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TREATMENT OF INTERNAL CONSTRAINTS BY MIXED FINITE ELEMENT METHODS: UNIFICATION OF CONCEPTS

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ABSTRACT

A general method to generate assumed stress and strain fields in the context of mixed finite element methods is presented. The assumed fields are constructed in such a way that internal constraints are satisfied *a priori*. Consequently, the locking behavior commonly observed in finite element solutions, when subjected to internal constraints, is avoided. This objective is obtained by requiring the assumed fields to possess a structure compatible with that of the equilibrium equations.

1. INTRODUCTION

1.1 Motivation and Objectives

The classical finite element method (i.e., isoparametric displacement formulation using full integration) fails to provide adequate solutions to problems involving internal constraints (e.g., isochoric motion or bending of thin plates based on theories that account for shear deformations). Using this approach often results in severe locking. Locking is characterized by poor results obtained when coarse meshes are used. The solution, however, converges slowly with mesh refinment (see e.g., convergence obtained when using four-node isoparametric displacement formulation elements (2×2) in Figure 1.1).

During the past two decades many methods have been proposed to overcome problems associated with the locking phenomena. Techniques devised to avoid locking, formulated within the framework of mixed finite element methods, often were based upon the separation of the "problematic" terms and the introduction of special treatments of those terms. These approaches fail to provide insight into the development of solutions based on the finite element method. Indeed, methods devised to overcome one problem do not carry over to a different problem (e.g., one could not use the same techniques to alleviate locking at the nearly incompressible limit and shear locking in plate bending problems).

The objective of this paper is to provide a general approach within the framework of mixed finite element methods that may be applied to different problems regardless of the nature of the particular internal constraints. Figures 1.1 and 1.2 are examples of the application of the proposed method to problems involving internal constraints. Elements formulated via this approach should exhibit the following properties:

- Resolve locking resulting from the presence of internal constraints at the element level.
- Recover stresses that are variationally consistent with the formulation used to obtain the displacement field.
- Produce reliable stresses at the element level (without recourse to smoothing).
- Posses correct rank (no spurious zero-energy modes).
- Have performance independent of coordinate system or user input data.

1.2 Background

Classical finite element formulations exhibit severe locking when applied to model problems involving internal constraints. Much research has been devoted to overcome this setback. Initial approaches utilized Reduced Integration (RI) and Selective Reduced Integration (SRI) schemes (Hughes [1977]). However, these approaches often lead to unstable elements (Bicanic & Hinton [1979]). Malkus & Hughes [1978] showed that the SRI and RI schemes fall within the concept of a mixed finite element method.

Hughes [1980] refined the SRI scheme into a general method, known as the B-bar method, for three-dimensional and axisymmetric elements. More recently, Simo, *et al.* [1985] showed that it is possible to derive a B-bar method from the Hu-Washizu variational principle.

Wilson, *et al.* [1973] observed that when the four-node element, based upon isoparametric displacement formulation, is subjected to a pure bending loading it deforms in shear rather than in bending. To improve the behavior in bending they introduced a set of incompatible displacements. It was soon realized, however, that the distorted element did not pass the constant strain patch test. Taylor, *et al.* [1976] modified the incompatible modes element so that the resulting element passed the constant strain patch test. This result was obtained by replacing the derivatives associated with the incompatible modes with their values at the center of the element. Furthermore, in addition to the improved behavior in bending, when applied to model plane strain problems at the nearly incompressible limit locking is avoided (it must be noted that this point was not observed when the paper first appeared in 1973). The use of incompatible displacements was discussed in detail by Strang & Fix [1973]. Their discussion was focused on presentation of conditions for convergence.

Pian & Sumihara [1984] presented a four-node plane stress element based upon the Hellinger-Reissner variational principle. This element has excellent characteristics in bending applications. Also, when modified to account for the plane strain constitutive relations, it has excellent characteristics at the nearly incompressible limit. The assumed independent stress field was subjected to a set of constraint equations. These equations were interpreted as satisfying the equilibrium equations in a weak sense relative to a set of incompatible displacement modes. Initially, the formulation required a quadratic perturbation of the element shape in order to obtain the required number of independent constraint equations. Recently, Pian & Wu [1988] presented a new formulation that avoids element perturbation. The constraint equations are obtained by constraining the stress field to perform no work along the element boundary when subjected to a set of assumed incompatible displacement modes. A general formulation to generate incompatible element functions was presented by Wu, *et al.* [1987].

