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Head Internal Relative Clauses, Quantifier Float, the Definiteness Effect and the Mathematics of Determiners

S.-Y. Kuroda
University of California, San Diego

In memory of Tanya Reinhart

Keenan's account of the DEFINITENESS EFFECT associated with the English *there*-construction based on the notion of INTERSECTIVE DETERMINER is well-known. In Part 1 of this paper, I will consider a similar kind of effect in Japanese constructions. In particular, I shall show that a very general form of Quantifier Float and the Head Internal Relative Clause, two phenomena particularly prominent in Japanese syntax, allow us to extend the idea of the Definiteness Effect to predicates with more than one argument. I will then show in Part 2 that this idea provides an empirical motivation for extending Keenan's idea of intersective determiners to 2-dimensional (transitive) spaces. Part 2 thus concerns a mathematical extension of the classical mathematical theory of determiners to 2-dimensional spaces with its empirical grounding in Japanese syntax. Part 3 generalizes the mathematical theory introduced in Part 2 to n-dimensional spaces in general, without any empirical concern.

Part 1, Part 2 and Part 3 can in principle be read independently. Part 1 primarily concerns Japanese and is empirical and descriptive. The reader who is not particularly interested in mathematics may wish to read only Part 1. Those who are interested in the "mathematics of language" but not particularly in details of Japanese syntax may wish to skim through Part 1 and start careful reading from Part 2. On the other hand, those who are only interested in mathematics and do not care about, or wish not to be bothered by, empirical facts may wish to read only Part 3. Due to the intended relative independence of the three Parts, the reader who wishes to read through the paper from Part 1 through Part 3 may encounter some redundancy through the paper.

Part 1, sections 1-5, is a slightly revised version of the first five sections of Kuroda (2007). I wish to express my gratitude to the CSLI Publications for granting me a permission to reproduce this portion in this paper.

1. Part I: Existential Sentences and the Definiteness Effect in Japanese

1.1 Preliminaries

The definiteness effect of the English *there* construction is well known.[1]

(1) Definiteness effect. Post-copula DPs in *there* sentences are indefinite.

The contrast between a and b, and between c and d in (2) illustrates this generalization:

[1] This is an expanded version of the paper I read at the 16th Japanese/Korean Linguistics Conference held in Kyoto in October, 2006 and published in the Proceedings of the Conference, Takubo (2007), under the title "HIRC, QF and the Definiteness Effect."

(2) a. There are three apples that I bought at the market on the table.
b. *There are the three apples that I bought at the market on the table
c. There are lots of apples that I bought at the market on the table
d. *There are most of/all the apples that I bought yesterday on the table

The term 'indefinite' is used in (1) without any formal definition in mind for the convenience of stating the generalization. Commonly, following Milsark (1974), the technical terms WEAK and STRONG are defined in terms of the observed definiteness effect:

(3) *Definition 1.* Weak and strong determiners (English)
The determiners that can modify post copula DPs in *there* sentences are by definition WEAK; those that cannot are STRONG.

We have some examples of weak and strong determiners in English in (4) and (5):

(4) Weak determiners: *three, a lot of, some, many, no*, …..

(5) Strong determiners: *the, the three, all, most,* .....

Edward Keenan gave a formal characterization of weak determiners:

(6) Determiners are weak iff they are *intersective.* (Keenan 1987)

The definition of INTERSECTIVE is given as follows in (7):

(7) *Definition 2.* Intersective determiners
A determiner D is INTERSECTIVE iff $D(A)(P) = D(A \cap P)(E)$.

Take A as a common noun like *student*, P as one place predicate like *be diligent*. Let D be the determiner *three.* Then, D(A) in the left hand side of the formula in (7) reads 'three students'; combine it with P *be diligent*, then, D(A)(P) reads as 'three students are diligent'. In the right hand side, $A \cap P$ stands for 'students who are diligent'; the symbol E is the predicate *exist* and stands for anything that exists. Thus, the right hand side of the formula reads 'three students who are diligent exist' or 'three diligent students exist'. Definition 2 says that the determiner *three* is by definition intersective if and only if the first equivalence in (8) holds:

(8) Informal illustration of *intersective*:

a. [Three (students)] are diligent	$<=>$	[Three (diligent students)] exist
b. [Every (student)] is diligent	$<\neq>$	[every (diligent student)] exists

The equivalence (8)a holds, hence *three* is an intersective determiner. In contrast, the determiner *every* is not intersective, since the corresponding equivalence does not hold as shown in (8)b. If the left hand side is true, the right hand side is also true, but even if the left hand side is not true, the right hand side is still true, thus the equivalence fails.

Keenan thus has given a formal account of the definiteness effect. Assertion (6) may be reformulated as follows:

(9) A *there* sentence is well-formed iff the post-copula DP has an intersective determiner.

About the definiteness effect, Keenan in fact obtained a much more general result than (9): a formal characterization of the type of noun phrases, simple as well as complex, that can occupy the post-copula position of the *there* construction. Assertion (9) concerns with simple DPs and is only a part, but the basic part, of Keenan's account of the definiteness effect. For our present purposes (9) suffices.

A mathematical theory of determiners, whose scope goes much beyond the definiteness issue, has been developed by Keenan in last 15 years or so. It has revealed much about interesting formal properties of determiners in general and of intersective determiners in particular. What strikes me as remarkable about this theory is that not only does it provide empirically adequate accounts of generalizations obtained in natural language studies, it also concerns a highly interesting and significant structure in its own right as an object of mathematics. I believe Keenan's work is one of the greatest achievements in formal linguistics in recent years. See Keenan (1987, 2002) among others.

It is significant that there is a grammatical effect in natural language, the definiteness effect, that relates grammar to mathematics: the *there* construction, so to speak, embodies a concept rooted in mathematics such as intersective determiner. But the definiteness effect, as we have introduced above, is defined in terms of the *there* construction, an English particular structure. Is this state of affairs to be understood as a peculiar feature of English, and perhaps of some other few languages where we can find a direct counterpart to the *there* construction? I would rather like to test the hypothesis that it is a common feature of natural languages, sanctioned by universal grammar, that they contain a construction which embodies intersective determiners, that is, which is licensed by intersective determiners.

And we might let ourselves drift in more fanciful imagination and suppose that human language has ways to have contacts with mathematical reality, as it does with mental and physical reality. Universal grammar might also be determined by how it interacts with this reality. We might hypothesize that along with A-P (Articulatory-Perceptual) and C-P (Conceptual-Intentional) Interface Conditions there could be another type of interface conditions, say, F-M (formal-mathematical) Interface Conditions, that mediate between language and mathematical reality. The definiteness effect in the usual sense defined in terms of English grammar must then be interpreted as an instance of an F-M interface condition that is actualized in English, and we might expect that a definiteness effect in a certain generalized sense be formulated in terms of universal grammar free of reference to constructions in some particular languages. Whether and/or how we can understand mathematical reality intelligibly would be, and would remain as, one of the most puzzling problems. Nonetheless, we might put the above thought in a form of a thesis:

(10) Thesis of mathematico-linguistic Platonism:
Human language interfaces with Mathematical Reality. A definiteness effect formulated in a certain generalized and universal form is an instance of a Formal-Mathematical Interface Condition in universal grammar.

This thesis could be interpreted as taking a position for linguistic Platonism, but it should not be meant as taking a simplistic form of realism according to which E-languages in Chomsky's sense are entitled to existence in Platonic reality.[2]

[2] I assume that mathematics is concerned with a reality which transcends nature, the reality science is concerned with, and in this sense it is Platonic. But I would not maintain here anything about this Platonic reality beyond the fact that it imposes conditions on nature; in particular, I would not commit myself to any propositions that might imply the EXISTENCE of any particular ENTITIES in a Platonic world, be it language or other abstract objects. For the Platonist view that takes natural languages to be abstract objects, see Katz (1981, 1984), Langendoen and Postal (1984, 1985) and Katz and Postal (1991). For a succinct exposition of a Platonist position from a different perspective, see Koster (2006).

Be that as it may, we must now set out to search for an English-independent definition of "existential sentences" where the "definiteness effect" holds in analogy to (9). But before doing so, let us first look back to English and introduce a couple of terms in preparation for our exploration outside English.

1.2 Sentence types in English

1.2.1 The plain sentence S

First of all, for now we restrict our consideration to sentences with a one- place (one-argument) predicate.

(11) PLAIN SENTENCE:
Let D be a determiner, A a common noun and P a one-place predicate; we call a sentence of the following form a plain sentence:
Syntax: $S = [[D\ A]_{DP}\ P]_{CP}$
Semantics: $\langle S \rangle = [D(A)](P)$

(12) Examples of plain sentences:
a. Three students are diligent
b. Most students are diligent

1.2.2 The there transform ${}^tS$

Next, let us agree to call sentences of the following form *there* transforms:

(13) *There* transforms:
Syntax: ${}^tS$ = [there be $[D[A\ [\text{that } P]_{CP}]\ ]_{DP}]$
Semantics: $\langle {}^tS \rangle = [D(A \cap P)](E)$

To relate *there* transforms to plain sentences, we have the following statement:

(14) A plain sentence S and its *there* transform ${}^tS$ are equivalent (${}^tS$ paraphrases S) if and only if D is intersective.

Indeed, if D is not intersective, a *there* transform is ungrammatical, according to assertion (9), hence the equivalence fails. On the other hand, if D is intersective, the plain sentence is equivalent to the corresponding *there* transform. These facts are exemplified by (15) and (16) below:

(15) most (of the) students are diligent <≠> *there are most (of the) students who are diligent

(16) three students are working <=> there are 3 students who are working
three students are available <=> there are three students available
three students are diligent <=> there are three students who are diligent
three students are tall <=> there are three students who are tall

Some of the sentences like those on the left hand side of <=> with individual level predicates may be judged awkward in English, but for the sake of space I ignore this aspect of complication and assume that they are grammatical without further comment.

Let us summarize: The initial definiteness effect, informally formulated, states that the D in the *there* transform defined in (13) must be indefinite, but Keenan asserts that *indefinite* can be replaced by a formally defined term *intersective*. Statement (14), then, says something very close to the original definiteness effect. However, it is, on the one hand, stronger than the definiteness effect: if D is intersective, (14) not only implies that the *there* transform is grammatical but also states that it is equivalent to the plain sentence. On the other hand, (14) is weaker than the definiteness effect in another respect, because it does not say that the *there* transform is ungrammatical if D is not intersective; it only states that the *there* transform cannot be equivalent to the plain sentence.

1.2.3 The existential transform ^S

Let us go back to the syntax of the *there* transform in (13); the existential predicate *(there) is/are* precedes the post-copula noun phrase. If we put the existential predicate after the noun phrase, we get sentence forms with the canonical word order without expletive *there* as exemplified below in (17).

(17) most (of the) diligent students are

(18) three students who are working are

These forms are hardly acceptable in English, but we could substitute *exist* for *are*. Then, we would get better acceptability:

(19) most (of the) diligent students exist

(20) three students who are working exist

Let us agree to call the sentence form illustrated by (17)-(20) the existential transform of S, with the notation ^S:

(21) EXISTENTIAL TRANSFORMS:
Syntax: $\hat{}S = D(A\cap P)$be/exist
Semantics: $\langle \hat{}S\rangle = [D(A\cap P)](E)$

The semantics of the existential transform is the same as that of the *there* transform: $\langle \hat{}S\rangle = \langle {}^{t}S\rangle$. Hence, if D is intersective, by the definition of *intersective,* ^S paraphrases ${}^{t}S$, and hence it does S, too. On the other hand, if D is not intersective, ${}^{t}S$ is not grammatical, and it is senseless to ask if ^S paraphrases ${}^{t}S$ or not. But we can, and must ask if $\hat{}S$ paraphrases S or not; as a matter of fact, ^S does not paraphrase S: (19) does not imply (12)b. Thus, we can make the same statement for ^S as for ${}^{t}S$:

(22) The Plain Sentence S and its existential transform $\hat{}S$ are equivalent ($\hat{}S$ paraphrases S) if and only if D is intersective.

1.3 In search of an English-independent definition of the definiteness effect

We are now in a position to leave English behind. First of all, let us agree that the concept *plain sentence* can be used beyond English in tact. Now, we wish to extend the idea of the *definiteness effect* beyond English and we do this by turning statement (14) into a definition:

(23) *Definition 3*: The generalized definiteness effect
A construction with a one-place predicate has the DEFINITENESS EFFECT if it satisfies the following condition: it paraphrases the corresponding plain sentence if and only if the determiner D associated with the subject is intersective.

And on the basis of this definition, let us define *existential sentence*:

(24) *Definition 4*: Existential sentence
A sentence consisting of a determiner D, a common noun A and a one-place predicate P is EXISTENTIAL if it has the definiteness effect.

According to this definition, both the *there* transform and the existential transform are existential sentences in English. Note, however, that the above two definitions do not imply that an existential sentence is ungrammatical if D is not intersective, as is the case with the *there* transform.

1.4 Sentence types and determiners in Japanese

We are now going to examine Japanese sentences. But before doing so, let me make one disclaimer. Certain determiners are subject to partitive and non-partitive interpretations. For example, consider the following three sentence forms with the determiner *san-nin* 'three persons':

(25) a. san-nin no gakusei ga asoko de hataraite-iru
three-CLS GEN student NOM there at working are
b. gakusei ga san-nin asoko de hataraite-iru
c. gakusei ga asoko de san-nin hataraite-iru
'three (of the) students are working there'

In (25)a the determiner *san-nin* is inside a DP, while it floats in (25)b and (25)c. In all of these forms this determiner can be interpreted either as a partitive or not a partitive, as the translations indicate, perhaps with different degrees of preference to one or the other interpretation. Whether we should take these sentences as ambiguous between these two readings or simply as vague about these readings is a moot question. For the sake of space, let us ignore this issue and agree to disregard partitive readings in the examples that follow.

1.4.1 The plain sentence

The definition of plain sentence is the same as before, which I repeat:

(26) PLAIN SENTENCE
Syntax: $S = [[D\ A\ (ga)]_{DP}\ P]_{CP}$
Semantics: $\langle S \rangle = [D(A)](P)$

Examples follow.

(27) a. san-nin no gakusei ga kinben-da
three-CLS GEN students NOM diligent-are
'three (of the) students are diligent'

b. hotondo no gakusei ga kinben-da
almost GEN student NOM diligent-are
'most (of the) students are diligent'

c. san-nin no gakusei ga hataraite-iru
three-CLS GEN students NOM working-are
'three (of the) students are working'

d. hotondo no gakusei ga hataraite-iru
almost GEN student NOM working-are
'most (of the) students are working'

Sentences with individual level predicates and indefinite subjects like (27)a may sound awkward, but again for the sake of space, I ignore this issue here without further comment.

1.4.2 The Q-float plain sentence

In Japanese most determiners, in particular, numerals, can float. We have Q-float plain sentences:

(28) Q-FLOAT PLAIN SENTENCE
Syntax: ${}^{f}S = [[A\text{-}ga]_{DP}\ [D\ P]]_{CP}$
Semantics: $\langle {}^{f}S \rangle = \langle S \rangle = [(DA)](P)$

I assume without argument here that the "floating" determiner in a Q-float plain sentence is an adverb adjoined to the verb phrase. The semantics of the Q-float plain sentence is the same as that of the plain sentence. Q-float keeps meaning invariant.

(29) a. san-nin no gakusei ga kinben-da (= (27)a)
<=> gakusei ga san-nin kinben-da
'students are, three of them, all diligent'

b. hotondo no gakusei ga kinben-da (= (27)b)
<=> gakusei ga hotondo kinbenda
'students are, most of them, diligent'

We conclude:

(30) ${}^{f}S$ paraphrases S.

(31) The Q-float Plain Sentence does not have the definiteness effect.

1.4.3 The existential transform

1.4.3.1 THE HE (head-external) existential transform ^S

Now observe the following examples. Sentences that follow <=> or <≠> are the plain sentences corresponding to those that precede them and illustrate HE existential transforms. Inserting *koko ni/de* 'here' may help process these examples, but *here* must be understood as referring to the entire model/universe, not to a particular location in the model, one place 'here' as opposed to another place 'there'.

(32) a. san-nin no kinben-na gakusei ga (koko-ni) iru
<=> san-nin no gakusei ga kinben da
three-CLS GEN diligent-be student NOM (here-at) are
'three students that are diligent are (here)'
<=> 'three students are diligent'

b. hotondo no kinben-na gakusei(-tati) ga (koko-ni) iru
    <≠> hotondo no gakusei(-tati) ga kinben da
    almost GEN diligent-are student(PL) NOM (here-at) are
    'most of the diligent students are (here)'
    <≠> 'most of the students are diligent'

These examples illustrate the fact that the construction *....ga aru/iru* shows the definiteness effect, that is to say, if D is not intersective, ^S does not paraphrase S; indeed, ^S (may not be ill-formed but) is a tautology or contradiction, modulo existential presupposition, and may sound odd. If D is intersective, ^S is well-formed and paraphrases S. Hence this construction is existential. We agree to called this construction the HEAD EXTERNAL (HE) EXISTENTIAL TRANSFORM: in the construction the subject of the main verb *aru/iru* is modified by the head external relative clause derived from the corresponding plain sentence:

(33) HE EXISTENTIAL TRANSFORM
    Syntax: ^S = [[D[[e]^P]$_{CP}$A]$_{D'}$]$_{DP}$ (ga) aru/iru]$_{CP}$,
        where [[e]^P]$_{CP}$ is a head-external relative clause, whose head is A and [e] is an empty subject coindexed with A.

    Semantics: <^S> = [D[[P]A]]]aru/iru = D(A ∩ P)(E)

The determiner D modifies the complex noun phrase, not just common noun A. It follows that the HE existential transform ^S is equivalent to the plain sentence S iff D is intersective, since the condition D(A)(P) = D(A ∩ P)(E) is exactly the one that defines the intersective determiner. Thus, we have the following observation:

(34) The HE existential transform has the definiteness effect.

Put it another way, according to our definition of *existential sentence*:

(35) The HE existential transform is an existential sentence.

1.4.3.2 The HI (Head Internal) existential transform

Next let us substitute head internal relative clauses for head external relative clauses in existential transforms. Again, the sentences that follow <=> are the plain sentences that correspond to those forms that precede them. In this case existential transforms are equivalent to the corresponding plain sentences whether the determiners contained in them are intersective or not:

(36) a. san-nin no gakusei ga kinben-na no ga iru
    three-CLS GEN students NOM diligent-are COMP NOM are
    <=> san-nin no gakusei ga kinben-da
    'Three students that are diligent are/exist'
    <=> 'three students are diligent'

b. san-nin no gakusei ga hataraite-iru no ga iru
    three-CLS GEN students NOM working-are COMP NOM are
    <=> san-nin no gakusei ga hataraite-iru
    'three students that are working are/exist'
    <=> 'three students are working'

c. hotondo no gakusei ga hataraite-iru no ga (koko-ni) iru
most GEN student NOM working-are COMP NOM (here-at) are
<=> hotondo no gakusei ga hataraite-iru
'most of the students, who are working, are/exist (here)'
<=> 'most of the students are working'

Let us call the construction illustrated by these examples the HI (Head Internal) existential transform:

(37) HEAD INTERNAL EXISTENTIAL TRANSFORM
Syntax: $\oplus S = [[[DA]_{DP}P]_{CP}]_{DP}$ ga aru/iru,
where $[[[DA]_{DP}P]_{CP}]_{DP}$ is a head-internal relative clause with DA being its semantic head.
Semantics: $\langle \oplus S \rangle = [[DA]_{DP}P]_{CP}$ & pro_e ga aru/iru,
where pro_e is an e-type pronoun coindexed with $[DA]_{DP}$.

Whether the determiner is intersective or not, the HI existential transform paraphrases the associated plain sentence; thus, it does not have the definiteness effect and is not an existential sentence. We have the following observation:

(38) The HI existential transform is not existential.

1.4.3.3 The HI (head internal) existential transform with a FQ
Let us now consider the following examples.

(39) a. gakusei ga kinben-na no ga san-nin iru
student NOM diligent-be COMP NOM three-CLS are
<=> san-nin no gakusei ga kinben-da
'Students who are diligent are/exist three in number'
<=> 'three students are diligent'
b. gakusei ga hataraite-iru no ga san-nin iru
students NOM work-are COMP NOM three-CLS are
`<=> san-nin no gakusei ga hataraite-iru
'students who are working are/exist three in number'
<=> 'three students are working'

c. gakusei ga hataraite-iru no ga hotondo (koko-ni) iru
student NOM working-are COMP NOM most (here) are
<≠> hotondo no gakusei ga hataraite-iru
'students who are working are/exist (here), most of them'
(= 'most of the students who are working are/exist (here)')
<≠> 'most of the students are working'

These examples are similar to HI existential transforms in that the subjects are HIRCs, but the determiners are outside of HIRC and are adjoined to the matrix existential verb. At the first sight D might appear to float from (that is, be associated with) the embedded subject, but it does not. Instead, it is associated with the matrix subject, which is a HIRC. This difference is crucial; it means that D must be associated with the E-type pronoun pro in the semantics given above for the HI existential transform. Thus, we have the following syntax and semantics for the HI existential transform with a FQ:

(40) HEAD INTERNAL EXISTENTIAL TRANSFORM WITH a FQ
Syntax: $^{f}\oplus S = [[[A]_{DP}P]_{CP}]_{DP}$ ga D aru/iru,
where $[[A]_{DP}P]_{CP}]_{DP}$ is a HIRC with A its semantic head.
Semantics: $<^{f}\oplus S> = [[A]_{DP}P]_{CP}$ & D(pro_e)(be),
where pro_e is an e-type pronoun coindexed with $[A]_{DP}$.

Note that the E-type pronoun pro_e here does not have the same denotation as *A*, but rather it denotes those *A*'s that satisfy the condition defined by the HIRC. Take (39)c above, for example. The semantics of this sentence would look like as follows:

(41) students are working and [most of them] are (here).

Here *them* is meant to refer, not to 'students' but 'those students who are working'. Hence meaning is not kept invariant unless the determiner is intersective. We can confirm this situation from the above examples and we have the following observation:

(42) $^{f}\oplus S$ has the definiteness effect.

(43) $^{f}\oplus S$ is an existential sentence.

1.5 Summary

Let us summarize the observations we had above.

