Wonderful Loop Group Embeddings and Applications to the Moduli of G-bundles on curves

by

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Abstract

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Moduli problems have become a central area of interest in a wide range of mathematical fields such as representation theory and topology but particularly in the geometries (differential, complex, symplectic, algebraic). In addition, studying moduli problems often requires utilizing tools from other mathematical fields and creates unexpected bridges within mathematics and between mathematics and other fields.

A notable example came in 1991 when the mathematical physicists Edward Witten made a conjecture [Wit91] connecting the partition function for quantum gravity in two dimensions with numbers associated to the cohomology of the moduli space of stable curves $\overline{\mathcal{M}_{g,n}}$, a space that was already of independent interest to algebraic geometers.

We study a related moduli problem $\mathcal{M}_G$ of principal G-bundle on stable curves for G a simple algebraic group. A defect of $\mathcal{M}_G$ over singular curves is that it is not compact and thus more difficult to study. We focus specifically on nodal singularities and examine how to compactify $\mathcal{M}_G$ over nodal curves.

The approach we present relies on two main mathematical objects: the loop group and the wonderful compactification of a semisimple adjoint group. For an algebraic group G the loop group LG is the group of maps $D^\times \to G$ where $D^\times$ is a punctured formal disk, see 2.2 for a precise definition. The connection between LG and $\mathcal{M}_G$ is that G-bundles can be described by transition functions and roughly speaking any such transition function comes from an element of LG.

The wonderful compactification is a particularly nice way of comapactifying a semi simple group. Then in a sentence, the aim this dissertation is to (1) extend the construction of the wonderful compactification for semi simple group to LG and (2) use this compactification to compactify $\mathcal{M}_G$ over nodal curves.

We give a brief introduction in Chapter 1. Chapter 2 addresses (1) and Chapter 3 addresses (2).

We begin in Chapter 2 with a discussion of the classical wonderful compactification of an adjoint group given by De Concini and Procesi in [DCP83]. Because the group LG is infinite dimensional many of the elements in De Concini and Procesi's construction do not immedi-

ately extend or have more than one possible generalization. The technical heart of the paper is developing the appropriate analogs of all the elements needed to make the construction possible for LG. Also building on work of Brion and Kumar we give an enhancement of the compactificaiton from schemes to stacks that we utilize in Chapter 3.

Chapter 3 returns to the problem of compactifying $\mathcal{M}_G$ over nodal curves. We begin by carefully studying the points in the boundary of the compactification of LG and relating them to moduli problems over nodal curves. The moduli problems that appear in this way are closely related to flag varieties for the loop group and can be identified as moduli of torsors for a particular group scheme determined by parabolic subgroups of the loop group.

We go on to show that the moduli problem of torsors on nodal curves is isomorphic to a moduli problem of G-bundles on *twisted* nodal curve; these are orbifolds that are isomorphic to the original nodal curve on the smooth locus. Finally, building on related work of Kausz [Kau00, Kau05a] and Thaddeus and Martens [MTa] and the results of Chapter 2 we introduce a larger moduli problem $\mathcal{X}_G$ of G bundles on twisted curves which compactifies $\mathcal{M}_G$.

Contents

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Chapter 1

Introduction

The wonderful compactification of a complex semi simple adjoint group G was constructed and named by De Concini and Procesi in [DCP83]. The compactification is a smooth projective variety containing G as a dense open sub variety and the boundary is a normal crossing divisor whose structure is determined by the root datum of G. The wonderful compactification has seen a wide range of applications particularly in spherical geometry.

Here we construct an analogue of the wonderful compactification for the loop group of G. The loop group LG is the group of maps from a punctured formal disc to the group G; in algebraic geometry it's $\mathbb{C}$ points are $G(\mathbb{C}((z)))$ where $\mathbb{C}((z))$ is the field of formal Laurent series. The precise meaning of "analogue of the wonderful compactification for loop groups" requires further comment and is clarified in the subsection below on loop groups.

The main application of our construction concerns the moduli stack of G bundles on a family of nodal curves. This moduli stack is not compact and we use the wonderful compactification of LG to compactify this stack [1]

In fact, one of the earliest attempted applications of De Concini and Procesi's compactification was precisely to compactify the aforementioned moduli problem. For the special case of $G = GL_n(\mathbb{C})$ see the work of Kausz [Kau05b] and Seshadri[Ses00]. However, for general semi simple G, no satisfactory construction that worked for families of curves and for general semi simple G has been obtained.

Our second main result demonstrates that the wonderful compactification of the loop group always carries enough information to compactify the moduli stack of G bundles on a family of nodal curves. This application makes use of parabolic subgroups of LG which for general Dynkin type have no counterpart for G itself. In other words, for general Dynkin type, passing to the loop group is necessary.

The approach here is quite general; it works for simple, simply connected G of arbitrary Dynkin type and for curves of arbitrary genus. This is a significant improvement from other approaches which only dealt with either low genus or only certain Dynkin types: [MTa](any

[1] We only show one can obtain a complete moduli stack; to get something separated a stability condition must be imposed.

reductive G, genus 0), [RF] (G simple simply connected, genus 1), [Pan96, Fal96] (GL_n, SO_n, SP_{2n}, any genus).

There are two obvious questions not addressed here. The first is if all the construction presented here can be made to work for a general reductive group. The second concerns finding a good stability condition on the modular compactification of the stack of G bundles. The stability condition is necessary to obtain a coarse moduli space.

Further, we expect the modular compactification given here can find applications in gauged Gromov-Witten theory. For more in this direction see [TFT]. We aim to pursue these directions in future work.

The loop group and the moduli of G bundles have themselves found many applications in geometry and representation theory and we believe the wonderful compactification of LG given here has potential for further application in these areas as well.

Loop groups

For any ring R we form the ring $R((z))$ of formal Laurent series with coefficients in R. Elements of $R((z))$ are formal sums $\sum_{i=i_0}^{\infty} r_i z^i$ with $r_i \in R$ and where the start of the sum i_0 can be any integer.

If G is an algebraic group over $\mathbb{C}$ then the loop group is a functor from $\mathbb{C}$-algebras to groups which assigns to a $\mathbb{C}$-algebra R the group $G(R((z)))$ of $R((z))$ valued points of G.

For concreteness we consider $G = SL_2(\mathbb{C})$ and $R = \mathbb{C}$. Then the group $LSL_2(\mathbb{C})$ consists of 2×2 matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where the entires are in $\mathbb{C}((z))$ and $ad - bc = 1$. These can be equivalently represented as formal sums

$$\gamma(z) = \sum_{i=i_0}^{\infty} \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} z^i \qquad a_i, b_i, c_i, d_i \in \mathbb{C}.$$

The determinant condition translates into a sequence of polynomial conditions on the coefficients a_i, b_i, c_i, d_i.

Some geometric intuition can be gained by considering the smooth loop group which is defined for any Lie group G: $L^{sm}G = C^\infty(S^1, G)$. The approximate parallel between LG and $L^{sm}G$ is seen by setting $z = \exp(i\theta)$ (θ a coordinate on S^1) and imposing a convergence condition on the above sum.

The loop group is studied through its representations. The remarkable fact is that LG admits a class of projective representations which behave in many ways like the finite dimensional representations of a semi simple group. These projective representations V, called positive energy representations, are honest representations of a central extension $\tilde{L}G$ of LG by $\mathbb{C}^\times$. The representations V are infinite dimensional but have a finiteness property. Specifically introduce a conjugation action of $u \in \mathbb{C}^\times$ on $\gamma(z) \in LG(\mathbb{C})$ by $u\gamma(z)u^{-1} := \gamma(uz)$. Then one can form the semi-direct product $\mathbb{C}^\times \ltimes LG(\mathbb{C})$ and this lifts to the central extension $\mathbb{C}^\times \ltimes \tilde{L}G(\mathbb{C})$. The finiteness property of V is that $\mathbb{C}^\times$ acts on V in such a way that V becomes a representation of $\mathbb{C}^\times \ltimes \tilde{L}G(\mathbb{C})$ and all the weight spaces of $\mathbb{C}^\times$ are finite dimensional.

So it is actually the slightly more complicated group $G^{aff}(\mathbb{C}) := \mathbb{C}^\times \ltimes \tilde{L}G(\mathbb{C})$ whose representation theory mimics that of a finite dimensional semi simple group. The group $G^{aff}(\mathbb{C})$ is called a Kac-Moody group and it is associated to an affine Dynkin diagram in the same way a semi simple group is associated to a Dynkin diagram.

We can now say more precisely what we mean by a loop group analogue of the wonderful compactification. Take G to be simple and simply connected group. For any group H let $Z(H)$ denote it's center. Then De Concini and Procesi's construction produces a compactification of the adjoint group $G/Z(G)$. The construction we give here gives a partial compactification of $G^{aff}(\mathbb{C})/Z(G^{aff}(\mathbb{C})) = \mathbb{C}^\times \ltimes LG(\mathbb{C})/Z(G)$. Further the boundary of the partial compactification is a normal crossing divisor whose structure is determined by root datum of $LG(\mathbb{C})$ in a manner analogous to the wonderful compactifcation of $G/Z(G)$.

Example 1. The wonderful compactification of $SL_3(\mathbb{C})/Z(SL_3(\mathbb{C})) = PGL_3(\mathbb{C})$ has a boundary with 2 components D_1, D_2. For $i = 1, 2$ there is a fibration $\pi_i \colon D_i \to SL_3(\mathbb{C})/P_i^- \times SL_3(\mathbb{C})/P_i$ with P_i, P_i^- opposite maximal parabolic subgroups. The fiber of π_i is the wonderful compactification of $PGL_2(\mathbb{C})$. The intersection $B = P_1 \cap P_2$ is the subgroup of upper triangular matrices, B^- is the transpose of B and $D_1 \cap D_2$ is the product $SL_3(\mathbb{C})/B^- \times SL_3(\mathbb{C})/B$.

Example 2. The analogue of the wonderful compactification of $\mathbb{C}^\times \ltimes LSL_2(\mathbb{C})/Z(SL_2(\mathbb{C}))$ also has two boundary component D_1, D_2. There are fibrations $\pi_i \colon D_i \to LG/\mathcal{P}_i^- \times LG/\mathcal{P}_i$ for $\mathcal{P}_i, \mathcal{P}_i^-$ opposite parabolic (or parahoric) subgroups of LG. The fiber of π_i is again the wonderful compactification of $PGL_2(\mathbb{C})$. In this case $D_1 \cap D_2 = LG/\mathcal{B}^- \times LG/\mathcal{B}$ where $\mathcal{B} = \mathcal{P}_1 \cap \mathcal{P}_2$ and with $\mathcal{B}^-$ defined similarly.

In both examples the appearance of $PGL_2(\mathbb{C})$ reflects the fact that the Dynkin diagram for $\mathbb{C}^\times \ltimes \tilde{L}SL_2(\mathbb{C})$ and $SL_3(\mathbb{C})$ both contain the Dynkin diagram of $PGL_2(\mathbb{C})$ as a sub diagram.

Moduli Problems

Here we provide some background for the main application of the loop group wonderful compactification.

Moduli problems concern the geometry of spaces whose points themselves correspond to geometric objects. A basic but important example is complex projective space $\mathbb{P}^n(\mathbb{C})$. By definition, each $x \in \mathbb{P}^n(\mathbb{C})$ corresponds to a line $l \subset \mathbb{C}^{n+1}$.

The particular moduli problem this work deals with is the moduli of principal G bundles on a compact Riemann surface, or more precisely on a projective algebraic curve C. Here G is a simple algebraic group; the classic examples are $G = SL_r(\mathbb{C}), SO_r(\mathbb{C}), SP_{2r}(\mathbb{C})$. Geometrically, one can study either the moduli *stack* $\mathcal{M}_G(C)$ or the coarse moduli *space* $M_G(C)$ which requires a stability condition. We work with $\mathcal{M}_G(C)$ and specifically when the curve C has a nodal singularity. It is known in this case that $\mathcal{M}_G(C)$ is not compact which makes it harder to work with. We provide a way to compactify $\mathcal{M}_G(C)$.

The study of $\mathcal{M}_G(C)$ for a general reductive G began with the study of $M_{GL_r}(C)$[2]. In

[2]For GL_r, one must also fix the degree of the bundles to get a connected moduli space.

turn, the latter arose in Mumford's geometric invariant theory (GIT) [MFK94] and naturally generalizes the Jacobian of C. The stack $\mathcal{M}_G(C)$ appeared in the work of Atiyah and Bott [AB83] in gauge theory, although they worked in an analytic framework. The algebraic version of $\mathcal{M}_G(C)$, which we study here, attracted more attention with the work of Beauville, Laszlo [BL94a] and Faltings [Fal94] where it was used to connect $M_G(C)$ to conformal field theory in physics.

In fact by that time $M_G(C)$ had already been studied from the perspective of number theory by Harder and Narasimhan [HN75] and also in the context of symplectic geometry by Hitchin [Hit87] and shortly after also from the perspective of representation theory by, for example, Kumar [Kum97] and Teleman [Tel98].

The results above, with the exception of [Fal94], apply only to a fixed smooth curve C. However, it is natural to consider $\mathcal{M}_G(C_b)$ for a curve C_b varying with a parameter b. Let B be the set of all possible values of b, then we use the shorthand C_B and say C_B is a *family of curves over* B. It is of interest to study $\mathcal{M}_G(C_B)$ when, for some special values $b_0 \in B$, the curve C_{b_0} becomes singular. This is known as *degeneration* and is a standard technique in algebraic geometry.

In Chapter 3 we describe a degeneration of $\mathcal{M}_G(C)$ with good compactness properties which until now had only been provided for $\mathcal{M}_{GL_r}(C_B)$ by Kausz [Kau05b].

Example 3. Here we consider $G = \mathbb{C}^\times$ and the curve

$$C = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) | zy^2 - x^3 + xz^2 = 0\}.$$

Then $\mathcal{M}_G(C)$ is a group and its connected component is a well studied space called the Jacobian $Jac(C)$ of C. In this specific example $C \cong Jac(C)$. The isomorphism occurs by identifying principal $\mathbb{C}^\times$ bundles with line bundles and in turn identifying these with rank 1 locally free sheaves. Then the isomorphism $C \cong Jac(C)$ identifies $p \in C$ with the ideal sheaf $\mathcal{O}_C(-p)$. If $p \neq [0 : 1 : 0]$ then this can be made more transparent on the affine curve $C' = C - [0 : 1 : 0] = \{y^2 - x^3 + x = 0\} \subset \mathbb{C}^2$. In this case $\mathcal{O}_C(-p)$ can be identified with the maximal ideal $m_p \subset \mathbb{C}[x, y]/(y^2 - x^3 + x)$ of elements that vanish at p.

Introduce a parameter t and form $C_t = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) | zy^2 - x(x - z)(x + tz) = 0\}$. For any nonzero value $t \in \mathbb{C}^\times$ the curve C_t is smooth and projective. The limit of as $t \to 0$ yields the singular curve $C_0 = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) | zy^2 - x^3 + x^2z = 0\}$. This is depicted in figure 1.1. In contrast to C, we have $Jac(C_0) \cong C_0 - [0 : 0 : 1]$. This is an illustration that $\mathcal{M}_G(C_0)$ is not compact when C_0 is nodal. In this specific example $Jac(C_0)$ seems to be missing the point $[0 : 0 : 1]$. However there is no way to associate in any natural way a line bundle on C_0 to the point $[0 : 0 : 1]$. Indeed *any* line bundle (up to degree shift) is already naturally assigned to the points of $C_0 - [0 : 0 : 1]$. What is missing is evidently the object corresponding to the maximal ideal $(x, y) \subset \mathbb{C}[x, y]/(y^2 - x^3 + x^2)$.

The ideal (x, y) is not locally free but it is torsion free. Geometrically it determines a line bundle on $C_0 - [0 : 0 : 1]$ but the fiber over $[0 : 0 : 1]$ is two dimensional. The process of enlarging the moduli problem from line bundles on C_0 to torsion free sheaves is a prototypical example of compactification.

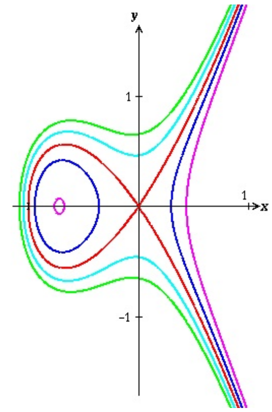

Figure 1.1: Degeneration of a smooth curve into a nodal curve.

As has already been indicated, throughout we will use the language of algebraic geometry. This means among other things that we will speak of sheaves, schemes, morphisms of schemes and their properties and eventually stacks. However we hope the example provides some appreciation of the problem to an audience with at least a familiarity with manifolds and vector bundles.

As for the rest of the work, much of Chapter 2 is approachable with familiarity with representation theory. Chapter 3 is the most technically demanding using advanced algebraic geometry, including algebraic stacks and criteria for representability.

1.1 Bird's Eye View

Section 2.1 describes the classical construction of the wonderful compactification of the semisimple group G of adjoint type given by De Concini and Procesi in [DCP83]. Briefly one takes a regular dominant weight λ and considers the associated irreducible representation $V = V(\lambda)$. The wonderful compactification is the $G \times G$ orbit closure of $[id] \in \mathbb{P}End(V)$:

$$\overline{G} = \overline{G \times G.[id]} \subset \mathbb{P}End(V) = \mathbb{P}\Bigg(V \otimes V^*\Bigg). \tag{1.1}$$

The simplest example is $G = PGL_2(\mathbb{C})$ and $V = \mathbb{C}^2$. Then G is embedded in $\mathbb{P}End(\mathbb{C}^2) = \mathbb{P}^3$ as $\{[a : b : c : d] | ad - bc \neq 0\}$; the closure is $\overline{G} = \mathbb{P}^3$.

Section 2.2 defines the loop group and defines its root datum which is analogous to the root datum of a semisimple group. In particular we discuss how to view the loop group as an infinite dimensional ind-scheme: an increasing union of infinite dimensional schemes.

Section 2.3 begins to develop the necessary generalizations to make sense of (1.1) for LG. First the representation V is taken to be a positive energy or highest weight representation. Then V is countable dimensional and admits only a projective action of LG. Neither of these differences pose a problem, however the appearance of V^* in (1.1) does pose a technical challenge.

The challenge arises because in algebraic geometry we need definitions that make sense for any $\mathbb{C}$-algebra R. For example $LG(\mathbb{C})$ will act on the projectivzation of V^* but $LG(R)$ will not act on the projectivization of $V^* \otimes_{\mathbb{C}} R$. We give a definition of $\mathbb{P}V^*$ that behaves well under base change and study embeddings of flag varieties of LG in $\mathbb{P}V$ and $\mathbb{P}V^*$.

Section 2.4 deals with similar technicalities this time to give a good definition of $\mathbb{P}End(V)$. In this setting $End(V)$ is much bigger than $V \otimes V^*$; for example $id \notin V \otimes V^*$. Using properties of the action of LG on $\mathbb{P}V$, we identify an ind-scheme $\mathbb{P}End^{ind}(V) \subset \mathbb{P}End(V)$. Further, for $Z(G)$ the center of G we show that the recipe of (1.1) defines an embedding of $LG/Z(G)$, or rather the semidirect product $\mathbb{C}^\times \ltimes LG/Z(G) \subset \mathbb{P}End^{ind}(V)$. The closure gives the partial compactfication which we denote X^{aff}. The remainer of the section proves that X^{aff} has analogous properties to the wonderful compactification (the main theorem is 2.4.3). In particular, X^{aff} is formally smooth and the boundary is a normal crossing divisor.

Section 2.5 has two subsections one on the $LG \times LG$ orbits in X^{aff} and a second demonstrating that the boundary divisor and another naturally defined divsor are actually Cartier. These results are needed to prove the final statement of the main theorem (2.4.3(d)) which states that the Picard group X^{aff} is free of finite rank in analogy with the classical wonderful compactification. We also prove that the boundary of X^{aff} is stratified by fibrations over flag varieties for LG (corollary 2.5.3), providing another analogy with X^{aff} and the wonderful compactification of a semisimple adjoint group.

Section 2.6 describes a partial compactification for the polynomial loop group $L_{poly}G(\mathbb{C}) = G(\mathbb{C}[z^{\pm}])$. All that is needed to modify slightly the definitions of $\mathbb{P}V^*$ and $\mathbb{P}End(V)$.

Section 2.7 briefly returns to studying compactification of G itself. The technical issue that arises in the above constructions is that the embeddings always take place in some $\mathbb{P}End(V)$ and the center $Z(G)$ always acts trivially. This is a defect that must be remedied to study G as opposed $G/Z(G)$ or LG as opposed to $LG/Z(G)$. This section uses a construction of Brion and Kumar [BK05b, 6.2.4] blended together with the theory of stacks to provide a wonderful smooth stacky compactification of G. Then blends this with the construction of Section 2.4 to give a different, stacky enhancement of the boundary divisor of X^{aff}.

In Chapter 3 the focus returns to studying the moduli of G -bundles on a curve.

Sections 3.1 - 3.3 contain some history of the development of this problem and previous results as well as a survey on the connection between LG and the moduli $\mathcal{M}_G(C)$ of G bundles on a curve C.

Section 3.4 gives a modular interpretation to each of the orbits in the boundary X^{aff}. One can interpret the boundary as a moduli space for more general objects known as torsors $\mathcal{F}$ on a curve C. In turn these torsors can be identified with ordinary G-bundles but one must replace C with an orbifold or twisted curve.

Section 3.5 enlarges the moduli problem $\mathcal{M}_G(C)$ of G bundles on C to include the objects of 3.4. We show the enlarged moduli problem $\mathcal{X}_G(C)$ is an algebraic stack and further $\mathcal{X}_G(C)$ compactifies $\mathcal{M}_G(C)$. More precisely we show $\mathcal{M}_G(C) \subset \mathcal{X}_G(C)$is open and dense and that $\mathcal{X}_G(C)$ satisfies the valuative criterion for completeness.

Chapter 2

The wonderful embedding of the loop group

In 1983 De Concini and Procesi studied the symmetric space G/H where G is a complex Lie group and H is the fixed point set of an involution σ of G; see [DCP83]. They constructed a "wonderful" compactification $\overline{G/H}$ of G/H; see definition 2.1.1. After De Concini and Procesi's result the properties of their compactification were axiomatized and such varieties were called *wonderful*. A paticular case is $G = G \times G/\Delta(G)$; when G is of adjoint type this gives a wonderful compactification of G. De Concini and Procesi's original construction utilized a regular dominant weight λ of G:

$$\overline{G} = \overline{G \times G.[id]} \subset \mathbb{P}End(V(\lambda)) \tag{2.1}$$

where $[id]$ is the class of the identity endomorphism.

In fact smooth compactifications $\overline{G}$ for all reductive groups G exist [BK05a, 6.2.4] but in general lack certain combinatorial properties required to be wonderful. Additionally, there is a so called canonical embedding of a semi simple group but this compactification is generally not smooth unless $Z(G) = 1$. The only exceptions occur when $G = Sp_{2n}(\mathbb{C})$ [1].

The canonical embedding for semi simple G has finite quotient singularities and the singularities can be resolved by working with Deligne-Mumford stacks. In [MTa], Martens and Thaddeus carry this out explicitly by constructing certain moduli problems about G-bundles on chains of $\mathbb{P}^1$s that *represent* the compactification. In this chapter, we give a different approach using representation theory. Namely for a regular dominant weight λ there is a quasiprojective variety Y with an action of a torus T such that the global quotient $\mathcal{X} = [Y/T]$ contains G as a dense open subvariety. Additionally, $\mathcal{X}$ contains a dense open substack $\mathcal{X}_0$ which is the closure of the open cell U^-TU of G and

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[1] Several places in the literature mistakenly state that the only exception is $G = Sp_2(\mathbb{C}) = SL_2(\mathbb{C})$. I thank Johan Martens for preventing another mistake here.

Theorem 2.7.3. *Let G be a semi simple, connected, simply connected group. Then there is a stacky compactification $\mathcal{X}$ of G such that:*

(a) $\mathcal{X}$ is smooth and proper.

(b) $\mathcal{X} - \mathcal{X}_0$ is of pure codimension 1 and we have an exact sequence

$$0 \to \mathbb{Z}^r \to Pic(\mathcal{X}) \to \hom(Z(\beta), \mathbb{C}^\times) \to 0$$

where $Z(\beta)$ if a finite group and $\mathbb{Z}^r$ is generated by the irreducible components of $\mathcal{X} - \mathcal{X}_0$.

(c) The boundary $\mathcal{X} - G$ consists of r divisors $D_1, \ldots, D_r$ with simple normal crossings and the closure of the $G \times G$-orbits are in bijective correspondence with subsets $I \subset \{1, \ldots, r\}$ in such a way that to I we associate $\cap_{i \in I} D_i$.

Though this result is not new the fact that is can be proved using only representation theory will be important when we turn to the study of loop groups.

Loop groups

The algebraic loop group is $LG = G((z)) = G(\operatorname{Spec} \mathbb{C}((z)))$. In fact these are just the $\mathbb{C}$ points; see section 2.2 for the full definition.

In fact, the group of interest is a semi-direct product $\mathbb{C}^\times \ltimes LG$; this means for $u \in \mathbb{C}^\times$ we have $u\gamma(z)u^{-1} = \gamma(uz)$. The object X^{aff} is an ind-scheme constructed using a regular dominant weight λ of LG.

Theorem 2.4.3. *Let G be a simple, connected and simply connected group over $\mathbb{C}$ and set $r = rk(G)$. The ind-scheme X^{aff} contains $L^\ltimes G/Z(G)$ as a dense open sub-ind scheme and further*

(a) X^{aff} is formally smooth and independent of the choice of regular dominant weight $(0, \lambda, l)$.

(b) The boundary $X^{aff} - L^\ltimes G/Z(G)$ is a Cartier divisor with $r+1$ components $D_0, \ldots, D_r$. The $L^\ltimes G \times L^\ltimes G$ orbits closures are in bijection with subsets $I \subset \{0, \ldots, r\}$ in such a way that to I we associate $\cap_{i \in I} D_i$.

(c) Each D_i is formally smooth and $\cup_{i=0}^r D_i$ is locally a product $S \times Z$ where S is an ind-scheme and Z is the union of hyperplanes in $\mathbb{A}^{r+1}$.

(d) $X^{aff} - X_0^{aff}$ is a Cartier divisor and with $r+1$ components which freely generate $Pic(X^{aff})$.

We use the word embedding because X^{aff} is not compact. However we do have a completeness result for the polynomial loop group $G(\mathbb{C}[z^{\pm}])$; see theorem 2.7.5.

The strategy for proving 2.4.3 is to use the representation theory of G^{aff}. More precisely, take a highest weight representation $V(\lambda)$ of G^{aff}. $V(\lambda)$ is an infinite dimensional vector space which is a direct sum of weight spaces V_μ for a maximal torus $\mathbb{C}^\times \times T$ of $\mathbb{C}^\times \ltimes LG$.

Now we consider

$$X^{aff} = \overline{G^{aff} \times G^{aff}.[id]} \subset \mathbb{P}End(V(\lambda))$$

Where, because $V(\lambda)$ is infinite dimensional, $\mathbb{P}End(V(\lambda))$ must be appropriately defined. This is carried out in section 2.4.

Compactifications to moduli spaces

In this subsection we indicate with a small example for $G = PGL_2(\mathbb{C})$ how to pass from group compactifications to completions of moduli spaces of bundles on curves. The more elaborate procedure of using X^{aff} to complete moduli spaces will be carried out in the next chapter.

The wonderful compactfication of G is $\mathbb{P}^3(\mathbb{C})$. If a, b, c, d are homogeneous coordinate in $\mathbb{P}^3(\mathbb{C})$ then G is identified with the open sub variety where $ad - bc \neq 0$.

We give G a modular interpretation as follows. Fix a smooth connected curve $\tilde{C}$ and two points p, q. Let C be the nodal curve obtained by identifying p and q. Fix a principal G-bundle E on $\tilde{C}$ and framings $f_p\colon E|_p \cong G$, $f_p\colon E|_q \cong G$. Then every $g \in G$ now defines an isomorphism $\phi_g\colon E|_p \xrightarrow{f_p} G \xrightarrow{x \mapsto gx} G \xrightarrow{f_q^{-1}} E|_q$ which allows us to descend E to a principal G bundle $E(\phi_g)$ on C. Thus we have a family of G-bundles on C parametrized by G.

We make use of an alternative description of $E(\phi_g)$. Namely by identifying $G \cong G \times G/\Delta(G)$ we can equivalently identify $E(\phi_g)$ with the bundle E on $\tilde{C}$ together with a reduction of the $G \times G$ bundle $E_p \times E_q$ over $p \times q$ to the subgroup $\Delta(G)$.

Now any morphism $\operatorname{Spec} \mathbb{C}[t^{\pm}] = \mathbb{C}^\times \to G$ gives a family of G-bundles on C parametrized by $t \in \mathbb{C}^\times$. If the morphism does not extend to $\operatorname{Spec} \mathbb{C}[t] \to G$ (such as $\mathbb{C}^\times \xrightarrow{t \mapsto diag(t,1/t)} SL_2 \to G$ where $diag(t, 1/t)$ is a diagonal 2×2 matrix and $SL_2(\mathbb{C}) \to G$ is the natural quotient) the we can ask if there is a limit as t tends to 0.

A limit exists in $\mathbb{P}^3(\mathbb{C})$ and so we can provide an answer by giving a modular interoperation to points of $\mathbb{P}^3(\mathbb{C}) \backslash G = \{ad - bc = 0\} \cong \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) = G/B \times G/B$ where B is the subgroup of upper triangular matrices in G. Consequently each $(s_p, s_q) \in G/B \times G/B$ can be considered (through the use of the framings f_p, f_q) as sections of $E|_p/B$ and $E|_q/B$ respectively; that is, a reduction of $E_p \times E_q$ to $B \times B$. Thus $\mathbb{P}^3(\mathbb{C})$ provides a compactification of the moduli problem of bundles on $\tilde{C}$ with $\Delta(G)$ reductions at p, q by allowing principal G bundles on $\tilde{C}$ together with $B \times B$-reductions at p, q.

The same logic applies to X^{aff}. Namely, we start by understanding the boundary $X^{aff} - \mathbb{C}^\times \ltimes LG$. Then we endow the boundary with a modular interpretation and create a larger moduli problem where principal G-bundles on a nodal curve are allowed to limit to objects parametrized by the boundary $X^{aff} - \mathbb{C}^\times \ltimes LG$. The full details of this procedure is described in the next chapter.

Summary

In section 2.1, we recall the construction of the wonderful compactification of an adjoint group. In section 2.2 we give a quick introduction to loop groups. The subsequent sections 2.3,2.4 discuss the representation theory of loop groups and deal with technicalities of infinite dimensional projective spaces in algebraic geometry. Also the main theorem of the wonderful embedding of LG is stated and proved (2.4.3). Section 2.5 studies the orbits of $LG \times LG$ in the embedding and finishes a technical detail needed for 2.4.3. Section 2.6 describes a partial compactification for the polynomial loop group $L_{poly}G(\mathbb{C}) = G(\mathbb{C}[z^{\pm}])$. Section 2.7 is where the first theorem cited (theorem 2.7.3) is proved and an analogous result for loop groups is also established.

2.1 Wonderful Compactification of an Adjoint Group

This section largely follows chapter 6 of [BK05a].

Definitions

Let G be a semisimple group. It has associated subgroups: a maximal torus T, opposite Borels B, B^-, their unipotent radicals U, U^-. The character lattice we denote as Λ_T, the co-character lattice we denote as V_T and if $\mu \in \Lambda_T, \eta \in V_T$ then the integer $\mu \circ \eta$ we denote as $\langle \mu, \eta \rangle$, $\langle \mu, \eta \rangle$, or $\mu(\eta)$. Let $r = rk(G)$ and let $\alpha_1, \ldots, \alpha_r$ be the positive simple roots. Let $\omega_1, \ldots, \omega_r$ be the fundamental weights.

For dominant weight λ let $V(\lambda)$ denote the highest weight representation (HWR) of highest weight (HW) λ. By $V(\lambda)_\mu$ we denote the weight space of $V(\lambda)$ with weight μ. When no confusion is likely to arise write simply V_μ. We can decompose

$$End(V(\lambda)) = \bigoplus_{\mu,\chi \in \Lambda_T} V_\mu \otimes V_\chi^*.$$

Let $\mathbb{P}End(V(\lambda))_0 \subset \mathbb{P}End(V(\lambda))$ be the open subset consisting of points whose projection to $V_\lambda \otimes V_\lambda^*$ is not zero. Define for a *regular* dominant weight λ

$$\begin{aligned} X &:= \overline{G \times G.[id]} \in \mathbb{P}End(V(\lambda)) = \mathbb{P}\big[V(\lambda) \otimes V(\lambda)^*\big] \\ X_0 &:= X \cap \mathbb{P}End(V(\lambda))_0 \end{aligned} \tag{2.2}$$

Let $Z(G)$ denote the center of G. It is routine to see that $\mathrm{Stab}([id])$ is $Z(G) \times Z(G) \cdot \Delta(G)$ and consequently X contains $G_{ad} := G/Z(G) = \frac{G \times G}{Z(G) \times Z(G) \Delta(G)}$ as a dense open subset. X is the *wonderful compactification* of G_{ad} and X_0 is the *open cell* of X. A maximal torus for G_{ad} is $T_{ad} = T/Z(G)$. By $\overline{T_{ad}}$ we mean the closure of T_{ad} in X and $\overline{T_{ad,0}} := X_0 \cap \overline{T_{ad}}$.

Lastly, let H be a reductive group and Y a normal H-variety

Definition 2.1.1. Y is *wonderful of rank* r if Y is smooth, proper and has r normal corssing divisors $D_1, \ldots, D_r$ such that the H-orbit closures are give by intersections $\cap_{i\in I} D_i$ for any subset $I \subset \{1, \ldots, r\}$.

The notation of $X, X_0, \overline{T_{ad}}, \overline{T_{ad,0}}$ does not reflect the dependence on λ; this is justified by theorem 2.1.2.

Theorem 2.1.2. *Let $X = \overline{G}_{ad}$ be as in* (2.2). *Then*

(a) X is independent of λ.

(b) X is smooth.

(c) $X - X_0$ is of pure codimension 1*; it consists of divisors that freely generate the Picard group.*

(d) The boundary $X - G_{ad}$ consists of r normal crossing divisors $D_1, \ldots, D_r$ and the closure of the $G \times G$-orbits are in bijective correspondence with subsets $I \subset [1, r]$ in such a way that to I we associate $\cap_{i\in I} D_i$.

(e) Any G equivariant $X' \to X$ determines and is determined by a fan supported in the negative Weyl chamber.

Proof. [BK05a, 6.1.8] □

The proof cited for theorem 2.1.2 exploits the representation theory of G and follows from propositions 2.1.3 below.

Let $t^{-\alpha}$ represent the regular function on T given by the character $-\alpha$.

Proposition 2.1.3. *(a) $\overline{T}_0 \cong \operatorname{Spec} \mathbb{C}[t^{-\alpha_1}, \ldots, t^{-\alpha_r}] \cong \mathbb{A}^r$*

(b) The action morphism $U^- \times U \times \overline{T} \to X$ sending $(u_1, u_2, t) \mapsto u_1 t u_2^{-1} \in X$ maps isomorphically onto X_0.

(c) $X = \cup_{g\in G\times G} g X_0$.

(d) $\overline{T} = \cup_{w\in W} w \overline{T}_0 w^{-1}$.

Proof. See [BK05a, 6.1.6,6.1.7] □

Theorem 2.4.3 is an analogue of theorem 2.1.2 for the loop group LG and we will be able to prove theorem 2.4.3 once we prove the loop group analogue of propositions 2.1.3.

2.2 Preliminaries on Loop Groups

Let G be a simple algebraic group over $\mathbb{C}$ with $\pi_1(G) = \pi_0(G) = 1$. In general we say $H \subset G$ is a subgroup to mean that H is an algebraic group over $\mathbb{C}$ and there is an closed embedding $H \to G$ such that for every $\mathbb{C}$-algebra R the set map $H(R) \to G(R)$ is a group homomorphism.

The loop group LG is the functor from $\mathbb{C}$-algebras to groups given by

$$R \mapsto LG(R) := G(R((z))) := G(\operatorname{Spec} R((z)))$$

We call elements of $LG(R)$ *loops*. The functor LG has an ind-scheme structure. For $G = SL_n$ this is shown in [BL94b, 1.2]; in this case loops can be represented by matrices and the ind-scheme structure $LSL_n = \cup_k (LSL_n)_k$ is defined so that $(LSL_n)_k(R)$ consists of loops γ such that both γ, γ^{-1} have entries in $R((z))$ with poles of order $\leq k$; that is each entry is of the form $\sum_{i=-k}^{\infty} a_i z^i$ where each $a_i \in R$.

