We define the crossed product of M by α to be $M \rtimes_\alpha G := (\pi_\alpha(M) \cup \lambda(G))''$.

We remark that the unitaries λ_g in the crossed product satisfy the following imporant covariance relation:

$$\lambda_g \pi_\alpha(x) \lambda_g^* = \pi_\alpha(\alpha_g(x)).$$

By the above covariance relation, we see that $M \rtimes_\alpha G$ is generated by finite sums of the form $\sum_i \pi_\alpha(x_i)\lambda_{g_i}$, with $x_i \in M, g_i \in G$. Due to the faithfulness of π_α, we may abuse notation slightly and identify M with its image $\pi_\alpha(M)$, suppressing π_α from the notation. The covariance relation then reads

$$\lambda_g x \lambda_g^* = \lambda_g x \lambda_{g^{-1}} = \alpha_g(x).$$

In the presence of τ, $M \rtimes_\alpha G$ has a conditional expectation E_G onto $L(G)$ satisfying $E_G(\sum_i x_i \lambda_{g_i}) = \sum \tau(x_i)\lambda_{g_i}$ for finite sums with $x_i \in M, g_i \in G$. Similarly, $M \rtimes_\alpha G$ has an expectation onto M determined by $E_M(\sum_g x_g \lambda_g) = x_e \lambda_e = x_e$ for finite sums $\sum_g x_g \lambda_g$, where e again denotes the neutral element of G. The composition of these expectations (in either order) gives a faithful normal trace state $\widehat{\tau}$ on $M \rtimes_\alpha G$ satisfying $\widehat{\tau}(\sum x_g \lambda_g) = \tau(x_e)$.

Example 1.5.16 (Trivial Actions) Let (M, τ) be a tracial von Neumann algebra and G a discrete group. If we let $\alpha : G \to \mathrm{Aut}(M)$ be the trivial action, so that $\alpha_g(x) = x$ for all $g \in G, x \in M$, then $M \rtimes_\alpha G$ is easily seen to be (spatially, in terms of our constructions) isomorphic to $M \overline{\otimes} L(G)$.

We now discuss some conditions we can place on the action α to produce corresponding structural properties of $M \rtimes_\alpha G$. In particular, we discuss what conditions are necessary to ensure that $M \rtimes_\alpha G$ is a II_1 factor.

1.5.2.3 Factoriality for the Crossed Product

First, a few definitions concerning automorphisms and the action α:

Definition 1.5.17 An automorphism σ of M is called *inner* if there exists a unitary $u \in M$ such that $\sigma(x) = uxu^*$ for all $x \in M$. The subgroup of $\mathrm{Aut}(M)$ of inner automorphisms of M is denoted $\mathrm{Inn}(M)$. An automorphism is called *outer* if it is not inner.

Definition 1.5.18 An automorphism is σ of M is called *properly outer* if there is no nonzero $v \in M$ satisfying $\sigma(x)v = vx$ for all $x \in M$.

Remark 1.5.19 Clearly any properly outer automorphism is outer. The converse holds if M is a factor. To see this, note that if $0 \neq v$ satisfies $\sigma(x)v = vx$ for all x, then v^*v and vv^* are central in M. Thus, by positivity of v^*v and factoriality of M, $v^*v = \|v^*v\| = \|vv^*\| = vv^*$. Rescaling v therefore gives a unitary u such that $\sigma = \mathrm{Ad}\ u$, a contradiction.

Given an automorphism σ, we write M^σ for the fixed point algebra $\{x \in M : \sigma(x) = x\}$. It is a von Neuman subalgebra of M. Similarly, if α is an action, we write M^α for the fixed point algebra of the action, i.e. $\{x \in M : \alpha_g(x) = x\ \forall g \in G\}$.

Definition 1.5.20 (Freeness and Ergodicity of an Action)
Let α be an action of G on M. We say that α is:

1.) ...*free* (or *properly outer*) if α_g is properly outer for all $g \in G \backslash \{e\}$.

2.) ...*ergodic* if $M^\alpha = \mathbb{C}$ (i.e. the fixed point algebra of the action α is trivial).

We now give the example of the group measure space construction and its connection to the definitions above.

Example 1.5.21 (Group Measure Space Construction) Let G be a countable discrete group, and let (X, μ) be a probability measure space. A *probability measure preserving* (or *p.m.p.*) action of G on X is a group homomorphism $\underline{\alpha} : G \to \mathrm{Aut}(X, \mu)$, where $\mathrm{Aut}(X, \mu)$ is the group under composition of (almost-everywhere defined) automorphisms of X which preserve μ. Such an action induces an action $\alpha : G \curvearrowright L^\infty(X, \mu)$ via the relation $\alpha_g(f)(x) = f(\underline{\alpha}_g x)$. Note that because the action $\underline{\alpha}$ is measure preserving, the induced action α preserves the natural trace $\tau(f) = \int f\, d\mu$ on $L^\infty(X, \mu)$.

We define the group measure space algebra for $\underline{\alpha}$ to be the crossed product $L^\infty(X, \mu) \rtimes_\alpha G$ by the corresponding induced action α on $L^\infty(X, \mu)$.

We say that the action $\underline{\alpha}$ on (X, μ) is *free* if $\{x \in X : \underline{\alpha}_g(x) = x\}$ (i.e. the set of fixed points for $\underline{\alpha}_g$) is null for all $e \neq g \in G$. We say that $\underline{\alpha}$ is *ergodic* if the only measurable subsets E for which the symmetric difference $\underline{\alpha}_g(E) \Delta E$ is null for all $g \in G$ are either null or co-null. This is the same as saying that there are no non-trivial subsets that are invariant under the G-action.

Under some mild assumptions on (X, μ) (e.g. (X, μ) is standard or countably separated), one can check that $\underline{\alpha}$ on (X, μ) is free (resp. ergodic) if and only if the induced action α is free (resp. ergodic) on $L^\infty(X, \mu)$, which motivates the terminology.

The following lemma combines the above conditions to guarantee factoriality for the crossed product.

Lemma 1.5.22 *Let $\alpha : G \to Aut(M, \tau)$ be a free and ergodic (trace-preserving) action of a discrete group G on a tracial von Neumann algebra M. Then $M \rtimes_\alpha G$ is a factor.*

Proof. Suppose $x \in M' \cap (M \rtimes_\alpha G)$. Write $x = \sum x_g \lambda_g$ with $x_g \in M$. (Note: This can be done uniquely for any $x \in M \rtimes_\alpha G$, with $x_g = E_M(x\lambda_g^*)$. We may not have strong convergence of this sum, but we can still write x in this form with convergence in the L^2 sense.) We have that for any $y \in M$:

$$\sum y x_g \lambda_g = yx = xy = \sum x_g \alpha_g(y) \lambda_g.$$

So we must have that $yx_g = x_g \alpha_g(y)$ for all $g \in G, y \in M$. From this, we see that if the action is free, we must have $x_g = 0$ for all $g \neq e$, so that $x = x_e$ and $M' \cap (M \rtimes_\alpha G) \subset M$. So, in particular, freeness of the action implies that the center $\mathcal{Z}(M \rtimes_\alpha G)$ of $M \rtimes_\alpha G$ is contained in M. Furthermore, by combining this with the ergodicity of the action, we have $\mathcal{Z}(M \rtimes_\alpha G) \subset L(G)' \cap M = M^\alpha = \mathbb{C}$, so that the $M \rtimes_\alpha G$ is a factor.

Remark 1.5.23 As we've seen, $M \rtimes_\alpha G$ possesses a faithful normal trace. So, if it is a factor, it is necessarily of Type II_1 (assuming it's not finite-dimensional). Therefore, free and ergodic actions give us a source of examples for II_1 factors. We give a few examples of such actions below.

Example 1.5.24 (Irrational Rotation) Let α be irrational and let $\mathbb{Z}$ act on $\mathbb{T}$ via $n.z := e^{2\pi i \alpha n} z$. More generally, if G is a countable, dense subgroup of some compact group (X, μ) equipped with its Haar measure, then the natural action of G on X is both free and ergodic.

Example 1.5.25 (Bernoulli Shift) Let G be a countable group and let (Y, ν) be a probability measure space such that ν is not a Dirac mass at a point. Then the *Bernoulli shift* action of G on $(Y^G, \nu^{\otimes G})$ is given by $g(y_h)_{h \in G} = (y_{g^{-1}h})_{h \in G}$. It can be shown that Bernoulli shifts of this type are always free and ergodic.

1.6 Semifinite Algebras

1.6.1 Weights and Traces

We will occasionally need to consider semifinite algebras in what follows, so we briefly describe the notions generalizing states/traces for these algebras. We write M_+ for the cone of positive elements of a von Neumann algebra M.

Definition 1.6.1 Let M be a von Neumann algebra. A *weight* on M is an $\mathbb{R}_{\geq 0}$-linear map ω from M_+ to $\mathbb{R}_{\geq 0} \cup \{\infty\}$.

If a weight ω satisfies $\omega(1) < \infty$, then it extends by linearity to a positive linear functional (still denoted ω) on the full algebra M. A weight with $\omega(1) = 1$ is, in this way, a state. We say that a weight is *tracial* or a *trace* if $\omega(x^*x) = \omega(xx^*)$ for all $x \in M$. For weights with $\omega(1)$ finite, this tracial condition is equivalent to the usual one with $\omega(xy) = \omega(yx)$ for all $x, y \in M$.

We'll mostly be concerned with tracial weights. In particular, we'll want our weights to be semifinite, as defined below:

Definition 1.6.2 We say that ω is:

1.) ...*faithful* if $\omega(x^*x) > 0$ for all $0 \neq x \in M$.

2.) ...*normal* if $\omega(\sup_\lambda x_\lambda) = \sup_\lambda \omega(x_\lambda)$.

3.) ...*semifinite* if for all $x \in M_+$, there exists a $y \in M_+$ such that $y \leq x$ and $\omega(y) < \infty$.

Remark 1.6.3 We remark that condition 2 above is equivalent to ω being lower semicontinuous with respect to the restriction of the ultraweak topology to M_+. If $\omega(1) < \infty$, these conditions are equivalent to normality in the previously defined sense (i.e. ultraweak continuity).

Definition 1.6.4 (Semifiniteness) We say that M is *semifinite* if there is a faithful, normal, semifinite trace on M_+.

It can be shown that M is finite if and only if it has a faithful, normal tracial state, so all finite von Neumann algebras are semifinite. Semifinite algebras are strictly more general, though, as the following example shows.

Example 1.6.5 Let H be a Hilbert space of countably infinite dimension. If $\{\xi_1, \xi_2, ...\}$ is an orthonormal basis for H, we can define $\mathrm{Tr}(x) = \sum_{i \in \mathbb{N}} \langle \xi_i, x\xi_i \rangle$ for $x \in B(H)$. Then Tr is a faithful, normal semifinite trace on $B(H)$. So $B(H)$ is a semifinite (but not finite) factor of type I. Furthermore, Tr is independent of the choice of orthonormal basis.

Recall that finite factors have a unique faithful normal tracial state. Similarly, it can be shown that semifinite factors possess a unique faithful, normal semifinite weight up to rescaling by $\lambda \in \mathbb{R}_{>0}$. (One way of showing this is to use the fact that if M is semifinite, we can write 1_M as a strong limit of an increasing family of finite projections $p_n \to 1_M$ and leverage the uniqueness of the trace on the II_1 factor $p_n M p_n$.) In the presence of minimal projections (i.e. the type I case), the natural choice of scaling satisfies $\mathrm{Tr}(p) = 1$ for a minimal projection p, as above. In the absence of such projections, there is no such canonical choice.

Motivated by the above example, we will often denote a faithful, normal tracial weight by Tr.

1.6.2 Type II_∞ von Neumann algebras

We have already considered the Type II algebras whose unit is a finite projection. In this subsection, we consider the other Type II von Neumann algebras. Such algebras possess finite projections, but their unit is an infinite projection. Using the constructions of the previous sections, it's easy to produce examples of factors of this type.

Example 1.6.6 (Tensoring with $B(H)$)

Let (M, τ) be a II_1 factor, and consider the tensor product $N = M \overline{\otimes} B(H)$, where H is a separable, infinite-dimensional Hilbert space. As we saw in the previous section, $B(H)$ comes equipped with a faithful, normal semifinite trace Tr. If we let $\overline{\tau} = \tau \otimes \mathrm{Tr}$ on N, then $\overline{\tau}$ is again a faithful, normal semifinite trace. Note that N is a factor since

$$\mathcal{Z}(N) = (M\overline{\otimes}B(H)) \cap (M\overline{\otimes}B(H))' = (M\overline{\otimes}B(H)) \cap (M' \otimes \mathbb{C}1_H) = (M \cap M') \otimes \mathbb{C}1_H = \mathbb{C}.$$

If $B(H) \ni p < 1_H$ is a strict subprojection of the identity on H such that $p \; 1_H$, then $1 \otimes p$ is a strict subprojection equivalent to 1_N. Thus, 1_N is an infinite projection, and N is a factor of type II_∞.

Remark 1.6.7 In fact, the above is the most general example of a II_∞ factor: every II_∞ factor can be written as a tensor product of a II_1 factor with $B(H)$, albeit non-canonically.

In an earlier section, we defined group measure space actions of discrete groups and studied when they produced II_1 factors. We can make an analogous study in the case of locally compact groups. First, we will need to extend the earlier definitions in the discrete case to the more general case of locally compact groups, which we do in the next section.

1.7 Crossed Products and Actions of Locally Compact Groups

For a discrete group, all actions $G \curvearrowright M$ are obviously continuous, so we avoided discussion of the topology on $\mathrm{Aut}(M)$. But in the general case, we need the following:

Let $\alpha \in \mathrm{Aut}(M)$, and let $\phi_1, ..., \phi_n$ be normal linear functionals on M (equivalently, $\phi_i \in M_*$ for all i, where M_* denotes the Banach space predual of M). Note that $\phi_i \circ \alpha$ is again a normal linear functional on M. Following the exposition by Takesaki in [Tak13], we denote by $U_\alpha(\phi_1, ..., \phi_n)$ the collection $\{\beta \in \mathrm{Aut}(M) : \max\{\|\phi_i \circ \beta - \phi_i \circ \alpha\|, \|\phi_i \circ \beta^{-1} - \phi_i \circ \alpha^{-1}\|\} < 1\}$. Then the collection $\{U_\alpha(\phi_1, ..., \phi_n) : n \in \mathbb{N}, \alpha \in \mathrm{Aut}(M), \phi_i \in M_*(1 \leq i \leq n)\}$ forms a neighborhood basis for a topology on $\mathrm{Aut}(M)$ that makes it a topological group. Unless otherwise stated, we'll topologize $\mathrm{Aut}(M)$ in this way. Equipped with this topology, we can now define continuous actions of locally compact groups on M.

Definition 1.7.1 (Action)

Let G be a locally compact group. We define an *action* of G on a von Neumann algebra M to be a continuous group homomorphism $\alpha : G \to \mathrm{Aut}(M)$. If M is tracial, we will assume that G takes its image in the trace-preserving automorphisms of M. We use the same notation introduced for the discrete case for such actions ($G \curvearrowright_\alpha M$, etc.).

Example 1.7.2 (Unitary Representations) Let $\lambda : G \to \mathcal{U}(B(H))$ be a unitary representation of G (always assumed continuous with respect to the strong operator topology on $B(H)$). If $M \subset B(H)$ is a von Neumann algebra that is globally invariant under conjugation by λ_g, i.e. $\lambda_g M \lambda_g^* = M$ for all $g \in G$, then $\alpha_g = \mathrm{Ad}\ \lambda_g$ defines an action of G on M.

In fact, every action can be realized in the above way using the standard implementation, which relies in part on the (essential) uniqueness of the standard representation.

Theorem 1.7.3 *(Standard Unitary Implementation, see Thm IX.1.15 [Tak13])*

Suppose $\{M, H, J, P\}$ is a standard form. Then the subgroup $\Gamma \subset \mathcal{U}(B(H))$ of unitaries u that satisfy $uMu^* = M$, $uJu^* = J$, and $uP = P$ is isomorphic to $Aut(M)$ via the map $u \mapsto Ad\ u$. Furthermore, this map is a homeomorphism from Γ, equipped with the strong-operator topology, to $Aut(M)$ with the topology defined above.

Using this isomorphism (or, rather, its inverse), we can take an automorphism α of M and produce a unitary $u_\alpha \in B(L^2(M))$, conjugation by which implements α. We'll call u_α the *standard (unitary) implementation* of α. Similarly, given an action $G \curvearrowright_\alpha M$, there's a unique unitary representation $u : G \to \mathcal{U}(B(L^2(M)))$ satisfying the following for all $g \in G$, $x \in M$:

$$\alpha_g(x) = u_g x u_g^*$$

$$J u_g = u_g J$$

$$u_g P = P,$$

where P is the canonical positive cone. We call u the *standard implementation* (or standard form) of the action.

We can now mimic our construction of the crossed product by a discrete group in the present setting. Given $G \curvearrowright_\alpha M$ with $M \subset B(H)$, let $L^2(G, H)$ denote the Hilbert space of square-integrable H-valued functions on G (modulo almost-everywhere equivalence) with respect to a left Haar measure μ on G, so that $L^2(G, H) = \{\xi : \int_G \|\xi_g\|^2 \, d\mu < \infty\}$, equipped with the inner product $\langle \xi, \eta \rangle_{L^2(G,H)} = \int_G \langle \xi_g, \eta_g \rangle_H \, d\mu$.

We can again define faithful representations $\pi_\alpha : M \to B(L^2(G, H))$ and $\lambda : G \to B(L^2(G, H))$ via the following relations:

$$(\pi_\alpha(x)\xi)_g = \alpha_g^{-1}(x)\xi_g$$

$$(\lambda_g \xi)_h = \xi_{g^{-1}h},$$

for all $x \in M, g, h \in G$, and $\xi \in L^2(G, H)$.

Definition 1.7.4 (Crossed Product of M by α) We define the *crossed product of M by α*, denoted $M \rtimes_\alpha G$, by $M \rtimes_\alpha G := (\pi_\alpha(M) \cup \lambda(G))''$.

It's easy to check that the definition given here extends the previous definition in the discrete case, and, furthermore, the unitaries λ_g in the crossed product algebra still satisfy the important covariance relation:

$$\lambda_g \pi_\alpha(x) \lambda_g^* = \pi_\alpha(\alpha_g(x)).$$

We will again abuse notation slightly and use the faithfulness of π_α to identify M with $\pi_\alpha(M)$, suppressing π_α from the notation. So, for example, we see that the covariance relation can be written $\lambda_g x \lambda_g^* = \alpha_g(x)$ (conjugation by λ_g implements the action α), and this relation implies that $M \rtimes_\alpha G$ is generated by finite sums of the form $\sum x_i \lambda_{g_i}$, with $x_i \in M$, $g_i \in G$.

Remark 1.7.5 We mention that the above construction is independent of the representing Hilbert space H in the following sense: if $M \subset B(H)$ has a faithful representation $\rho : M \to B(K)$ for some Hilbert space K, then there's a unique isomorphism from $M \rtimes_\alpha G$ to

$\rho(M) \rtimes_{\rho\circ\alpha} G$ taking $x \in M$ (technically, $\pi_\alpha(x)$) to $\rho(x)$ and λ_g to λ_g^K, the corresponding canonical unitary in $\rho(M) \rtimes_{\rho\circ\alpha} G$. See Thm. X.1.7.(i) in [Tak13] for details.

Now that we've seen locally compact group actions and their associated crossed product construction, we will use the next section to give us a helpful equivalence relation on such actions.

1.7.1 Cocycle Conjugacy

Throughout this section, let G be a locally compact group, and let $\alpha : G \to \mathrm{Aut}(M)$ be an action of G (as defined in the previous section).

Definition 1.7.6 (Cocycle) Given an action α of G on M, we define an *α-cocycle* (or *α-one cocycle*) to be a function $u : G \to \mathcal{U}(M)$, continuous with respect to the strong operator topology on $\mathcal{U}(M)$, which satisfies the following *cocycle condition* for all $g, h \in G$:

$$u_{gh} = u_g \alpha_g(u_h).$$

The collection of all α-cocycles will be denoted $Z^1(\alpha)$. Cocycles are of interest, in part, because we can use them to perturb an action α by a cocycle u into a new action α^u by defining

$$\alpha_g^u(x) := u_g \alpha_g(x) u_g^*.$$

In other words, $\alpha_g^u = \mathrm{Ad}\ u_g \circ \alpha_g$. We see that α^u is indeed an action:

$$\begin{aligned}
\alpha_g^u(\alpha_h^u(x)) &= \alpha_g^u(u_h \alpha_h(x) u_h^*) \\
&= u_g(\alpha_g(u_h \alpha_h(x) u_h^*))u_g^* \\
&= (u_g \alpha_g(u_h))\alpha_{gh}(x)(\alpha_g(u_h)^* u_g^*) \\
&= u_{gh}\alpha_{gh}(x)u_{gh}^* = \alpha_{gh}^u(x),
\end{aligned}$$

where the first equality of the last line follows from the cocycle condition. We call α^u the *perturbation* of α by u.