(44) Sentence types (English):

Plain sentence:	$S = [[D\ A]_{DP}\ P]_{CP}$
There transform:	^{t}S = [there be $[D[A$ [that $P]_{CP}]\ ]_{DP}]$
Existential transform:	$^{\wedge}S = [[D[A$ [that $P]_{CP}]]_{DP}$ be/exist]

(45) Sentence types (Japanese):

Plain sentence:	$S = [[D\ A\ (ga)]_{DP}\ P]_{CP}$
Q float plain sentence:	$^{f}S = [[A\text{-}ga]_{DP}\ [D\ P]_{CP}$
HE Existential transform:	$^{\wedge}S = [[D[[e]^{\wedge}P]_{CP}A]_{D'}]_{DP}$ (ga) aru/iru$]_{CP}$
HI Existential transform :	$\oplus S = [[[DA]_{DP}P]_{CP}]_{DP}$ ga aru/iru
HI Existential transform with a FQ:	$^{f}\oplus S = [[[A]_{DP}P]_{CP}]_{DP}$ ga D aru/iru

In the following tables (46) and (47) <=> and <≠> mean that the forms in the first column are equivalent to and not equivalent to S, respectively.

(46) The definiteness effect (English):

Determiner	Intersective	Non-intersective	Def Effect
S			
^{t}S	<=>	*	+DF
$^{\wedge}S$	<=>	<≠>	+DF

(47) The definiteness effect (Japanese):

Determiner	Intersective	Non-intersective	Def Effect
S			
^{f}S	<=>	<=>	-DF
^S	<=>	<≠>	+DF
⊕S	<=>	<=>	-DF
^{f}⊕S	<=>	<≠>	+DF

Let us conclude. To begin with, the HE existential transform exists both in English and in Japanese, with the definiteness effect. The English *there* transform is peculiar in that it has a stronger form of the definiteness effect. Not only is it the case that the *there* transform is equivalent to the plain sentence if and only if D is intersective, but also it is well-formed only if D is intersective. This is the original, stronger form of the definiteness effect.

English, on the other hand, lacks HIRC as well as a general form of quantifier float. HIRC keeps the function of determiners invariant, and hence yields an existential transform without the definiteness effect, i.e., ⊕S, in Japanese. I am here using the term "existential transform" for ⊕S, but this sentence is not "existential" in the sense I defined this term in (24).

Now, note the difference between the HI existential transform ⊕S and the HI existential transform with a QF, i.e., ^{f}⊕S. The latter form has the definiteness effect. This form is of a particular interest for us, since it leads us to the extension of the theory of determiners I am going to pursue in the next section. I withhold further comment on this form here. For now, let me continue by making a comment on the term *quantifier float*.

I employ here the term *quantifier float* as a descriptive term, informally referring to a phenomenon, not to any exactly defined grammatical process. A number of delicate issues are involved with this phenomenon: the distinction between partitive and non-partitive readings of determiners mentioned above at the beginning of section 4; the distinction between genuinely adverbial "floating" quantifiers and in some sense genuinely floating "floating" quantifiers. Besides, there is an issue of quantifiers that only look like "floating." A comprehensive treatment of the phenomenon of "quantifier float" is thus beyond the scope of this paper. However, although I have not worked out any detail, my feeling is that only intersective determiners should be taken as capable of being genuinely adverbial. If this hypothesis holds and if we agree to understand floating quantifiers in the definitions of all the constructions prefixed with a superscript *f* in the above table as genuinely adverbial, then ^{f}⊕S, and even ^{f}S, too, should turn out to have the definiteness effect in the *strong* form, i.e., they are *grammatical* if and only if D is intersective, just like the English *there* transform. We could claim, then, that it would bring Japanese to the same status as English with respect to the fact that the mathematical concept of *intersective* is embodied in sentence structures of natural language, by the *there* transform in English, and the constructions with "floating" *adverbial* quantifiers in Japanese.

But I have to leave these matters aside for now. We do not make any distinction among "floating" determiners as to whether they are in fact floating and originate in an argument position in some sense or genuinely adverbial. This supposition suffices for our claim that ^{f}⊕S satisfies the definiteness effect in the generalized sense defined in (23). I would now like to proceed to another claim concerning HIRC, and demonstrate that it can contribute to providing empirical motivation for a significant generalization of Keenan's mathematical theory to a new direction.

1.6 Binary determiners (determiners in 2-dimensional spaces)

I now wish to extend our study to sentences with transitive predicates. Observe the following examples in (48). (48)a is a *plain* sentence S with a transitive verb, *oi-atumete-iru* 'to chase and gather, herd' with both the subject and the object accompanied by intersective determiners. A sentence with multiple determiners is in general subject to more than one interpretation, but in what follows we are interested only in the group reading. The choice of the predicate *oi-atumete-iru* 'herd' is thus deliberate; the semantics of the predicate excludes distributive readings. We also exclude possible partitive readings of indefinite determiners; *go-hiki no inu*, for example, is to be interpreted as 'five dogs,' excluding the possible reading 'five of the'.

Since the verb has two arguments, we can derive two HERC from S, with the subject and the object, respectively, as heads. So, we can construct two HE EXISTENTIAL TRANSFORMS, ^^{1}S and ^^{2}S, as shown in (48)b and c. (48)d may sound somewhat, or maybe considerably, awkward, as two quantifiers in the plain sentence S are forced to float, but I would take it acceptable. (48)e illustrates a structure impossible in English: ⊕S, a HI EXISTENTIAL TRANSFORM. Finally, we have (48)f, ^{f}⊕S a HI EXISTENTIAL TRANSFORM WITH FLOATING QUANTIFIERS, an example instrumental to my argument below, where both determiners associated with the two arguments of the original plain sentence are floating outside the HIRC. Again, this sentence is unusual, but I would accept it and would take it grammatical. Note that the HIRCs in (48)e and f are both meant to have split heads, *go hiki no inu* and *sanzyu-too no usi*, as indicated by the translation given to (48)e and f.

(48) a. S go-hiki no inu ga sanzyut-too no usi o oi-atumete-iru
five-CLF GEM dog NOM 30-CLF GEN cow ACC chase-gather-are
'five dogs are herding thirty cows'

b. ^^{1}S sanzyut-too no usi o oi-atumete-iru go-hiki no inu ga iru
30-CLF GEN cow ACC chase-gather-are 5-CLF GEN dog NOM are
'there are five dogs that are herding thirty cows'

c. ^^{2}S go-hiki no inu ga oi-atumete-iru sanzyut-too no usi ga iru
5-CLF GEN dog NOM chase-gather-are 30-CLF GEN cow NOM are
'there are thirty cows that five dogs are herding'

d. fS inu ga usi o go-hiki ga sanzyut-too o oi-atumete-iru
dogs NOM cow ACC 5-CLF NOM thirty-CLF ACC chase-gather-are
'dogs are herding cows, five and thirty in number, respectively'

e. ⊕S go-hiki no inu ga sanzyut-too no usi o oi-atumete-iru no ga iru
5-CL GEN dog NM 30-CF GN cow AC chase-gather-are CMP NM are
'5 dogs are herding thirty cows and they all exist'

f. ^{f}⊕S inu ga usi o oi-atumete-iru no ga 5-hiki to sanzyut-too iru
dog NM cow AC herding-are CMP NM 5-CLF and thirty-CLF be
'dogs are herding cows, and there are, 5 and 30, respectively'

Displayed in (49) below is the same set of examples with non-intersective determiners, *daibubun* 'most' and *zenbu* 'all', substituting for intersective determiners *go-hiki* and *zyut-too*, respectively.

(49) Examples with non-intersective determiners:

a. S daibubun no inu ga zenbu no usi o oi-atumete-iru
most GEN dog NOM all GEN cow ACC herd-are
'most of the dogs are herding all of the cows'

b. ^^{1}S zenbu no usi o oi-atumete-iru daibubun no inu ga iru
all GEN cow ACC herd-are most GEN dog NOM are
'most of the dogs that are herding all of the cows exist'

c. ^^{2}S daibubun no inu ga oi-atumete-iru zenbu no usi ga iru
most GEN dog NOM herd-are all GEN cow NOM are
'all the cows that most of the dogs are herding exist'

d. ^{f}S inu ga usi o daibubun ga zenbu o oi-atumete-iru
dog NOM cow ACC most NOM all ACC herding-are
'dogs, most of them, are herding cows, all of them'

e. ⊕S daibubun no inu ga zenbu no usi o oi-atumete-iru no ga iru
most GEN dog NOM all GEN cow ACC herding-are CMP NOM are
'most of the dogs are herding all of the cows and they both exist'

f. ^{f}⊕S inu ga usi o oi-atumete-iru no ga daibubun to zenbu iru
dog NOM cow ACC herding-are CMP NOM most and all are
'dogs are herding cows and most of the dogs and all of the cows exist'

Let us extend the definition (23) of *definiteness effect* to these sentence constructions with transitive verbs as follows:

(50) *Definition 3'.* The generalized definiteness effect
A construction with a two-place predicate has the DEFINITENESS EFFECT if it satisfies the following condition: it paraphrases the corresponding plain sentence if and only if the determiners D associated with the subject and the object are both intersective.

Then, we get the following result concerning the definiteness effect.

(51) Summary. Japanese transitive sentence constructions 1:

D S	Intersective	Non-intersective	Definiteness Effect
fS	<=>	<=>	-DF
^^{1}S	<=>	<≠>	+DF
^^{2}S	<=>	<≠>	+DF
⊕S	<=>	<=>	-DF
^{f}⊕S	<=>	<≠>	+DF

If we compare this result about the transitive constructions, (51), with the earlier result about the intransitive constructions, (47), we attained basically the same result, provided that we ignore the superscripted prefixes ^{1} and ^{2}. Thus, if we extend the definition (24) of *existential sentence* directly to transitive sentences at this point, we get two sets of existential sentences, and would get the impression we are in the same situation as with intransitive sentences, as shown in (52).

But it is yet premature for us to draw such a conclusion. With transitive sentences, we have two arguments as independent parameters. We can, and we must, substitute a non-intersective determiner for an intersective one for one argument, leaving the other determiner invariant, and try to see what effects we get. We obtain the set of sentences in (53) below by substituting a non-intersective determiner for an intersective one in subject position.

(52) Summary. Japanese transitive sentence constructions 2:

D	Intersective	Non-intersective	Definiteness Effect
S			
^{f}S	<=>	<=>	-DF
^^{1}S	<=>	<≠>	+DF
^^{2}S	<=>	<≠>	+DF
⊕S	<=>	<=>	- DF
^{f}⊕S	<=>	<≠>	+DF

(53) Examples with partially intersective determiners:

a. S daibubun no inu ga sanzyut-too no usi o oi-atumete-iru
most GEN dog NOM thirty-CLF GEN cow ACC herding-are
'most of the dogs are herding thirty cows'

b. ^^{1}S sanzyut-too no usi o oi-atumete-iru daibubun no inu ga iru
thirty-CLF GEN cow ACC herding-are most GEN dog NOM are
'most of the dogs that are herding thirty cows exist'

c. ^^{2}S daibubun no inu ga oi-atumete-iru sanzyut-too no usi ga iru
most GEN dog NOM herding-are thirty-CLF GEN cow NOM are
'there are thirty cows that most of the dogs are herding'

d. ^{f}S inu ga usi o daibubun ga sanzyut-too o oi-atumete-iru
dog NOM cow ACC most NOM thirty-CLF ACC herding-are
'dogs, most of them, are herding cows, thirty of them'

e. ⊕S daibubun no inu ga 30-too no usi o oi-atumete-iru no ga iru
most GEN dog NOM 30-CLF GEN cow ACC herd-are CMP NOM be
'most of the dogs are herding thirty cows and they both exist'

f. ^{f}⊕S inu ga usi o oi-atumete iru no ga daibubun to sanzyut-too iru
dog NOM cow ACC herding-are CMP NOM most and thirty-CLF are
'dogs are herding cows and most of the dogs and 30 of the cows exist'

With sentences like those in (53) we get the result summarized in the following table (54). The heading "intersective" at the second column means that both arguments are associated with intersective determiners; the heading at the third column "∂^2-intersective" means that the second argument, i.e., object position, is associated with an intersective determiner. An entry <=> in the third column means that an intersective determiner in object position entails that the sentence form in the first column of the same row is equivalent to, that is, paraphrases, the plain sentence form S; and an entry <≠> in the same column means that we have no such entailment. Notice that we have now no definiteness effect for ^^{2}S.

(54) Summary. Japanese transitive sentence constructions 3:

D	Intersective	∂^2-intersective	Definiteness Effect
S			
$\wedge^1$S	<=>	<≠>	+DF
$\wedge^2$S	<=>	<=>	- DF
$\oplus$S	<=>	<=>	- DF
^{f}S	<=>	<=>	- DF
$^f\oplus$S	<=>	<≠>	+DF

By substituting a non-intersective determiner for an intersective one at object position, instead of subject position, we get a similar result, symmetric with respect to the permutation of superscripts ^{1} and ^{2}. Combining this predicted result with (54) above, we can now draw the following final conclusion for the transitive structure as below in table (58). We redefine the term EXISTENTIAL, its derivatives and DEFINITENESS EFFECT as follows:

(55) *Definition 5*: Existential sentence
A construction built on a transitive predicate is EXISTENTIAL by definition in case it is equivalent to the plain sentence iff both determiners associated with the subject and object are intersective.

(56) *Definition 6*: Partial existential sentence
A construction built on a transitive predicate is PARTIALLY EXISTENTIAL by definition in case it is equivalent to the plain sentence iff the determiner associated with one of its argument is intersective; more specifically, it is ∂^1 existential (∂^2 existential) if it is equivalent to the plain sentence in case its subject (object) has an intersective determiner.

(57) *Definition 7*: Definiteness effect
A sentence construction with a transitive predicate has the DEFINITENESS EFFECT (and is EXISTENTIAL by definition) if it satisfies the following condition: it is equivalent to the corresponding plain sentence iff both determiners that are associated with the subject and the object are intersective.

(58) Japanese transitive sentence constructions 3

D	Intersective	∂^1-Intersective	∂^2-Intersective	non-Intersective	Existentiality
S	<=>	<=>	<=>	<=>	non-existential
$\wedge^1$S	<=>	<=>	<≠>	<≠>	$\partial^{1\text{-}}$existential
$\wedge^2$S	<=>	<≠>	<=>	<≠>	∂^2-existential
$\oplus$S	<=>	<=>	<=>	<=>	non-existential
^{f}S	<=>	<=>	<=>	<=>	non-existential
$^f\oplus$S	<=>	<≠>	<≠>	<≠>	existential

According to (58) only the HI existential transform with FQs, $^f\oplus$S, has the definiteness effect and is existential:

(59) The HI existential transform with FQs, $^f\oplus$S, has the definiteness effect and by definition is existential.

2. PART II: An Extension of Keenan's Mathematical Theory of Intersective Determiners to Two Dimensional Spaces

I would now like to discuss how our findings about Japanese determiners relate to Keenan's mathematical theory of determiners. We are going to extend the notion of determiners to "higher dimensions." Let us agree to use the term "unary determiner" for the determiner in the commonly understood sense. We are going to discuss binary determiners below, and, more generally, n-ary determiners in Part III.

2.1 Introduction: the theory of unary determiners

In this section I summarize the concepts and results from the formal theory of determiners. A unary determiner is defined as a function that maps common nouns A to functions from the set of one-place predicates P to the truth values {0, 1}. We agree to denote truth and falsehood by 1 and 0, respectively:

(60) D: A → (P → {0,1})
(i.e., D is an entity of type <<e, t>, <<e, t>, t>>)

According to this definition, D(A)(P) takes the value 1 or 0, depending on whether it is true or false.

Definition 1. A determiner is called *conservative* if it satisfies the following condition:

(61) D(A)(P) = D(A)(A∩P).
(Barwise & Cooper 1981, Keenan 1981, 2002)

Assertion 1. (Barwise & Cooper): Determiners of human languages are conservative.

This is an empirical assertion. It can be illustrated by the following examples, where the a-sentences are logically equivalent to the corresponding b-sentences:

(62) a. subete no gakusei ga kinben-da
all GEN student NOM diligent-are
'all students are diligent'

b. subete no gakusei ga gakusei de kinben-da
all GEN student NOM student and diligent-are
'all students are students and diligent'

(63) a. san-nin no gakusei ga hataraite-iru
three CLF GEN student NOM working-are
'three students are working'

b. san-nin no gakusei ga gakusei-de hataraite-iru
three CLF GEN student NOM student and working-are
'three students are students and working'

Keenan characterizes those determiners that can occupy the post-copula position of the *there* construction in English (weak determiners) as INTERSECTIVE. (Keenan 1987, 2002) An INTERSECTIVE DETERMINER is defined as follows:

Definition 2. A determiner D is *intersective* if it satisfies the following condition:

(64) $D(A)(P) = D(A \cap P)(E)$.

For example *(exactly) three*, is weak and intersective: (65) is grammatical and (66) and (67) are equivalent and shows that *three* satisfies (64).

(65) there are three students who are working

(66) three students are working

(67) three students who are working exist.

In contrast, (68) below is ungrammatical and (69) and (70) are not equivalent:

(68) *there are almost all students who are working

(69) almost all students are working

(70) almost all students who are working exist

almost all is not weak or intersective.

We can easily confirm that the following proposition holds:

Proposition 1. Intersective determiners are conservative.

For, if D is intersective we have the following equations:

(71) $D(A)(A \cap P) = D(A \cap (A \cap P))(E) = D(A \cap P)(E) = D(A)(P)$.

From the definitions of intersective and conservative determiners, one can also easily confirm the following propositions:

Proposition 2. D is intersective iff the following condition holds:

(72) If $A \cap P = A' \cap P'$, then $D(A)(P) = D(A')(P')$.

Proposition 3. D is conservative iff the following condition holds:

(73) If $A \cap P = A \cap P'$, then $D(A)(P) = D(A)(P')$.

Conditions (72) and (73) would serve to make *intersective* and *conservative* appear as a conceptual minimal pair. We could use these conditions as definitions of *intersective* and *conservative* instead of Definition 1 and Definition 2, respectively. In fact, for extending the concept of intersective determiners from unary determiners to determiners of higher dimensions, the definition based on (72) can for some cases serve as a better anchor for such an extension. So let me repeat Proposition 2 as a second version of the definition of intersective determiners:

Definition 2'. D is intersective iff the following condition holds:

(74) If $A \cap P = A' \cap P'$, then $D(A)(P) = D(A')(P')$.

Now, let **Int** be the set of intersective determiners. The following theorem is important for understanding the significance of the concept of intersective determiner:

Theorem 1. (Keenan) **Int** is isomorphic to $\mathbf{P}(\mathbf{P}(E))$, the set of sets of subsets of E. The isomorphism is given by $\iota: D \to \{X: D(X)(E) = 1\}$.

Proof: Let $D \neq D'$; take A, A', X, X' such that $D(A)(X) \neq D'(A')(X')$. Since D and D' is intersective, we have

$$D(A)(X) = D(A \cap X)(E) \neq D'(A')(X') = D'(A' \cap X')(E),$$
$$D(A \cap X)(E) \neq D'(A' \cap X')(E).$$

It follows that $\iota(D) \neq \iota(D')$; ι is one-to-one.

Conversely, take a set $\mathbf{X}$ of subsets of E and define a determiner D as follows: $D(X)(E) = 1$ if $X \in \mathbf{X}$ and $= 0$ if $X \notin \mathbf{X}$; more generally, for any pair X, $Y \subset E$, let $D(X)(Y) = 1$ if $X \cap Y \in \mathbf{X}$ and $= 0$ if $\notin \mathbf{X}$. Then, D is an intersective determiner. Hence, ι is an isomorphism.

As a matter of fact, "isomorphic" in the theorem should not mean simply "one-to-one onto", but more deeply "isomorphic with respect to boolean algebra. But I leave this matter aside in this paper; cf: Keenan (2002).

By way of example, take D = *(exactly) three*. Then,

$$\iota(D) = \{X: D(X)(E) = 1\} = \{X: |X| = 3\}.$$

That is, $\iota(D)$ is the set of those subsets of E that consist of exactly three elements of E. Thus, we could interpret this as signifying that the determiner *three* serves, via ι, as (a name for) a higher order predicate for these subsets, characterizing them by the property that they consist of 3 elements. Furthermore, again by way of example, let X be the set of skyscrapers of roof-top height more than 400m; there exist three of them as of the time this line is written. We might say that the manner that skyscrapers of height more than 400 meters exist in the world is *three*-wise. Thus, in general, one might say that an intersective determiner D describes the manner of existence, in the world E, of the entities denoted by the predicate X, in case $D(X)(E) = 1$. More generally, if $D(A)(X) = D(A \cap X)(E) = 1$, one might say that the manner A's (i.e., entities predicated of A) are distributed among X's is D-*wise*. This is a cognitive significance of the definition given in (64) and Theorem 1.

We might also note in this context that in general (that is, unless E is very small) there are formally just much too much more *intersective determiners* available than a natural language can exploit them resourcefully, just in the same way that there are just much too many *predicates* formally available as elements of $\mathbf{P}(E)$ for a natural language to exploit them; the cardinality of the set of intersective determiners is 2 to the power of 2 to the power of |E| while the cardinality of the set of predicates is already 2 to the power of |E|. There are also some cognitively crucial differences between these two cases. Entities of a certain quite arbitrary subset of E could be conceived of by means of a derived predicate expression such as *skyscrapers of height more than*

400m, or for that matter, *skyscrapers of height more than 264m.* Such would not be the case for determiners in natural languages. We are here facing the issue of open vs. closed grammatical categories.

2.2 The intersective determiner and Japanese grammar

I am going to demonstrate how the equivalence in (64), which distinguishes intersective from non-intersective determiners, manifests itself in the Japanese language.

2.2.1
Consider the following PLAIN SENTENCE:

(75) san-nin no gakusei ga hataraite-iru
three-CLF GEN student NOM work-are
'three students are working'

We transform this plain sentence into THE HEAD-EXTERNAL (HE) EXISTENTIAL TRANSFORM:

(76) san-nin no hataraite-iru gakusei ga iru
three-CLF GEN work-are student NOM are
'three students that are working are/exist'

The main predicate of (76) is the existential verb *iru*, and its subject is a noun phrase with the head external relative clause derived from (75).

The semantics of (75) and (76) can be expressed by the following forms:

(77) (3)(gakusei)(hataraite-iru)

(78) (3)(gakusei∩hataraite-iru)(iru)

The Japanese sentences (75) and (76) have the same truth values. We have the following equivalence:

(79) (3)(gakusei)(hataraite-iru) ⇔ (3)(gakusei∩hataraite-iru)(iru).

This equivalence illustrates the equivalence in (64) and shows that that the determiner *san-nin* 'three persons' is intersective.