Simo & Rifai [1989] presented a method based upon the Hu-Washizu functional which introduced "enhanced strains". They applied the method to generate four-node plane stress/strain, axisymmetric, and plate bending elements. The plane elements' performance is almost identical to that of the Pian & Sumihara [1984] element. The plate bending elements' performance is close to that of the T1 element (Hughes & Tezduyar [1981]).

1.3 Paper Overview

In Section 2 a general framework for mixed finite element methods developed from the Hu-Washizu variational principle is reviewed. The optimal number of independent parameters in the assumed stress and strain fields is obtained from the mixed patch test (Zienkiewicz, *et al.* [1986]).

A method that can be used to obtain the assumed independent stress and strain fields is developed in Section 3. The assumed strain field is taken as the sum of two independent fields termed "compatible" and "incompatible." The stress and compatible strain fields are each initially chosen to possess more independent variables than the minimal (optimal) number prescribed by the mixed patch test. The stress field is obtained by constraining the complimentary strain energy associated with the incompatible strain to vanish in a weak sense over each element's domain. The strain field is obtained by constraining the strain energy resulting from the coupling of the compatible and incompatible strain fields to vanish in a weak sense over each element's domain. It follows from the variational structure that the incompatible strain field vanishes pointwise. Thus, the reduced fields (i.e., those that satisfy the constraint equations) are used as the independent fields in the finite element approximation.

Concluding remarks are presented in Section 4. The numerical implementation of the proposed scheme is considered in detail in subsequent papers.

2. MIXED FINITE ELEMENTS

A general framework for mixed finite element methods is laid out. The full mechanism involved is described in order to demonstrate that the choice of independent fields is the only degree of freedom left in developing mixed finite elements. Consequently, the performance of these elements depends solely on the particular choice of the assumed independent fields.

Mixed finite element methods are based on formulations that involve different types of independent variables, e.g:

1. Displacement and stress - Hellinger-Reissner functional.
2. Displacement, stress and strain - Hu-Washizu functional.

It is useful to note that the Hellinger-Reissner functional may be recovered from the Hu-Washizu functional by enforcing the constitutive equations pointwise to express the strain field in terms of the stress field. Furthermore, the classical strain energy functional, which forms the basis for classical displacement formulation of the finite element method, may be obtained from the Hu-Washizu functional by enforcing the pointwise strain-displacement relations as well as the constitutive relation. Consequently, the Hu-Washizu functional is used to setup the framework for a mixed finite element method.

2.1 Weak Form

As was pointed out above, the Hu-Washizu functional involves three independent fields, namely the displacement, stress and strain fields. Consequently, in order to proceed with the development a structure must be imposed on the spaces of admissible functions. To this end, let the following classes of functions be introduced:

Trial displacement solutions:*

$$U := \left\{ \mathbf{U} \mid \mathbf{U} \in H^1(\bar{\Omega}), \mathbf{U} = \mathbf{U}^a \text{ on } C_U \right\}. \tag{2.1}$$

Displacement weighting functions:

$$\dot{\bar{U}} := \left\{ \mathbf{U} \mid \mathbf{U} \in H^1(\bar{\Omega}), \mathbf{U} = \mathbf{0} \text{ on } C_U \right\}. \tag{2.2}$$

Trial stress solutions:

$$\Sigma := \left\{ \sigma \mid \sigma \in H^0(\Omega) \right\}. \tag{2.3}$$

* A function G is said to be a member of H^n if the function and its first n derivatives are members of L_2. A function F is said to be a member of L_2 if it is square integrable, i.e., $\int_\Omega F^2 \, d\Omega < \infty$ where Ω is the domain of interest. Thus, $H^0 = L_2$.

Trial strain solutions:

$$\Psi := \left\{ \epsilon \mid \epsilon \in H^0(\Omega) \right\}. \tag{2.4}$$

Note that the space of functions introduced as the stress solution space is not required to satisfy the traction boundary conditions. As a result the stress and strain solution spaces may also be used as the stress and strain weight functions, respectively. This is made possible since, as will be shown below, the traction boundary conditions evolve naturally from the weak statement of the problem, which is the starting point for the finite element approximation. Also note that the functions contained in the displacement solution and weight function spaces require higher order continuity than the stress and strain solution spaces. This is motivated by the appearance of the derivatives of the displacement functions as opposed to the stress and strain functions themselves in the weak form (see below).