For general G one gets an ind scheme structure on LG via an embedding $G \to SL_n$. The ind-scheme structure does not depend on the choice of representation [Sor00, 3.7].

To study other aspects of LG, such as it's representation theory it is helpful to introduce two closely related groups. First is $L^{\ltimes}G(R) := \mathbb{G}_m(R) \ltimes LG(R)$ which, for $u \in \mathbb{G}_m(R), \gamma \in LG(R)$ is defined by $u\gamma(z)u^{-1} = \gamma(uz)$. In fact $L^{\ltimes}G$ will be the group we are primarily concerned with.

Finally the representations of $L^{\ltimes}G$ we consider will only be projective representations; or in other words they are representations of a central extension G^{aff}:

$$1 \to \mathbb{G}_m \to G^{aff} \to L^{\ltimes}G \to 1$$

As we are working over $\mathbb{C}$ we'll write $\mathbb{C}^\times$ for $\mathbb{G}_m$. As (ind-)schemes, $G^{aff} = \mathbb{C}^\times \times LG \times \mathbb{C}_c^\times$ where the subscript c indicates the factor is central.

The maximal torus of G^{aff} is $T^{aff} := \mathbb{C}^\times \times T \times \mathbb{C}_c^\times$ and characters are denoted as $(n, \mu, l) \in \mathbb{Z} \oplus \Lambda_T \oplus \mathbb{Z}$; sometimes we abbreviate $T^{\ltimes} := \mathbb{C}^\times \times T \subset L^{\ltimes}G$. As in the case of semi simple groups, we can use the maximal torus to decompose the Lie algebra. The Lie algebra is identified with the $\mathbb{C}[\epsilon]/\epsilon^2$ points that extend the identity $\mathbb{C}$-point. We denote $Lie(LG(\mathbb{C})) = L\mathfrak{g}(\mathbb{C})$, $Lie(L^{\ltimes}G(\mathbb{C})) = L^{\ltimes}\mathfrak{g}(\mathbb{C})$ and $Lie(G^{aff}) = \mathfrak{g}^{aff}(\mathbb{C})$.

We have $L\mathfrak{g}(\mathbb{C}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((z))$, $L^{\ltimes}\mathfrak{g}(\mathbb{C}) = \mathbb{C}d \oplus \mathfrak{g} \otimes \mathbb{C}((z))$ where $\mathfrak{g} \otimes \mathbb{C}((z))$ inherits a Lie algebra structure from $\mathfrak{g}$ and $[d, -]$ acts by $z\frac{d}{dz}$. Then we have

$$\mathfrak{g}^{aff}(\mathbb{C}) = \mathbb{C}d \oplus \mathfrak{t} \bigoplus_{k \neq 0} z^k \mathfrak{t} \bigoplus_{(k,\alpha)} z^k X_\alpha.$$

The weights appearing here are the *roots* of $L^{\ltimes}G$; the weight of $z^k\mathfrak{t}$ shows for $L^{\ltimes}G$ not all roots spaces are 1-dimensional. If $\alpha_1, \dots, \alpha_r$ are the simple roots of G then the *simple affine roots* of G^{aff} are $(0, \alpha_1, 0), \dots, (0, \alpha_r, 0), \alpha_0 = (1, -\theta, 0)$ where θ is the longest root of G. By abuse of notation we denote $(0, \alpha_i, 0)$ simply by α_i so the simple roots of G^{aff} are $\alpha_0, \dots, \alpha_r$.

The *positive* roots of $L^{\ltimes}G$ are those of the form (k, α) with $k > 0$ or with $k = 0$ and α positive.

We now define loop group analogues of the Weyl group, Weyl chamber and parabolic subgroups of a semi simple group. The affine Weyl group is $W^{aff} := N(T^{\ltimes}/T^{\ltimes})$. W^{aff} is isomorphic to $W \ltimes \hom(\mathbb{C}^{\times}, T)$; we can define the Weyl chamber as for semi simple groups however it is standard practice to introduce instead Weyl alcoves. Namely, the roots are linear forms on $\mathbb{Q} \oplus \mathfrak{t}_{\mathbb{Q}}$ and we can identify them with affine linear forms on $\mathfrak{t}_{\mathbb{Q}}$ by identifying the Lie algebra with $1 \oplus \mathfrak{t}_{\mathbb{Q}}$.

For $\alpha \neq 0$ we can define affine hyperplane in $\mathfrak{t}_{\mathbb{Q}}$ via

$$H_{k,\alpha} = \{\zeta \in \mathfrak{t}_{\mathbb{Q}} | \alpha(\zeta) = -k\}$$

The complement of all the $H_{k,\alpha}$ is known as the *Weyl alcove decomposition* of $\mathfrak{t}_{\mathbb{Q}}$. W^{aff} acts freely on $\mathfrak{t}_{\mathbb{Q}}$ and permutes the alcoves in the decomposition. A fundamental domain is given by the *positive Weyl alcove*

$$Al_0 := \{\zeta \in \mathfrak{t}_{\mathbb{Q}} | \alpha_i(\zeta) \geq 0, \ i = 0, \ldots, r\}$$

The *vertices* η_i of Al_0 are defined by $\alpha_j(\eta_i) = \delta_{i,j}$.

Next we introduce the analogue of parabolic subgroups $P \subset G$ for LG. Let $p_1 \colon T^{\ltimes} = \mathbb{C}^{\times} \times T \to \mathbb{C}^{\times}$ be the projection. For any co-character $\eta \colon \mathbb{C}^{\times} \to T^{\ltimes}$ wesay η is *positive* if $p_1 \circ \eta > 0$ and negative if $p_1 \circ \eta < 0$ and *nonzero* if it is either positive or negative. For a nonzero η set

$$P_{\eta}^{\ltimes} = \{\gamma \in L^{\ltimes}G | \lim_{s \to 0} \eta(s)\gamma\eta(s)^{-1} \text{ exists } \}.$$

We can define P_{η} even for $\eta \in \hom(\mathbb{C}^{\times}, T^{\ltimes}) \otimes_{\mathbb{Z}} \mathbb{Q}$; this is because the limit in the definition of P_{η} doesn't depend on any integrality properties; in particular P_{η} is defined for $\eta \in Al_0$.

A *parahoric subgroup* of $L^{\ltimes}G$ is any subgroup $P^{\ltimes} \subset L^{\ltimes}G$ that is conjugate in $L^{\ltimes}G$ to some $P_{\eta}^{\ltimes}$.

$$\mathcal{P}_{\eta}^{\ltimes} = \{\gamma \in L^{\ltimes}G | \lim_{s \to 0} \eta(s)\gamma\eta(s)^{-1} \text{ exists } \}$$

We set $\mathcal{P}_{\eta} = \mathcal{P}_{\eta}^{\ltimes}/\mathbb{C}^{\times} = \mathcal{P}_{\eta}^{\ltimes} \cap LG$ and $P \subset LG$ is a *parahoric* subgroup if it is conjugate to some $\mathcal{P}_{\eta}$.

The groups $P_{\eta}^{\ltimes}$ come with a natural Levi decomposition

$$\begin{aligned} L_{\eta} &= \{\gamma \in \mathcal{P}_{\eta}^{\ltimes} | \lim_{s \to 0} \eta(s)\gamma\eta(s)^{-1} = \gamma\} \\ U_{\eta} &= \{\gamma \in \mathcal{P}_{\eta}^{\ltimes} | \lim_{s \to 0} \eta(s)\gamma\eta(s)^{-1} = id\} \end{aligned} \tag{2.3}$$

and these project to give Levi decompositions of P_{η}.

If η is positive then P_{η} is an infinite dimensional group scheme. If η is negative then P_{η} is an ind-group which is a union of finite dimensional varieties. In either case, the Levi factor L_{η} is always a finite dimensional algebraic group.

Example 4. The vertex η_0 of Al_0 defines $\mathbb{C}^\times \xrightarrow{\eta_0} T^\ltimes$ by $\eta_0(s) = (s, id)$. Then for $\gamma \in G(R((z)))$ we have $\eta(s)\gamma(z)\eta(s)^{-1} = \gamma(sz)$ and the limit $s \to 0$ exists if and only if $\gamma \in G(R[[z]])$. Therefore $P_{\eta_0}(R) = G(R[[z]])$. We also use the notation $L^+G(R) = G(R[[z]])$. In this case the Levi factor is G and if we replaced η_0 with it's inverse $-\eta_0$ (we use additive notation) we would get $P_{-\eta_0}(R) = G(R[z^{-1}])$ which we denote $L^-G(R)$.

We also have the parahorics P_{η_i} associated to the other vertices; these are maximal parahoric subgroups of LG.

More generally to any subset $I \subset \{0, \dots, r\}$ we can associate a parahoric P_I and $P_I = P_\eta$ for some $\eta \in Al_0$ but η is not unique. In fact the set $Al_0 = C \cap 1 \oplus \mathfrak{t}_\mathbb{Q}$ where $C = \oplus_{i=0}^r \mathbb{Q}_{\geq 0}\eta_i$ and C is the union of its sub cones $C_I = \oplus_{i \in I \subset \{0,\dots,r\}} \mathbb{Q}_{\geq 0}\eta_i$ which have interior $C_I^{int} = \oplus_{i \in I \subset \{0,\dots,r\}} \mathbb{Q}_{>0}\eta_i$. For $I \subset \{0, \dots, r\}$ the *cone* $Al_{0,I}$ of Al_0 is $C_I \cap Al_0$ and it's interior is $Al_{0,I}^{int} = Al_0 \cap C_I^{int}$. Then Al_0 is stratified by $Al_{0,I}^{int}$ and $P_\eta = P_{\eta'}$ whenever $\eta, \eta' \in Al_{0,I}^{int}$. Therefore we can set $P_I = P_\eta$ for any $\eta \in Al_{0,I}^{int}$, we can take for example $\eta_I = \frac{1}{|I|}\sum_{i \in I} \eta_i \in Al_{0,I}^{int}$. Similarly, we define $P_I^- = P_{-\eta}$ for $\eta \in Al_{0,I}^{int}$. In both cases the Levi decomposition $P_I^\pm = L_I U_I^\pm$ (2.3) is independent of the choice of η.

The group $P_{\{0,\dots,r\}}$ can be described as follows. Let $B \subset G$ be a Borel subgroup. Define a subgroup scheme $\mathcal{B} \subset LG$ by $\mathcal{B}(R) = \{\gamma(z) \in L^+G(R) | \gamma(0) \in B(R)\}$. If η is a point in the interior of Al_0, such as $\frac{1}{r+1}\sum_i \eta_i$ then $P_\eta = \mathcal{B}$. Similarly, we have $\mathcal{B}^-(R) = \{\gamma(z^{-1}) \in L^-G(R) | \gamma(0) \in B^-(R)\}$ where B^- is the opposite Borel. The groups $\mathcal{U}$ and $\mathcal{U}^-$ are the respective subgroup which specialize at 0 to the unipotent subgroups U, U^- of B, B^-.

The Bruhat decomposition $G = \cup_{w \in W} U^- wB$ generalizes to the Birkhoff factorization [Kum02, pg.142] for G^{aff}

$$G^{aff} = \bigsqcup_{w \in W^{aff}} \mathcal{U}^- w \mathbb{C}^\times \ltimes \mathcal{B} \times \mathbb{C}_c^\times \tag{2.4}$$

Of course this restrict to give a similar decomposition for LG and $L^\ltimes G$. Also we have an analogous group G_{poly}^{aff} where LG is replaced by $L_{poly}G := G(\mathbb{C}[z^\pm])$.

2.3 Representation Theory and Flag Varieties

Here we discuss a class projective representations of $L^\ltimes G$ that have similar properties to highest weight representations of a semi simple group. These representations come from honest representations of G^{aff} and are labeled by weights (n, λ, m) of the maximal torus $T^\ltimes \times \mathbb{C}_c^\times \subset G^{aff}$.

If $\omega_1, \dots, \omega_r$ are the fundamental weights of G then the *fundamental weights* of G^{aff} are $\omega_0 = (0, 0, 1)$, $(0, \omega_1, 1) \dots, (0, \omega_r, 1)$. A *dominant* weight is any weight of the form $n_0\omega_0 + \sum_{i=1}^r n_i(0, \omega_i, 1)$ with $n_i \geq 0$. A dominant weight is *regular* if all $n_i > 0$.

Associated to any dominant weight $(0, \lambda, l)$ is a representation $V(0, \lambda, l)$ of the group $G^{aff}(\mathbb{C})$. The representation $V(0, \lambda, n)$ is infinite dimensional but decomposes under $T^\ltimes \times \mathbb{C}_c^\times$ into a direct sum of 1-dimensional weight spaces $V(0, \lambda, l) = \bigoplus_{n,\mu} V_{(n,\mu,l)}$. Any vector in $v_{(0,\lambda,l)}$ is called a highest weight.

The *level* of the representation is the integer l. We define an ind-variety structure on $V(0,\lambda,l)$ as follows. Set $V_k = \oplus_{\mu\in\hom(T,\mathbb{C}^\times)}V_{(k,\mu,l)}$ then only finitely many $\oplus_\mu V_{(k,\mu,l)}$ are non zero and $V_{\leq k} = \oplus_{n\leq k}V_n$ is a finite dimensional vector space and $V = \cup_k V_{\leq k}$.

The representation $V(0,\lambda,n)$ has the following properties. Recall the η_i are the vertices of Al_0.

Proposition 2.3.1. *Set* $(0,\lambda,l) = n_0(0,0,1) + \sum_{i=1}^r n_i(0,\omega_i,1)$ *and* $\eta = \sum_{i=0}^r n_i\eta_i$. *Let* $P = P(\eta)$ *and* $V = V(0,\lambda,l)$. *Let* $Z(G^{aff})$ *denote the center of* G^{aff}. *Then*

(a) If (n,μ,l) *is any other wight of* L *then* $\lambda - \mu$ *is a sum of positive roots.*

(b) If $(0,\lambda,l)$ *is regular then* $(0,\lambda,l) - \alpha_i$ *is a weight of* V *for all* i.

(c) The stabilizer of the weight space $V_{(0,\lambda,l)}$ *in* $\mathbb{P}V$ *is* P.

(d) If $(0,\lambda,l)$ *is regular then* $P = \mathcal{B}$.

(e) The morphism $LG/P \to \mathbb{P}(V)$ *given by* $\gamma P \mapsto \gamma V_\lambda$ *is injective and gives* LG/P *the structure of a projective ind variety; in particular* LG/P *is closed in* $\mathbb{P}(V)$.

(f) The action of G^{aff} *on* $\mathbb{P}(V)$ *factors through a faithful action of* $G^{aff}/Z(G^{aff}) = L^\ltimes G/Z(G)$.

(g) For any $V' \subset V$ *with* $\dim V' < \infty$ *and any* P_{η_i} *there is a finite dimensional* P_{η_i}*-representation* $W \supset V'$ *and a normal subgroup* $N \subset P_{\eta_i}$ *such that* P_{η_i}/N *is a finite dimensional algebraic group,* N *acts trivially on* W *and the induced action* $P/N \times W \to W$ *is a morphism of schemes.*

(h) The induced representation $\mathcal{U}^- \times V \to V$ *is a morphism of ind-schemes.*

Proof. All of these results are proven in [Kum02]. For (a)-(b) see [Kum02, 2.2.1]; for (c) - (e) see [Kum02, 7.1.2]; for (f) see [Kum02, 13.2.8]; for (g) see [Kum02, 4.4.22,6.2.3]. □

We now recall some algebraic properties of action of LG on $\mathbb{P}V$ for $V = V(0,\lambda,l)$ from [BL94b]. For any field k let Aff/k denote the site of k-algebras equipped with the $fppf$-topology. In [BL94b], a k-*space* is defined as a sheaf of sets on $Aff/\mathbb{C}$ and a k-*group* is defined as a sheaf of groups. A morphism between k-spaces $F \to G$ is a map of sheaves. The category of schemes over k forms a full subcategory of the category of k-spaces.

If G is defined over k then LG is a k-group. If V is a vector space over k then we can consider V as a k-space via $V(R) := V \otimes_k R$. Another k-space is $End(V)(R) := End_R(V \otimes_k R)$ the latter being the set of R-module endormorphisms of $V(R)$. The k-space $End(V)$ has a monoid structure and the taking the group of units gives a k-group $GL(V)$. The k-group $PGL(V)$ is the quotient $GL(V)/\mathbb{G}_m$ by the scaling action. Ind-schemes $Y = \cup_i Y_i$ are another source of k-spaces with $Y(R) = \cup_i Y_i(R)$. If the vector space V has the structure of an ind-scheme $V = \cup_i V_i$ with each V_i a vector space then we can form the ind-scheme $\mathbb{P}V = \cup_i \mathbb{P}V_i$.

One more example that appears in the proof of proposition 2.3.2 comes from Lie algebra. If H is a k-group and R is a k-algebra then we can can define k-space $Lie(H)(R) = \ker(H(R[\epsilon]/\epsilon^2) \to H(R))$.

We now return to working over $\mathbb{C}$ although it is enough to be over an algebraically closed field of characteristic 0. In [BL94b, appendix 7] it is shown that the representations $V = V(0,\lambda,l)$ of G^{aff} are algebraic representations in the sense that there is a morphism of $\mathbb{C}$-groups $LG \to PGL(V)$. The action map $PGL(V) \times \mathbb{P}V \to \mathbb{P}V$ is a morphism of $\mathbb{C}$-spaces hence gives a morphism of $LG \times \mathbb{P}V \to \mathbb{P}V$. For each $k \geq 0$ this restrict to a morphism of $\mathbb{C}$-spaces $(LG)_k \times \mathbb{P}V_{\leq k} \to \mathbb{P}V$ and because if S is a scheme then $\hom(S, \cup_i X_i) = \cup_i \hom(S, X_i)$, it follows that $LG \times \mathbb{P}V \to \mathbb{P}V$ is a morphism of ind-schemes.

We now discuss a dual representation. For V as above define a $\mathbb{C}$-space $\mathbb{P}V^*$ as follows. If R is a $\mathbb{C}$-algebra then

$$\mathbb{P}V^*(R) = \{V \otimes_{\mathbb{C}} R \xrightarrow{\phi} \mathcal{L} \to 0 | \mathcal{L} \text{ is a projective } R \text{ module}\}/\sim \tag{2.5}$$

and the equivalence relation $\sim$ is such that two quotients $\mathcal{L}, \mathcal{L}'$ are equivalent if they are isomorphic. Then $\mathbb{P}V^*$ is a scheme, in fact setting $S_V = \cup_k Sym^*(V_{\leq k})$ then $\mathbb{P}V^*$ is isomorphic to Proj S_V.

If R is a $\mathbb{C}$-algebra and $g \in PGL(V)(R)$ then g determines an equivalence class of elements in $GL(V)(R)$. More precisely there is a faithfully flat extension $R \to R'$ and an element $g' \in GL(V)(R')$ such that for $R'' = R' \otimes_R R'$ the two different pull backs of $g_1'', g_2'' \in GL(V)(R'')$ differ by an element of $\mathbb{G}_m(R'')$. If $(\mathcal{L}, \phi) \in \mathbb{P}V^*(R)$ let $(\mathcal{L} \otimes_R R', \phi') \in \mathbb{P}V^*(R')$ denote the pullback then from the diagram

$$V \otimes_{\mathbb{C}} R' \xrightarrow{(g')^{-1}} V \otimes_{\mathbb{C}} R' \xrightarrow{\phi'} \mathcal{L} \otimes_R R'$$

we obtain the an element $(\mathcal{L} \otimes_R R', \phi' \circ (g')^{-1}) \in \mathbb{P}V^*(R')$. The two different pull backs $(\mathcal{L} \otimes_R R'', \phi'' \circ (g_i'')^{-1})$ represent the same element in $\mathbb{P}V^*(R'')$ and thus determine an element denoted $(\mathcal{L}, \phi \circ g^{-1})$ of $\mathbb{P}V^*(R)$.

Thus we have a morphism of $\mathbb{C}$-spaces $PGL(V) \times \mathbb{P}V^* \to \mathbb{P}V^*$ which determines a morphism of ind-schemes $LG \times \mathbb{P}V^* \to \mathbb{P}V^*$ where $\mathbb{P}V^*$ is given the trivial ind-scheme structure.

Remark that one could define a dual of V as $\prod_k V_k^*$ and worked with the $\mathbb{C}$-space $R \mapsto \prod_k V_k \otimes_{\mathbb{C}} R$ and considered the projectivization as the $\mathbb{C}$-space $R \mapsto$ the set of rank 1 projective R-modules of $\prod_k V_k \otimes_{\mathbb{C}} R$ such that the quotient is projective. This gives a different $\mathbb{C}$-space than $\mathbb{P}V^*$ and the group $GL(V)$ does not act on it; this is a consequence of the fact that tensor products do not commute with inverse limits. Nevertheless $\mathbb{P}V^*(\mathbb{C})$ is the set of lines in $\prod_k V_k^*$.

In the case of the representation V, the orbit of the highest determines a morphism of ind-schemes $LG/P \to \mathbb{P}V$ which is a closed embedding. To study the boundary of the embedding of $L^{\ltimes}G/Z(G)$ we need to establish a similar result for the orbit of the lowest weight in $\mathbb{P}V^*$. Recall the definition of η_i in example 4.

The precise statement is

Proposition 2.3.2. *Let $(0,\lambda,l) = n_0\omega_0 + \sum_{i=1}^r n_i(0,\omega_i,1)$ be a dominant weight. Set $\eta = \sum_{i=0}^r \eta_i$ and $P^- = P_{-\eta}$ and $V = V(0,\lambda,l)$ and $[v^*]$ the class of a lowest weight vector in $\mathbb{P}V^*(\mathbb{C})$. The restriction of $LG \times \mathbb{P}V^* \to \mathbb{P}V^*$ to $LG \times v^*$ factors through LG/P^- and gives a closed embedding of LG/P^- in $\mathbb{P}V^*$.*

Proof. We must show 1) the stabilizer of $[v^*]$ in $LG(\mathbb{C})$ is P^-, 2) the induced map $LG/P^- \to \mathbb{P}V^*$ is injective on tangent spaces and 3) LG/P^- is closed in $\mathbb{P}V^*$.

The Lie algebra $Lie(Stab([v^*]))$ is the completion of the sub algebra $\mathfrak{s}$ which is contained in $\mathfrak{g} \otimes \mathbb{C}[z^{\pm}]$. Under the involution of $\mathfrak{g} \otimes \mathbb{C}[z^{\pm}]$ that sends a root space to it's negative, $\mathfrak{s}$ corresponds to the Lie algebra of a parahoric subgroup of $L_{poly}G$ determined by a subset $I \subset \{0,\ldots,r\}$ [Kum02, 6.1.10]. Under the aforementioned involution $P_\eta \mapsto P_{-}\eta$. Further $P_I = P_\eta$ and if follows that $Stab([v^*]) = P_{-}\eta$.

For 2) we observe LG acts transitively on LG/P^- so it is enough to check the condition at the identity. The tangent space of LG/P^- is readily identified with $Lie(U_\eta)$ where $P_\eta = L_\eta U_\eta$ is the Levi factorization and the injectivity of this map follow because each root subgroup of U_η acts non trivially on $[v^*]$.

For 3) we need to study the $\mathbb{C}((t))$ points LG/P^- and show any points which extends to $\mathbb{C}[[t]]$ point in $\mathbb{P}V^*$ actually lands in LG/P^-. The proof utilizes a decomposition theorem (W_P is a subset of W^{aff} determined by P, see the paragraph before proposition 2.3.3):

$$LG(\mathbb{C}((t))) = \bigsqcup_{w \in W_P'} P^-(\mathbb{C}((t)))wU(\mathbb{C}((t))),$$

which we prove in proposition 2.3.3. We first observe that $LG \to LG/P^-$ is Zariski locally trivial P^--bundle and so any $\mathbb{C}((t))$ point γ_t of LG/P^- can be lifted to LG. Then by the factorization we can assume $\gamma_t = wU(\mathbb{C}((t)))$.

The $\mathbb{C}((t))$ points $[v^*]$ of $\mathbb{P}V^*$ is represented by projection onto the highest weight space $V_{(0,\lambda,l)}$, that is by the class of the projection $V \otimes_{\mathbb{C}} \mathbb{C}((t)) \xrightarrow{v^*} V_{(0,\lambda,L)} \otimes_{\mathbb{C}} \mathbb{C}((t)) \cong \mathbb{C}((t))$. The vector $[v^*w]$ corresponds to the composition $V \otimes_{\mathbb{C}} \mathbb{C}((t)) \xrightarrow{w} V \otimes_{\mathbb{C}} \mathbb{C}((t)) \xrightarrow{v^*}$. By the previous paragraph we are reduced to studying the $U(\mathbb{C}((t))$ orbit of $[v^*w]$.

Assume first $\gamma_t = \exp f(t)X_\alpha$ with $f(t) \in \mathbb{C}((t))[[z]]$. Then γ_t is an infinite product of $\exp f_n(t)z^nX_\alpha$ where $f_n(t) \in \mathbb{C}((t))$ and $n \geq 0$.The action $\exp f_n(t)z^nX_\alpha$ is determined by the Lie algebra action of $f_n(t)z^nX_\alpha$ on $[v^*w]$. We note $z^nX_\alpha, z^{-n}X_{-\alpha}$ generate a Lie algebra $\mathfrak{s}_{n,\alpha}$ isomorphic to $\mathfrak{sl}_2$ in the central extension $\widetilde{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}c$ of $L\mathfrak{g}$ where all the Lie algebras in question are being considered as $\mathbb{C}$-spaces. The Cartan is spanned by the element $[z^nX_\alpha, z^{-n}X_{-\alpha}] = H_\alpha + nc$ which we identify with a co-character $(0,\alpha^\vee,n)$ of T^{aff}; see [Kum02, pg.483,eq.(2)] for Lie algebra structure on the $\widetilde{L\mathfrak{g}}$.

The T^{aff} weight space $[v^*w]$ has a weight of the form $(m,-\lambda+\beta,-l)$ with $m \geq 0$ and β a sum of positive roots of $\mathfrak{g}$. Then the vector $v^*wz^{-n}X_{-\alpha}$ has a weight of the form

$(m-n, \lambda+\beta-\alpha, -l)$. Thus for sufficiently large n the action must be trivial and v^*w will generate an irreducible representation of $\mathfrak{s}_{n,\alpha}$ of lowest weight $\langle (m, \lambda+\beta, -l), (0, \alpha^\vee, n)\rangle = \lambda(\alpha^\vee) + \beta(\alpha^\vee) - ln =: \lambda_{n,\alpha,w}$ which we denote $V(\lambda_{n,\alpha,w})$. From $\mathfrak{sl}_2$ representation theory if $v_{n,\alpha,w}$ is a highest weight vector in $V(\lambda_{n,\alpha,w})$ then $v^*w(\exp f z^n X_\alpha)v_{n,\alpha,w} = f^i$ where i is the highest power such that $(z^n X_\alpha)^i \neq 0$ on $V(\lambda_{n,\alpha,w})$. The value i increases linearly in n.

Now a $\mathbb{C}((t))$ point of $\mathbb{P}V^*$ is linear map $V \otimes_{\mathbb{C}} \mathbb{C}((t)) \xrightarrow{\phi} \mathbb{C}((t))$ and it extends to $\mathbb{C}[[t]]$ exactly when there is $u \in \mathbb{C}((t)) - 0$ such that the restriction of $\phi \circ u$ to $V \otimes_{\mathbb{C}} \mathbb{C}[[t]]$ lies in $\mathbb{C}[[t]]$ where u denotes the automorphism of $V \otimes_{\mathbb{C}} \mathbb{C}((t))$ that scales by u; it is enough to consider units of the form $u = t^j$.

Recall $\gamma_t = \exp f(t)X_\alpha$ with $f(t) = \sum_i f_i(t)z^i$, suppose for infinitely many i that $f_i(t) \in \mathbb{C}((t)) - \mathbb{C}[[t]]$. Then for any fixed j there is a sufficiently large n such that $v^*w\gamma_t v_{n,\alpha,w} = (f_n)^i$ with $i > j$. It follows that in order for γ_t to extend to a $\mathbb{C}[[t]]$ point only finitely many $f_i(t)$ can be in $\mathbb{C}((t)) - \mathbb{C}[[t]]$. This is also sufficient because then γ_t can be factored into a product $\gamma_t = \gamma_{t,1}\gamma_{t,2}$ with $\gamma_{t,1}$ mapping into the projective ind-variety $LG/(P^- \cap L^{poly}G)$ and $\gamma_{t,2} \in U(\mathbb{C}[[t]])$. Indeed $\gamma_{t,1}$ extends to a $\mathbb{C}[[t]]$ so by translating on the left by an element of $P^-(\mathbb{C}((t)))$ we get that $\gamma_{t,1}$ becomes a $\mathbb{C}[[t]]$ valued loop and conclude the same for γ_t. In particular the extension to the $\mathbb{C}[[t]]$ point lies in LG/P^- as desired.

For general γ_t the same argument applies because for any fixed n an arbitrary element of $U(\mathbb{C}((t)))$ can be written as $\gamma_1\gamma_2$ where $\gamma_1$is generated by elements $\exp z^i X_\alpha$ with $i < n$ and γ_2 is generated by elements of the form $\exp z^j X_\alpha$ with $j \geq n$. $\square$

Now we prove a decomposition theorem for $LG(\mathbb{C}((t)))$:

Proposition 2.3.3. *Set $\mathcal{K} = \mathbb{C}((t))$. Let $I \subset \{0, \dots, r\}$ and $P_I^-, P_I \subset LG$ be the parahoric subgroup defined in example 4 with Levi factorization $P_I = L_I U_I$, $P_I^- = L_I U_I^-$ (2.3). Then*

$$LG(\mathcal{K}) = \bigcup_{w \in W^{aff}} \mathcal{U}^-(\mathcal{K}) w P_I(\mathcal{K}) = \bigsqcup_{w \in W^{aff}} P_I^-(\mathcal{K}) w \mathcal{U}(\mathcal{K}) \tag{2.6}$$

In particular, for $I = \{0, \dots, r\}$ we have

$$LG(\mathcal{K}) = \bigcup_{w \in W^{aff}} \mathcal{U}^-(\mathcal{K}) w T(\mathcal{K}) \mathcal{U}(\mathcal{K})$$

Remark 1. When $\mathcal{K} = \mathbb{C}$ a stronger result [Kum02, 6.2.8] is proved using the axioms of a refined Tits system [Kum02, 5.2] and the representations $V = V(0, \lambda, l)$ of $LG(\mathbb{C})$. Namely, letting $I^c = \{0, \dots, r\} \backslash I$ one can identify a subset $W'_{I^c} \subset W^{aff}$ (see [Kum02, 1.3.17]) such that the union of the first equality in (2.6) becomes disjoint when ranging over W'_{I^c}.

In the case $\mathcal{K} = \mathbb{C}((t))$ the axioms of [Kum02, 5.2] can be verified using lemma 2.3.4 together with the fact that for any field k the group $LG(k)$ is generated by $\exp rX_\alpha$ where $r \in k((z))$ and α is a root of $\mathfrak{g}$ and using the fact that the representations V are algebraic. However, for our purposes we don't need the union to be disjoint or to identify the subset W'_{I^c} so we present a more direct argument that allows us to easily deduce the second equality in (2.6).

The next lemma is used repeatedly in proving proposition 2.3.3. Recall that $B = TU \subset G$ is a Borel subgroup and U is the unipotent radical and $B^- = TU^-$ is the opposite Borel with opposite unipotent radical. The element $z^i X_\alpha$ generates a root space of $L\mathfrak{g}$ and for $w \in W^{aff}$ let $z^{i_w} X_{w\alpha}$ denote the root space corresponding to the W^{aff} action on the roots.

Lemma 2.3.4. *Let R be a $\mathbb{C}$-algebra. For any fixed $g \in U(R[z])$ there is $n \geq 0$ such that $g \notin \ker(U(R[z]) \to U(R[z]/z^n))$ and $U(R[z]/z^n))$ embeds into a finite dimensional unipotent group over R. Moreover $U(R)$ is generated by $\exp rX_\alpha$ for α a root of $\mathfrak{g}$ and consequently $U(R[z]/z^n)$ is generated by $\exp rz^iX_\alpha$ for $r \in R, i \leq n$.*

For each fixed $\exp rz^iX_\alpha \in U(R[z])$ and $w \in W^{aff}$ the element $w \exp rz^iX_\alpha w^{-1}$ is either in $U(R[z])$ or in $U^-(R[z])$. Moreover, $w \exp rz^iX_\alpha w^{-1} = \exp rz^{i_w}X_{w\alpha}$ and for sufficiently large n the same holds over $R[z]/z^n$. In particular, for each simple reflection $s_0, \ldots, s_r \in W^{aff}$ we have $s_i(\alpha_i) = -\alpha_i$ and all other positive roots of $L\mathfrak{g}$ are permuted by s_i.

Proof. The first statement is obvious. The second statement, as well as the statement about being generated by $\exp rz^iX_\alpha$ follows because we can always choose a faithful finite dimensional representation $G \subset SL(V)$ such that U maps into the group of upper triangular matrices together with the fact that $\exp(R[z]/z^n)X_\alpha$ as a group over R is isomorphic to $(\exp RX_\alpha)^n$.

The final statement also follows from using an embedding $G \subset SL(V)$ and expanding the exponential and using that z^iX_α and $z^{i_w}X_{w\alpha}$ are nilpotent operators on V. $\square$

We remark that $X_{\alpha_0} = zX_{-\theta}$ where θ is the longest root of $\mathfrak{g}$.

proof of 2.3.3. Let $\gamma \in LG(\mathcal{K})$. Choose an isomorphism $\hat{\mathcal{O}}_{\mathbb{P}^1_{\mathcal{K}},0} \cong \mathcal{K}[[z]]$. Then we can use γ as gluing data on the trivial bundle over $\operatorname{Spec}\mathcal{K}[[z]] \sqcup \operatorname{Spec}\mathcal{K}[z^{-1}]$; as this is an fpqc cover the element γ determines a G-bundle E^γ on $\mathbb{P}^1_K$. Moreover E^γ is trivial at the $\mathcal{K}$-rational point $0 \in \mathbb{P}^1$. By [MS02, thm 4.3] we conclude that E^γ reduces to a T bundle. Further, $T(\mathcal{K}[z^\pm]) = T(\mathbb{C}[z^\pm])T(\mathcal{K})$ and$T(\mathbb{C}[z^\pm]) = \hom(\mathbb{C}^\times, T)$ hence

$$LG(\mathcal{K}) = \bigsqcup_{\mu \in \hom(\mathbb{C}^\times, T)} L^-G(\mathcal{K})\mu(z)L^+G(\mathcal{K}). \tag{2.7}$$

Remark that $\hom(\mathbb{C}^\times, T) = W^{aff}/W$ and $P_{\{0\}} = L^+G$ and so the case $I = \{0\}$ of the proposition can readily be deduced from (2.7).

For any $\gamma \in LG(\mathcal{K})$ let $[\gamma]$ denote the double coset $\mathcal{B}^-(\mathcal{K})\gamma\mathcal{B}(\mathcal{K})$. To obtain the case $I = \{0, \ldots, r\}$ it suffices to show for any $\gamma \in LG(\mathcal{K})$ there exists $w \in W^{aff}$ such that $[\gamma] = [w]$.

As $L^+G(\mathcal{K}) = G(\mathcal{K})\mathcal{U}(\mathcal{K})$ and similarly for $L^-G(\mathcal{K})$, by (2.7) we can assume $[\gamma] = [g_1\mu(z)g_2]$ where $g_i \in G(\mathcal{K})$. Use the Bruhat decomposition for $G(\mathcal{K})$ in the form $G(\mathcal{K}) = \sqcup_{w\in W}U^-(\mathcal{K})wB^-(\mathcal{K})$ and $G(\mathcal{K}) = \sqcup_{w\in W}U^-(\mathcal{K})wB(\mathcal{K})$ to write $[g_1\mu(z)g_2] = [w_1u_1\mu(z)u_2w_2]$ with $w_i \in W$ and $u_i \in U^-(\mathcal{K})$.