Given an α-cocycle $u : G \to \mathcal{U}(M)$ and a unitary $w \in \mathcal{U}(M)$, we can produce a new α-cocycle u^w by setting $u_g^w := wu_g\alpha_g(w^*)$. If u and v are α-cocycles such that $v = u^w$ for some $w \in \mathcal{U}(M)$, then we say that u and v are *equivalent* and write $u \sim v$. One easily checks that this is indeed an equivalence relation on $Z^1(\alpha)$, and the quotient $Z^1(\alpha)/\sim$ of $Z^1(\alpha)$ by this equivalence relation is denoted $H^1(\alpha)$ (the *one-cohomology space*).

Given two actions α and β of G on M, we will say that α and β are *equivalent* (written $\alpha \simeq \beta$) if there exists an automorphism Φ of M such that $\Phi(\alpha_g(x)) = \beta_g(\Phi(x))$ for all $g \in G$, $x \in M$. In view of the standard implementation, in the tracial case this is the same as saying that there is a unitary operator on $L^2(M)$ which intertwines the actions. This is a strict equivalence relation on our actions, requiring them to be the same up to an automorphism of M. But the use of cocycle perturbations will allow us to consider a less rigid notion of equivalence between our actions.

Definition 1.7.7 (Cocycle Conjugacy) Let α and β be actions of G on M. We say that α and β are *cocycle conjugate* (written $\alpha \sim \beta$) if $\alpha^u \simeq \beta$ for some $u \in Z^1(\alpha)$.

So two actions are cocycle conjugate if we can perturb one of them by a cocycle in such a way that it becomes equivalent to the other up to an automorphism of M. Our main reason for concerning ourselves with cocycle conjugacy is the following theorem, which guarantees that if $\alpha \sim \beta$, then $M \rtimes_\alpha G \simeq M \rtimes_\beta G$. For completeness, we'll reproduce the proof given in [Tak13]:

Theorem 1.7.8 *(Thm X.1.7.ii, [Tak13]) Let $u \in Z^1(\alpha)$. If we define $U \in \mathcal{U}(B(L^2(G, H)))$ by:*

$$(U\xi)_g = u_{g^{-1}}\xi_g$$

for $\xi \in L^2(G, H)$, then

$$U(M \rtimes_\alpha G)U^* = M \rtimes_{\alpha^u} G.$$

In particular, the crossed product is stable (up to isomorphism) under cocyclic perturbations.

Proof. Set $\beta = \alpha^u$. Since we're working with both $M \rtimes_\alpha G, M \rtimes_\beta G \subset B(L^2(G,H))$, we'll avoid our usual suppression of π_α and π_β from the notation. We compute:

$$(U\pi_\alpha(x)U^*\xi)_g = (u_{g^{-1}}\alpha_g^{-1}(x)u_{g^{-1}}^*)\xi_g = \beta_g^{-1}(x)\xi_g = (\pi_\beta(x)\xi)_g,$$

so that $U\pi_\alpha(x)U^* = \pi_\beta(x)$, and, similarly,

$$\begin{aligned}(U\lambda_g U^*\xi)_h &= u_{h^{-1}}u_{h^{-1}g}^*\xi_{g^{-1}h} = u_{h^{-1}}(u_{h^{-1}}\alpha_{h^{-1}}(u_g))^*\xi_{g^{-1}h} \\ &= u_{h^{-1}}\alpha_{h^{-1}}(u_g^*)u_{h^{-1}}^*\xi_{g^{-1}h} = (\pi_\beta(u_g^*)\lambda_g\xi)_h,\end{aligned}$$

which implies that $U\lambda_g U^* = \pi_\beta(u_g^*)\lambda_g$. Therefore, $U(M \rtimes_\alpha G)U^* \subset M \rtimes_\beta G$. Finally, we note that $u_{gh}^* = u_g^*\beta_g(u_h^*)$, which means that u^* is a β-cocycle, and $\alpha = \beta^{u^*}$. Therefore, a similar computation implies that $U^*(M \rtimes_\beta G)U \subset M \rtimes_\alpha G$, so that Ad U is an isomorphism from $M \rtimes_\alpha G$ to $M \rtimes_\beta G$.

1.7.2 The Modular Automorphism Group and Type III Factors

In this section, we describe another example of a locally compact group action, which arises naturally from a state on our von Neumann algebra. First, we make some definitions and mimic the construction of the modular conjugation J from the Type II_1 case. See [Bla06] III.4 or [Tak13] Chapter VII for more details on these constructions.

Definition 1.7.9 (One Parameter Automorphism Group) Let M be a von Neumann algebra. A *one parameter automorphism group* of M is a continuous homomorphism $\sigma : \mathbb{R} \to \text{Aut}(M)$.

Equivalently, a one parameter automorphism group is an action by $\mathbb{R}$ on M. In what follows, we will see how to produce a one parameter automorphism group from a state on M. One can make a similar construction given an arbitrary semifinite weight, but we'll restrict to (faithful) states for simplicity.

Let ϕ be a faithful, normal state on M, and let ξ denote the usual cyclic vector (i.e. $\widehat{1}$) in the GNS representation H_ϕ for ϕ. Then $x\xi \mapsto x^*\xi$ is a densely-defined conjugate-linear operator on H_ϕ. It is easily seen to be closeable, so we let S denote its closure.

Taking the polar decomposition of S, we have:

$$S = J\Delta^{1/2},$$

where J is an involutive anti-unitary (i.e. $J^2 = 1$), as in the Type II_1 case, and $\Delta = S^*S$ is an invertible (unbounded) densely defined positive operator. As before, we call J the *modular conjugation*, and we will call Δ the *modular operator* associated with ϕ. We may write J_ϕ or Δ_ϕ if we need to emphasize their dependence on ϕ. We also have that ξ is fixed by S, J, and Δ, and $J\Delta = \Delta^{-1}J$.

Example 1.7.10 Let G be a locally compact group with left Haar measure μ. If we consider the left regular representation of G on $L^2(G, \mu)$, then Δ_ϕ is simply multiplication by the usual modular function Δ_G on $L^2(G, \mu)$, where ϕ is the canonical Plancherel weight on $L(G)$. This motivates the choice of the term "modular operator." For details, see [Bla06] III.4.1.4 and III.3.3.1 (for the definition of the Plancherel weight).

Note that since ξ is cyclic and separating for M, it is also cyclic and separating for M'. One can similarly define F as the closure of the map $y\xi \mapsto y^*\xi$ for $y \in M'$, which makes F another densely-defined conjugate-linear operator. A computation verifies that $F = S^*$ in this case, so we will neglect further mention of F.

Now, since Δ is a positive (unbounded) operator, for any $t \in \mathbb{R}$, we have that $\Delta^{it} \in \mathcal{U}(H_\phi)$. Furthermore, $t \mapsto \Delta^{it}$ is strongly continuous, so we can define a one parameter automorphism group $\sigma : \mathbb{R} \to \mathrm{Aut}(B(H_\phi))$ via:

$$\sigma_t(x) := \mathrm{Ad}\ (\Delta^{it})x = \Delta^{it}x\Delta^{-it}.$$

Since $J\Delta = \Delta^{-1}J$ with J anti-linear, we have $J\Delta^{it} = \Delta^{it}J$. In particular, conjugation by J commutes with σ. The central theorem for these modular automorphisms can be stated as follows:

Theorem 1.7.11 *(Tomita-Takesaki) With M, ϕ as above, we have $JMJ = M'$ and $\sigma_t(M) = M$ for all t. Furthermore, for $x \in M \cap M'$, we have $JxJ = x^*$ and $\sigma_t(x) = x$.*

In particular, by the comment before the theorem, the same statement holds with M' instead of M. By the global invariance of M (or M') under the modular action, we have produced a one parameter automorphism group σ_t^ϕ of M from the state ϕ.

Remark 1.7.12 (ϕ is tracial iff $\Delta = 1$) If ϕ is a tracial state (or more generally, tracial weight), then it's easy to see that $S = F = J$, so the modular operator and its corresponding action are trivial. Conversely, if $\Delta_\phi = 1$, then a quick computation verifies that the vector state $\langle \xi_\phi, \cdot \xi_\phi \rangle = \phi$ is tracial. Since we presently want to use the modular operator to construct interesting one parameter subgroups, we will now restrict to the Type III case, where our algebras don't have traces and modular theory is especially useful for classification.

The following example gives a simple relation between modular automorphism groups in the case that $\psi = \phi(h\cdot)$ for some positive h.

Example 1.7.13 Let ϕ be a state on M. We write M^ϕ for M^{σ^ϕ}, the fixed point algebra for the modular action. If h is a positive operator affiliated with M^ϕ, then setting $\psi := \phi(h\cdot)$, we have

$$\sigma_t^\psi(x) = h^{it}\sigma_t^\phi(x)h^{-it}.$$

Connes proved the following theorem, which fully describes the relationship between modular automorphism groups for two faithful states.

Theorem 1.7.14 *([Tak13] Thm. 3.3, Connes Cocycle Derivative) Let ϕ and ψ be faithful semifinite normal weights on M. There exists a strongly continuous map $u : \mathbb{R} \to \mathcal{U}(M)$ such that:*

$$u_{s+t} = u_s \sigma_s^\phi(u_t)$$

(that is, u is a σ^ϕ-cocycle) and

$$\sigma_t^\psi(x) = u_t \sigma_t^\phi(x) u_t^*.$$

Proof. (Sketch) Let ϕ and ψ be as above, and define the *balanced weight* θ on $N = M_2(M)$ by:

$$\theta \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \phi(x_{11}) + \psi(x_{22}).$$

The faithfulness of ϕ and ψ implies the faithfulness of θ. Now let (e_{ij}) denote the usual matrix units, and let $u = e_{11} - e_{22}$. Note that $u = u^*$ is a unitary such that

$$u \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} u = \begin{pmatrix} x_{11} & -x_{12} \\ -x_{21} & x_{22} \end{pmatrix}.$$

Therefore, $\theta((\mathrm{Ad}\ u)(x)) = \theta(x)$ for all $x \in N$, which implies that u is fixed by the modular action for θ (i.e. $u \in N^\theta$). In particular, it follows that $\frac{1+u}{2} = e_{11}$ and $\frac{1-u}{2} = e_{22}$ are in N^θ.

Therefore, for any $t \in \mathbb{R}$, we have:

$$e_{11}\sigma_t^\theta \begin{pmatrix} x_{11} & 0 \\ 0 & 0 \end{pmatrix} e_{11} = \sigma_t^\theta \left(e_{11} \begin{pmatrix} x_{11} & 0 \\ 0 & 0 \end{pmatrix} e_{11} \right) = \sigma_t^\theta \begin{pmatrix} x_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

so that

$$\sigma_t^\theta \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_t(x) & 0 \\ 0 & 0 \end{pmatrix}$$

for some action α on M. But, since $\theta|_{e_{11}Ne_{11}} = \phi$, we must have $\alpha_t = \sigma_t^\phi$. We similarly have

$$\sigma_t^\theta \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_t^\psi(x) \end{pmatrix}.$$

Furthermore, we have $e_{11}\sigma_t^\theta(e_{21}) = \sigma_t^\theta(e_{11}e_{21}) = 0 = \sigma_t^\theta(e_{21}e_{22}) = \sigma_t^\theta(e_{21})e_{22}$, so that

$$\sigma_t^\theta(e_{21}) = \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix}$$

for some $u_t \in M$. Note that $\sigma_t^\theta(e_{21})$ is a partial isometry from e_{11} to e_{22}, which implies that u_t is a unitary.

Now we need only check that u is the desired cocycle. First, we have

$$\begin{pmatrix} 0 & 0 \\ 0 & \sigma_t^\psi(x) \end{pmatrix} = \sigma_t^\theta \left(e_{21} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} e_{12} \right) = \sigma_t^\theta(e_{21}) \begin{pmatrix} \sigma_t^\phi(x) & 0 \\ 0 & 0 \end{pmatrix} \sigma_t^\theta(e_{21})^* = \begin{pmatrix} 0 & 0 \\ 0 & u_t\sigma_t^\phi(x)u_t^* \end{pmatrix}$$

which verifies one of the desired relations. To see that u is a σ^ϕ-cocycle, we check:

$$\begin{aligned}\begin{pmatrix} 0 & 0 \\ u_{s+t} & 0 \end{pmatrix} &= \sigma_s^\theta \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix} \\ &= \sigma_s^\theta \left(e_{21} \begin{pmatrix} u_t & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix} \begin{pmatrix} \sigma_s^\phi(u_t) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ u_s \sigma_s^\phi(u_t) & 0 \end{pmatrix},\end{aligned}$$

which concludes the proof.

The above proof is more carefully given in [SZ79] Theorem 10.28. For a proof that doesn't require ψ to be faithful, see e.g. [Str81] Chapter 1.3. The cocycle u of the above theorem is called the *cocycle derivative* of ϕ with respect to ψ. It is sometimes denoted $(D\phi : D\psi)_t := u_t$. The cocycle u is unique if we additionally impose a certain analytic condition, related to the so-called KMS condition, but we won't need this uniqueness here. There is also a converse to this theorem (see [Tak13] 3.8):

Theorem 1.7.15 *Let ψ be a faithful semifinite normal weight on M. If u is a σ^ψ-cocycle, then there exists a semifinite normal weight ϕ such that $u = (D\phi : D\psi)$.*

Connes' cocycle theorem gives the following characterization of semifiniteness:

Lemma 1.7.16 *Let ϕ be a faithful weight on M. Then M is semifinite if and only if the modular action of ϕ is inner (i.e. $\sigma_t^\phi = Ad\, u_t$ for some $u : \mathbb{R} \to \mathcal{U}(M)$).*

Proof. (Sketch) By Connes' theorem, the innerness of the modular action for some ϕ implies the innerness of the action for all other faithful weights. So it suffices to show that M is semifinite if and only if there exists a weight whose modular action is inner.

If M is semifinite, then take ϕ to be a faithful tracial weight on M. By Remark 1.7.12, $\sigma^\phi = \mathrm{id}$, so that the modular action is trivially inner.

Conversely, if ϕ's modular action is inner, then Stone's theorem guarantees that we can find a (possibly unbounded) self-adjoint h affiliated with M such that $\sigma_t^\phi = \mathrm{Ad}\, h^{it}$. An application of Example 1.7.13 to the weight $\psi := \phi(h^{-1}\cdot)$ shows that σ^ψ is trivial, so that ψ is tracial, which implies that M is semifinite.

Example 1.7.17 (Crossed Product of Type III Factor by its Modular Automorphism Group) Let M be a Type III factor, and take ϕ any faithful state on M. Because the modular action for ϕ is cocycle conjugate (by the preceding theorem) to the modular action for any faithful ψ, the crossed product $M \rtimes_{\sigma^\phi} \mathbb{R}$ is canonically determined by M up to isomorphism. For this reason, we call $M \rtimes_{\sigma^\phi} \mathbb{R}$ "the" *(continuous) core* of M.

From ϕ, one can naturally construct a weight $\widehat{\phi}$ on $N = M \rtimes_{\sigma^\phi} \mathbb{R}$, called the *dual weight*. The modular action for $\widehat{\phi}$ is necessarily inner on N (and corresponds to the dual action of σ^ϕ), which implies by the preceding lemma that N is semifinite. So the continuous core of a Type III von Neumann algebra is semifinite. Furthermore, a duality theorem of Takesaki guarantees that $M \simeq N \rtimes_{\sigma^{\hat{\phi}}} \mathbb{R}$, which gives a canonical way of writing any Type III factor as a crossed product of a Type II_∞ factor (its core) by this dual action. We won't need this decomposition in what follows, so we omit further discussion of this duality. For details, see e.g. [Tak13] Chapter XII or [Bla06] III.4.8.

Remark 1.7.18 The statements of the previous example are still true when M is semifinite, but because semifinite algebras possess traces (whose modular actions are trivial), the core is only interesting in the Type III case.

It follows from Connes' cocycle derivative theorem that there is a canonical homomorphism $\delta : \mathbb{R} \to \mathrm{Out}(M)$, where $\mathrm{Out}(M) := \mathrm{Aut}(M)/\mathrm{Inn}(M)$, the quotient of the automorphism group by the inner automorphisms. Connes used δ to define his invariant $T(M) = \ker \delta$, a subgroup of $\mathbb{R}$, which is important in distinguishing factors. It can be shown that $T(M) = \mathbb{R}$ if and only if the factor is semifinite (i.e. Type I or Type II). In particular, every Type III factor has an outer automorphism. If $T(M)$ is trivial, we call M a Type III_1 factor. If $T(M)$ is a proper dense subgroup, M is a Type III_0 factor. If $T(M)$ is discrete, then it has a generator $t > 0$. In this case, we call M a Type III_λ factor, where $\lambda = e^{-2\pi/x}$.

CHAPTER 2

Free Products and Voiculescu's Free Gaussian Functor

2.1 The Full Fock Space, Free Product Construction, and Freeness

The goal of this section is to describe various free product constructions. Ultimately, the goal is to make sense of $*_{i\in I}(M_i, \phi_i)$, the *free product* of the von Neumann algebras M_i with normal states ϕ_i. To do this, we first define the full Fock space of a Hilbert space H and the free product of a family of Hilbert spaces with distinguished unit vectors. For a more detailed exposition of these constructions, see [VDN92]. We will try to match the notation used there whenever it's convenient.

The full Fock space construction is as follows:

Definition 2.1.1 Let H be a Hilbert space. We define the *full Fock space* of H, denoted $\mathcal{F}(H)$, via:

$$\mathcal{F}(H) := \mathbb{C}\Omega \bigoplus_{j\geq 1} H^{\otimes j}.$$

We remark that the unit vector $\Omega \in \mathcal{F}(H)$ is called the *vacuum vector*, and its corresponding vector state ω will be called the *vacuum state* or *vacuum expectation.*

We can define in a similar way the Hilbert space free product of a family of Hilbert spaces with distinguished unit vectors. We note that this construction is not a categorical coproduct. Instead, it will serve as the underlying space for the free product von Neumann algebra $*_{i\in I}M_i$.

Definition 2.1.2 Let $\{(H_i, \xi_i) : i \in I\}$ be a collection of Hilbert spaces with distinguished unit vectors $\xi_i \in H_i$, and define $H_i^\circ := H_i \ominus \mathbb{C}\xi_i$ (i.e. H_i° is the orthocomplement of the distinguished unit vector ξ_i in H_i. We define the Hilbert space free product $*_{i\in I}(H_i,\xi_i)$ to be (H,ξ), where

$$H := \mathbb{C}\xi \bigoplus_{j\geq 1} \left(\bigoplus_{\substack{i_1,\ldots,i_j\in I \\ i_1\neq i_2\neq\ldots\neq i_j}} H_{i_1}^\circ \otimes H_{i_2}^\circ \otimes \ldots \otimes H_{i_j}^\circ \right).$$

We note that the restriction $i_1 \neq i_2 \neq \ldots \neq i_j$ is shorthand for $i_1 \neq i_2$, $i_2 \neq i_3$, etc. In other words, an index is allowed to appear multiple times, but adjacent indices cannot be the same.

Remark 2.1.3 Let $\{H_i\}_{i\in I}$ be a family of Hilbert spaces with distinguished unit vectors. Note that we can identify $(\mathcal{F}(\oplus_i H_i),\Omega)$ with $*_i(\mathcal{F}(H_i),\Omega_i)$ in a natural way by using the identification: $(\oplus_i H_i)^{\otimes n} = \oplus_{i_1,\ldots,i_n} H_{i_1} \otimes \ldots \otimes H_{i_n}$.

Now we move toward defining the free product of a family of von Neumann subalgebras $M_i \subset B(H_i)$ on (H_i,ξ_i). If (H,ξ) is the Hilbert space free product of $\{(H_i,\xi_i) : i \in I\}$, we define

$$H(i) := \mathbb{C}\xi \bigoplus_{j\geq 1} \left(\bigoplus_{\substack{i_1\neq i_2\neq\ldots\neq i_j \\ i_1\neq i}} H_{i_1}^\circ \otimes H_{i_2}^\circ \otimes \ldots \otimes H_{i_j}^\circ \right) \subset H.$$

Recall that ξ_i is the distinguished unit vector in H_i. We note that $H_i \otimes H(i) \simeq H$ via the unitary operator U_i which satisfies:

$$\begin{aligned} U_i(\xi_i \otimes \xi) &= \xi \\ U_i(\xi_i \otimes (\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_j)) &= \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_j && \text{for } \eta_k \in H_{i_k}^\circ, i_1 \neq i \\ U_i(\eta \otimes \xi) &= \eta && \text{for } \eta \in H_i^\circ \\ U_i(\eta \otimes (\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_j)) &= \eta \otimes \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_j && \text{for } \eta \in H_i^\circ, \eta_k \in H_{i_k}^\circ, i_1 \neq i. \end{aligned}$$

Now suppose $T \in B(H_i)$. We can "amplify" T to an operator on $H_i \otimes H(i)$ by simply tensoring with the identity operator, i.e. by considering $T \otimes id_{H(i)} \in B(H_i \otimes H(i))$. Then, conjugating by the unitary defined above, we get a faithful representation $\lambda_i : B(H_i) \to B(H)$ via:

$$\lambda_i(T) := U_i(T \otimes id_{H(i)})U_i^*.$$

Now we define $*_{i\in I}(M_i, \xi_i) := (\cup_{i\in I}\lambda_i(M_i))''$, i.e. the von Neumann algebra generated by the images of M_i under their respective representations on the Hilbert space free product. [Note: we may abbreviate $*_{i\in I}(M_i, \xi_i)$ to $*_{i\in I}M_i$ if the vectors ξ_i are understood. See the following remark for the connection to the GNS representation.]