The Hu-Washizu functional, for the case of small deformations, linear elastic materials, is given by:

$$\Pi_H(\mathbf{U}, \sigma, \epsilon) := \int_\Omega [\, \frac{1}{2} \epsilon^T \mathbf{D} \epsilon - \epsilon^T \mathbf{D} \epsilon_0 + \epsilon^T \sigma_0 + \sigma^T (\mathbf{L}\mathbf{U} - \epsilon)\,] \, d\Omega - \Pi_{EXT}(\mathbf{U}) \tag{2.5}$$

where:

D is the elastic moduli coefficients matrix;

L is the strain displacement operator;

ϵ_0 is an initial strain vector;

σ_0 is an initial stress vector;

Ω, the domain of interest, is a bounded set with a "curve" C as the boundary of its closure; and

Π_{EXT} is the external work.

The finite element method is formulated from the weak, or variational, form of the problem. The weak form may be obtained from the energy functional by making it stationary. This is obtained by taking the first total variation of the energy functional (Hu-Washizu functional in the present discussion) and equating it to zero, as stated in Box 2.1.

Box 2.1: The Weak Form

Given $\mathbf{b}$, $\mathbf{t}^a$, σ_0, ϵ_0, and $\mathbf{U}^a$; find $\mathbf{U} \in U$, $\sigma \in \Sigma$ and $\epsilon \in \Psi$ such that for every $\delta\mathbf{U} \in \bar{\mathbf{U}}$, $\delta\sigma \in \Sigma$ and $\delta\epsilon \in \Psi$

$$0 = \int_\Omega [\delta\sigma^T (\mathbf{L}\mathbf{U} - \epsilon) + \delta\epsilon^T (\mathbf{D}\epsilon - \mathbf{D}\epsilon_0 + \sigma_0 - \sigma) + (\mathbf{L}\delta\mathbf{U})^T \sigma - \delta\mathbf{U}^T \mathbf{b}]\, d\Omega - \int_{C_t} \delta\mathbf{U}^T \mathbf{t}^a \, dS$$

where $\mathbf{b}$ are body forces, $\mathbf{t}^a$ are applied traction boundary conditions, $\mathbf{U}^a$ are the applied displacement boundary conditions and C_t is the part of C on which traction boundary conditions are specified.

Since $\delta\sigma$, $\delta\epsilon$ and $\delta\mathbf{U}$ are independent the weak form, as presented in Box 2.1, consists of three independent equations, known as the Euler-Lagrange equations. The strong form of the problem can be obtained from the weak form by integrating by parts the equation associated with $\delta\mathbf{U}$, to obtain the balance of momentum equations, and apply the fundamental lemma of the calculus of variations to obtain the local form. Thus, showing the equivalence of the strong and weak forms of the problem. For a detailed discussion of this equivalence see e.g. Hughes [1987].

2.2 Finite Element Approximation

The finite element approximation of the weak form is obtained by substituting into the weak form an approximation of the assumed fields, which are assumed at the element level. Consequently, the energy functional (2.5) is replaced by the sum over the number of elements of the functional which is now defined over each element domain. Let the assumed fields be approximated over each element by:

- Stress field: $\sigma = \mathbf{S}\,\mathbf{s}$
- Strain field: $\epsilon = \mathbf{E}\,\mathbf{e}$
- Displacement field: $\mathbf{U} = N_I\, \mathbf{d}_I$

where $\mathbf{s}$ is the vector of independent stress parameters, $\mathbf{S}$ the matrix of shape functions for the stresses, $\mathbf{e}$ is the vector of independent strain parameters, $\mathbf{E}$ the matrix of shape functions for the strain, $\mathbf{d}_I$ is the vector of nodal displacements at node I, N_I is the shape function associated with node I, and the usual summation convention is implied for repeated indices

Substituting the assumed fields into the Euler-Lagrange equations, and rewriting these equations in matrix form yields:

$$\begin{bmatrix} \mathbf{H} & -\mathbf{A}^T & \mathbf{0} \\ -\mathbf{A} & \mathbf{0} & \mathbf{G} \\ \mathbf{0} & \mathbf{G}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{s} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_0 \\ \mathbf{0} \\ \mathbf{f} \end{bmatrix} \tag{2.6}$$

where

$$\mathbf{H} := \int_\Omega \mathbf{E}^T \mathbf{D} \mathbf{E} d\Omega \quad ; \quad \mathbf{A} := \int_\Omega \mathbf{S}^T \mathbf{E} d\Omega \tag{2.7a}$$

$$\mathbf{G} := \int_\Omega \mathbf{S}^T \mathbf{B} d\Omega ; \quad ; \quad \mathbf{f}_0 := \int_\Omega [\epsilon^T \mathbf{D} (\epsilon_0 + \sigma_0)] d\Omega ; \tag{2.7b}$$

$\mathbf{f}$ is the force vector resulting from body forces and traction boundary conditions; and $\mathbf{B}$ is defined by $\mathbf{B} := \mathbf{L}\mathbf{N}$ with $\mathbf{N}$ the matrix of shape functions used for the displacement.