By conjugation by $\mu(z), w_1$, lemma 2.3.4 and left multiplication by $\mathcal{U}^-(\mathcal{K})$ we obtain $[w_1u_1\mu(z)u_2w_2] = [uw_1\mu(z)w_2]$ with $u \in \mathcal{U}(\mathcal{K})$ and moreover u is a finite product of elements of the form $\exp rz^iX_\alpha$ with α positive and $i \geq 0$. The case $I = \{0, \dots, r\}$ of the proposition follows from lemma 2.3.5.

Now $\mathcal{B} \subset P_I$ and $\mathcal{B}^- \subset P_I^-$ for every $I \subset \{0, \dots, r\}$ so the proposition follows in general from the case $I = \{0, \dots, r\}$. $\square$

Lemma 2.3.5. *Set $u_\alpha(r) = \exp rX_\alpha$ and $u_\alpha^-(r) = \exp rX_{-\alpha}$ where α is a root of $L\mathfrak{g}$ and $r \in \mathcal{K}$. If $w \in W^{aff}$ and u is a finite product of $u_{\alpha_i}(r)$ for $i = 0, \dots, r$ then $[uw] = [w']$ for some $w' \in W^{aff}$.*

Proof. We proceed by induction on the number factors in u. For the base case $u = u_{\alpha_i}(r)$ observe that if $w^{-1}uw \in \mathcal{U}(\mathcal{K})$ then evidently $[uw] = [w]$ so we can assume $w^{-1}uw \in \mathcal{U}^-(\mathcal{K})$. By appealing to lemma 2.3.4 we infer w admits a factorization $w_1s_jw_2$ such that $w_1^{-1}uw_1 = u_{\alpha_j}(r)$ and $w_2^{-1}u_{\alpha_j}^-(r)w_2 \in \mathcal{U}^-(\mathcal{K})$. Then from $r \neq 0$ and the explicit factorization

$$\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/r & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} 1 & -1/r \\ 0 & 1 \end{pmatrix}$$

we can factor $u_{\alpha_j}(r)s_j = u_{\alpha_j}^-(1/r)tu_{\alpha_j}(-1/r)$ where $t \in T(\mathcal{K})$. Thus we obtain

$$\begin{aligned}[uw_1s_jw_2] = [w_1\boldsymbol{u_{\alpha_j}(r)s_j}w_2] &= [w_1\boldsymbol{u_{\alpha_j}^-(1/r)tu_{\alpha_j}(-1/r)}w_2] \\ &= [\boldsymbol{u_{\alpha_i}^-(r)}w_1t\boldsymbol{u_{\alpha_j}(-1/r)}w_2] = [w_1t\boldsymbol{u_{\alpha_j}(-1/r)w_2}] \\ &= [w_1t\boldsymbol{w_2 \cdot w_2^{-1}u_{\alpha_j}(-1/r)w_2}] = [w_1tw_2] = [w_1w_2].\end{aligned}$$

We note also that this base case also proves the result if $u = u_\alpha(r)$ for α not necessarily a simple root.

For the inductive step write $u = u_\alpha(r)u'$ where by the induction hypothesis we assume $[u'w] = [w']$ with $w' \in W^{aff}$. In particular $u'w = u_1w'b_1$ where $u_1 \in \mathcal{U}^-(\mathcal{K})$ and $b_1 \in \mathcal{B}(\mathcal{K})$. Then $[uw] = [u_i(r)u_1w']$. Further, the element $u_i(r)u_1$ lies in the minimal parabolic P_I^- defined in example 4 with I the complement of $\{i\}$ in $\{0, \dots, r\}$. We can express $u_i(r)u_1 = u_1'g$ according to the Levi factorization $P_I^- = U_I^-L_I$; note also that $U_I^- \subset \mathcal{U}^-$. Then $[u_i(r)u_1w'] = [u_1'gw'] = [gw']$. The group L_I reductive with its adjoint being a rank 1 semi simple group. By the Bruhat decomposition we can write $g = u''w''v''t'' \in \mathcal{U}^-(\mathcal{K})W^{aff}\mathcal{U}(\mathcal{K})T(\mathcal{K})$ such that $w''v''(w'')^{-1} \in \mathcal{U}(\mathcal{K})$. Then $[gw'] = [u''w''v''t''w'] = [w''v''w'] = [w''v''(w'')^{-1} \cdot (w''w')] = [w''']$ for some $w''' \in W^{aff}$ where the last equality follows from the base case. $\square$

Let $V = V(0, \lambda, l)$ be a highest weight representation and let $V = \cup_k V_{\leq k}$ be it's ind-variety structure. Define the ind-varieties $End^{fin}(V) = \cup_k End(V_{\leq k})$ and $\mathbb{P}End^{fin}(V) = \cup_k \mathbb{P}End(V_{\leq k})$. It is straightforward to see that $L_{poly}G \times L_{poly}G$ acts on $\mathbb{P}End^{fin}(V)$. A final lemma we utilize in proposition 2.4.4 is

Lemma 2.3.6. *If $Z \subset \mathbb{P}End^{fin}(V)$ is closed and $L_{poly}G \times L_{poly}G$ stable then $[v \otimes v^*] \in Z$.*

Proof. Choose any point $p \in Z(\mathbb{C})$ then p is a finite sum of weight vectors $v' \otimes w^*$. By applying various root subgroups $\mathbb{G}_a$ for positive and negative roots α we can eventually specialize from $v \otimes w^*$ to $v \otimes v^*$. That is by taking limits of the form $\lim_{t\to 0} \begin{pmatrix} 1 & 1/t \\ 0 & 1 \end{pmatrix} [v' \otimes w^*] \begin{pmatrix} 1 & 0 \\ 1/t & 1 \end{pmatrix}$ we eventually get $[v \otimes v^*]$. □

2.4 The Embedding of $L^{\ltimes}G/Z(G)$

Following the construction outlined in 2.1 of the wonderful compactification of G_{ad} we must now construct an analogue of $\mathbb{P}End(V)$ for a representation $V = V(0, \lambda, l)$ of G^{aff}.

After the proof of 2.3.1 we defined the notion of $\mathbb{C}$-space and defined in particular $\mathbb{C}$-spaces $End(V), GL(V), PGL(V)$. For a $\mathbb{C}$-algebra R the set $End(V)(R) = End_R(V_R)$ has the structure of an R-module. The assignment

$$R \mapsto \{\mathcal{L} \subset End(V)(R) | \mathcal{L} \text{ is projective of rank 1 and the module } End(V)(R)/\mathcal{L} \text{ is projective}\}$$

defines a $\mathbb{C}$ space $\mathbb{P}End(V)$.

There is an action of $GL(V) \times GL(V)$ on $\mathbb{P}End(V)$ given by a morphism of $\mathbb{C}$-spaces $GL(V) \times GL(V) \times \mathbb{P}End(V) \to \mathbb{P}End(V)$ defined by left and right multiplication. Moreover the restriction to the scalar action $\mathbb{G}_m \times \mathbb{G}_m \subset GL(V) \times GL(V)$ is the trivial action hence induces an action $PGL(V) \times PGL(V) \times \mathbb{P}End(V) \to \mathbb{P}End(V)$. By [BL94b, appendix 7], for every representation $V = V(0, \lambda, l)$ we have a morphism of $\mathbb{C}$-space $LG \to PGL(V)$ and hence an action $LG \times LG \times \mathbb{P}End(V) \xrightarrow{a} \mathbb{P}End(V)$.

Let $[id] \in \mathbb{P}End(V)(\mathbb{C})$ be the class of $id : V \to V$. Then the orbit of $[id]$ yields a morphism of $\mathbb{C}$-spaces $LG \times LG \xrightarrow{a(-,-,[id])} \mathbb{P}End(V)$. We show that $LG \times LG \to \mathbb{P}End(V)$ factors as $LG \times LG \to \mathbb{P}End^{ind}(V) \to \mathbb{P}End(V)$ where $\mathbb{P}End^{ind}(V) = \cup_i \mathbb{P}End^{ind}(V)_i$ is an ind-scheme.

The action map $LG \times \mathbb{P}V \to \mathbb{P}V$ is a morphism of ind-schemes and hence for any $k \geq 0$ there is an integer $n(k)$ such that $(LG)_k \times \mathbb{P}V_{\leq k} \to \mathbb{P}V_{\leq n(k)}$ is a morphism of schemes. Fix $k \geq 0$ then for $i \geq 0$ we have a morphism of schemes

$$V_{\leq k} \times (LG)_i \subset V_{\leq k+i} \times (G^{aff})_{k+i} \to V_{n(k+i)}. \tag{2.8}$$

Observe also that for a $\mathbb{C}$-algebra R and $\mathcal{L} \subset \mathbb{P}End(V)(R)$ every $\phi \in \mathcal{L}$ defines a homomorphism $V_R \xrightarrow{\phi} V_R$. With these observations in mind we define a sub $\mathbb{C}$-space $\mathbb{P}End^{ind}(V)_i$ by

$$\mathbb{P}End^{ind}(V)_i(R) = \{\mathcal{L} \in \mathbb{P}End(V)(R) | \phi(V_{\leq k,R}) \subset V_{\leq n(k+i),R} \forall k \geq 0, \phi \in \mathcal{L}\} \tag{2.9}$$

The $\mathbb{C}$-space $\mathbb{P}End^{ind}(V)_i$ is a scheme. In fact set $H_{i,j} = \hom(V_j, V_{n(i+j)})$ and $S_{E(V),i} = \cup_{j\geq 0} Sym^*(H^*_{i,j})$ where $\cup_{j\geq 0}$ denotes the co-product of rings. Then one can verify that $\mathbb{P}End^{ind}(V)_i = \mathrm{Proj}\ S_{E(V),i}$. We thus obtain an ind-scheme $\mathbb{P}End^{ind}(V) = \cup_i \mathbb{P}End^{ind}(V)_i \subset$

$\mathbb{P}End(V)$. By (2.8) the morphism $a(-,-,[id])\colon LG\times LG\to\mathbb{P}End(V)$ factors through $\mathbb{P}End^{ind}(V)$.

We now identify the image of $a(-,-,[id])$ with $LG/Z(G)$ where $Z(G)$ is the center of G. For this we make a few definitions. Let v be a highest weight vector in V and v^* the dual vector in V^*. Then the classes $[v]\in\mathbb{P}V,[v^*]\in\mathbb{P}V^*$ and $[v\otimes v^*]\in\mathbb{P}End^{ind}(V)$ define a section of a line bundle and the nonvanishing of this section define open affine sub (ind)-scheme $\mathbb{P}_{[v]}V\subset\mathbb{P}V$, $\mathbb{P}_{[v^*]}V^*$, $\mathbb{P}_{[v\otimes v^*]}End^{ind}(V)$.

There is a morphism $\pi_v\colon\mathbb{P}_{[v\otimes v^*]}End^{ind}(V)\to\mathbb{P}_{[v]}V$ given by $\phi\mapsto\phi(v)$. Similarly there is a morphism $\pi_{v^*}\colon\mathbb{P}_{[v\otimes v^*]}End^{ind}(V)\to\mathbb{P}_{[v^*]}V^*$ given by $\phi\mapsto v^*\circ\phi$.

Lemma 2.4.1. *Suppose $V=V(0,\lambda,l)$ with $(0,\lambda,l)$ a dominant weight. The restriction of $a(-,-,[id])$ to $\mathcal{U}\times\mathcal{U}$ maps to a closed sub ind-scheme of $\mathbb{P}_{[v\otimes v^*]}End^{ind}(V)$ isomorphic to $\mathcal{U}$ where the latter is given the trivial ind-scheme structure. The restriction of $a(-,-,[id])$ to $\mathcal{U}^-\times\mathcal{U}^-$ maps to a closed sub ind-scheme of $\mathbb{P}_{[v\otimes v^*]}End^{ind}(V)$ isomorphic to $\mathcal{U}^-$.*

Proof. Assume first $(0,\lambda,l)$ is a regular dominant weight. Let $\mathbb{G}_a\subset\mathcal{U}$ be a root subgroup corresponding to a positive root α. Then $\mathbb{G}_a$ acts nilpotently and nontrivially on v^* and consequently gives a closed embedding $\mathbb{G}_a\subset\mathbb{P}_{[v^*]}V^*$. As $\mathcal{U}$ is generated by these root subgroups it follows that we obtain a closed embedding $\mathcal{U}\subset\mathbb{P}_{[v^*]}V^*$. By the same argument applied to any other weight vector $v\in V^*$ we conclude that $\mathcal{U}\times\mathcal{U}$ maps onto a closed sub ind scheme $Z_{\mathcal{U}}$ of $\mathbb{P}_{[v\otimes v^*]}End^{ind}(V)$. We claim $\mathcal{U}\xrightarrow{a(-,id,[id])}Z_{\mathcal{U}}$ is an isomorphism; indeed the map π_{v^*} defined before the lemma gives an inverse. The same argument works for $\mathcal{U}^-$ with π_v playing the role of π_{v^*}.

For general dominant $(0,\lambda,l)$ the same argument gives a closed sub ind scheme $Z_{\mathcal{U}}\subset\mathbb{P}_{[v\otimes v^*]}End^{ind}(V)$ but there may be finitely many root subgroups $\mathbb{G}_a\subset\mathcal{U}$ that act trivially on $[v^*]$ so π_{v^*} may not give an inverse. The problem comes from $\mathbb{G}_a$ corresponding to α for which $\langle(0,\lambda,l),\alpha\rangle=0$; but by general properties of the weight spaces of $V(0,\lambda,l)$, see e.g. [Seg81, 9.3.7,9.3.8], for such α there is some nonzero weight μ of $V^*(0,\lambda,l)$ such that $\langle\mu,\alpha\rangle<0$ (see also the end of the proof of 2.3.2). Therefore there is a finite number of weight vectors $v_0^*=v^*,v_1^*,\dots,v_m^*$ and analogous morphisms $\pi_{v_i^*}\colon\mathbb{P}_{[v\otimes v^*]}End^{ind}(V)\to\mathbb{P}_{[v_i^*]}V^*$ such that for $\mathbb{G}_a\subset\mathcal{U}$ act nil potently and nontrivially on some v_i^* and $\prod_{i=0}^m\pi_{v_i^*}\colon Z_{\mathcal{U}}\to\prod_{i=0}^m\mathbb{P}_{[v_i^*]}V^*$ maps isomorphically onto $\mathcal{U}$. The same argument applies to $\mathcal{U}$. $\square$

Assume V is as in the lemma then by the ind-scheme structure on V defined before proposition 2.3.1, we have that V_0 is highest weight representation of G for a regular dominant weight λ and $T\times T\xrightarrow{a(-,-,[id_{V_0}])}\mathbb{P}_{v\otimes v^*}End(V_0)$ has locally closed image $T_{ad}=T/Z(G)$ so the same holds for $T\times T\xrightarrow{a(-,-,[id])}\mathbb{P}_{v\otimes v^*}End^{ind}(V)$. Further $\mathcal{U}^-T\mathcal{U}\subset LG$ is an open sub ind scheme and consequently $\mathcal{U}^-\times T_{ad}\mathcal{U}\xrightarrow{a(-,-,[id])}\mathbb{P}_{[v\otimes v^*]}End^{ind}(V)$ maps isomorphically onto a locally closed sub ind-scheme $\Omega\subset\mathbb{P}_{[v\otimes v^*]}End^{ind}(V)$. Now as $LG/Z(G)=\cup_{g\in LG/Z(G)}g\mathcal{U}^-T_{ad}\mathcal{U}$ we obtain an isomorphism of ind-scheme $LG/Z(G)$ onto

$\cup_{g\in LG/Z(G)} g\Omega \subset \mathbb{P}End^{ind}(V)$. Clearly the map $LG \times LG \xrightarrow{a(-,-,[id])} \cup_{g\in LG/Z(G)} g\Omega$ is onto and so we have proved

Lemma 2.4.2. *Let $(0,\lambda,l)$ be a regular dominant weight of LG and $V = V(0,\lambda,l)$. The morphism $a(-,-,[id])$ maps $LG \times LG$ onto a locally closed sub ind-scheme of $\mathbb{P}End^{ind}(V)$ isomorphic to $LG/Z(G)$. Moreover the morphism $a(-,-,[id])$ extends to $L^{\ltimes}G \times L^{\ltimes}G$ mapping onto a locally closed sub ind-scheme of $\mathbb{P}End^{ind}(V)$ isomorphic to $L^{\ltimes}G/Z(G)$.*

Proof. We only need to check the second statement. This follows because the projective representation V of LG comes from an actual representation of G^{aff} which is a central extension $1 \to \mathbb{G}_m \to G^{aff} \to L^{\ltimes}G \to 1$. Therefore we also have a projective representation of $L^{\ltimes}G$. □

The naive guess of using the ind-scheme structure $V = \cup_k V_{\leq k}$ to define the ind-scheme $\cup_k End(V_{\leq k})$ is problematic because $id \notin End(V_{\leq k}) \ \ \forall k$.

Construction of X^{aff}

Mimicking a the construction in (2.2) we now construct an ind-scheme X^{aff} that contains $L^{\ltimes}G/Z(G)$ as a dense open sub ind-scheme. Let $Z(G^{aff})$ denote the center of G^{aff}; the quotient $L^{\ltimes}G/Z(G)$ is also $G^{aff}_{ad} = G^{aff}/Z(G^{aff})$. To emphasize the parallel with the wonderful compactification of G_{ad} we use G^{aff}_{ad} over $L^{\ltimes}G/Z(G)$.

The construction depends on a regular dominant weight $(0,\lambda,l)$ but we will prove that X^{aff} is independent of $(0,\lambda,l)$.

Construction 1. Let $(0,\lambda,l)$ be a regular dominant weight of $L^{\ltimes}G$ and let $V = V(0,\lambda,l)$. Lemma 2.4.2 gives an embedding of ind-schemes $G^{aff}_{ad} = L^{\ltimes}G/Z(G) \subset \mathbb{P}End^{ind}(V)$. In particular for every $k \geq 0$ we have a locally closed embedding $(G^{aff}_{ad})_k = \mathbb{C}^{\times} \times (LG)_k \subset \mathbb{P}End^{ind}(V)_{n(k)}$. Define

$$\begin{aligned} X^{aff}_k &= \overline{(G^{aff}_{ad})_k} \subset \mathbb{P}End^{ind}(V)_{n(k)} \\ X^{aff} &= \cup_k X^{aff}_k \end{aligned} \tag{2.10}$$

Recall the definition of the affine ind-scheme $\mathbb{P}_{[v\otimes v^*]}End(V)$ before lemma 2.4.1. Define

$$\begin{aligned} X^{aff}_{0,k} &= X^{aff}_k \cap \mathbb{P}_{[v\otimes v^*]}End^{ind}(V)_{n(k)} \\ X^{aff}_0 &= \cup_k X^{aff}_{0,k} \end{aligned} \tag{2.11}$$

The main theorem we prove is theorem 2.4.3. First let us mention a few technicalities. In the statement we mention Cartier divisors; we remark that for infinite dimensional object the notion of codimension is potentially problematic and the same goes for Weil divisors. However line bundles and Cartier divisors still make sense. Finally, the ind-scheme LG is not the union of smooth schemes [FGT08, 5.4]. However LG is formally smooth: if R is a $\mathbb{C}$ algebra and I is a nilpotent ideal then any R/I point of LG lifts to an R point. As such we can at most ask for embeddings of that are formally smooth.

In out development we work with the group $L^{\ltimes}G$ and it's quotient $L^{\ltimes}G/Z(G)$ which is the group that actually embeds. However we note that $G_{ad}^{aff} = L^{\ltimes}G/Z(G)$ and one can replace $L^{\ltimes}G$ and $L^{\ltimes}G/Z(G)$ with G^{aff} and G_{ad}^{aff} which makes the analogy between the wonderful compactification of G_{ad} slightly stronger.

Theorem 2.4.3. *Let G be a simple, connected and simply connected group over $\mathbb{C}$ and set $r = rk(G)$. The ind-scheme X^{aff} contains G_{ad}^{aff} as a dense open sub-ind scheme and further*

(a) X^{aff} is formally smooth and independent of the choice of regular dominant weight $(0, \lambda, l)$.

(b) The boundary $X^{aff} - L^{\ltimes}G/Z(G)$ is a Cartier divisor with $r+1$ components $D_0, \ldots, D_r$. The $L^{\ltimes}G \times L^{\ltimes}G$ orbits closures are in bijection with subsets $I \subset \{0, \ldots, r\}$ in such a way that to I we associate $\cap_{i \in I} D_i$.

(c) Each D_i is formally smooth and $\cup_{i=0}^{r} D_i$ is locally a product $S \times Z$ where S is an ind-scheme and Z is the union of hyperplanes in $\mathbb{A}^{r+1}$.

(d) $X^{aff} - X_0^{aff}$ is a Cartier divisor and with $r+1$ components which freely generate $Pic(X^{aff})$.

Remark for that Chevellay's theorem for constructible sets allows one to readily identify homogeneous spaces G/H as subschemes of $\mathbb{P}W$. The lack of a similar theorem that would apply to LG and is one technical hurdle to circumvent. But with this in mind the proofs of (a) - (c) follow closely the approach used for the wonderful compactificaiton of G_{ad}. The proof of (d) however is a bit more involved and we need to develop a series of intermediate results before we can prove it.

Proposition 2.4.4. *X^{aff} is an ind scheme containing G_{ad}^{aff} as a dense open sub ind-scheme.*

Proof. That G_{ad}^{aff} is dense is clear. Observe that $[v \otimes v^*] \in L^{\ltimes}G \times L^{\ltimes}G.[v \otimes v^*] \cap X_0^{aff}$ and $\cup_{\gamma_1, \gamma_2 \in L^{\ltimes}G} \gamma_1 X_0^{aff} \gamma_2$ is $L^{\ltimes}G \times L^{\ltimes}G$ stable and open. Therefore the complement Z of $\cup_{\gamma_1, \gamma_2 \in L^{\ltimes}G} \gamma_1 X_0^{aff} \gamma_2$ is $L^{\ltimes}G \times L^{\ltimes}G$ stable and closed and $[v \otimes v^*] \notin Z(\mathbb{C})$. If $p \in Z(\mathbb{C})$ than for a generic 1-psg $\rho \colon \mathbb{C}^{\times} \to T^{\ltimes}$ we have that $\lim_{t \to 0} \rho(t)p \in Z(\mathbb{C}) \cap \mathbb{P}End^{fin}(V)$ but then lemma 2.3.6 would imply that $[v \otimes v^*] \in Z(\mathbb{C})$, a contradiction. Therefore $X^{aff} = \cup_{\gamma_1, \gamma_2 \in L^{\ltimes}G} \gamma_1 X_0^{aff} \gamma_2$. Thus to prove the proposition it suffices to show that $G_{ad}^{aff} \cap X_0^{aff}$ is open in X_0^{aff}. This follows from proposition 2.4.6. $\square$

Let $t^{-\alpha_i}$ be the regular function on $T_{ad}^{\ltimes}$ given by the character $-\alpha_i$. Let $\overline{T_{ad}^{\ltimes}} \subset X^{aff}$ be the closure and $\overline{T_{ad,0}^{\ltimes}} = \overline{T_{ad}^{\ltimes}} \cap X_0^{aff}$.

Proposition 2.4.5. $\overline{T_{ad,0}^{\ltimes}} \cong \mathbb{C}[t^{-\alpha_0}, \ldots, t^{-\alpha_r}] \cong \mathbb{A}^{r+1}$. *In particular $\overline{T_{ad,0}^{\ltimes}}$ is smooth and its fan is given by the negative Weyl alcove $-Al_0$; the fan $\overline{T_{ad}^{\ltimes}}$ is given by the Weyl aclove decomposition of $\mathfrak{t}_{\mathbb{R}}^{\ltimes} = Lie(T_{ad}^{\ltimes})_{\mathbb{R}}$.*

Proof. Let S be the set of nonzero weight spaces of $V = V(0, \lambda, l)$. The image of $T^{\ltimes}/Z(G)$ in X_0^{aff} is $\prod_{\mu \in S} t^{\mu-(0,\lambda,l)}$. By proposition 2.3.1(b) we see that $t^{-\alpha_i}$ appear in this product and more over by 2.3.1(a) all other terms in the product are monomials in $t^{-\alpha_i}$. The first statement follows.

The second statement follows because

$$\overline{T_{ad}^{\ltimes}} = \cup_{w \in W^{aff}} w \overline{T_{ad,0}^{\ltimes}} w^{-1}$$

and $-Al_0$ is a fundamental domain for the action of W^{aff} on $\mathfrak{t}_{\mathbb{R}}^{\ltimes}$. □

The proof of proposition 2.4.6 is adapted from [BK05a, 6.1.7]

Proposition 2.4.6. *There is an $\mathcal{U} \times \mathcal{U}^-$ equivariant isomorphism*

$$\begin{aligned} \mathcal{U}^- \times \mathcal{U} \times \overline{T_{ad,0}^{\ltimes}} &\xrightarrow{a} X_0^{aff} \\ (l, u, t) &\mapsto l \cdot t \cdot u \end{aligned}$$

Proof. First note that the restriction to $\mathcal{U}^- \times \mathcal{U} \times T_{ad}^{\ltimes}$ is just the multiplication map and this is known to be open by the Birkhoff decomposition; consequently the morphism is birational. By lemma 2.4.1, we have a morphism $X_0^{aff} \xrightarrow{(\pi_v, \pi_{v^*})} \mathcal{U}^- \times \mathcal{U}$ which is moreover $\mathcal{U}^- \times \mathcal{U}$ equivariant.

The composition $\mathcal{U}^- \times \mathcal{U} \times \overline{T_{ad,0}^{\ltimes}} \to X_0^{aff} \to \mathcal{U}^- \times \mathcal{U}$ is given by $(l, t, u) \mapsto (l, u)$.

To finish we show $\mathcal{U}^- \times \mathcal{U}^- \times (\pi_v, \pi_{v^*})^{-1}(1,1) \xrightarrow{a} X_0^{aff}$ is bijective and $b^{-1}(1,1) = \overline{T_{ad,0}^{\ltimes}}$. For injectivity note that as $(\pi_v, \pi_{v^*})^{-1}(1,1)$ is a subset of X_0^{aff} it suffices to show that if $p \in (\pi_v, \pi_{v^*})^{-1}(1,1)$ and $lpu = l'pu'$ then $u = u', l = l'$. This follows

$$(l, u) = (\pi_v, \pi_{v^*}) lpu = (\pi_v, \pi_{v^*}) l'pu' = (l', u').$$

Now surjectivity. Let $p \in X_0^{aff}$ and $(l, u) = (\pi_v, \pi_{v^*})(p)$. Then $t := (l^{-1}, u^{-1}).p \in b^{-1}(1,1)$, hence (l, t, u) does the job.

It remains to show $b^{-1}(1,1) = \overline{T_{ad,0}^{\ltimes}}$. Clearly we have $\supset$ as $(\pi_v, \pi_{v^*})^{-1}$ is closed and contains $T_{ad}^{\ltimes}$ and as a is birartional it follows that they have the same dimension. Now $\pi_0(LG) = \pi_1(G) = 1$. Further, the map $G \to G/Z(G) =: G_{ad}$ induces a map $LG \to LG_{ad}$; the image is the connected component of the identity, in particular it is irreducible. It follows that X^{aff} and X_0^{aff} are irreducible hence so is $X_0^{aff}/\mathcal{U} \times \mathcal{U}^- \cong b^{-1}(1,1)$. Thus it must equal $\overline{T_{ad,0}^{\ltimes}}$ □

We can now prove (a)-(c) of theorem 2.4.3.

proof of theorem 2.4.3(a) - (c). Let $(0, \lambda, l), (0, \mu, l')$ be two regular dominant weights and denote $X_\lambda^{aff}, X_\mu^{aff}$ the respective embeddings associated to $(0, \lambda, l), (0, \mu, l')$. Let X_Δ^{aff} be the closure of $\Delta(L^{\ltimes}G/Z(G))$ in $X_\lambda^{aff} \times X_\mu^{aff}$. The projection $p_\lambda \colon X_\Delta^{aff} \to X_\lambda^{aff}$ is equivariant and proposition 2.4.6 implies $X_{\Delta,0}^{aff} := p_\lambda^{-1}(X_{\lambda,0}^{aff}) \cong \mathcal{U}^- \cdot \overline{T_{ad,0}^{\ltimes}} \cdot \mathcal{U}$; that is, the restriction of

p_λ to $X^{aff}_{\Delta,0}$ is an isomorphism and therefore induces a $L^{\ltimes}G \times L^{\ltimes}G$ equivariant isomorphism on

$$\bigcup_{g \in L^{\ltimes}G \times L^{\ltimes}G} g.X^{aff}_{\Delta,0} = X^{aff}_{\Delta} \to X^{aff}_{\lambda} = \bigcup_{g \in L^{\ltimes}G \times L^{\ltimes}G} g.X^{aff}_{\lambda,0}.$$

This shows X^{aff} is independent of the choice of regular dominant weight. Further $\mathcal{U}^-, \overline{T^{\ltimes}_{ad,0}}, \mathcal{U}$ are all formally smooth hence (a).

As the boundary $X^{aff} - L^{\ltimes}G/Z(G)$ is $L^{\ltimes}G(\mathbb{C}) \times L^{\ltimes}G(\mathbb{C})$ stable it is enough to verify (b) on X^{aff}_0. Indeed once it holds over X^{aff}_0 we can establish it for X^{aff} by translating by elements in $L^{\ltimes}G(\mathbb{C}) \times L^{\ltimes}G(\mathbb{C})$. We have by proposition 2.4.6

$$X^{aff} - L^{\ltimes}G/Z(G) \cap X^{aff}_0 = \mathcal{U}^- \times Hy \times \mathcal{U}$$

where Hydenotes the union of the hyperplanes in $\overline{T^{\ltimes}_{ad,0}} \cong \mathbb{A}^{r+1}$. In fact this proves both (b) and (c). □

2.5 The proof of 2.4.3(d)

Orbit stratification

The final statement of the main theorem 2.4.3(d) is proved in proposition 2.5.10. Unfortunately the proof is long and technical requiring many intermediate results. Here we begin with two 2.5.1, 2.5.2 related to the orbits in X^{aff}. We also present corollary 2.5.3 which provides another strong analogy between X^{aff} and the classical wonderful compactification of an adjoint group. Further 2.5.3 will play an important role when the moduli of G-bundles is discussed.

Fix a subset $I \subset \{0, \dots, r\}$. Let $P_I = L_I U_I$ and $P^-_I = L_I U^-_I$ be the parahoric subgroups with Levi decomposition described in example 4. Let $Z(L_I) \subset L_I$ denote the center and $L_{I,ad} = L_I/Z(L_I)$ the adjoint group.

For any $\mathbb{A}^n$ let $e_i \in \mathbb{A}^n(\mathbb{C})$ be the n-tuple with a 1 in the ith position and 0 elsewhere. Let $e_I = \sum_{i \in I} e_i \in \overline{T^{\ltimes}_{ad,0}} \cong \mathbb{A}^{r+1}$.

Lemma 2.5.1. *Let R be a $\mathbb{C}$-algebra and identify $e_I \in X^{aff}(\mathbb{C})$ with it's image in $X^{aff}(R)$. Let $T_I(R) = Z(L_I)(R) \times Z(L_I)(R)$ and $S_I(R) = \Delta(L_I)(R) \ltimes U_I(R) \times U^-_I(R)$. Let $Stab_R(e_I)$ denote the stabilizer of e_I in $L^{\ltimes}G(R) \times L^{\ltimes}G(R)$. Then $T_I(R)S_I(R)$ is a group and $T_I(R)S_I(R) = Stab_R(e_I)$*

Proof. Let $\pi_I \colon P_I = L_I \ltimes U_I \to L_I \to L_{I,ad}$ be the projection and define $\pi^-_I \colon P^-_I \to L_{I,ad}$ similarly. The first statement follows because we can identify $T_I(R)S_I(R) = \{(g_1, g_2) \in P_I \times P^-_I | \pi^-_I(g_1) = \pi_I(g_2)\}$ which defines a subgroup of $P^-_I \times P_I$.

We first show $T_I(R)S_I(R) \subset Stab_R(e_I)$. That $T(J)$ is in the stabilizer follows from the description of $\overline{T^{\ltimes}_{ad,0}}$ given in the proof proposition 2.4.5 and the fact that $Z(L_J) = \cap_{k \notin J} \ker \alpha_k$. So let us focus on the group $S(J)$.

The group $L^{\ltimes}G \times L^{\ltimes}G$ and in particular the standard parabolic subgroups are generated by the root subgroups $U(\alpha) \cong \mathbb{G}_a$ which it contains. So to check $S(J)$ is in the stabilzer it suffices to check it for the root subgroups it contains. These break up into two cases. Roots subgroups of the form $(U(\alpha), 1)$ or $(1, U(\alpha))$ and those of the form $\Delta(U(\alpha))$. We treat the first case; the second case follows similarly

In the first case α is not a root of $Lie(L_J)$ and we have that $\alpha = \alpha_i + \alpha'$ for some $i \notin J$. It suffices to check that $X_\alpha.e_J = 0$. Recall e_J is an idempotent of $End(V(\lambda))$ and we can express $e_J = \sum_j e_j \otimes e_j^*$ where j ranges over some subset of the weights of the representation. Therefore to show $X_\alpha.e_J = 0$ it suffices to show $X_\alpha.e_j = 0\ \forall j$. Assume that $X_\alpha.e_j \neq 0$ for some j. The weight j has the property that $\lambda - j \in \sum_{i \in J} n_i\alpha_i$ with $n_i \geq 0$. But if $e_\mu := X_\alpha.e_j$ is not zero then it is a weight vector of weight $\mu = \alpha + j$. But then $\lambda - \mu$ fails to be a sum of positive roots, contradiction.

To conclude that $T_I(R)S_I(R) \subset Stab_R(e_I)$ we first establish an intermediate result: $Stab_R(e_I) \subset P_I \times P_I^-$.

The action of $L_I(R)$ on $v_{(0,\lambda,l)} \subset V_R = V(0, \lambda, l)_R$ generates a finite dimensional $L_I(R)$-rep which we denote $V_{I,R}$ and moreover $e_I = id \in End(V_{I,\mathbb{C}})$. Let $m = \dim V_{I,\mathbb{C}})$. Set $V'_{\mathbb{C}} = \wedge^m V_{\mathbb{C}}$; this gives an in general reducible representation of $G^{aff}(\mathbb{C})$ and $\wedge^m V_{I,\mathbb{C}}$ generates an irreducible representation $V'_{I,\mathbb{C}}$. Consequently we have a well defined morphism

$$L^{\ltimes}G \times L^{\ltimes}G \xrightarrow{a(-,-,e_I)} \mathbb{P}End^{ind}(V) \xrightarrow{\wedge^m} \mathbb{P}End^{ind}(V') \tag{2.12}$$

Further $\wedge^m e_I = w \otimes w^*$ where w is a highest weight vector for V'_I and further that action of $X_{-\alpha_i}$ acts trivially on w for $i \notin I$ and nontrivially for $i \in I$ and hence the stabilizer of w is the parabolic P_I. It follows that the stabilizer of $\wedge^m e_I$ is $P_I \times P_I^-$. It follows that the image of $a(-,-,e_I)$ maps to $P_I \backslash L^{\ltimes}G \times L^{\ltimes}G/P_I^-$ and thus $Stab(e_I) \subset P_I \times P_I^-$.

We have $L^{\ltimes}G \times L^{\ltimes}G/T_I(R)S_I(R) \to L^{\ltimes}G \times L^{\ltimes}G/Stab_R(e_I) \to P_I \backslash L^{\ltimes}G \times L^{\ltimes}G/P_I^-$ and these map both realize the first two terms as $L_{I,ad}$-fibrations over $P_I \backslash L^{\ltimes}G \times L^{\ltimes}G/P_I^-$ hence we must have $T_I(R)S_I(R) = Stab_R(e_I)$. □

Lemma 2.5.2. *The orbit of e_I defines a sub ind-scheme $Orb(I) \subset X^{aff}$*

Proof. In the proof of lemma 2.5.1 we identified a finite dimensional irreducible L_I representation V_I such that $e_I = id \in End(V_I)$. It readily follows that the $L_I \times L_I$ orbits of $[e_I] \in \mathbb{P}End(V_I)$ is isomorphic to $L_{I,ad}$.