A straightforward computation (using the fact that $U_i^*(\xi) = \xi_i \otimes \xi$) shows that $\langle \xi, \lambda_i(T)\xi\rangle = \langle \xi_i, T\xi_i\rangle$ for all $T \in M_i$. Furthermore, if ξ_i is cyclic for M_i for all i, then ξ is cyclic for $*_{i\in I}M_i$.

Remark 2.1.4 We chose to define $U_i : H_i \otimes H(i) \to H$ with H_i on the left, as above, but we could have repeated the above construction with H_i on the "right" instead, defining a unitary $V_i : H(i) \otimes H_i \to H$ analogously. If we then defined $\rho_i : B(H_i) \to B(H)$ via $\rho_i(T) = V_i(id_{H(i)} \otimes T)V_i^*$, then one can check that λ_i and ρ_j have commuting images. In fact, assuming each ξ_i is cyclic for M_i, it can be shown that $(\cup\lambda_i(M_i))' = (\cup\rho_i(M_i))''$.

Remark 2.1.5 Suppose ξ_i is cyclic for M_i for all i. Because the GNS construction gives a correspondence between normal states $\phi : M \to \mathbb{C}$ and normal representations $\pi_\phi : M \to B(H_\phi)$ with distinguished cyclic unit vector ξ_ϕ, we could have equivalently started with a family $\{(M_i, \phi_i) : i \in I\}$ of von Neumann algebras equipped with (faithful) normal states and used the GNS representations to build $(H_{\phi_i}, \xi_{\phi_i})$ before proceeding with the above construction. In this case, if we identify M_i with its image under $\lambda_i : B(H_i) \to B(H)$, we remark that the vector state $\phi(x) = \langle \xi, x\xi\rangle$ satisfies $\phi|_{M_i} = \phi_i$ by the comment preceding this remark. Moreover, if $x_j \in M_{i_j}$ satisfies $\phi_{i_j}(x_j) = 0$ and $i_j \neq i_{j+1}$ for all j, then $\phi(x_1...x_k) = 0$, and this uniquely determines ϕ. We call ϕ the *free product state* of $\{\phi_i : i \in I\}$, and sometimes denote it by $*_{i\in I}\phi_i$. The free product state can be shown to be faithful if ϕ_i is faithful for all i. This discussion motivates the following definition.

Definition 2.1.6 Let (M, ϕ) be a von Neumann algebra equipped with a normal state ϕ. We say that a family $\{A_i : i \in I\}$ of subalgebras of M is *free* with respect to ϕ if $\phi(x_1x_2...x_k) = 0$ whenever $x_j \in A_{i_j}$ satisfies $\phi(x_j) = 0$ and $i_j \neq i_{j+1}$ for all j (that is, adjacent elements of $x_1...x_k$ come from different subalgebras). We say that the family is $*$-*free* with respect to ϕ if the same condition holds when $x_j \in A_{i_j}$ is replaced with $x_j \in A_{i_j} \cup A_{i_j}^*$. Note that if each A_i is already $*$-closed, then these notions are equivalent.

Remark 2.1.7 If the family $\{A_i : i \in I\}$ is free with respect to ϕ and each A_i is $*$-closed, then the normality of ϕ implies that the same is true of the family of von Neumann algebras they generate, $\{(A_i)'' : i \in I\}$.

Definition 2.1.8 (Freeness) Let (M, ϕ) be a von Neumann algebra equipped with a normal state ϕ. We say that a family $\{S_i : i \in I\}$ of subsets of M is *free* (or *freely independent*) (resp. $*$-*free* or $*$-*freely independent*) with respect to ϕ if the family of algebras (resp. $*$-algebras) generated by S_i are free (resp. $*$-free). We may omit the "with respect to ϕ" if ϕ is clear from context.

Example 2.1.9 Let $\{G_i\}_{i\in I}$ be a family of discrete groups, and write $G = *_{i\in I}G_i$ for their free product. We'll write $(L(G_i), \delta_{e_i})$ to denote the group von Neumann algebra of G_i together with the vector $\delta_{e_i} \in l^2(G_i)$ associated with the neutral element e_i of G_i. Recall that δ_{e_i} implements the canonical trace on $L(G_i)$. Let $\lambda_i : G_i \to B(l^2(G_i))$, $\lambda : G \to B(l^2(G))$ denote the left regular representations of G_i and G respectively. We identify $(l^2(G), \delta_e)$ with $*_{i\in I}(l^2(G_i), \delta_{e_i})$ in the natural way, so that $\delta_{g_1...g_n}$ is identified with $\delta_{g_1} \otimes ... \otimes \delta_{g_n}$. If $w = g_1...g_n$ is a word in G, then for any $g \in G_i$, one verifies that $\lambda(g)\delta_w = \delta_{gw} = U_i(\lambda_i(g) \otimes \mathrm{id}_{l^2(G)(i)})U_i^*\delta_w$ (using the notation above), so that the left regular representation of G is the free product of the left regular representations of $\{G_i\}$. In other words, we have a spatial isomorphism for the left regular representations that gives $(L(*_iG_i), \delta_e) = *_i(L(G_i), \delta_{e_i})$. Furthermore, this isomorphism shows that δ_e implements the free product of the canonical trace states on $\{G_i\}$.

Example 2.1.10 (Free Groups) The previous example implies in particular that $L\mathbb{F}_n \simeq *_1^n L\mathbb{Z}$ in such a way that the canonical group trace on the former corresponds to the free product of the group traces on the latter.

2.2 Freeness with Amalgamation

In this section, we discuss a generalization of the freeness of the previous section, where we replace the state ϕ with a more general conditional expectation.

Definition 2.2.1 Let $E : M \to A$ be a conditional expectation. Write $\langle A, S\rangle$ for the von Neumann algebra generated by A and a subset $S \subset M$. We say that a family $\{S_i : i \in I\}$ of subsets of M is *free with amalgamation over* A if

$$E(x_1...x_k) = 0$$

whenever $x_j \in \langle A, S_{i_j}\rangle$ with $i_1 \neq i_2 \neq ... \neq i_k$ and $E(x_j) = 0$ for all j.

Remark 2.2.2 Note that a normal state is a conditional expectation onto $\mathbb{C}$, so that this definition extends the previous definition of freeness, which corresponds to freeness with amalgamation over the scalars.

Now suppose that M is generated by $\{x_i : i \in I\}$ and A, and $E : M \to A$ is a faithful normal conditional expectation. Let $A\langle X_i : i \in I\rangle$ denote the $*$-algebra generated by A and $\{X_i\}$, where $\{X_i\}$ are non-commuting indeterminates. Then consider the A-linear map $\mu_A : A\langle X_i : i \in I\rangle \to A$ determined by

$$\mu_A(P(X_{i_1}, ..., X_{i_k}) = E(P(x_{i_1}, ..., x_{i_k})),$$

where P is a non-commutative $*$-polynomial in $X_{i_1}, ..., X_{i_k}$ with A-coefficients. We call μ_A the *A-valued joint law* (or *joint distribution*) of $\{x_i\}$.

We will often implicitly make use of the following theorem, which guarantees that M is determined up to isomorphism by the A-valued distribution of its generators.

Theorem 2.2.3 *Let $E^M_A : M \to A$, $E^N_A : N \to A$ be faithful normal conditional expectations, and suppose that M (respectively N) is generated by $A \cup \{x_i\}_{i\in I}$ (resp. $A \cup \{y_i\}_{i\in I}$). If the A-valued joint law of $\{x_i\}$ is the same as the A-valued joint law of $\{y_i\}$, then there exists a unique $*$-isomorphism $\Phi : M \to N$ such that $\Phi(x_i) = y_i$ for all i and $\Phi \circ E^M_A = E^N_A \circ \Phi$.*

For details on the proof, see e.g. [NS06] Theorems 4.10 and 4.11.

2.3 Free Creation/Annihilation Operators

Now that we've defined freeness of a family of subsets/subalgebras of a von Neumann algebra with fixed reference state, we return to the full Fock space, which gives a natural source of freely independent subalgebras.

Let H be a Hilbert space. For $\xi \in H$, we define the *left creation operator* $l(\xi) \in B(\mathcal{F}(H))$ as follows:

$$l(\xi)\Omega = \xi$$
$$l(\xi)(\xi_1 \otimes ... \otimes \xi_n) = \xi \otimes \xi_1 \otimes ... \otimes \xi_n.$$

Its adjoint $l^*(\xi)$ is the corresponding *left annihilation operator* and satisfies:

$$l^*(\xi)\Omega = 0$$
$$l^*(\xi)\eta = \langle \xi, \eta \rangle \Omega$$
$$l^*(\xi)(\xi_1 \otimes ... \otimes \xi_n) = \langle \xi, \xi_1 \rangle (\xi_2 \otimes ... \otimes \xi_n)$$

One can analogously define the right creation/annihilation operators, $r(\xi)$ and $r^*(\xi)$. If $\|\xi\| = 1$, it is easily verified that $l(\xi)$ is a non-unitary isometry satisfying $l^*(\xi)l(\xi) = 1$. If $(\xi_i)_{i\in\mathbb{N}}$ is an orthonormal basis for H, then $\sum_i l(\xi_i)l^*(\xi_i) = 1 - P_\Omega$, where P_Ω is the rank 1 projection onto $\mathbb{C}\Omega$ and the sum is convergent in the strong operator topology.

Now let ϕ be the vector state corresponding to the vacuum vector, i.e. $\phi(x) = \langle \Omega, x\Omega \rangle$. Our next goal is to understand the distribution of $l(\xi)$ (and that of its real part) with respect to the vacuum state ϕ. For this, we briefly describe the notion of Dyck paths and their connection to Catalan numbers.

2.3.1 Dyck Paths

Consider a walk γ on $\mathbb{Z}^2$ beginning at the origin and taking steps of type $(1, \pm 1)$. If we write λ_i for the ith step's y-coordinate, so that $\lambda_i \in \{1, -1\}$, we will say that γ is a *Dyck path* of length k if we have:

$$\sum_{i=1}^{j} \lambda_i \geq 0 \qquad\qquad \forall 1 \leq j \leq k$$
$$\sum_{i=1}^{k} \lambda_i = 0.$$

This corresponds to a walk which moves right from the origin and stays above or touches the x-axis without ever dipping beneath it. In other words, the y-coordinates of the walk remain nonnegative. Note that if γ is a Dyck path of length k, then k is necessarily even, since $|\{i : \lambda_i = 1\}| = |\{i : \lambda_i = -1\}|$. We will abuse the terminology slightly and identify a Dyck path with the corresponding tuple of its λ_i's in $\{-1, 1\}^k$.

Now we want to count the number of Dyck paths of a fixed length. Let D_{2k} denote the number of Dyck paths of length $2k$ (we set $D_{2k+1} = 0$ since there are no Dyck paths of odd length). We'll determine D_{2k} by using a recurrence relation. First, let's define $D_0 = 1$, and note that $(\lambda_1, \lambda_2) = (1, -1)$ is the only Dyck path of length 2. So $D_0 = D_2 = 1$. Now let's subdivide D_{2k} into a sum

$$D_{2k} = \sum_{j=1}^{k} D_{2k}(2j),$$

where $D_{2k}(2j)$ is the number of Dyck paths of length $2k$ whose first return to the x-axis occurs at position $2j$. Thus, $\sum_{i=1}^{2j} \lambda_i = 0$ and $\sum_{i=1}^{n} \lambda_i > 0$ if $1 \leq n < 2j$.

Now note that $D_{2k}(2k)$ counts the number of Dyck paths whose only zeros are at the endpoints of the path. We will call these paths *positive* Dyck paths. Such a path must be of the form $(\lambda_1, \lambda_2, ..., \lambda_{2k-1}, \lambda_{2k}) = (1, \lambda_2, ..., \lambda_{2k-1}, -1)$, where $(\lambda_2, ..., \lambda_{2k-1})$ is a Dyck path of length $2(k-1)$. So we have $D_{2k}(2k) = D_{2(k-1)}$. Similarly, since a path that contributes to $D_{2k}(2j)$ has its first zero at position $2j$, by considering these paths to be a concatenation of a positive Dyck path of length $2j$ with an arbitrary Dyck path of length $2(k-j)$, we have $D_{2k}(2j) = D_{2j}(2j)D_{2(k-j)} = D_{2(j-1)}D_{2(k-j)}$. So we may rewrite the sum above as follows:

$$D_{2k} = \sum_{j=1}^{k} D_{2k}(2j) = \sum_{j=1}^{k} D_{2(j-1)}D_{2(k-j)}.$$

This recurrence relation determines D_{2k} in terms of D_{2j} with $j < k$.

Reindexing, we see that the numbers D_{2k} are determined by $D_0 = 1$ and $D_{2(k+1)} = \sum_{j=0}^{k} D_{2j} D_{2(k-j)}$. If we let $C_n := \frac{(2n)!}{n!(n+1)!}$ denote the nth *Catalan number*, then an easy inductive argument verifies that $C_0 = 1$, $C_{k+1} = \sum_{j=0}^{k} C_j C_{k-j}$, so that C_k satisfies the same recurrence relation as D_{2k}. Therefore, $D_{2n} = C_n$ for all n.

2.3.2 Distribution of $l(\xi)$ and $l(\xi) + l^*(\xi)$

We now have the tools to determine the distribution of l and $l + l^*$ with respect to the vacuum expectation. Suppose $\xi \in H$ with $\|\xi\| = 1$ and denote its corresponding left creation/annihilation operators by l and l^* respectively. Let ω denote the vacuum expectation on $B(\mathcal{F}(H))$. For the purpose of determining the distribution of l (resp. $l + l^*$), we may without loss of generality assume that $H = \mathbb{C}\xi$, and we note that $\xi^{\otimes n}$ is orthogonal to the vacuum vector for all $n \geq 1$. If we let $\sigma(1) = *$ and $\sigma(-1) = 1$, then, using the relations $l^* l = 1$ and $l l^* = 1 - P_\Omega$, we have:

$$\omega(l^{\sigma(\lambda_1)} l^{\sigma(\lambda_2)} ... l^{\sigma(\lambda_k)}) = \begin{cases} 1 & \text{if } (\lambda_1, ..., \lambda_k) \text{ is a Dyck path} \\ 0 & \text{otherwise.} \end{cases}$$

This determines the $*$-distribution of l with respect to the vacuum state. We can use this information to get the distribution of the self-adjoint element corresponding to its real part as follows:

$$\begin{aligned} \omega((l + l^*)^k) &= \sum_{(\lambda_1,...,\lambda_k) \in \{-1,1\}^k} \omega(l^{\sigma(\lambda_1)} ... l^{\sigma(\lambda_k)}) \\ &= \sum_{\text{Dyck paths } (\lambda_1,...,\lambda_k)} 1 \\ &= D_k. \end{aligned}$$

To conclude this section, we describe the distribution of $l + l^*$ as a measure on $\mathbb{R}$. That is, we describe the measure μ on $\mathbb{R}$ whose moments agree with the moments of $l + l^*$ with respect to the vacuum expectation: $\omega((l + l^*)^k) = \int x^k \, d\mu$.

Lemma 2.3.1 *The distribution of $l + l^*$ with respect to the vacuum expectation is supported* *on* $[-2,2]$ *and is given by*

$$d\mu(x) = \frac{1}{2\pi}\sqrt{4-x^2}dx.$$

More generally, if we let $\mu_{c,r}$ *denote the measure supported on* $[c-r, c+r]$ *satsfying*

$$d\mu_{c,r}(x) = \frac{2}{\pi r^2}\sqrt{r^2-(x-c)^2}dx,$$

then for any $\eta \in H$*, the distribution of* $l(\eta) + l^*(\eta)$ *is given by* $\mu_{0,2\|\eta\|}$*. We call* $\mu_{c,r}$ *the semicircular law centered at* c *of radius* r*.*

Proof. Renormalizing, it clearly suffices to prove the claim when $\|\eta\| = 1$, in which case an easy computation verifies that $\int x^k \, d\mu = D_k = \omega((l+l^*)^k)$, by the preceding discussion.

Remark 2.3.2 We note that, like Gaussian distributions, semicircular laws are completely determined by knowledge of their first and second moments, which satisfy: $\int x \, d\mu_{c,r} = c$, $\int x^2 \, d\mu_{c,r} = c^2 + \frac{r^2}{4}$. Equivalently, a semicircular element (i.e. a self-adjoint element x of some C^*/W^*-algebra whose law with respect to a reference state is the semicircular law) has its law $\mu_{c,r}$ determined by its mean c and variance $\frac{r^2}{4}$.

Now that we've established the semicircular distribution of $l(\xi) + l^*(\xi)$, we may use the following notation for convenience: for $\xi \in H$, we write $s(\xi) := \Re(l(\xi)) = \frac{l(\xi)+l^*(\xi)}{2}$ for the associated semicircular element (with respect to the vacuum state) on the full Fock space of H. Note that $\|s(\xi)\| = \|\xi\|$ and that the distribution of $s(\xi)$ is $\mu_{0,\|\xi\|}$, the centered semicircular distribution of radius $\|\xi\|$.

We conclude this section with the following lemma, which characterizes the algebra generated by a single semicircular element.

Lemma 2.3.3 *Let* $\xi \in H$*. Then* $s(\xi)'' \simeq L(\mathbb{Z})$*. Furthermore, this isomorphism can be chosen so that* ω *on the former algebra corresponds to the canonical group trace on the latter.*

Proof. (See [VDN92], Lemma 2.6.5) Without loss of generality, if we let $H = \mathbb{C}\xi$ with $\|\xi\| = 1$, then Ω is cyclic and separating for $s(\xi)''$ (see the next section for a proof).

Therefore Ω induces a unitary $U : \mathcal{F}(H) \to L^2([-1,1],\mu_{0,1})$ such that $U(\Omega) = 1$ and $Us(\xi)U^* = x \in L^\infty([-1,1],\mu_{0,1}) \subset B(L^2([-1,1],\mu_{0,1}))$. Conjugation by U thus gives an isomorphism between $s(\xi)''$ and $L^\infty([-1,1],\mu_{0,1})$ which takes the vacuum expectation to integration against $\mu_{0,1}$.

Finally, if we let $\phi : [-1,1] \to \mathbb{S}^1$ denote the map $x \mapsto \exp(2i(\arcsin x + x\sqrt{1-x^2}))$, one can check that pre-composition with ϕ gives an unitary V from $L^2(\mathbb{S}^1)$ to $L^2([-1,1],\mu_{0,1})$ that takes integration against the Haar measure to integration against $\mu_{0,1}$. Therefore conjugation by V^*U is a spatial isomorphism between $s(\xi)'' \subset B(\mathcal{F}(H))$ and $L^\infty(\mathbb{S}^1) \subset L^2(\mathbb{S}^1)$ which takes the vacuum expectation to integration against the Haar measure on $\mathbb{S}^1$. Since we've already seen that the latter corresponds to $L(\mathbb{Z})$ with its canonical trace, this concludes the proof.

This lemma guarantees, for example, that we can find a Haar unitary (with respect to ω) that generates $s(\xi)''$.

2.4 The Free Gaussian Functor

We've now built up enought machinery to discuss Voiculescu's free Gaussian functor. First, we discuss the connection between orthogonality in H and freeness in $B(\mathcal{F}(H))$:

Lemma 2.4.1 *(see Prop. 1.5.10 in [VDN92]) Let $H = \oplus_{i\in I} H_i$ be a direct sum of Hilbert spaces. Write $l(H) := \{l(\xi) : \xi \in H\}'' \subset B(\mathcal{F}(H))$. If ω (resp. ω_i) denotes the vacuum expectation on $B(\mathcal{F}(H))$ (resp. $B(\mathcal{F}(H_i))$), then $l(H) \simeq *_i l(H_i)$ in such a way that ω is identified with $*_i\omega_i$. In particular, if ξ, η are orthogonal in H, then $l(\xi), l(\eta)$ are freely independent with respect to the vacuum expectation.*

Proof. (Sketch) By Remark 2.1.3, we can identify $*_i(\mathcal{F}(H_i), \Omega_i)$ with the full Fock space for H, $(\mathcal{F}(\oplus_i H_i), \Omega) = (\mathcal{F}(H), \Omega)$. Using this identification, one checks (using the earlier notation for the free product representation) that for $\xi \in H_i$,

$$\lambda_i(l_{H_i}(\xi)) = l(\xi)$$

as operators in $B(\mathcal{F}(H))$. Furthermore, this relation clearly implies that $\omega_i(l_{H_i}(\xi)) = \omega \circ \lambda_i(l_{H_i}(\xi)$. In particular, since H is spanned by such ξ, we have that $*_i\lambda_i : *_i l(H_i) \to l(H)$ is an isomorphism, and the free product state on $*_i l(H_i)$ (or, rather, its image) is implemented by Ω, i.e. $\omega = *_i\omega_i$.

We now describe Voiculescu's free Gaussian functor and some of its properties, with the help of the above lemma. The main properties of this functor are described by the following theorem and remark, which are a paraphrasing of Theorem 2.6.2 in [VDN92].