Eliminating the stress and strain coefficients yields:

$$\mathbf{K}\mathbf{d} = \mathbf{F} \tag{2.8}$$

where $\mathbf{K}$ is the element stiffness matrix, given by:

$$\mathbf{K} := \mathbf{G}^T (\mathbf{A}\mathbf{H}^{-1}\mathbf{A}^T)^{-1} \mathbf{G} \tag{2.9}$$

and $\mathbf{F}$ is the element load vector, given by:

$$\mathbf{F} := \mathbf{f} + \mathbf{G}^T \mathbf{A}^{-T} \mathbf{f}_0 \tag{2.10}$$

If $\mathbf{A}$ is invertible, equation (2.9) can be replaced by:

$$\mathbf{K} := \mathbf{G}^T \mathbf{A}^{-T} \mathbf{H} \mathbf{A}^{-1} \mathbf{G} \tag{2.11}$$

which requires only one matrix inversion.

To satisfy the stability conditions for every admissible displacement (Zienkiewicz et al. [1986]):

$$n_e + n_d \geq n_s \quad ; \quad n_s \geq n_d \tag{2.12}$$

where n_s is the number of stress parameters, n_e is the number of strain parameters and n_d is equal to the number of nodal degrees of freedom minus the number of rigid body modes. Note that the dimension of $\mathbf{A}$ is $n_e \times n_s$. It follows that if $\mathbf{A}$ is required to be invertible then $n_e = n_s$ and thus, the first inequality equation $(2.12)_1$, is satisfied. Furthermore, in view of the desire to minimize computations, the "optimal" number of independent stress and strain parameters is obtained by replacing the inequality $(2.12)_2$ by an equality.

3. ASSUMED STRESS AND STRAIN FIELD GENERATION PROCEDURE

The assumed displacement field was given an isoparametric structure. Indeed the selection of this structure is now considered "natural" in the context of the finite element method. However, there is no similar "natural" structure for the assumed independent stress and strain fields. A trial and error method may be used, and indeed as long as the stability conditions are observed a solution will be obtained. Unfortunately, in the presence of internal constraints such a naive approach will often result in elements that exhibit severe locking. Consequently, a methodology that may be used to generate these assumed fields is necessary.

Throughout this section all quantities are assumed at the element level. Recall that the objective is to obtain stress and strain fields that lead to elements free of locking independent of the presence of internal constraints. This objective can be obtained if the assumed fields are constructed in such a way that they *a priori* satisfy the internal constraint. Moreover, this requirement is necessary in order to avoid the limitation principle as put forth by Fraeijs de Veubeke [1965]. Such fields can be constructed by imposing a structure compatible with the equilibrium equations. To this end let the strain field be given as the sum of two independent fields as follows:

$$\epsilon := \bar{\epsilon}^c + \epsilon^i \tag{3.1}$$

and let ϵ^i be given by:

$$\epsilon^i := \mathbf{L}\,\mathbf{U}^i \tag{3.2}$$

where $\mathbf{U}^i$ is a displacement field not contained in U^h, the finite element approximation of the admissible solution space, U. The motivation for this restriction will become clear as the method is presented. In the finite element literature, approximations of this class are commonly referred to as incompatible displacements. For this reason, ϵ^i are termed incompatible strains and, consequently, $\bar{\epsilon}^c$ are termed compatible strains.

The initially assumed stress field $\bar{\sigma}$ and compatible strain field $\bar{\epsilon}^c$ are chosen to possess more independent variables than the minimal, or optimal, number of independent variables prescribed by the mixed patch test. These fields may be chosen as the complete polynomial terms to any desired order in the Pascal triangle (e.g., for bilinear elements the complete linear terms). Substituting equation (3.1) into equation (2.5) yields:

$$\Pi_H(\mathbf{U},\bar{\sigma},\epsilon) := \int_\Omega [\, \frac{1}{2}(\bar{\epsilon}^c + \epsilon^i)^T \mathbf{D}(\bar{\epsilon}^c + \epsilon^i) - (\bar{\epsilon}^c + \epsilon^i)^T \epsilon_0 +$$

$$(\bar{\epsilon}^c + \epsilon^i)^T \sigma_0 + \bar{\sigma}^T(\mathbf{L}\,\mathbf{U} - \bar{\epsilon}^c - \epsilon^i)\,] - \Pi_{EXT}(\mathbf{U}) \tag{3.3}$$