Let $W_I \subset \mathbb{P}End^{ind}(V)$ be the open subset on which projection to $\mathbb{P}End(V_I)$ is defined. The inverse image of $L_I \times L_I.[e_I]$ in W_I defines a subscheme we also $Orb_{L_I}(e_I)$. Set $T_{L_I} = T \cap L_I$ and let $B_{L_I} = T_{L_I} \ltimes U_{L_I} \subset L_I$ be a Borel subgroup. And $\Omega_{L_{I,ad}} = U_{L_I}^- T_{L_I}/Z(L_I)U_{L_I}$ the open cell in $L_{I,ad}$.

As $U_I \subset \mathcal{U}$ and $U_I^- \subset \mathcal{U}^-$, lemma 2.4.1 implies we $U_I^- \times U_I\colon \xrightarrow{a(-,-,[id])}$ maps onto a closed subscheme of W_I and lemma 2.5.1 implies $U_I^- \Omega_{L_{I,ad}} U_I$ is a locally closed subscheme of W_I.

Moreover for every $\mathbb{C}$-algebra R, the R points of $U_I^- \Omega_{L_{I,ad}} U_I$ is the inverse image in $L^{\ltimes}G \times L^{\ltimes}G/T_I S_I$ of the R points of the open sub ind-scheme $U_I^- \times U_I \subset P_I \backslash L^{\ltimes}G \times$

$L^\ltimes G/P_I^-$. Hence $L^\ltimes G \times L^\ltimes G/T_I S_I$ acquires an ind scheme structure by realizing it as the union $\cup_{g_1,g_2 \in L^\ltimes G} g_1 U_I^- \Omega_{L_{I,ad}} U_I g_2$. It follows that the image of $L^\ltimes G \times L^\ltimes G \xrightarrow{a(-,-,e_I)} W_I \subset \mathbb{P}End^{ind}(V)$ is a sub ind-scheme isomorphic to $L^\ltimes G \times L^\ltimes G/T_I S_I$. □

Corollary 2.5.3. *Let $Orb(I) \subset X^{aff}$ be the sub ind-scheme which is the orbit of e_I defined before lemma 2.5.1. Then $X^{aff} = \bigsqcup_{I \subset \{0,\dots,r\}} Orb(I)$. Moreover $\overline{Orb(I)} = \bigsqcup_{J \supset I} Orb(J)$ and if $\overline{L_{I,ad}}$ is the wonderful compactification of $L_{I,ad}$ then $\overline{Orb(I)} \xrightarrow{\pi_I} P_I \backslash L^\ltimes G \times L^\ltimes G/P_I^-$ is an $\overline{L_{I,ad}}$ fibration over $P_I \backslash L^\ltimes G \times L^\ltimes G/P_I^-$.*

Proof. From the equality $X^{aff} = \cup_{\gamma_1,\gamma_2 \in L^\ltimes G} \gamma_1 X_0^{aff} \gamma_2^{-1}$ it follows that every $L^\ltimes G \times L^\ltimes G$ orbit intersects $\underline{X_0^{aff}}$. Further from proposition 2.4.6 it follows that every $L^\ltimes G \times L^\ltimes G$ in fact intersects $\overline{T^\ltimes_{ad,0}}$. Hence $L^\ltimes G \times L^\ltimes G$ orbits form a subset of the $T^\ltimes \times T^\ltimes$ orbits in $\overline{T^\ltimes_{ad,0}}$. Representatives for the latter are the e_I.

The parahorics $P_I, P_J \subset L^\ltimes G$ are not conjugate for $I \neq J$ hence e_I, e_J lie in distinct $L^\ltimes G \times L^\ltimes G$ orbits. Moreover $e_J \subset \overline{T^\ltimes \times T^\ltimes . e_I}$ exactly when $J \supset I$ hence the claim about $\overline{Orb(I)}$.

Further, for $J \supset I$ we have $P_J \subset P_I$ and hence $Stab(e_J) \subset P_I \times P_I^-$ and hence the morphism $Orb(I) \to P_I \backslash L^\ltimes G \times L^\ltimes G/P_I^-$ extends to $\overline{Orb(I)}$. To conclude that the fibers are the wonderful compactification one can note e_I is the identity in $End(V(\lambda_I))$ for a regular dominant weight λ_I of L_I.

□

We end with a more refined description of the orbits. In addition to being used in the proof of 2.4.3(d) we think the result is interesting in its own right.

Proposition 2.5.4. *Let $R = \mathbb{C}$ or $\mathbb{C}((t))$ and $J \subset \{0,\dots,r\}$ and $e_J \subset X^{aff}(R)$ as in lemma 2.5.1. Let W_J denote the Weyl group of the Levi factor $L_J \subset P_J$. Then*

$$Orb(J)(R) = \bigsqcup_{(w_1,w_2) \in W^{aff}/W_J \times W_J \backslash W^{aff}} \ \bigsqcup_{w_3 \in W_J} \mathcal{B}^-(R) w_1 . e_J . w_3 w_2 \mathcal{B}(R)$$

Note when $J = \emptyset$ we get $P_J^\pm = L_J = L^\ltimes G$ and $W_J = W^{aff}$ and the disjoint union becomes the usual Birkhoff decomposition. When $J = \{0,\dots,r\}$ then $P_J = \mathcal{B}$ and $P_J^- = \mathcal{B}^-$ and $W_J = 1$ and the disjoint union becomes the stratification by Shubert cells of $L^\ltimes G/\mathcal{B} \times \mathcal{B}^- \backslash L^\ltimes G$.

Proof. The given expression is stable under the action of $\mathcal{B}^-(R) \times \mathcal{B}(R)$ and disjointness of the expression is implied by the fact that $\mathcal{B}^-(R) \times \mathcal{B}(R) \cap W^{aff} = 1$. For $p \in Orb(J)(R)$ let $[p]$ denote the $\mathcal{B}^-(R) \times \mathcal{B}(R)$ class of p. To prove the result it suffices to show that for any $g_1, g_2 \in L^\ltimes G(R)$ that $[g_1 . e_J . g_2] = [w_1 w_3 . e_J . w_1]$ where w_i are as in the statement.

By proposition 2.3.3, we can write $g_i = v_i w_i l_i u_i \in U_J^-(R) W^{aff} L_J(R) U_J(R)$. This together with lemma 2.5.1 implies $[g_1 . e_J . g_2] = [w_1 l_3 . e_J . w_2]$ for some $l_3 \in L_J(R)$ and moreover we can substitute and $w_1' \in w_1 W_J$ and $w_2' \in W_J w_2$ for w_i. Let $B_{L_J} \subset L_J$ be a Borel subgroup,

U_{L_J} its unipotent radical and $U^-_{L_J}$ the opposite radical. Applying the Bruhat decomposition $L_J = U^-_{L_J} W_J B_{L_J}$ to l_3 we get $[w_1 l_3.e_J.w_2] = [w_1 v_3.e_J.w_3 u_3.w_2]$ with $v_3 w_3 u_3 \in U^-_{L_J} W_J U_{L_J}$.

We claim we can arrange for $w_1^{-1} v_3 w_1 \in \mathcal{U}^-$ and $w_2^{-1} u_3 w_2 \in \mathcal{U}$ which implies $[w_1 v_3.e_J.w_3 u_3.w_2]$ $= [w_1.e_J.w_3 w_2]$.

The claim follows from the claim': if $w \in W^{aff}$ and $-\alpha_i$ is a negative root then $w(-\alpha_i)$ is a positive root $\leftrightarrow$ there exists an reduced expression $w = s_{i_1} \cdots s_i$. To see that claim' implies the claim note that v_3 can be expressed a finite product of root subgroups $\exp X_{-\alpha_i} \cong \mathbb{G}_a$ with $\alpha_i \in Lie(L_J)$ and the claim holds for v_3 if it holds for each of these $\mathbb{G}_a$. But by replacing w_1 with w_1' as described above we ensure there is not a reduced expression for w_1 ending in s_i and similarly for w_2.

To prove claim' note the $\Leftarrow$ direction follows from [BK05a, pg.61]. By abuse of notation let w denote a reduced word for w. If $w'' = ws_i$ is not reduced then by the exchange property for reflection groups we can find a reduced expression for w ending in s_i hence we must show w'' is not reduced. If it is reduced then $w''(-\alpha_i)$ is a positive root by the $\Leftarrow$ direction. But $w''(-\alpha_i) = w(\alpha_i) = -w(-\alpha_i)$ and $w(-\alpha_i)$ was assumed to be positive, a contradiction. $\square$

Cartier Results

The next step towards proving theorem 2.4.3(d) requires developing some results about divisors in LG and in X^{aff}.

Proposition 2.5.5. *Both $X^{aff} - L^{\ltimes}G/Z(G)$ and $X^{aff} - X_0^{aff}$ consists of Cartier divisors*

Proof. Let $D_i \subset X^{aff} - L^{\ltimes}G/Z(G)$ be a component. It's is evidently $L^{\ltimes}G \times L^{\ltimes}G$ stable hence the Cartier condition is uniquely determined by $D_i|_{X_0^{aff}}$ as the $L^{\ltimes}G \times L^{\ltimes}G$ translates cover X^{aff}. Recall $t^{-\alpha_i}$ is the ith coordinate function on $\overline{T^{\ltimes}_{ad,0}} = \mathbb{A}^{r+1}$; let also $t^{-\alpha_i}$ denote the pullback function to X_0^{aff} then $D_i|_{X_0^{aff}} = \{t^{-\alpha_i} = 0\}$.

Let F_i denote the components of $X^{aff} - X_0^{aff}$. Using proposition 2.5.4 (and other) we conclude that $F_i = \overline{\mathcal{U}^- s_i \mathbb{C}^\times \ltimes \mathcal{B}}$ and hence the F_i are $\mathcal{B}^- \times \mathcal{B}$ stable. For $w_1, w_2 \in W^{aff}$ we show F_i is Cartier on $w_1 X_0^{aff} w_2^{-1}$ and by 2.5.4 we be able to show that $\mathcal{B}^- \times \mathcal{B}$ translates of all of the $w_1 X_0^{aff} w_2^{-1}$ cover X^{aff}.

We have

$$w_1 X_0^{aff} w_2^{-1} = w_1 \mathcal{U}^- w_1^{-1} w_1 \overline{T^{\ltimes}_{ad,0}} w_2^{-1} w_2 \mathcal{U} w_2^{-1}$$

Further by [Kum02, pg.169(7),pg.227(1)] we have the decompositions as (ind)-schemes

$$w_1 \mathcal{U}^- w_1^{-1} = \left(w_1 \mathcal{U}^- w_1^{-1} \bigcap \mathcal{U}^-\right) \cdot \left(w_1 \mathcal{U}^- w_1^{-1} \bigcap \mathcal{U}\right)$$

And moreover $\left(w_1 \mathcal{U}^- w_1^{-1} \bigcap \mathcal{U}\right) \cong \mathbb{A}^{n_1}$ is a finite dimensional unipotent group. A similar statement holds for $w_2 \mathcal{U} w_2^{-1}$. And because F_i is preserved by $\mathcal{B}^- \times \mathcal{B}$ it is enough to show $F_i|_{\mathbb{A}^{n_1} w_1 \overline{T^{\ltimes}_{ad,0}} w_2^{-1} \mathbb{A}^{n_2}}$ is Cartier which holds because this is a smooth finite dimensional variety.

Finally, to see that the W be the union of all $\mathcal{B}^- \times \mathcal{B}$ translates of all $w_1 X_0^{aff} w_2^{-1}$. It is an open sub ind-scheme of X^{aff} which by proposition 2.5.4 contains all the closed points. We have $Z = X^{aff} - W$ is closed, intersects some $g_1 X_0^{aff} g_2^{-1}$ and has no closed points. But the latter is an affine ind-scheme hence so is $Z \cap g_1 X_0^{aff} g_2^{-1}$ and any affine scheme has a closed point hence $Z = \emptyset$. □

Let s_i for $i = 0, \dots, r$ denote the simple reflections in W^{aff}. The product $\mathcal{U}^- s_i \mathcal{B}$ defines a sub ind-scheme of LG and more over the closure E_i of $\mathcal{U}^- s_i \mathcal{B}$ in LG is given by $\bigsqcup_{v \geq s_i} \mathcal{U}^- v \mathcal{B}$ where $\geq$ is the Bruhat order on W^{aff}, see [Kum02, 1.3.15].

In fact E_i is a Cartier divisor on LG. To see this let P_i be the maximal parahoric subgroup defined in example 4. There is a highest weight representation $V(i)$ with highest weight line $[v] \subset \mathbb{P}V(i)$ such that $P_i = Stab([v])$. Let v^* be the dual vector to v and then $\gamma \mapsto v^*(\gamma v)$ is a section of a line bundle on LG whose vanishing set is exactly E_i. By abuse of notation we let E_i denote the Cartier divisors on $L^{\ltimes} G$ obtained by pullback.

Let $\overline{E_i}$ denote the closure of E_i in X^{aff}.

Lemma 2.5.6. *For $i = 0, \dots, r$ the $\overline{E_i}$ are $\mathcal{B}^- \times \mathcal{B}$ stable and $X^{aff} - X_0^{aff} = \cup_{i=0}^r \overline{E_i}$.*

Proof. From $E_i = \bigsqcup_{v \geq s_i} \mathcal{U}^- v \mathcal{B}$ it follows that E_i is $\mathcal{B}^- \times \mathcal{B}$ stable hence so is $\overline{E_i}$. This together with proposition 2.5.4 allows us to reduce $p \in X^{aff} - X_0^{aff}$ of the form $p = w_1.e_J.w_2$ for e_J as in lemma 2.5.1 and $w_i \in W^{aff}$ with $w_1 w_2 \neq 1$. There is a co-character $\eta \colon \mathbb{C}^\times \to T^{\ltimes}$ such that $p = \lim_{t \to 0} w_1 \eta(t) w_2$. Then $p(t) = w_1 w_2 (w_2^{-1} \eta(t) w_2) \in \mathcal{U}^- w_1 w_2 \mathcal{B} \subset E_i$ for some i. Hence $p \in \overline{E_i}$ for some i. □

Proposition 2.5.7. *For $i = 0, \dots, r$ the sub ind-schemes $\overline{E_i}$ are Cartier divisors on X^{aff}.*

Proof. We begin by showing $\overline{E_i}$ is Cartier on $w_1 X_0^{aff} w_2^{-1}$ for $w_i \in W^{aff}$. For any $w \in W^{aff}$ lemma 2.3.4 implies that

$$w \mathcal{U} w^{-1} = \Big(w \mathcal{U} w^{-1} \cap \mathcal{U}^- \Big) \cdot \Big(w \mathcal{U} w^{-1} \cap \mathcal{U} \Big) =: N_w \cdot N'_w \tag{2.13}$$

$$w \mathcal{U}^- w^{-1} = \Big(w \mathcal{U}^- w^{-1} \cap \mathcal{U}^- \Big) \cdot \Big(w \mathcal{U}^- w^{-1} \cap \mathcal{U} \Big) =: M_w \cdot M'_w. \tag{2.14}$$

Note we also have $\mathcal{U} = M'_w N'_w$ and $\mathcal{U}^- = M_w N_w$ and N_w, M'_w are finite dimensional normal schemes. As $\overline{E_i}$ is $\mathcal{B}^- \times \mathcal{B}$ stable it is in particular $M_w \times N'_w$ stable. Using proposition 2.4.6 and the above decompositions we get

$$\begin{aligned} w_1 X_0^{aff} w_2^{-1} &\cong M_{w_1} M'_{w_1} w_1 \overline{T^{\ltimes}_{ad,0}} w_2 N_{w_2} N'_{w_2} \\ w_1 X_0^{aff} w_2^{-1} \cap \overline{E_i} &= w_1 X_0^{aff} w_2^{-1} \cap \overline{w_1 X_0^{aff} w_2^{-1} \cap E_i} \\ E_i &= \bigsqcup_{v \geq s_i} M_{w_1} N_{w_1} v T^{\ltimes}_{ad} M'_{w_2} N'_{w_2} \end{aligned}$$

Altogether we conclude that there is a projection $w_1 X_0^{aff} w_2^{-1} \xrightarrow{\pi} M'_{w_1} w_1 \overline{T^{\ltimes}_{ad,0}} w_2 N_{w_2}$ and $w_1 X_0^{aff} w_2^{-1} \cap \overline{E_i}$ agrees with the pullback along π of $M'_{w_1} w_1 \overline{T^{\ltimes}_{ad,0}} w_2 N_{w_2} \cap \overline{E_i}$. The latter is the closure of a Cartier divisor in a finite dimensional normal variety hence also Cartier.

This shows that $\overline{E_i}$ is Cartier on all the open sub ind-scheme of all $\mathcal{B}^- \times \mathcal{B}$ translates of all the $w_1 X_0^{aff} w_2^{-1}$. We claim this is all of X^{aff}. If not then the complement Z is a closed subscheme which by proposition 2.5.4 contains no closed point. However Z must intersect some $\gamma_1 X_0^{aff} \gamma_2^{-1}$ for $\gamma_i \in L^{\ltimes} G(\mathbb{C})$. But this intersection defined a closed sub ind-scheme of an affine ind-scheme which necessarily contains a closed point hence Z must be empty. □

Remark using the map $V \to \wedge^m V$ in (2.12) one can easily obtain that some multiple of D_i is Cartier; however to show D_i itself is Cartier we don't know of a simpler argument.

The following result is crucial for proving theorem 2.4.3(d); the argument given below was conveyed to me by Sharwan Kumar

Proposition 2.5.8. *$Pic(\mathcal{U}^-) = 0$.*

Proof. For any $w \in W^{aff}$ we have a Schubert variety $\mathcal{B} w \mathcal{B}/\mathcal{B} \subset LG/\mathcal{B}$; set $\mathcal{U}^-_w = \mathcal{U}^- \cap \mathcal{B} w \mathcal{B}/\mathcal{B}$. In fact $\mathcal{U}^- \subset LG/\mathcal{B}$ and we get an ind-structure on $\mathcal{U}^- = \bigcup_n \mathcal{U}^-_n$ where $\mathcal{U}^-_n = \bigcup_{l(w) \leq n} \mathcal{U}^-_w$. We show $Pic(\mathcal{U}_w) = 0$ for all w.

Fix w and abbreviate $Y = \mathcal{U}^-_w$. For any $k \in \mathbb{N}$ we have a short exact sequence $\mathbb{Z}/k \to \mathcal{O}^*_Y \xrightarrow{f \mapsto f^k} \mathcal{O}^*_Y$; using that $H^1_{et}(Y, \mathcal{O}^*_Y) \cong Pic(Y)$ and looking at the long exact sequence in étale cohomology we get

$$\cdots \to H^1_{et}(Y, \mathbb{Z}/k) \to Pic(Y) \to Pic(Y) \to H^2_{et}(Y, \mathbb{Z}/k) \to \cdots$$

By the proof of [Kum02, 7.4.17], Y is contractible and because $H^*_{et}(-, \mathbb{Z}/k) = H^*_{singular}(-, \mathbb{Z}/k)$ it follows that the outer terms vanish and $Pic(Y) \xrightarrow{L \mapsto L^{\otimes k}} Pic(Y)$ is an isomorphism for any k. We now show $Pic(Y)$ is finitely generated and together with the previous statement it will follow that $Pic(Y) = 0$.

Y is a normal variety with $\dim Y = l(w)$ so by [Ful98, 2.1.1] $Pic(Y)$ embeds in the Chow group $Pic(Y) \subset A_{l(w)-1}(Y)$. So reduce to showing $A_{l(w)-1}(Y)$ is finitely generated. By [Ful98, 1.8] there is a surjection $A_{l(w)-1}(\mathcal{B} w \mathcal{B}/\mathcal{B}) \to A_{l(w)-1}(Y)$. By [Ful98, 19.1.11b] $A_{l(w)-1}(Y) = H_{2(l(w)-1)}(\mathcal{B} w \mathcal{B}/\mathcal{B}, \mathbb{Z})$ and finally the Bruhat decomposition implies the latter group is finitely generated. □

Corollary 2.5.9. $Pic(\mathcal{U}) = Pic(Pic(\overline{T^{\ltimes}_{ad,0}}\mathcal{U}) = Pic(X_0^{aff}) = 0$

Proof. The group $\mathcal{U}$ is a pro-unipotent pro group [Kum02, 4.4]. In particular there is a family $\mathcal{F}$ or normal subgroups $N \subset \mathcal{U}$ such that for $N \in \mathcal{F}$ the quotient $\mathcal{U} \xrightarrow{\pi_N} \mathcal{U}/N$ is a morphism and $\mathcal{U}/N$ is a finite dimensional unipotent group. Moreover, the open sets $\pi_N^{-1}(W) \subset \mathcal{U}$ for $W \subset \mathcal{U}/N$ open form a base for the topology of $\mathcal{U}$.

Write $\mathcal{U}/N = \operatorname{Spec} A_N$. If $N, N' \in \mathcal{F}$ then $N'' = N \cap N' \in \mathcal{F}$. Therefore we have injective ring homomorphisms $A_N \to A_{N''}$ and $A_{N'} \to A_{N''}$. In this way $\bigcup_{N \in \mathcal{F}} A_N$ inherits a ring structure. We can present $\mathcal{U} = \operatorname{Spec} \bigcup_{N \in \mathcal{F}} A_N$.

Let L be a line bundle on $\mathcal{U}$. We can trivialize L on finitely many affine open subschemes and so L is determined by finitely many transition functions t_{ij}. Making the open sets smaller we can t_{ij} to be defined on some $\pi_N^{-1}(W)$. Let N' be the intersections of all N that appear in this way. Then L is pulled back from a line bundle on $\operatorname{Spec} A_{N'}$ and as A_N is polynomial ring we have $Pic(\operatorname{Spec} A_{N'}) = 0$ hence $Pic(\mathcal{U}) = 0$.

This immediately implies that $Pic(\overline{T^{\ltimes}_{ad,0}}\mathcal{U}) = 0$ as $\overline{T^{\ltimes}_{ad,0}} \cong \mathbb{A}^{r+1}$.

By the previous proposition we $Pic(\mathcal{U}_n^-) = 0$. Writing $\mathcal{U}_n^- = \operatorname{Spec} B_n$ and replacing A_N with $B_n \otimes A_N$ in the argument above allows us to conclude that $Pic(\mathcal{U}_n^- \overline{T^{\ltimes}_{ad,0}}\mathcal{U}) = 0$ for all n from which it follows that $Pic(X_0^{aff}) = 0$. $\square$

Proposition 2.5.10. *Theorem 2.4.3(d) holds: $X^{aff} - X_0^{aff}$ is a Cartier divisor and with $r+1$ components which freely generate $Pic(X^{aff})$.*

Proof. Since for each n the components of $X_n^{aff} - X_{0,n}^{aff}$ are Cartier and $Pic(X_{0,n}^{aff}) = Pic(\mathcal{U}_n^- \overline{T^{\ltimes}_{ad,0}}\mathcal{U}) = 0$. We get that the components of $X_n^{aff} - X_{0,n}^{aff}$ generate $Pic(X_n^{aff})$.

A relation among these generators is a principal divisor (f) which is invertible on X_0^{aff}. As the boundary is $\mathcal{B}^- \times \mathcal{B}$ stable the function the values of f on X_0^{aff} is determined by its restriction to $\overline{T^{\ltimes}_{ad,0}}$. It follows that f must be constant c and $f - c$ is zero on a dense open set hence there are no relations. $\square$

2.6 Polynomial Loop Group

We can establish a result analogous to theorem 2.4.3 for the polynomial loop group. In this setting everything is easier. Fix a highest representation $V = V(0, \lambda, l) = \oplus_k V_k$. Define the restricted dual $V_{res}^* := \oplus V_k^*$.

Recall for an integer k we use $n(k)$ to indicate that $(LG)_k$ maps $\mathbb{P}V_{\leq k}$ into $\mathbb{P}V_{\leq n(k)}$

Let R be a $\mathbb{C}$-algebra. Define a $\mathbb{C}$ space V^* by the assignment $R \mapsto \hom_R(V_R, R) =: V_R^*$. Then every $\phi \in End(V)(R)$ defines a dual map $\phi^* \colon V_R^* \to V_R^*$. Define a$\mathbb{C}$-space $End^{res}(V)$ by

$$End^{res}(V)(R) = \{\phi \in End(V)(R) | \phi^*(V_{res,R}^*) \subset V_{res,R}^*\}$$

For each i define a sub $\mathbb{C}$ space $End^{poly}(V)_i$ by

$$End^{poly}(V)_i(R) = \{\phi \in End^{res}(V)(R) | \phi(V_{\leq k,R}) \subset V_{\leq n(k+i),R} \text{ and } \phi^*(V_{\leq k,R}^*) \subset V_{\leq n(k+i),R}^*\}. \tag{2.15}$$

Then the R values points of these End spaces have R-module structures and we can form their projectivizations by looking at rank 1 projective submodules that have projective quotients. Moreover $End^{poly}(V)_i$ is a scheme and $id \in End^{poly}(V)_i$ for every i. Let $\mathbb{P}End^{poly}(V)$ be the ind-scheme $\cup_i \mathbb{P}End^{poly}(V)_i End^{poly}(V)_i$. Then the $L_{poly}^{\ltimes}G \times L_{poly}^{\ltimes}G$ orbits of $[id]$ gives an embedding of $L_{poly}^{\ltimes}G/Z(G)$ in $\mathbb{P}End^{poly}(V)$. The ind-scheme structure on $L_{poly}G$ comes from choosing an embedding $G \to SL(V)$ and writing elements as finite sums $\sum_{i=n}^m a_i z^i$ with $a_i \in End(V)$. Then $(L_{poly}G)_k$ is the scheme where the sums range from $n = -k$ to $m = k$.

Construction 2. Let $(0,\lambda,l)$ be a regular dominant weight of $L_{poly}G$ and let $V = V(0,\lambda,l)$. For every $k \geq 0$ we have a locally closed embedding $(L^{\ltimes}_{poly}G/Z(G))_k = \mathbb{C}^\times \times (L_{poly}G)_k \subset \mathbb{P}End^{poly}(V)_{n(k)}$. Define

$$\begin{aligned} {X^{aff,poly}}_k &= \overline{(L^{\ltimes}_{poly}G/Z(G))_k} \subset \mathbb{P}End^{poly}(V)_{n(k)} \\ X^{aff,poly} &= \cup_k {X^{aff,poly}}_k \end{aligned} \tag{2.16}$$

Recall the definition of the affine ind-scheme $\mathbb{P}_{[v\otimes v^*]}End(V)$ before lemma 2.4.1. Define

$$\begin{aligned} {X^{aff,poly}}_{0,k} &= {X^{aff,poly}}_k \cap \mathbb{P}_{[v\otimes v^*]}End^{ind}(V)_{n(k)} \\ {X^{aff,poly}}_0 &= \cup_k {X^{aff,poly}}_{0,k} \end{aligned} \tag{2.17}$$

Corollary 2.6.1. *theorem 2.4.3 holds for* $G[z^{\pm}]$ *Let* G *be a simple, connected and simply connected group over* $\mathbb{C}$ *and set* $r = rk(G)$. *The ind-scheme* X^{aff} *contains* G^{aff}_{ad} *as a dense open sub-ind scheme and further*

(a) $X^{aff,poly}$ *is formally smooth and independent of the choice of regular dominant weight* $(0,\lambda,l)$.

(b) The boundary $X^{aff,poly} - L^{\ltimes}_{poly}G/Z(G)$ *is a Cartier divisor with* $r+1$ *components* $D_0,\dots,D_r$. *The* $L^{\ltimes}_{poly}G \times L^{\ltimes}_{poly}G$ *orbits closures are in bijection with subsets* $I \subset \{0,\dots,r\}$ *in such a way that to* I *we associate* $\cap_{i\in I}D_i$.

(c) Each D_i *is formally smooth and* $\cup_{i=0}^r D_i$ *is locally a product* $S \times Z$ *where* S *is an ind-scheme and* Z *is the union of hyperplanes in* $\mathbb{A}^{r+1}$.

(d) $X^{aff,poly} - {X^{aff,poly}}_0$ *is a Cartier divisor and with* $r+1$ *components which freely generate* $Pic(X^{aff})$.

(e) Let $Orb(I) \subset X^{aff,poly}$ *be the sub ind-scheme which is the orbit of* e_I *defined before lemma 2.5.1. Then* $X^{aff,poly} = \bigsqcup_{I\subset\{0,\dots,r\}} Orb(I)$. *Moreover* $\overline{Orb(I)} = \bigsqcup_{J\supset I} Orb(J)$ *and if* $\overline{L_{I,ad}}$ *is the wonderful compactification of* $L_{I,ad}$ *then* $\overline{Orb(I)} \xrightarrow{\pi_I} P_I\backslash L^{\ltimes}G \times L^{\ltimes}G/P_I^-$ *is an* $\overline{L_{I,ad}}$ *fibration over* $P_I\backslash L^{\ltimes}_{poly}G \times L^{\ltimes}_{poly}G/P_I^-$.

Proof. As $L_{poly}G \subset LG$ and all the $\mathbb{C}$-utilized above are sub $\mathbb{C}$-spaces utilized to embed $L^{\ltimes}G$ the proof of thm 2.4.3 and corollary 2.5.3 applies to $L^{\ltimes}_{poly}G$. □

2.7 Stacky Modification

We would like to use the boundary $\partial X^{aff,poly} = X^{aff} - L^{\ltimes}G/Z(G)$ to study the moduli of G-bundles on nodal curves. However to be able to establish results about about G-bundles as opposed to G_{ad}-bundles we need to make a technical modification.

Here we work exclusively with the polynomial loop group and the polynomial embedding $X^{aff,poly}$. By abuse of notation we use the same notation $H \subset LG$ for a subgroup to denote $H \cap L_{poly}G$.

To be more precise from 2.6.1(e) we have that $\partial X^{aff,poly}$ has $r+1$ components. And the ith component D_i fibers over $P_i \backslash L_{poly}G \times L_{poly}G/P_i^-$ with fiber the wonderful compactification $\overline{L_{i,ad}}$ of a Levi factor L_i of P_i.

In this section we construct and ind-stack $\partial \mathcal{X}^{aff}_{poly}$ with $r+1$ components such that the ith component $\mathcal{D}_i$ fibers over $P_i \backslash L_{poly}G \times L_{poly}G/P_i^-$ with the fiber being a stacky compactification of L_i.

Remark 2. Even when $G = G_{ad}$, as happens for $G = E_8$ this construction is still needed because there are maximal parahorics $P_i \subset L_{poly}E_8$ that have Levi factors $L_i \neq L_{i,ad}$; specifically A_8, D_8 appear among the L_i.

Remark 3. For any semi simple G has a so called canonical embedding which is an equivariant compactification of G. In general the canonical embedding is not smooth but it is smooth for $G = Sp_{2n}$. Thus for $G = Sp_{2n}$ we can take $\partial \mathcal{X}^{aff}_{poly}$ to be and ind-scheme.

Brion and Kumar's Toroidal compactification

This first step is to construct stacky compactification of semi simple groups. We do this by modifying a construction of Brion and Kumar [BK05b, 6.2.4].

Remark 4. The construction here gives an alternative construction of stacky wonderful compactifications of G of Johan and Martens [MTa]. Moreover we calculate the Picard group of this compactification 2.7.3(b) which seems to be a new result.

Construction 3. Let σ be a cone with suppose in the negative Weyl chamber. To σ we associate a quasi-projective $G \times G$-equivariant embedding $Y(\sigma)$ of G with an equivariant map $Y(\sigma) \to \overline{G_{ad}} = X$.

The construction depends on a choice $\mu_1, \dots, \mu_m$ of generators for the dual cone $\Lambda_T \cap \sigma^\vee$ and a choice of a regular dominant weight λ such that $\mu_i + \lambda$ are also regular dominant weights. Consider the closure

$$\overline{G \times G.[id_\lambda \oplus_i id_{\lambda+\mu_i}]} \subset \mathbb{P}\Big(End(V(\lambda)) \bigoplus_i End(V(\lambda+\mu_i))\Big)$$

and the rational $G \times G$-equivariant map $p_\sigma \colon \mathbb{P}\Big(End(V(\lambda)) \bigoplus_i End(V(\lambda+\mu_i))\Big) \dashrightarrow \mathbb{P}(End(V(\lambda)))$.

Let W be the maximal open subscheme on which p_σ is defined. We set $Y(\sigma, \mu_i, \lambda) = \overline{G \times G.[id_\lambda \oplus_i id_{\lambda+\mu_i}]} \cap p_\sigma^{-1}(X)$; it is closed subscheme of W hence quasi-projective. Set $p_{\sigma,\mu_i,\lambda} \colon Y(\sigma, \mu_i, \lambda) \to X$ be the restriction of p_σ to $Y(\sigma, \mu_i, \lambda)$.

In the appendix to this section we show in lemma 2.7.4 that $Y(\sigma, \mu_i, \lambda)$ is does not depend on the choice of μ_i, λ and we simply denote it as $Y(\sigma)$ and similarly we write map to X simply as $Y(\sigma) \xrightarrow{p_\sigma} X$.

Construction 4. Let $\tau \subset \sigma$ be cones with support in the negative Weyl chamber. Then $\sigma^\vee \subset \tau^\vee$. Let $\mu_1, \ldots, \mu_m$ be generators for $\sigma^\vee \cap \Lambda_T$ and complete this set to a set of generators $\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n$ for $\tau^\vee \cap \Lambda_T$. Choose a regular dominant weight λ according to construction 3. The projection

$$\mathbb{P}\bigg(End(V(\lambda)) \bigoplus_i End(V(\lambda+\mu_i)) \bigoplus_j End(V(\lambda+\nu_j))\bigg) \dashrightarrow \mathbb{P}\bigg(End(V(\lambda)) \bigoplus_i End(V(\lambda+\mu_i)) \bigoplus_j)\bigg)$$

gives a map $i_{\tau,\sigma} \colon Y(\tau) \to Y(\sigma)$. This map is an open immersion: from proposition 2.1.3(d), it suffices to check this on $p_\tau^{-1}(X_0) = U^- \overline{T}_\tau U$ and here the morphism is determined by $\overline{T}_\tau \to \overline{T}_\sigma$ which is the open immersion determined by the morphism of toric varieties corresponding to the inclusion of cones $\tau \subset \sigma$.

Let $\sigma_1, \sigma_2, \sigma_3$ be cones with support in the negative Weyl chamber such that $\tau_{i,j} := \sigma_i \cap \sigma_j \neq \emptyset$. Then the $Y(\sigma_i)$ can be glued along $Y(\tau_{i,j})$; that is, the maps $\phi_{i,j} = i_{\tau_{i,j},\sigma_j} \circ i^{-1}_{\tau_{i,j},\sigma_i}$ satisfy the co-cycle condition $\phi_{2,3} \circ \phi_{1,2} = \phi_{1,3}$ where the domain of these functions is the image of $Y(\sigma_1 \cap \sigma_2 \cap \sigma_3) \to Y(\sigma_1)$. The same steps as in the previous paragraph reduce the assertion to showing $\overline{T}_{\sigma_i}$ can be glued together along $\overline{T}_{\tau_{i,j}}$ which follows from the general theory of toric varieties.

Finally, for any fan Σ with support in the negative Weyl chamber we can glue together all the embeddings $Y(\sigma)$ for $\sigma \subset \Sigma$ a cone and form an embedding $Y(\Sigma) = \cup_{\sigma \subset \Sigma} Y(\sigma)$.

Let G be a connected reductive group. In [BK05a, 6.2], Brion and Kumar define a G-embedding Y to be a normal $G \times G$ variety containing $G = (G \times G)/diag(G)$ as an open orbit. They call X *toroidal* if the quotient map $G \to G/Z(G) = G_{ad}$ extends to a map $Y \to \overline{G_{ad}} =: X=$ the wonderful compactification of G_{ad}. In fact, toroidal has a more general definition in the theory of spherical varieties but we will not need this level of generality.

The embeddings $Y(\Sigma)$ give examples of toroidal embeddings; in fact these are all of them:

Proposition 2.7.1. *Let Y be a toroidal G-embedding and $\overline{T} \subset Y$ the associated toric variety. Then the fan of $\overline{T}$ is of the form $\cup_{w \in W} w \cdot \Sigma$ where W is the Weyl group and Σ is a fan with support in the negative Weyl chamber and Y is $G \times G$ equivariantly isomorphic to $Y(\Sigma)$.*

Further, for each cone $\sigma \subset \Sigma$ we have a morphism $p_\sigma \colon Y(\sigma) \to X$ and a $U^- \times U$ equivariant isomorphism $p_\sigma^{-1}(X_0) \cong U^- \times U \times \overline{T}_\sigma$.