Theorem 2.4.2 *Let $H_{\mathbb{R}}$ be a separable real Hilbert space, and let H be its complexification. We write $s(H_{\mathbb{R}})$ for the von Neumann algebra generated by $\{s(\xi) : \xi \in H_{\mathbb{R}}\}$ in $B(\mathcal{F}(H))$ and ω for the vacuum state. Then we have:*

1.) The vacuum vector Ω is cyclic and separating for $s(H_{\mathbb{R}})$, so that ω is a faithful, normal trace state.

2.) If $\dim(H_{\mathbb{R}}) = n \in \{1, 2, ...\} \cup \{\infty\}$, then $s(H_{\mathbb{R}}) \simeq L\mathbb{F}_n$.

Proof. The cyclicity of Ω follows from the fact that if (ξ_i) is an orthonormal basis for $H_{\mathbb{R}}$, then $s(\xi_{i_1})...s(\xi_{i_k})\Omega = \xi_{i_1} \otimes ... \otimes \xi_{i_k} + \eta_{k-1}$, where $\eta_{k-1} \in \oplus_{i=0}^{k-1} H^{\otimes i}$. The fact that Ω is separating for $s(H_{\mathbb{R}})$ is equivalent to its being cyclic for $s(H_{\mathbb{R}})' = \{r(\xi) + r^*(\xi) : \xi \in H_{\mathbb{R}}\}''$, where $r(\xi)$ is the right creation operator corresponding to ξ, which follows similarly.

The one-dimensional case of the second claim was discussed in the previous section. For the multi-dimensional case, note that by the preceding lemma:

$$\begin{aligned} s(H_{\mathbb{R}}) &= *_{i=1}^{n} s(\xi_i)'' \\ &= *_i L(\mathbb{Z}) \\ &= L(*_i\mathbb{Z}) = L\mathbb{F}_n. \end{aligned}$$

Remark 2.4.3 It follows from the faithfulness in the preceding theorem and Lemma 2.3.1 that $\operatorname{spec}(l + l^*) = \operatorname{supp}(\mu) = [-2, 2]$. See, for example, Prop. 3.15 in [NS06].

Remark 2.4.4 (Functoriality) We won't prove these properties here (see [VDN92] for a complete discussion), but we'll briefly describe the sense in which the map $H_{\mathbb{R}} \mapsto s(H_{\mathbb{R}})$ is functorial. First, let $T : H \to K$ be a bounded operator between Hilbert spaces H and K. Then T induces a map between the associated full Fock spaces, denoted $\mathcal{F}(T) : \mathcal{F}(H) \to \mathcal{F}(K)$, via $\mathcal{F}(T) := \mathrm{id}_\Omega \oplus_{i=1}^{\infty} T^{\otimes i}$.

Now consider the category $\mathcal{C}$ of real Hilbert spaces with contractions as morphisms, and let $\mathcal{D}$ be the category whose objects are pairs (M, τ) of von Neumann algebras with specified faithful, normal tracial states and whose morphisms are trace-preserving, unital, completely positive maps.

Given a contraction $T : H_\mathbb{R} \to K_\mathbb{R}$, we still write T for its complexification $T : H \to K$. There is a map $s(T) : s(H_\mathbb{R}) \to s(K_\mathbb{R})$ uniquely determined by:

$$(s(T)x)\Omega = \mathcal{F}(T)(x\Omega)$$

for all $x \in s(H_\mathbb{R})$.

The free Gaussian functor $s(\cdot)$ is then a functor $\mathcal{C} \xrightarrow{s} \mathcal{D}$. If $V : H_\mathbb{R} \to K_\mathbb{R}$ is an isometry, then $s(V) : s(H_\mathbb{R}) \to s(K_\mathbb{R})$ is an injective homomorphism. If $P : H_\mathbb{R} \to H_\mathbb{R}$ is a projection, then $s(P)$ is a conditional expectation from $s(H_\mathbb{R})$ onto $s(P(H_\mathbb{R}))$.

2.5 Free Bogoliubov Actions

2.5.1 Definitions and Outerness

Let $\alpha : G \to \mathcal{O}(H_\mathbb{R})$ be an action of a group G on a real Hilbert space $H_\mathbb{R}$. We'll denote by $\overline{\alpha}$ the complexified action on $H = H_\mathbb{R} \otimes \mathbb{C}$ given by $\alpha \otimes id : G \to \mathcal{U}(H)$.

We can now define a new action of G on the full Fock space $\mathcal{F}(H)$, which we will denote by $\mathcal{F}(\alpha) : G \to \mathcal{U}(\mathcal{F}(H))$. It is determined by the following relations:

$$\mathcal{F}(\alpha)_g(\Omega) = \Omega$$
$$\mathcal{F}(\alpha)_g(\xi_1 \otimes \xi_2 \otimes ... \otimes \xi_n) = \overline{\alpha}_g(\xi_1) \otimes \overline{\alpha}_g(\xi_2) \otimes ... \otimes \overline{\alpha}_g(\xi_n),$$

for all $\xi_i \in H$ and $g \in G$.

We can now define a new vacuum state-preserving action, called the *free Bogoliubov* *action* (or *shift*) induced by α of G on $\Gamma(H) \subset B(\mathcal{F}(H))$ via:

$$\sigma_g^\alpha = \mathrm{Ad}(\mathcal{F}(\alpha)_g).$$

We note that the free Bogoliubov action satisfies for any $\xi \in H$:

$$\sigma_g^\alpha(s(\xi)) = \mathcal{F}(\alpha)_g s(\xi) \mathcal{F}(\alpha)_g^* = s(\overline{\alpha}_g(\xi)).$$

Actions obtained in this way are almost guaranteed to be outer. More precisely, we have:

Theorem 2.5.1 *(Thm. 5.1, [HS11]) Let $\alpha : G \to \mathcal{O}(H)$ be a strongly continuous representation of a locally compact group G on a real Hilbert space H. Then σ_g^α is an inner automorphism of $s(H)$ if and only if $\alpha_g = 1$. In particular, if α is faithful, then the free Bogoliubov action σ^α is free (i.e. properly outer).*

Remark 2.5.2 The last sentence of this theorem isn't part of the statement of Theorem 5.1 in [HS11], but we include it for later reference. It follows for trivial reasons when $\dim H = 1$ and otherwise from the factoriality of $s(H)$, which forces any outer automorphism to be properly outer (as we saw in Remark 1.5.19).

2.5.2 Mixing Representations and (Strong) Solidity

We now discuss some structural results about free Bogoliubov actions and their corresponding crossed products. First, we define mixingness of a representation.

Definition 2.5.3 Let G be a countable group. Consider an orthogonal/unitary representation $\alpha : G \to \mathcal{O}(H)$ (resp. $\alpha : G \to \mathcal{U}(H)$). We say that α is *mixing* if

$$\lim_{g \to \infty} \langle \alpha_g \xi, \eta \rangle = 0$$

for all $\xi, \eta \in H$.

Similarly, we have mixingness for actions on von Neumann algebras.

Definition 2.5.4 Given an action $\alpha : G \to \mathrm{Aut}(M)$ on a tracial von Neumann algebra (M, τ), we say that α is *mixing* if

$$\lim_{g \to \infty} \tau(\alpha_g(x)y) = 0$$

for all $x, y \in M \ominus \mathbb{C}$, where $M \ominus \mathbb{C}$ denotes the set of traceless (i.e. $\tau(x) = 0$) elements of M.

Remark 2.5.5 (Mixing $\implies$ Ergodic) It's easy to see that mixingness of an action implies ergodicity, because if $0 \neq x \in M^\alpha \ominus \mathbb{C}$, we have $\lim_{g\to\infty} \tau(\alpha_g(x)x^*) = \tau(xx^*) > 0$.

A straightforward computation using the fact that $\sigma_g^\alpha(s(\xi)) = s(\overline{\alpha}_g(\xi))$ (see also Proposition 2.8 in [HS11]) yields:

Lemma 2.5.6 *The following are equivalent:*

(1) The representation $\alpha : \mathbb{Z} \to \mathcal{O}(H)$ is mixing.

(2) The associated free Bogoliubov action $\sigma^\alpha : \mathbb{Z} \curvearrowright s(H)$ is mixing.

Before describing the relevant structural result of Houdayer and Shlyakhtenko on the crossed products associated to such actions, we need a few definitions. The following definition was given originally by Ozawa in [Oza04], in which Ozawa established the solidity of $L\Gamma$ for any hyperbolic group Γ. In particular, the free group factors are solid.

Definition 2.5.7 (Solidity) A von Neumann algebra M is called *solid* if for any diffuse von Neumann subalgebra $N \subset M$, the relative commutant $N' \cap M$ is amenable.

Remark 2.5.8 We give Connes' characterization of amenability in the next chapter, but for now we'll say that a von Neumann algebra $M \subset B(H)$ is *amenable* if there is a conditional expectation $E : B(H) \to M$. We will see, by Arveson's theorem, that this condition is independent of the choice of representing Hilbert space H.

There is also a useful strengthening of solidity, i.e. *strong solidity*, first defined and used by Ozawa and Popa in [OP10]. For the definition, we need the notion of the normalizer of an inclusion.

Definition 2.5.9 (Normalizer) Let $N \subset M$ be a von Neumann subalgebra. The *normalizer* of N in M, denoted $\mathcal{N}_M(N)$, is the set of unitaries in M that leave N globally invariant under conjugation. That is, $\mathcal{N}_M(N) = \{u \in \mathcal{U}(M) : uNu^* = N\}$.

Definition 2.5.10 (Strong Solidity) Let M be a diffuse von Neumann algebra. We say that M is *strongly solid* if for any diffuse amenable subalgebra $N \subset M$ with expectation (i.e. such that M has a faithful, normal expectation onto N), we have $\mathcal{N}_M(N)''$ is also amenable.

Remark 2.5.11 Note that if u is a unitary in N', then $u \in \mathcal{N}_M(N)$. Since $N' \cap M$ is generated by its unitaries, we have $N' \cap M \subset \mathcal{N}_M(A)''$ for any subalgebra $A \subset N$. In particular, if M is tracial and strongly solid, then it is solid, because amenability passes to subalgebras (at least in the tracial setting, as we see in the next chapter).

We can now state Houdayer and Shlyakhtenko's result on strong solidity for free Bogoliubov crossed products:

Theorem 2.5.12 *(Thm. 3.10, [HS11]) Let $\alpha : G \to \mathcal{O}(H)$ be faithful mixing representation. Then the non-amenable II_1 factor $s(H) \rtimes_{\sigma^\alpha} G$ is strongly solid.*

Remark 2.5.13 The result here is the strong solidity. It's easy to see that $M = s(H) \rtimes_{\sigma^\alpha} G$ is a non-amenable II_1 factor. Mixingness implies both ergodicity of the action and that $\dim H > 1$, so that $s(H) \subset M$ is non-amenable (being a free group factor), and factoriality of M then follows from 2.5.1 (outerness for Bogoliubov actions). In fact, for the groups we'll be interested in ($G = \mathbb{R}, \mathbb{Z}$), it's easy to see that mixingness implies faithfulness, which makes the assumption of faithfulness (equivalently, the assumption of outerness for the Bogoliubov action) superfluous.

In what follows, we won't need the full strength of this theorem. We'll only use it to guarantee the solidity of our Bogoliubov crossed products. But first we'll address some preliminaries on bimodules and completely positive maps in the next chapter.

CHAPTER 3

Bimodules and Completely Positive Maps

3.1 Complete Positivity and Stinespring Dilation

Let M, N be von Neumann algebras, and let $\psi : M \to N$ be a linear map. Then ψ induces a map $\psi_n : M^n \to N^n$, called the *amplification* of ψ, via

$$\psi_n(x \otimes A) = \psi(x) \otimes A$$

for all $x \in M, A \in M_n(\mathbb{C})$. In other words, if we think of M^n as being $n \times n$ matrices with entries in M, then the amplification of ψ is obtained by entry-wise application of ψ. We note that ψ_n is normal if ψ is normal.

There are positive maps $\phi : M \to N$ such that ϕ_n is no longer positive for some $n > 1$. For example, if ϕ is the transpose map on $M = N = M_2(\mathbb{C})$, then ϕ is positive, but ϕ_2 is not. This motivates the following definitions:

Definition 3.1.1 A map $\psi : M \to N$ is called *n-positive* ($n \in \mathbb{N}$) if ψ_i is positive for all $1 \leq i \leq n$.

Definition 3.1.2 A map $\psi : M \to N$ is called *completely positive* if it is n-positive for all $n \in \mathbb{N}$.

In other words, a map is completely positive if all of its amplifications are positive. We immediately have some important examples of completely positive maps.

Example 3.1.3 (Homomorphisms) Suppose $\pi : M \to N$ is a homomorphism of von Neumann algebras (or of C*-algebras). Then π is already positive, since $\pi(x^*x) = \pi(x)^*\pi(x) \geq 0$

for all $x \in M$. Since $\pi_n = \pi \otimes \mathrm{id}$ is still a homomorphism (from M^n to N^n), π is completely positive.

Example 3.1.4 Let $V \in B(H, K)$ be a bounded operator from a Hilbert space H to a Hilbert space K. The map $B(H) \ni T \mapsto VTV^* \in B(K)$ is completely positive by an elementary computation.

Completely positive maps are closed under composition, so a combination of the two above examples gives that if $\pi : A \to B(H)$ is a homomorphism and $V \in B(H, K)$, then $\psi(a) = V\pi(a)V^*$ is a completely positive map from A to $B(K)$. In fact, all completely positive maps are of this form for some V and π (in the same way that all states are vector states via the GNS representation) by the Stinespring construction later in this section.

Conditional expectations provide another important example of completely positive maps. First, some definitions:

Definition 3.1.5 A map $\pi : A \to B$ between C*-algebras A and B is called *contractive* if $\|\pi(a)\| \leq \|a\|$ for all $a \in A$.

Definition 3.1.6 (Conditional Expectation) Let $B \subset A$ be an inclusion of C*-algebras. A *conditional expectation* from A to B is a contractive, completely positive B-bimodule map $E : A \to B$ satisfying $E(b) = b$ for all $b \in B$.

The B-bimodular condition is the same as saying that $E(bab') = bE(a)b'$ for all $a \in A$ and $b, b' \in B$. If A and B are von Neumann algebras, we also require our conditional expectations to be normal unless otherwise stated.

Since we will often make use of conditional expectations in later sections, we record a few useful facts about them here. The following characterization is due to Tomiyama. See [BO08] Chapter 1 for an elementary proof.

Theorem 3.1.7 *Let $B \subset A$ be C*-algebras and suppose $E : A \to B$ is a linear map satisfying $E(b) = b$ for all $b \in B$. Then the following conditions are equivalent:*

(i) E is a conditional expectation

(ii) E is contractive and completely positive

(iii) E is contractive.

In other words, every contractive projection from A onto B is automatically a conditional expectation.

The following lemma guarantees the existence of conditional expectations in the tracial von Neumann algebraic setting. Recall that a map $\phi : M \to M$ is called *trace-preserving* if $\tau(\phi(x)) = \tau(x)$ for all $x \in M$.

Lemma 3.1.8 *Let (M, τ) be a von Neumann algebra with faithful normal tracial state τ, and let $N \subset M$ be a von Neumann subalgebra with the same unit as M. Then there is a unique τ-preserving normal conditional expectation $E : M \to N$.*

Proof. Let e_N denote the orthogonal projection $e_N : L^2(M, \tau) \to L^2(N, \tau)$, and define $E : M \to B(L^2(N, \tau))$ via $E(x) = e_N x e_N$. Noting that $E(x)$ commutes with the right N-action on $L^2(N)$ and recalling that $N = \rho(N)'$, we see that E takes its image in N and satisfies $E(y) = y$ for all $y \in N$. Therefore E is a trace-preserving conditional expectation $E : M \to N$. To see that such an expectation must be unique, note that if $\widehat{E}$ is another such expectation, then

$$\tau(\widehat{E}(x)y) = \tau(xy) = \tau(E(x)y),$$

for all $x \in M, y \in N$. So $\widehat{E} = E$.

By definition, conditional expectations are completely positive maps. We give a couple of examples of these expectations.

Example 3.1.9 (Group Case) Let $H \subset G$ be a subgroup of a discrete group G. The canonical trace $\tau(\sum c_g \lambda_g) = c_e$ on the group von Neumann algebra $L(G)$ induces (by the above lemma) a unique trace-preserving conditional expectation $E : L(G) \to L(H)$. It satisfies the following relation:

$$E(\sum_{g \in G} c_g \lambda_g) = \sum_{h \in H} c_h \lambda_h.$$

Example 3.1.10 (Abelian Case) Let $\mathcal{F}$ be a sub-σ-algebra of the σ-algebra $\mathcal{G}$ of μ-measurable subsets on a probability measure space (X, μ). As in the proof of the above lemma, we consider the orthogonal projection $E : L^2(X, \mathcal{G}, \mu) \to L^2(X, \mathcal{F}, \mu) \subset L^2(X, \mathcal{G}, \mu)$. This orthogonal projection restricts to a conditional expectation $E : L^\infty(\mathcal{G}, \mu) \to L^\infty(\mathcal{F}, \mu)$. In fact, every conditional expectation on an abelian von Neumann algebra is of this form in the sense that if $\widehat{E} : A \to B$ is such an expectation, then there exists an expectation $E : L^\infty(\mathcal{G}, \mu) \to L^\infty(\mathcal{F}, \mu)$ and an isomorphism $\phi : A \to L^\infty(\mathcal{G}, \mu)$ such that $\phi \circ \widehat{E} = E \circ \phi$ for some choice of $\mathcal{G}$ and $\mathcal{F}$.

We remark that a state is a special case of a conditional expectation. It can be shown that any positive map into or out of an abelian algebra is automatically completely positive. In particular, states are both completely positive and contractive, making them conditional expectations onto the scalars. And in the same way that we can use a state to run the GNS construction, we can take a completely positive map $\psi : M \to B(K)$ and produce a representation $\pi : M \to B(H)$ and operator $V \in B(H, K)$ so that $\psi(x) = V\pi(x)V^*$ for all $x \in M$. We give a sketch of Stinespring's construction below, but see [BO08] Chapter 1 for more details.

Theorem 3.1.11 *(Stinespring Dilation) Let $\psi : M \to N \subset B(K)$ be a completely positive map. There exists a Hilbert space H, a representation $\pi : M \to B(H)$, and an operator $V \in B(H, K)$ satisfying*

$$\psi(x) = V\pi(x)V^*$$

for all $x \in M$. If ψ is unital, then V^* is an isometry.

Proof. (Sketch) Consider the algebraic tensor product $M \otimes K$ equipped with the sesquilinear form satisfying $\langle x \otimes \xi, y \otimes \eta \rangle = \langle \psi(y^* x)\xi, \eta \rangle_K$. As in the GNS construction, one checks that this is non-negative definite and quotients by the zero-length vectors to obtain a positive-definite form on the quotient H_0. The left multiplication $\pi_0 : M \to M \otimes K$, defined by $\pi_0(x)(y \otimes \eta) := xy \otimes \eta$, descends to a representation $\pi : M \to B(H)$ on the completion H of H_0 with respect to $\langle \cdot, \cdot \rangle$. We define $V^* \in B(K, H)$ by extending $V^*(\xi) = 1 \widehat{\otimes} \xi$ (where we write $x\widehat{\otimes}\xi$ for the image of $x \otimes \xi \in M \otimes K$ in H). Then for any $x \in M$, $\xi, \eta \in K$ we can see that

$$\langle V\pi(x)V^*\xi, \eta\rangle_K = \langle \pi(x)(1\widehat{\otimes}\xi), 1\widehat{\otimes}\eta\rangle = \langle x\widehat{\otimes}\xi, 1\widehat{\otimes}\eta\rangle = \langle \psi(x)\xi, \eta\rangle_K,$$

Thus, as operators on K, we get that $\psi(x) = V\pi(x)V^*$. Finally, if ψ is unital, then $\psi(1) = 1 = VV^*$. So V^* is an isometry from K to H.

We may call (π, H, V) a *Stinespring dilation* of the completely positive map ψ. If we also require that $\pi(M)V^*K$ is dense in H (as in the above construction), then a Stinespring dilation of ψ (i.e. a triple (π, H, V) satisfying the conditions of the theorem) is unique up to unitary equivalence. Furthermore, the constructed π is normal if ψ was normal.

The next section gives an important definition (that of injectivity) related to the existence of extensions of completely positive maps.

3.2 Injectivity and Connes' Characterization of Amenability

Using the notion of complete positivity, we now consider injectivity in a certain category, which enables us to give Connes' characterization of injectivity (or amenability) for von Neumann algebras. First we set up the appropriate category.

Definition 3.2.1 An *operator system* is a $*$-closed subspace $E \subset A$ of a unital C*-algebra A such that $1 \in E$.

Note that any von Neumann algebra (or C*-algebra) is an operator system. Now let $\mathcal{C}$ denote the category of operator systems whose morphisms are contractive (i.e. $\|\psi\| \leq 1$) completely positive maps. We will say that a von Neumann algebra is *injective* if it is an injective object in this category. In other words:

Definition 3.2.2 (Injectivity) A von Neumann algebra M is *injective* if every contractive completely positive map $\psi : E \to M$ from an operator system $E \subset A$ in a unital C*-algebra A extends to a contractive completely positive map $\widehat{\psi} : A \to M$.