The desired reduced stress field, σ (possessing the optimal number of independent parameters), is obtained by constraining the complimentary strain energy associated with the incompatible strain to vanish in a weak sense over the element's domain as follows:

$$\int_\Omega [\epsilon^{iT}(\bar{\sigma}-\sigma_0)]\,d\Omega = 0 \tag{3.4}$$

The desired reduced compatible strain field ϵ^c (possessing the optimal number of independent parameters), is obtained by constraining the strain energy resulting from the coupling of the compatible and the incompatible strain fields to vanish in a weak sense over the element's domain. Accordingly,

$$\int_\Omega [\epsilon^{iT}\,\mathbf{D}(\bar{\epsilon}^c-\epsilon_0)]\,d\Omega = 0 \tag{3.5}$$

Substituting ϵ^c and σ for $\bar{\epsilon}^c$ and $\bar{\sigma}$, respectively, and noting equations (3.4) and (3.5), the energy functional given by equation (3.3) is reduced to:

$$\Pi_H(\epsilon,\sigma,\mathbf{U}) = \int_\Omega [\,\frac{1}{2}(\epsilon^{cT}\mathbf{D}\epsilon^c + \epsilon^{iT}\mathbf{D}\epsilon^i) - \epsilon^{cT}\mathbf{D}\epsilon_0 +$$

$$\epsilon^{cT}\sigma_0 + \sigma^T(\mathbf{LU}-\epsilon^c)\,]\,d\Omega - \Pi_{EXT}(\mathbf{U}) \tag{3.6}$$

The first variation of the energy functional Π_H with respect to ϵ^i is given by:

$$D\,\Pi_H\cdot\delta\epsilon^i = \int_\Omega \delta\epsilon^i\,\mathbf{D}\epsilon^i\,d\Omega = 0 \tag{3.7}$$

where $\delta\epsilon^i$ denotes a virtual incompatible strain field. It follows that at a solution ϵ^i is zero pointwise, provided the elastic coefficient matrix, **D**, is positive definite and all terms in equation (3.2) are linearly independent. Consequently, $\epsilon \equiv \epsilon^c$. Thus, the reduced stress field, σ, and the reduced compatible strain field, ϵ^c, can be used as the assumed stress and strain fields.

It follows from the structure of the constraint equations that the assumed stress and strain fields approximation is now of the following form:

- Stress field: $\sigma = \mathbf{S}\,\mathbf{s} + \bar{\sigma}$
- Strain field: $\epsilon = \mathbf{E}\,\mathbf{e} + \bar{\epsilon}$

where $\bar{\sigma}$ and $\bar{\epsilon}$ are known vectors resulting from the presense of the initial stress and strain fields in the constraint equations for the stress and strain fields, respectively. It follows that the Euler-Lagrange equations, in matrix form, equation (2.6), are now given by:

$$\begin{bmatrix} \mathbf{H} & -\mathbf{A}^T & \mathbf{0} \\ -\mathbf{A} & \mathbf{0} & \mathbf{G} \\ \mathbf{0} & \mathbf{G}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{s} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{f} - \mathbf{f}_2 \end{bmatrix} \tag{3.8}$$

where

$$\mathbf{f}_0 := \int_\Omega [\mathbf{E}^T \mathbf{D} (\epsilon_0 - \bar{\epsilon}) - \mathbf{E}^T (\sigma_0 - \bar{\sigma})] \, d\Omega \tag{3.9}$$

$$\mathbf{f}_1 = \int_\Omega \mathbf{S}^T \bar{\epsilon} \, d\Omega \tag{3.10}$$

and

$$\mathbf{f}_2 = \int_\Omega \mathbf{B}^T \bar{\sigma} \, d\Omega \tag{3.11}$$

Comparing equations (3.8) with equations (2.6) reveals that the element stiffness matrix, K, is left unchanged while the element load vector F is now given by:

$$\mathbf{F} := \mathbf{f} - \mathbf{f}_2 + \mathbf{G}^T \mathbf{A}^{-T} \mathbf{f}_0 + \mathbf{G}^T \mathbf{A}^{-T} \mathbf{H} \mathbf{A}^{-1} \mathbf{f}_1 \tag{3.12}$$

The method presented above involves the use of incompatible displacements, to generate the incompatible strains. A set of properties that these displacements must possess is presented in Box 3.1 and motivated below.