Proof. See [BK05a, 6.2.4]. For the last statement see lemma 2.7.4(a). □

By a stack we mean a stack in the $fpqc$ topology. We could work with the étale topology in this section but the $fpqc$ topology is needed to discuss moduli of bundles so for uniformity we stick with the $fpqc$.

Using proposition 2.7.1 we construct a stack $\mathcal{Y}$ with an action of $G \times G$ containing G as a dense open subscheme. In analogy with 2.7.1 the construction of $\mathcal{Y}$ will be determined by the closure $\mathcal{S} = \overline{T}$ in $\mathcal{Y}$. In turn there is an open affine sub stack $\mathcal{S}_0$ such that $\mathcal{S} = \cup_{w \in W} w \mathcal{S} w^{-1}$.

In fact $\mathcal{S}_0$ will be an affine toric stack as described in [GSa]. We use some notation from [GSa] but we don't need the full machinery of toric stacks to describe $\mathcal{S}_0$. Indeed $\mathcal{S}_0 \cong [\mathbb{A}^r/Z(\beta)]$ where $Z(\beta)$ is a finite group.

Let G be semi simple, $T \subset G$ and fix an ordering $\alpha_1, \ldots, \alpha_r$ of the simple roots. The toric stack $\mathcal{S}_0$ is a stacky resolution of the affine toric variety associated to the Weyl chamber $C = \{v \in \hom(\mathbb{C}^\times, T) \otimes \mathbb{Q} | \alpha_i(v) \geq 0\}$. Let $u_1, \ldots, u_r$ be generators of the rays of $C \cap \hom(\mathbb{C}^\times, T)$. Let $e_i \in \mathbb{Z}^r$ be the vector with 1 in the ith entry and 0 elsewhere. We get a morphism of lattices

$$\beta \colon \mathbb{Z}^r \xrightarrow{e_i \mapsto u_i} \hom(\mathbb{C}^\times, T). \tag{2.18}$$

Noting that $T \cong \hom(\hom(T, \mathbb{C}^\times), \mathbb{C}^\times)$ we see β induces a morphism $\beta \colon T' \to T$ where $T' := \hom(\hom(\mathbb{Z}^r, \mathbb{Z}), \mathbb{C}^\times)$. Let $Z(\beta) = \ker(T' \xrightarrow{\beta} T)$. Let $\mathbb{A}^r_b$ denote the affine toric variety corresponding to the cone in $\hom(\mathbb{C}^\times, T') \otimes \mathbb{Q}$ generated by e_i. The map $T' \to T$ is surjective and as $\dim T' = \dim T$ it follows that $Z(\beta)$ is finite.

Proposition 2.7.2. *Let G be semi simple and $T \subset G$ a maximal torus and C the Weyl chamber. The affine toric variety corresponding to C is isomorphic to the GIT quotient $\mathbb{A}^r_b//Z(\beta)$. When $G = E_8, Sp_{2n}$ the toric variety $\mathbb{A}^r_b//Z(\beta)$ is smooth. Let $(\mathbb{C}^\times)^r$ act on $T \times \mathbb{A}^r_b$ via $(t, v) \xrightarrow{u} (\beta(u)t, uv)$. Then there is an isomorphism of toric varieties $(T \times \mathbb{A}^r_b)//(\mathbb{C}^\times)^r \cong \mathbb{A}^r_b//Z(\beta)$*

Proof. The first statement follows from the Cox construction of toric varieties [Cox95]. When $G = E_8$ the group $Z(\beta) = 1$ because E_8 is an adjoint group. For $G = Sp_{2n}$ see [Hur, pg. 14].

For the final statement we construct an isomorphism from the associated coordinate rings. The invariants in the coordinate ring $\mathbb{C}[\mathbb{A}^r_b]$ are generated by monomial invariants. Let m be such an invariant. Let ν be the weight by which T acts on m. Then $\nu^{-1} \otimes m$ is a monomial invariant in $\mathbb{C}[T \times \mathbb{A}^r_b]$ for the action of $(\mathbb{C}^\times)^r$. This defines the required isomorphism. □

Construction 5. Let G be a semi simple, connected, and simply connected group of rank r and T a maximal torus. Set $T' = (\mathbb{C}^\times)^r$ and let $e_i \in \hom(\mathbb{C}^\times, T')$ and $\beta \colon \hom(\mathbb{C}^\times, T') \to \hom(\mathbb{C}^\times, T)$ be as in (2.18) and $\beta^* \colon \hom(T, \mathbb{C}^\times) \to \hom(\mathbb{C}^\times, T')$ the dual map. Set $H = T' \times G$.

Let $N \subset \hom(\mathbb{C}^\times, T')$ be the cone generated by the e_i. So that the toric variety associated to N is $\mathbb{A}^r_b$ with coordinate functions $a_1, \ldots, a_r \in \hom(T', \mathbb{C}^\times)$. Let $C_\Delta \subset \hom(\mathbb{C}^\times, T') \otimes \mathbb{Q} \oplus \hom(\mathbb{C}^\times, T) \otimes_{\mathbb{Q}}$ be the cone generated by all $(v, \beta(v))$ for $v \in N$. The dual cone is then generated by $(a_i, 0)$ and $(\beta^*(\omega), \omega)$ for $\omega \in \hom(T, \mathbb{C}^\times)$ and the toric variety $\overline{T' \times T}_{C_\Delta} \cong \mathbb{A}^r_b \times T$.

According to proposition 2.7.1 there is toroidal embedding $Y(C_\Delta)$ of $T' \times G$. Under the left and right action of T' the diagonal $\Delta(T)$ acts trivially so we identify T' with $T' \times T'/\Delta(T')$ and we define an embedding of G as the global quotient

$$\mathcal{X} = \overline{G} = [Y(C_\Delta)/T'] \tag{2.19}$$

As the morphism $Y(C_\Delta) \xrightarrow{p_{C_\Delta}} X$ is $T' \times T'$ equivariant this morphism descends to $\mathcal{X} \to X$ which we abusively also denote by p_{C_Δ}. We denote by $\mathcal{X}_0$ the pre image of X_0 under this map.

Theorem 2.7.3. *Let G be a semi simple, connected, simply conneted group and $\mathcal{X}$ as in definition* (2.19)*. Then:*

(a) $\mathcal{X}$ is smooth and proper.

(b) $\mathcal{X} - \mathcal{X}_0$ is of pure codimension 1 *and we have an exact sequence*

$$0 \to \mathbb{Z}^r \to Pic(\mathcal{X}) \to \hom(Z(\beta), \mathbb{C}^\times) \to 0$$

where the subgroup $\mathbb{Z}^r$ is generated by the irreducible components of $\mathcal{X} - \mathcal{X}_0$.

(c) The boundary $\mathcal{X} - G$ consists of r divisors $D_1, \dots, D_r$ with simple normal crossings and the closure of the $G \times G$-orbits are in bijective correspondence with subsets $I \subset \{1, \dots, r\}$ in such a way that to I we associate $\cap_{i \in I} D_i$.

Proof. By proposition 2.7.1 for $p_{C_\Delta} \colon Y(C_\Delta) \to X$ we have $p_{C_\Delta}^{-1}(X_0) \cong U^- \times \mathbb{A}_b^r \times T \times U$. Thus we have

$$\begin{aligned} \mathcal{X}_0 :=& [(U^- \times \mathbb{A}_b^r \times T \times U)/T'] \\ \cong& U^- \times [(\mathbb{A}_b^r \times T)/T'] \times U \\ \cong& U^- \times [\mathbb{A}_b^r / Z(\beta)] \times U. \end{aligned}$$

As $[\mathbb{A}_b^r/Z(\beta)]$ is the quotient of a smooth scheme by a smooth group it is smooth and thus $\mathcal{X}_0$ is smooth and its translates cover $\mathcal{X}$

Further the map $\mathcal{X}_0 \to X_0$ is given by $U^- \times [\mathbb{A}_b^r/Z(\beta)] \times U \xrightarrow{(id,f,id)} U^- \times \overline{T_{ad,0}} \times U$. The map f is the composition $[\mathbb{A}_b^r/Z(\beta)] \to \overline{T}_c \xrightarrow{f'} \overline{T_{ad,0}}$ where $\overline{T}_c$ is the course moduli space of $[\mathbb{A}^r/Z(\beta)]$. We will show f is affine and finite hence proper; this reduces to showing f' is affine and finite; the first property is clear; finiteness follows because the roots of G are finite index in the weights of G. We conclude that $\mathcal{X}$ is finite over the projective scheme X hence (a).

Let X_c denote the coarse moduli space for $\mathcal{X}$; it comes with a surjection $X_c \xrightarrow{p_c} X$. To prove (b) we use that $Pic(\mathcal{X}) = Cl(X_c)$ where the latter denotes the Weil divisor class group; this is shown in remark 3.4 of [GSb]. Let $X_{c,0}$ denote the coarse moduli space of $\mathcal{X}_0$. Then $X_{c,0} = U^- \times \overline{T_c} \times U$ is affine and the complement is pure codimension 1 and we have an exact sequence

$$\mathbb{Z}^r \xrightarrow{i} Cl(X_c) \to Cl(X_{c,0}) \cong Cl(\overline{T_c}). \tag{2.20}$$

After tensoring (2.20) with $\mathbb{Q}$ we get a surjection $\mathbb{Q}^r \xrightarrow{i_\mathbb{Q}} Pic(X_c)_\mathbb{Q}$ as $\mathbb{Q}$ is flat over $\mathbb{Z}$. In fact $i_\mathbb{Q}$ must be an isomorphism and consequently i must be injective.

Now $Cl(\overline{T}_c) = Pic([\mathbb{A}^r/Z(\beta)])$ and the latter is equal to the group of $Z(\beta)$-equivariant line bundles on $\mathbb{A}^r$. Any line bundle on $\mathbb{A}^r$ is trivial and an equivariant structure is uniquely determined by an element of $\hom(Z(\beta), \mathbb{C}^\times)$. Altogether this gives (b).

As the map $\mathcal{X} \to X$ is $G \times G$ equivariant the statement about orbits follows from the corresponding statement for X. That the boundary divisor has simple normal crossings is a local question that can be checked on $\mathcal{X}_0$. Here it reduces to the boundary in $[\mathbb{A}^r_b/Z(\beta)]$ and here the claim follows because it holds for the atlas $\mathbb{A}^r_b$. □

Appendix

Lemma 2.7.4. *Let $Y(\sigma, \mu_i, \lambda) \xrightarrow{p_\sigma} X$ be as in construction 3. Let $\overline{T}_\sigma$ be the toric variety associated to the cone σ.*

(a) $\overline{T}_\sigma \subset Y(\sigma, \mu_i, \lambda)$ and the action map $U^- \times U \times \overline{T}_\sigma \to Y(\sigma, \mu_i, \lambda)$ sending $(u_1, u_2, t) \to u_1 t u_2^{-1}$ maps isomorphically onto $p^{-1}_{\sigma,\mu_i,\lambda}(X_0)$.

(b) If λ' is another regular dominant weight such that $\lambda' + \mu_i$ are all regular dominant weights then there exists a $G \times G$-equivariant isomorphism $Y(\sigma, \mu_i, \lambda) \to Y(\sigma, \mu_i, \lambda')$.

(c) If $\nu_1, \dots, \nu_l \Lambda_T \cap \sigma^\vee$ are generators and λ'' is a regular dominant weight such that $\lambda'' + \nu_i$ are regular dominant weights then there is a $G \times G$ equivariant isomorphism is another choice of generators for $\Lambda_T \cap \sigma^\vee$ and λ'' is $Y(\sigma, \mu_i, \lambda) \to Y(\sigma, \nu_i, \lambda'')$.

Proof. The proof of (a) appears in the proof of [BK05a, 6.2.4]. For (b) define set $Y_\Delta = \overline{\Delta(G)} \subset Y(\sigma, \mu_i, \lambda) \times Y(\sigma, \mu_i, \lambda')$. Let $p_\lambda \colon Y_\Delta \to Y(\sigma, \mu_i, \lambda)$ be the projection and define $p_{\lambda'}$ similarly. Then $p_\lambda, p_{\lambda'}$ are $G \times G$ equivariant.

Consider the product morphism $Y(\sigma, \mu_i, \lambda) \times Y(\sigma, \mu_i, \lambda') \xrightarrow{f := (p_{\sigma,\mu_i,\lambda}, p_{\sigma,\mu_i,\lambda'})} X \times X$. We have $Y_\Delta \subset f^{-1}(X)$ because the latter is closed and contains $\Delta(G) \subset Y(\sigma, \mu_i, \lambda) \times Y(\sigma, \mu_i, \lambda')$. We also have a commutative diagram

$$\begin{tikzcd} Y_\Delta \arrow[r, "f := (p_{\sigma,\mu_i,\lambda}, p_{\sigma,\mu_i,\lambda'})"] \arrow[d, "p_\lambda"] & \Delta(X) \arrow[d, "p_1"] \\ Y(\sigma, \mu_i, \lambda) \arrow[r, "p_{\sigma,\mu_i,\lambda}"] & X \end{tikzcd}$$

therefore $p_\lambda^{-1}(p^{-1}_{\sigma,\mu_i,\lambda}(X_0)) = f^{-1}(p_1^{-1}(X_0)) = f^{-1}(\Delta(X_0))$. By (a) we can conclude $p_\lambda^{-1}(p^{-1}_{\sigma,\mu_i,\lambda}(X_0)) \subset \Delta(U^-) \times (\overline{T}_\sigma \times \overline{T}_\sigma) \times \Delta(U)$ but the only in the latter space that can belong to Y_Δ are those inside $\Delta(U^- \overline{T}_\sigma U)$.

Therefore p_λ defines an isomorphism from $p_\lambda^{-1}(p^{-1}_{\sigma,\mu_i,\lambda}(X_0))$ onto $p^{-1}_{\sigma,\mu_i,\lambda}(X_0)$. Using equivariance and proposition 2.1.3(c) we conclude that p_λ is a $G \times G$ equivariant isomorphism; the same argument applies to $p_{\lambda'}$ and together this shows (b).

For part (c) we note that because the μ_i and the ν_i are both generators for $\sigma^\vee \cap \Lambda_T$ then expressing μ_i in terms of ν_i defines an isomorphism $\operatorname{Spec} \mathbb{C}[\mu_i] \to \operatorname{Spec} \mathbb{C}[\nu_i] = \overline{T}_\sigma$. This together with the argument for part (b) gives (c). $\square$

Henceforth we now will write $p_\sigma \colon Y(\sigma) \to X$ for $p_{\sigma,\mu_i,\lambda} \colon Y(\sigma, \mu_i, \lambda) \to X$ as we are only concerned with the isomorphism class of $Y(\sigma)$. The embeddings $Y(\sigma)$ can be used to produce embeddings $Y(\Sigma)$ for any fan Σ with support in the negative Weyl chamber.

Stacky Modification for the loop group

For each $i \in \{0, \dots, r\}$ we have the maximal parahoric $P_i = P_{\eta_i} \subset L_{poly}G$ with η_i defined in example 4. Let $L_i \subset P_i$ be it's Levi factor. Denote the 1 parameter subgroup generated by η_i as $\eta_i(\mathbb{C}^\times) \subset T^\ltimes$. The map $1 \times T \subset T^\ltimes \to T^\ltimes/\eta_i(\mathbb{C}^\times)$ is an isomorphism and defines a maximal torus in L_i.

Let C_i denote the Weyl chamber of L_i and $\mu_{i,j} \in C_i^\vee \cap \hom(T, \mathbb{C}^\times)$ a list of generators. Also let $e_1, \dots, e_r \in \hom(T', \mathbb{C}^\times)$ be the elements that define the map $T' \xrightarrow{\beta} T$ from (2.18) and $\beta^* \colon \hom(T, \mathbb{C}^\times) \to \hom(T', \mathbb{C}^\times)$ the associated dual map.

Remark that here we can actually work with $V \otimes V^*$ because we are starting at the boundary and working with the polynomial loop group.

Observe for any $\nu \in \hom(T', \mathbb{C}^\times)$ and any regular dominant weight $(0, \lambda, l)$ of $L_{poly}G$ we can form the representation $\mathbb{C}_\nu \otimes V(0, \lambda, l)$. For notation convenience we denote $End(V(0, \lambda, l)) = E(\lambda, l)$ and $End(\mathbb{C}_\nu \otimes V(0, \lambda, l)) = E(\nu, \lambda, l)$.

Construction 6. Following construction 5 choose a regular dominant weight $(0, \lambda, l)$ such that for each $\mu \in F = \cup_{i=0}^r \{\mu_{i,j}\}$ we have $(0, \lambda + \mu, l)$ is a regular dominant weight.

We are going to define $r + 1$ elements $[id(i)]$ of

$$\mathbb{P}\left(End(\lambda, l) \bigoplus_{j=1}^{r} E(e_j, \lambda, l) \oplus E(e_j, \lambda, l) \bigoplus_{\mu \in F} E(\beta^*(\lambda + \mu), \lambda + \mu, l) \right)$$

.

For each $(0, \lambda + \mu, l)$ let $V_i(0, \lambda + \mu, l)$ denote the finite dimensional L_i representation generated by the highest weight vector. For $\nu \in \hom(T', \mathbb{C}^\times)$ let $id_i^{\nu, \lambda+\mu}$ denote the identity in $\mathbb{C}_\nu \otimes V_i(0, \lambda + \mu, l)$, write simply $id_i^{\lambda+\mu}$ if $\nu = 0$.

Let $F_i \subset F$ be the union of $\{\mu_{i,j}\}$ and those $\mu \in S$ which are in the monoid generated by the $\mu_{i,j}$. Then we set $[id(i)] = [id_i^\lambda \bigoplus id_i^{-e_i, \lambda} \bigoplus_{j \neq i} id_i^{e_j, \lambda} \bigoplus_{\mu \in F_i} id_i^{\beta^*(\lambda+\mu), \lambda+\mu}]$.

Let $\overline{Orb'(i)}$ be the $T' \times (L^\ltimes_{poly}G) \times T' \times (L^\ltimes_{poly}G)$ orbit closure of $[id(i)]$. Then set

$$\begin{aligned} \mathcal{D}_i &= [Orb'(i)/T'] \\ \partial \mathcal{X}^{aff,poly} &= [\cup_{i=0}^r Orb'(i)/T'] \end{aligned} \tag{2.21}$$

Remark 5. One is able to obtain a full embedding of $L^\ltimes G$ however the boundary will be the same as the boundary of X^{aff}. The problem is that the base points $[id(i)]$ are not specializations of the identity.

Theorem 2.7.5. *Each $\mathcal{D}_i$ is a formally smooth ind-stack. Each closed point $p \in \mathcal{D}_i$ has finite isotropy group. Further $\mathcal{D}_i$ fibers over $P_i \backslash L_{poly}G \times L_{poly}G/P_i$ and further $\mathcal{X}^{aff,poly}$ satisfies the valuative criterion for completeness.*

Proof. By 2.6.1(e) $\mathcal{D}_i$ contains an open substack isomorphic to the product of a formally group and a smooth toric stack which has finite isotropy group. For each i the $L_i \times L_i$ orbit closure of $[id(i)]$ in $\mathcal{D}_i$ is isomorphic to the compactification $\overline{L_i}$ constructed in 2.7.3. Then by 2.6.1(e) again it follows that we have a morphism $\mathcal{D}_i \to P_i \backslash L_{poly}G \times L_{poly}G/P_i$ with fibers the desired compactification.

To check the valuative criterion note any morphism $\operatorname{Spec} \mathbb{C}[[t]] \to \mathcal{X}^{aff,poly}$ must map into some $\mathcal{D}_i$ and then the result follows because $P_i \backslash L_{poly}G \times L_{poly}G/P_i$ is a projective ind-scheme and $\mathcal{D}_i$ is proper over $P_i \backslash L_{poly}G \times L_{poly}G/P_i$. $\square$

This result will be used in the next chapter.

Chapter 3

Degeneration

This chapter is adapted from a pre-print.

3.1 Introduction

This paper introduces a moduli problem $\mathcal{X}_G$ of G-bundles on twisted curves that "compactifies" the moduli space of principal G-bundles on a family of smooth curves degenerating to a nodal curve. More precisely, we show the moduli functor $\mathcal{X}_G$ satisfies the valuative criterion for completeness, which is a compactness statement for non separated spaces.

To motivate this problem we give a brief history of the subject starting with geometric invariant theory. Fix two positive integers r, d. One of the first moduli problems which was intensely studied using geometric invariant theory was the moduli space $M_{r,d}(C)$ of semistable rank r vector bundles of degree d on a smooth curve C of genus $g \geq 2$. Mumford showed the locus of stable bundles is always a smooth quasi projective variety [NS64, Mum63]. Seshadri then showed in [Ses67] that including the semistable bundles always yields a normal projective variety and hence a modular compactification when there are strictly semi stable bundles (which can happen if $(r, d) > 1$).

In [Ram75], Ramanathan extended the notion of semistability to principal G-bundles; there he also constructed moduli spaces for stable G-bundles on a curve. When G is semisimple it was shown by Balaji, Seshadri [BS02] and Faltings in [Fal93] that there is a projective coarse moduli space $M_G(C)$ of semistable G-bundles providing a modular compactification of the moduli space of strictly stable bundles.

Interest increased in these moduli spaces after a 1994 result of Faltings (for G semisimple) and Beauville, Lazlo (for $G = SL_n$) regarding the global sections a particular line bundle L on $M_G(C)$. The result states that $H^0(M_G(C), L)$ coincides with the vector space of conformal blocks appearing in conformal field theory. A crucial idea in establishing this result it to work with the moduli *stack* $\mathcal{M}_G(C)$ parametrizing all G-bundles on C. The stack $\mathcal{M}_G(C)$ is not proper but is complete which means it satisfies the existence (but not uniqueness) part of the valuative criterion for properness.

The connection with conformal field theory effectively computed the dimension of $H^0(M_G(C), L)$ using a result called the Verlinde formula. The proof of the Verlinde involves degenerating C to a nodal curve where computations are easier. The work of Faltings and Beauville,Lazlo suggested, at the very least, of considering degenerations of both $M_G(C)$ and $\mathcal{M}_G(C)$.

In fact the idea of degeneration had already proven useful a decade before in 1984, when Gieseker had used degeneration techniques on $M_{2,2n+1}(C)$ to prove a conjecture of Newstead and Ramanan [Gie84]. In 1993 Caporaso used Giesker's approach to give a compactification of the moduli space of $\mathbb{C}^\times$-bundles over the moduli space of stable curves $\overline{M}_g$. Just a year later, Pandharipande [Pan96] gave a compactification over $\overline{M}_g$ of $M_{r,d}$ using torsion free sheaves. In 1996, Faltings [Fal96], used torsion free sheaves to give degenerations of $M_{r,d}(C)$ and $M_G(C)$ for $G = SP_r, O_r$. Then in the 1999 paper [NS99], Nagaraj and Seshadri extended Gieseker's approach to give a different degeneration for $M_{r,d}(C)$.

One advantage of the Gieseker approach is that the resulting singularities are milder; indeed the boundary of the degeneration (the locus not parameterizing GL_r-bundles on the original nodal curve) is a divisor with simple normal crossings [Ses00, §5]; in contrast the singularities for the torsion free sheaf approach are worse [Fal96, sect. 3] (they are formally smooth to the singularity at the zero matrices in the variety $\{XY = YX = 0\}$ with X, Y square matrices). Nagaraj and Seshadri's work seemed to solidify the Gieseker approach as a standard alternative to using torsion free sheaves.

The remaining developments in this summary include mostly results using the Gieseker approach. Let $\mathcal{M}_{r,d}$ be the moduli stack of rank r vector bundles of degree d and set $\mathcal{M}_{GL_r} = \sqcup_{d\in\mathbb{Z}}\mathcal{M}_{r,d}$. In 2005, Kausz [Kau05a] provided a degeneration of $\mathcal{M}_{GL_r}(C)$ using a compactification KGL_r of GL_r. In 2009, Tolland [Tol09], gave a Gieseker comapactification for the moduli of $\mathbb{C}^\times$ bundles over $\overline{M}_{g,n}$. Recently, Martens and Thaddeus [MTa] gave compactifications of arbitrary reductive groups using a Gieseker like approach to studying degenerations of G bundles on genus 0 curves. On the other hand, Schmitt [Sch05] has provided a torsion free sheaf approach for an arbitrary semisimple group G although it should be noted that the approach depends on a non canonical embedding $G \to SL(V)$.

The contribution we make here is to offer a Gieseker-like degeneration for $\mathcal{M}_G$ with G a simple group. Let us now state the main theorem of this paper more precisely. Let $S = \operatorname{Spec}\mathbb{C}[[s]]$ and let C_S be a projective curve over S such that the generic fiber $C_{\mathbb{C}((s))}$ is smooth and the special fiber C_0 is a nodal curve with a single node. Let G be a connected, simple and simply connected algebraic group. We define a moduli stack $\mathcal{X}_G(C_S)$ parametrizing G-bundles on what we call twisted modifications of C_S. Then $\mathcal{X}_G(C_S)$ contains $\mathcal{M}_G(C_{\mathbb{C}((s))})$ as an open substack and

Theorem 3.5.4. *The stack $\mathcal{X}_G(C_S)$ satisfied the valuative criterion for completeness: let $R = \mathbb{C}[[s]]$ and $K = \mathbb{C}((s))$; for a finite extension $K \to K'$ let R' denote the integral closure of R in K'. Given the right commutative square below, there is finite extension $K \to K'$ and* *a dotted arrow making the entire diagram commute:*

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r, "h^*"] \arrow[d] & \mathcal{X}_G(C_S) \arrow[d] \\
\operatorname{Spec} R' \arrow[r] \arrow[urr, dashed, "h"'] & \operatorname{Spec} R \arrow[r, "f"] & S
\end{tikzcd}$$

The approach of this paper is to use the connection between loop groups and the moduli of principal bundles on curves as well as the embedding of the loop group from Chapter 2. Further, because we work with stacks, this approach works in all genus and works for both reducible and irreducible nodal curves.

We now elaborate on the notion of a twisted modification. Specifically a twisted modification $\mathcal{C}'_S$ of C_S is a curve over S with a map $\mathcal{C}' \xrightarrow{f} C$ such that if $C_S^* = C_S \backslash \{p\}$ with p the node, then $f^{-1}(C_S^*) \to C_S^*$ is an isomorphism and $f^{-1}(p)$ is $[R_n/\mu_k]$ where R_n is a connected chain of $\mathbb{P}^1$s (see figure 3.1), μ_k is the group of kth roots of unity and the value of k is determined by G. The stack $\mathcal{X}_G(C_S)$ parametrizes G-bundles on twisted modifications

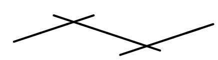

Figure 3.1: A chain of $\mathbb{P}^1$s of length 3.

$\mathcal{C}'$ of C_S where the G-bundle has prescribed equivariant structure on the fixed points of the μ_k action on $f^{-1}(p)$. We call such an object a twisted Gieseker bundle on C_S.

The restriction of the G bundle to the chain $[R_n/\mu_k]$ is a μ_k equivariant G-bundle. The use of equivariant G-bundles on chains is an idea introduced by Martens and Thaddeus in [MTa]. In fact they worked with $\mathbb{C}^\times$-equivariant bundles but both Martens and Thaddeus had mentioned to me that they considered working with μ_k-equivariants and that it could be a viable alternative.

At the same time I was lead to consider μ_k equivariant G-bundles for an entirely different reason. Namely, under the base change $S \xrightarrow{s \mapsto s^k} S$, the standard genus 0 degeneration to a node $\mathbb{C}[x,y,s]/(xy-s)$ becomes $\mathbb{C}[x,y,s]/(xy-s^k)$ which can be identified with μ_k invariants in $\mathbb{C}[u,v]$ where $\zeta \in \mu_k$ acts by $\zeta(u,v) = (\zeta u, \zeta^{-1} v)$. This observation had been made and used by both Faltings and Seshadri. The step taken here was to combine this observation with equivariant bundles on chains to arrive at the definition of $\mathcal{X}_G(C_S)$. Finally, I relate the geometry of $\mathcal{X}_G(C_S)$ to the geometry of the loop group embedding constructed in Chapter 2 to show the valuative criterion of completeness for $\mathcal{X}_G(C_S)$.

The basic idea for the proof of theorem 3.5.4 is as follows. Working in a neighborhood of the node, the moduli space of G-bundles on these equivariant chains is naturally isomorphic to a certain orbits closure $\partial\mathcal{X}^{aff,poly}$ of the polynomial loop group from Chapter 2. This allows one to show that the objects in $\mathcal{X}_G(C_S)$ degenerate in way that corresponds to a $L_{poly}G$-orbit stratification of $\partial\mathcal{X}^{aff,poly}$ and consequently deduce the completeness statement form a corresponding completeness statement for $\partial\mathcal{X}^{aff,poly}$.

The outline of the paper is as follows. Section 3 contains a discussion of some of the subtler points about the moduli spaces M_G and $\mathcal{M}_G$. It also contains some standard arguments used throughout the paper. Section 4 develops results on G bundles on twisted curves. When C is a fixed smooth curve there is some overlap with [VB]. We then proceed to a fixed nodal curve with a single node, and then to fixed curve where the node has been replaced with a μ_k equivariant chain. In section 5 we define precisely the moduli problem $\mathcal{X}_G(C_S)$ and prove the main theorem.

3.2 Basic constructions, conventions and notation

Here we pin down conventions for various tools, construction and other notation used throughout the paper. This is an attempt to delegate notation building here and have the other sections focused on proving the main theorem.

Groups and Lie algebras

We use G to denote a simple, connected and simply connected algebraic group G over $\mathbb{C}$ and $T \subset G$ a maximal torus. Let $\mathfrak{g} = Lie(G)$, $\mathfrak{t} = Lie(T)$ and let $\Delta \subset \mathfrak{t}^*$ be the roots so that $\mathfrak{g} = \mathfrak{t} \oplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$. Let Δ^+ be a choice of positive roots so that $\Delta = \Delta^+ \cup -\Delta^+$. Let $r = \dim T$ and $\alpha_1, \ldots, \alpha_r$ denote an ordered choice of simple roots.

We have a parallel set of conventions for the loop group LG. As a functor, the loop groups is defined on $\mathbb{C}$-algebras via $LG(R) := G(R((z)))$. Similarly, the polynomial loop group is $L_{poly}G(R) := G(R[z^{\pm}])$.

There is a strong parallel between LG and G which is best seen by introducing $L^{\ltimes}G := \mathbb{C}^\times \ltimes LG$ or $L^{\ltimes}_{poly}G := \mathbb{C}^\times \ltimes L_{poly}G$. The group structure is given by

$$(u_1, \gamma_1(z)) \cdot (u_2, \gamma_2(z)) = (u_1 u_2, u_2^{-1}\gamma_1(z) u_2 \gamma_2(z))$$
$$u_2^{-1}\gamma_1(z) u_2 = \gamma_1(u_2^{-1} z)$$

A maximal torus for $L^{\ltimes}G$ is $\mathbb{C}^\times \times T$ for any maximal torus $T \subset G$. In sections 4,5 we work with $L_{poly}G$ and it's Lie algebra $Lie(L_{poly}G) = \mathfrak{g} \otimes \mathbb{C}[z^{\pm}] =: \mathfrak{g}[z^{\pm}]$. Define d by $Lie(\mathbb{C}^\times \times T) = \mathbb{C}d \oplus \mathfrak{t}$. We are now in a position to set up analogous root notation for $\mathfrak{g}[z^{\pm}]$ and it is conventional to use the term affine to differentiate it from the notation for $\mathfrak{g}$. The root spaces for $\mathfrak{g}[z^{\pm}]$ are of the form $z^i\mathfrak{g}_\alpha$ and $z^j\mathfrak{t}$. Let $\Delta^{aff} \subset (\mathbb{C}d \oplus \mathfrak{t})^*$ be the subset so that

$$\mathbb{C}d \oplus \mathfrak{g}[z^{\pm}] = \mathbb{C}d \oplus \mathfrak{t} \bigoplus_{(n,\alpha)\in\Delta^{aff}} z^n \mathfrak{g}_\alpha.$$

Then the elements of Δ^{aff} are called the *affine roots*. Let $z^i\Delta$ stand for the roots of the form (i, α) for $\alpha \in \Delta$. A choice of positive roots is $\Delta^{aff,+} = \Delta^+ \bigcup_{i\geq 1} z^i\Delta \cup \{(i, 0)\}$.

Let θ denote the longest root in $\mathfrak{g}$. The simple roots for $\mathbb{C}^\times \ltimes L_{poly}G$ are $(0, \alpha_1), \dots, (0, \alpha_r), (1, -\theta)$. All of this notation also applies to $L^\ltimes G = \mathbb{C}^\times \ltimes LG$. By abuse of notation we denote $(0, \alpha_i)$ with α_i and set $\alpha_0 = (1, -\theta)$.

(Co-)Characters, Parabolic and Parahoric Subgroups

For any torus T we have the lattice of characters $\hom(T, \mathbb{C}^\times)$ and co-characters $\hom(\mathbb{C}^\times, T)$. Further, for $(\eta, \chi) \in \hom(\mathbb{C}^\times, T) \times \hom(T, \mathbb{C}^\times)$ we set $\langle \eta, \chi \rangle := \chi \circ \eta \in \mathbb{Z}$.

For $T \subset G$ a maximal torus and for $\eta \in \hom(\mathbb{C}^\times, T)$ the set $P(\eta) := \{g \in G | \lim_{t \to 0} \eta(t) g \eta(t)^{-1} \text{ exists}\}$ is a subgroup. A *parabolic* subgroup is any subgroup $P \subset G$ conjugate to some $P(\eta)$.

We can apply the same construction for $\eta \in \hom(\mathbb{C}^\times, \mathbb{C}^\times \times T)$ to get a subgroup $P(\eta) \subset L^\ltimes G$. A *parahoric* subgroup is any group conjugate to one of the $P(\eta)$. By abuse of notation, we use $P(\eta)$ to denote its image under the projection $L^\ltimes G \to LG$. Parahoric subgroups of LG are any subgroups conjugate to one of the $P(\eta)$.

Parabolic and parahoric subgroups come with natural factorizations $P(\eta) = L(\eta)U(\eta)$ known as a Levi decomposition: $L(\eta) = \{g \in G | \lim_{t \to 0} \eta(t) g \eta(t)^{-1} = g\}$ and $U(\eta) = \{g \in G | \lim_{t \to 0} \eta(t) g \eta(t)^{-1} = 1\}$. A simple is example comes from $\eta_0 \colon \mathbb{C}^\times \to \mathbb{C}^\times \times T$ defined by $\eta_0(t) = (t, 1)$. Then $\eta_0(t) g(z) \eta_0(t)^{-1} = g(tz)$ and $P(\eta_0) = G[[z]] = G(\mathbb{C}[[z]]) =: L^+G$. The Levi factorization is $G \cdot N$ where N is the kernel of the map $G[[z]] \xrightarrow{z \mapsto 0} G$.

By $\mathfrak{t}_\mathbb{Q}$ we denote $\hom(\mathbb{C}^\times, T) \otimes_\mathbb{Z} \mathbb{Q}$. The Weyl chamber is defined as $Ch := \{\eta \in \mathfrak{t}_\mathbb{Q} | \langle \alpha_i, \eta \rangle \geq 0\}$. It is a simplicial cone whose faces are given by $\{\langle \alpha, \eta \rangle = 0 | \alpha \in I\}$ for subsets of $I \subset \{\alpha_1, \dots, \alpha_r\}$.

Similarly, we have the affine Weyl chamber $Ch^{aff} = \{\eta \in \mathbb{Q} \oplus \mathfrak{t}_\mathbb{Q} | \langle \alpha_i, \eta \rangle > 0\}$; now the faces are in bijection with subsets $\{\alpha_0, \dots, \alpha_r\}$. It is convention to instead work with the affine Weyl alcove $Al := Ch^{aff} \cap 1 \oplus \mathfrak{t}_\mathbb{Q} = \{\eta \in \mathfrak{t}_\mathbb{Q} | 0 \leq \langle \alpha_i, \eta \rangle, \langle \theta, \eta \rangle \leq 1\}$. A *face* F of Al is $F' \cap 1 \oplus \mathfrak{t}_\mathbb{Q}$ where F' is a face of Ch^{aff}.