In contrast, if we substitute *hotondo* 'almost all' for *san-nin* 'three' in (75) and (76), we get a pair of non-equivalent sentences:

(80) hotondo no gakusei ga hataraite-iru
almost GEN student NOM work-are
'almost all students are working'

(81) hotondo no hataraite-iru gakusei ga iru
almost GEN work-are student NOM are
'almost all students who are working are/exist'

The semantics of (80) and (81) can be expressed by the following forms:

(82) (hotondo)(gakusei)(hataraite-iru)

(83) (hotondo)(gakusei$\cap$hataraite-iru)(iru)

(81), under the existential presupposition of working students, must be taken tautologically true, provided that we can agree to ignore an implausible pragmatic implicature that "but not the small number of the rest do not exist", which would have the effect of making the statement a contradiction. (80), on the other hand, is not a tautology or a contradiction; (80) and (81) are not synonymous. We have the non-equivalence of (82) and (83); the determiner *hotondo* is not intersective:

(84) (hotondo)(gakusei)(hataraite-iru)
$\not\Leftrightarrow$ (hotondo)(gakusei$\cap$hataraite-iru)(iru).

2.2.2
The subjects of (76) and (81) are noun phrases with head external relative clauses (HERC) derived from (75) and (80), respectively. Replace these subjects with the head internal relative clauses (HIRC) derived from (69) and (80), respectively, and we get the following sentences:

(85) san-nin no gakusei ga hataraite-iru no ga iru
three-CLF GEN student NOM work-are COMP NOM are
'three students are working: they exist'

(86) hotondo no gakusei ga hataraite-iru no ga iru
almost GEN student NOM working-are CMP NOM are
'almost all students are working: they are/exist'

We call this form of construction the HI EXISTENTIAL TRANSFORM. Here, the main predicates are *iru* ‘exist’. Note that both (85) and (86) are equivalent to the corresponding plain sentences, (75) and (80), respectively. The HI EXISTENTIAL TRANSFORM does not serve to distinguish intersective vs. non-intersective determiners. Hence, this form is outside of our present interest.

2.2.3
However, we can derive a construction that would interest us from the HI existential transform. Let us “float” the determiner out of the HIRC and put it directly to the left of the main existential predicate. We call the construction we thus obtain the HI EXISTENTIAL TRANSFORM WITH QF (QUANTIFIER FLOAT). We derive the following HI existential transform with QF from (85) and (86):

(87) gakusei ga hataraite-iru no ga san-nin iru
student NOM working -are CMP NOM three-CLF are
'students are working: they are/exist three in number'

(88) gakusei ga hataraite-iru no ga hotondo (zenbu) iru
student NOM working -are CMP NOM almost (all) are
'students are working: they are/exist, almost all of them'
(I suggest to insert *zenbu* in (88) above to enhance acceptability.)

Let us engage ourselves in maneuvering the formalism of these expressions. First we consider (87). We may take the determiner *san-nin* 'three-CLF' as a VP adverbial adjoined to the main verb *iru*. Let us prepose this adverbial to the left:

(89) (san-nin)(gakusei ga hataraite-iru no)(iru)

Now, we have a form that looks closer to D(X)(E), where D = *san-nin* and E = *iru*. In order to interpret (89) in terms of the form D(X)(E), we must take *gakusei ga hataraite-iru no* as an expression of an argument. This HIRC has the same denotation as the noun phrase with a HERC *hataraite-iru gakusei.* Hence, *gakusei ga hataraite-iru no* taken as an argument can be coded in the same way as we code *hataraite-iru gakusei,* i.e., $X = A \cap P$, where A = *gakusei* and P = *hataraite-iru.* We can conclude, then, that the semantics of (87) can be expressed by (78). In fact, we see that (75) and (87) are logically synonymous and hence can confirm the equivalence of (77) and (78). This equivalence thus also serves as an illustration of (64), the defining equivalence for intersective determiners.

Next, we consider (88). We get (90) below, as we got (89) for (87) above:

(90) (hotondo)(gakusei ga hataraite-iru no)(iru).

By the same consideration as above, we can conclude that the semantics of (88) can be expressed by (83). In contrast to the above case, however, we see that (80) and (88) are not logically equivalent and confirm that (82) and (83) are not equivalent; *hotondo* is not intersective.

2.2.4

To sum up, both the HE existential transform and the HI existential transform with QF may be considered as embodiments of the right hand side $D(A \cap P)(E)$ of the equation (64) in Japanese. The equivalence between the plain sentence, on the one hand, and the HE existential transform or the HI transform with QF, on the other, can serve for testing if the determiner involved is intersective or not. In contrast, the HI existential transform without QF cannot serve for this purpose, since it is equivalent to the corresponding plain sentence, whether the determiner involved is intersective or not.

2.3 Binary determiners

After this preparation, we are now in a position to shift our attention to the constructions with transitive predicates and we attempt to extend Keenan's theory of determiners to 2- and higher dimensional spaces. Consider an English sentence of the form:

(91) $S = (D_1A_1)\ P\ (D_2A_2)$,

where D_1 and D_2 are determiners, A_1 and A_2 are nouns and P is a transitive verb. Take, for example:

(92) 5-hiki no inu ga 30-too no usi o otte-iru
5-CLF GEN dog NOM 30-CLF GEN cow driveing-are
'five dogs are driving (away) 30-cows'

Let us conceive of the pair of nouns $A = (A_1, A_2)$ as the 2-dimensional *argument* of the transitive verb P, and the pair $D = (D_1, D_2)$ as the 2-*dimensional determiner* modifying this argument. Then, we can represent the semantics of S by the same formula as before with the intransitive sentence:

(93) $<S> = D(A)(P)$.

P is a subset of two-dimensional Cartesian product $E^2 = E \times E$. $A = (A_1, A_2)$ is an ordered pair of subsets A_1 and A_2 of E; D(A) can be interpreted as a function that maps P to a truth value **1** (true) or **0** (false), and hence D as a function that maps A to such a function.

(94) $D: A \rightarrow (P \rightarrow \{0,1\})$.

We can express the semantics of sentence (92) as in (95):

(95) D(A)(P) = (5, 30)(inu, usi)(otte-iru).

But two remarks are in order here. First of all, sentences with two determiners like (92) are subject to more than one construal depending on how two determiners interact scope-wise; 5 may take wider scope than 30: for each of the five dogs, there are thirty cows, not necessarily the same ones, that it is driving away; or 30 may take scope over 5: for each of the 30 cows there are five dogs, not necessarily same ones, that drive it away. there are also two symmetric possible readings, collective and group readings. Five dogs altogether are driving 30 cows away, (i) each dog driving away one or more cows, but not necessarily all 30 (collective reading), or (ii) five dogs are each driving all of the cows (group reading).

Let us agree that (5, 30) in (95) is meant to express a collective reading. In general, we introduce the following convention:

(96) Given two ordinary determiners D_1 and D_2, we agree to let (D_1, D_2) stand for the binary determiner defined by D_1 and D_2 with a collective reading.

In broader terms, we must assume that D is a function $\delta(D_1, D_2)$ determined by two variables, D_1 and D_2. For example, we might define $\delta_>(D_1, D_2)$ as a 2-dimensional determiner defined by D_1 taking wide scope over D_2. More generally, for that matter, D in (94) may not necessarily be dependent on unary determiners quantifying the subject and object nouns, that is, may not be reducible to the form $D = \delta(D_1, D_2)$. For the moment, however, we are concerned only with the collective reading.

2.4 Binary ε-intersective determiners

At this point, let us consider how the concept of intersective determiner may be extended to binary determiners. The unary intersective determiner is defined in terms of the equivalence (64). Given D(A)(P) we took the intersection $A \cap P$ in (64). But in the present case, we cannot directly intersect A and P; P is a subset of the 2-dimensional space $E^2 = E \times E$, but $A = (A_1, A_2)$ is a pair of subsets of E. But we can first take the Cartesian product of A_1 and A_2 and intersect it with P and denote this set by P|A (P cut by A) :

(97) $P|A = (A_1 \times A_2) \cap P$.

This is the set of pairs (x, y) related by P, under the condition that x and y are restricted to elements of A_1 and A_2, respectively. Project this set onto the first and the second components of E^2, respectively:

(98) n $\quad prj_1P|A = prj_1((A_1 \times A_2) \cap P)$
$\qquad\qquad = \{x: \exists y \text{ such that } (x, y) \in (A_1 \times A_2) \cap P)\}$

(99) $\quad prj_2P|A = prj_2((A_1 \times A_2) \cap P)$
$\qquad\qquad = \{y: \exists x \text{ such that } (x, y) \in (A_1 \times A_2) \cap P)\}$.

Recall the above example (92) of dogs driving away cows. Then, $prj_1P|A$ is the set of those 5 dogs that are driving away some cows and $prj_2P|A$ is the set of those 30 cows that are being driven away by some of the 5 dogs.

We might consider $(prj_1P|A, prj_2P|A)$ as a binary argument obtained from the binary argument A by "intersecting" it with a binary predicate P; it would be an analog to the intersection $A \cap P$ in the Keenan equation, but we would rather call it *A cut by P* and use the notation $A|P$:

Definition 3. An argument A cut by a predicate P is defined as follows: $A|P = (prj_1P|A, prj_2P|A)$.

That is to say, $A|P = (prj_1((A_1 \times A_2) \cap P), prj_2((A_1 \times A_2) \cap P))$. For the sake of convenience, we may also write $(A|P)_i = prj_iP|A$, i = 1, 2; hence we have $A|P = ((A|P)_1, (A|P)_2)$. We could now extend the definition of unary intersective determiner to the case of binary determiners by the following formula. Anticipating that we extend the concept of intersective determiners in more than one way, we call this extension ε-intersective:

Definition 4. A binary determiner is ε-intersective iff

(100) $\quad D(A)(P) = D(A|P)(E^2)$.

Consider again (92), where D = (5, 30) with the collective reading and see if D qualifies as ε-intersective according to this definition. The right hand side of (100) is given as follows:

(101) $\quad (5, 30)(A|P)(E^2) = (5, 30)(prj_1P|A, prj_2P|A)(E^2)$.

In fact, we have

(102) $\quad (5, 30)(prj_1P|A, prj_2P|A)(E^2) = 1$ iff $\#prj_1P|A = 5$ and $\#prj_2P|A = 30$
where # means "the cardinality of"

Hence (101) holds; the example illustrates the fact that (5, 30) is ε-intersective.

Indeed, we have the following theorem. This theorem licenses our definition of ε-intersective as a natural extension of the concept of INTERSECTIVE to binary determiners:

Theorem 2. Let D_1 and D_2 be two unary *conservative* determiners. Then, the binary determiner $D = (D_1, D_2)$ with the collective reading is ε-intersective iff D_1 and D_2 are intersective.

Proof. Assume D_1 and D_2 are intersective.
$D(A)(P) = 1$
iff $D_1(A_1)(prj_1P|A) = 1$ and $D_2(A_2)(prj_2P|A) = 1$
(Def. of the collective reading)
iff $D_1(A_1 \cap prj_1P|A)(E) = 1$ and $D_2(A_2 \cap prj_2P|A)(E)$
(D_1 and D_2 are intersective)

iff $D_1(prj_1P|A)(E) = 1$ and $D_2(prj_2P|A)(E) = 1$
$(prj_1P|A \subset A_1, prj_2P|A \subset A_2)$
iff $D_1(prj_1P|A \cap prj_1P|A)(E) = 1$ and $D_2(prj_2P|A \cap prj_2P|A)(E) = 1$
iff $D_1(prj_1P|A)(prj_1P|A) = 1$ and $D_2(prj_2P|A)(prj_2P|A) = 1$
(D_1 and D_2 are intersective)
iff $D_1(prj_1P|A)(prj_1E|(prj_1P|A, prj_2P|A)) = 1$
and $D_2(prj_2P|A)(prj_2E|(prj_1P|A, prj_2P|A)) = 1$
iff $D(prj_1P|A, prj_2P|A)(E) = 1$ (Def. of the collective reading)

Conversely, assume that $D = (D_1, D_2)$ is ε-intersective.
Let A_1 and P_1 be subsets of E, and let $A = (A_1, E)$ and $P = P_1 \times E$.
$D(A)(P) = (D_1, D_2)(A_1, E)(P_1 \times E)$
$= (D_1, D_2)(prj_1(P_1 \times E)|(A_1, E), prj_2(P_1 \times E)|(A_1, E))(E \times E)$
(D is ε-intersective by assumption)
$= (D_1, D_2)(P_1 \cap A_1, E)(E \times E)$
Now, by the definition of the collective reading,
$(D_1, D_2)(A_1, E)(P_1 \times E) = 1$
iff $D_1(A_1)(prj_1((P_1 \times E) \cap (A_1 \times E))) = 1$
and $D_2(E)(prj_2((P_1 \times E) \cap (A_1 \times E))) = 1$
iff $D_1(A_1)(P_1 \cap A_1) = 1$ and $D_2(E)(E) = 1$
iff $D_1(A_1)(P_1) = 1$ (D_1 is conservative)
$(D_1, D_2)(P_1 \cap A_1, E)(E \times E) = 1$
iff $D_1(P_1 \cap A_1)(prj_1((E \times E) \cap (P_1 \cap A_1 \times E)) = 1$
and $(D_2)(E)(prj_2((E \times E) \cap (P_1 \cap A_1 \times E)) = 1$
iff $D_1(P_1 \cap A_1)(P_1 \cap A_1) = 1$ and $D_2(E)(E) = 1$
iff $D_1(P_1 \cap A_1)((P_1 \cap A_1) \cap E) = 1$ and $D_2(E)(E) = 1$.
iff $D_1(P_1 \cap A_1)(E) = 1$ and $D_2(E)(E) = 1$
(D_1 is conservative).
Hence, $D_1(A_1)(P_1) = 1$ iff $D_1(P_1 \cap A_1)(E) = 1$.
Similarly, $D_2(A_2)(P_2) = 1$ iff $D_2(P_2 \cap A_2)(E) = 1$,
where A_2 and P_2 are arbitrary subsets of E.
Hence D_1 and D_2 are intersective. QED.

We have seen above by inspection that (92) provides an instance for illustrating the assertion of this theorem for the case where both D_1 and D_2 are intersective. If D_1 and/or D_2 are not intersective, D is not ε-intersective. The equation (100) $D(A)(P) = D(A|P)(E^2)$ should fail for any of the following sentences:

(103) 5-hiki no inu ga daibubun no usi o otte-iru
5-CLF GEN dog NOM most-of-the GEN cow ACC driving-are
'five dogs are driving most of the cows'

(104) hotondo no inu ga 30-too no usi o otte-iru
almost-all GEN dog NOM 30-clf GEN cow ACC driving-are
'almost all the dogs are driving 30 cows'.

(105) hotondo no inu ga daibubun no usi o otte-iru
almost-all GEN dog NOM most-of-the GEN cow ACC driving-are
'almost all the dogs are driving most of the cows'

Let us confirm this prediction with (103). We have

(106) $D(A|P)(E^2)$
$= (5, d)(A|P)(E^2)$
$= (5, d)(prj_1P|A, prj_2P|A)(E^2)$
where $prj_1P|A$ = *dogs that are driving cows*,
$prj_2P|A$ = cows driven by dogs
and d stands for *daibubun* 'most of the'.

By the definition of the collective reading,

(107) $D(A|P)(E^2) = 1$ iff
$5(prj_1P|A)(prj_1[E^2 \cap (prj_1P|A \times prj_2P|A)]) = 5(prj_1P|A)(prj_1P|A) = 1$
and $d(prj_2P|A)(prj_2[E^2 \cap (prj_1P|A \times prj_2P|A)]) = d(prj_2P|A)(prj_2P|A) = 1$

Now, $5(prj_1P|A)(prj_1P|A) = 1$ means that the number of dogs that are driving cows is 5, and $d(prj_2P|A)(prj_2P|A) = 1$ means 'most of the cows that are being driven by dogs are cows that are being driven by dogs'. On the other hand, D(A)(P) means (103), i.e., 'five dogs are driving most of the cows'. These two are obviously different. Hence (100) $D(A)(P) = D(A|P)(E^2)$ fails.

2.5 The ε-intersective binary determiner and Japanese grammar

2.5.1

Let us recall that, in the case of unary determiners, Japanese grammar gives evidence for the equation $D(A)(P) = D(A \cap P)(E)$ by the semantic equivalence of two sentence constructions; the plain sentence on the one hand and either the HE existential transform or the HI existential transform with QF, on the other. The previous examples of these constructions are repeated below:

(108) san-nin no gakusei ga hataraite iru (=(75))

(109) san-nin no hataraite-iru gakusei ga iru (= (76))

(110) gakusei ga hataraite-iru no ga san-nin iru (= (87))

These three sentences are logically equivalent to each other. In contrast, if we substitute *hotondo* 'almost' for *san-nin* we obtain sentences that are not equivalent to the corresponding plain sentence; neither (81) nor (86) is equivalent to (80).

For the binary case, the theorem we have proved above states that the collective reading of a binary determiner derived from unary intersective determiners provides an ε-intersective determiner. In conformity with this theorem, we have indeed seen that (5, 30) is an ε-intersective determiner; we did this by directly inspecting that the meaning of (92) matches with what the right hand side of the equation (100) signifies. We would wonder, however, whether Japanese grammar provides us with a direct witness to the defining equation (100) for ε-intersectivity for binary determiners, as we have seen it does for the defining equation (64) for the intersectivity of unary determiners; that is, whether Japanese grammar provides a construction (i) whose syntax mirrors more or less transparently the right hand side of equation (100), i.e., $D(A|P)(E^2)$ and (ii) which is logically equivalent to the corresponding plain sentence if and only if D is ε-intersective.

2.5.2

There are two HE existential transforms corresponding to a plain transitive sentence construction; the head may be either the subject or the object of the original sentence. From (92) we can derive the following two HE transforms:

(111) 5-hiki no [30-too no usi o otte-iru inu] ga iru
5-CLF GEN 30-CLF GEN cow ACC driving-are dog NOM are
'five dogs that are driving 30 cows are/exist'

(112) 30-too no [5-hiki no inu ga otte-iru usi] ga iru
30-CLF GEN 5-CLF GEN dog NOM driving-are cow NOM are
'30 cows that five dogs are driving are/exist'

However, neither of these sentence constructions could serve for providing evidence for ε-intersectivity for binary determiners. There are two points to note. First of all, it would be difficult, or perhaps, impossible, to argue that the semantics of these constructions could be expressed in the form $D(A|P)(E^2)$, the right hand side of the defining equation in Definition 4. But, this point turns out to be irrelevant any way. Thus, secondly, these sentences, (111) and (112), are indeed equivalent to the plain sentence (92). However, even if we substitute a non-intersective determiner for one of the intersective determiners in (111) and (112), we still get a sentence that is equivalent to the corresponding plain sentence, an unwelcome result for our purpose. Observe the following sentence where *daibubun* 'most of the' substitutes for 30 in (111):

(113) 5-hiki no [daibubun no usi o otte-iru inu] ga iru
5-CLF GEN most-of-the GEN cow ACC driving-are dog NOM are
'five dogs that are driving most of the cows are/exist'

This sentence is equivalent to (103), which I repeat here:

(114) 5-hiki no inu ga daibubun no usi o otte-iru
5-CLF GEN dog NOM most-of-the GEN cow ACC driving-are
'five dogs that are driving most of the cows are/exist'

But (5, daibubun) cannot be ε-intersective, due to Theorem 2. A similar situation obtains with (112).

The *HI existential transform* does not serve for our purpose, either; for, this construction is logically equivalent to the corresponding plain sentence no matter what determiners they have for subject and object. Observe the following forms:

(115) 5-hiki no inu ga 30-too no usi o otte-iru no ga iru
5-CLF GEN dog NOM 30-CLF GEN cow ACC driving-are CMP NOM are
'five dogs are driving (away) 30 cows: they are/exist'

(116) 5-hiki no inu ga daibubun no usi o otte-iru no ga iru
5-CLF GEN dog NOM most GEN cow ACC driving-are CMP NOM are
'five dogs are driving (away) most of the cows: the dogs and cows are/exist'

(117) hotondo no inu ga 30-too no usi o otte-iru no ga iru
almost GEN dog NOM 30-CLF GEN cow ACC driving-are CMP NOM are
'almost all the dogs are driving (away) 30 cows: the dogs and the cows are/exist'

(118) hotondo no inu ga daibibun no usi o otte-iru no ga iru
almost GEN dog NOM most GEN cow ACC driving-are CMP NOM are
'almost all the dogs are driving (away) most of the cows: the dogs and the cows are/exist'

A remark may be in order for the intended interpretation of these sentences. The head internal relative clauses in them involve "split heads." The "semantic head" of the head internal relative clause in (115) is the subject and the object of the clause conjoined together and they together function as the subject of the main verb *aru* 'be'. The *they* in the translation given to (115) means '5 dogs that are driving cows and 30 cows that are driven by dogs.' Similarly for (116), (117), (118).

The crucial fact for us at this point is that the HI existential transform keeps semantics invariant; (115), (116), (117) and (118) are equivalent to the plain sentences (92), (103), (104) and (105), respectively. But in order to find a construction that simulates ε-intersectivity, we are looking for a transform that keeps meaning invariant only if the two involved determiners are both intersective. Thus again an unwelcome result.

2.5.3

Happily, however, the HI EXISTENTIAL TRANSFORM WITH FLOATING QUANTIFIERS is a construction we are looking for. First of all, this construction is equivalent to the plain sentence if both determiners are intersective, but it is not if either one or both determiners are not intersective. Observe the following sentence:

(119) inu ga usi o otte-iru no ga 5-hiki to 30-too iru
dog NOM cow ACC driving-are CMP NOM 5-CLF and 30-CLF are
'dogs are driving (away) cows: they exit, five and thirty in number'

This sentence is equivalent to the plain sentence (92). In contrast, if we substitute *daibubun* 'most of the' for *30-too* in the above sentence, we get a sentence form of dubious grammaticality:

(120) inu ga usi o otte-iru no ga 5-hiki to daibubun iru
dog NOM cow ACC driving-are CMP NOM 5-CLF and most-of-them are

One might judge the acceptability to improve if we insert *sorezore* 'respectively' before *5-hiki*; a formulaic translation could be given as below:

(121) inu ga usi o otte-iru no ga sorezore 5-hiki to daibubun iru
'dogs are driving cows; they exist, (the dogs) 5 in number and (the cows) most of them, respectively'

Note, however, the *they* above should mean 'the dogs that are driving cows and the cows that are being driven by dogs', not just 'dogs and cows' and, crucially, *most (of them)* should mean 'most of those cows that are being driven by dogs', not 'most of the cows'. Hence, (121) is not equivalent to (114), whether it is judged grammatical or not. For similar reasons, the following forms, even if they are judged grammatical, are not equivalent to (115) and (116), either:

(122) inu ga usi o otte-iru no ga hotondo to 30-too iru
dog NOM cow ACC driving-are CMP NOM almost and 30-CLF are

(123) inu ga usi o otte-iru no ga hotondo to daibubun iru
dog NOM cow ACC driving-are CMP NOM almost and 30-CLF are

Secondly, I claim that the semantics of (119) can naturally be expressed in the form $D(A|P)(E^2)$, the right hand side of equation (100). As we did with a floating quantifier in (87), we may consider the floating determiners in (119) as adjuncts to the main verb *iru*. Let us prepose the determiners in (119) to the left as we did in (89):

(124) (5-hiki to 30-too)(inu ga usi o otte-iru no ga)(iru)

As split heads of the HI relative clause *inu ga usi o otte-iru no*, *inu* and *usi* denote those dogs and cows that satisfy the relation P = *X* DRIVES *Y*. In order to get the set of pairs of A_1 *dog* and A_2 *cow* that satisfy this relation, first, take the intersection $(A_1 \times A_2) \cap P$ and project this intersection onto the first and the second axes, respectively. We get (125) as a representation of a two-dimensional argument:

(125) $(prj_1[(A_1 \times A_2) \cap P], prj_2[(A_1 \times A_2) \cap P])$.