The fact that $B(H)$ is injective is the content of Arveson's theorem. See [BO08] Theorem 1.6.1 for a proof of the following.

Theorem 3.2.3 *(Arveson's Theorem) Let $E \subset A$ be operator systems. Any contractive completely positive map $\psi : E \to B(H)$ extends to a contractive completely positive map $\widehat{\psi} : A \to B(H)$.*

Recall our earlier definition of amenability:

Definition 3.2.4 (Amenability) A von Neumann algebra $M \subset B(H)$ is *amenable* if there exists a conditional expectation $E : B(H) \to M$.

Equipped with Arveson's theorem, it's easy to see that injectivity and amenability are equivalent. If M is injective, then any contractive, completely positive extension to $B(H)$ of the identity map $\mathrm{id} : M \to M$ is a conditional expectation onto M. Conversely, if M has such an expectation, then we can take a morphism $\psi : E \to M \subset B(H)$ and use Arveson's theorem to produce an extension $\widehat{\psi} : A \to B(H)$. The composition $E_M^{B(H)} \circ \widehat{\psi}$ then witnesses the injectivity of M. The equivalence of injectivity with amenability also shows that the amenability of M doesn't depend on the choice of representing Hilbert space H.

Connes' theorem expands this equivalence - notably showing that amenability is equivalent to hyperfiniteness. We'll state the theorem in the tracial case. Proving that (ii) through (v) are equivalent is relatively straighforward (see e.g. [AP17] Chapter 10). The difficult part is showing that amenability implies hyperfiniteness. See [Tak03] Chapter XVI for a full proof.

Theorem 3.2.5 *(Connes) Let (M, τ) be a separable tracial von Neumann algebra. The following are equivalent:*

(i) M is hyperfinite

(ii) M is amenable

(iii) M is injective

(iv) there exists a state ϕ on $B(L^2(M))$ extending τ such that $\phi(xT) = \phi(Tx)$ for all

$x \in M$, $T \in B(L^2(M))$.

(v) there exists a net (ξ_i) of unit vectors in $L^2(M) \otimes L^2(M)$ satisfying $\|x\xi_i - \xi_i x\|_2 \to 0$ and $\langle x\xi_i, \xi_i \rangle \to \tau(x)$ for all $x \in M$.

Remark 3.2.6 The above conditions are also equivalent to the condition that the trivial bimodule ${}_M L^2(M)_M$ is weakly contained in the coarse bimodule ${}_M L^2(M) \otimes L^2(M)_M$ (defined in the next section), but we won't explore weak containment further here.

We can, with the help of Connes' theorem, give more examples of amenable von Neumann algebras.

Example 3.2.7 (Amenable Groups) It's easy to show that $L(G)$ is amenable as a von Neumann algebra if and only if G is an amenable group. Thus, for example, any finitely generated group of subexponential growth produces an amenable group von Neumann algebra.

Example 3.2.8 Let $N \subset (M, \tau)$ be an arbitrary von Neumann subalgebra of an amenable tracial von Neumann algebra M. Composing the unique trace-preserving expectation $E_N : M \to N$ with an expectation $E_M : B(H) \to M$ gives the amenability of N. So, at least in the tracial setting, it's easy to see that amenability passes to subalgebras.

Example 3.2.9 (Abelian/Type I) Since abelian von Neumann algebras are easily seen to be hyperfinite, we see that any abelian algebra is amenable. More generally, any Type I von Neumann algebra is hyperfinite, thus amenable.

Now that we've seen the general structure and some important uses/examples of completely positive maps, we can discuss the correspondence between completely positive maps and bimodules.

3.3 Bimodules

Before establishing the connection to completely positive maps, we first discuss the definition and common examples of bimodules over von Neumann algebras. See e.g. [AP17] for more detailed exposition. Throughout this section, let M and N be separable von Neumann algebras.

3.3.1 Definitions and Examples

Definition 3.3.1 (Left/Right Modules) We define a *left M-module* to be simply a (normal) representation $\pi : M \to B(H)$. Sometimes we refer to H itself as the left module and write $x\xi$ instead of $\pi(x)\xi$ when π is understood. In this case, we may write ${}_MH$ to emphasize that H is a left M-module and suppress π from the notation. Similarly, a *right* M-module is a left M^{op} module. Given a right M-module $\rho : M^{op} \to B(H)$, we may write ξx for $\rho(x^{op})\xi$. Similarly, we may use H_M to emphasize that H is a right M-module if ρ is understood.

Definition 3.3.2 (Bimodules) An *M-N-bimodule* is a pair of representations $\{\pi : M \to B(H),\ \rho : N^{op} \to B(H)\}$ on the same Hilbert space H such that $\pi(M)$ commutes with $\rho(N^{op})$. In other words, H is simultaneously a left M-module and a right N-module such that these two actions commute. When the representations π and ρ are understood, we may suppress them from our notation and write ${}_MH_N$ for the bimodule.

We remark that the commutativity of the left and right actions is equivalent to the associativity relation

$$(x\xi)y = x(\xi y)$$

for $x \in M, y \in N$. We will say that two left M-modules (H_1, π_1) and (H_2, π_2) are *isomorphic* if there is a unitary operator $U : H_1 \to H_2$ such that $U\pi_1(x) = \pi_2(x)U$ for every $x \in M$. We similarly define isomorphism of right M-modules and M-N-bimodules in the obvious way (i.e. by the existence of a unitary that intertwines the actions).

We've already seen a few important examples of (bi)modules. We will postpone the examples of bimodules over abelian and group algebras until we've discussed the correspondence between bimodules and completely positive maps in the next section.

Example 3.3.3 $L^2(M, \tau)$

Let (M, τ) be a tracial von Neumann algebra. Then $L^2(M)$, equipped with its natural left- and right-multiplication operations, is an M-M-bimodule. It is sometimes called the *identity* or *trivial* M-M-bimodule for reasons explained below.

Example 3.3.4 (Direct Sums) If H, K are M-N-bimodules, then $H \oplus K$ is again an M-N-bimodule with the natural diagonal M- and N-actions.

Example 3.3.5 (Coarse Bimodule) Given tracial von Neumann algebras M and N, the *coarse bimodule* is the M-N-bimodule $L^2(M) \otimes L^2(N)$ with left- and right- actions satisfying

$$x(\xi \otimes \eta)y = x\xi \otimes \eta y.$$

The space of Hilbert-Schmidt operators, which we will denote $\mathcal{L}^2(L^2(N), L^2(M))$, is defined to be the collection of operators $T : L^2(N) \to L^2(M)$ with $\mathrm{Tr}(T^*T)$ finite. This space is a Hilbert space with $\langle S, T\rangle := \mathrm{Tr}(S^*T)$ and is endowed with an M-N-bimodular structure via composition, i.e. for $x \in M$, $y \in N$,

$$xTy = x \circ T \circ y.$$

One can check that the coarse bimodule is isomorphic to the latter bimodule via the map that takes $\xi \otimes \eta$ to $\langle J\eta, \cdot\rangle\xi : L^2(N) \to L^2(M)$. See [AP17] for more details.

Remark 3.3.6 (Classification of left/right M-modules)

We can use the trivial module to build larger M-modules in the following way. Consider the right M-module $l^2(\mathbb{N}) \otimes L^2(M)$ where $x \in M$ acts by $\mathrm{id}_{l^2(\mathbb{N})} \otimes \rho(x)$ (i.e. the diagonal right action). The commutant of $1 \otimes \rho(M)$ in $B(l^2(\mathbb{N}) \otimes L^2(M))$ is $B(l^2(\mathbb{N}))\overline{\otimes}M$ (where we write $M \subset B(L^2(M))$ for the left multiplication operators). Thus, given any projection $p \in B(l^2(\mathbb{N}))\overline{\otimes}M$, we have the associated right M-module given by $p(l^2(\mathbb{N}) \otimes L^2(M))$. It can be shown that two such right M-modules are isomorphic if and only if the associated projections are equivalent in $B(l^2(\mathbb{N}))\overline{\otimes}M$. In fact, all (separable) right M-modules are isomorphic to $p((l^2(\mathbb{N}) \otimes L^2(M))$ for some p, which establishes a bijective correspondence between right M-modules and equivalence classes of projections in $B(l^2(\mathbb{N}))\overline{\otimes}M$. The situation for left modules is completely analogous.

Using the preceding classification, we can now define a notion of size (or dimension) of a module. Recall that $B(l^2(\mathbb{N}))$ has a canonical semifinite trace given by Tr. The tensor product of these traces $\mathrm{Tr} \otimes \tau$ gives a faithful, normal semifinite trace on $B(l^2(\mathbb{N}))\overline{\otimes}M$. We define the dimension of a module as follows:

Definition 3.3.7 (Dimension of a left/right M-module) Let (M, τ) be a tracial von Neumann algebra. We define the *M-dimension* of a right (resp. left) M-module H to be $\dim_M(H) := (\mathrm{Tr} \otimes \tau)(p)$, where p is any projection in $B(l^2(\mathbb{N}))\overline{\otimes}M$ (resp. $B(l^2(\mathbb{N}))\overline{\otimes}\rho(M)$) such that $p((l^2(\mathbb{N}) \otimes L^2(M))$ is isomorphic to H as M-modules.

Remark 3.3.8 Unfortunately, the above definition of dimension for a module H is not intrinsic to the module structure. It depends on the choice of the faithful trace τ, which is not unique except in the case that M is a II_1 factor.

Example 3.3.9 (Dual Bimodule)

Given a bimodule ${}_MH_N$, the conjugate Hilbert space $\overline{H}$ is canonically an N-M bimodule via the following action:

$$y\bar{\xi}x = \overline{x^*\xi y^*},$$

for $x \in M, y \in N$. We call ${}_N\overline{H}_M$ the *dual* or *contragredient* bimodule of ${}_MH_N$. The identity bimodule ${}_ML^2(M)_M$ is self-dual in the sense that ${}_M\overline{L^2(M)}_M$ is isomorphic to ${}_ML^2(M)_M$ as M-M-bimodules.

Before proceeding to the discussion of composition (or fusion tensor products) of bimodules, we need to define left- and right-boundedness of vectors in a module.

Definition 3.3.10 (Left- and Right-Bounded Vectors) Let ${}_MH_N$ be an M-N-bimodule. A vector $\xi \in H$ is called *left M-bounded* (resp. *right N-bounded*) if the map $x \mapsto x\xi$ (resp. $y \mapsto \xi y$) extends to a bounded operator $R_\xi : L^2(M) \to {}_MH$ ($L_\xi : L^2(N) \to H_N$).

The collection of all left-bounded vectors is denoted ${}_MH^\circ$, and similarly H_N° for right-bounded vectors. Popa showed in [Pop86] that ${}_MH^\circ \cap H_N^\circ$ is dense in ${}_MH_N$.

Now let $\xi, \eta \in H_N^\circ$. One can check that $L_\eta^* L_\xi : L^2(N) \to L^2(N)$ commutes with the right N-action. Since the commutant of the right N-action on the trivial bimodule is simply the left N-action, $L_\eta^* L_\xi \in N$. We will write $\langle \eta, \xi \rangle_N$ for $L_\eta^* L_\xi$. Clearly $\langle \eta, \xi \rangle_N^* = \langle \xi, \eta \rangle_N$. On H_N°, $\langle \cdot, \cdot \rangle_N$ defines an N-valued inner product which is right N-linear in the second variable, i.e.

$$\langle \eta, \xi_1 x + \xi_2 \rangle_N = \langle \eta, \xi_1 \rangle_N x + \langle \eta, \xi_2 \rangle_N$$

for all $x \in N$, $\xi_1, \xi_2, \eta \in H_N^\cdot$. One verifies that $\tau(\langle \xi, \xi \rangle_N) = \|\xi\|^2$, so that $\langle \cdot, \cdot \rangle_N$ is in fact positive definite.

Similar considerations hold for $R_\eta^* R_\xi$ with $\xi, \eta \in {}_M H^\circ$. In this case, $R_\eta^* R_\xi$ defines an element of $M' = JMJ$, so one conjugates by J to get an element of M. Because of the anti-linearity of J, $\langle \cdot, \cdot \rangle_M$ is M-linear in the first variable.

Example 3.3.11 (Connes' Fusion Tensor Product) Suppose we have a right M-module H_M and a left M module ${}_M K$. We define the *fusion* or *relative* tensor product $H \otimes_M K$ as follows. Consider the algebraic tensor product $H_M^\circ \otimes K$ (note the restriction to right-bounded vectors in H), together with the pre-inner product defined by extending the relation

$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle_{H_M^\circ \otimes K} := \langle \xi_1, \langle \eta_1, \eta_2 \rangle_M \eta_2 \rangle_K.$$

Just as in the GNS construction, we define $H \otimes_M K$ to be the completion of the separation (i.e. the quotient by the ideal I of zero-length vectors) of $H_M^\circ \otimes K$ with respect to this pre-inner product. Denote by $\xi \otimes_M \eta$ the image of $\xi \otimes \eta \in H_M^\circ \otimes K$ after separation and completion.

Note that for any $x \in M$, we have $\xi x \otimes_M \eta = \xi \otimes_M x\eta$. To see this, we compute the length of their difference $\xi x \otimes_M \eta - \xi \otimes_M x\eta$ with respect to the pre-inner product. For readability, we suppress the subscripts for the scalar-valued inner products.

$$\begin{aligned} \|\xi x \otimes_M \eta - \xi \otimes_M x\eta\|_{H_M^\circ \otimes K}^2 &= \langle \xi x \otimes_M \eta - \xi \otimes_M x\eta, \xi x \otimes_M \eta - \xi \otimes_M x\eta \rangle \\ &= \langle \eta, \langle \xi x, \xi x \rangle_M \eta \rangle - \langle \eta, \langle \xi x, \xi \rangle_M x\eta \rangle \\ &\quad - \langle x\eta, \langle \xi, \xi x \rangle_M \eta \rangle + \langle x\eta, \langle \xi, \xi \rangle_M x\eta \rangle \\ &= 0, \end{aligned}$$

where the last equality follows from the M-linearity of the M-valued inner product.

One can check that any left L-action on H_M and right N-action on ${}_MK$ descend from $H_M^\circ \otimes {}_MK^\circ$ (equipped with the action from the coarse bimodule) to actions on $H \otimes_M K$. In this way, the fusion of ${}_LH_M$ and ${}_MK_N$ yields an L-N-bimodule ${}_LH \otimes_M K_N$. We note that if $M = \mathbb{C}$, then ${}_LH \otimes_M K_N$ is simply the coarse bimodule ${}_LH \otimes K_N$.

Remark 3.3.12 In the above definition, we used left-bounded vectors from H to define the inner product on $H_M^\circ \otimes {}_MK$. One could have instead used the M-valued inner product coming from right-bounded vectors in K, yielding a similarly defined pre-inner product on $H_M \otimes_M K^\circ$. By Proposition 12.2.2 in [AP17], these pre-inner products coincide on $H_M^\circ \otimes_M K^\circ$ and the above construction applied to any of these three spaces yields the same Hilbert space (or L-N-bimodule) $H \otimes_M K$.

The relative tensor product satisfies the following properties:

Lemma 3.3.13 *Let ${}_LH_M$, ${}_MK_N$, and ${}_NQ_P$ be bimodules. The following are canonically isomorphic (as bimodules):*

(i) (dual) $\overline{K} \otimes_M \overline{H} \simeq \overline{H \otimes_M K}$

(ii) (associativity) $H \otimes_M (K \otimes_N Q) \simeq (H \otimes_M K) \otimes_N Q$

(iii) (identity) $L^2(M) \otimes_M H \simeq H \simeq H \otimes_N L^2(N)$

For proofs of the above, see [AP17], [Con94]. We will see another justification for (ii) and (iii) when we discuss the relationship between bimodules and completely positive maps, where the fusion tensor product corresponds to composition of maps. For this reason, the fusion tensor product is sometimes called the *composition* of H_M and ${}_MK$.

We conclude this section with a final example of a common M-M bimodule, coming from the basic construction.

Example 3.3.14 (Jones' Basic Construction)

Let $N \subset (M, \tau)$ be a unital inclusion of tracial von Neumann algebras. There is a corresponding natural inclusion of $L^2(N, \tau|_N) \subset L^2(M, \tau)$. Let e_N be the orthogonal projection from $L^2(M)$ onto $L^2(N)$, viewed as an operator in $B(L^2(M))$. The *basic construction* of the inclusion $N \subset M$ is (the inclusion of M into) the algebra $\langle M, e_N \rangle$, the von Neumann subalgebra of $B(L^2(M))$ generated by M and e_N. Because compression by e_N implements the conditional expectation onto N, we have $e_N x e_N = E_N(x) e_N$ for all $x \in M$, where E_N denotes the conditional expectation. Therefore, using the fact that the central support of e_N in $\langle M, e_N \rangle$ is 1, we see that $\langle M, e_N \rangle$ is generated (as a von Neumann algebra) by elements of the form $x e_N y$ with $x, y \in M$.

The basic construction comes equipped with a faithful, normal semifinite trace Tr defined on the generators $x e_N y$ as follows:

$$\mathrm{Tr}(x e_N y) = \tau(xy),$$

for all $x, y \in M$.

Concerning the basic construction, we will occasionally make use of the following theorem. The map described below was first defined by Popa (see [Pop95]). For a more complete discussion of the following theorem and its proof, see Theorem 4.5.3 in [SS08].

Theorem 3.3.15 *(The Pull-down Map) Let $N \subset (M, \tau)$ a unital inclusion. There is an M-M-bimodular map $T_M : L^1(\langle M, e_N \rangle, Tr) \to L^1(M, \tau)$ such that*

$$T_M(x e_N y) = xy$$

for all $x, y \in M$ and satisfying the following conditions for all $X \in L^1(\langle M, e_N \rangle)$:

(i) T_M respects $*$ and the respective traces: $\tau(T_M(X)) = Tr(X)$

(ii) $e_N(T_M(e_N X)) = e_N X$

(iii) $\|T_M(X)\|_1 \leq \|X\|_1$ (where each 1-norm comes from the appropriate trace)

(iv) T_M maps $e_N M$ onto M isometrically for the 2-norm (i.e. $\|T_M(e_N X)\|_2 = \|X\|_2$).

We will call T_M the *pull-down map*. It can equivalently be described as the unique faithful, normal operator-valued weight T_M from the basic construction $\langle M, e_N \rangle$ to M satisfying condition (i) above.

3.3.2 Correspondence Between Bimodules and Completely Positive Maps

We have already examined the relationship between cyclic representations (H, ξ) with designated cyclic vector ξ and states ϕ, where $\phi = \langle \xi, \cdot \xi \rangle$ is the recipe for producing a state from the former, and the GNS representation gives the recipe for producing a cyclic representation from the latter. In this section, we discuss the generalization of this correspondence to that of cyclic bimodules and completely positive maps.

3.3.2.1 From Cyclic Bimodules to Completely Positive Maps

Definition 3.3.16 (Cyclic M-N-Bimodule) An M-N-bimodule ${}_M H_N$ is *cyclic* if there is a vector $\xi \in H$ such H is densely spanned by $M\xi N$. We call ξ a *cyclic vector.*

Given a cyclic bimodule $({}_M H_N, \xi)$ with ξ left N-bounded, recall from the previous section that we defined $L_\xi : L^2(N) \to H$ to be the extension of the map $y \mapsto \xi y$, and we saw that $\langle \eta, \xi \rangle_N := L_\eta^* L_\xi \in N$ for any left-bounded $\xi, \eta \in H$. It's easy to see that, for any $x \in M$ and left N-bounded ξ, $x\xi$ is still left-bounded, and $L_{x\xi} = xL_\xi$. So, given such a cyclic bimodule, we define for $x \in M$:

$$\Phi(x) := \langle \xi, x\xi \rangle_N = L_\xi^* x L_\xi.$$

From the last term, it's clear (using Example 3.1.4) that we've defined a completely positive map $\Phi : M \to N$.

Remark 3.3.17 We note the similarity between the vector state recipe $\phi(x) = \langle \xi, x\xi \rangle$ and the above definition of $\Phi(x) = \langle \xi, x\xi \rangle_N$. It's easy to see that the notion of cyclicity defined above corresponds, in the case $N = \mathbb{C}$, to a cyclic representation of M and that $\Phi = \phi$ in this case.

3.3.2.2 From Completely Positive Maps to Bimodules

Now let $\Phi : M \to N$ be a normal, (unital) completely positive map between tracial von Neumann algebras. Using Φ, we can define a pre-inner product on $M \otimes L^2(N)$ satisfying:

$$\langle y \otimes \eta, x \otimes \xi \rangle := \langle \eta, \Phi(y^* x)\xi \rangle,$$

where the latter inner product is that of $L^2(N)$, coming from its trace. Complete positivity of Φ gives positive semidefiniteness, and we will denote by H_Φ (or simply H) the separation and completion of $M \otimes L^2(N)$ with respect to this pre-inner product. We note that H_Φ is a Stinespring dilation for Φ.

We will again abuse notation and write $x \otimes \xi$ for its image in H. Then H becomes an M-N-bimodule with the actions satisfying

$$x(y \otimes \xi)z = xy \otimes \xi z$$

for all $x, y \in M$, $\xi \in L^2(N)$, and $z \in N$.