Any $\eta \in Ch$ determines a fractional co-character $\mathbb{C}^\times \to T$ but nevertheless a well defined parabolic $P(\eta)$. Any parabolic is conjugate to some $P(\eta)$ and if η, η' are in the interior of the same face then $P(\eta) = P(\eta')$. Similarly any $\eta \in Al$ determines a parahoric $P(\eta) \subset LG$. Any parahoric is conjugate either to $P(\eta)$ or to $P(-\eta)$. Let $Al_e = \{\eta \in Al | \langle \theta, \eta \rangle = 1\}$. If $\eta \in Al_e$ the resulting parahoric is called *exotic*. Alternatively, the inclusion $\{\alpha_1, \dots, \alpha_r\} \subset \{\alpha_0, \dots, \alpha_r\}$ defines a map from faces of Ch to those of Al. The faces missed by Ch are exactly those contained in Al_e.

The exotic parahorics give rise to moduli spaces of torsors on curves which are not isomorphic with moduli spaces of G-bundles. Informally then the exotic parahorics can be viewed as geometry only visible to LG. Exotic parahorics are studied in depth in [VB]; there they are called nonhyperspecial maximal parahoric subgroups.

The ordered simple roots $\{\alpha_0, \alpha_1, \dots, \alpha_r\}$ determine ordered vertices $\{\eta_0, \dots, \eta_r\}$ determined by the conditions $\langle \eta_i, \alpha_j \rangle = 0$ for $i \neq j$ and $\langle \eta_0, \alpha_0 \rangle = 1$. If we write $\theta = \sum_{i=1}^r n_i \alpha_i$ and set $n_0 = 1$ then one can check these condition can be expressed as

$$\langle \alpha_i, \eta_j \rangle = \frac{1}{n_i} \delta_{i,j} \tag{3.1}$$

Now for each $I \subset \{0, \ldots, r\}$ we define $\eta_I = \sum_{i \in I} \eta_i$. Then η_I lies in the face of Al associated to the complement of I; if $I = \emptyset$ we take η_I to be the trivial co-character. Finally, we set

$$\begin{aligned} \mathcal{P}_I &= P(\eta_I) & \mathcal{P}_I^- &= P(-\eta_I) \\ \mathcal{U}_I &= U(\eta_I) & \mathcal{U}_I^- &= U(-\eta_I) \\ & L_I = L(\eta_I) = L(-\eta_I) \end{aligned} \tag{3.2}$$

One can check that $\mathcal{P}_I = \cap_{i \in I} P(\eta_i)$. It is sufficient to establish this at the level of Lie algebras because $P(\eta)$ is connected $\forall \eta$ (the map $g \mapsto \lim_{t \to 0} \eta(t) g \eta(t)^{-1}$ defines a retraction onto the Levi factor which is connected). Returning to Lie algebras, we note the $\supset$ direction is routine to verify. Going the other way we have $Lie(\mathcal{P}_I)$ is spanned by $\mathbb{C}d \oplus \mathfrak{t}$ and those X_α for which $\langle \alpha, \eta_I \rangle \geq 0$. It is suffices to work with α negative so that $0 \geq \langle \alpha, \eta_i \rangle \forall i$. Then we have $0 \geq \sum_{i \in I} \langle \alpha, \eta_i \rangle = \langle \alpha, \eta_I \rangle \geq 0$ which is only possible if each term is equal to 0; i.e. $X_\alpha \in Lie(\mathcal{P}_i) \forall i \in I$.

(Equivariant)-Bundles, Quotient Stacks and torsors

Let H be a linear algebraic group over $\mathbb{C}$. A principal H-bundle over a base scheme B is scheme P with a smooth map $P \to B$ such that any $p \in B$ has an fppf neighborhood B' such that $P \times_B B' \cong B' \times H$. Because all of our group schemes are smooth we can equivalently require local triviality in the étale topology but below we generally work on curves with fppf covers coming from formal neighborhoods of points.

Given a scheme B equipped with an action of an algebraic group H we can form the quotient stack $[B/H]$. By definition a morphism $B' \to [B/H]$ is the data of a principal H-bundle P over B' together with an H-equvariant map $P \to B$. Quotient stacks play a prominent role in our use of twisted curves defined in the next section.

Given a base B with the action of a group Π an equivariant H-bundle on B is a bundle $P \to B$ together with an action of Π making the following diagram commute

$$\begin{tikzcd} \Pi \times P \arrow[r] \arrow[d] & P \arrow[d] \\ \Pi \times B \arrow[r] & B \end{tikzcd} \tag{3.3}$$

Equivalently, or by definition, an equivariant H-bundle is a H-bundle on $[B/\Pi]$. For $b \in B$ let Π_b denote the stabilizer of b in Π. Then the above diagram produces an action of Π_b on the fiber of P over b. The action is determined by a representation $\rho \colon \Pi_b \to H$. In general we summarize this situation by saying that *the equivariant structure of P at b is given by ρ*.

Let G be a connected, simply connected simple group over $\mathbb{C}$. The basic source of equivariant bundles in this paper are G-bundle on $[\operatorname{Spec} \mathbb{C}[[z]]/\mu_k]$ where $\zeta \in \mu_k$ acts by $z \mapsto \zeta z$. Any G-bundle on $\operatorname{Spec} \mathbb{C}[[z]]$ is trivial and so an equivariant bundle is determined by its equivariant structure $\mu_k \to G$ at the closed point of $\operatorname{Spec} \mathbb{C}[[z]]$.

We also utilize torsors for a sheaf of groups $\mathcal{G}$. In general, given a curve C and a sheaf of groups $\mathcal{G}$ on C we define a *$\mathcal{G}$-torsor* to be a sheaf of sets $\mathcal{F}$ on C together with a right action of $\mathcal{G}$ such that (1) there is a fppf cover $\{C_i \to C\}$ such that $\mathcal{F}(C_i) \neq \emptyset$ and (2) the action map $\mathcal{G} \times \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is an isomorphism.

Given G as above, we can form the sheaf $U \mapsto \hom_{sch}(U, G) =: \mathcal{G}^{std}(U)$. Generally our sheafs of groups agree with $\mathcal{G}^{std}$ on an open set $U \subset C$ but in general have more intricate behavior $C \backslash U$. Torsors for $\mathcal{G}^{std}$ can be identified with G bundles and so the notion is most relevant when working with a sheaf of groups $\mathcal{G} \neq \mathcal{G}^{std}$; we often write simply *torsor* to indicate a torsor for a sheaf of groups $\mathcal{G} \neq \mathcal{G}^{std}$ to be specified later. Examples of torsors are given in 3.3.

Conventions on Curves

Generally we work over $\operatorname{Spec}\mathbb{C}$ and a scheme will mean a scheme over $\operatorname{Spec}\mathbb{C}$. Let S be a scheme. We denote a flat family of curves $C \to S$ as C_S. If B is an S-scheme then $C_B := C_S \times_S B$. For affine schemes $\operatorname{Spec} R \to S$ we write C_R for $C_{\operatorname{Spec} R}$.

Generally we work with a fixed curve over $\operatorname{Spec}\mathbb{C}$ or with a family of curves over $S = \operatorname{Spec}\mathbb{C}[[s]]$. Set $S^* = \operatorname{Spec}\mathbb{C}((s))$ and $S_0 = \operatorname{Spec}\mathbb{C} = \operatorname{Spec}\mathbb{C}[[s]]/(s)$ the closed point. Then C_S always denotes a curve with generic fiber C_{S^*} smooth and special fiber $C_0 := C_{S_0}$ nodal with unique node p. We write $C_S - p$ for the open subscheme $C_S \backslash \{p\}$. We also assume C_S is a regular surface as scheme over $\operatorname{Spec}\mathbb{C}$.

For any closed point p in a scheme Z we denote by $\hat{\mathcal{O}}_{Z,p}$ the completion of $\mathcal{O}_{Z,p}$ with respect to the maximal ideal. We often use D to denote a formal neighborhood of a point in a curve. The cases that will arise are

- $p \in C$ a smooth curve, $\hat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[z]]$ and we set $D = \operatorname{Spec}\mathbb{C}[[z]]$
- $p \in C_0$ is the node, $\hat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[x,y]]/xy$ and we set $D_0 = \operatorname{Spec}\mathbb{C}[[x,y]]/(xy)$
- $p \in C_S$ is the node, $\hat{\mathcal{O}}_{C_S,p} \cong \frac{\mathbb{C}[[s,x,y]]}{(xy-s)} \cong \mathbb{C}[[x,y]]$ and we set $D_S = \operatorname{Spec}\mathbb{C}[[x,y]]$
- for $k \geq 2$ and kth roots u, v of x, y we set $D_S^{\frac{1}{k}} = \operatorname{Spec}\mathbb{C}[[u,v]]$

The last case arises as follows. We first notice that if we base change D_S under $s \mapsto s^k$ then D_S becomes $\operatorname{Spec}\mathbb{C}[[x,y,s]]/(xy - s^k)$. If we let μ_k denote the kth roots of unity then $\operatorname{Spec}\mathbb{C}[[x,y,s]]/(xy - s^k) = D_S^{\frac{1}{k}}//\mu_k$ where $\zeta \in \mu_k$ acts by $\zeta(u,v) = (\zeta u, \zeta^{-1} v)$for $\zeta \in \mu_k$. A basic strategy we employ is to replace the curve $\operatorname{Spec}\mathbb{C}[[x,y,s]]/(xy - s^k)$ with the orbifold or twisted curve $[D_S^{\frac{1}{k}}/\mu_k]$.

In section 5 we utilize results on twisted curves from [AOV11]. We now recall the definition of a twisted curve (with no marked points) in characteristic 0. A twisted nodal curve $\mathcal{C} \to S$ is a proper Deligne-Mumford stack such that

(i) The geometric fibers of $\mathcal{C} \to S$ are connected of dimension 1 and such that the coarse moduli space C of $\mathcal{C}$ is a nodal curve over S.

(ii) If $\mathcal{U} \subset \mathcal{C}$ denotes the complement of the singular locus of $\mathcal{C} \to S$ then $\mathcal{U} \to C$ is an open immersion.

(iii) Let $p\colon \operatorname{Spec} k \to C$ be a geometric point mapping to a node and let $s \in S$ denote the image of $\operatorname{Spec} k$ under $C \to S$ and let $m_{S,s}$ denote the maximal ideal of the local ring $\mathcal{O}_{S,s}$. Then there is an integer k and an element $t \in m_{S,s}$ such that

$$\operatorname{Spec} \mathcal{O}_{C,p} \times_C \mathcal{C} \cong [D^{sh}/\mu_k]$$

where D^{sh} denotes the strict henselization of $D := \operatorname{Spec} \mathcal{O}_{S,s}[u,v]/(uv - t)$ at the point $(m_{S,s}, u, v)$ and $\zeta \in \mu_k$ acts by $\zeta(u,v) \mapsto (\zeta u, \zeta^{-1} v)$.

We did not mention markings because largely we will not make use of them except for one exception. If C is a smooth curve we can twist at a marked point p as described below. Let $p \in C$ and $D = \operatorname{Spec} \mathbb{C}[[z]]$ as in the first bullet point above and fix a positive integer k and a kth root w of z. We have $\operatorname{Spec} \mathbb{C}((w))/\mu_k = \operatorname{Spec} \mathbb{C}((z))$ so let $C_{[k]}$ denote $C - p \cup_{\operatorname{Spec} \mathbb{C}((z))} [\operatorname{Spec} \mathbb{C}[[w]]/\mu_k]$. It is a twisted curve whose coarse moduli space is C.

In a similar fashion, with C_0, C_S as in the bullet points, we can construct twisted curves $C_{0,[k]}$ and $C_{S,[k]}$ with coarse moduli space C_0, C_S and such the the fiber of the node is $[pt/\mu_k]$.

3.3 Survey of Facts about $\mathcal{M}_G(C)$

The problem of compactifying G-bundle on nodal curves involves some subtleties that are well known to the experts but are nevertheless worth stating explicitly. These subtleties include coarse moduli spaces vs stacks, issues on nodal curves, Gieseker bundles vs torsion free sheaves, and the connection with the loop group.

$\mathcal{M}_G$, M_G, completeness and compactness

Let H be reductive group over $\mathbb{C}$. If C is a smooth curve of genus g over $\operatorname{Spec} \mathbb{C}$ then there is a stack $\mathcal{M}_H(C)$ parametrizing principal H-bundles on C. It is a smooth algebraic stack of dimension $\dim H(g-1)$. Further there is a universal bundle $P^{univ} \to C \times \mathcal{M}_H(C)$ such that if $P \to C \times B$ is any H-bundle then there is a morphism $B \xrightarrow{f} \mathcal{M}_H(C)$ such that $P \cong (id, f)^* P^{univ}$.

Let us now specialize to groups G as in 3.2. It is known that $Pic(\mathcal{M}_G(C)) = \mathbb{Z}$ and there is a generator L which is ample. Using L, one constructs the coarse moduli space of semistable G-bundles $M_G(C) = \operatorname{Proj} \bigoplus_n \Gamma(\mathcal{M}_G(C), L^{\otimes n})$ [Tel00, §8]. This is not the conventional construction but illustrates how $M_G(C)$ can be recovered from $\mathcal{M}_G(C)$. On the other hand, $M_G(C)$ has the advantage of being a projective variety and hence compact whereas $\mathcal{M}_G(C)$ is not separated and thus not compact.

The case of $\mathcal{M}_{SL_2}(\mathbb{P}^1)$ is an instructive example. As a set, $\mathcal{M}_{SL_2}(\mathbb{P}^1) = \mathbb{N}$ where n corresponds to the bundle $\mathcal{O}(n) \oplus \mathcal{O}(-n)$ where we abbreviate $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1}$. Further, the ample generator $L \in Pic(\mathcal{M}_{SL_2}(\mathbb{P}^1))$ satisfies $H^0(\mathcal{M}_{SL_2}(\mathbb{P}^1), L^{\otimes n}) = \mathbb{C}$ so $M_{SL_2}(\mathbb{P}^1) = \text{Proj } \mathbb{C}[t] = \operatorname{Spec}\mathbb{C}$ which corresponds to $\mathcal{O} \oplus \mathcal{O}$, the unique semistable bundle.

Further there is a vector bundle $E \to \mathbb{P}^1 \times Ext^1(\mathcal{O}(1), \mathcal{O}(-1))$ [NS65, Lemma 3.1] such that $E|_{\mathbb{P}^1 \times v}$ corresponds to the extension $v \in Ext^1(\mathcal{O}(1), \mathcal{O}(-1)) = H^1(\mathcal{O}(-2)) = \mathbb{C}$. For $v \neq 0$ this extension is the Euler sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(-1) \to 0$$

Comparing with the trivial family $p_1^*(\mathcal{O} \oplus \mathcal{O})$ on $\mathbb{P}^1 \times \mathbb{A}^1$ we get two maps $\mathbb{A}^1 \xrightarrow{f_1, f_2} \mathcal{M}_{SL_2}(\mathbb{P}^1)$ that agree on $\mathbb{C}^\times$ such that $f_1(0) = \mathcal{O} \oplus \mathcal{O}$ and $f_2(0) = \mathcal{O}(1) \oplus \mathcal{O}(-1)$. This shows $\mathcal{M}_{SL_2}(\mathbb{P}^1)$ is not separated and further $0, 1 \in \mathcal{M}_{SL_2}(\mathbb{P}^1)$ are in the same connected component; this construction generalizes to show $\mathcal{M}_{SL_2}(\mathbb{P}^1)$ is connected. More generally, $\pi_0(\mathcal{M}_G(C)) = \pi_1(G)$.

Because of this behavior, we can at most ask for $\mathcal{M}_G(C)$ to satisfy the existence part of the valuative criterion for properness; this is called completeness. Specifically, a morphism of stacks $X \to Y$ is complete if for every complete discrete valuation ring R with fraction field K and every diagram with solid arrows there exists a dotted arrow making the diagram commute.

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & Y
\end{tikzcd}$$

where $K \to K'$ is a finite extension R' is the integral closure of R in K'.

If C is a smooth curve, then $\mathcal{M}_G(C) \to \operatorname{Spec}\mathbb{C}$ is complete. Completeness fails when C is nodal as is discussed in the next section.

Nodal Curves and the case of GL_n

If C is a nodal curve then $\mathcal{M}_G(C)$ may be complete. For the group $\mathbb{C}^\times$ this holds on any curve of compact type. If C is a chain of $\mathbb{P}^1$s and H is reductive then $\mathcal{M}_H(C)$ is discrete and naturally isomorphic to $\mathcal{M}_H(\tilde{C})$ hence complete [MTb, Variation 4]. But as soon as the irreducible components of C have genus ≥ 2 then $\mathcal{M}_H(C)$ will not be complete. Even if the genus is 1 we will run into trouble as the next example shows.

Consider the curve $C = \{y^2 - x^2(x+1) = 0\} \subset \mathbb{A}^2$. Consider the divisor defined on $C \times \mathbb{C}^\times$ defined by the section $t \mapsto (t^2 + 2t, (t^2 + 2t)(t+1), t)$. This defines a line bundle on $C \times \mathbb{C}^\times$. The limit as $t \mapsto 0$ is the nodal point which doesn't define a line bundle but a rather a torsion free sheaf. By enlarging the moduli problem to parametrize torsion free sheaves one can get a compact coarse moduli space [Fal96, Ses00].

A key insight originally due to Gieseker [Gie84] is that torsion free sheaves can be replaced by vector bundles on modified curves. Specifically, on any nodal curve C with nodes $\{p_1, \dots, p_m\}$, a torsion free sheaf $\mathcal{F}$ on C can be realized as the pushforward of a vector bundle F on a modification $C' \xrightarrow{\pi} C$ where $\pi^{-1}(C - \{p_1, \dots, p_m\}) \to C$ is an isomorphism and

$\pi^{-1}(p_i)$ is a chain of projective lines of length at most the rank of $\mathcal{F}$. Further, if $\pi_*(F) = \mathcal{F}$ then for each $\mathbb{P}^1 \subset \pi^{-1}(p_i)$ it is necessary that $F|_{\mathbb{P}^1} = \mathcal{O}(1)^{\oplus i} \oplus \mathcal{O}^{rk(\mathcal{F})-i}$ with $i > 1$ and that $H^0(\pi^{-1}(p_i), F|_{\pi^{-1}(p_i)} \otimes \mathcal{O}(-p_i' - p_i'')) = 0$ where p_i', p_i'' denote the extreme points on the chain $\pi^{-1}(p_i)$.

Torsors versus G bundles

See 3.2 for the definition of a torsor for a sheaf of groups $\mathcal{G}$. The point of discussing $\mathcal{G}$-torors is that a family of G bundles over a nodal curve can limit to a $\mathcal{G}$ torsor which cannot be identified with a G-bundle.

Starting with G we can form the sheaf of groups $\mathcal{G}^{std}(U) := \hom_{Sch}(U, G)$. Any principal bundle F on C defines a torsor $\mathcal{F}$ for $\mathcal{G}^{std}$ by $\mathcal{F}(U) \mapsto Sect(U, F|_U)$. In fact in much the same way vector bundles can be identified with locally free sheaves, G bundles can be identified with $\mathcal{G}^{std}$-torsors.

More generally let $P \subset G$ be a parabolic subgroup. Let $L_P^+G = \{\gamma \in G[[z]] \,|\gamma(0) \in P\}$. Construct a sheaf of groups $\mathcal{G}^P$ on $\operatorname{Spec}\mathbb{C}[[z]] = \{(z), (0)\}$ by $\mathcal{G}^P(\{(z),(0)\}) = L_P^+G$ and $\mathcal{G}^P(\{0\}) = G((z))$. Given a smooth curve C and a point p we notice that $\mathcal{G}^{std}|_{C-p}$ and $\mathcal{G}^P$ agree over $\operatorname{Spec}\mathbb{C}((z)) \cong C - p \times_C \operatorname{Spec}\hat{\mathcal{O}}_p$ and thus define a sheaf of group which we also denote $\mathcal{G}^P$. Clearly we can iterate over $(x_i) = x_1, \ldots, x_m \in C$ with parabolics $(P_i) = P_1, \ldots, P_m$. Call the resulting sheaf of groups $\mathcal{G}^{(x_i),(P_i)}$. Then $\mathcal{G}^{(x_i),(P_i)}$-torsors are exactly quasi parabolic bundles: G-bundles on C with reduction of structure group to P_i at x_i.

In the examples mentioned thus far all the $\mathcal{G}$-torsors can be identified with G-bundles potentially with additional structure; this is not always the case. The groups L_P^+G are parahoric subgroups and we can apply the same construction to any parahoric subgroup $\mathcal{P}$ (see 3.2 in particular for the definition of (exotic) parahorics). Specifically, given a set $(\mathcal{P}_i)$ of parahoric subgroups we can analogously construct a sheaf of groups $\mathcal{G}^{(x_i),(\mathcal{P}_i)}$. When the parahorics are exotic the resulting moduli spaces are not isomorphic to moduli spaces of G-bundles on C; see remark 7 after corollary 3.4.3.

The double coset construction

There is a close connection between the loop group LG and the moduli stack $\mathcal{M}_G(C)$ for a smooth curve C. Notice any $\gamma \in G(C-p) = \hom_{Sch}(C-p, G)$ can be Laurent expanded around p to produce an element in LG. This realizes $G(C-p)$ as a subgroup of LG which we denote L_CG. Let $m_p \subset \mathcal{O}_{C,p}$ be the maximal ideal then choosing a basis $z \in m_p/m_p^2$ determines an isomorphism $\operatorname{Spec}\hat{\mathcal{O}}_{C,p} \cong \operatorname{Spec}\mathbb{C}[[z]] = D$.

To make the connection between LG and $\mathcal{M}_G(C)$ we introduce two functors. Let $\mathbb{C}Alg$ denote the category of $\mathbb{C}$-algebras. Let $T'\colon \mathbb{C}Alg \to \mathbf{Set}$ be defined by setting $T'(R)$ to be the set of isomorphism classes of triples (P, τ_C, τ_D) where P is principal G-bundle on C_R, $\tau_D\colon G \times D_R \xrightarrow{\sim} P|_{D_R}$, $\tau_C\colon G \times (C-p)_R \xrightarrow{\sim} P|_{(C-p)_R}$ are trivializations. Let T be the functor defined by setting $T(R)$ to be isomorphism classes of pairs (P,τ_C) defined as above. We have forgetful functors $T' \xrightarrow{f_D} T \xrightarrow{f_C} \mathcal{M}_G(C)$ defined by $(P,\tau_C,\tau_D) \xrightarrow{f_D} (P,\tau_C) \xrightarrow{f_C} P$.

Let τ_D^* denote the restriction of τ_D to $D_R^* = \operatorname{Spec} R((z))$ and define τ_C^* similarly. Then we get a map

$$\begin{aligned} T' &\xrightarrow{\Theta_{C,D}} LG \\ (P,\tau_C,\tau_D) &\mapsto (\tau_C^*)^{-1} \circ \tau_D^*. \end{aligned} \tag{3.4}$$

Of course we also have $\Theta_{C,D}^{-1}\colon T' \to LG$ given by $(P,\tau_C,\tau_D) \mapsto (\tau_D^*)^{-1} \circ \tau_C^*$. For definiteness we work with $\Theta_{C,D}$ but this choice is inconsequential.

Denote by LG/L^+G the sheaf associated to the pre sheaf $R \mapsto LG(R)/L^+G(R)$ in the fppf topology. Then, for example, if $R \to R'$ is faithfully flat, $\gamma \in LG(R')$, γ_1,γ_2 denote the images of γ under the two maps $R' \rightrightarrows R' \otimes_R R' =: R''$ and $\gamma_1\gamma_2^{-1} \in L^+G(R'')$, then letting $\overline{\gamma}$ denote the class of γ in $LG(R')/L^+G(R')$ we have $\overline{\gamma_1} = \overline{\gamma_2} \in LG(R'')/L^+G(R'')$. By definition this determines a point of $(LG/L^+G)(R)$ which we denote $\gamma \rightrightarrows (\gamma_1,\gamma_2)$.

We define a map $T \xrightarrow{\Theta_C} LG/L^+G$ as follows. If $(P,\tau_C) \in T(R)$ then there is faithfully flat base extension $R \to R'$ such that $P|_{D_{R'}}$ admits a trivialization τ_D and hence a point of $T'(R')$. Let $\gamma(\tau_D) = \Theta_{C,D}(P,\tau_C,\tau_D) \in LG(R')$. With $\gamma_i(\tau_D)$ as above we set $\Theta_C((P,\tau_C))$ $= \gamma(\tau_D) \rightrightarrows (\gamma_1(\tau_D),\gamma_2(\tau_D))$. If τ_D' is another trivialization $\gamma(\tau_D'),\gamma_i(\tau_D')$ differ from the unprimed version by elements in L^+G hence define the same element in the quotient.

Similarly, we define a map $\mathcal{M}_G(C) \xrightarrow{\Theta} L_CG\backslash LG/L^+G$. Let $P \in \mathcal{M}_G(C)(R)$ then by [DS95], there is a faithfully flat (in fact étale) base change $R \to R'$ such that $P|_{(C-p)_{R'}}$ admits a trivialization τ_C and hence a point of $T(R')$. Let $\gamma(\tau_C) = \Theta_C((P,\tau_C)) \in (LG/L^+G)(R')$ and $\gamma_i(\tau_C)$ denote the two images of $\gamma(\tau_C)$ in $(LG/L^+G)(R' \otimes_R R')$. Let $\overline{\gamma(\tau_C)}$ denote the class of γ in $L_CG(R')\backslash(LG/L^+G)(R')$ and define $\overline{\gamma_i(\tau_C)}$ similarly. One checks $\overline{\gamma_1(\tau_C)} = \overline{\gamma_2(\tau_C)}$ and we set $\Theta(P) = \gamma(\tau_C) \rightrightarrows (\gamma_1(\tau_C),\gamma_2(\tau_C))$; as in the definition of Θ_C, the map Θ is independent of the choice τ_C.

Let $\pi_D\colon LG \to LG/L^+G$ and $\pi_C\colon LG/L^+G \to L_CG\backslash LG/L^+G$ be the quotient maps. Summarizing, we have a commutative diagram

$$\begin{tikzcd} T' \arrow[r, "\Theta_{D,C}"] \arrow[d, "f_D"] & LG \arrow[d, "\pi_D"] \\ T \arrow[r, "\Theta_C"] \arrow[d, "f_C"] & LG/L^+G \arrow[d, "\pi_C"] \\ \mathcal{M}_G(C) \arrow[r, "\Theta"] & L_CG\backslash LG/L^+G. \end{tikzcd} \tag{3.5}$$

We stress that while $\Theta_{D,C},\Theta_C$ are easy to construct, in order to construct Θ we need to use the non trivial result [DS95] of Drinfeld and Simpson. The construction of the maps $\Theta_{D,C},\Theta_C,\Theta$ we refer to collectively as the double coset construction (DCC). The connection between $\mathcal{M}_G(C)$ and LG can then be stated as

Theorem 3.3.1. *All the horizontal maps in the diagram* (3.5) *are isomorphisms.*

Proof. See [BL94b, Prop.3.4] for details. For each map one constructs a map in the other directions and checks it is the required inverse. For the inverse to $\Theta_{C,D}$, we construct for every $\gamma \in LG(R)$ a G-bundle on C_R with trivializations on $(C-p)_R, D_R$. If R is Noetherian then $(C-p)_R \sqcup D_R$ form an fppf cover of C_R and standard descent allows us to glue the trivial G-bundles on $(C-p)_R$ and D_R over $D_R^* = R((z))$ using γ as a transition function. In [BL95] Beauville and Lazlo show such gluing is possible for an arbitrary $\mathbb{C}$-algebra R. Alternatively, for any $\mathbb{C}$-algebra R we can write $R = \varinjlim R_i$ with R_i Noetherian. By fixing an embedding $G \subset SL_n(\mathbb{C})$ we can realize any $\gamma \in LG(R)$ as an $n \times n$ matrix with entries in R. Each entry lies in some R_i and it follows there is a single R_i such that $\gamma \in LG(R_i)$. Then we can apply fppf descent to obtain a G-bundle with trivializations on C_{R_i} and pull everything back to C_R.

For the inverse to Θ_C, let $\overline{\gamma} \in (LG/L^+G)(R)$. Then there is a faithfully flat base change $R \to R'$ such that we can present $\overline{\gamma}$ as $\gamma \rightrightarrows (\gamma_1, \gamma_2)$ with $\gamma \in LG(R')$ as discussed below (3.4). Set $\lambda = \gamma_1\gamma_2^{-1}$. Let $(P', \tau_C, \tau_D) = \Theta_{C,D}^{-1}(\gamma)$ and let $(P_i'', \tau_{i,C}, \tau_{i,D}) = \Theta_{C,D}^{-1}(\gamma_i)$ for $i = 1, 2$ be the two different pull backs to $C_{R''}$ where $R'' = R' \otimes_R R'$. The group $L^+G(R'')$ acts on $(P_i'', \tau_{i,C}, \tau_{i,D})$ by changing the trivialization $\tau_{i,D}$. We have $\lambda := \gamma_2^{-1}\gamma_1 \in L^+G(R'')$ and evidently $(P_1'', \tau_{1,C}, \tau_{1,D}) = (P_2'', \tau_{2,C}, \tau_{2,D})\lambda$. Applying the forgetful map f_D we see $(P_1'', \tau_{1,C}) = (P_2'', \tau_{2,C})$ in $T(R'')$ and therefore this data descends to $(P, \tau_C) \in T(R)$. The argument for Θ is similar and omitted. $\square$

We now describe a few variants of the DCC. The descent lemma [BL95] of Beauville and Lazlo in general will not apply to these variants but we can still argue by filtering by Noetherian subrings as in the proof above.

Suppose $\mathcal{M}$ is a moduli space of sheaves of sets on a smooth curve C with a marked point p such that for all $P \in \mathcal{M}$ we have $P|_{C-p} \in \mathcal{M}_G(C-p)$. Suppose further that all objects are isomorphic over D and the set of automorphisms of $P|_D$ is a subgroup $H \subset LG$. Let $T_{\mathcal{M}}$ denote the moduli of space of pairs (P, τ) where τ is a trivialization of $P|_{C-p}$. Then the DCC yields maps

$$\begin{aligned} T_{\mathcal{M}} &\xrightarrow{\Theta_C^H} LG/H \\ \mathcal{M} &\xrightarrow{\Theta^H} L_C G \backslash LG/H. \end{aligned} \tag{3.6}$$

For example, we can take $\mathcal{M}$ be the moduli space of quasi parabolic bundles with a reduction to a parabolic $Q \subset G$ at $p \in C$. Then $H = L_Q^+G = \{\gamma \in L^+G | \gamma(0) \in Q\}$.

Consider a nodal curve C_0 with single node p; we have $\operatorname{Spec} \hat{\mathcal{O}}_{C_0,p} \cong \operatorname{Spec} \mathbb{C}[[x,y]]/xy = D_0$ so $D_0^* = \operatorname{Spec} \mathbb{C}((x)) \times \mathbb{C}((y))$ and $LG \times LG$ takes the roles of LG and $G^\Delta = G(\mathbb{C}[[x,y]]/xy)$ takes the role of L^+G. The DCC yields

$$\begin{aligned} T &\xrightarrow{\Psi_{C_0}} L_xG \times L_yG/G^\Delta \\ \mathcal{M}_G(C_0) &\xrightarrow{\Psi} L_{C_0}G \backslash L_xG \times L_yG/G^\Delta. \end{aligned} \tag{3.7}$$

We can generalize as before to a moduli stack $\mathcal{M}$ of sheaves of sets on the nodal curve C_0 such that for all $P \in \mathcal{M}$ we have $P|_{C_0 - p} \in \mathcal{M}_G(C_0 - p)$ and all objects are isomorphic over D_0 and $Aut(P|_{D_0})$ is a subgroup of $L_xG \times L_yG$. Defining $T_{\mathcal{M}}$ in an analogous manner we obtain

$$\begin{aligned} T_{\mathcal{M}} &\xrightarrow{\Psi^H_{C_0}} L_xG \times L_yG/H \\ \mathcal{M}_G(C_0) &\xrightarrow{\Psi^H} L_{C_0}G\backslash L_xG \times L_yG/H. \end{aligned} \tag{3.8}$$

For example, we could take $\mathcal{M}$ to be the moduli of quasi parabolic G-bundles with a reduction of the structure group to a parabolic $Q \subset G$ at the node p. Then $H = \{(\gamma_1, \gamma_2) \in L^+_QG \times L^+_QG | \gamma_1(0) = \gamma_2(0)\}$ where $L^+_QG = \{\gamma \in L^+G | \gamma(0) \in Q\}$.

Another variant is to take a twisted curve $C_{[k]}$ with twisted point p and a smooth coarse moduli space C. We choose a kth root w of z so that $C_{[k]} \times_C D = [\operatorname{Spec} \mathbb{C}[[w]]/\mu_k]$ where $\zeta \in \mu_k$ acts by $w \mapsto \zeta w$. Let $\mu_k \xrightarrow{\eta} G$ be a homomorphism; the proof of lemma 3.4.4 shows we can take this to be the restriction of a co-character $\mathbb{C}^\times \xrightarrow{\eta} G$. Then $\zeta \in \mu_k$ acts on $L_wG = G((w))$ by $g(w) \xrightarrow{\eta} \eta(\zeta)^{-1} g(\zeta w) \eta(\zeta)$; this action is explained in the proof of proposition 3.4.2. Let $(L_wG)^{\mu_k}$ denote the invariants. Then for $g(z) \in LG = G((z))$ the assignment $g(z) \mapsto g_\eta(w) := \eta(w) g(w^k) \eta(w)^{-1}$ defines an isomorphism $LG \xrightarrow{\eta(w)(\quad)\eta(w)^{-1}} (L_wG)^{\mu_k}$ and in this way allows us to consider $L_CG \subset LG$ as a subgroup of $(L_wG)^{\mu_k}$.

Let $\mathcal{M}_{G,\eta}(C_{[k]})$ be the moduli stack of G-bundles on $C_{[k]}$ with equivariant structure at p determined by η. Let $T_{G,\eta}$ be the moduli of paris (P, τ) with $P \in \mathcal{M}_{G,\eta}(C_{[k]})$ and τ a trivialization of P over $C_{[k]} - p$. The DCC yields

$$\begin{aligned} T_{G,\eta} &\xrightarrow{\Theta^\eta_C} (L_wG)^{\mu_k}/(L^+_wG)^{\mu_k} \\ \mathcal{M}_{G,\eta}(C_{[k]}) &\xrightarrow{\Theta^\eta} (L_CG)^{\mu_k}\backslash(L_wG)^{\mu_k}/(L^+_wG)^{\mu_k}. \end{aligned} \tag{3.9}$$

Finally, we can consider a fixed twisted nodal curve $C_{0,[k]}$ with twisted node p and coarse moduli space C_0. We choose kth roots u, v of x, y so that $C_{0,[k]} \times_{C_0} D_0 = [\operatorname{Spec} \frac{\mathbb{C}[[u,v]]}{uv}/\mu_k]$ where $\zeta \in \mu_k$ acts by $(u, v) \mapsto (\zeta u, \zeta^{-1} v)$. We define $\mathcal{M}_{G,\eta}(C_{0,[k]}), T_{G,\eta}$ similarly as above. The DCC yields

$$\begin{aligned} T_{G,\eta} &\xrightarrow{\Psi^\eta_{C_0}} (L_uG \times L_vG)^{\mu_k}/(G^\Delta)^{\mu_k} \\ \mathcal{M}_G(C_0) &\xrightarrow{\Psi^\eta} (L_{C_0}G)^{\mu_k}\backslash(L_uG \times L_vG)^{\mu_k}/(G^\Delta)^{\mu_k}. \end{aligned} \tag{3.10}$$

3.4 Bundles on twisted curves and twisted chains

Here we investigate G-bundles on twisted nodal curves. The motivation to consider these objects comes from the valuative criterion for completenss. Specifically it comes from the following local calculation.

Let C_S be as in 3.2 and $f\colon S \to S$ any morphism. Let $C_{S,f}$ denote the base change and $C_{S^*,f} := C_{S,f} \times_S S^*$. The valuative criterion requires that we provide, for any G-bundle P

on the smooth curve $C_{S^*,f}$, an object F on $C_{S,f}$ such that $F|_{C_{S^*,f}}$ is P; we assume that F is at least a sheaf of sets.