Thus, the semantics of (119) can be given as follows:

(126) $(5, 30)(prj_1[(\text{DOG} \times \text{COW}) \cap \text{DRIVE}], prj_2[(\text{DOG} \times \text{COW}) \cap \text{DRIVE}])(E \times E)$
$= (5, 30)((\text{DOG}, \text{COW})|\text{DRIVE})(E^2)$.

To sum up, Japanese grammar provides an embodiment of the defining equation for ε-intersectivity (100) in the form of the equivalence between the plain sentence and the HI existential transform with QF:

(127) D is ε-intersective if and only if the following equivalence holds:
The plain sentence ⇔ the HI existential transform with QF

This equivalence is illustrated by the following equivalence between (92) and (119):

(128) 5-hiki no inu ga 30-too no usi o otte-iru ⇔
inu ga usi o otte-iru no ga 5-hiki to 30-too iru

2.6 A further development of the mathematics of binary determiners

The following definition is an extension to binary determiners of the second version of the definition of intersective determiners for unary determiners (*Definition 2′*).

Definition 5. A binary determiner D is ι-intersective iff the following condition holds:

(129) For any binary arguments A and A' and binary predicates P and P', if $A|P = A'|P'$,
then $D(A, P) = D(A', P')$.

The following theorem is an extension of Proposition 2:

Theorem 3. A binary determiner D is ε-intersective iff D is ι-intersective.

Proof. Assume D is ε-intersective. Let A and A' binary arguments and P and P' binary predicates, and assume $A|P = A'|P'$. We want to show $D(A, P) = D(A', A')$. Since D is ε-intersective, we have

$$D(A, P) = D(A|P, E^2),\ D(A', P') = D(A'|P', E^2).$$

Since $A|P = A'|P'$ by assumption, we have $D(A|P) = D(A'|P')$.

Conversely, assume D is ι-intersective. But, then, since we have

$$A|P = (A|P)|E$$
$$D(A, P) = D(A|P, E)$$

D is ε-intersective. QED.

To proceed, let me introduce some definitions:

Definition 6. Let P be a binary predicate. $\mathrm{rec}(P) = \mathrm{prj}_1P \times \mathrm{prj}_2P$.

Definition 7. A binary predicate P is rectangular if $P = \mathrm{rec}P$.

Definition 8. Let μ be a function from the set of binary arguments **A** into the set of binary predicates **P** defined by

$$\mu(A_1, A_2) = A_1 \times A_2.$$

We have another characterization of ι-intersective (and ε-intersective) determiners.

Theorem 4. A binary determiner D is ι-intersective iff the following condition holds:

(130) For any binary arguments A and A' and binary predicates P and P', if $\mathrm{rec}(P|A) = \mathrm{rec}(P'|A')$, then $D(A, P) = D(A', P')$.

Proof. $\mathrm{rec}(P|A) = \mathrm{prj}_1P|A \times \mathrm{prj}_2P|A = \mathrm{prj}_1P \cap (A_1 \times A_2) \times \mathrm{prj}_2P \cap (A_1 \times A_2)$

$\mathrm{rec}(P'|A') = \mathrm{prj}_1P'|A' \times \mathrm{prj}_2P'|A' = \mathrm{prj}_1P' \cap (A'_1 \times A'_2) \times \mathrm{prj}_2P' \cap (A'_1 \times A'_2)$.

Hence, $\mathrm{prj}_1P|A \times \mathrm{prj}_2P|A = \mathrm{prj}_1P'|A' \times \mathrm{prj}_2P'|A'$

$$\Leftrightarrow \mathrm{prj}_1P \cap (A_1 \times A_2) = \mathrm{prj}_1P' \cap (A'_1 \times A'_2) \;\&\; \mathrm{prj}_2P \cap (A_1 \times A_2) = \mathrm{prj}_2P' \cap (A'_1 \times A'_2).$$
$$\Leftrightarrow (\mathrm{prj}_1P \cap (A_1 \times A_2), \mathrm{prj}_2P \cap (A_1 \times A_2)) = (\mathrm{prj}_1P' \cap (A'_1 \times A'_2), \mathrm{prj}_2P' \cap (A'_1 \times A'_2))$$
$$\Leftrightarrow A|P = A'|P'$$

Hence, (129) <=> (130). QED.

Theorem 5. The set of ε-intersective determiners $\mathbf{Int}^{\varepsilon}$ is isomorphic to the set **Rec** of sets of rectangular subsets of E^2.

Proof. Let D be an ε-intersective determiners and let

$\rho(D) = \{X: X = \mu(A|P)$ such that $D(A, P) = D(A|P, E) = 1\}$.

(i) ρ is one-to-one from $\mathbf{Int}^{\varepsilon}$ to **Rec**.

For: First of all, note that μ is one-to-one and if $\mu(A|P) = \mu(A'|P')$, then $A|P = A'|P'$. Hence, if $D(A|P, E) \neq 1$ and $\mu(A|P) = \mu(A'|P')$, $D(A'|P', E) = D(A|P, E) \neq 1$. Thus, we can conclude that if $D(A|P, E) = 1$, $\mu(A|P) \in \rho(D)$ and if $D(A|P, E) \neq 1$, $\mu(A|P) \notin \rho(D)$.

Now, assume $D \neq D'$, and take (A, P) and (A', P') such that $D(A, P) \neq D'(A', P')$. Then, $D(A|P, E) \neq D'(A'|P', E)$; assume $D(A|P, E) = 1$ and $D'(A'|P', E) \neq 1$. Then, $\mu(A|P) \in \rho(D)$ and $\mu(A|P) \notin \rho(D)$, hence $\rho(D) \neq \rho(D')$. ρ is one-to-one.

(ii) ρ is onto.

Take a set of rectangular predicates **R**. Define D:

$D(A, P) = 1$ iff $A = ((\mathrm{prj}_1(P), \mathrm{prj}_2(P))$ and $P \in \mathbf{R}$.

Then, $\rho(D) = \{X: X = \mu(A|P)$ such that $A = ((prj_1(P), prj_2(P))$ and $P \in \mathbf{R}\}$
$= \{X: X = \mu((prj_1(P), prj_2(P)), P \in \mathbf{R}\} = \mathbf{R}$. QED.

Definition 9. A binary determiner D is μ-intersective if the following condition holds:
(131) If for any binary arguments A and A' and binary predicates P and P', if P|A = P'|A' then D(A, P) = D(A', P').

Theorem 6. If D is ι-intersective, D is μ-intersective.

Proof. If P|A = P'|A', A|P = (prj_1P|A, prj_2P|A) = (prj_1P|'A', prj_2P'|A') = P'|A'. Then, if D is ι-intersective, D(n, P) = D(A', P'). Hence, D is μ-intersective.

The converse does not hold:

Example 1. Consider the following two sentences:

(132) sihuku ga 8-nin ayasii otoko o 5-nin kannsisite-ita
plainclothes NOM 8-CLF suspicious men ACC 5-CLF observe were
'8 plainclothes agents were observing 5 suspicious men'

(133) keikan ga 8-nin issyoni yakuza o 5-nin taihosita
policemen NOM 8-CLF together yakuza ACC 5- CLF arrested
'eight policemen together arrested 5 yakuzas'

Let us assume first that '8 plainclothes agents' and '8 policemen' denote the same sets of men and so do '5 suspicious men' and '5 yakuzas'. Secondly, let us assume that (8-nin, 5-nin) is construed in (132) collectively as before and in fact assume that not every of the plainclothes men observed every suspicious men. On the other hand, we assume that the pair (8-nin, 5-nin) has a *group* reading in (133) and that all the eight policemen jointly arrested all the five yakuzas. We represent the collective reading by D = (8, 5) and the group reading by D' = (8*5).

D' is μ-intersective. Now, let A_1 = *plainclothes agents,* A'_1 = *policeman,* A_2 = *suspicious man* A'_2= *yakuza*, P = *observe* and P' = *arrest*. By assumption D'((A_1, A_2), P) = 0. We have A_1|P = A'_1|P' and A_2|P = A'_2|P' but D'((A_1, A_2), P) ≠ D'((A_1, A_2), P) = 1. Hence, D' is not ι-intersective.

That the converse of Theorem 6 fails in fact follows from Theorem 5 and the following theorem:

Theorem 7. The set $\mathbf{Int}^{\mu}$ of μ-intersective binary determiners is isomorphic to the set of sets of subsets of E^2.

Proof: Given a μ-intersective binary determiner D, let s(D) be defined as follows:

(134) $s(D) = \{X: D(E, E)(X) = 1\}$.

First we show that s is one-to-one from $\mathbf{Int}^{\mu}$ to $\mathbf{P}(E^2)$. Assume D ≠ D' and D(A, P) ≠ D'(A, P). Without loss of generality, we can assume that D(A, P) = 1, D'(A, P) ≠ 1. As P|A = (P|A)|(E, E) and D is μ-intersective, we have D(A, P) = D((E, E), P|A) = 1. Hence P|A ∈ s(D). Similarly, D'(A, P) = D'((E, E), P|A) ≠1, hence P|A ∉ s(D'). It follows that s(D) ≠ s(D'); s is one-to-one.

Next, we show s is onto. Let $\mathbf{X} \subset \mathbf{P}(E^2)$. Let D be defined as follows. D(A, P) = 1 iff $P|A \in \mathbf{X}$. Then,

$$\begin{aligned} s(D) &= \{X: D((E, E)(X)) = 1\} \\ &= \{X: X|(E, E) \in \mathbf{X}\} \\ &= \{X: X \in \mathbf{X}\} \\ &= \mathbf{X}. \qquad\qquad \text{QED.} \end{aligned}$$

Now, assume $X \in s(D)$, i.e. D(E, E)(X) = 1. $(E, E)|X = (prj_1(X), prj_2(X))$. Consider rec(X) = $prj_1(X) \times prj_2(X)$. We have $(E, E)|X = (E, E)|recX$. Then, if D is ι-intersective, D((E, E), X) = D((E, E), recX) = 1. Hence $recX \in s(D)$. Hence, we have

Corollary 1. If D is ι-intersective, s(D) is closed under rec, i.e., if $X \in s(D)$, $recX \in s(D)$.
Corollary 2. The set of ι-intersective determiners $\mathbf{Int^{\iota}}$ is a proper subset of $\mathbf{Int^{\mu}}$.

The reader would have noticed that the above argument virtually provides us with a second proof of Theorem 5.

2.7 Partial intersective determiners

2.7.1
Consider now the following sentence:

(135) hotondo no inu ga 30-too no usi o otte-iru (= (104))
almost-all GEN dog NOM 30-clf GEN cow ACC driving-are
'almost all the dogs are driving 30 cows'.

The subject is quantified by a non-intersective determiner *hotondo* (*zenbu*) *'almost all'*. (For the interest of space, I suppress *zenbu* at determiner position inside Noun Phrase without losing acceptability; I retain it only at floating position where acceptability seems to require it.) The HI existential transform with QF (136) below corresponding to this sentence is not logically equivalent to it.

(136) inu ga usi o otte-iru no ga *hotondo zenbu* to *30-too* iru

This fact illustrates the defining characteristic of the HI existential transform as an existential sentence. In contrast, an HE EXISTENTIAL TRANSFORMS with QF (137) below is equivalent to (136), with the intended collective reading of the two quantifiers:

(137) *hotonto no* inu ga otte iru usi ga *30-too* iru

To recall, the equivalence of the plain sentence and its HI existential transform with QF when both subject and the object are quantified with intersective determiners provides an empirical evidence for the equation in (41) (repeated here):

(100) $D(A)(P) = D(A|P)(E \times E)$

We are now trying to determine the mathematical significance of the equivalence of (135) to the HE existential transform with QF (137). For this purpose I will manipulate the sentence form of (137). I first convert the HERC in (137) to a HIRC:

(138) *hotondo no* inu ga usi o otte iru no ga *30-too* iru

Here the intended semantic head of the HIRC is the object *usi*. Next, we follow the standard semantic interpretation of HIRCs by means of e-type pronouns and we attempt to translate literally the semantic representation according to this interpretation back into Japanese:

(139) *hotondo no* inu ga usi o otte-ite, sore ga 30-too iru

sore is intended to be coreferential with *hotonto no inu ga otte iru usi* 'cows that almost all the dogs are driving'. We then float the determiner *hotondo* and quantify the second occurence of *usi* by the relative clause *inu ni owarete-iru* 'that are being driven':

(140) inu ga *hotondo zenbu* usi o otte-ite, [inu ni owarete-iru] usi ga *30-too* iru

Taking the advantage of the fact that the determiner *hotondo zenbu* is conservative, we get the following form:

(141) inu ga *hotondo zenbu* **[**usi o otte-iru] inu de, [inu ni owarete-iru] usi ga *30- too* iru

We again qualify *usi* redundantly by the relative clause *inu ni owarete-iru*:

(142) inu ga *hotondo zenbu* **[**inu ni owarete-iru usi o otte-iru] inu de, [inu ni owarete-iru usi] ga *30-too* iru

(142) is a sentence that conjoins two intransitive sentences, that is, two one-argument sentences. The semantics of this sentence may be represented as follows:

(143) $(\text{HOTONDO})(\text{INU})(prj_1(\text{OTTE-IRU}\cap(\text{INU}\times\text{USI}))$
$\& (30)(prj_2(\text{OTTE-IRU}\cap(\text{INU}\times\text{USI})))(E)$

Now, we must refer to the following lemma:

Lemma. Let $D_1$and D_2 be unary conservative determiners and $D = (D_1, D_2)$ be a binary determiner defined by the collective reading of $D_1$and D_2; let $A = (A_1, A_2)$ and $P = P_1\times P_2$. Then,

(144) $D(A)(P) = 1$ iff $D_1(A_1)(P_1) = 1$ & $(D_2)(A_2)(P_2) = 1$.

Proof. By the definition of collective reading, we have

(145) $(D_1, D_2)(A_1, A_2)(P_1\times P_2) = 1$
iff $D_1(A_1)(prj_1(P\cap (A_1\times A_2))) = 1$
& $D_2(A_2)(prj_2(P\cap (A_1\times A_2))) = 1$

Since

(146) $\mathrm{prj}_1((P_1 \times P_2) \cap (A_1 \times A_2) = P_1 \cap A_1$
$\mathrm{prj}_2((P_1 \times P_2) \cap (A_1 \times A_2) = P_2 \cap A_2$,
we have

(147) $(D_1, D_2)(A_1, A_2)(P_1 \times P_2) = 1$
iff $D_1(A_1)(P_1 \cap A_1) = 1$ & $(D_2)(A_2)(P_2 \cap A_2) = 1$.

Since D_1 and D_1 and D_2 are conservative, we have

(148) $(D_1, D_2)(A_1, A_2)(P_1 \times P_2) = 1$
iff $D_1(A_1)(P_1) = 1$ & $(D_2)(A_2)(P_2) = 1$ QED.

Let us apply the equivalence (144) of this lemma to (143) from right to left; we get the following representation, which is equivalent to (143):

(149) (HOTONDO, 30)(INU, prj_2(OTTE-IRU∩(INU×USI)))(prj_1(OTTE-IRU∩(INU×USI))×E).

To conclude: The transformations from (137) to (142) keep the truth value invariant; hence, in particular, (137) is equivalent to (142). In addition, the representations (143) and (149) are equivalent. Thus, the equivalence of Japanese plain sentence (135) and its HE existential transform (137) can be represented by the equivalence of the following formula and (149):

(150) (HOTONDO, 30)(inu, usi)(otte-iru)

Now, we draw a generalization from the equivalence between (150)and (149) and formulate the following equivalence in a general form:

(151) $D(A)(P) = D((A_1, \mathrm{prj}_2((A_1, A_2) \cap P)), \mathrm{prj}_1((A_1, A_2) \cap P), E)$.

Examples (135) and (138) have a non-intersective determiner at subject position and an intersective determiner at object position. Reversing this asymmetry to the opposite direction, we get a sentence like the following:

(152) 5-hiki no inu ga daibubun no usi o otte-iru (=(103))
5-CLF GEN dog NOM most-of-the GEN cow ACC driving-are
'five dogs are driving most of the cows'

(153) daibubun no usi o otte-iru inu ga 5-hiki iru
most-of-the GEN cow ACC driving-are dog NOM 5-CLF are
'dogs that are driving most of the cows are/exist five in number '

By the same argument as above *mutatis mutandis* we derive the equivalence between the following two formulas:

(154) (5, DAIBUBUN)(INU, USI)(OTTE-IRU)

(155) (5, DAIBUBUN)(prj_1(OTTE-IRU∩(INU×USI)), USI)(E×prj_2(OTTE-IRU∩(INU×USI))).

We draw a similar generalization from the equivalence between (154) and (155):

(156) $D(A)(P) = D(prj_1((A_1 \times A_2) \cap P), A_2)(E \times prj_2((A_1 \times A_2) \cap P))$.

In analogy to the concept of ε-intersectivity, we define partial ε-intersectivity by means of the equivalences (151) and (156). For the sake of concision I rewrite these equivalences by means of the abbreviated notation introduced above:

Definition 10. A binary determiner D is partial ε-intersective if for any binary argument A and predicate P it satisfies either one of the following equivalences:

(157) $D(A)(P) = D((A_1, (A|P)_2)), (A|P)_1 \times E)$.

(158) $D(A)(P) = D(((A|P)_1, A_2)), E \times (A|P)_2)$.

The equivalences between the Japanese sentences (135) and (137) on the one hand and between (152) and (153) on the other are illustrative instances of (157) and (158), respectively.

2.7.2
The following definition and the theorem are analogues to Definition 5 and Theorem 3:

Definition 11. A binary determiner D is partial ι-intersective iff either one of the following conditions holds:

(159) For any binary arguments A and A' and binary predicates P and P', if $A_1 = A'_1$ and $A|P = A'|P'$ then $D(A)(P) = D(A')(P')$.

(160) For any binary arguments A and A' and binary predicates P and P', if $A_2 = A'_2$ and $A|P = A'|P'$ then $D(A)(P) = D(A')(P')$.

If we spell out the abbreviated notation, we have the following condition:

If $A_1 = A'_1$ (or $A_2 = A'_2$) and
$prj_1((A_1 \times A_2) \cap P) = prj_1((A'_1 \times A'_2) \cap P')$
$prj_2((A_1 \times A_2) \cap P) = prj_2((A'_1 \times A'_2) \cap P')$,
then $D(A)(P) = D(A')(P')$.

Theorem 8. A binary determiner D is partial ε-intersective iff D is partial ι-intersective.

Proof. Assume D is partial ε-intersective. Let $A_1 = (A_1, A_2)$ and $A' = (A'_1, A'_2)$ be binary arguments and P and P' be binary predicates. Since D is partial ε-intersective, we assume without losing generality:

(161) $D(A)(P) = D((A_1, prj_2((A_1 \times A_2) \cap P)), prj_1((A_1 \times A_2) \cap P) \times E)$.

(162) $D(A')(P') = D((A'_1, prj_2((A'_1 \times A'_2) \cap P')), prj_1((A'_1 \times A'_2) \cap P') \times E)$.

Assume

$A_1 = A'_1$
$prj_1((A_1 \times A_2) \cap P) = prj_1((A'_1 \times A'_2) \cap P')$

$$\mathrm{prj}_2((A_1{\times}A_2)\cap P) = \mathrm{prj}_2((A'_1{\times}A'_2)\cap P').$$

It follows that D(A, P) = D(A', A').

To prove the converse, note first that trivially we have:

$$\mathrm{prj}_1(A_1{\times}\mathrm{prj}_2((A_1{\times}A_2)\cap P)) \supset \mathrm{prj}_1((A_1{\times}A_2)\cap P))$$

Hence we have

(163)
$$\begin{aligned}&\mathrm{prj}_1((A_1{\times}A_2)\cap P))\\ &= \mathrm{prj}_1[A_1{\times}\mathrm{prj}_2((A_1{\times}A_2)\cap P)] \cap \mathrm{prj}_1((A_1{\times}A_2)\cap P))\\ &= \mathrm{prj}_1[(A_1{\times}\mathrm{prj}_2((A_1{\times}A_2)\cap P)] \times \mathrm{prj}_1[\mathrm{prj}_1((A_1{\times}A_2)\cap P){\times}E]\\ &= \mathrm{prj}_1[(A_1{\times}\mathrm{prj}_2((A_1{\times}A_2)\cap P) \times (\mathrm{prj}_1((A_1{\times}A_2)\cap P){\times}E)]\end{aligned}$$

Similarly,

(164)
$$\begin{aligned}&\mathrm{prj}_2((A_1{\times}A_2)\cap P)\\ &= \mathrm{prj}_2[(A_1{\times}\mathrm{prj}_2((A_1{\times}A_2)\cap P) \cap (\mathrm{prj}_1((A_1{\times}A_2)\cap P){\times}E)]\end{aligned}$$

Assume now D is partial ι-intersective. Then, from (163) and (164) we have

$$D(A)(P) = D((A_1, \mathrm{prj}_2(A_1{\times}A_2)), (\mathrm{prj}_1((A_1{\times}A_2)\cap P){\times}E)).$$

That is, D is partial ε-intersective. QED.

Theorem 8 provides a good justification for the introduction of partial ε- and ι-intersectivity by means of Definition 10 and Definition 11.