As in the GNS representation, we can recover the completely positive map Φ from the cyclic vector $\widehat{\xi} = 1 \otimes \widehat{1}$. It is easy to check that $\widehat{\xi}$ is left N-bounded, and we have $\Phi(x) = \langle \widehat{\xi}, x\widehat{\xi} \rangle_N$. It follows that if Φ is trace-preserving, then $\langle \cdot \widehat{\xi}, \widehat{\xi} \rangle$ and $\langle \widehat{\xi} \cdot, \widehat{\xi} \rangle$ (the scalar-valued inner products) are the traces on M and N respectively.

3.3.2.3 More Bimodule Examples

We can now justify some of the previous nomenclature and give examples of important bimodules and their corresponding completely positive maps.

Example 3.3.18 (Identity/Homomorphisms) Using the construction of the previous section, it's easy to see that if $\pi : M \to N$ is a $*$-homomorphism, then the corresponding bimodule H_π is isomorphic to ${}_{\pi(M)}L^2(N)_N$, where the left M-action is twisted by π (i.e. $x \cdot \widehat{y} = \widehat{\pi(x)y}$) for $x \in M$ and $y \in N$, and the right action is the usual one. In particular, an automorphism α of M corresponds to ${}_{\alpha(M)}L^2(M)_M$, and the identity automorphism corresponds to the identity bimodule. If $\pi : M \to N$ is an isomorphism, one can check that the contragredient bimodule satisfies $\overline{H_\pi} \simeq H_{\pi^{-1}}$.

Example 3.3.19 (Conditional Expectations) Let $E_N : M \to N \subset M$ be a conditional expectation, regarded as a completely positive map from M to itself. Then $x \otimes y \mapsto xe_N y$ induces an isomorphism between $H_{E_N} \simeq L^2(M) \otimes_N L^2(M)$ and $L^2(\langle M, e_N \rangle)$ as M-M-bimodules, where e_N is the Jones projection corresponding to the inclusion $N \subset M$. In particular, $\tau : M \to \mathbb{C}$ corresponds to the coarse bimodule $L^2(M) \otimes L^2(M)$, and we again see that $\mathrm{id} : M \to M$ corresponds to $L^2(M) \otimes_M L^2(M) \simeq L^2(M)$.

Example 3.3.20 (Abelian Case) Recall that any abelian von Neumann algebra on a separable Hilbert space can be written as $L^\infty(X, \mu_X)$ for some standard measure space (X, μ_X). (In fact, we may take $X = [0, 1]$.) Now let $A \simeq L^\infty(X, \mu_X)$, $B \simeq L^\infty(Y, \mu_Y)$ be abelian algebras. Then any A-B-bimodule is isomorphic to one of the form $H = \int H_{(x,y)} d\mu(x, y)$, where μ is a measure on $X \times Y$ whose marginals are absolutely continuous with respect to μ_X, μ_Y and with $n(x, y) := \dim H_{(x,y)}$ μ-measurable. The actions are given by

$$f \cdot \xi(x, y) \cdot g = f(x)g(y)\xi(x, y)$$

for all $f \in A$, $g \in B$, $\xi \in H$. See [Con94] Appendix V.B for more details.

We now restrict our attention to A-A-bimodules with $A \simeq L^\infty(X, \nu)$. Given a measure η on $X \times X$ whose marginals are absolutely continuous with respect to ν, we may define a completely positive map $\widehat{\eta} : A \to A$ by:

$$\widehat{\eta}(f)(x) = \int f(y)\eta(x, y).$$

More precisely, $\widehat{\eta}(f)$ is the element of A satisfying $\int_X \widehat{\eta}(f)(x)g(x)d\nu = \int_{X\times X} f(y)g(x)d\eta$ for all $g \in A$. Conversely, every A-A-bimodule is of this form for some η and multiplicity function n. See [Shl99] Examples 2.8, 3.4 for more details.

In terms of the construction from the previous section, the completely positive map $\widehat{\eta}$ corresponds to $H_{\widehat{\eta}} \simeq L^2(X \times X, \eta)$ with the left and right A-actions being multiplication by $f(x)$ and $g(y)$ respectively and with $1_{X\times X}$ being the associated cyclic vector.

In particular, with $n \equiv 1$, the identity bimodule corresponds to the η given by the push-forward of ν under the diagonal embedding $x \mapsto (x, x)$, and the coarse bimodule corresponds to the product measure $\eta = \nu \times \nu$. More generally, with A and B as above, the product measure on $X \times Y$ corresponds to the coarse bimodule $L^2(A) \otimes L^2(B)$.

Example 3.3.21 (Composition) Let $\Phi : M \to N$, $\Psi : N \to P$ be normal and completely positive. Note that $\Psi \circ \Phi : M \to P$ is again completely positive. The following are isomorphic as M-P-bimodules:

$$H_{\Psi\circ\Phi} \simeq H_\Phi \otimes_N H_\Psi.$$

Thus, the fusion tensor product of bimodules corresponds to the composition of completely positive maps. See Proposition 17 in Appendix V.B from Connes' [Con94] for a proof.

3.3.3 A-valued Semicircular Families

3.3.3.1 Construction and Definitions

Before proceeding, we need the notion of an A-A bimodule with an A-valued inner product. More precisely:

Definition 3.3.22 (Hilbert-A-bimodule) Let A be a von Neumann algebra (or C*-algebra), and let K be an A-A-bimodule (in ring theoretic sense, i.e. K is not required to be a Hilbert space). We say that a $\mathbb{C}$-sesquilinear form $\langle\cdot,\cdot\rangle_A : K \times K \to A$ is an A-*valued inner product* if it satisfies

$$\langle a\xi, \psi b\rangle_A = \langle \xi, a^*\psi\rangle_A b$$
$$\langle \xi, \psi\rangle_A^* = \langle \psi, \xi\rangle_A$$
$$\text{and } \langle \xi, \xi\rangle_A \geq 0$$

for all $a, b \in A$ and $\xi, \psi \in K$. The inner product is called *non-degenerate* if $\langle \xi, \xi\rangle_A > 0$ whenever $\xi \neq 0$.

If K is equipped with a (possibly degenerate) A-valued inner product, we call K a *Hilbert-A-bimodule.*

Note that a non-degenerate Hilbert-$\mathbb{C}$-bimodule is just a (pre-)Hilbert space. One can compose the A-valued inner product with a state on A to produce a scalar-valued pre-inner product. For more details on these bimodules, see e.g. [Spe98] Chapter IV.

Now let A be a semifinite von Neumann algebra. Throughout this section, we use $[n]$ to denote the set $\{1, 2, ..., n\}$ if $n \in \mathbb{N}$ and $[n] = \mathbb{N}$ if n is infinite. Now let $n \in \mathbb{N} \cup \{\infty\}$ and suppose we're given a family of maps $\eta_{ij} : A \to A$ with $i, j \in [n]$ such that the associated map $\eta : A \to A \otimes B(H)$ is normal and completely positive. Here, H is a Hilbert space of dimension n (separable if $n = \infty$), and η_{ij} are the entrywise components of the map η, where we view $A \otimes B(H)$ as matrices with A-valued entries.

We now describe Shlyakhtenko's construction of an A-valued semicircular family of covariance η, following [Shl99]. Given an η as above, Shlyakhtenko showed that there is a unique A-A bimodule K, spanned as a bimodule by vectors ξ_i ($i \in [n]$), with an A-valued inner product satisfying:

$$\langle \xi_i, a\xi_j \rangle_A = \eta_{ij}(a)$$

for all $a \in A$.

Using this bimodule K, we mimic the construction of the full Fock space but using the Connes fusion tensor product for A-bimodules in place of the usual tensor product. We obtain the following Fock space bimodule:

$$\mathcal{F}(K) = A \oplus K \oplus (K \otimes_A K) \oplus ... \oplus K^{\otimes_A m} \oplus ...$$

We will sometimes write $\mathcal{F}$ instead of $\mathcal{F}(K)$ if K is clear from the context. We define another A-valued inner product on $\mathcal{F}(K)$ by linearly extending the following relations:

$$\langle a, b \rangle_{\mathcal{F}} = a^* b$$

$$\langle \xi_1 \otimes_A ... \otimes_A \xi_j, \psi_1 \otimes_A ... \otimes_A \psi_k \rangle_{\mathcal{F}} = \delta_{jk} \langle \xi_j, \langle \xi_{j-1}, ..., \langle \xi_1, \psi_1 \rangle_A ... \psi_{j-1} \rangle_A \psi_j \rangle_A$$

for all $a, b \in A$, $\xi_i, \psi_i \in K$. One can check that this defines a (possibly degenerate) Hilbert-A-module structure on $\mathcal{F}(K)$. See [Spe98] 4.6 for details.

For $\xi \in K$, we consider the left creation operator $l(\xi) : \mathcal{F} \to \mathcal{F}$ defined by:

$$l(\xi) 1_A = \xi$$

$$l(\xi)(\xi_1 \otimes_A \xi_2 \otimes_A ... \otimes_A \xi_n) = \xi \otimes_A \xi_1 \otimes_A \xi_2 \otimes_A ... \otimes_A \xi_n.$$

Here, we note that $1_A \in \mathcal{F}(K)$ lives in the first summand of the above decomposition of $\mathcal{F}(K)$ and takes the place of the usual vacuum vector Ω of the full Fock space.

As described in [Shl99], $l(\xi)$ has an adjointable (with respect to the A-valued inner product) extension, which we still denote by $l(\xi)$. This extension and its adjoint, the left annihilation operator $l(\xi)^*$, satisfy the following relations for all $\xi, \psi \in K$ and $a, b \in A$:

$$l(\xi)^* l(\psi) = \langle \xi, \psi \rangle_A$$
$$a l(\xi) b = l(a\xi b).$$

Now let ϕ be a faithful, normal state on A. By composing the natural A-valued inner product of $\mathcal{F}(K)$ with ϕ, we get a pre-inner product on $\mathcal{F}$. By separating and completing with respect to this inner product, we obtain a Hilbert space $\mathcal{F}_\phi$. The actions of A and the creation/annihilation operators on $\mathcal{F}$ descend to actions on $\mathcal{F}_\phi$, satisfying the same relations as above.

Matching the notation used in [Shl99], we define $\Phi_\phi(A, \eta)$ to be the von Neumann algebra in $B(\mathcal{F}_\phi)$ generated by A and $\{X_i : i \in [n]\}$, where $X_i = s(\xi_i) := l(\xi_i) + l(\xi_i)^*$. We will call $\Phi_\phi(A, \eta)$ the von Neumann algebra generated by a family of A-valued semicirculars of covariance η. We note that the compression of $\Phi_\phi(A, \eta)$ by the projection onto the Hilbert space closure of A in $\mathcal{F}_\phi$ gives a normal conditional expectation $E_\phi : \Phi_\phi(A, \eta) \to A$. Finally, Shlyakhtenko showed that $\Phi_\phi(A, \eta)$ and E_ϕ do not depend on the particular choice of faithful, normal state ϕ in the following sense: if ψ is another faithful, normal state on A, there's an isomorphism of $\Phi_\phi(A, \eta)$ with $\Phi_\psi(A, \eta)$ which intertwines the expectations. So in what follows we will suppress ϕ from the notation, writing instead $\Phi(A, \eta)$ and $E : \Phi(A, \eta) \to A$ for the von Neumann algebra generated by a family of A-valued semicirculars of covariance η with its expectation onto A.

Lastly, we mention that $\Phi(A, \eta)$ encodes η in the following way:

$$E(X_i a X_j) = \langle \xi_i, a\xi_j \rangle_A = \eta_{i,j}(a).$$

For this reason and because X_i is distributed as a semicircular element with respect to $\phi \circ E$, we will call $\{X_i : i \in [n]\}$ a *semicircular family of covariance* η.

Remark 3.3.23 (Scalar-valued Case) Here, we briefly describe the specialization to $A = \mathbb{C}$ in the construction above. The requisite input data $\eta : \mathbb{C} \to \mathbb{C} \otimes B(H) = B(H)$ is determined by $\eta(1)$, which can be taken to be any positive element of $B(H)$. The associated $\mathbb{C}$-$\mathbb{C}$ bimodule H is simply a Hilbert space. If η is given in terms of component functions η_{ij}, then we get a corresponding basis (ξ_i) of H whose Gram matrix is $\eta(1)$. The canonical expectation E onto A in this case corresponds to the vector state ω associated to the vacuum vector $\Omega \in \mathcal{F}(H)$. In particular, if $\{\xi_i : i \in [n]\}$ is an orthonormal basis for H, then $\Phi(\mathbb{C}, \eta) = s(H) = \{s(\xi_i) : i \in [n]\}''$ is the von Neumann subalgebra of $B(\mathcal{F}(H))$ corresponding to the free Gaussian functor on H with faithful, normal trace state ω. The completely positive map η in the above construction is given in this case by:

$$\eta_{ij} = \langle \xi_i, \xi_j \rangle = \delta_{ij}.$$

3.3.3.2 Freeness with Amalgamation and Examples

Earlier, we saw that the orthogonality of ξ, η in H corresponds to $l(\xi), l(\eta)$ being free. In other words, direct summands of a Hilbert space correspond to freely independent algebras under the free Gaussian functor. Here, we mention the A-valued analogue of this result and give a couple of related examples.

Definition 3.3.24 (Vacuum Expectation) Let K be a Hilbert-A-module. Let $l(K)$ denote the $*$-algebra generated by A and $\{l(\xi) : \xi \in K\}$ on $\mathcal{F}(K)$. We define the *vacuum expectation* on $l(K)$ by:

$$\omega_A(x) := \langle 1_A, x 1_A \rangle_{\mathcal{F}}.$$

Remark 3.3.25 Note that here 1_A is taking the place of the usual vacuum vector Ω, and the expectation is A-valued. It's easy to see that this expectation corresponds (after separation and completion) to the conditional expectation $E : \Phi(A, \eta) \to A$ of the previous section. Thus, to establish the A-valued analogue of the earlier orthogonality result, it's enough to show the following (we mimic the proof of Theorem 4.6.15 in [Spe98]).

Lemma 3.3.26 *Let K_1 and K_2 be Hilbert-A-bimodules. Then $l(K_1)$ is free with amalgamation over A from $l(K_2)$ in $(l(K_1 \oplus K_2), \omega_A)$, i.e. if $x_j \in l(K_{i_j})$ with $i_j \neq i_{j+1}$ and $\omega_A(x_j) = 0$ for all j, then $\omega_A(x_1 ... x_n) = 0$.*

Proof. Let x_j be as above. Since $\omega_A(x_n) = 0$, we have

$$x_n 1_A \in \oplus_{m \in \mathbb{N}_+} K_{i_n}^{\otimes_A m} \subset \mathcal{F}(K_1 \oplus K_2) \ominus A.$$

Similarly, we have a corresponding inclusion for x_{n-1}. Since $i_n \neq i_{n-1}$, we get:

$$x_{n-1} x_n 1_A \in \left(\oplus_m K_{i_{n-1}}^{\otimes_A m}\right) \otimes_A \left(\oplus_m K_{i_n}^{\otimes_A m}\right) \subset \mathcal{F}(K_1 \oplus K_2) \ominus A.$$

Continuing inductively, we get

$$x_1 ... x_n 1_A \in \left(\oplus_m K_{i_1}^{\otimes_A m}\right) \otimes_A ... \otimes_A \left(\oplus_m K_{i_n}^{\otimes_A m}\right) \subset \mathcal{F}(K_1 \oplus K_2) \ominus A.$$

Thus, $x_1 ... x_n 1_A \perp A$, and $\omega_A(x_1 ... x_n) = 0$ as desired.

It follows from this lemma that direct sums of A-A-bimodules correspond to algebras of creation/annihilation operators which are free with amalgamation over A. Specializing $A = \mathbb{C}$ recovers the earlier scalar-valued case.

We now give a couple of corollaries concerning the A-valued semicircular systems of the previous subsection. See [Shl99] for more details/examples.

Example 3.3.27 Suppose η_{ij} (the coefficiencts of the completely positive map $\eta : A \to A \otimes B(H)$) form a block diagonal matrix (i.e. $\eta_{ij} \equiv 0$ off the block diagonal). If the blocks are enumerated by i and η_i denotes the restriction of η to the ith block, then it's easy to see that the associated $K_\eta = \oplus_i K_{\eta_i}$ as A-A-bimodules, so the above gives

$$\Phi(A, \eta) \simeq *_A(\Phi(A, \eta_i)).$$

Example 3.3.28 Let $A = L^\infty(X, \mu)$. By Example 3.3.20, we can obtain a completely positive map $\eta : A \to A$ from a finite positive measure ν on $X \times X$. If $\nu = \nu_1 + \nu_2$, with ν_1 disjointly supported from ν_2, then we have $K_\nu = K_{\nu_1} \oplus K_{\nu_2}$, and

$$\Phi(A, \eta_\nu) \simeq \Phi(A, \eta_{\nu_1}) *_A \Phi(A, \eta_{\nu_2}).$$

Example 3.3.29 Let $\eta = E_B : A \to B$ be a conditional expectation. As we've seen, the associated A-A-bimodule is $L^2(\langle A, e_B \rangle)$, with $\xi = e_B$ implementing η, i.e. $\langle e_B, a e_B \rangle_A = E_B(a)$.

In this case, one can show (see [Shl99] Example 3.3) that

$$\Phi(A, \eta) = \Phi(A, E_B) \simeq (L^\infty[0,1] \otimes B) *_B A.$$

In particular, if $\eta = \phi$ is a state on A,

$$\Phi(A, \phi) \simeq L^\infty[0,1] * A,$$

and if $\eta = \mathrm{id}_A$,

$$\Phi(A, \mathrm{id}) \simeq L^\infty[0,1] \otimes A.$$

3.3.4 Intertwining by Bimodules

In [Pop06a], [Pop06b], Popa introduced the following characterization, which will be useful in studying the relative position of two subalgebras A and B of an ambient von Neumann algebra M. We outline Popa's proof of the characterization below. More details can be found in [Pop06b] Section 2.

Theorem 3.3.30 *(Popa)*

Let (M, τ) be a tracial von Neumann algebra with von Neumann subalgebras $A, B \subset M$ satisfying $1_A \leq 1_M$, $1_B = 1_M$ (so that $A \subset M$ is not necessarily a unital inclusion). Then the following are equivalent:

(i) There is no sequence of unitaries $(u_k)_{k \in \mathbb{N}}$ in A such that

$$\lim_{k \to \infty} \|E_B(x^* u_k y)\|_2 = 0$$

for all $x, y \in 1_A M$, where $\|x\|_2 := \tau(x^* x)^{1/2}$.

(ii) There is a nonzero subbimodule ${}_A H_B \subset_A L^2(1_A M)_B$ of finite dimension over B, i.e.

$$\dim_B(H_B) < \infty.$$

(ii') There exists a projection $0 \neq p \in A' \cap 1_A \langle M, e_B \rangle_+ 1_A$ with $Tr(p) < \infty$, where Tr is the canonical semifinite trace on $\langle M, e_B \rangle$.

(iii) There exist $n \geq 1$, a homomorphism $\pi : A \to M_n(B)$ (possibly non-unital), and a partial isometry $0 \neq v \in M_{1,n}(1_A M)$ such that for all $x \in A$:

$$xv = v\pi(x).$$

If $A, B \subset M$ satisfy the above conditions, then we write $A \preceq_M B$ and say that A *embeds into B inside M.*

Proof. First, we note that (ii) and (ii') are easily seen to be equivalent: Given ${}_A H_B$ as in (ii), let p be the projection $p : L^2(1_A M) \to H$. Since H is a right B-module, p commutes with the right B action, i.e. $p \in \langle M, e_B \rangle_+$. Furthermore, $\mathrm{Tr}(p) = \dim_B(H) < \infty$. Similarly, since H is a left A-module, $p \in A'$. In particular, $p = 1_A p = p 1_A$. Putting all of this together, we see that $0 \neq p \in A' \cap 1_A \langle M, e_B \rangle_+ 1_A$. Conversely, any such projection commutes with the left A- and right B-actions, and therefore takes as its range a subbimodule of finite dimension over B.

Now we show (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

(i) $\implies$ (ii'): Since no sequence of unitaries of the above form exists, we can find an $\epsilon > 0$ and $F \subset 1_A M$, a finite subset such that $\|E_B(x^* u y)\|_2^2 \geq \epsilon$ for all $x, y \in F$, $u \in \mathcal{U}(A)$. Now let $x = \sum_{y \in F} y e_B y^* \in 1_A \langle M, e_B \rangle_+ 1_A$, and let $C \subset 1_A \langle M, e_B \rangle_+ 1_A$ denote the ultraweakly-closed convex hull of $\{uxu^* : u \in \mathcal{U}(A)\}$. Then, for any $c' \in C$, since $\mathrm{Tr}(x) = \sum_{y \in F} \tau(yy^*) < \infty$, we have $\|c'\|_2 \leq \|x\|_2 < \infty$. (Note: the $\|\cdot\|_2$ norm here is with respect to Tr in the basic construction.) So C is closed and bounded in $L^2(\langle M, e_B \rangle)$.