By abuse of notation let f denote also the map on rings $\mathbb{C}[[s]] \xrightarrow{f} \mathbb{C}[[s]]$. Assuming $f(s) \neq 0$ we can, after suitable change of coordinates, normalize f so that $f(s) = s^k$, with $k \neq 0$. We can further restrict to $k \geq 1$, otherwise f maps to the generic point and the base change is a family of smooth curves. When $k \geq 2$, let $C_{S,[k]}$ denote the twisted curve obtained from C_S by removing a formal disc D_S around the node and gluing in the quotient stack $[D_S^{\frac{1}{k}}/\mu_k]$; see 3.2 for definitions. Then there is a map $C_{S,[k]} \to C_{S,f}$ realizing the latter as the coarse moduli space of the former. By abuse of notation let $p \in C_{S,[k]}$ also denote the twisted node, then $C_{S,[k]} \to C_{S,f}$ restricts to an isomorphism $C_{S,[k]} - [p] \cong C_{S,f} - p$.

Proposition 3.4.1. *Let p be the node in $C_{S,f}$. Let P be a G-bundle on $C_{S^*,f}$. There is a G-bundle P' on $C_{S,f} - p$ extending P. If $k = 1$ then P' extends to a G-bundle on $C_{S,f}$. If $k > 1$ then there is a G-bundle P'' on $C_{S,[k]}$ that restricts to P' under the isomorphism $C_{S,[k]} - [p] \cong C_{S,f} - p$.*

Proof. Let K be the generic point of C_{S^*}. In [dJHS11, Corollary 1.5] it is shown that $H^1(K, G) = 1$. Therefore we can extend P over the generic point of C_0 by taking it to be trivial in a neighborhood of this point. Thus we have extended P on the complement of a codimension 2 subset. The surface $C_{S,f} - p$ is always smooth and by [CTS79, Thm 6.13] the G-bundle extends to all of $C_{S,f} - p$. When $k = 1$ the surface $C_{S,f}$ is smooth and applying again [CTS79] covers this case.

We now assume $k \geq 2$. By the above, it suffices to study F in a neighborhood of the node. So we restrict to $D_S = \operatorname{Spec} \mathbb{C}[[x, y]]$ and then $D_{S,f} = \operatorname{Spec} \mathbb{C}[[x, y, s]]/(xy - s^k)$. The basic observation is that $\mathbb{C}[[x, y, s]]/(xy - s^k)$ is the ring of μ_k invariants in $\mathbb{C}[[u, v]]$ with action given by $\zeta(u, v) = (\zeta u, \zeta^{-1} v)$ and we identify $u^k = x, v^k = y, uv = s$. By [BF10, Prop. 3.7], every G-bundle on $D_{S,f} - p = \operatorname{Spec} \mathbb{C}[[x, t]][x^{-1}] \cup \operatorname{Spec} \mathbb{C}[[y, t]][y^{-1}]$ is the restriction of a G-bundle on $[\operatorname{Spec} \mathbb{C}[[u, v]]/\mu_k]$; the result is stated for $\operatorname{Spec} \mathbb{C}[x, y, s]/(xy - s^k)$ but the same proof works in our case. Consequently there is a G-bundle on $[D_S^{\frac{1}{k}}/\mu_k]$ that extends P over the node. □

Fixed curve

We now enter into an analysis of G-bundles on twisted curves. Let $C_{[k]}$ denote a twisted curve with smooth coarse moduli space C and a single twisted point p with stabilizer group μ_k. We show G-bundles P on $C_{[k]}$ can be identified with torsors $\mathcal{F}$ on C and that the moduli of such $\mathcal{F}$ on C is not isomorphic to $\mathcal{M}_G(C)$. This represents an obstruction to completing $\mathcal{M}_G(C_S)$ by only parametrizing degenerations of G-bundles on C_0; one should include degenerations of G-bundles on $C_{0,[k]}$ or degenerations of torsors on C_0.

Let $\eta \in \hom(\mathbb{C}^\times, T) \otimes_{\mathbb{Z}} \mathbb{Q}$ be an exotic co-character so that the associated parahoric $\mathcal{P} = P(\eta)$ is exotic (see 3.2). Let $\mathcal{G}^{std}$ be the sheaf of groups defined by $C \supset U \mapsto \hom_{Sch}(U, G)$. Let $\mathcal{G}^{\mathcal{P}}$ be the sheaf of group constructed in 3.3; namely $\mathcal{G}^{\mathcal{P}}|_{C-p} = \mathcal{G}^{std}$ and $\mathcal{G}^{\mathcal{P}}(\hat{\mathcal{O}}_{C,p}) = \mathcal{P}$.

Let $T_{\mathcal{G}^{\mathcal{P}}}(C)$ be the moduli space of pairs $(\mathcal{F},\tau)$ consisting of a $\mathcal{G}^{\mathcal{P}}$-torsor $\mathcal{F}$ on C together with a trivialization τ over $C-p$. Similarly, let $T_{G,\eta}(C_{[k]})$ be the moduli space of pairs (P,τ) consisting of a G bundle P on $C_{[k]}$ with equviariant structure determined by η (see (3.3) in 3.2 and the paragraph below it) and a trivialization τ on $C-p$. Define $T_{\mathcal{G}^{\mathcal{P}}}(D)$ and $T_{G,\eta}([D^{\frac{1}{k}}/\mu_k])$ similarly with $C-p$ replaced by $D-p$.

Proposition 3.4.2. *Suppose $k\eta \in \hom(\mathbb{C}^\times, T)$. Let $\mathcal{G}^{\mathcal{P}}$,$C$,$C_{[k]}$, $D = \operatorname{Spec}\mathbb{C}[[z]]$,$[D^{\frac{1}{k}}/\mu_k]$ be as above. Choose a kth root w or z so that $D^{\frac{1}{k}} = \operatorname{Spec}\mathbb{C}[[w]]$. Let $i_{[k]}\colon [D^{\frac{1}{k}}/\mu_k] \to C_{[k]}$ and $i\colon D \to C$ be the natural maps. Then we have isomorphisms*

$$\begin{tikzcd} T_{\mathcal{G}^{\mathcal{P}}}(D) & T_{\mathcal{G}^{\mathcal{P}}}(C) \arrow[l, "i^*"'] \arrow[r, "\Xi_C"] \arrow[d, "\Theta^{\mathcal{P}}_C"'] & T_{G,\eta}(C_{[k]}) \arrow[r, "i^*_{[k]}"] \arrow[d, "\Theta^{\eta}_C"] & T_{G,\eta}([D^{\frac{1}{k}}/\mu_k]) \\ & LG/\mathcal{P} \arrow[r, "\eta(\ )\eta^{-1}"] & (L_wG)^{\mu_k}/(L^+_wG)^{\mu_k} & \end{tikzcd}$$

where Ξ_C is defined to be $(\Theta^{\eta}_C)^{-1}\circ\eta(\ \ )\eta^{-1}\circ\Theta^{\mathcal{P}}_C$ and $\Theta^{\mathcal{P}}_C$ is the map in (3.6), Θ^{η}_C the map in (3.9), and $\eta(\ \ )\eta^{-1}$ is $g(z)\mathcal{P}\mapsto \eta(w)g(w^k)\eta(w)^{-1}(L^+_wG)^{\mu_k}$.

Proof. Using descent theory as in the proof of theorem 3.3.1 we construct inverses to $i^*, i^*_{[k]}$. Let $(P_R,\tau)\in T_{\mathcal{G}^{\mathcal{P}}}(D_R)$; after a flat base change $R\to R'$, the pullback of P_R become trivial and comparing with τ defines a loop $\in LG(R')$. By gluing with the trivial bundle over $C-p$, we obtain a bundle with a fixed trivialization over $C-p\times\operatorname{Spec}R$. Again by descent theory this is well defined and gives an inverse map $T_{\mathcal{G}^{\mathcal{P}}}(D)\to T_{\mathcal{G}^{\mathcal{P}}}(C)$ similarly we have an inverse map $T_G([D^{\frac{1}{k}}/\mu_k])\to T_G(C_{[k]})$.

To establish that $\Theta^{\mathcal{P}},\Theta^{\eta}$ are isomorphisms it suffices to show their restrictions $T_{\mathcal{G}^{\mathcal{P}}}(D)\to LG/\mathcal{P}$ and $T_{G,\eta}([D^{\frac{1}{k}}/\mu_k])\to (L_wG)^{\mu_k}/(L^+_wG)^{\mu_k}$ define isomorphisms. In the first case this follows because a point $LG/\mathcal{P}$ defines descent data for an object in $T_{\mathcal{G}^{\mathcal{P}}}(D)$.

To handle the equivariant case we need to compute the μ_k equivariant automorphisms of $\operatorname{Spec}\mathbb{C}((w))\times G$ over $\operatorname{Spec}\mathbb{C}((w))$. In order for $\gamma\in L_wG = G((w))$ to define an equiviariant automorphism of $\operatorname{Spec}\mathbb{C}((w))\times G$ (and thus determine an element of $(L_wG)^{\mu_k}$) we need for $\zeta\in\mu_k$

$$\begin{tikzcd} (w,g) \arrow[r, "\zeta"] \arrow[d, "\gamma"] & (\zeta w, \eta(\zeta)g) \arrow[d, "\gamma"] \\ (w,\gamma(w)g) \arrow[r, "\zeta"] & (\zeta w, \eta(\zeta)\gamma(w)g) = (\zeta w, \gamma(\zeta w)\eta(\zeta)g) \end{tikzcd}$$

Or $\gamma(w) = \eta(\zeta)^{-1}\gamma(\zeta w)\eta(\zeta)$. Thus we are concerned with invariants for the action of μ_k given by $\gamma(w)\xrightarrow{\zeta}\eta(\zeta)^{-1}\gamma(\zeta w)\eta(\zeta)$.

We can now argue as before to establish $T_{G,\eta}([D^{\frac{1}{k}}/\mu_k])\to G((w))^{\mu_k}/G[[w]]^{\mu_k}$ is an isomorphism. It remains to check $LG/\mathcal{P}\to (L_wG)^{\mu_k}/(L^+_wG)^{\mu_k}$ is an isomorphism.

Let $\gamma \in G((z))$ and set $\gamma_\eta(w) := \eta(w)\gamma(w^k)\eta(w)^{-1}$; the following shows $\gamma_\eta(w) \in (L_wG)^{\mu_k}$:

$$\begin{aligned}\gamma_\eta(w) &\xrightarrow{\zeta} \eta(\zeta)^{-1}\eta(\zeta w)\gamma(w^k)\eta(\zeta w)^{-1}\eta(\zeta)\\ &= \gamma_\eta(w)\end{aligned}$$

where we have used that $(\zeta w)^k = w^k$ and $\eta(\zeta w) = \eta(\zeta)\eta(w)$. Similarly, one can check for any $g(w) \in G((w))^{\mu_k}$ that $g^\eta(w) = \eta(w)g(w)\eta(w)^{-1} \in G((z))$ by checking it is invariant under the action $g^\eta(w) \mapsto g^\eta(\zeta w)$; thus $LG \xrightarrow{\eta(\ )\eta^{-1}} ((L_wG)^{\mu_k})$ is an isomorphism.

Now let $\gamma \in P(\eta)$. We show in this case $\gamma_\eta \in G[[w]]$. It is sufficient to do this at the level of Lie algebras again because the groups involved are connected. In particular, $Lie(P(\eta))$ has a basis consisting of elements of the form $X_\alpha z^i$ where X_α is the root space associated to α. We have $\eta(w)X_\alpha z^i\eta(w)^{-1} = X_\alpha w^{k\langle\eta,\alpha\rangle}z^i$. Now the value of $\langle\eta,\alpha\rangle$ is a rational number between -1 and 1. We can check if this is in $\mathfrak{g}[[w]]$ by checking if $k\langle\eta,\alpha\rangle + ki \geq 0$. But this is equivalent to $\langle\eta,\alpha\rangle + i \geq 0$. Finally, $X_\alpha z^i \in Lie(P(\eta))$ implies that $\lim_{t\to 0} t^{\langle\eta,\alpha\rangle+i}X_\alpha z^i$ exists which guarantees that $\langle\eta,\alpha\rangle + i \geq 0$. Altogether, we see $LG \xrightarrow{\eta(\ )\eta^{-1}} ((L_wG)^{\mu_k})$ descends to an isomorphism as in the statement of the proposition. $\square$

Let $\mathcal{M}_{G^{\mathcal{P}}}(C)$ be the moduli stack of $\mathcal{G}^{\mathcal{P}}$-torsors on C and $\mathcal{M}_{G,\eta}(C_{[k]})$ be the moduli space of G bundle on $C_{[k]}$ with equivariant structure determined by η.

Corollary 3.4.3. *Suppose $k\eta \in \hom(\mathbb{C}^\times, T)$. The isomorphism $\Xi_C \colon T_{\mathcal{G}^{\mathcal{P}}}(C) \to T_{G,\eta}(C_{[k]})$ of proposition 3.4.2 descends to an isomorphism $\Xi \colon \mathcal{M}_{G^{\mathcal{P}}}(C) \to \mathcal{M}_{G,\eta}(C_{[k]})$*

Proof. In light of the previous proposition, the argument is purely formal and follows as in the proof of theorem 3.3.1; see also [BL94b, prop.3.4].

Let P be a G-bundle on $C_{[k]}$. The restriction of P to $C - p$ is a G-bundle. By [DS95] it is trivial. Consequently the forgetful map $T_{G,\eta}(C_{[k]}) \to \mathcal{M}_{G,\eta}(C_{[k]})$ is essentially surjective and equivariant for the action of $L_CG = G[C-p]$ which changes the trivialization. It descends to give a map $L_CG\backslash T_{G,\eta}(C_{[k]}) \to \mathcal{M}_{G,\eta}(C_{[k]})$ and one can construct an inverse by associating to P the set of trivializations over $C - p$. The same argument holds for a $G^{\mathcal{P}}$-torsor $\mathcal{F}$ on C. We obtain isomorphisms $\mathcal{M}_{G^{\mathcal{P}}}(C) \xrightarrow{\sim} L_CG\backslash LG/\mathcal{P}$ and $\mathcal{M}_{G,\eta}(C_{[k]}) \xrightarrow{\sim} L_CG\backslash(L_wG)^{\mu_k}/(L_w^+G)^{\mu_k}$. Finally, the isomorphism $LG/\mathcal{P} \xrightarrow{\eta(\ )\eta^{-1}} (L_wG)^{\mu_k}/(L_w^+G)^{\mu_k}$ gives an isomorphism $L_CG\backslash LG/\mathcal{P} \to L_CG\backslash(L_wG)^{\mu_k}/(L_w^+G)^{\mu_k}$ which establishes the result. $\square$

Remark 6. In [VB], Balaji and Seshadri develop similar results in the context of Bruhat-Tits group schemes. For corollary 3.4.3 see specifically [VB, Thm 5.2.2].

Remark 7. For $G = SL_n$ all the parahorics of LG are conjugate by elements in LGL_n to subgroups $L_Q^+G \subset L^+G$ where $Q \subset G$ is a parabolic. Consequently the resulting moduli spaces can be identified with moduli spaces of vector bundles with in general nontrivial determinant. However in general the parahorics will no longer be even abstractly isomorphic and thus neither will be the resulting moduli spaces. For example, SP_4 has a parahoric whose Levi factor is $SL_2 \times SL_2$ which distinguishes it from the standard parahoric $SP_4[[z]]$.

Remark 8. Let η_i be the vertices of Al. Define k_i as the minimum integer such that $k_i \cdot \eta_i \in \hom(\mathbb{C}^\times, T)$ and set $k_G = lcm(k_i)$. The η_i correspond to the maximal parahorics $\mathcal{P}_i$ of LG and further any parahoric $\mathcal{P}$ is conjugate to a subgroup of some $\mathcal{P}_i$. It follows readily that $k = k_G$ is the minimum value of k for which the statement of corollary 3.4.3 holds for any particular parahoric $\mathcal{P}$.

In section 3.5 we will need to fix the value of k; this is possible by remark 8 and lemma 3.4.4. To state it we introduce some notation. Let i be a positive integer and set $C_i^* = \operatorname{Spec} \mathbb{C}[[x, y, s]]/(xy - s^i) - (0, 0, 0)$. For any two positive integers i, j we can obtain C_{ij}^* as a base change either from C_i^* or C_j^*, that is the left commutative diagram induces the right commutative diagram:

$$\begin{tikzcd}
\mathbb{C}[[s]] & \mathbb{C}[[s]] \arrow[l, "s^j \mapsfrom s"'] \\
\mathbb{C}[[s]] \arrow[u, "s^i \mapsfrom s"] & \mathbb{C}[[s]] \arrow[l, "s^j \mapsfrom s"'] \arrow[u, "s^i \mapsfrom s"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C_{ij}^* \arrow[r] \arrow[d] & C_i^* \arrow[d] \\
C_j^* \arrow[r] & C_1^*
\end{tikzcd}$$

Lemma 3.4.4. *Let $k = k_G$ be as in remark 8 and let l be any positive integer. Let P be a G-bundle on C_l^*. Then there is a G-bundle P' on C_k^* such that $P \cong P'$ on C_{lk}^*.*

Proof. The curve C_l^* has an l-fold cover $C_1^* \to C_l^*$. By [BF10, Prop. 3.7], a G-bundle P on C_l^* is equivalent to a μ_l equivariant G-bundle on C_1^*, which in turn is determined by a homomorphism $\mu_l \to G$.

Let $\zeta \in \mu_l$ be a generator and $\mu_l \to G$ a homomorphism; by abuse of notation let ζ also denote the image in G. Then $\zeta \in G$ is a semisimple element any by [Hum75, Thm. 22.2], ζ lies in a Borel subgroup; by[Hum75, Cor. 19.3] it follows that ζ lies in a maximal torus T and thus we can take $\mu_l \to G$ to be the restriction of a co-character $\eta \in \mathbb{C}^\times \to T \subset G$, but for any such η, the co-character η^l will always define the trivial action. Thus, setting $\mathfrak{t}_\mathbb{Z} = \hom(\mathbb{C}^\times, T)$ and $\mathfrak{t}_\mathbb{Q} = \hom(\mathbb{C}^\times, T) \otimes_\mathbb{Z} \mathbb{Q}$, we can take $\eta \in \mathfrak{t}_\mathbb{Z}/l \cdot \mathfrak{t}_\mathbb{Z} \subset \mathfrak{t}_\mathbb{Q}/\mathfrak{t}_\mathbb{Z}$ where the inclusion is given by $\eta \mapsto \frac{1}{l}\eta$.

Further, identifying η with $\mathbb{C}^\times \xrightarrow{(id,\eta)} \mathbb{C}^\times \times T \subset L^\ltimes G$, we can also transform by the affine Weyl group $W^{aff} := N_{L^\ltimes G}(\mathbb{C}^\times \times T)/(\mathbb{C}^\times \times T)$ and thus assume $\frac{1}{l}\eta \in Al$. For some subset $I \subset \{0, 1, \ldots, r = rk(G)\}$, we can express $\eta = \sum_{i \in I} a_i \eta_i$ with $a_i \in (0, 1) \cap \mathbb{Q}$ and $\eta_i \in \mathfrak{t}_\mathbb{Q}$ the vertices of Al.

Consider $\eta_I = \sum_{i \in I} \eta_i$; because $k\eta_i \in \hom(\mathbb{C}^\times, T) \forall i$ we have that η_I determines a G-bundle P' on C_k^*. We claim P, P' pull back to isomorphic bundles on C_{kl}^*. For this it suffices to show $T_{G,\frac{1}{l}\eta}([D_S^{\frac{1}{kl}}/\mu_{kl}]) \cong T_{G,\eta_I}([D_S^{\frac{1}{kl}}/\mu_{kl}])$ and this in turn reduces to showing that the framed automorphism groups of P, P' coincide. The automorphism groups are connected so it reduces to a Lie algebra calculation. As these are subgroups of $G[[u, v]]$ (with $x = u^{kl}, y = v^{kl}$) the Lie algebra is spanned by formal sums of $X_\alpha u^i v^j$. If $i \geq j$ then this is $X_\alpha u^{i-j}(uv)^j$ and uv is fixed by μ_{kl} so we are reduced to the one variable case; we can argue analogously if $j \geq i$. Then the claim about automorphism groups follows because η_I

and $\frac{1}{l}\eta$ lie in the interior of the same face of Al; follow the argument in the paragraph after (3.2) in section 3.2. □

Fixed Nodal curve

Let $C_{0,[k]}$ be a twisted nodal curve with a single twisted node p. Let C_0 be it's coarse moduli space and by abuse of notation we also write $p \in C_0$ for the node. The stabilizer of $p \in C_{0,[k]}$ is μ_k and in particular $C_{0,[k]} \times_{C_0} D_0 \cong [D_0^{\frac{1}{k}}/\mu_k] =$.

For an parahoric $\mathcal{P}$ let $\mathcal{LU}$ be its Levi decomposition and set $\mathcal{P}^\Delta = \Delta(\mathcal{L}) \ltimes (\mathcal{U} \times \mathcal{U})$. Similarly as in 3.3 one can construct a sheaf of groups $\mathcal{G}^\Delta$ over C_0 such that $\mathcal{G}^\Delta(\hat{\mathcal{O}}_{C_0,p}) = \mathcal{P}^\Delta$ and $\mathcal{G}^\Delta|_{C_0 - p} = \mathcal{G}^{std}$. Let $\mathcal{M}_{\mathcal{G}^\Delta}(C_0)$ denote the moduli stack of $\mathcal{G}^\Delta$ torsors on C_0 and let $T_{\mathcal{G}^\Delta}(C_0)$ denote the moduli space of pairs $(\mathcal{F}, \tau)$ where $\mathcal{F} \in \mathcal{M}_{\mathcal{G}^\Delta}(C_0)$ and τ a trivialization of $\mathcal{F}$ over $C_0 - p$. Define $T_{\mathcal{G}^\Delta}(C_0)$ similarly.

Let $\eta \in \hom(\mathbb{C}^\times, T) \otimes_\mathbb{Z} \mathbb{Q}$ and consider the moduli stack $\mathcal{M}_{G,\eta}(C_{0,[k]})$ of G-bundles on $C_{0,[k]}$ with equivariant structure at p determined by η. Let $T_{G,\eta}(C_{0,[k]})$ denote the moduli space of pairs (P, τ) with $P \in \mathcal{M}_{G,\eta}(C_{0,[k]})$ and τ a trivialization of P on $C_{0,[k]} - p$. Define $T_{G,\eta}([D_0^{\frac{1}{k}}/\mu_k])$ similarly.

The arguments of proposition 3.4.2 and corollary 3.4.3 readily extend to nodal curves and we obtain

Proposition 3.4.5. *Suppose $k\eta \in \hom(\mathbb{C}^\times, T)$ and set $\mathcal{P} = \mathcal{P}(\eta)$. Let $\mathcal{G}^\Delta$,C_0,$C_{0,[k]}$, $D_0 = \operatorname{Spec} \mathbb{C}[[x,y]]/xy$, $[D_0^{\frac{1}{k}}/\mu_k]$ be as above. Choose kth roots u, v of x, y so that $D_0^{\frac{1}{k}} = \mathbb{C}[[u,v]]/uv$. Let $i_{0,[k]}\colon [D_0^{\frac{1}{k}}/\mu_k] \to C_{0,[k]}$ and $i_0\colon D_0 \to C_0$ be the natural maps. Let $G^\Delta_{u,v} = \{(g_1, g_2) \in L_u^+G \times L_v^+G | g_1(0) = g_2(0)\}$. Then we have isomorphisms*

$$\begin{tikzcd}
T_{\mathcal{G}^\Delta}(D_0) & T_{\mathcal{G}^\Delta}(C_0) \arrow[l, "i_0^*"'] \arrow[r, "\Xi_{C_0}"] \arrow[d, "\Psi_C^{\mathcal{P}^\Delta}"'] & T_{G,\eta}(C_{0,[k]}) \arrow[r, "i_{0,[k]}^*"] \arrow[d, "\Psi_C^\eta"] & T_{G,\eta}([D_0^{\frac{1}{k}}/\mu_k]) \\
& LG \times LG/\mathcal{P}^\Delta \arrow[r, "\eta(\ )\eta^{-1}"] & (L_uG \times L_vG)^{\mu_k}/(G^\Delta_{u,v})^{\mu_k} &
\end{tikzcd}$$

where Ξ_{C_0} is defined to be $(\Psi_C^\eta)^{-1} \circ \eta(\ )\eta^{-1} \circ \Psi_C^{\mathcal{P}^\Delta}$ and $\Psi_C^{\mathcal{P}^\Delta}$ is the map in (3.8)*, Ψ_C^η the map in* (3.10)*, and the last map is the product of $g(z)\mathcal{P} \mapsto \eta(w)g(w^k)\eta^{-1}(w)(L_w^+G)^{\mu_k}$. The isomorphism Ξ_C descends to an isomorphism $\Xi\colon \mathcal{M}_{\mathcal{G}^\mathcal{P}}(C_0) \to \mathcal{M}_{G,\eta}(C_{0,[k]})$.*

□

Connection with $\overline{L^\ltimes_{poly}G}$

In Chapter 2 a stacky orbit closure $\partial\mathcal{X}^{aff,poly}$ was constructed which is analogous to the boundary of the wonderful compactification of a semisimple adjoint group. In particular, the components are smooth and intersect transversely. Let $r = rk(G)$. There are $2^{r+1} - 1$

orbits $\mathbf{O}_I$ labeled by the subsets I of $\{0, \dots, r+1\}$ with $I \neq \emptyset$ and $\mathbf{O}_I$ is further described by:

Proposition 3.4.6. *Let $L_I, \mathcal{P}_I^{\pm}, \mathcal{U}_I^{\pm}$ be as in (3.2) in section 3.2 and let $Z_0(L_I)$ be the connected component of the center $Z(L_I)$ of L_I. Define $\mathcal{P}_I^{\Delta,\pm} = \Delta(L_I) \ltimes (\mathcal{U}_I \times \mathcal{U}_I^-)$. We have*

$$\mathbf{O}_I = \frac{L_{poly}G \times L_{poly}G}{Z_0(L_I) \times Z_0(L_I) \cdot \mathcal{P}_I^{\Delta,\pm}}$$

In particular, the orbit $\mathbf{O}_I$ fibers over $LG/\mathcal{P}_I \times LG/\mathcal{P}_I^-$ with fiber $L_{I,ad} = L_I/Z_0(L_I)$. Further, when I is a singleton set the group $Z_0(L_I)$ is trivial and when I has cardinality > 1 we have $Z_0(L_I) = Z(L_I)$.

The isomorphisms of proposition 3.4.5 allows us to identify $\mathbf{O}_{\{i\}}$ with $T_{G,\eta_i}([D_0^{\frac{1}{k}}/\mu_k])$, $T_{G,\eta_i}(C_{0,[k]})$; here η_i is the ith vertex of Al. The natural expectation is that $T_{G,\eta_i}([D_0^{\frac{1}{k}}/\mu_k])$ can further degenerate to a moduli problem parametrized by the higher co-dimensional orbits in $\overline{\mathbb{C}^\times \ltimes L_{poly}G}$ and similarly with $T_{G,\eta_i}(C_{0,[k]})$. We show that this is indeed the case in the next sub section.

G-bundles on Twisted Chains

In the previous section we saw that associated to the singleton sets $\{i\} \subset \{0, r+1\}$ there is a moduli space parametrizing G-bundles on a twisted nodal curve and further the moduli space can be identified with an orbit of the wonderful embedding of the loop group. In this section we introduce a more general moduli problem which we show is isomorphic to the orbit $\mathbf{O}_I$ in the wonderful embedding for any $I \subset \{0, \dots, r+1\}$.

Let R_n denote the rational chain of projective lines with n-components; figure 3.1 in the introduction depicts a chain of length 3. There is an action of $\mathbb{C}^\times$ on R_n which scales each component. Let $p_0, \dots, p_n$ denote the fixed points of this action.

Recall u, v are kth roots of x, y which are our coordinates near a node. Let p', p'' be denote the closed points of $\operatorname{Spec}\mathbb{C}[[u]]$, $\operatorname{Spec}\mathbb{C}[[v]]$ and finally let $D_n^{\frac{1}{k}}$ be the curve obtained from $\operatorname{Spec}\mathbb{C}[[u]] \sqcup R_n \sqcup \operatorname{Spec}\mathbb{C}[[v]]$ by identifying p' with p_0 and p'' with p_n.

The group μ_k acts on $D_n^{\frac{1}{k}}$ through its usual action on u, v and through the inclusion $\mu_k \subset \mathbb{C}^\times$ on the chain R_n. For an n-tuple $(\beta_0, \dots, \beta_n) \in \hom(\mathbb{C}^\times, T)^n$, we can speak about the equivariant G-bundles on $D_n^{\frac{1}{k}}$ with equivariant structure at p_i determined by β_i. We refer to this equivalently as a G-bundles on $[D_n^{\frac{1}{k}}/\mu_k]$ of type $(\beta_1, \dots, \beta_n)$.

Further, we can also glue $[D_n^{\frac{1}{k}}/\mu_k]$ to $C_0 - p_0$ to obtain a curve $C_{n,[k]}$. Let C_n denote the coarse moduli space of $C_{n,[k]}$.

We call C_n a *modification* of C_0 and $C_{n,[k]}$ a *twisted modification* of C_0. (3.11)

Recall the specific co-characters $\eta_0, \dots, \eta_r$ defined in (3.1) in 3.2. For $I = \{i_1, \dots, i_n\} \subset \{0, \dots, r\}$, let $T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$ denote the moduli space of pairs (P, τ) where P is a G-bundles on $[D_n^{\frac{1}{k}}/\mu_k]$ of type $(\eta_{i_1}, \dots, \eta_{i_n})$ and τ is a trivialization on $[\operatorname{Spec} \mathbb{C}((u)) \times \mathbb{C}((v))/\mu_k]$. Let $H = Aut(P)$ then restriction to $\operatorname{Spec} \mathbb{C}[[u]]$ and $\operatorname{Spec} \mathbb{C}[[v]]$ realizes $H \subset (L_u G)^{\mu_k} \times (L_u G)^{\mu_k}$.

Theorem 3.4.7. *Let $I \subset \{0, \dots, r\}$ and $T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$ be as above. Then there is an isomorphism*

$$T_{G,I}(C_{0,[k]}) \xrightarrow{\Psi^{\eta_I}} (L_u G)^{\mu_k} \times (L_u G)^{\mu_k}/H \xrightarrow{\eta_I^{-1}(\ \ )\eta_I} \frac{L_{poly}G \times L_{poly}G}{Z(L_I) \times Z(L_I) \cdot \mathcal{P}_I^{\Delta,\pm}}.$$

where Ψ^{η_I} is as in (3.10) and $\eta_I^{-1}(\ \ )\eta_I$ is described in proposition 3.4.5. Let $i \colon [D_n^{\frac{1}{k}}/\mu_k] \to C_{0,[k]}$ be the natural map. Then $i^* \colon T_{G,I}(C_{0,[k]}) \to [D_n^{\frac{1}{k}}/\mu_k]$ is an isomorphism. In particular, $T_{G,I}(C_{0,[k]})$, $T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$ are isomorphic to an orbit in the wonderful embedding of $\overline{L_{poly}^{\ltimes} G}$.

Proof. That i^* is an isomorphism follows formally so we focus on showing that $T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$ is isomorphic to the stated homogeneous space. We suppress the isomorphism $\eta_I^{-1}(\ \ )\eta_I$ and work inside $G((x)) \times G((y))$ with the help of the identification $[\operatorname{Spec} \mathbb{C}((u)) \times \mathbb{C}((v))/\mu_k] = \operatorname{Spec} \mathbb{C}((x)) \times \mathbb{C}((y))$.

The strategy is the same as in the proof of proposition 3.4.2 and corollary 3.4.3 above. Namely, fix an object (P, τ) of $T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$. The restriction of P to $\operatorname{Spec} \mathbb{C}[[x]] \sqcup \operatorname{Spec} \mathbb{C}[[y]]$ is necessarily trivial and comparing with τ produces loops in $G((x)) \times G((y)) = LG \times LG$. Loops are identified that differ by an automorphism of P over $[D_n^{\frac{1}{k}}/\mu_k]$; that is, an element of H. We will show $H \cong Z(L_I) \times Z(L_I) \cdot \mathcal{P}_I^{\Delta} \subset LG \times LG$. Then we notice that

$$\frac{LG \times LG}{Z(L_I) \times Z(L_I) \cdot \mathcal{P}_I^{\Delta}} \cong \frac{L_{poly}G \times L_{poly}G}{Z(L_I) \times Z(L_I) \cdot \mathcal{P}_I^{\Delta,\pm}},$$

The above isomorphism holds because for $\mathcal{P}_{I,poly} = \mathcal{P}_I \cap L_{poly}G$ we have $LG/\mathcal{P}_I = L_{poly}G/\mathcal{P}_{I,poly} \cong L_{poly}G/\mathcal{P}^-_{I,poly}$; these statements are proved in [Kum02, 7.4].

We turn now to computing $H = Aut(P)$. Let $H_u = Aut(P|_{[\operatorname{Spec} \mathbb{C}[[u]]/\mu_k]})$, $H_n = Aut(P|_{[R_n/\mu_k]})$ and $H_v = Aut(P|_{[\operatorname{Spec} \mathbb{C}[[v]]/\mu_k]})$. Let $ev_u \colon H_u \to G$ be the restriction of an automorphism to the special point; define ev_v similarly. Finally let $ev_{0,n} \colon H_n \to G \times G$ be the restriction of an automorphism to the two extreme points of $[R_n/\mu_k]$. Then we have $H = \{(f_u, f_n, f_v) | (ev_u(f_u), ev_v(f_v)) = ev_n(f_n)\} \subset H_u \times H_n \times H_v$.

By 3.4.2, we have $H_u \times H_v = P(\eta_{i_0}) \times P(\eta_{i_n})$.

We now compute H_n. Let $E = E(\eta_{i_1}, \dots, \eta_{i_n})$ denote $P|_{[R_n/\mu_k]}$. In fact, automorphisms of G-bundles on $[R_n/\mathbb{C}^\times]$ have been computed by Martens and Thaddeus in [MTa]. They consider a slightly different situation where they fix $\eta_{i_0} = \eta_{i_n} = 0$, but we can still use the same methods to handle our case.

Then results of [MTb] imply H_n is connected so we pass to $Lie(H_n) = H^0([R_n/\mu_k], adE)$. Let $ev_{0,n}$ denote also the map on Lie algebras $ev_{0,n}\colon H^0([R_n/\mu_k], adE) \to \mathfrak{g} \oplus \mathfrak{g}$. The map $ev_{0,n}$ is embedding because $\ker ev_{0,n} = H^0([R_n/\mu_k], adE(\eta_I) \otimes \mathcal{O}(-p_0 - p_n)) = 0$ by lemma 3.4.8. For a tuple of integers $(b_0, \ldots, b_n)$ let $\mathcal{O}(b_0, \ldots, b_n)$ denote line bundle on $[R_n/\mu_k]$ with equivariant structure at the fixed points p_i given by b_i. Then $adE \cong \oplus_{i=1}^{rk(G)} \mathcal{O}(0, \ldots, 0) \oplus_{\alpha \in \Delta} \mathcal{O}(\alpha \circ \eta_{i_1}, \ldots, \alpha \circ \eta_{i_n})$ and we can compute separately for each α.

For α a root of L_I we have $\alpha \cdot \eta_{i_j} = 0$ and these roots contribute a factor of $\Delta(Lie(L_I))$ to the image of $ev_{0,n}$. If α is negative, $\langle \alpha, \eta_{i_0} \rangle = 0$, and some other $\langle \alpha, \eta_{i_j} \rangle < 0$ then there is a consecutive subset $\{i_0, i_1, \ldots, i_{j'}\}$ such that $0 = \langle \eta_{i_0}, \alpha \rangle = \ldots = \langle \eta_{i_{j'-1}} \rangle$ and $\langle \eta_{i_{j'}}, \alpha \rangle < 0$. Then [MTa, 1.2(c)] implies $(X_\alpha, 0) \in \mathfrak{g} \oplus \mathfrak{g}$ lies in the image of $ev_{0,n}$; in fact [MTa, 1.2(c)] ensures there is such a $\mathbb{C}^\times$-invariant section which is then necessarily μ_k invariant. Similarly, if α is positive, $\langle \alpha, \eta_{i_n} \rangle = 0$ and some other $\langle \alpha, \eta_{i_j} \rangle > 0$ then the image contains $(0, X_\alpha)$.

There is a second contribution to the group H_n. Namely, we can lift $Aut(R_n) = (\mathbb{C}^\times)^n$ to E. Describe $R_n = \cup_{i=1}^n C_i$ as a chain of $\mathbb{P}^1$s going from left to right with fixed points $p_{j-1}, p_j \in C_j$ on the jth component. Let $(\mathbb{C}^\times)_j$ be the jth $\mathbb{C}^\times$ factor in $Aut(R_n)$. Then lifting $(\mathbb{C}^\times)_j$ to E requires a homomorphism $\mathbb{C}^\times \to G$ for each fixed point. This homomorphism must be $\eta_{i_{j-1}}$ at p_{j-1} and by continuity it must also be $\eta_{i_{j-1}}$ on all C_i with $i < j$. Similarly the lifting is determined by η_{i_j} on all C_i with $i > j$.