3. PART 3: The Mathematical Theory of Determiners in n-Dimensional Spaces

3.1 General definitions, notations, conventions

3.1.1 Basic symbols

$(\ )^n$, $<\ >^n$ indicate "n-tuple".
X^n : Cartesian product of n copies of X.
$\prod_{i\in I}X_i$: Cartesian product of X_i, $i\in I$.
1 = {0, 1}, where 0 and 1 may be interpreted as "false" and "true," respectively.
N = <1, 2, ..., n>: a *naturally* ordered subset {1, 2, …, n} of natural numbers.
$\prod$ = <I, J>: a *partition* of N, where I are J are (naturally ordered) subsets of N ($I, J \subset N$) and $I\cap J = \varnothing$, $I\cup J = N$.

Remark 1. Without fear of confusion, $\prod$ is used ambiguously both as a sign for Cartesian product and as a general symbol for a partition.

$\prod < \prod'$: Given two partitions $\prod$ = <I, J> and $\prod'$ = <I', J'>, $\prod < \prod'$ iff $I \subset I'$ (and hence $J' \subset J$).

Notational conventions:

(1) Where there is no fear of confusion in the context, we may let I and J stand for subsets of N as elements of partition <I, J> without specifically stating so.

(2) Where there is no fear of confusion, letters h and k may be used as follows: $h = |I|$, $k = |J|$, where $\prod = \langle I, J\rangle$ is a partition of $N = \{1, 2, ..., n\}$, $h + k = n$.

3.1.2 Universe, predicates, arguments

E: A non-empty set; a contextualized universe.
$E^N = \prod_{i \in N} E_i$, $E_i = E$: n-ary universe.

I-subspace E^I: Given $I \subset N$, $E^I = \prod_{i \in I} E_i$. In particular, $E^{\{i\}} = E_i$.

Remark 2. E^I is referred to as a subspace, but it is not to be considered as a subset of E^N.
If $I = \{i_1, i_2,.., i_h\}$, $I' = \{i'_1, i'_2,.., i'_h\}$, set-theoretically $E^I \cong E^h$ and $E^I \cong E^h$, but as subspaces of E, $E^{I'} \neq E^h$, unless $I = I'$.

Predicates
n-ary predicate or N-predicate (or, simply, predicate) P: $P \subset E^N$.
I-predicate P: $P \subset E^I$.
$P\perp P'$: the join of I-predicate P and I'-predicate P'.
Given $I, I' \subset N$ and $I \cap I' = \varnothing$ and $P \subset E^I$, $P \subset E^{I'}$,
$P\perp P' = \{(x_i)_{i \in I\cup I'}: (x_i)_{i \in I} \in P, (x_{i'})_{i' \in I'} \in P'\}$.

In conformity with this definition of $\perp$, given $I \subset N$ and $J \subset N$, we have
$E^I \perp E^J = E^{I\cup J}$, $E^I = \{(x_i)_{i \in I} : x_i \in E_i\} = \perp_{i \in I} E_i$, where E_i is the i-th component of E^N.

We may furthermore extend the use of $\perp$ and write $I\cup I' = I\perp I'$, where $I\cup I'$ is considered as an ordered subset of N and $I\cap I' = \varnothing$. Then, $E^I \perp E^J = E^{I\cup I'} = E^{I\perp I'}$.

Arguments
n-ary argument A: $A = (A_i)_{i\in N}$, $A_i \subset E_i$.
I-argument A_I: Given $I \subset N$, $A_I = (A_i)_{i\in I}$, $A_i \subset E_i$.
$A_I \perp A_J$ the join of an I-argument A_I and a J-argument A_J: Given $I, J \subset N$ and $I\cap J = \varnothing$
and an I-argument $A_I = (A_i)_{i\in I}$ and a J-argument $A_J = (A_i)_{i\in J}$, $A_I \perp A_J = (A_i)_{i\in I\cup J}$.

$\mathfrak{P}_N = \mathbf{P}(E^N)$: The set of n-ary predicates (n-place predicates) = the power set of E^N.
$\mathcal{A}_N = \prod_{i \in N} [\mathbf{P}(E_i)] = \{A: A = (A_i), 1\leq i\leq n, A_i \subset E_i\}$ (the set of n-ary arguments).

Notational convention. Where there is no fear of confusion in the context, we may suppress the subscript $_N$: $\mathcal{A} = \mathcal{A}_N$ and $\mathfrak{P} = \mathfrak{P}_N$. We also let A and P generically stand for elements of $\mathcal{A}$ and $\mathfrak{P}$, respectively.

The canonical embedding μ of $\mathcal{A}$ to $\mathfrak{P}$:
$\mu: \mathcal{A} \rightarrow \mathfrak{P}$ defined by $\mu A = \prod A_i$, where $A = (A_i)$, $1\leq i\leq n$.

P|A, a predicate P cut by an argument A: $P|A = \mu A\cap P$. P|A is a predicate: $P|A \in \mathfrak{P}$.

Example 1. n=1.
$\mathfrak{P}_{\{1\}} = \mathbf{P}(E)$. $\mathcal{A}_{\{1\}} = \mathbf{P}(E)$.

There are two possible partitions: $\prod_1 = <\{1\}, \varnothing>$, $\prod_2 = <\varnothing, \{1\}>$.
μ: $\mathbf{A} \rightarrow \mathbf{P}(E)$. $\mu A = A$, $A \subset E$. (μ is the identity.)
$P|A = A \cap P$.

Example 2. n = 2.
$\mathcal{P}_{\{1, 2\}} = \mathbf{P}(E^2)$. $\mathcal{A}_{\{1, 2\}} = [\mathbf{P}(E), \mathbf{P}(E)]$.
There are four possible partitions:
$\prod_1 = <\{1, 2\}, \varnothing>$, $\prod_2 = <\{1\}, \{2\}>$, $\prod_3 = <\{2\}, \{1\}>$, $\prod_4 = <\varnothing, \{1, 2\}>$.
μ: $\mathbf{A} = [\mathbf{P}(E), \mathbf{P}(E)] \rightarrow \mathbf{P}(E^2)$. $\mu A = \mu((A_1, A_2)) = A_1 \times A_2$.
$P|A = (A_1 \times A_2) \cap P$.

3.1.3 Projections

$prj_I(P) = \{x\text{: } x = (x_i)_{i \in I}\ \exists x' = (x'_i)_{1 \leq i \leq n},\ x' \in P \text{ and for } i \in I,\ x_i = x'_i\}$.

In particular, $prj_{\{i\}}(P) = prj_i(P)$.
We have $prj_I(P) \subset E^I$, $prj_{\{i\}}(P) \subset E^{\{i\}}$.
$prj_I(A, P)$: $prj_I(A, P) = prj_I(P|A) = prj_I(\mu A \cap P)$.

$Prj_{\prod}(P)$: Given a partition $\prod = <I, J>$, $Prj_{\prod}(P) = Prj_{<I, J>}(P) = prj_I(P) \perp E^J$.

Defined explicitly,
$Prj_{\prod}(P) = \{x\text{: } x = (x_i)_{1 \leq i \leq n}\ \exists\ x' = (x'_i)_{1 \leq i \leq n},\ x' \in P \text{ and for } i \in I,\ x_i = x'_i \text{ and for } j \in J,\ x_j \in E_j\}$.
Thus, $Prj_{\prod}(P)$ is the projection of P to subspace E^I join the co-subspace E^J of E^I in E^N.

In particular, $Prj_{<\varnothing, N>}(P) = E^N$; $Prj_{<N, \varnothing>}(P) = prj_N(P) = P$.

$Prj_{\prod}(A, P)$: Given $A \in \mathcal{A}$ and $P \in \mathcal{P}$, $Prj_{\prod}(A, P) = Prj_{\prod}(P|A) = prj_I(P|A) \subset E^J$.

Remark 3. $prj_I(P) \subset E^I$. But, $prj_I(P) \not\subset E^N$ unless I = N. In contrast, $Prj_{\prod}(P) \subset E^N$.

Remark 4. If for any $i \in I$ and $j \in J$, $i<j$ (i.e., if I is an initial section of N), then $Prj_{\prod}(P) \cong prj_I(P) \times \prod_J E \cong prj_I(P) \times E^k \subset E^h \times E^k$. If I is not an initial section of N, one could still define a canonical isomorphism of $Prj_{\prod}(P)$ into $E^h \times E^k \subset E^n$, but the intended canonical embedding is much harder to define: Let $I = \{i_1, i_2, ..., i_h\}$ and $J = \{j_1, j_2, ..., j_k\}$ and let f be a permutation of N such that $f(1, 2, ..., n) = (i_1, i_2, ..., i_h, j_1, j_2, ..., j_k)$, and let $\phi = f^{-1}$. Let $x = (x^i)_{i \in N}$ and define a transformation ϕ of E^N by $\phi(x) = (x_{\phi(1)}, x_{\phi(2)}, ..., x_{\phi(n)})$. Then, $\phi Prj_{\prod}(P) \cong prj_{\phi(I)} P \times E^k \subset E^h \times E^k$.

$(A)_I$: Argument projection (restriction) to E^I. Given an n-ary argument $A = (A_i)_{1 \leq i \leq n}$, $(A)_I = (A_i)_{i \in I}$.
$(P)_I$: Projecting $\mathcal{P}$ to $\mathcal{A}$ (Dissolving a predicate to an argument). Given a predicate P and $I \subset N$, $(P)_I = (prj_i P)_{i \in I}$.
$A|P$: an argument A cut by a predicate P. $A|P = (P|A)_N$.

Remark 5. $(A|P)_I = (P|A)_I$.

Example 1. n = 1.
$prj_{\{1\}} = \iota$ (identity map).
$Prj_{<\varnothing, \{1\}>}(P) = \varnothing \perp E = E$.
$Prj_{<\{1\}, \varnothing>}(P) = prj_{\{1\}}(P) \perp E^{\varnothing} = P$.

$\mathrm{Prj}_{<\varnothing, \{1\}>}(A, P) = \mathrm{Prj}_{<\varnothing, \{1\}>}(\mu A \cap P) = E.$
$\mathrm{Prj}_{<\{1\}, \varnothing>}(A, P) = \mathrm{Prj}_{<\{1\}, \varnothing>}(\mu A \cap P) = A \cap P.$

Example 2. n = 2.
$I = \{1, 2\}, \{1\}. \{2\}, \varnothing.$
$\mathrm{prj}_I(P)$:
$\mathrm{prj}_{\{1, 2\}}(P) = P.$
$\mathrm{prj}_{\{1\}}(P) = \mathrm{prj}_1(P) = \{x: \exists(x, y) \in P\},$
$\mathrm{prj}_{\{2\}}(P) = \mathrm{prj}_2(P) = \{y: \exists(x, y) \in P\}.$
$\mathrm{prj}_{\varnothing}(P) = \varnothing.$
$\mathrm{prj}_I(A, P)$:
$\mathrm{prj}_{\{1\}}(A, P) = \mathrm{prj}_{\{1\}}(\mu A \cap P) = \mathrm{prj}_{\{1\}}((A_1 \times A_2) \cap P)$. [the set of those A_1s that do P some A_2.]
$\mathrm{prj}_{\{2\}}(A, P) = \mathrm{prj}_{\{2\}}(\mu A \cap P) = \mathrm{prj}_{\{2\}}((A_1 \times A_2) \cap P)$. [set of those A_2's that are P-ed by some A_1.]

$\Pi = <\{1, 2\}, \{\varnothing\}>, <\{1\}, \{2\}>, <\{2\}, \{1\}>, <\{\varnothing\}, \{1,2\}>.$
$\mathrm{Prj}_{\Pi}(P)$:
$\mathrm{Prj}_{<\{1, 2\}, \varnothing>}(P) = \mathrm{prj}_{\{1, 2\}}(P) \perp E^{\varnothing} = P \perp \varnothing = P.$
$\mathrm{Prj}_{<\{1\}, \{2\}>}(P) = \mathrm{prj}_{\{1\}}(P) \perp E^{\{2\}} = \{(x, y): \exists(x, z) \in P\}.$
[the set of x's that do P paired with any y.]
$\mathrm{Prj}_{<\{2\}, \{1\}>}(P) = \mathrm{prj}_{\{2\}}(P) \perp E^{\{1\}} = \{(x, y): \exists(z, y) \in P\}.$
[the set of those y's that are P-ed paired with any x.]
$\mathrm{Prj}_{<\varnothing, \{1, 2\}>}(P) = \mathrm{prj}_{\varnothing}(P) \perp E^{\{1, 2\}} = \varnothing \perp E^2 = E^2.$
[2-dimensional existence predicate]
$\mathrm{Prj}_{<\{1, 2\}, \varnothing>}(P|A) = \mathrm{prj}_{\{1, 2\}}(\mu A \cap P) \perp E^{\varnothing} = \iota(\mu A \cap P) \perp \varnothing = \mu A \cap P.$
$\mathrm{Prj}_{<\{1\}, \{2\}>}(P|A) = \mathrm{prj}_{\{1\}}(\mu A \cap P) \perp E^{\{2\}} = \{(x, y): \exists(x, z) \in \mu A \cap P\}.$
[the set of those A_1s that P some A_2 paired any y.]
$\mathrm{Prj}_{<\{2\}, \{1\}>}(P|A) = \mathrm{prj}_{\{2\}}(\mu A \cap P) \perp E^{\{1\}} = \{(x, y): \exists(z, y) \in \mu A \cap P\}.$
[the set of those A_2s that are P-ed by some A_1 paired with any x.]
$\mathrm{Prj}_{<\varnothing, \{1, 2\}>}(P|A) = \mathrm{prj}_{\varnothing}(\mu A \cap P) \perp E^{\{1, 2\}} = \varnothing \perp E^2 = E^2.$

3.1.4 Intersections
3.1.4.1
$\alpha_{\Pi}(A, P) = (A)_I \perp (P|A)_J$

Explicitly put $\alpha_{\Pi}(A, P) = (A'_i)_{i \in N}$, where $A'_I = A_i$ for $i \in I$ and $A'_j = \mathrm{prj}_j(P|A) = \mathrm{prj}_j(\mu A \cap P)$ for $j \in J$.

Proposition 1. In particular,
$\alpha_{<\varnothing, N>}(A, P) = (P|A)_N = A|P = (\mathrm{prj}_j(\mu(A) \cap P))_{i \in N}$
$\alpha_{<N, \varnothing>}(A, P) = A.$

Example 1. n = 1.
$\alpha_{<\varnothing, \{1\}>}(A, P) = \varnothing \perp (\mu A \cap P)_{\{I\}} = A \cap P$
$\alpha_{<\{1,\}, \varnothing>}(A, P) = A \perp \varnothing = A$

Example 2. n = 2.
$\alpha_{<\{1, 2\}, \varnothing>}(A, P) = A \perp \varnothing = A$
$\alpha_{<\{1\}, \{2\}>}(A, P) = A_1 \perp (\mu A \cap P)_{\{2\}} = (A_1, \mathrm{prj}_2((A_1 \times A_2) \cap P))$
$\alpha_{<\{2\}, \{1\}>}(A, P) = A_2 \perp (\mu A \cap P)_{\{1\}} = (\mathrm{prj}_1((A_1 \times A_2) \cap P), A_2)$
$\alpha_{<\varnothing, \{1, 2\}>}(A, P) = \varnothing \perp (\mu A \cap P)_{\{1, 2\}} = (\mathrm{prj}_1((A_1 \times A_2) \cap P), \mathrm{prj}_2((A_1 \times A_2) \cap P))$

3.1.4.2

$\iota_{\prod}(A, P) = (\alpha_{\prod}(A, P), Prj_{\prod}(A, P)) = ((A)_I \perp (P|A)_J, prj_I(P|A) \perp E^J)$

Proposition 2. In particular,

$\iota_{<\varnothing, N>}(A, P) = ((P|A)_N, E^N) = (A|P, E^N)$;

$\iota_{<N, \varnothing>}(A, P) = (A, prj_N(P|A))) = (A, P|A)$.

Example 1. n= 1.

$\iota_{<\varnothing, \{1\}>}(A, P) = (A\cap P, E)$

$\iota_{<\{1\}, \varnothing>}(A, P) = (A, A\cap P)$

Example 2. n= 2.

$\iota_{<\varnothing, \{1, 2\}>}(A, P) = (prj_1((A_1\times A_2\cap P), prj_2((A_1\times A_2\cap P)), E^2)$

$\iota_{<\{1\}, \{2\}>}(A, P) = ((A_1, prj_2((A_1\times A_2) \cap P)), prj_1((A_1\times A_2) \cap P) \times E)$

$\iota_{<\{2\}, \{1\}>}(A, P) = (prj_1((A_1\times A_2\cap P), A_2), E \times prj_2((A_1\times A_2\cap P))$

$\iota_{<\{1, 2\}, \varnothing>}(A, P) = (A, (A_1\times A_2) \cap P)$

3.1.4.3

$\mu_{\prod}(A, P) = \mu\alpha_{\prod}(A, P)\cap Prj_{\prod}(A, P)$.

Proposition 3. $\mu_{\prod}(A, P) = prj_I(P|A) \perp \mu(P|A)_J$.

Proof. $\mu_{\prod}(A, P)$

$= \mu((A)_I \perp (P|A)_J) \cap (prj_I(P|A)\perp E^J)$

$= (\mu(A)_I \cap prj_I(P|A)) \perp \mu(A|P)_J$ [cf: $(P|A)_J = (A|P)_J$]

$= prj_I(P|A)) \perp \mu(P|A)_J$. [since $\mu(A)_I \supset prj_I(P|A)$]

Corollary. In particular,

$\mu_{<\varnothing, N>}(A, P) = \mu(P|A)_N$;

$\mu_{<\square N, \varnothing>}(A, P) = \mu(P|A)$.

Example 1. n= 1.

$\mu_{<\varnothing, \{1\}>}(A, P) = A\cap P$

$\mu_{<\{1\}, \varnothing>}(A, P) = A\cap(A\cap P) = A\cap P$

Example 2. n= 2.

$\mu_{<\varnothing, \{1, 2\}>}\square A, P) = (prj_1((A_1\times A_2\cap P), prj_2(A_1\times A_2\cap P))$

$\mu_{<\{1\}, \{2\}>}(A, P) = (prj_1((A_1\times A_2\cap P), prj_2(A_1\times A_2\cap P))$

$\mu_{<\{2\}, \{1\}>}(A, P) = (prj_1((A_1\times A_2\cap P), prj_2 (A_1\times A_2\cap P))$

$\mu_{<\{1, 2\}, \varnothing>}(A, P) = \mu A\cap(\mu A\cap P) = (A_1\times A_2)]\cap P$

Remark 6.

$\alpha_{\prod}(A, P)\in \mathcal{A}$.

$\iota_{\prod}(A, P) \in \mathcal{A}\times\mathfrak{P}$.

$\mu_{\prod}(A, P)\in \mathfrak{P}$.

3.2 Determiners

Definition 1. D is an n-ARY determiner iff D: $\mathcal{A}\times\mathfrak{P} \rightarrow \mathbf{1}$.

Definition 2. D is an n-DIMENSIONAL determiner iff D: $\mathfrak{P}\times\mathfrak{P} \rightarrow \mathbf{1}$.

Remark 7. For n = 1, the two concepts collapse.

Definition 3. An n-ary determiner is $\prod$-ε-intersective iff for any $(A, P) \in \mathcal{A}\times\mathfrak{P}$, $D(A, P) = D(\iota_{\prod}(A, P))$.

Proposition 4. In particular,
An n-ary determiner is <∅, N>-ε-intersective
iff for any $(A, P)\in \mathcal{A}\times\mathfrak{P}$, $D(A, P) = D(A|P, E^N)$.
An n-ary determiner is <N, ∅>-ε-intersective
iff for any $(A, P)\in \mathcal{A}\times\mathfrak{P}$, $D(A, P) = D(A, P|A)$.

Definition 4. An n-ary determiner D is $\prod$-ι-intersective iff the following condition holds:
For any pair $(A, P)\in \mathcal{A}\times\mathfrak{P}$, $(A', P') \in \mathcal{A}\times\mathfrak{P}$, if $\iota_{\prod}(A, P) = \iota_{\prod}(A', P')$, then $D(A, P) = D(A', P')$.

Proposition 5. In particular,
An n-ary determiner is <∅, N>-ι-intersective iff the following condition holds:
For any $(A, P)\in \mathcal{A}\times\mathfrak{P}$, $(A', P')\in \mathcal{A}\times\mathfrak{P}$, if $A|P = A'|P'$, then $D(A, P) = D(A', P')$.
An n-ary determiner is < N, ∅>-ι-intersective iff the following condition holds:
For any $(A, P)\in \mathcal{A}\times\mathfrak{P}$, $(A', P')\in \mathcal{A}\times\mathfrak{P}$, if $P|A = P'|A'$, then $D(A, P) = D(A, P')$.

Definition 5. An n-ary determiner D is $\prod$-μ-intersective iff the following condition holds:
For any pair $(A, P)\in \mathcal{A}\times\mathfrak{P}$, $(A', P')\in \mathcal{A}\times\mathfrak{P}$, if $\mu_{\prod}(A, P) = \mu_{\prod}(A', P')$, then $D(A, P) = D(A', P')$.

Proposition 6. In particular,
An n-ary determiner is <∅, N>-μ-intersective iff the following condition holds:
for any $(A, P)\in \mathcal{A}\times\mathfrak{P}$, $(A', P')\in \mathcal{A}\times\mathfrak{P}$, if $\mu(A|P) = \mu(A'|P')$, then $D(A, P) = D(A', P')$.
An n-ary determiner is < N, ∅>-μ-intersective iff the following condition holds:
for any $(A, P)\in \mathcal{A}\times\mathfrak{P}$, $(A', P')\in \mathcal{A}\times\mathfrak{P}$, if $P|A = P'|A'$, then $D(A, P) = D(A, P')$.

Proposition 7. An n-ary determiner D is $\prod$-μ-intersective iff the following condition holds:
For any pair $(A, P)\in \mathcal{A}\times\mathfrak{P}$, $(A', P')\in \mathcal{A}\times\mathfrak{P}$, if $prj_I(P|A) = prj_I(P'|A')$ & $(P|A)_J = (P'|A')_J$ [i.e., $prj_j(P|A) = prj_j(P'|A')$ for $j\in J$], then $D(A, P) = D(A', P')$.

Proof. By Proposition 3,
$\mu_{\prod}(A, P) = prj_I(P|A) \perp \mu(P|A)_J$.
Hence,
$\mu_{\prod}(A, P) = \mu_{\prod}(A', P')$
$\Leftrightarrow prj_I(P|A) \perp (P|A)_J = prj_I(P'|A') \perp \mu(P'|A')_J$
$\Leftrightarrow prj_I(P|A) = prj_I(P'|A')$ & $\mu(P|A)_J = \mu(P'|A')_J$. QED.