Now let c be the unique element of smallest $\|\cdot\|_2$ norm in C. By the uniqueness of c and the fact that $\|ucu^*\|_2 = \|c\|_2$ for all $u \in \mathcal{U}(A)$, we see that $c \in A'$. Now note that for any $u \in \mathcal{U}(A)$, we have:

$$\begin{aligned}
\sum_{y \in F} \mathrm{Tr}(e_B y^* (u^* x u) y e_B) &= \sum_{y,z \in F} \mathrm{Tr}(e_B (z^* u y)^* e_B (z^* u y) e_B) \qquad (\text{since } x = \sum_{z \in F} z e_B z^*) \\
&= \sum_{y,z \in F} \tau(E_B(z^* u y)^* E_B(z^* u y)) \\
&= \sum_{y,z \in F} \|E_B(z^* u y)\|_2^2 \geq \epsilon.
\end{aligned}$$

Taking convex combinations, we see that $\sum_{y\in F} \mathrm{Tr}(e_B y^* c' y e_B) \geq \epsilon$ for all $c' \in C$, in particular for c. So $0 \neq c \in A' \cap 1_A\langle M, e_B\rangle_+ 1_A$. Finally, if we take p to be a spectral projection of the form $1_{[\delta, \|c\|]}(c)$ with $0 < \delta < \|c\|$, then p satisfies the conditions of (ii').

(ii) $\implies$ (iii): By the classification of right B-modules, there exists an $n \geq 1$, a projection $p \in M_n(B)$, and an isomorphism of right B-modules $\psi : p(\mathbb{C}^n \otimes L^2(B)) \to {}_AH_B$. The left action of A on H induces a $*$-homomorphism $\pi : A \to pM_n(B)p$ via ψ defined by

$$x\psi(\xi) = \psi(\pi(x)\xi)$$

for all $x \in A, \xi \in p(\mathbb{C}^n \otimes L^2(B))$. We will write $\pi(x) = (\pi_{i,j}(x))_{i,j\in\{1,\ldots,n\}})$ for the component entries in $M_n(B)$.

We now let $e_i \in \mathbb{C}^n \otimes L^2(B)$ denote the vector $(0, ..., 0, \widehat{1_B}, 0, ..., 0)$ with $\widehat{1_B}$ in its ith entry. Consider the vector $\xi = (\psi(pe_1), ..., \psi(pe_n)) \in \mathbb{C}^n \otimes H$. We aim to show that $x\xi = \xi\pi(x)$ for all $x \in A$. So let $x \in A$.

In the jth component, we have:

$$\begin{aligned}
(x\xi)_j &= x\psi(pe_j) \\
&= \psi(\pi(x)pe_j) = \psi(p(\pi(x)e_j)) = \psi(p(\sum_{i=1}^{n} e_i \pi_{i,j}(x))) \\
&= \sum_i \psi((pe_i)\pi_{i,j}(x)) \\
&= \sum_i \psi(pe_i)\pi_{i,j}(x) \quad \text{(since } \psi \text{ respects the right } B \text{ action)} \\
&= (\xi\pi(x))_j.
\end{aligned}$$

So $x\xi = \xi\pi(x)$.

Now consider the operator in $1_{A^{n+1}} M_{n+1}(M) 1_{A^{n+1}} \subset B(\mathbb{C}^{n+1} \otimes L^2(M))$ given by:

$$X = \begin{pmatrix} x & 0 \\ 0 & \pi(x) \end{pmatrix},$$

and consider the vector

$$\eta = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \in 1_{A^{n+1}}(\mathbb{C}^{n+1} \otimes L^2(M)) 1_{A^{n+1}}.$$

Because $x\xi = \xi\pi(x)$, we have

$$\begin{pmatrix} 0 & x\xi \\ 0 & 0 \end{pmatrix} = X\eta = \eta X = \begin{pmatrix} 0 & \xi\pi(x) \\ 0 & 0 \end{pmatrix}.$$

Let Y_η be the unbounded operator affiliated with $M_{n+1}(M)$ that corresponds to $\eta \in 1_{A^{n+1}}(\mathbb{C}^{n+1} \otimes L^2(M))1_{A^{n+1}}$, with polar decomposition $Y_\eta = V|Y_\eta|$. Note that Y_η (and therefore V) depends only on ξ and not on the choice of $x \in A$. Then V is a partial isometry in $1_{A^{n+1}}M_{n+1}(M)$ that commutes with X (because $X\eta = \eta X$).

If we write

$$V = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$$

to be compatible with the matrix description of X, then $v \in M_{1,n}(1_A M)$, and a quick computation shows that $XV = VX$ implies $xv = v\pi(x)$. A similar computation verifies that v is again a partial isometry (from $\ker z$ to $\ker u^*$), and, in particular, $v \neq 0$. Since v doesn't depend on x, we've found a partial isometry v and homomorphism π which satisfy the requirements of (iii).

(iii) $\implies$ (i): Suppose for contradiction that we can find a sequence $(u_k) \subset \mathcal{U}(A)$ satisfying the conditions in (i). Then a component-wise computation yields that

$$\|(\mathrm{id}_{M_n(\mathbb{C})} \otimes E_B)v^*u_k v\|_2 \to_{k\to\infty} 0.$$

On the other hand, since $u_k \in A$, using the intertwining relation in (iii), we see that $v^*u_k v = \pi(u_k)v^*v$. Let $p = \pi(1_A)$. Then $\pi(u_k) \in \mathcal{U}(pM_n(B)p)$, and, since $v^*vp = v^*1_A v = v^*v$, we see that $v^*v \leq p$. Therefore we have:

$$\begin{aligned}\|(\mathrm{id}_{M_n(\mathbb{C})} \otimes E_B)(v^*v)\|_2 &= \|\pi(u_k)(\mathrm{id}_{M_n(\mathbb{C})} \otimes E_B)(v^*v)\|_2 \\ &= \|(\mathrm{id}_{M_n(\mathbb{C})} \otimes E_B)(\pi(u_k)v^*v)\|_2 \\ &= \|(\mathrm{id}_{M_n(\mathbb{C})} \otimes E_B)(v^*u_k v)\|_2 \to 0.\end{aligned}$$

So, by faithfulness of E_B, we conclude that $v^*v = 0$ and thus that $v = 0$, which is a contradiction.

Our main use of Popa's intertwining techniques will be to mimic the characterization obtained in Theorem 3.1 of [HSV16] in the setting of our free Bogoliubov actions. In [HSV16], the authors are able to characterize the unitary conjugacy of some corners of the states ϕ, ψ on M in terms of the existence of an intertwining bimodule between (the images of) the group algebras $L_\phi(\mathbb{R})$ and $L_\psi(\mathbb{R})$ in the core $c(M) \simeq M \rtimes_{\sigma^\phi} \mathbb{R} \simeq M \rtimes_{\sigma^\psi} \mathbb{R}$. We obtain a similar characterization for the equivalence of some corners of our Bogoliubov actions (see Theorem 4.1.6).

We also make repeated use of the following theorem of Houdayer and Ueda, which allows us to locate algebras with non-amenable relative commutant inside an amalgamated free product. More precisely, we have:

Theorem 3.3.31 *(Theorem 4.4, [HU16]) Let* $(B \subset M_i)_{i\in I}$ *be a family of inclusions with expectation, with* M_i σ*-finite and* B *amenable, and let* $(M, E) = *_B(M_i, E_i)$ *be the associated amalgamated free product over* B*. If* $A \subset M$ *is a (not necessarily unital) inclusion with expectation and* A *is finite, then at least one of the following holds:*

1.) The von Neumann algebra $A' \cap 1_A M 1_A$ *is amenable.*

2.) There exists $i \in I$ *such that* $A \preceq_M M_i$*.*

Now take A to be a corner of $L(\mathbb{R})$ inside of the crossed product for our free Bogoliubov action $M = L(\mathbb{F}_\infty) \rtimes_{\sigma^\alpha} \mathbb{R}$. We will use the fact that the spectral measure of α has a mass at 0 to guarantee that $A' \cap 1_A M 1_A$ is non-amenable. Thus, if we realize M as a certain amalgamated free product, this theorem will force the embedding of A into one of the factors in the free product decomposition. See the proof of 4.2.1 for details.

CHAPTER 4

Classification Results

4.1 Technical Results and Preliminaries

4.1.1 Rajchman Measures

In order to apply some of the strong solidity results of Houdayer and Shlyakhtenko in [HS11], we will need to have free Bogoliubov actions which are mixing. Recall that an orthogonal/unitary representation $\alpha : G \to \mathcal{O}(H)$ (resp. $\alpha : G \to \mathcal{U}(H)$) is called mixing if

$$\lim_{g\to\infty} \langle \alpha_g \xi, \eta \rangle = 0$$

for all $\xi, \eta \in H$. (In what follows, we'll mainly be concerned with mixingness for $G = \mathbb{Z}$.)

By Proposition 2.8 in [HS11], the following are equivalent:

(1) The representation $\alpha : \mathbb{Z} \to \mathcal{O}(H)$ is mixing.

(2) The associated free Bogoliubov action $\sigma^\alpha : \mathbb{Z} \curvearrowright \Gamma(H)$ is mixing.

So if we want our Bogoliubov actions to be mixing, it will be sufficient to restrict our attention to orthogonal actions of $\mathbb{Z}$ on a real Hilbert space H which are mixing. To do this, we first reframe the mixingness condition above in terms of the spectral measure associated to the representation α. Recall that a representation of $\mathbb{Z}$ is determined by the data $([\mu], n)$, where $[\mu]$ is a measure class and n a multiplicity function on S^1. Choosing a representative probability measure μ for $[\mu]$, we can form the associated representation $\pi^\mu : \mathbb{Z} \to \mathcal{U}(L^2(S^1, \mu))$ given by $\pi^\mu_n(f(z)) = z^n f(z)$ for $f \in L^2(S^1, \mu)$. We note that the representation associated to $([\mu], n)$ is mixing if and only if π^μ is mixing, i.e. iff $\int z^n f(z)\, d\mu \to 0$ for all $f \in L^2(S^1, \mu)$ as $n \to \infty$. But since $1 \in L^2(S^1, \mu)$ is a cyclic vector for π^μ, this latter condition is equivalent to $\int z^n \, d\mu \to 0$ as $n \to \infty$. In other words, our representation will be mixing if and only if the Fourier coefficients of μ decay to zero at infinity.

We define a *Rajchman measure* to be a measure on S^1 whose Fourier coefficients $\hat{\mu}(n) = \int z^n \, d\mu$ go to zero as $n \to \infty$. Menshov was the first to construct examples of Rajchman measures that were singular with respect to Lebesgue measure. For a survey, see [Lyo95]. We will need the following facts about Rajchman measures:

Lemma 4.1.1 *Let μ be a Rajchman measure.*

(i) If $\nu \ll \mu$, then ν is a Rajchman measure.

(ii) Any translation of μ is a Rajchman measure.

(iii) A finite periodic extension of μ is a Rajchman measure.

Remark 4.1.2 Our use for (iii) is to take a Rajchman measure supported on $[-\frac{1}{2}, \frac{1}{2}]$ and extend it periodically to $[-1, 1]$. We only need the lemma to ensure that this extension is still Rajchman. This is what we mean by "finite periodic extension."

Proof. For (i), let $\nu = f\mu$ with $f \in L^1(\mu)$. For any polynomial $p(z) = a_n z^n + ... + a_0$, we have $\limsup_n |\widehat{f\mu}(n)| = \limsup_n |\widehat{f\mu - p\mu}(n)| \le \|f - p\|_1$. Since the last term can be made arbitrarily small, we see that $\nu = f\mu$ is Rajchman.

Part (ii) follows from the fact that translation of the measure μ corresponds to modulation of its Fourier coefficients, which doesn't affect their convergence to zero.

Finally, (i) and (ii) imply (iii), since a finite periodic extension of μ is a finite sum of translations of (restrictions of) μ.

Remark 4.1.3 In the following sections, we may identify $\mathbb{R}/\mathbb{Z}$ with S^1 in the usual way, via the map $t \mapsto e^{2\pi i t}$, with $t \in [0, 1]$. We will implicitly use this identification when we say that a $[0, 1]$-supported measure is Rajchman.

4.1.2 Corners Retain Spectral Data

Throughout this section we restrict our attention to an $\mathbb{R}$-action implemented by $(U_t)_{t \in \mathbb{R}}$ (on a real Hilbert space $\mathcal{H}$) whose infinitesimal generator has spectral measure $(\delta_0, \infty) + (\mu, \infty)$

with μ non-atomic. Under these assumptions, we can write $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where each summand is infinite-dimensional, and with $U_t = id \oplus U_t'$ where U_t' has no eigenvectors (since μ is non-atomic). Note that μ being non-atomic also implies that the only finite-dimensional invariant subspaces of $\mathcal{H}$ for U_t are those fixed by the action (i.e. where the action is id). Applying the free Gaussian functor $\mathcal{F}$ to this action, which turns direct sums into free products (see e.g. [VDN92]), we get a von Neumann algebra $M = s(\mathcal{H}_0) * s(\mathcal{H}_1) = A * B$ with $A \simeq B \simeq L(\mathbb{F}_\infty)$ (since $s(\mathcal{H}_i) \simeq L(\mathbb{F}_\infty)$), with corresponding free Bogoliubov action α_t arising from $\mathcal{F}(U_t) = id * \mathcal{F}(U_t')$. Note that $\mathcal{F}(U_t')$ has no eigenvectors since U_t' had none, and therefore, denoting by M^α the fixed point algebra of the action, the above discussion implies the following:

Lemma 4.1.4 *We have $M^\alpha = A$.*

Now, given a projection $0 \neq p \in A$, since p is fixed by the action, we can consider the restriction of α to the corner pMp. Note that A is a II_1 factor, so, taking a smaller p if necessary we can assume $\tau(p) = \frac{1}{n}$ for some $n \in \mathbb{N}$ with τ the canonical trace on A. We now relate the spectral data from α with the spectral data from the compression $(pMp, p\alpha_t p)$ as follows:

Lemma 4.1.5 *If $p \in A$ satisfies $\tau(p) = \frac{1}{n}$, we have:*

$$(pMp, \alpha_t|_{pMp}) \simeq (M^{*n^2}, \alpha_t^{*n^2}) \simeq (M, \alpha_t).$$

Proof. Let $\{S_i\}_{i\in\mathbb{N}}$ be a semicircular family of generators for B corresponding to an orthonormal basis $\{\xi_i\}$ for $\mathcal{H}_1$. Note that, on these generators, the Bogoliubov action α satisfies $\alpha_t(s(\xi)) = s(U_t'(\xi))$.

Since A is a factor and $\tau(p) = \frac{1}{n}$, we can find partial isometries $v_i \in A$, $i = 1, ..., n$, such that $v_i v_i^* = p$ for all i and $\sum_i v_i^* v_i = 1$. Note that M is generated by $A \cup \{S_i\}_{i\in\mathbb{N}}$, so that pMp is generated by pAp and $\{v_i S_k v_j^*\}_{1\leq i,j\leq n, k\in\mathbb{N}}$ [VDN92, Lemma 5.2.1].

For $i, j \in 1, ..., n$ and $k \in \mathbb{N}$, denote $S_{ikj} = n^{1/2} v_i S_k v_j^*$. The normalization is chosen so that in the compressed W*-probability space $(pMp, n\tau|_{pMp})$ we have that $\{S_{iki}\}_{1\leq i\leq n, k\in\mathbb{N}}$ is a free semicircular family (with semicircular law supported on $[-1,1]$), and $\{S_{ikj}\}_{1\leq i<j\leq n, k\in\mathbb{N}}$ is a free circular family $*$-free from $\{S_{iki}\}$ [VDN92, Prop. 5.1.7].

So, all together, pMp is generated $*$-freely by pAp, the semicircular family $\{S_{iki}\}_{1\leq i\leq n, k\in\mathbb{N}}$, and the circular family $\{S_{ikj}\}_{1\leq i<j\leq n, k\in\mathbb{N}}$.

Now we note that $(pAp, \alpha_t|_{pMp}) = (pAp, id) \simeq (A, id)$. The first isomorphism of the claim then follows, since for any $i, j \in \{1, ..., n\}$ and any k, we have $\alpha_t(S_{ikj}) = \alpha_t(n^{1/2}v_i(s(\xi_k))v_j^*) = n^{1/2}v_i(s(U_t'(\xi_k)))v_j^*$. Finally, the latter isomorphism follows from the infinite multiplicities of our representations, since an m-fold free product of the Bogoliubov action corresponds to multiplying the multiplicity function for the $\mathcal{H}$-representation by m.

4.1.3 Embedding gives Corner Conjugacy of Actions

By mimicking the proof of Theorem 3.1 in [HSV16], we can relate the embeddability of $L_\beta(\mathbb{R})$ into $L_\alpha(\mathbb{R})$ inside the crossed product with the existence of a corner on which the two actions are conjugate. More precisely, we have:

Theorem 4.1.6 *Let M be a tracial von Neumann algebra with a fixed faithful normal trace τ. Suppose $\alpha, \beta : \mathbb{R} \to Aut(M)$ are two actions of $\mathbb{R}$ on M which are cocycle conjugate, and suppose that the only finite-dimensional α-invariant subspaces of $L^2(M)$ are those on which α acts trivially. Fix any $q \in M^\beta$ a nonzero projection. The following are equivalent:*

(a) There exists a nonzero projection $r \in L_\beta(\mathbb{R})$ such that

$$\Pi_{\alpha,\beta}(L_\beta(\mathbb{R})qr) \prec_{M\rtimes_\alpha\mathbb{R}} L_\alpha(\mathbb{R})$$

(b) There exists a nonzero partial isometry $v \in M$ such that $v^*v \in qM^\beta q$, $vv^* \in M^\alpha$, and for all $x \in M$,

$$\alpha_t(vxv^*) = v\beta_t(x)v^*.$$

Proof. To see that (a) implies (b), take r as in (a), so that $\Pi_{\alpha,\beta}(L_\beta(\mathbb{R})qr) \prec_{M\rtimes_\alpha\mathbb{R}} L_\alpha(\mathbb{R})$, and take $w_t \in M$ with $\mathrm{Ad}\ w_t \circ \alpha_t = \beta_t$.

First, we claim that there's a $\delta > 0$ for which there exist $x_1, ..., x_k \in qM$ with

$$\sum_{i,j=1}^{k} |\tau(x_i^* w_t \alpha_t(x_j))|^2 \geq \delta$$

for all t. Suppose for a contradiction that no such δ exists. Then we can find a net $(t_i)_{i\in I}$ such that

$$\lim_i \tau(x^* w_{t_i} \alpha_{t_i}(y)) = 0$$

for any $x, y \in qM$.

But then for any p, p' finite trace projections in $L_\alpha(\mathbb{R})$, $s, s' \in \mathbb{R}$, and $x, y \in M$, we have (in the 2-norm from the trace on $M \rtimes_\alpha \mathbb{R}$):

$$\begin{aligned}\|E_{L_\alpha(\mathbb{R})}(p\lambda_\alpha(s)^* x^* \Pi_{\alpha,\beta}(\lambda_\beta(t_i)q)y\lambda_\alpha(s')p')\|_2 &= \|\lambda_\alpha(s)^* p E_{L_\alpha(\mathbb{R})}(x^* q \Pi_{\alpha,\beta}(\lambda_\beta(t_i)qy)p'\lambda_\alpha(s'))\|_2 \\ &= \|pE_{L_\alpha(\mathbb{R})}((qx)^* w_{t_i}\alpha_{t_i}(qy))p'\lambda(s'+t_i)\|_2 \\ &= \|E_{L_\alpha(\mathbb{R})}((qx)^* w_{t_i}\alpha_{t_i}(qy))pp'\|_2 \to 0,\end{aligned}$$

where the last equality follows from the fact that $((qx)^* w_{t_i}\alpha_{t_i}(qy)) \in M$, so

$$E_{L_\alpha(\mathbb{R})}((qx)^* w_{t_i}\alpha_{t_i}(qy)) = \tau((qx)^* w_{t_i}\alpha_{t_i}(qy)),$$

and the latter term goes to zero by supposition for any $x, y \in M$.

Now note that linear combinations of terms of the form $x\lambda_\alpha(s)p$ (resp. $y\lambda_\beta(s')p'$) as above are dense in $L^2(M \rtimes_\alpha \mathbb{R}, Tr)$, so by approximating $\Pi_{\alpha,\beta}(r)a$, (resp. $\Pi_{\alpha,\beta}(r)b$) with such sums for any $a, b \in M \rtimes_\alpha \mathbb{R}$, it follows from the above estimate that

$$\|E_{L_\alpha(\mathbb{R})}(a^*\Pi_{\alpha,\beta}(\lambda_\beta(t_i)qr)b\|_2 \to 0.$$

But this contradicts $\Pi_{\alpha,\beta}(L_\beta(\mathbb{R})qr) \prec_{M\rtimes_\alpha\mathbb{R}} L_\alpha(\mathbb{R})$, so the $\delta > 0$ of our above claim exists.

We can thus find $\delta > 0$, $x_1, ..., x_k \in qM$ such that $\sum_{i,j=1}^{k} |\tau(x_i^* w_t \alpha_t(x_j))|^2 \geq \delta$ for all t.