Box 3.1: Properties of the Incompatible Displacements

- Frame invariant,
- Do not bias the element in any direction,
- Simple functions (lowest order polynomial admissible),
- Higher order than the assumed compatible displacements (e.g., for the bilinear elements, at least quadratic),
- First derivatives vanish in a weak sense, and
- Preserve the sign convention used in the strong form (plates, shells and beams).

The first three requirements are obvious, and are primary objectives for every finite element development.

The fourth requirement is introduced in order to obtain a stable algorithm. If the order of the incompatible functions were of the same order as the compatible functions, the orthogonalization procedure introduced in equations (3.4) and (3.5) will render unstable elements. This is because the null kernel of the G matrix with respect to the solution space will be augmented by the incompatible functions. By requiring the incompatible functions to be of higher order polynomials, the null kernel with respect to the solution space is not modified.

The fifth requirement is introduced in order to pass the constant strain/stress patch test. Provided the initially assumed stress and strain fields possess the ability to model a constant, a necessary requirement in order to pass the constant strain patch test, this constraint decouples the constant terms from the nonconstant terms. Consequently, provided the assumed displacement field can model the associated displacements the constant strain patch test is satisfied.

The last constraint pertains to application of the proposed method to plates, shells and beams. In these cases the equilibrium equations involve in-plane resultants, moment resultants and shear resultants. Consequently, the reduced fields will be coupled (e.g., in plate bending problems the shear resultant and moments will be coupled (Weissman & Taylor [1990])). If the reduction procedure does not preserve the sign conventions used, the contribution from the coupling terms will have the wrong orientation inside the element.

The fifth constraint was introduced in order to maintain the constant terms decoupled from the nonconstant terms. The constraint equation for the strain field, equation (3.5), involves the elastic coefficients matrix, $\mathbf{D}$. If it is not constant over the element domain the constant terms in the strain field will be coupled with the nonconstant terms. To avoid this problem the $\mathbf{D}$ matrix in equation (3.5), associated with $\bar{\epsilon}^c$, is replaced by its mean value, $\bar{\mathbf{D}}$, defined by:

$$\bar{\mathbf{D}} := \int_\Omega \mathbf{D}\, d\Omega \Big/ \int_\Omega d\Omega \tag{3.13}$$

Recall that this procedure is performed at the element level, and hence in the limit case under mesh refinement $\mathbf{D}$ converges to $\bar{\mathbf{D}}$. Consequently, σ and ϵ^c (obtained using $\bar{\mathbf{D}}$) are used as the assumed stress and strain fields for equation (2.5).

The proposed method is summarized in Box 3.2.

4. CONCLUDING REMARKS

The main objective for this paper is to provide, within the framework of mixed finite element methods, a general methodology that may be applied to the solution of different problems regardless of the presence of internal constraints. This objective is achieved by constructing the assumed stress and strain fields so that they *a priori* satisfy the internal constraints.

Box 3.2: Proposed Method

1. Select initial assumed (enriched) strain field, $\bar{\epsilon}^c$, and stress field, $\bar{\sigma}$ (Pascal triangle).
2. Select the incompatible strain field, ϵ^i, (U^i).
3. Reduce the number of independent variables in $\bar{\epsilon}^c$ and $\bar{\sigma}$ by introducing the following constraints:

$$\int_\Omega U^i_{,i} \, d\Omega = 0$$

$$\int_\Omega [\epsilon^{iT} (\bar{\sigma} - \sigma_0)] \, d\Omega = 0 \quad \rightarrow \sigma$$

$$\int_\Omega [\epsilon^{iT} (\bar{\mathbf{D}} \bar{\epsilon}^c - \mathbf{D} \epsilon_0)] \, d\Omega = 0 \quad \rightarrow \epsilon^c$$

4. Substitute the reduced fields into the variational principle.

Subsequent work will be concerned with the application of the proposed method to problems involving internal constraints. In particular, attention will be directed to the numerical implementation of the following problems:

1. **Plane stress/strain**: The development of plane elements that avoid locking at the nearly incompressible limit. Four-node elements are developed using the minimal number of incompatible modes, in order to reduce computational effort. These elements are shown to be among the best four-node elements presented in the literature. Furthermore, it is proved that the trace of the strain field vanishes at the nearly incompressible limit.