Definition 6. An n-*dimensional* determiner D is π-intersective iff the following condition holds:
For any pair $(P_1, P_2), (P'_1,P'_2) \in \mathfrak{P}\times\mathfrak{P}$, if $P_1\cap P_2=P'_1\cap P'_2$, then $D(P_1, P_2)=D(P'_1, P'_2)$.

Definition 7. An n-ary determiner D is conservative if the following condition holds: If $P|A = P'|A$, then $D(A, P) = D(A, P')$.

Proposition 8. [Keenan's theorem] D is conservative iff for any $(A,P) \in \mathcal{A} \times \mathfrak{P}$, $D(A,P)=D(A, P|A)$.

Proof.
ONLY IF: $P|A = \mu A \cap P = \mu A \cap (\mu A \cap P) = (\mu A \cap P)|A$.
Hence $D(A, P) = D(A, (\mu A \cap P)) = D(A, P|A)$.

IF: Assume $P\|A = P'\|A$.	(1)
$D(A, P) = D(A, P\|A)$	by assumption
$\quad = D(A, P'\|A)$.	by (1)
$D(A, P') = D(A, P'\|A)$.	by assumption

Hence, $D(A, P) = D(A, P')$. QED.

3.3 Sets of determiners

Det = $\mathbf{1}^{\mathcal{A} \times \mathfrak{P}}$ (the set of n-ary determiners)
$^{\wedge}$**Det** = $\mathbf{1}^{\mathfrak{P} \times \mathfrak{P}}$ (the set of n-dimensional determiners)

Con = the set of conservative determiners

$\mathbf{Int}^{\varepsilon}_{\prod}$ = the set of $\prod$-ε-intersective determiners
$\mathbf{Int}^{\iota}_{\prod}$ = the set of $\prod$-ι-intersective determiners
$\mathbf{Int}^{\mu}_{\prod}$ = the set of $\prod$-μ-intersective determiners
$\mathbf{Int}^{\pi}_{\prod}$ = the set of $\prod$-π-intersective determiners

$\mathbf{Int}^{\varepsilon}$ = the class of $\prod$-ε-intersective determiners
$\mathbf{Int}^{\iota}$ = the class of $\prod$-ι-intersective determiners
$\mathbf{Int}^{\mu}$ = the class of $\prod$-μ-intersective determiners

Definition 8. Rectangularizer rec: For an n-ary predicate P, $rec(P) = \Pi_{i \in N} prj_i P$.

Definition 9. An n-ary predicate P is rectangular iff $P = rec(P)$.

Proposition 9. An n-ary predicate P is rectangular iff there is an n-ary argument A such that $P=\mu(A)$.

The proposition follows directly from the definitions.

Definition 10. $^{\wedge}\mu$:**Det**→$^{\wedge}$**Det** is defined as follows: For D, D ∈ **Det** and for (P, P'), $P \in \mathfrak{P}$, $P' \in \mathfrak{P}$, if $P = rec(P)$, take $A \in \mathcal{A}$ such that $P = \mu(A)$, and let $^{\wedge}\mu D(P, P') = {}^{\wedge}\mu D(\mu(A), P') = D(A, P')$; otherwise, let $A = (prj_i(rec(P \cup P')))_{i \in N}$ (i.e., $A = (rec(P \cup P'))_N$) and $^{\wedge}\mu D(P, P') = D(A, P \cap P')$.

Proposition 10. $^{\wedge}\mu$ is one-to-one from **Det** into $^{\wedge}$**Det**.

Proof. If $D \neq D'$, take A, P such that $D(A, P) \neq D'(A, P)$. Then $^{\wedge}\mu D((\mu(A), P) = D(A, P) \neq D'(A, P) = {}^{\wedge}\mu D'((\mu(A), P)$. Hence $^{\wedge}\mu D \neq {}^{\wedge}\mu D'$.

Convention. We may take $^{\wedge}\mu$ as a canonical embedding of **Det** into $^{\wedge}$**Det** and we may regard a set **Det'** of n-ary determiners (i.e., **Det'** $\subset$ **Det**) as a subet of $^{\wedge}$**Det**. In particular, we can write **Det** $\subset$ $^{\wedge}$**Det** instead of $^{\wedge}\mu$**Det** $\subset$ $^{\wedge}$**Det**.

3.4 Families of determiners

Notation. **Int** $\subset$ **Int'** if for any $\prod$, $\mathbf{Int}_{\prod} \subset \mathbf{Int'}_{\prod}$.
If **Int** or **Int'** are classes independent of $\prod$, like **Con**, we agree to understand **Int** $\subset \mathbf{Int'}_{\prod}$ or $\mathbf{Int}_{\prod} \subset$ **Int'** to mean that for any $\prod$, **Int** $\subset \mathbf{Int'}_{\prod}$ or $\mathbf{Int}_{\prod} \subset$ **Int'**, respectively.

Proposition 11. $\mathbf{Int}^{\mu} \subset \mathbf{Int}^{\iota}$.

The proposition follows directly from the definitions of μ- and ι-intersectivity.

Proposition 12. $\mathbf{Int}^{\mu}{}_{<\emptyset, N>} = \mathbf{Int}^{\iota}{}_{<\emptyset, N>}$.

Proof.

$\mu_{<\emptyset,N>}(A, P) = \mu(P|A)_N$ (Proposition 6).

$\iota_{<\emptyset,N>}(A, P) = (A|P, E^N)$ (Proposition 5).

Hence, $\mu_{<\emptyset,N>}(A, P) = \mu_{<\emptyset,N>}(A', P') \Leftrightarrow \mu(P|A)_N = \mu(P'|A')_N \Leftrightarrow \mu(A|P) = \mu(A'|P')$.

$\iota_{<\emptyset,N>}(A, P) = \iota_{<\emptyset,N>}(A', P') \Leftrightarrow A|P = A'|P'$.

Hence, $\mu_{<\emptyset,N>}(A, P) = \mu_{<\emptyset,N>}(A', P') \Leftrightarrow \iota_{<\emptyset,N>}(A, P) = \iota_{<\emptyset,N>}(A', P')$. Hence the proposition.

Theorem 1. For $D \in$ **Det**, $D \in \mathbf{Int}^{\pi}$ if and only if the following condition holds:
For any A, A' and P, P', if $P|A = P'|A'$ then $D(A, P) = D(A', P')$. (1)

Proof. $D \in \mathbf{Int}^{\pi}$ means, according to the above convention, $^{\wedge}\mu D \in \mathbf{Int}^{\pi}$.
Hence, $D \in \mathbf{Int}^{\pi}$ iff

$$P_1 \cap P_2 = P_1' \cap P_2' \Rightarrow {}^{\wedge}\mu D(P_1, P_2) = {}^{\wedge}\mu D(P_1', P_2'). \quad (2)$$

Now assume $D \in \mathbf{Int}^{\pi}$ and assume $\mu(A) \cap P = \mu(A') \cap P'$. Then, by (2),

$$^{\wedge}\mu D(\mu(A), P) = {}^{\wedge}\mu D(\mu(A'), P')$$

and by the definition of $^{\wedge}\mu$,

$$D(A, P) = D(A', P'). \quad .$$

Conversely, assume (1) and assume

$$P_1 \cap P_2 = P_1' \cap P_2'. \quad (3)$$

(i) Assume both P_1 and P_1' are rectangular, that is, for some A and A',

$$P_1 \cap P_2 = \mu(A) \cap P_2 \text{ and } P_1' \cap P_2' = \mu(A') \cap P_2'. \quad (4)$$

By the definition of $^{\wedge}\mu$,

$$^{\wedge}\mu D(P_1, P_2) = D(A, P_2),\ {}^{\wedge}\mu D(P_1', P_2') = D(A', P_2').$$

On the other hand, by (3) and (4),

$$\mu(A) \cap P_2 = \mu(A') \cap P_2'$$

Hence, by (1),

$$D(A, P) = D(A', P').$$

(ii) Assume P_1 is not rectangular, and P_2 is. Then, by the definition of $\hat{}\mu$

$$\hat{}\mu D(P_1, P_2) = D((rec(P_1 \cup P_2))_N, P_1 \cap P_2). \qquad (5)$$

Now, since $rec(P_1 \cup P_2) \supset P_1 \cup P_2$,

$$P_1 \cap P_2 \ = rec(P_1 \cup P_2) \cap (P_1 \cap P_2)$$
$$= \mu(rec(P_1 \cup P_2))_N \cap (P_1 \cap P_2)$$

By (3) and (4),

$$\mu(rec(P_1 \cup P_2))_N \cap (P_1 \cap P_2) = \mu(A') \cap P_2'.$$

It follows by (1)

$$D((rec(P_1 4 P_2))_N, P_1 \cap P_2) = D(A', P_2').$$

Hence, by (4), (5),

$$\hat{}\mu D(P_1, P_2) = \hat{}\mu D(P_1', P_2') .$$

(iii) Similarly for the case where P_1 is rectangular and P_2 is not.

(iv) Assume both P_1 and P_2 are not rectangular. Then, we have

$$\hat{}\mu D(P_1, P_2) = D((rec(P_1 \cup P_2))_N, P_1 \cap P_2) \qquad (5)$$

and

$$\hat{}\mu D(P_1', P_2') = D((rec(P_1' \cup P_2'))_N, P_1' \cap P_2').$$

As in (ii) above, we have

$$P_1 \cap P_2 = \mu(rec(P_1 \cup P_2))_N \cap (P_1 \cap P_2)$$

And by a similar process as in (ii), we get

$$P_1' \cap P_2' = \mu(rec(P_1' \cup P_2'))_N \cap (P_1' \cap P_2'),$$

and by (3),

$$\mu(rec(P_1 \cup P_2))_N \cap (P_1 \cap P_2) = \mu(rec(P_1' \cup P_2'))_N \cap (P_1' \cap P_2').$$

Hence, by (1),

$$D((rec(P_1 \cup P_2))_N, P_1 \cap P_2) = D((rec(P_1' \cup P_2'))_N, P_1' \cap P_2'),$$

and by (4), (5),

$$\hat{}\mu D(P_1, P_2) = \hat{}\mu D(P_1', P_2').$$ QED.

Corollary. $\mathbf{Int}^{\mu}_{<N,\emptyset>} = \mathbf{Det} \cap \mathbf{Int}^{\pi}$.

Proof. From Proposition 6 we have

$D \in \mathbf{Int}^{\mu}_{<N,\emptyset>}$ iff $P|A = P'|A' \Rightarrow D(A, P) = D(A', P'')$.

Hence, the corollary follows from the theorem. QED.

We know that the canonical embedding $\hat{}\mu$ is one-to-one from **Det** into $\hat{}$**Det** (Proposition 10).

Theorem 2. $\hat{}\mu$ is an isomorphism onto $\mathbf{Int}^{\pi}$.

Proof. Assume $D \in \mathbf{Int}^{\pi}$. For any P_1, P_2 , since $rec(P_1 \cup P_2) \cap (P_1 \cap P_2) = P_1 \cap P_2$, we have

$$D(rec(P_1 \cup P_2), P_1 \cap P_2) = D(P_1, P_2,). \qquad (1)$$

Define an n-ary determiner D(A, P) as follows:

$$^{\vee}D(A, P) = D(\mu(A), P).$$

We will see that the following equation holds and hence $\hat{}\mu$ is a map onto Int^{π}**.**

$$\hat{}\mu^{\vee}D = D.$$

Indeed, if P is rectangular, for some A, $P = \mu(A)$ and by the definition of $\hat{}\mu$,

$$\hat{}\mu^{\vee}D(P_1, P_2) = {}^{\vee}D(A, P_2) = D(\mu(A), P_2) = D(P_1, P_2).$$

On the other hand, if P is not rectangular,

$$\begin{aligned} {}^\wedge\mu^\vee D(P_1, P_2) &= {}^\vee D((\mathrm{rec}(P_1 \cup P_2))_N, P_1 \cap P_2) \\ &= D(\mu(\mathrm{rec}(P_1 \cup P_2))_N, P_1 \cap P_2) \\ &= D((\mathrm{rec}(P_1 \cup P_2))_N, P_1 \cap P_2) \\ &= D(P_1, P_2) \qquad \text{by (1). QED.} \end{aligned}$$

This theorem means that we can restate the corollary of the preceding theorem in a stronger form:

Corollary. $\mathbf{Int}^{\mu}_{<N,ø>} = \mathbf{Int}^{\pi}$

This corollary shows that the n-ary determiner D(A, P) have the same expressive power as the n-dimensional determiner D(P, P') in spite of the fact that the range of the first argument A for the former appears to be more limited than that of the first argument P for the latter.

Theorem 3. $\mathbf{Int}^{\varepsilon} \subset \mathbf{Con}$.

Proof. Assume D is ∏-ε-intersective:

$$D(A, P) = D(\iota_{\prod}(A, P)) = D(\alpha_{\prod}(A, P), \mathrm{Prj}_{\prod}(A, P)) \tag{1}$$

We wish to show that D is conservative, that is, according to Theorem 1,

$$D(A, P) = D(A, P|A). \tag{2}$$

Applying (1) with P|A in place of P,

$$D(A, P|A) = D(\alpha_{\prod}(A, P|A), \mathrm{Prj}_{\prod}(A, P|A)). \tag{3}$$

Now,

$$\alpha_{\prod}(A, P|A) = (A_i)_I \perp ((P|A)|A)_J = (A)_I \perp (P|A)_J = \alpha_{\prod}(A, P). \tag{4}$$

From (3), (4),

$$D(A, P|A) = D(\alpha_{\prod}(A, P), \mathrm{Prj}_{\prod}(A, P|A)). \tag{5}$$

From (1) and (5), to show (2) is equivalent to showing

$$\mathrm{Prj}_{\prod}(A, P) = \mathrm{Prj}_{\prod}(A, P|A). \tag{6}$$

But $\mathrm{Prj}_{\prod}(A, P) = \mathrm{prj}_I(P|A) \perp E^J$ and $\mathrm{Prj}_{\prod}(A, P|A) = \mathrm{prj}_I((P|A)|A) \perp E^J = \mathrm{prj}_I((P|A) \perp E^J$.
Hence (6). QED.

Theorem 4. $\mathbf{Int}^{\iota}_{<N,ø>} = \mathbf{Con}$.

Proof. Assume D is $<N, ø>$-ι-intersective, that is,

$$\iota_{<N,ø>}(A, P) = \iota_{<N,ø>}(A', P') \Rightarrow D(A, P) = D(A', P') \tag{1}$$

But

$$\iota_{<N,ø>}(A, P) = (A, P|A),\ \iota_{<N,ø>}(A', P') = (A', P'|A'). \qquad \text{(cf: Proposition 5)}$$

Hence, (1) amounts to

$$P|A = P'|A \Rightarrow D(A, P) = D(A, P'). \text{ QED.}$$

Theorem 5. $\mathbf{Int}^{\varepsilon} = \mathbf{Int}^{\iota}$

Proof:
ONLY IF: Assume D is ∏-ε-intersective. Assume

$$\iota_{\prod}(A, P) = \iota_{\prod}(A', P'). \tag{1}$$

We want to show

$$D(A, P) = D(A', P'). \tag{2}$$

Since D is $\prod$-ε-intersective,

$D(A, P) = D(\iota_{\prod}(A, P))$

$D(A', P') = D(\iota_{\prod}(A', P'))$

But, then, from (1), we have (2).

IF: Assume D is $\prod$-ι-intersective.

Lemma. $\iota_{\prod}$ is idenpotent: $\iota_{\prod}\bullet\iota_{\prod} = \iota_{\prod}$.
The proof of the lemma is given in Appendix 1. The lemma means for any $A\in\mathcal{A}$ and $P\in\mathfrak{P}$,

$\iota_{\prod}(A, P) = \iota_{\prod}(\iota_{\prod}(A, P))$.

Then, if D is $\prod$-ι-intersective,

$D(A, P) = D(\iota_{\prod}(A, P))$.

But this means that D is $\prod$-ε-intersective. QED.

Theorem 6. If $\prod < \prod'$, then $\mathbf{Int}^{\iota}_{\prod} \subset \mathbf{Int}^{\iota}_{\prod'}$.

Proof:

Lemma. If $\prod < \prod'$, then

$\iota_{\prod'}(A, P) = \iota_{\prod'}(A', P') \Rightarrow \iota_{\prod}(A, P) = \iota_{\prod}(A', P')$.

The proof of the lemma is given in the Appendix 1. Assume $D(A, P) \in \mathbf{Int}^{\iota}_{\prod}$, and assume

$\iota_{\prod'}(A, P) = \iota_{\prod'}(A', P')$.

By Lemma 2,

$\iota_{\prod}(A, P) = \iota_{\prod}(A', P')$

Then,

$D(A, P) = D(A', P')$, by definition of $\mathbf{Int}^{\iota}_{\prod}$.

Hence $D(A, P) \in \mathbf{Int}^{\iota}_{\prod'}$, by definition of $\mathbf{Int}^{\iota}_{\prod'}$. QED.

Theorem 7. If $\prod < \prod'$, $\mathbf{Int}^{\mu}_{\prod} \subset \mathbf{Int}^{\mu}_{\prod'}$.

Proof.

Lemma

Assume $\Pi = \langle I, J\rangle < \Pi' = \langle I', J'\rangle$. Then, if

$prj_{I'}(P)\perp \mu(P)_{J'} = prj_{I'}(P')\perp \mu(P')_{J'}$

then

$prj_{I}(P)\perp \mu(P)_{J} = prj_{I}(P')\perp \mu(P')_{J}$.

The proof of the lemma is given in Appendix 1. Now, assume

$\mu_{\prod'}(A, P) = \mu_{\prod'}(A', P')$. (1)

Since

$\mu_{\prod'}(A, P) = prj_{I'}(P|A)\perp\mu(P|A)_{J'}$ (2)

$\mu_{\prod'}(A', P') = prj_{I'}(P'|A')\perp\mu(P'|A')_{J'}$ (3)

$\mu_{\prod}(A, P) = prj_{I}(P|A)\perp\mu(P|A)_{J}$ (4)

$\mu_{\prod}(A', P') = prj_{I}(P'|A')\perp\mu(P'|A')_{J}$, (5) by applying the lemma to P|A and P'|A', from (1) and (2)-(5) we have

$$\mu_{\prod}(A, P) = \mu_{\prod}(A', P'). \tag{6}$$

Now assume

$$D \in \mathbf{Int}^{\mu}_{\prod}.$$

Then, from (6) we have

$$D(A, P) = D(A', P'). \tag{7}$$

Hence, we see that (7) follows from (1), i.e., $D \in \mathbf{Int}^{\mu}_{\prod'}$. Q.E.D.

From the above Propositions and Theorems we have established the following hierarchy of intersective determiners:

Theorem 8. Hierarchy of Intersective determiners. Given $\prod < \prod'$, we have the following relations among sets of determiners:

$$\begin{array}{ccccccccccc}
 & & & & \mathbf{Int}^{\varepsilon}_{\langle N,\emptyset\rangle} & & \mathbf{Int}^{\varepsilon}_{\prod'} & & \mathbf{Int}^{\varepsilon}_{\prod} & & \mathbf{Int}^{\varepsilon}_{\langle \emptyset, N\rangle} \\
 & & & & \| & & \| & & \| & & \| \\
\mathbf{Det} & \supset & \mathbf{Con} & = & \mathbf{Int}^{\iota}_{\langle N,\emptyset\rangle} & \supset\ldots\supset & \mathbf{Int}^{\iota}_{\prod'} & \supset\ldots\supset & \mathbf{Int}^{\iota}_{\prod} & \supset\ldots\supset & \mathbf{Int}^{\iota}_{\langle \emptyset, N\rangle} \\
 & & & & \cup & & \cup & & \cup & & \| \\
{}^\wedge\mathbf{Det} & \supset & \mathbf{Int}^{\pi} & \cong & \mathbf{Int}^{\mu}_{\langle N,\emptyset\rangle} & \supset\ldots\supset & \mathbf{Int}^{\mu}_{\prod'} & \supset\ldots\supset & \mathbf{Int}^{\mu}_{\prod} & \supset\ldots\supset & \mathbf{Int}^{\mu}_{\langle \emptyset, N\rangle}
\end{array}$$

Diagram 1

Appendix 1. Proofs of Lemmas.

Lemma to Theorem 5. $\iota_{\prod}(A, P) = \iota_{\prod}(\iota_{\prod}(A, P))$.