Let $P_I \subset G$ be the parabolic associated to the co-character $\eta_I = \sum_{i_j \in I} \eta_{i_j}$ and let $L_I U_I$ be its Levi decomposition. Each η_{i_j} maps into $Z(L_I)$ and under $ev_{0,n}$ generates a complement to $\Delta(Z(L_I)) \subset Z(L_I) \times Z(L_I)$. Also U_I consists of those X_α with $\alpha > 0$ such that $\langle \eta_{i_j}, \alpha \rangle > 0$ for some i_j. Altogether, we get that $H_n = Z(L_I) \times Z(L_I) \cdot \Delta(L_I) \ltimes (U_I^- \times U_I)$.

Consulting (3.2) in section 3.2 and comparing the computations of H_u, H_v, H_n, we conclude that $H = Z(L_I) \times Z(L_I) \cdot L_I \ltimes (\mathcal{U}_I^- \times \mathcal{U}_I) \cong Z(L_I) \times Z(L_I) \cdot \mathcal{P}_I^\Delta$. □

Lemma 3.4.8. *For $\{i_0, \ldots, i_n\} \subset \{0, \ldots, r\}$ let $E = E(\eta_{i_0}, \ldots, \eta_{i_n})$ be the G-bundle with splitting type $(\eta_{i_0}, \ldots, \eta_{i_n})$. Then $H^0([R_n/\mu_k], adE \otimes \mathcal{O}(-p_0 - p_n)) = 0$*

Proof. For a tuple of integers $(b_0, \ldots, b_n)$ let $\mathcal{O}(b_0, \ldots, b_n)$ denote line bundle on $[R_n/\mu_k]$ with equivariant structure at p_i given by b_i. Then $\mathcal{O}(-p_0 - p_n) = \mathcal{O}(-1, 0, \cdots, 0, 1)$. We remind the reader that a single subscript η_l denotes a specific co-character with l ranging from $\{0, \ldots, r\}$ and double subscripts η_{i_j} are used to denote ordered subsets $\{i_1, \ldots, i_n\} \subset \{0, \ldots, r\}$.

We have $adE = \oplus_{i=1}^{rk(G)} \mathcal{O}(0, \ldots, 0) \oplus_{\alpha \in \Delta} \mathcal{O}(\alpha \circ \eta_{i_0}, \ldots, \alpha \circ \eta_{i_n})$. Clearly the trivial summand poses no problem. By symmetry we can focus on α positive, in which case we show that

$$H^0([R_n/\mu_k], \mathcal{O}(\alpha \circ \eta_{i_0}, \ldots, \alpha \circ \eta_{i_n}) \otimes \mathcal{O}(-p_0 - p_n)) = H^0([R_n/\mu_k], \mathcal{O}(\alpha \circ \eta_{i_0} - 1, \ldots, \alpha \circ \eta_{i_n} + 1)) = 0.$$

Because all the η_i are in the Weyl alcove we have all $\alpha \circ \eta_{i_j} \geq 0$. Also for the longest root $\theta = \sum_i n_i \alpha_i$ we have $1 = \theta \circ \eta_j = \sum_i n_i(\alpha_i \circ \eta_j)$ and all the $n_i \geq 1$. This implies $\alpha_i \circ \eta_j = \frac{1}{n_i} \delta_{i,j}$.

Express our fixed $\alpha = \sum_i m_i \alpha_i$ with $0 \leq m_i \leq n_i$. Let us first establish some properties of $\mathcal{O}(\alpha \circ \eta_l, \alpha \circ \eta_{l'})$ on $[\mathbb{P}^1/\mu_k]$. The degree of $\mathcal{O}(\alpha \circ \eta_l, \alpha \circ \eta_{l'})$ is

$$d = k\alpha \circ (\eta_l - \eta_{l'}) = k(\frac{m_I}{n_l} - \frac{m_{l'}}{n_{l'}}) \tag{3.12}$$

and we note d is an integer with $|d| \leq k$. Provided $d \geq 0$ the global sections are spanned by the μ_k-invariant monomials $x_0^{d-d'} x_1^{d'}$ where x_0 has weight $\frac{k}{n_l}$ and x_1 has weight $\frac{k}{n_{l'}}$.

We examine the restriction of $\mathcal{O}(\alpha \circ \eta_{i_0}, \ldots, \alpha \circ \eta_{i_n}) \otimes \mathcal{O}(-p_0 - p_n)$ to various components. Clearly we can restrict to those components where the degree is $d > 0$. Below, when we restrict to the j component $[\mathbb{P}^1/\mu_k]$ we set $l = i_{j-1}$ and $l' = i_j$.

Suppose we restrict to a component with $d, m_{l'} > 0$ then $n_l \geq m_l > d$ so all monomials of degree d have nonzero weight provided the weight is less than k. This holds because x_0^d has the highest weight and it is $d\frac{k}{n_l}$ which is less than k. Consequently there are no sections.

We now assume $m_{l'} = 0$. If we restrict to the first component then tensoring with $\mathcal{O}(-p_0 - p_n)$ lowers the degree by 1 and again there are no sections. Otherwise, if the degree of the jth component of the bundle is k then there are two sections x_0^k, x_1^k that we must show cannot extend to a global section. Assume x_1^k is non vanishing at p_j. The degree on the $j+1$ component is either 0 or negative. If the degree is 0 then on the $j+2$ component the degree is either 0 or negative. Thanks to tensoring with $\mathcal{O}(-p_0 - p_n)$ we are certain to eventually get a negative bundle which has no sections. Therefore the section x_1^k cannot extend. But to extend the section x_0^k on the $j-1$ component we need a section on bundle with degree $d > 0$ and $m_{l'} > 0$ which is impossible by the previous paragraph. □

Remark 9. For comparison with the $\mathbb{C}^\times$ equivariant automorphisms see [MTa, 2.13,2.19].

Remark 10. In (3.12) we concluded that the degrees of the bundle on the chain have to be bounded by k. It is worth noting that this recovers the moduli problem considered for GL_r by Kausz [Kau05a]. In this case one can work on non twisted curves; that is, with $k = 1$. Then Gieseker bundles are exactly vector bundles on modifications of the curve such that the restriction to a chain splits as a direct sum $\mathcal{O}$ and $\mathcal{O}(1)$ and $H^0(R_n, E(-p' - p'')) = 0$. The latter condition implies $H^0(R_n, adE(-p' - p'')) = 0$.

3.5 Twisted Gieseker Bundles

In this section we begin with a curve C_S as in section 3.2 and construct an algebraic S-stack $\mathcal{X}_G(C_S)$ such that $\mathcal{M}_G(C_S) \subset \mathcal{X}_G(C_S)$ is a dense open substack and the boundary is a divisor with normal crossings. Further we show the morphism $\mathcal{X}_G(C_S) \to S$ is complete.

For the remainder of this section we fix a simple group G as in section 3.2 and further fix an integer $k = k_G$ as in remark 8. The only exception is proposition 3.5.1 where k can be any integer ≥ 1.

For convenience, we recall some of the notation from 3.2. Namely, $S = \operatorname{Spec} \mathbb{C}[[s]]$, $S^* = \operatorname{Spec} \mathbb{C}((s))$, $S_0 = \operatorname{Spec} \mathbb{C}[[s]]/(s) = \operatorname{Spec} \mathbb{C}$, $C_0 = C_{S_0}$. For B an S-scheme we set $B^* = B \times_S S^*$, $B_0 = B \times_S S_0$. We also have $D_S = \operatorname{Spec} \mathbb{C}[[x, y]]$ considered as an S-scheme via $s \mapsto xy$ and $D_0 = \operatorname{Spec} \mathbb{C}[[x, y]]/(xy)$. Further, we set $D_S^{\frac{1}{k}} := \mathbb{C}[[u, v]]$ where $u^k = x$ and $v^k = y$. Then $D_{S,[k]} = [D_S^{\frac{1}{k}}/\mu_k]$; the coarse moduli space of $D_{S,[k]}$ is $\operatorname{Spec} \mathbb{C}[[x, y, s]]/(xy - s^k)$. We further fix $p \in C_S$ to be the node.

To define $\mathcal{X}_G(C_S)$ we need to define twisted modifications of C_S; this is a relative version of (3.11). Then in subsection 3.5 we define $\mathcal{X}_G(C_S)$ to be the moduli stack parametrizing G-bundles on twisted modifications. There we prove the main theorem which shows that $\mathcal{X}_G(C_S)$ satisfies the valuative criterion for completeness.

Twisted Modifications

Let C_S be a nodal curve. A *modification of length* $\leq n$ of C_S over B is a curve C'_B over B with a morphism $C'_B \xrightarrow{\pi} C_B$ such that

- C'_B is flat over B and π is finitely presented and projective
- $C'_{B^*} \xrightarrow{\pi} C_{B^*}$ is an isomorphism
- for $b \in B_0$ the map of curves $C'_b \xrightarrow{\pi} C_b$ is a modification; that is the fiber $\pi^{-1}(p_b)$ over the unique node $p_b \in C_b$ is a rational chain of $\mathbb{P}^1$s with at most n components and there is $b \in B_0$ such that $\pi^{-1}(p_b)$ has exactly n components.

Example 1. In [Gie84, 4.2], Gieseker constructs modifications of length n over $B = \mathbb{C}[[t_1, \ldots, t_{n+1}]]$ mapping to $\mathbb{C}[[s]]$ via $s \mapsto t_1 \cdots t_{n+1}$ and where the maximum number of components in the modification is reached only over $(0, \ldots, 0)$. Further, the ith node is locally described by $B[[x, y]]/(xy - t_i)$.

We recall the construction for $n = 1$; it is sufficient for our purposes to work with the curve $D_S = \mathbb{C}[[x, y]]$. The base change to $\mathbb{C}[[t_1, t_2]]$ is $D_{[[t_1,t_2]]} := \operatorname{Spec} \mathbb{C}[[t_1, t_2, x, y]]/(xy - t_1 t_2)$ and the modification $D'_{[[t_1,t_2]]}$ is the blow up of $D_{[[t_1,t_2]]}$ along the ideal (x, t_1). The fibers of the map $D'_{[[t_1,t_2]]} \to \operatorname{Spec} \mathbb{C}[[t_1, t_2]]$ agree with those of $D_{[[t_1,t_2]]}$ except over $(0, 0)$ where the node has been replaced by a chain of length 1.

By a series of analogous blowups we obtain a modification $D'_{[[t_1,\ldots,t_{n+1}]]} \xrightarrow{f} D_{[[t_1,\ldots,t_{n+1}]]}$ of D_S over $\operatorname{Spec} \mathbb{C}[[t_1, \ldots, t_{n+1}]]$ such that for $\emptyset \neq I \subset \{1, \ldots, n+1\}$ the fiber of f over $\{t_i = 0\}_{i \in I}$ is a modification $D_{|I|-1}$ of D_0 of length $|I| - 1$. This local construction extends to give a modification $C'_{[[t_1,\ldots,t_{n+1}]]}$ of C_S over $\operatorname{Spec} \mathbb{C}[[t_1, \ldots, t_{n+1}]]$. Gieseker in fact proves this construction gives a versal deformation of the curve C_n in (3.11). We utilize this in the proof of theorem 3.5.2.

Let $(g_1, \ldots, g_n) \in (\mathbb{C}^\times)^n$ act on $\mathbb{C}[[t_1, \ldots, t_{n+1}]]$ by $(t_1, \ldots, t_n) \xrightarrow{(g_1,\ldots,g_n)} (g_1 t_1, \frac{g_2}{g_1} t_2, \ldots, \frac{g_n}{g_{n-1}} t_n, \frac{1}{g_n} t_{n-1})$. This action extends to $C'_{[[t_1,\ldots,t_{n+1}]]}$ such that for every closed point $q \in \operatorname{Spec} \mathbb{C}[[t_1, \ldots, t_{n+1}]]$ the stabilizer of q in $(\mathbb{C}^\times)^n$ coincides with $Aut(C'_q/C_q)$. We set $Mdf_n = [\mathbb{C}[[t_1, \ldots, t_{n+1}]]/(\mathbb{C}^\times)^n]$. This is an algebraic S-stack that comes equipped with a curve $[C'_{[[t_1,\ldots,t_{n+1}]]}/(\mathbb{C}^\times)^n]$ and the modifications of C_S over B that arise from S-maps $B \to Mdf_n$ we call *local modifications* of length $\leq n$.

A *twisted modification* of length $\leq n$ of C_S over B is a twisted curve $\mathcal{C}'_B$ such that its coarse moduli space C'_B is a modification of length $\leq n$ of C_S over B. A twisted modification is of *order* k if the order of the stabilizer group of every twisted point has order exactly k.

Similarly, a twisted modification is of *order* $\leq k$ if the order of the stabilizer of every twisted point has order $\leq k$. A *local twisted modification* $\mathcal{C}'_B$ is a twisted modification whose coarse moduli space C'_B is a local modification. In the rest of this paper we work primarily with (twisted) local modifications.

Remark 11. Restricting to local modifications is probably unnecessary but it simplifies our arguments and is sufficient to prove the main theorem.

Let Mdf_n^{tw} denote the functor that assigns to $B \to S$ the groupoid of twisted local modifications of C_S over B of length $\leq n$. Let $Mdf_n^{tw,k} \subset Mdf_n^{tw,\leq k}$ be the functors of twisted local modifications of order k and order $\leq k$ respectively.

Proposition 3.5.1. *Let $k \geq 1, n \geq 0$ be integers. The functors Mdf_n^{tw} and $Mdf_n^{tw,\leq k}$ are algebraic stacks. Further $Mdf_n^{tw,\leq k} \subset Mdf_n^{tw}$ is an open substack and $Mdf_n^{tw,k} \subset Mdf_n^{tw}$ is a closed algebraic substack. Further, all of these stacks are locally of finite type.*

Proof. The basic tool is to use the stack of all genus g curves. For an integer g let $\mathcal{S}_g$ denote the functor on Sch which to any scheme B assigns the groupoid of all (not necessarily stable) genus g nodal curves $C \to B$. In [AOV11, A] it is shown that $\mathcal{S}_g$ is an algebraic stack locally of finite type; see also [Ols07, §5]. If $C'_B \to C_B$ is a local modification then forgetting the map to C_B defines a morphism $Mdf \to \mathcal{S}_g$.

Let $\mathfrak{M}_g^{tw}$ be the functor which to any scheme B assigns a genus g twisted curve $\mathcal{C} \to B$. In [AOV11, A] it is shown that $\mathfrak{M}_g^{tw}$ is algebraic with a representable map to $\mathcal{S}_g$. Further the sub functor $\mathfrak{M}_g^{tw,\leq k} \subset \mathfrak{M}_g^{tw}$ of twisted curves with twisting or order $\leq k$ is an open algebraic substack. Then the result follows from

$$Mdf_n^{tw} = Mdf_n \times_{\mathcal{S}_g} \mathfrak{M}_g^{tw}$$
$$Mdf_n^{tw,\leq k} = Mdf_n \times_{\mathcal{S}_g} \mathfrak{M}_g^{tw},$$

and that $Mdf_n^{tw,k}$ is the closed substack $Mdf_n^{tw,\leq k} \backslash Mdf_n^{tw,\leq k-1}$ where we have used [dJea, 06FJ,0509] to conclude that open and closed substacks behave as expected. ☐

Given a twisted modification $\mathcal{C}'_B$, we define subschemes p_B, p'_B and p'_b for $b \in B_0$ by the fiber product diagrams:

$$\begin{tikzcd} p_B \arrow[r] \arrow[d] & p \arrow[d] \\ C_B \arrow[r] \arrow[d] & C_S \arrow[d] \\ B \arrow[r] & S \end{tikzcd} \qquad \begin{tikzcd} p'_B \arrow[r] \arrow[d] & p \arrow[d] \\ \mathcal{C}'_B \arrow[r] \arrow[d] & C_S \arrow[d] \\ B \arrow[r] & S \end{tikzcd} \qquad \begin{tikzcd} p'_b \arrow[r] \arrow[d] & p_0 \arrow[d] \\ \mathcal{D}'_b \arrow[r] \arrow[d] & D_0 \arrow[d] \\ b \arrow[r] & S_0 \end{tikzcd}$$

Where $\mathcal{D}'_B := \mathcal{C}'_B \times_{C_B} D_B$ and $\mathcal{D}'_b$ is the restriction to $b \in B$.

Notice that p'_b is nothing other than the rational chain of $\mathbb{P}^1$s that appear in a modification over the fiber of the node. Further, p_B and p'_B are defined so that the map $\mathcal{C}'_B - p'_B \to C_B - p_B$ is an isomorphism.

The definition of twisted Gieseker bundles and the completeness statement

Let $r = rk(G)$; if $\mathcal{C}'_B$ is a twisted modification of length $\leq r$, then a G-bundle on $\mathcal{C}'_B$ is called *admissible* if the co-characters determining the equivariant structure at all nodes are linearly independent over $\mathbb{Q}$ and are given by a subset of $\{\eta_0 \ldots, \eta_r\}$; see (3.1) in section 3.2 for the definition of the η_i.

Let B be an S-scheme. Define a groupoid $\mathcal{X}_G(C_S)$ over S-schemes by the assignment

$$\mathcal{X}(C_S)(B) = \left\langle \begin{array}{l} P_B \\ \downarrow \\ \mathcal{C}'_B \longrightarrow C_B \end{array} \right\rangle$$

where $\mathcal{C}'_B$ is a twisted local modification of C_B and P_B is an admissible G-bundle on $\mathcal{C}_B$. Isomorphisms are commutative diagrams

$$\begin{tikzcd} P_B \arrow[rr, "\cong"] \arrow[d] & & Q_B \arrow[d] \\ \mathcal{C}'_B \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{C}''_B \arrow[dl] \\ & C_B & \end{tikzcd}$$

For notational convenience we abbreviate $\mathcal{X}_G(C_S)(B)$ as $\mathcal{X}_G(B)$.

Theorem 3.5.2. *The functor $\mathcal{X}_G = \mathcal{X}_G(C_S)$ is an algebraic stack locally of finite type. It contains $\mathcal{M}_G(C_S), \mathcal{M}_G(C_{S^*})$ as dense open substacks and the complement of $\mathcal{M}_G(C_{S^*})$ is a divisor with normal crossings.*

Proof. We first show that $\mathcal{X}_G$ is a stack fibered in groupoids. Namely, we show (1) for $x, y \in \mathcal{X}_G(B)$ that $U \to Isom(x|_U, y|_U)$ is a sheaf on Sch/B and (2) descent data is effective.

Objects x, y as above consist of G-bundles on twisted modifications of order k of some fixed length. By proposition 3.5.1, (1) and (2) holds for twisted local modifications and so it's enough to check (1) and (2) on the additional data of G-bundles on a twisted modification. By definition, G-bundles are determined by local gluing data (so (2) holds). Further, given two G-bundles P, Q we can identify the isomorphisms $P \to Q$ as the sections of $P \times Q/G$ over the base and this forms a sheaf so (1) holds.

To show $\mathcal{X}_G$ is algebraic we adapt a proof [Hei10, Prop.1] of Heinloth; namely we will verify Artin's axioms [dJea, 07Y3]. First we recall some deformation theory of G-bundles. Let A be a local Artin $\mathbb{C}[[s]]$-algebra with maximal ideal m and residue field k. Let $I \subset A$ be a nilpotent ideal such that $mI = 0$. An object $x \in \mathcal{X}_G(A/I)$ can be identified with a G-bundle $\overline{P}$ on a twisted curve $\mathcal{C}'_{A/I}$. If P is an extension of $\overline{P}$ over A then the auomorphisms of P inducing the identity on $\overline{P}$ are classified by $H^0(\mathcal{C}'_{A/I}, ad(\overline{P}) \otimes_{A/I} I)$. The possible extensions are classified by $H^1(\mathcal{C}'_{A/I}, ad(\overline{P}) \otimes_{A/I} I)$ and obstructions lie in $H^2(\mathcal{C}'_{A/I}, ad(\overline{P}) \otimes_{A/I} I) = 0$; see [Har10] for the case of GL_r and for general G this can be deduced from the proof of [Hei10, Prop.1].

Artin's axioms can be stated as (1) $\Delta\colon \mathcal{X}_G \to \mathcal{X}_G \times \mathcal{X}_G$ is representable by algebraic spaces, (2) If $B = \varprojlim B_i$ with B, B_i affine then $\varinjlim \mathcal{X}_G(B_i) \to \mathcal{X}_G(B)$ is an equivalence, (3) $\mathcal{X}_G$ satisfies the Rim-Schlessinger (RS) condition, (4) $H^i(\mathcal{C}'_{A/I}, ad(\overline{P}) \otimes_{\mathbb{C}} I)$ for $i = 0, 1$ are finite dimensional where $I = (\epsilon) \subset \mathbb{C}[\epsilon]/\epsilon^2 = A$, (5) formal objects come from Noetherian complete local rings $R \supset m$ with R/m finite type over S, and (6) $\mathcal{X}_G$ satisfies openness of versality. We elaborate on (3),(5),(6) when we verify them below.

We also use that any algebraic stack locally of finite type over a locally noetherian base automatically satisfy (1) - (6); see [dJea, 07SZ]. In particular, the algebraic stack $Mdf_n^{tw,k}$ of proposition 3.5.1 satisfies (1) - (6).

By [LMB00, Cor.3.13], we can verify (1) by showing $Isom(x, y)\colon Sch/U \to Sets$ is representable by an algebraic space for every $x, y \in \mathcal{X}_G(U)$. The objects x, y can be identified with G-bundles P, Q over a fixed curve $\mathcal{C}'_U$. Then $Isom(x, y)$ can be identified with the sheaf of sections of $P \times_G Q = P \times Q/G$ crossed with $Aut(\mathcal{C}'_U/C_U)$ which is an algebraic space by [KM97, thm 1.1].

Statement (2) amounts to showing for any $P \to \mathcal{C}'_B$ there is an index j, a modification $\mathcal{C}'_{B_j}$ and a G-bundle $P_j \to \mathcal{C}'_{B_j}$ such that $P \to \mathcal{C}'_B$ is pulled back from $P_j \to \mathcal{C}'_{B_j}$. Because twisted local modifications form an algebraic stack we can reduce to showing this for the G-bundles. That is there is a fixed k such that if we define $\mathcal{C}'_{B_{j+k}}$ as the pull back of $\mathcal{C}'_{B_j}$ under $B_{j+k} \to B_j$ then $\mathcal{C}'_B = \varprojlim \mathcal{C}'_{B_{j+k}}$. We must then show there is a j such that $P \to \mathcal{C}'_B$ is pulled back from $P_{j+k} \to \mathcal{C}'_{B_{j+k}}$ and this follows because G-bundles are finitely presented.

For the RS condition suppose we have a pushout $Y' = Y \sqcup_X X'$ with (1)X, X', Y, Y' spectra of local Artin rings of finite type over S and (2) $X \to X'$ a closed immersion. Then the RS condition states that the functor $\mathcal{X}_G(Y') \to \mathcal{X}_G(Y) \times_{\mathcal{X}_G(X)} \mathcal{X}_G(X')$ is an equivalence of categories. We show the functor is essentially surjective; that it is fully faithful is a formal argument we omit.

The condition holds with $\mathcal{X}_G$ replaced with $Mdf_n^{tw,k}$ so we can assume the following situation

$$\begin{tikzcd} \mathcal{C}'_X \arrow[r] \arrow[d] & \mathcal{C}'_{X'} \arrow[d] \\ \mathcal{C}'_Y \arrow[r] & \mathcal{C}'_{Y'} \end{tikzcd}$$

where all curves are pulled back from $\mathcal{C}'_{Y'}$. We further have G-bundles $P_X, P_{X'}, P_Y$ on the respective curves such that $P_{X'}, P_Y$ extend P_X. We can consider $P_{X'} \in \mathcal{M}_G(\mathcal{C}'_{Y'})(X')$ and similarly for P_X, P_Y. The stack $\mathcal{M}_G(\mathcal{C}'_{Y'})$ is algebraic by lemma 3.5.3. The latter satisfies the RS condition so there is a G-bundle $P_{Y'}$ extending all others and it is necessarily admissible because otherwise the bundles $P_X, P_{X'}, P_Y$ would not be admissible.

Statement (4) follows readily because we work with twisted curves which have projective coarse moduli spaces.

A formal object is a triple $\zeta = (R, \zeta_n, f_n)$ where (R, m) is a Noetherian complete ring, $\zeta_n \in \mathcal{X}_G(\operatorname{Spec} R/m^n)$ and $\zeta_n \xrightarrow{f_n} \zeta_{n+1}$ are morphisms over $\operatorname{Spec} R/m^n \to \operatorname{Spec} R/m^{n+1}$. There is a notion of morphisms of formal objects and they form a category. Any $\psi \in \mathcal{X}_G(R)$ gives rise to a formal object by restriction along $\operatorname{Spec} R/m^n \to \operatorname{Spec} R$; this is a functor from $\mathcal{X}_G(R)$ to formal objects over R. We must show this is an equivalence. We show it is essentially surjective; that it is fully faithful follows formally.

The argument is similar to the verification of (3). Assume now (R, ζ_n, f_n) is a formal object of $\mathcal{X}_G$. Forgetting the data of the G-bundle produces a formal object of $Mdf_l^{tw,k}$ where l is the length of modification at the closed point of $\operatorname{Spec} R$. Because $Mdf_l^{tw,k}$ is algebraic, the formal objects comes from a twisted modification $\mathcal{C}'_R$. Now the original data of the G-bundles on the various $\mathcal{C}'_{R/m^n} = \mathcal{C}'_{\operatorname{Spec} R} \times_{\operatorname{Spec} R} \operatorname{Spec} R/m^n$ define a formal object of the algebraic stack $\mathcal{M}_G(\mathcal{C}'_R)$ and hence there is a G-bundle extending them which, as in the verification of condition (3), is necessarily admissible.

Openness of versality is explained precisely in [dJea, 07XP] but using the Kodoira-Spencer map [Har10, 2.7], as in [Hei10, Prop.1], the statement can be simplified. Let $P_R \to \mathcal{C}'_R$ be an object of $\mathcal{X}_G(\operatorname{Spec} R)$ and let P_{univ} be the universal bundle over $\mathcal{C}'_R \times_R \mathcal{M}_G(\mathcal{C}'_R)$ and let π be the projection to $\mathcal{M}_G(\mathcal{C}'_R)$. Then P_R gives a map $\operatorname{Spec} R \xrightarrow{f} \mathcal{M}_G(\mathcal{C}'_R)$ and there is an induced Kodoira-Spencer map $\mathcal{T}_R \to f^*(R^1_{\pi,*} ad(P_{univ}))$ where $\mathcal{T}_R$ denote the tangent sheaf of $\operatorname{Spec} R$. Openness of versality means that this map being surjective is an open condition which follows because the locus where a map of coherent sheaves is surjective is open. We conclude that $\mathcal{X}_G(C_S)$ is algebraic.

Let η_i be the vertices of Al, then $D_i := \mathcal{M}_{G,\eta_i}(C_{0,[k]}) \subset \mathcal{X}_G - \mathcal{M}_G(C_{S^*})$ and because we have fixed the value of k, D_i appears only once in the boundary. Further, the proof of theorem 3.5.4 below shows any object $\in \mathcal{X}_G - \mathcal{M}_G(C_{S^*})$ is in the closure of some D_i hence $\mathcal{X}_G - \mathcal{M}_G(C_{S^*}) = \cup_{i=0}^r \overline{D_i}$ and thus $\mathcal{M}_G(C_{S^*})$ is an open sub stack. Using theorem 3.4.7 we conclude that $\cup_{i=0}^r \overline{D_i}$ can be presented as a quotient of $\mathcal{X}^{aff,poly}$ from Chapter 2; the former has simple normal crossing singularities. Finally, $\mathcal{M}_G(C_S) = \mathcal{X}_G - \cup_{i \neq 0} \overline{D_i}$, which is open. $\square$

Lemma 3.5.3. *Let $\mathcal{C} \to B$ be a twisted curve over a locally noetherian base $\mathbb{C}$-scheme B and let H be an affine algebraic group over $\mathbb{C}$. Then the functor $\mathcal{M}_H(\mathcal{C}_B)$ which assigns to any $B' \to B$ the groupoid of principal H-bundles on $\mathcal{C}_B \times_B B'$ is an algebraic stack locally of finite type.*

Proof. Writing $pt = \operatorname{Spec} \mathbb{C}$, we have $[pt/H] \to pt$ is a morphism of finite presentation hence so is $[pt/H] \times B \to B$. We observe that $\mathcal{M}_H(\mathcal{C}_B) = Hom_B(\mathcal{C}, [pt/H] \times B)$ and apply [Aok06b] to conclude the result. Note we must check an additional condition from [Aok06a]; namely that $\mathcal{M}_H(\mathcal{C}_B)$ satisfies condition (5) stated in the proof of theorem 3.5.2:

$$Hom_R(\mathcal{C}_R, [pt/H] \times \operatorname{Spec} R) \to \varprojlim Hom_{R/m^n}(\mathcal{C}_{R/m^n}, [pt/H] \times \operatorname{Spec} R/m^n) \tag{3.13}$$

is an equivalence for any $\operatorname{Spec} R \to B$ with R a complete local Noetherian ring R. By [Ols07], after an étale extension on the base, there is a finite flat morphism $Z \to \mathcal{C}$ over B with Z

a projective scheme. As in [Aok06b, pg. 50], we can verify (3.13) after replacing $\mathcal{C}$ with Z. Then $Hom_B(Z, [pt/H] \times B) = \mathcal{M}_H(Z_B)$ which is an algebraic stack locally of finite type by [Wan]; in particular (3.13) holds by [dJea, 07SZ]. □

We now come to the main theorem

Theorem 3.5.4. *Let $R = \mathbb{C}[[s]]$ and $K = \mathbb{C}((s))$; for a finite extension $K \to K'$ let R' denote the integral closure of R in K'. Given the right commutative square below, there is finite extension $K \to K'$ and a dotted arrow making the entire diagram commute:*

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r, "h^*"] \arrow[d] & \mathcal{X}_G(C_S) \arrow[d] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr, dashed, "h"'] & \operatorname{Spec} R \arrow[r, "f"] & S \end{tikzcd}$$

Proof. If f factors through $S^* \subset S$ then the morphism h^* determines a G-bundle on the smooth curve C_{S^*} and completeness of $\mathcal{M}_G(C_{S^*})$ assures we can extend this to a G-bundle over $C_{S^*} \times_{S^*} \operatorname{Spec} R'$ which produces the required morphism h.

Assume now f is surjective. As in proposition 3.4.1, normalize f so it is given by $s \mapsto s^l$ for $l \geq 1$. Also by proposition 3.4.1, the map h^* amounts to a G-bundle P on $\mathbb{C}[[x, y, s]]/(xy - s^l) - (0, 0, 0)$. By lemma 3.4.4, after a finite base change, we can identify P with the restriction of a G-bundle on twisted curve of order k. Moreover, we can further suppose the equivariant structure of the bundle is determined by a co-character η which lies in a face of Al. If η happens to be one of the vertices η_i of Al then we've determined an objects of $\mathcal{X}_G(C_S)$ extending E.

In general η lies in a higher dimensional face of Al and there is a subset $I = \{\eta_{i_1}, \ldots, \eta_{i_n}\} \subset \{0, \ldots, r\}$ such that $P(\eta) = \mathcal{P}_I$ where $\mathcal{P}_I$ is defined in (3.2) section 3.2. Let $D_S^{'\frac{1}{k}}$ be the iterated blowup of $D_S^{\frac{1}{k}} = \operatorname{Spec} \mathbb{C}[[u, v]]$ such that $D_S^{'\frac{1}{k}} \xrightarrow{\pi} \operatorname{Spec} \mathbb{C}[[u, v]]$ is a modification of length $\leq n - 1$ and let $E(\eta_I)$ be the bundle on $[D_S^{'\frac{1}{k}}/\mu_k]$ determined by fixing the equivariant structure at the jth node in $\pi^{-1}(0, 0)$ to be η_{i_j}. Because η, η_I lie in the same face of Al they determine isomorphic bundles hence $E(\eta_I)$ yields an object in $\mathcal{X}_G(C_S)$ extending E.

Finally, if f is the map $s \mapsto 0$ then the map h^* define an admissible G bundles P_{h^*} on a curve $C_{n,[k]}$ as in (3.11). By definition, there is a subset $I = \{i_j\} \subset \{0, \ldots, r\}$ of cardinality $n + 1$ such that $P_{h^*}|_{D_{n,[k]}}$ is an equivariant bundle with equivariant structure at $p_j \in D_{n,[k]}$ determined by η_{i_j}.

After potentially a faithfully flat base change $\operatorname{Spec} R' \to \operatorname{Spec} R$ the bundle is trivial on the complement of the chain $\cong C_0 - p_0$. By theorem 3.4.7, fixing a trivialization defines a morphism $\operatorname{Spec} K' \to T_{G,I}(D_{n,[k]}) = \frac{L_{poly}G \times L_{poly}G}{Z(L_I)\Delta(L_I) \ltimes (\mathcal{U}_I^- \times \mathcal{U}_I)}$. Further we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K'^{*} \arrow[r, "\phi"] \arrow[d] & \frac{L_{poly}G\times L_{poly}G}{Z(L_I)\Delta(L_I)\ltimes(\mathcal{U}_I^-\times\mathcal{U}_I)} \arrow[d] \\ \operatorname{Spec} R' \arrow[r, "\overline{\phi}"] & L_{poly}G/\mathcal{P}_I^- \times L_{poly}G/\mathcal{P}_I. \end{tikzcd} \tag{3.14}$$

This follows because $L_{poly}G/\mathcal{P}_I^- \times L_{poly}G/\mathcal{P}_I$ is a projective ind variety. Let $H = L_{poly}G \times L_{poly}G$, $H_1 = \Delta(L_I) \ltimes (\mathcal{U}_I^- \times \mathcal{U}_I)$ and $H_2 = \mathcal{P}_I^- \times \mathcal{P}_I$. Identifying $K' = \mathbb{C}((s))$ and using that $\cup_{m\geq 1}\mathbb{C}((s^{1/m}))$ is the algebraic closure of $\mathbb{C}((s))$ we conclude that after another base change $\operatorname{Spec} K'' \to \operatorname{Spec} K'$ the element $\phi \in H/H_1((s))$ lifts to an element $\phi' \in H((s^{1/m}))$ for some m. The fact that ϕ extends to a map $\overline{\phi}$ on $\mathbb{C}[[s]]$ means that ϕ' has a factorization $\phi' = \phi''\psi$ where $\phi'' \in H[[s^{1/m}]]$ and $\psi \in H_2((s^{1/m}))$. By applying a change of trivialization over the normalization $\widetilde{D_0}$ we can replace ϕ' with ψ. Then using the Levi decomposition of H_2 we can factor $\psi = \psi_L \times \psi_U$ where $\psi_L \in L_I((s^{1/m})) \times L_I((s^{1/m}))$ and $\psi_U \in \mathcal{U}_I^-((s^{1/m})) \times \mathcal{U}_I((s^{1/m}))$. Finally by applying a suitable automorphism over $D_{n,[k]}$ we can replace ψ simply with ψ_L. Altogether the map ϕ induces a morphism $\psi_L\colon \mathbb{C}((s^{1/m})) \to H_2 \to H_2/H_1 \cong L_I$. By abuse of notation let the composition also be denoted ψ_L. Since we have only changed ϕ by automorphisms and extensions of the variable, the map ψ_L is in the same isomorphism class of ϕ.

Using the Bruhat decomposition for loop groups we conclude $\psi_L \in L[[s^{1/m}]]\eta'(s^{1/m})L[[s^{1/m}]]$, where we again can take η' to be in the affine Weyl alcove. Then as in the previous case we find a subset $I' \subset \{0,\dots,r\}$ such that $P(\eta') = P(\eta_{I'})$ where $\eta_{I'} = \sum_{i_j\in I'}\eta_{i_j}$ and use this to construct an object in $\mathcal{X}_G(C_S)$ extending h^*.

This degeneration terminates when the subset $I = \{0,\dots,r\}$ because then the right vertical map in (3.14) is an isomorphism. $\square$

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