We now pass to the basic construction $\langle M, e_\tau\rangle$, where e_τ is the rank-one Jones projection corresponding to the trace on M. Denote by $\hat{\tau}$ the canonical trace on $\langle M, e_\tau\rangle$ which satisfies $\hat{\tau}(xe_\tau y) = \tau(xy)$ for all $x, y \in M$. We denote by T_M the faithful normal operator-valued weight from the basic construction to M satisfying $\hat{\tau} = \tau \circ T_M$ (i.e. T_M is the pull-down map). Here, we consider the positive element

$$X = \sum_{i=1}^{k} x_i e_\tau x_i^*,$$

together with the following normal positive linear functional on $\langle M, e_\tau \rangle$:

$$\psi(T) = \sum_{i=1}^{k} \hat{\tau}(e_\tau x_i^* T x_i e_\tau).$$

Note that $T_M(X) = \sum_{i=1}^{k} x_i x_i^* \in M$, so in particular $\|T_M(X)\| < \infty$.

For every $t \in \mathbb{R}$, we have:

$$\begin{aligned}\psi(\beta_t(X)) &= \sum_{i,j} \hat{\tau}(e_\tau x_i^* w_t \alpha_t(x_j) e_\tau \alpha_t(x_j)^* w_t^* x_i e_\tau) \\ &= \sum_{i,j} |\tau(x_i^* w_t \alpha_t(x_j)|^2 \geq \delta > 0\end{aligned}$$

Now consider K the ultraweak closure of the convex hull of $\{\beta_t(X) : t \in \mathbb{R}\}$ inside $q\langle M, e_\tau \rangle q$. Note that by normality of ψ, $\psi(x) \geq \delta$ for any $x \in K$.

Since K is convex and $\|\cdot\|_2$-closed, there exists a unique $X_0 \in K$ of minimal 2-norm. But since the 2-norm is invariant under β, we must have that $\|\beta_t(X_0)\|_2 = \|X_0\|_2$ for all t, so by uniqueness of the minimizer, X_0 is itself fixed by the extended β action (and nonzero since $\phi(X_0) \geq \delta$). Also, by ultraweak lower semicontinuity of T_M, we know that $\|T_M(X_0)\| \leq \|T_M(X)\| < \infty$.

Take a nonzero spectral projection e of X_0. Then e is still β-invariant and satisties $\|T_M(e)\| < \infty$. But this means that $\hat{\tau}(e) = \tau(T_M(e)) < \infty$, so e must be a finite rank projection, since $\hat{\tau}$ corresponds to the usual trace Tr on the trace-class operators in $B(L^2(M), \tau)$.

Now since e_τ has central support 1 in $\langle M, e_\tau \rangle$ (and because e_τ is minimal), we have that there exists V a partial isometry in $\langle M, e_\tau \rangle$ such that $V^*V = f \leq e$ and $VV^* = e_\tau$. We remark that f remains β-invariant, since e was finite rank, and our finite-dimensional invariant subspaces are all fixed by the action. Note also that $e \leq q$ (since $X_0 \in q\langle M, e_\tau \rangle q$), so that $V = Vq = e_\tau V$.

Applying the pull-down lemma, we see that:

$$V = e_\tau V = e_\tau(T_M(e_\tau V)) = e_\tau T_M(V).$$

Set $v = T_M(V)$ and note that since $\|T_M(V^*V)\| = \|T_M(e)\| < \infty$, we have $v \in M$, and $V = e_\tau v$.

Since $e_\tau \langle M, e_\tau \rangle e_\tau = \mathbb{C}e_\tau$, and since V is left-supported by e_τ, we have that for each t there exists a $\lambda_t \in \mathbb{C}$ such that $\lambda_t e_\tau = V w_t \hat{\alpha}_t(V^*)$. Note that since $V w_t \hat{\alpha}_t(V^*)(V w_t \hat{\alpha}_t(V^*))^* = V w_t \hat{\alpha}_t(V^* V) w_t^* V^* = V \hat{\beta}_t(e) V^* = V V^* = e_\tau$, the last equality of the previous sentence implies that $\lambda_t \overline{\lambda_t} = 1$. We also have:

$$\begin{aligned} e_\tau \lambda_t \alpha_t(v) &= \lambda_t e_\tau \alpha(v) = \lambda_t e_\tau \hat{\alpha}(V) \\ &= V w_t \hat{\alpha}_t(V^* V) = V \hat{\beta}_t(e) w_t = V w_t \\ &= e_\tau v w_t. \end{aligned}$$

Thus, applying the pull-down map, we have that $\lambda_t \alpha_t(v) = v w_t$, and, replacing v by its polar part if necessary, we've found a partial isometry in M, conjugation by which intertwines the actions. We have for any $x \in M$:

$$\alpha_t(v x v^*) = \alpha_t(v)\alpha_t(x)\alpha_t(v^*) = \overline{\lambda_t} v w_t \alpha_t(x) w_t^* v^* \lambda_t = v \beta_t(x) v^*.$$

Furthermore, with some applications of $\alpha_t(v) = \overline{\lambda_t} v w_t$, we see that

$$\beta_t(v^* v) = w_t \alpha_t(v^* v) w_t^* = w_t(w_t^* v^* \lambda_t)(\overline{\lambda_t} v w_t) w_t^* = v^* v,$$

and

$$\alpha_t(v v^*) = (\overline{\lambda_t} v w_t)(w_t^* v^* \lambda_t) = v v^*,$$

so we've found the promised intertwiner.

Conversely, assume that we have $v \in M$ such that $v^* v \in q M^\beta q$, $v v^* \in M^\alpha$, and satisfying $\alpha_t(v x v^*) = v \beta_t(x) v^*$ for all $x \in M$. Take $w_t \in M$ with $\mathrm{Ad}\, w_t \circ \alpha_t = \beta_t$. Then, as above, we have $v w_t = \lambda_t \alpha_t(v)$, for some $\lambda_t \in S^1$. Multiplying both sides by $\overline{\lambda_t}$ and absorbing this factor into w_t, we may assume without loss of generality that $\lambda_t = 1$ for all t, so we have $v w_t = \alpha_t(v)$.

Now let λ_t^α (resp. λ_t^β) denote the canonical unitaries that implement the respective actions on M in the crossed product $M \rtimes_\alpha \mathbb{R}$ ($M \rtimes_\beta \mathbb{R}$). Then the relation $vw_t = \alpha_t(v)$ implies $v\Pi_{\alpha,\beta}(\lambda_t^\beta) = \lambda_t^\alpha v$. Furthermore, for any finite trace projection $r \in L_\beta(\mathbb{R})$, we have $v\Pi_{\alpha,\beta}(qr) = vq\Pi_{\alpha,\beta}(r) = v\Pi_{\alpha,\beta}(r) \neq 0$, so v^* is a partial isometry that witnesses $\Pi_{\alpha,\beta}(L_\beta(\mathbb{R})qr) \prec_{M \rtimes_\alpha \mathbb{R}} L_\alpha(\mathbb{R})$ (by condition (iii) in 3.3.30). Thus, (b) implies (a).

4.2 The Main Result

We can now prove our main result after fixing some notation. Since orthogonal representations of $\mathbb{R}$ are characterized by their spectral measure class and multiplicity function (see e.g. [HSV16, Section 1]), we'll assume in what follows that our mutliplicity function $m : \mathbb{R} \to \mathbb{N} \cup +\infty$ satisfies $m \equiv +\infty$. Further, we assume that our representation π of $\mathbb{R}$ is of the following type: $\pi = \pi_0 \oplus \pi_\lambda \oplus \pi_\mu$, where π_0 is the trivial representation, π_λ the left regular representation, and μ is a Rajchman measure on $[0, 1]$. Here, we let π_μ denote the representation of $\mathbb{R}$ whose spectral measure is given by periodic extension of μ. Again, all summands here are representations of infinite multiplicity. Let $\mathfrak{C}$ denote the set of all spectral measure classes on $\mathbb{R}$ associated to representations of this type. We'll use $[\pi]$ to denote the spectral measure class associated to the representation π.

Following the usual recipe, we have, associated to π, a free Bogoliubov action α of $\mathbb{R}$. Let α, β be two Bogoliubov actions of $\mathbb{R}$ on $L\mathbb{F}_\infty$ obtained in this way, and (with a slight abuse of notation), denote by $[\alpha]$, $[\beta]$ the associated spectral measure classes in $\mathfrak{C}$. We can now show the following:

Theorem 4.2.1 *With the notation above, if α and β are cocycle conjugate, then $[\alpha] = [\beta]$.*

Proof. Note that cocycle conjugacy implies that $L\mathbb{F}_\infty \rtimes_\alpha \mathbb{R} \cong L\mathbb{F}_\infty \rtimes_\beta \mathbb{R}$. We'll denote by $A \subset L\mathbb{F}_\infty \rtimes_\alpha \mathbb{R} = M$ the algebra $L_\alpha\mathbb{R}$, which is isomorphic to $L^\infty(\mathbb{R})$ via the Fourier transform. We know that this crossed product is generated by a family of A-valued semicirculars, with covariance maps $\eta_i^{(k)} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ (making use of the identification in the previous sentence) of the following type:

$$\eta(f)(x) = \int f(y) K_\eta(x,y)$$

where K_η is a measure on $\mathbb{R}^2$ corresponding to η (see, e.g. [Shl99, Examples 2.8, 5.2]).

In our setting, we have covariance maps $\eta_i^{(k)}$, with $k \in \mathbb{N}$ accounting for our infinite multiplicities and $i = 0, 1, 2$ corresponding to the trivial, left regular, and μ summands of our orthogonal representation respectively. For all k, the associated measures on $\mathbb{R}^2$ are given by:

$$\begin{aligned} K_0^{(k)}(x,y) &= \delta_{x=y} \\ K_1^{(k)}(x,y) &= e^{-(x^2+y^2)}\, dx\, dy \\ K_2^{(k)}(x,y) &= \mu(x-y). \end{aligned}$$

[We remark that the (arbitrary) choice of Gaussian measure instead of Lebesgue measure for $K_1^{(k)}$ was simply for finiteness of $K_1^{(k)}$ and for definiteness. It follows from [Shl99, Prop. 2.19] that the W*-algebra generated by A and the A-valued semicirculars associated to η (i.e. $\Phi(A,\eta)$ in the notation of [Shl99]) depends only on the absolute continuity class of K_η.]

Now we remark that if $K_\eta = K'_\eta + K''_\eta$ with K'_η, K''_η disjoint and satisfying the same self-adjointness condition as K_η (i.e. $K_\eta(x,y) = \overline{K_\eta(y,x)}$), then, setting $\eta'(f)(x) = \int f(y) K'_\eta(x,y)$ (respectively, $\eta''(f)(x) = \int f(y) K''_\eta(x,y)$), we obtain completely positive maps $\eta', \eta'' : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$. Recalling the notation $\Phi(A,\eta)$ from [Shl99] for the W*-algebra generated by the A-valued semicircular family of covariance η and using our choice of $A = L\mathbb{R}$ and η above, we have the following isomorphism:

$$\Phi(A,\eta) \cong \Phi(A,\eta') \underset{A}{*} \Phi(A,\eta'').$$

Let $\Omega_j = \left(\cup_{k=-\infty}^{\infty} [jk, j(k+1)] \times [jk, j(k+1)]\right) \cap \{(x,y) : |x-y| \leq \frac{j}{2}\}$ and set $K_{2,j} = 1_{\Omega_j} K_2$, with K_2 as above. Now we note that, by the preceding discussion and the infinite multiplicity of our representations, we may decompose our algebra $L\mathbb{F}_\infty \rtimes_{\alpha_t} \mathbb{R}$ as an amalgamated free product over A in the following way:

$$L\mathbb{F}_\infty \rtimes_{\alpha_t} \mathbb{R} \cong \mathop{*}_{\substack{A\\ j\in\mathbb{Z}}} \left[\Phi(A,\eta_0) \mathop{*}_{A} \Phi(A,\eta_1) \mathop{*}_{A} \Phi(A,\eta_{2,j})\right],$$

where η_0 and η_1 are, as before, the completely positive maps corresponding to the trivial and left regular representations, still with infinite multiplicities.

Now let B be the image in M of $L_\beta(\mathbb{R})$ under an isomorphism of the crossed products $M = L\mathbb{F}_\infty \rtimes_\alpha \mathbb{R} \cong L\mathbb{F}_\infty \rtimes_\beta \mathbb{R}$. We first want to establish the following claim:

Claim 4.2.2 *With A, B, and M as above, we have $B \prec_M A$.*

Proof. Since B has nonamenable relative commutant in M, we may apply [HU16, Theorem 4.4] to the amalgamated free product decomposition preceding the claim to conclude that $B \prec_M \Phi(A,\eta_0) \mathop{*}_{A} \Phi(A,\eta_1) \mathop{*}_{A} \Phi(A,\eta_{2,j})$ for some j. Thus, restricting to a corner of B if necessary, it suffices to establish the claim with M replaced by $\Phi(A,\eta_0) \mathop{*}_{A} \Phi(A,\eta_1) \mathop{*}_{A} \Phi(A,\eta_{2,j})$. Furthermore, we may assume without loss of generality that $j = 1$, since the following argument will apply (mutatis mutandis) for all j.

Thus we've reduced to the case where M is the W*-algebra generated by $X_i^{(k)}$, a family of A-valued semicirculars of covariance $\eta_i^{(k)}$ (corresponding to the measures $K_i^{(k)}$), with $k \in \mathbb{Z}$, $i \in \{0, 1, 2\}$, and:

$$\begin{aligned} K_0^{(k)} &= \delta_{x=y} \\ K_1^{(k)} &= e^{-(x^2+y^2)}\,dx\,dy \\ K_2^{(k)} &= \omega(x,y), \end{aligned}$$

where $\omega(x,y) = \mu(x-y)$ for $(x,y) \in \cup_{k=-\infty}^{\infty} [k,k+1] \times [k,k+1] \cap \{(x,y) : |x-y| \leq \frac{1}{2}\}$ and $\omega(x,y) = 0$ otherwise. Recall that μ is some representative measure for the third summand in our representation, as defined at the beginning of this section.

Recalling our identification of A with $L^\infty(\mathbb{R})$, we consider the projections $p_j = 1_{[j,j+1]} \in A$. By considering the polar decomposition of $X_1^{(0)}$, we obtain partial isometries $v_j \in M$ such that

$$v_j v_j^* = p_0, \qquad v_j^* v_j = p_j.$$

First we note that we have $v_j A v_j^* \cong p_j A p_j \cong L^\infty[0,1]$ in a natural way. For later use, let's give a name to this isomorphism: $\beta_j : v_j A v_j^* \to L^\infty[0,1]$. We also observe that since $K_0^{(k)}$ and $K_2^{(k)}$ are supported on $\cup_{k\in\mathbb{Z}}[k,k+1]\times[k,k+1]$, then if $j \neq j'$, we have $p_j X_0^{(k)} p_{j'} = p_j X_1^{(k)} p_{j'} = 0$ for all k.

Therefore, if we consider the compression $p_0 M p_0$ (call it N), we see that N is generated by the following with $j,k \in \mathbb{Z}$:

$$\begin{aligned}&(1)\, A_j := v_j A v_j^* \\ &(2)\, v_j X_0^{(k)} v_j^* \\ &(3)\, v_j X_2^{(k)} v_j^* \\ &(4)\, v_j X_1^{(k)} v_{j'}^*, \quad j' \in \mathbb{Z}.\end{aligned}$$

But because $X_1^{(k)}$ comes from the left regular representation of $\mathbb{R}$, the terms in item (4) are semicircular (if $j = j'$) or circular (if $j \neq j'$) elements, $*$-free from the terms corresponding to $(1), (2)$ and (3). Furthermore, since the terms from item (1) above are also pairwise $*$-free from each other, if we write Q for the algebra they generate, we have $Q = W^*(A_j : j \in \mathbb{Z}) \cong *_{\mathbb{Z}} L^\infty[0,1]$. Writing $Y_{i,j}^{(k)}$ for $p_j X_i^{(k)} p_j^*$ ($i \in \{0,2\}$, $j,k \in \mathbb{Z}$), we see that $Y_{i,j}^{(k)}$ form a Q-valued semicircular family of covariance $\eta_{i,j} : Q \to Q$, with $\eta_{i,j}$ as follows:

$$\begin{aligned}\eta_{0,j}(q) &= E^Q_{A_j}(q) \\ \eta_{2,j}(q) &= \beta_j^{-1}\left(\eta_2(\beta_j \circ E^Q_{A_j}(q))\right),\end{aligned}$$

where $E^Q_{A_j}$ is the unique trace-preserving expectation from Q onto A_j.

It follows that $\{W^*(A_j, Y_{0,j}^{(k)}, Y_{2,j}^{(k)}) : k \in \mathbb{Z}\}_{j\in\mathbb{Z}}$ are free in N. Therefore, we see that

$$N \cong *_{j\in\mathbb{Z}} W^*(A_j, Y_{0,j}^{(k)}, Y_{2,j}^{(k)} : k \in \mathbb{Z}) * L\mathbb{F}_\infty.$$

Now let B be an abelian subalgebra of N with nonamenable relative commutant, as before. Making another application of [HU16, Theorem 4.4] to the free product decomposition above, we see that $B \prec_N W^*(A_j, Y_{0,j}^{(k)}, Y_{2,j}^{(k)} : k \in \mathbb{Z})$ for some j or $B \prec_N L\mathbb{F}_\infty$. The latter is impossible by the solidity of $L\mathbb{F}_\infty$, so we may assume without loss of generality that $B \prec_N W^*(A_j, Y_{0,j}^{(k)}, Y_{2,j}^{(k)} : k \in \mathbb{Z})$ for some fixed j.

But by the above description of the covariance maps $\eta_{i,j}$, we know that

$$\begin{aligned} W^*(A_j, Y_{0,j}^{(k)}, Y_{2,j}^{(k)} : k \in \mathbb{Z}) &\cong W^*(A_j, Y_{0,j}^{(k)} : k \in \mathbb{Z}) \underset{A_j}{*} W^*(A_j, Y_{2,j}^{(k)} : k \in \mathbb{Z}) \\ &\cong (A_j \otimes L\mathbb{F}_\infty) \underset{A_j}{*} W^*(A_j, Y_{2,j}^{(k)} : k \in \mathbb{Z}). \end{aligned}$$

Another application of [HU16] gives that $B \prec_N A_j \otimes L\mathbb{F}_\infty$ or $B \prec_N W^*(A_j, Y_{2,j}^{(k)} : k \in \mathbb{Z})$. If the former happens, then we are done with the proof of the claim, since the solidity of $L\mathbb{F}_\infty$ yields that $B \prec_N A_j$ and therefore that $B \prec_N A$, as desired. Therefore, to finish the proof of the claim, it suffices to show that the latter embedding cannot happen, i.e. $B \not\prec_N W^*(A_j, Y_{2,j}^{(k)} : k \in \mathbb{Z})$. So it's enough to show that $W^*(A_j, Y_{2,j}^{(k)} : k \in \mathbb{Z})$ is solid.

The preceding discussion implies that $W^*(A_j, Y_{2,j}^{(k)} : k \in \mathbb{Z}) \cong W^*(L^\infty[0,1], Y^{(k)} : k \in \mathbb{Z})$, where $Y^{(k)}$ are $L^\infty[0,1]$-valued semicircular elements whose covariance as a measure on $[0,1] \times [0,1]$ is given by

$$K(x, y) = 1_{|x-y|<\frac{1}{2}} \mu(x - y).$$

We extend the restriction of μ to $[-\frac{1}{2}, \frac{1}{2}]$ to a periodic measure $\widetilde{\mu}$ on $\mathbb{R}$ of period 1, and then we consider the measure $\widetilde{K}$ on $[0,1] \times [0,1]$ defined by:

$$\widetilde{K}(x, y) = \widetilde{\mu}(x - y).$$

Note that K is absolutely continuous with respect to $\widetilde{K}$, so that if $\eta, \widetilde{\eta}$ are the respective associated completely positive maps on $L^\infty[0,1]$, we have that $\Phi(L^\infty[0,1], \eta) \subset \Phi(L^\infty[0,1], \widetilde{\eta})$. But $\Phi(L^\infty[0,1], \widetilde{\eta}) \cong L\mathbb{F}_\infty \rtimes_\gamma \mathbb{Z}$ via an isomorphism that identifies $L^\infty[0,1]$ with $L(\mathbb{Z})$, where γ is the Bogoliubov action of $\mathbb{Z}$ associated with the measure $\widetilde{\mu}$. Note that γ is mixing by our earlier lemma 4.1.1 and its preceding discussion. Therefore, by [HS11] Theorem 3.10, $L\mathbb{F}_\infty \rtimes_\gamma \mathbb{Z}$ is solid, and therefore $\Phi(L^\infty[0,1], \eta)$ is solid also, which concludes the proof of the claim.

Now that we've established that $B \prec_M A$, we may apply Theorem 2.3 above to find a nonzero partial isometry $v \in M$ such that $vv^* \in M^\alpha$, $v^*v \in M^\beta$, and

$$\alpha_t(vxv^*) = v\beta_t(x)v^*.$$

Therefore, shrinking the support of v if necessary, we may assume that $\tau(vv^*) = \frac{1}{n}$ for some $n \in \mathbb{N}$ and apply Lemma 4.1.5 to conclude that $[\alpha] = [\alpha^{*n^2}] = [\beta^{*n^2}] = [\beta]$, as desired.

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