2. **Plate bending**: The development of plate bending elements based upon the Reissner-Mindlin plate theory.. It is shown that the proposed method leads to the introduction of coupling between the bending and transverse shear terms. It is proved that as a result of this coupling shear locking at the thin plate limit is avoided at the element level. Four-node elements are developed using the minimal number of incompatible modes. These elements are shown to be among the best four-node elements presented in the literature.

3. **Nonlinear formulation**: The present formulation can be extended to account for non-linear behavior (material and geometrical). The presense of initial stresses and strains

in the present formulation makes the extension to an iterative scheme straight forward for nonlinear elastic materials. When inelastic material properties are present, the present formulation may be incorporated within the framework of the return map algorithms (e.g., Simo & Hughes [1990]). These and related issues will be addressed in subsequent work.

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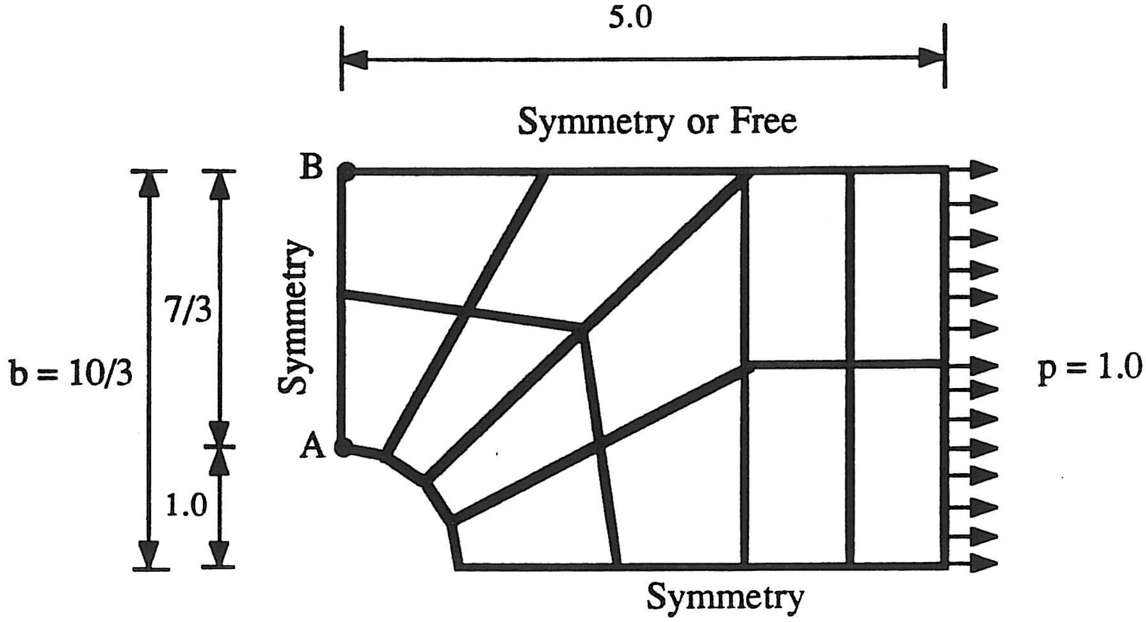


Figure 1.1a: Strip with a hole problem; plain strain ν = 0.4999; sample mesh.