Proof. Let $\prod = \langle I, J\rangle$.

$$\iota_{\langle I,J\rangle}(A, P) = (\alpha_{\langle I,J\rangle}(A, P), \mathrm{Prj}_{\langle I,J\rangle}(A, P)). \tag{1}$$

$$\alpha_{\langle I,J\rangle}(A, P) = ((A)_I \perp (P|A)_J). \tag{2}$$

Hence,

$$\begin{aligned}
&\iota_{\prod}(\iota_{\prod}(A, P)) \\
&\quad = \iota_{\langle I, J\rangle}(\alpha_{\langle I, J\rangle}(A, P), \mathrm{Prj}_{\langle I, J\rangle}(A, P)) \\
&\quad = \iota_{\langle I,J\rangle}((A)_I \perp (P|A)_J, \mathrm{Prj}_{\langle I, J\rangle}(A, P)).
\end{aligned} \tag{3}$$

We substitute for A and P in the right hand side of (1) thus:

$$\begin{aligned}
&A \Rightarrow (A)_I \perp (P|A)_J \\
&P \Rightarrow \mathrm{Prj}_{\langle I,J\rangle}(A, P) \\
&\quad = (\alpha_{\langle I,J\rangle}((A)_I \perp (P|A)_J, \mathrm{Prj}_{\langle I, J\rangle}(A, P)), \mathrm{Prj}_{\langle I, J\rangle}[(A)_I \perp (P|A)_J, \mathrm{Prj}_{\langle I,J\rangle}(A, P)]).
\end{aligned} \tag{4}$$

Next, we execute the same substitution for (2):

$$\begin{aligned}
&\alpha_{\langle I,J\rangle}((A)_I \perp (P|A)_J, \mathrm{Prj}_{\langle I,J\rangle}(A, P)) \\
&\quad = (((A)_I \perp (P|A)_J)_I \perp (\mathrm{Prj}_{\langle I,J\rangle}(A, P)|[(A)_I \perp (P|A)_J])_J) \\
&\quad = ((A)_I \perp ([(\mathrm{prj}_I(P|A) \perp E^J)| [(A)_I \perp (P|A)_J])_J \\
&\quad = ((A)_I \perp ([(\mathrm{prj}_I(P|A) \perp E^J) \cap \mu[(A)_I \perp (P|A)_J])_J \\
&\quad = ((A)_I \perp ([(\mathrm{prj}_I(P|A) \cap \mu(A)_I] \perp E^J \cap (P|A)_J])_J \\
&\quad = ((A)_I \perp ((\mathrm{prj}_I(P|A) \perp (P|A)_J)_J \\
&\quad = (A)_I \perp (P|A)_J.
\end{aligned} \tag{5}$$

Next,

$$\begin{aligned}
&\mathrm{Prj}_{\langle I, J\rangle}[(A)_I \perp (P|A)_J, \mathrm{Prj}_{\langle I,J\rangle}(A, P)] \\
&\quad = \mathrm{Prj}_{\langle I, J\rangle}[\mu((A)_I \perp (P|A)_J) \cap \mathrm{Prj}_{\langle I, J\rangle}(A, P)] \\
&\quad = \mathrm{Prj}_{\langle I, J\rangle}[\mu((A)_I \perp (P|A)_J) \cap \mathrm{Prj}_{\langle I,J\rangle}(P|A)] \\
&\quad = \mathrm{Prj}_{\langle I, J\rangle}[\mu((A)_I \perp (P|A)_J) \cap (\mathrm{prj}_I(P|A) \perp E^J)] \\
&\quad = \mathrm{Prj}_{\langle I, J\rangle}[(\mu(A)_I \cap \mathrm{prj}_I (P|A)) \perp (\mu(P|A)_J \cap E^J)] \\
&\quad = (\mu(A)_I \cap \mathrm{prj}_I (P|A)) \perp E^J.
\end{aligned} \tag{6}$$

Since

$$\mathrm{prj}_I(P|A) \subset \mu(A)_I,$$

we have

$$\mu(A)_I \cap (\mathrm{prj}_I (P|A) = \mathrm{prj}_I(P|A).$$

Hence,

$$(6) = \mathrm{prj}_I(P|A) \perp E^J. \tag{7}$$

From (4), (5) and (7),

$$\begin{aligned}
&\iota_{\prod}(\iota_{\prod}(A, P)) \\
&\quad = ((A)_I \perp (P|A)_J, \mathrm{Prj}_{\langle I, J\rangle}(A, P)) \\
&\quad = ((A)_I \perp (P|A)_J, \mathrm{prj}_I(P|A) \perp E^J) \\
&\quad = (\alpha_{\prod}(A, P), \mathrm{Prj}_{\prod}(A, P) \\
&\quad = \quad \iota_{\prod}(A, P). \qquad\qquad \text{QED.}
\end{aligned}$$

Lemma to Theorem 6. If $\prod = \langle I, J\rangle < \prod' = \langle I', J'\rangle$, then
$\iota_{\langle I',J'\rangle} (A, P) = \iota_{\langle I',J'\rangle} (A', P') \Rightarrow \iota_{\langle I,J\rangle} (A, P) = \iota_{\langle I,J\rangle} (A', P')$.

Proof. Let I' = I∪K, I∩K = ø. (Hence, J = J'4K, J'∩K = ø.)

$\iota_{<I', J'>}(A, P) = (\alpha_{<I',J'>}(A, P), Prj_{<I',J'>}(A, P))$

$\alpha_{<I',J'>}(A, P) = ((A)_{I'}\perp(P|A)_{J'}) = ((A)_{I}\perp(A)_{K}\perp(P|A)_{J'})$

$Prj_{<I',J'>}(A, P) = prj_{I'}(P|A)\perp E^{J'}$.

Hence,

$\iota_{<I', J'>}(A, P) = \iota_{<I', J'>}(A', P')$

$\iff$ $(A)_I = (A)'_I$

$(A)_k = (A')_K$

$(P|A)_{J'} = (P'|A')_{J'}$

$prj_{I'}(P|A) = prj_{I'}(P'|A')$.

Similarly,

$\iota_{<I, J>}(A, P) = (\alpha_{<I, J>}(A, P), Prj_{<I, J>}(A, P))$

$\alpha_{<I, J>}(A, P) = (A)_I\perp(P|A)_J$

$= (A)_I\perp(P|A)_K\perp(P|A)_{J'}$

$Prj_{<I,J>}(A, P) = prj_I(P|A)\perp E^J$.

Hence,

$\iota_{<I, J>}(A, P) = \iota_{<I, J>}(A', P')$

$\iff$ $(A)_I = (A)'_I$

$(P|A)_K = (P'|A')_K$

$(P|A)_{J'} = (P'|A')_{J'}$

$prj_I(P|A) = prj_I(P'|A')$.

Hence, it suffices to show that if

$(A)_K = (A')_K$ and $prj_{I'}(P|A) = prj_{I'}(P'|A')$

then

$(P|A)_K = (P'|A')_K$ and $prj_I(P|A) = prj_I(P'|A')$.

Proof of $(P|A)_K = (P'|A')_K$:

$y \in prj_k(P|A)$

$\iff$ $\exists x = (x_i)_{i\in N}$: $x\in P$, $x_i\in A_i$ $(i\in N)$ and $y = x_k$

$\iff$ $\exists x = (x_i)_{i\in N}$: $(x_i)_{i\in I'}\in prj_{I'}(P)$, $x\in P$, $x_i\in A_i$ $(i\in N)$ and $y = x_k\in A_k$

$\iff$ $\exists x = (x_i)_{i\in N}$: $(x_i)_{i\in I'}\in prj_{I'}(P|A)$, $x\in P$, $y = x_k\in A_k$.

Since $prj_{I'}(P|A) = prj_{I'}(P'|A')$ and $(A)_K = (A')_K$ by assumption,

$\Rightarrow$ $\exists x = (x_i)_{i\in N}$: $(x_i)_{i\in I'}\in prj_{I'}(P'|A')$, $y = x_k\in A'_k$

$\iff$ $y \in prj_k(P'|A')$.

Hence,

$prj_k(P|A) \subset prj_k(P'|A')$.

Symmetrically,

$prj_k(P|A) \supset prj_k(P'|A')$.

Hence,

$prj_k(P|A) = prj_k(P'|A')$

$prj_I(P|A) = prj_I(P'|A')$

$prj_I(P|A)$

$= prj_I prj_{I'}(P|A)$ (since $I \subset I'$)

$= prj_I prj_{I'}(P'|A')$ (since $prj_{I'}(P|A) = prj_{I'}(P'|A')$ by assumption)

$= prj_I(P'|A')$. QED.

Lemma to Theorem 7.

Assume

$\Pi = \langle I, J \rangle < \Pi' = \langle I', J' \rangle$. Then, if

$$\mathrm{prj}_{I'}(P) \perp \mu(P)_{J'} = \mathrm{prj}_{I'}(P') \perp \mu(P')_{J'} \tag{1}$$

then

$$\mathrm{prj}_{I}(P) \perp \mu(P)_{J} = \mathrm{prj}_{I}(P') \perp \mu(P')_{J}. \tag{2}$$

Proof.
First, we prove the following proposition (A):

(A) If $\mathrm{prj}_I(P) = \mathrm{prj}_I(P')$ and $I \supset J$, then
$\mathrm{prj}_J(P) = \mathrm{prj}_J(P')$.

Proof of Proposition (A).
Let $I = I' \perp J$.
Assume $x \in \mathrm{prj}_J(P)$; by the definition of prj, we have
$\exists z$: $z \in P$, $x = (z)_J$.
Put $y = (z)_I$. Then, by the assumption of the lemma,
$y \in \mathrm{prj}_I(P) = \mathrm{prj}_I(P')$.
Hence,
$\exists z'$: $z' \in P'$, $y = (z')_I$
Hence
$y = (z)_I = (z')_I$ and hence $x = (z)_J = (z')_J$.
That is, $\exists z'$: $z' \in P'$, $x = (z')_J$,
$x \in \mathrm{prj}_J(P')$,
$\mathrm{prj}_J(P) \subset \mathrm{prj}_J(P')$.
Likewise,
$\mathrm{prj}_J(P') \subset \mathrm{prj}_J(P)$.
Hence, (A).
Next, we prove the following proposition (B):

(B) If $\mathrm{prj}_I(P) = \mathrm{prj}_I(P')$, then $\mu(P)_I = \mu(P')_I$.

Indeed, if $\mathrm{prj}_I(P) = \mathrm{prj}_I(P')$, since $I \supset \{\iota\}$, by (A), we have
$\mathrm{prj}_i P = \mathrm{prj}_i P'$.
Then,
$\mu(P)_I = \mu(\perp_{i \in I} \mathrm{prj}_i P) = \mu(\perp_{i \in I} \mathrm{prj}_i P') = \mu(P')_I$.
Hence (B).
We return to the proof of the Lemma.
Let K be such that $I' = I \, 4 \, K$, $I \cap K = \varnothing$; $J = K \, 4 \, J'$. Assume (1). Then, we have

$$\mathrm{prj}_{I'}(P) = \mathrm{prj}_{I'}(P') \tag{3}$$

$$\mu(P)_{J'} = \mu(P')_{J'}. \tag{4}$$

By Proposition (A),

$$\mathrm{prj}_{I}(P) = \mathrm{prj}_{I}(P') \tag{5}$$

$$\mathrm{prj}_{K}(P) = \mathrm{prj}_{K}(P') \tag{6}$$

From (6) by Proposition (B),

$$\mu(P)_K = \mu(P')_K. \tag{7}$$

Then, we have by (7) and (4)

$$\begin{aligned} \mu(P')_J &= \mu(P')_{K4J'} \\ &= \mu(P')_K \perp \mu(P')_{J'} \\ &= \mu(P)_K \perp \mu(P')_{J'} \\ &= \mu(P')_J. \end{aligned} \tag{8}$$

By (5) and (8), we have (2). QED.

Appendix 2. Application 1: n = 1. "Intransitive" determiners.

$\mathfrak{P} = \mathbf{P}(E)$: The set of predicates.
$\mathcal{A} = \mathbf{P}(E)$
$\mu: \mathcal{A} \to \mathfrak{P}$. Identity map.

Partitions:
$\prod_1 = <ø,\{1\}>$
$\prod_2 = <\{1\},ø>$

Projections:
$Prj_{<ø, \{1\}>}(P) = prj_{ø}(P)\times E = E$
$Prj_{<\{1\},ø>}(P) = prj_{\{1\}}(P)\times\prod_{i\in ø}E = P$
$Prj_{<ø, \{1\}>}(A, P) = prj_{ø}(A, P)\times E = E$
$Prj_{<\{1\},ø>}(A, P) = prj_{\{1\}}(A\cap P)\times\prod_{i\in ø}E = A\cap P$

Intersections:
$\alpha_{<ø, \{1\}>}(A, P) = \{(A'_i): A'_i = A_i \text{ for } i\in ø \text{ and } A'_j = prj_j(\mu A\cap P) \text{ for } j = 1\} = A\cap P$
$\alpha_{<\{1\},ø>}(A, P) = \{(A'_i): A'_i = A_i \text{ for } i=1 \text{ and } A'_j = prj_j(\mu A\cap P) \text{ for } j\in ø\} = A$

$\iota_{<ø, \{1\}>}(A, P) = (\alpha_{<ø, \{1\}>}(A, P),\ Prj_{<ø, \{1\}>}(A, P)) = (A\cap P, E)$
$\iota_{<\{1\},ø>}(A, P) = (\alpha_{<\{1\},ø>}(A, P),\ Prj_{<\{1\},ø>}(A, P)) = (A, A\cap P)$

$\mu_{<ø, \{1\}>}(A, P) = \mu\alpha_{<ø, \{1\}>}(A, P) \cap Prj_{<ø, \{1\}>}(A, P) = (A\cap P)\cap E = A\cap P$
$\mu_{<\{1\},ø>}(A, P) = \mu\alpha^{\iota}_{<\{1\},ø>}(A, P) \cap Prj^{\iota}_{<\{1\},ø>}(A, P) = A\cap P$

$$\begin{array}{ccccccccc}
 & & & & \mathbf{Int}^{\varepsilon}_{<\{1\},ø>} & \supset & \mathbf{Int}^{\varepsilon}_{<ø, \{1\}>} \\
 & & & & \| & & \| \\
\mathbf{Det} & \supset & \mathbf{Con} & = & \mathbf{Int}^{\iota}_{<\{1\},ø>} & \supset & \mathbf{Int}^{\iota}_{<ø, \{1\}>} \\
 & & & & \cup & & \| \\
\hat{}\mathbf{Det} & \supset & \mathbf{Int}^{\pi} & \cong & \mathbf{Int}^{\mu}_{<\{1\},ø>} & \supset & \mathbf{Int}^{\mu}_{<ø, \{1\}>}
\end{array}$$

Diagram 1 for n = 1.

Since $\iota_{<ø, \{1\}>}(A, P) = (A\cap P, E)$, $\iota_{<ø, \{1\}>}(A, P) = \iota_{<ø, \{1\}>}(A', P')$ is equivalent to $A\cap P = A'\cap P'$, and this in turn is equivalent to $\mu_{<\{1\},ø>}(A, P) = \mu_{<\{1\},ø>}(A', P')$. Hence,

Proposition 13. $\mathbf{Int}^{\pi} = \mathbf{Int}^{\iota}_{<ø,\{1\}>}$.

With the above 2 propositions, the above Diagram 1 for n=1 is reduced to the following diagram:

$$\begin{array}{ccccccccc}
 & & & & \mathbf{Int}^{\varepsilon}_{<\{1\},ø>} & \supset & \mathbf{Int}^{\varepsilon}_{<ø, \{1\}>} \\
 & & & & \| & & \| \\
\mathbf{Det} & \supset & \mathbf{Con} & = & \mathbf{Int}^{\iota}_{<\{1\},ø>} & \supset & \mathbf{Int}^{\iota}_{<ø, \{1\}>} \\
 & & & & \cup & & \| \\
\hat{}\mathbf{Det} & \supset & \mathbf{Int}^{\pi} & \cong & \mathbf{Int}^{\mu}_{<\{1\},ø>} & = & \mathbf{Int}^{\mu}_{<ø, \{1\}>}
\end{array}$$

A reduced Diagram 1 for n=1.

Now, we note that for n = 1, **Det** and $^\wedge$**Det** are canonically isomorphic by identifying $D \in$ **Det** and $^\wedge D \in {}^\wedge$**Det** by $D(A, P) = {}^\wedge D(\mu A, P)$. Hence, we can identify **Det** and $^\wedge$**Det**.

Proposition 14. For n = 1, **Det** = $^\wedge$**Det**.

It remains to see if **Con** = $\mathbf{Int}^{\iota}_{<\{1\},\varnothing>}$ is a proper subset of **Det** = $^\wedge$**Det** and if $\mathbf{Int}^{\iota}_{<\varnothing, \{1\}>} = \mathbf{Int}^{\pi} \cong \mathbf{Int}^{\mu}_{<\{1\},\varnothing>} = \mathbf{Int}^{\mu}_{<\varnothing, \{1\}>}$ is a proper subset of **Con** = $\mathbf{Int}^{\iota}_{<\{1\},\varnothing>}$.

Proposition 15. **Con** = $\mathbf{Int}^{\iota}_{<\{1\},\varnothing>}$ is a proper subset of **Det** = $^\wedge$**Det**.

Proof: Let $D(A, P) = 1 \Leftrightarrow |A \cap P| << |P|$, where << reads "proportionately sufficiently small number of," say $|A \cap P|/|P| < 1/10$. (We may agree to read "rarely A is P" with the intended sense.) If $D \in$ **Con** = $\mathbf{Int}^{\iota}_{<\{1\},\varnothing>}$, we have

$$D(A, P) = D(A, A \cap P).$$

But $|A \cap (A \cap P)| = |A \cap P| << |A \cap P|$ cannot hold, a contradiction. QED.

Proposition 16. $\mathbf{Int}^{\iota}_{<\varnothing, \{1\}>}$ is a proper subset of **Con** = $\mathbf{Int}^{\iota}_{<\{1\},\varnothing>}$.

Proof: Define D(A, P) by

$$D(A, P) = 1 \Leftrightarrow |A \cap P| << |A|.$$

Then,

$$A \cap P = A \cap P' \Rightarrow D(A, P) = D(A, P').$$

Hence,

$$D \in \mathbf{Con}.$$

Next, assume

$$A \cap P = A' \cap P' \text{ and } |A \cap P| << |A|.$$

From this it does not follow that

$$|A' \cap P'| << |A'|.$$

Hence,

$$D \notin \mathbf{Int}^{\iota}_{<\varnothing, \{1\}>}.$$

From Proposition 15 and Proposition 16 Diagram 1 for n=1 further reduces to the following diagram.

$$\begin{array}{ccccccc}
{}^\wedge\mathbf{Det} = & \mathbf{Det} & \supsetneq & \mathbf{Con} = & \mathbf{Int}^{\iota}_{<\{1\},\varnothing>} & \supsetneq & \mathbf{Int}^{\iota}_{<\varnothing, \{1\}>} \\
 & & & & & & \| \\
 & & & \mathbf{Int}^{\pi} = & \mathbf{Int}^{\mu}_{<\{1\},\varnothing>} & = & \mathbf{Int}^{\mu}_{<\varnothing, \{1\}>}
\end{array}$$

Diagram 2.

We can summarize the above result as follows:

Theorem 9. For n = 1, there are three sets of intersective determiners, which we may call **Det** (determiners), **Con** (conservative determiners) and **Int** (intersective determiners):

$$\mathbf{Det} \supsetneq \mathbf{Con} \supsetneq \mathbf{Int}.$$

(I.e., **Det** $\supset$ **Con** $\supset$ **Int** and **Det** $\neq$ **Con** $\neq$ **Int**.)

Appendix 3. Application 2: n = 2. "Transitive" determiners.

$\mathfrak{P} = \mathbf{P}(E^2)$: The set of predicates.
$\mathcal{A} = [\mathbf{P}(E)]^{2:}$ the set of argument pairs.
$\mu: \mathcal{A} \rightarrow \mathfrak{P}.\ \ \mu A = A_1 \% A_2$

Partitions:
$\prod_1 = <ø,\{1.2\}>$
$\prod_2 = <\{1\}.\{2\}>$
$\prod_3 = <\{2\},\{1\}>$
$\prod_4 = <\{1, 2\},ø>$

Projections:
$Prj_{<ø,\ \{1,\ 2\}>}(A, P) = E^2$
$Prj_{<\{1\},\{2\}>}(A, P) = prj_1((A_1 \times A_2) \cap P) \times E$
$Prj_{<\{2\},\{1\}>}(A, P) = E \times prj_2((A_1 \times A_2) \cap P)$
$Prj_{<\{1,\ 2\},ø>}(A, P) = (A_1 \times A_2) \cap P$

Intersections:
$\alpha_{<ø,\ \{1,\ 2\}>}(A, P) = (prj_1((A_1 \times A_2)3P), prj_2((A_1 \times A_2) \cap P))$
$\alpha_{<\{1\},\{2\}>}(A, P) = (A_{1,}\ prj_2((A_1 \times A_2) \cap P))$
$\alpha_{<\{2\},\{1\}>}(A, P) = (prj_1((A_1 \times A_2) \cap P)), A_2)$
$\alpha_{<\{1,\ 2\},\ ø>}(A, P) = (A_1, A_2)$

$\iota_{<ø,\ \{1,\ 2\}>}(A, P) = (prj_1((A_1 \times A_2) \cap P), prj_2((A_1 \times A_2) \cap P)), E^2)$
$\iota_{<\{1\},\{2\}>}(A, P) = ((A_1, prj_2((A_1 \times A_2) \cap P)), prj_1((A_1 \times A_2) \cap P) \times E)$
$\iota_{<\{2\},\{1\}>}(A, P) = ((prj_2((A_1 \times A_2) \cap P), A_2), E \times prj_2((A_1 \times A_2) \cap P))$
$\iota_{<\{1,\ 2\},ø>}(A, P) = ((A_1, A_2), (A_1 \% A_2)3P))$

$\mu_{<ø,\{1,\ 2\}>}(A, P) = prj_1((A_1 \times A_2) \cap P) \times prj_2((A_1 \times A_2) \cap P) \cap E^2$
$\qquad = prj_1((A_1 \times A_2) \cap P) \times prj_2((A_1 \times A_2) \cap P)$
$\mu_{<\{1\},\{2\}>}(A, P) = [A_1 \times prj_2((A_1 \times A_2) \cap P)] \cap [prj_1((A_1 \times A_2) \cap P) \times E]$
$\qquad = [A_1 \cap prj_1((A_1 \times A_2) \cap P)] \times [prj_2((A_1 \times A_2)3P)3E]$
$\qquad = [A_1 \cap prj_1((A_1 \times A_2) \cap P)] \times [prj_2((A_1 \times A_2)3P)]$
$\qquad = prj_1((A_1 \times A_2) \cap P) \times prj_2((A_1 \times A_2) \cap P)$
$\mu_{<\{2\},\{1\}>}(A, P) = prj_1((A_1 \times A_2)) \cap P) \times p_2((A_1 \times A_2) \cap P)$
$\mu_{<\{1,\ 2\},ø>}(A, P) = (A_1 \times A_2) \cap P$

Conditions for intersective determiners

Int$^{\iota}_{<ø,\ \{1,\ 2\}>}$: $\quad prj_1((A_1 \times A_2) \cap P) = prj_1((A'_1 \times A'_2) \cap P')$
$\qquad prj_2((A_1 \times A_2) \cap P) = prj_2((A'_1 \times A'_2) \cap P')$

Int$^{\iota}_{<\{1\},\ \{2\}>}$: $\quad A_1 = A'_1,$
$\qquad prj_1((A_1 \times A_2) \cap P) = prj_1((A'_1 \times A'_2) \cap P')$
$\qquad prj_2((A_1 \times A_2) \cap P) = prj_2((A'_1 \times A'_2) \cap P')$