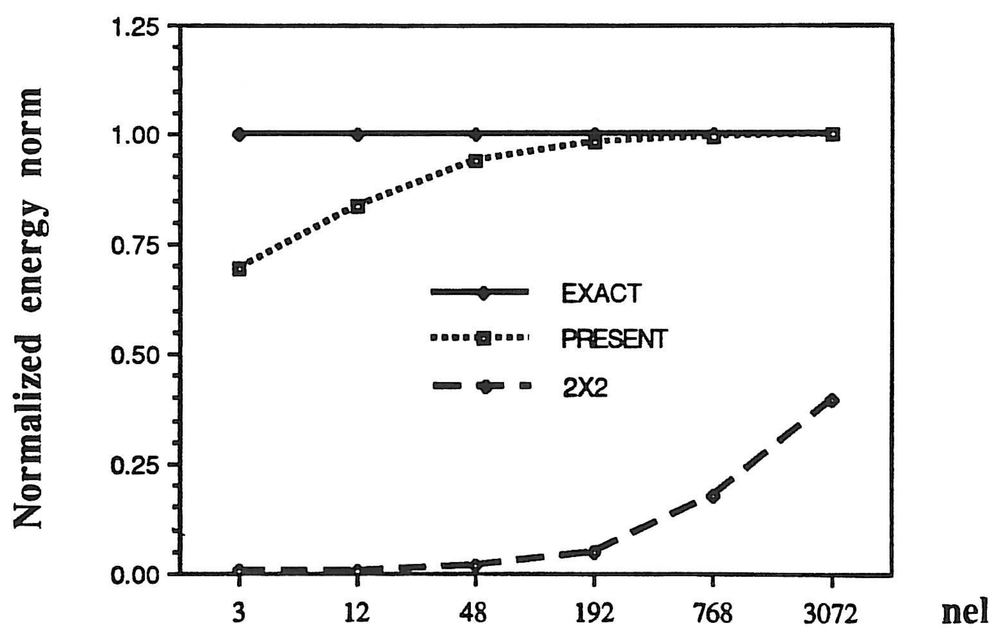
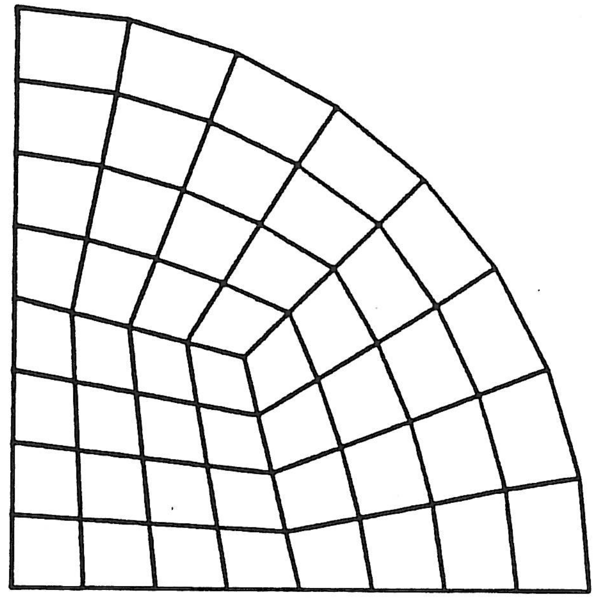


Figure 1.1b: Strip with a hole; plane strain, ν = 0.4999;
convergence in the energy norm.



Fugure 1.2a: Circular thin plate - sample mesh; r= 5.0, t = 0.1, ν = 0.3. Due to symmetry only one quadrant is discretized.

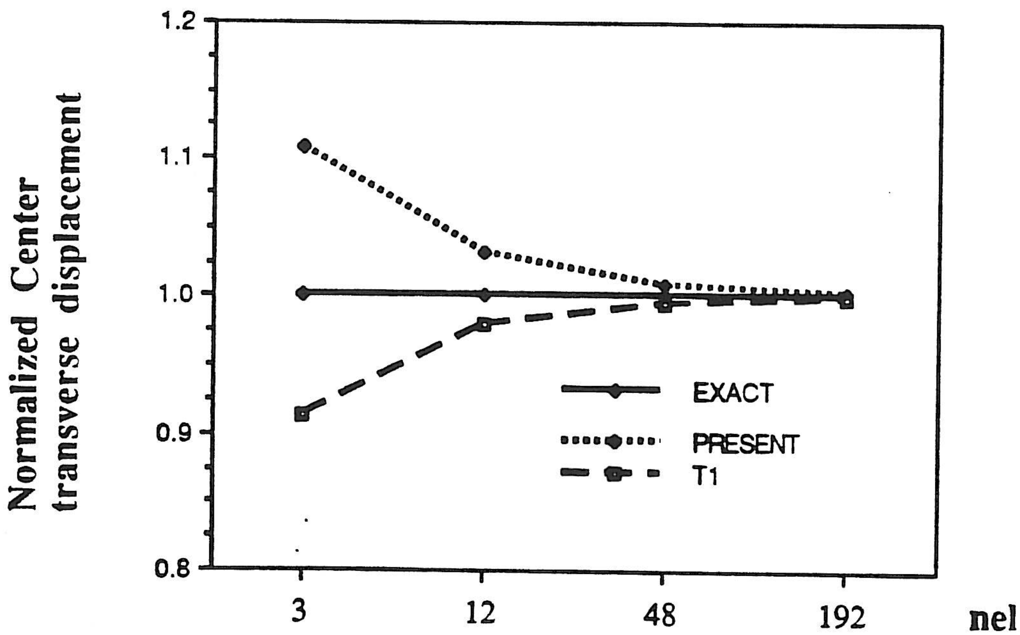


Figure 1.2b: Simply supported (SS1) thin circular plate; uniform load; convergence of center transverse displacemenet.