Int$^{\iota}_{<\{2\},\ \{1\}>}$: $\quad A_2 = A'_2$
$\qquad prj_1((A_1 \times A_2) \cap P) = prj_1((A'_1 \times A'_2) \cap P')$

$$
\begin{array}{ll}
 & prj_2((A_1\times A_2)\cap P) = prj_2((A'_1\times A'_2)\cap P') \\
\mathbf{Int}^{\iota}_{<\{1,2\},\varnothing>}: & A_1 = A'_1,\ A_2 = A'_2 \\
 & (A_1\times A_2)\cap P = (A'_1\times A'_2)\cap P'
\end{array}
$$

$$
\begin{array}{ll}
\mathbf{Int}^{\varepsilon}_{<\varnothing,\{1,2\}>}: & D(A, P) = D((prj_1((A_1\times A_2)\cap P), prj_2((A_1\times A_2)\cap P)), E^2) \\
\mathbf{Int}^{\varepsilon}_{<\{1\},\{2\}>}: & D(A, P) = D((A_1, prj_2((A_1\times A_2)\cap P)), prj_1((A_1\times A_2)\cap)\times E) \\
\mathbf{Int}^{\varepsilon}_{<\{2\},\{1\}>}: & D(A, P) = D((prj_1((A_1\times A_2)\cap P), A_2), E\times prj_2((A_1\times A_2)\cap P)) \\
\mathbf{Int}^{\varepsilon}_{<\{1,2\},\varnothing>}: & D(A, P) = D((A_1, A_2), (A_1\%A_2)\cap P) \\
\mathbf{Int}^{\mu}_{<\varnothing,\{1,2\}>}: & prj_1((A_1\times A_2)\cap P) = prj_1((A'_1\times A'_2)\cap P') \\
 & prj_2((A_1\times A_2)\cap P) = prj_2((A'_1\times A'_2)\cap P') \\
\mathbf{Int}^{\mu}_{<\{1\},\{2\}>}: & prj_1((A_1\times A_2)\cap P) = prj_1((A'_1\times A'_2)\cap P') \\
 & prj_2((A_1\times A_2)\cap P) = prj_2((A'_1\times A'_2)\cap P') \\
\mathbf{Int}^{\mu}_{<\{2\},\{1\}>}: & prj_1((A_1\times A_2)\cap P) = prj_1((A'_1\times A'_2)\cap P') \\
 & prj_2((A_1\times A_2)\cap P) = prj_2((A'_1\times A'_2)\cap P') \\
\mathbf{Int}^{\mu}_{<\{1,2\},\varnothing>}: & (A_1\times A_2)\cap P = (A'_1\times A'_2)\cap P'
\end{array}
$$

From these conditions for $\mathbf{Int}^{\mu}$, we have:

Proposition 17. $\mathbf{Int}^{\mu}_{<\varnothing,\{1,2\}>} = \mathbf{Int}^{\mu}_{<\{1\},\{2\}>} = \mathbf{Int}^{\mu}_{<\{2\},\{1\}>} = \mathbf{Int}^{\iota}_{<\varnothing,\{1,2\}>}$

Due to this proposition, we have the following diagram for n = 2.

$$
\begin{array}{ccccccccc}
 & & & & \mathbf{Int}^{\varepsilon}_{<\{1,2\},\varnothing>} & & \mathbf{Int}^{\varepsilon}_{<\{1\},\{2\}>} & & \mathbf{Int}^{\varepsilon}_{<\varnothing,\{1,2\}>} \\
 & & & & \| & & \| & & \| \\
\mathbf{Det} & \supset & \mathbf{Con} & = & \mathbf{Int}^{\iota}_{<\{1,2\},\varnothing>} & \supset & \mathbf{Int}^{\iota}_{<\{1\},\{2\}>} & \supset & \mathbf{Int}^{\iota}_{<\varnothing,\{1,2\}>} \\
 & & & & \cup & & \cup & & \| \\
{}^\wedge\mathbf{Det} & \supset & \mathbf{Int}^{\pi} & \cong & \mathbf{Int}^{\mu}_{<\{1,2\},\varnothing>} & \supset & \mathbf{Int}^{\mu}_{<\{1\},\{2\}>} & = & \mathbf{Int}^{\mu}_{<\varnothing,\{1,2\}>}
\end{array}
$$

Diagram 1 for n = 2:

EXAMPLE 1. $D_{(n, m)}$

DEFINITION: $D_{(n, m)}(A, P) = 1$

$\iff |prj_1((A_1\times A_2)\cap P)| = n, |prj_2((A_1\times A_2)\cap P)| = m.$

Interpretation: "$D_{(n, m)}(A, P)$" = "n A_1s collectively P mA_2s." ("collectively" is understood to imply "not necessarily each one of the A_1s Ps each one of the A_2s.")

For example, '5 dogs are driving 30 cows collectively'.

ASSERTION. $D_{(n, m)} \in \mathbf{Int}^{\iota}_{<\varnothing,\{1,2\}>}$.

Proof: $D_{(n, m)} \in \mathbf{Int}^{\varepsilon}_{<\varnothing,\{1,2\}>}$

iff $D(A, P) = D(prj_1((A_1\times A_2)\cap P), prj_2((A_1\times A_2)\cap P), E^2)$

iff $D(A, P) = 1$

$\iff D(prj_1((A_1\times A_2)\cap P), prj_2((A_1\times A_2)\cap P)), E^2) = 1$

iff $|prj_1((A_1\cap A_2)\times P)| = n, |prj_2((A_1\times A_2)\cap P)| = m$

$\iff |prj_1(prj_1((A_1\times A_2)\cap P)\times prj_2((A_1\times A_2)\cap P)\cap E^2)| = n,$

$|prj_2(prj_1((A_1\times A_2)3P)\times prj_2((A_1\times A_2)\cap P)\cap E^2)| = m$

But $\quad prj_1(prj_1((A_1 \times A_2) \cap P \times prj_2((A_1 \times A_2) \cap P) \cap E^2) = prj_1((A_1 \times A_2) \cap P$

Similarly,

$$prj_2(prj_1((A_1 \times A_2) \cap P) \times prj_2((A_1 \times A_2) \cap P)) = prj_2((A_1 \times A_2) \cap P). \text{ QED.}$$

EXAMPLE 2. $D_{(n*m)}$

DEFINITION: $\quad D_{(n*m)}(A, P) = 1$

$\iff \quad D_{(n, m)}(A, P) = 1$ &

$prj_1((A_1 \times A_2) \cap P) \times prj_2((A_1 \times A_2) \cap P) = (A_1 \times A_2) \cap P.$

INTERPRETATION. "$D_{(n*m)}(A, P)$" = "n A_1s together P m A_2s". ("Together" is understood to imply "all n A_1s P all m A_2s.")

For example, '5 dogs are (each) driving (the same) 30 cows'.

ASSERTION 1. $D_{(n*m)} \notin \mathbf{Int}^{\varepsilon}_{<ø, \{1, 2\}>} = \mathbf{Int}^{\mu}_{<\{1\}, \{2\}>}$.

Proof: Assume $D_{(n*m)} \in \mathbf{Int}^{\varepsilon}_{<ø, \{1, 2\}>}$ and derive a contradiction. By assumption,

$$D_{(n*m)}(A, P) = D_{(n*m)}(\iota_{<ø,\{1, 2\}>}(A, P))$$
$$= D_{(n*m)}((prj_1((A_1 \times A_2) \cap P), prj_2((A_1 \times A_2) \cap P), E^2)$$

By definition of $D_{(n*m)}$,

$$D_{(n*m)}(\iota_{<ø, \{1, 2\}>}(A, P)) = 1$$
$$\text{iff } D_{(n, m)}(\iota_{<ø, \{1, 2\}>}(A, P)) = 1 \text{ and}$$
$$prj_1[(prj_1((A_1 \times A_2) \cap P) \times prj_2((A_1 \times A_2) \cap P)) \cap E^2]$$
$$\times prj_2[(prj_1((A_1 \times A_2) \cap P) \times prj_2((A_1 \times A_2) \cap P)) \cap E^2] \quad (1)$$
$$= (prj_1((A_1 \times A_2) \cap P) \times prj_2((A_1 \times A_2) \cap P)) \cap E^2. \quad (2)$$

Now,

$$prj_1[([prj_1((A_1 \times A_2) \cap P)] \times [prj_2((A_1 \times A_2) \cap P)] \cap E^2] = prj_1((A_1 \times A_2) \cap P)$$
$$prj_2[([prj_1((A_1 \times A_2) \cap P] \times [prj_2((A_1 \times A_2) \cap P)] \cap E^2] = prj_2((A_1 \times A_2) \cap P).$$

Hence,

$$(1) = (2).$$

Hence,

$$D_{(n*m)}(\iota_{<ø, \{1, 2\}>}(A, P)) = 1 \text{ iff } D_{(n, m)}(\iota_{<ø, \{1, 2\}>}(A, P)) = 1. \quad (3)$$

Since $D(n*m) \in \mathbf{Int}^{\varepsilon}_{<ø, \{1, 2\}>}$,

we have from (3)

$$D_{(n*m)}(A, P) = 1 \text{ iff } D_{(n, m)}(A, P) = 1.$$

But certainly there can be A such that $D_{(n*m)}(A, P) \neq D_{(n, m)}(A, P)$, hence a contradiction. QED.

ASSERTION 2. $D_{(n*m)} \in \mathbf{Int}^{\mu}_{<\{1, 2\}, ø>}$.

Proof: Assume

$$\mu_{<\{1, 2\},ø>}(A, P) = \mu_{<\{1, 2\},ø>}(A', P').$$

Then,

$$(A_1 \times A_2)3P = (A'_1 \times A'_2) \cap P'.$$

Hence,

$$D_{(n*m)}(A, P) = D_{(n*m)}(A', P').$$

From EXAMPLE 2, Assertion 1 and Assertion 2, we have:

Proposition 18. $\mathbf{Int}^{\mu}_{<\{1, 2\}, ø>} \neq \mathbf{Int}^{\mu}_{<\{1\}, \{2\}>} = \mathbf{Int}^{\iota}_{<ø, \{1, 2\}>} = \mathbf{Int}^{\mu}_{<ø, \{1, 2\}>}$.

ASSERTION 3. $D_{(n*m)} \notin \mathbf{Int}^{\iota}_{<\{1\}, \{2\}>}$.

Proof: Take A_1, A_2, and P such that
$D_{(n*m)}((A_1, A_2), P) = 1$.
Then,
$D_{(n, m)}((A_1, A_2), P) = 1$.
Take P' such that
$D_{(n, m)}((A_1, A_2), P') = 1$.
On the other hand,
$D_{(n*m)}((A_1, A_2), P) = 0$.
Then, by definition of $D_{(n, m)}$,
$|prj_1((A_1{\times}A_2){\cap}P)| = |prj_1((A_1{\times}A_2){\cap}P')| = n$,
$|prj_2((A_1{\times}A_2){\cap}P)| = |prj_2((A_1{\times}A_2){\cap}P')| = m$.
It follows that
$prj_1((A_1{\times}A_2){\cap}P) = prj_1((A_1{\times}A_2){\cap}P') = n$,
$prj_2((A_1{\times}A_2){\cap}P) = prj_2((A_1{\times}A_2){\cap}P') = m$.
Then, if $D_{(n*m)} \in \mathbf{Int}^{\iota}_{<\{1\},\{2\}>}$, we have
$D_{(n*m)}((A_1, A_2), P) = D_{(n*m)}((A_1, A_2), P')$, a contradiction.

From ASSERTION 3, we have

Proposition 19. $\mathbf{Int}^{\mu}_{<\{1, 2\}, ø>} \not\subset \mathbf{Int}^{\iota}_{<\{1\}, \{2\}>}$.

EXAMPLE 3. $D_{(M, m)}(A, P)$
DEFINITION. $D_{(M, m)}(A, P) = 1$
$\iff |prj_1((A_1{\times}A_2){\cap}P)| > 1/2|A_1|, |prj_2((A_1{\times}A_2){\cap}P)| = m$.
INTERPRETATION: "most of the A_1s collectively P m A_2s."
For example, 'most of the dogs are driving 30 cows collectively'.

ASSERTION 1 $D_{(M, m)} \in \mathbf{Int}^{\varepsilon}_{<\{1\}, \{2\}>} = \mathbf{Int}^{\iota}_{<\{1\}, \{2\}>}$

Proof:
$D_{(M, m)}(\iota_{<\{1\},\{2\}>}(A, P)) = D((A_1, prj_2((A_1{\times}A_2){\cap}P)), prj_{\{1\}}((A_1{\times}A_2){\cap}P){\times}E)$.
Put
$B = (B_1, B_2) = (A_1, prj_2((A_1{\times}A_2){\cap}P), Q = prj_{\{1\}}((A_1{\times}A_2){\cap}P){\times}E$.
Now,
$(B_1{\times}B_2){\cap}Q$
$= [A_1{\times}prj_2((A_1{\times}A_2){\cap}P)]{\cap}[prj_{\{1\}}((A_1{\times}A_2){\cap}P){\times}E]$
$= [A_1{\cap}prj_{\{1\}}((A_1{\times}A_2){\cap}P)]{\times}[prj_{\{2\}}((A_1{\times}A_2){\cap}P){\cap}E]$.
Hence,
$prj_1((B_1{\times}B_2){\cap}Q) = A_1{\cap}prj_{\{1\}}((A_1{\times}A_2){\cap}P) = prj_{\{1\}}((A_1{\times}A_2){\cap}P)$
$prj_2((B_1{\times}B_2){\cap}Q) = prj_2((A_1{\times}A_2){\cap}P)$
$D_{(M, m)}(\iota_{<\{1\},\{2\}>}(A, P)) = D_{(M, m)}(B, Q) = 1$
$\iff |prj_{\{1\}}((A_1{\times}A_2){\cap}P)| > 1/2|B_1|, |prj_2((A_1{\times}A_2){\cap}P)| = m$.
$\iff |prj_{\{1\}}((A_1{\times}A_2){\cap}P)| > 1/2|A_1|, |prj_2((A_1{\times}A_2){\cap}P)| = m$.
$\iff D(A, P) = 1$.
i.e., $D_{(M, m)}(A, P) = D_{(M, m)}((\iota_{<\{1\},\{2\}>}(A, P))$.

Hence,

$D_{(M,\,m)} \in \mathbf{Int}^{\iota}{}_{<\{1\},\{2\}>} = \mathbf{Int}^{\varepsilon}{}_{<\{1\},\,\{2\}>}$. QED.

ASSERTION 2. $D_{(M,\,m)} \notin \mathbf{Int}^{\varepsilon}{}_{<\emptyset,\,\{1,\,2\}>} = \mathbf{Int}^{\iota}{}_{<\emptyset,\,\{1,\,2\}>}$.

Proof: Assume $D_{(M,m)} \in \mathbf{Int}^{\varepsilon}{}_{<\emptyset,\{1,\,2\}>}$ and derive a contradiction.

$D_{(M,m)}(\iota_{<\emptyset,\,\{1,\,2\}>}(A, P))$

$= D_{(M,m)}((prj_{\{1\}}((A_1\times A_2)\cap P),\ prj_{\{2\}}((A_1\times A_2)\cap P)),\ E^2)$

$prj_1(prj_1(A_1\times A_2)\cap P)\times prj_2((A_1\times A_2)\cap P) = prj_1((A_1\times A_2)\cap P)$

Hence, if $D_{(M,\,m)} \in \mathbf{Int}^{\varepsilon}{}_{<\emptyset,\{1,\,2\}>}$,

$|prj_1((A_1\times A_2)\cap P)\cap E| = |prj_1((A_1\times A_2)\cap P)| > 1/2\ |prj_1((A_1\times A_2)\cap P)|$,

a contradiction.

From Assertions 1 and 2, we have:

Proposition 20. $\mathbf{Int}^{\iota}{}_{<\{1\},\,\{2\}>} \neq \mathbf{Int}^{\iota}{}_{<\emptyset,\,\{1,\,2\}>}$.

Corollary. $\mathbf{Int}^{\iota}{}_{<\{1\},\,\{2\}>} \supsetneq \mathbf{Int}^{\mu}{}_{<\{1\},\,\{2\}>}$.

For, since $\mathbf{Int}^{\mu}{}_{<\{1\},\,\{2\}>} = \mathbf{Int}^{\iota}{}_{<\emptyset,\,\{1,\,2\}>}$ by Proposition 17, if $\mathbf{Int}^{\iota}{}_{<\{1\},\,\{2\}>} = \mathbf{Int}^{\mu}{}_{<\{1\},\,\{2\}>}$, then we have $\mathbf{Int}^{\iota}{}_{<\{1\},\,\{2\}>} = \mathbf{Int}^{\iota}{}_{<\emptyset,\,\{1,\,2\}>}$, contradicting the proposition.

EXAMPLE 4. $D_{(\exists\subset n,\,m)}(A, P)$

DEFINITION. $D_{(\exists\subset n,\,m)}(A, P) = 1 \Leftrightarrow |A_1| = n,\ |prj_{\{2\}}((A_1\times A_2)\cap P)| = m$.

Interpretation. "some of the n A_1s P m A_2s."

For example, 'some of the 5 dogs are driving m cows'.

ASSERTION 1. $D_{(\exists\subset n,\,m)} \in \mathbf{Int}^{\iota}{}_{<\{1\},\,\{2\}>}$.

Proof:

$\iota_{<\{1\},\,\{2\}>}(A, P) = \iota_{<\{1\},\,\{2\}>}(A', P')$

iff $A_1 = A'_1$,

$prj_1((A_1\times A_2)\cap P) = prj_1((A'_1\times A'_2)\cap P')$

$prj_2((A_1\times A_2)\cap P) = prj_2((A'_1\times A'_2)\cap P')$.

Then, if $\iota_{<\{1\},\,\{2\}>}(A, P) = \iota_{<\{1\},\,\{2\}>}(A', P')$

$|A_1| = |A'_1|$

$|prj_2((A_1\times A_2)\cap P)| = |prj_2((A'_1\times A'_2)\cap P')|$

and hence,

$D_{(\exists\subset n,\,m)}(A, P) = D_{(\exists\subset n,\,m)}(A', P')$.

It follows that $D_{(\exists\subset n,\,m)}\in\mathbf{Int}^{\iota}{}_{<\{1\},\,\{2\}>}$. QED.

ASSERTION 2. $D_{(\exists\subset n,\,m)} \notin \mathbf{Int}^{\mu}{}_{<\{1,\,2\},\,\emptyset>}$.

Proof: Assume $\mu_{<\{1,\,2\},\,\emptyset>}(A, P) = \mu_{<\{1,\,2\},\,\emptyset>}(A', P')$.

Then,

$(A_1\times A_2)\cap P = (A'_1\times A'_2)\cap P'$. (1)

There are A_1, A_2, A'_1, A'_2 such that (1) holds but not $|A_1| = |A_2|$.
Hence, from $\mu_{<\{1,2\},\emptyset>}(A,P)=\mu_{<\{1,2\},\emptyset>}(A',P')$ it does not follow that $D_{(\exists\subset n,\ m)}(A,P)=D_{(\exists\subset n,\ m)}(A',P')$.
QED.

We can conclude from Assertions 1 and 2.

Proposition 21. $\mathbf{Int}^{\mu}_{<\{1,2\},\emptyset>} \not\supset \mathbf{Int}^{\iota}_{<\{1\},\{2\}>}$.

Finally, from Propositions 19 and 21 we derive

Proposition 22. $\mathbf{Int}^{\iota}_{<\{1,2\},\emptyset>} \supsetneq \mathbf{Int}^{\iota}_{<\{1\},\{2\}>}$ and $\mathbf{Int}^{\iota}_{<\{1,2\},\emptyset>} \supsetneq \mathbf{Int}^{\mu}_{<\{1,2\},\emptyset>}$.

Proof: Since

$\mathbf{Int}^{\iota}_{<\{1,2\},\emptyset>} \supset \mathbf{Int}^{\iota}_{<\{1\},\{2\}>}$,

if

$\mathbf{Int}^{\iota}_{<\{1,2\},\emptyset>} = \mathbf{Int}^{\mu}_{<\{1,2\},\emptyset>}$,

we have

$\mathbf{Int}^{\mu}_{<\{1,2\},\emptyset>} \supset \mathbf{Int}^{\iota}_{<\{1\},\{2\}>}$, contradicting Proposition 19.

Furthermore, since

$\mathbf{Int}^{\iota}_{<\{1,2\},\emptyset>} \supset \mathbf{Int}^{\mu}_{<\{1,2\},\emptyset>}$,

if

$\mathbf{Int}^{\iota}_{<\{1,2\},\emptyset>} = \mathbf{Int}^{\iota}_{<\{1\},\{2\}>}$,

we have

$\mathbf{Int}^{\iota}_{<\{1\},\{2\}>} \supset \mathbf{Int}^{\mu}_{<\{1,2\},\emptyset>}$, contradicting Proposition 21.

Hence the proposition.

Diagram 1 for n = 2 is now reduced as below in Diagram 3.

$$
\begin{array}{ccccccccc}
 & & & & \mathbf{Int}^{\varepsilon}_{<\{1,2\},\emptyset>} & & \mathbf{Int}^{\varepsilon}_{<\{1\},\{2\}>} & & \mathbf{Int}^{\varepsilon}_{<\emptyset,\{1,2\}>} \\
 & & & & \| & & \| & & \| \\
\mathbf{Det} & \supset & \mathbf{Con} & = & \mathbf{Int}^{\iota}_{<\{1,2\},\emptyset>} & \supsetneq & \mathbf{Int}^{\iota}_{<\{1\},\{2\}>} & \supsetneq & \mathbf{Int}^{\iota}_{<\emptyset,\{1,2\}>} \\
 & & & & \cup\!\not\| & & \cup\!\not\| & & \| \\
{}^\wedge\mathbf{Det} & \supset & \mathbf{Int}^{\pi} & \cong & \mathbf{Int}^{\mu}_{<\{1,2\},\emptyset>} & \supsetneq & \mathbf{Int}^{\mu}_{<\{1\},\{2\}>} & = & \mathbf{Int}^{\mu}_{<\emptyset,\{1,2\}>}
\end{array}
$$

Diagram 3. n = 2.

In this diagram the case for $\prod = <\{2\},\{1\}>$ is not included. Since we have symmetry between $<\{1\},\{2\}>$ and $<\{2\},\{1\}>$, we can draw a symmetric diagram for $<\{2\},\{1\}>$. To conclude, we can state the following theorem:

Theorem 10. There are five binary sets of intersective determiners:

$\mathbf{Con} = \mathbf{Int}^{\iota}_{<\{1,2\},\emptyset>}$:	Conservative determiners
$\mathbf{Int}^{\pi} \cong \mathbf{Int}^{\mu}_{<\{1,2\},\emptyset>}$:	Intersective set-determiners
$\mathbf{Int}^{\iota}_{<\{1\},\{2\}>}$:	{1}-polarized partial intersective determiners
$\mathbf{Int}^{\mu}_{<\{2\},\{1\}>}$:	{2}-polarized partial intersective determiners
$\mathbf{Int}^{\mu}_{<\{1\},\{2\}>} = \mathbf{Int}^{\mu}_{<\emptyset,\{1,2\}>} = \mathbf{Int}^{\iota}_{<\emptyset,\{1,2\}>}$:	Intersective argument-determiners.

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S.-Y. Kuroda
Department of Linguistics
University of California, San Diego
sykuroda@ucsd.edu