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Nonlinear Stability Criteria for Elastic Rod Structures

by

Daniel Martínez Peters

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Engineering − Mechanical Engineering

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Oliver M. O'Reilly, Chair
Professor Benson H. Tongue
Professor Steven L. Lehman

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Abstract

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Branched elastic rod structures are abundant in engineering and nature, in applications ranging from MEMS devices to human spine models. While buckling is well-understood for problems of this type, stability is often difficult to assess, especially when the model is derived from a nonlinear rod theory. The purpose of this research is to establish criteria for determining nonlinear stability, based upon the minimization of an energy functional. By utilizing variational principles, and Legendre's classical work in particular, a new necessary condition for stability featuring the existence of bounded solutions to a set of Riccati differential equations is established. For a single rod, building on classical results, this condition is also shown to be sufficient for stability.

The stability criteria are demonstrated on a number of examples using a simple, planar rod theory. These examples range from a classical strut under axial load to a branched tree-like structure composed of several rods. In the branched model, the stability analysis consists of finding bounded solutions to a set of Riccati equations, which are coupled at branching points. The number of Riccati equations corresponds to the number of rods in the structure. The resulting condition is only necessary for stability of a branched structure, as a sufficient condition could not be established. However, this is the first instance of a stability criterion for branched structures that is based on the second variation of the total energy. The advantage is that this method provides a systematic means of identifying unstable, and therefore physically unrealizable, configurations of a branched structure. Finally, an extension of the stability criteria to other rod theories is discussed.

To my parents, Lou and Lee,

for their unconditional love, support and guidance,

without which I could not have completed this work.

&

To my brother, David,

for serving as a shining example

and inspiring me to never settle for less than my best effort.

Contents

Acknowledgments

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I also wish to thank the other members of my committee, Professors Benson Tongue and Steven Lehman, for agreeing to serve in this capacity. The time they devoted to reading this dissertation and, most importantly, the feedback they provided was crucial to its success.

Most of the research covered in this dissertation was the result of two journal papers [42, 43]. That work was partially supported by grant number CMMI-0726675 from the U. S. National Science Foundation. I would also like to thank the following researchers for their helpful comments during the completion of those papers: Prof. David Steigmann, Prof. George Leitmann, Prof. Robert S. Manning, Prof. John Prussing, Prof. Florian Wagener, Prof. Vera M. Zeidan and Prof. Alexandr Ivanov. A special thanks goes to the two anonymous referees for their constructive criticisms and suggestions on an earlier version of the paper [42].

My labmates also deserve a special mention, for they were sometimes closely involved in the work that went into this dissertation. I especially want to thank Tim Tresierras for assisting with the branched stability work and for contributing immensely to the numerical simulations that are contained here and in both papers [42, 43]. I also thank Bayram Orazov, Nur Adila Faruk Senan, David Moody, Miguel Christophy and Lucie Huet for their assistance and inspiration.

Finally, I must also thank my extended family for their love, support and prayers, which helped sustain me throughout graduate school. And last, but by no means least, I must thank the friends who not only helped keep me focused, but also kept me sane during the last 5 years of my life: Ryan Sochol, Chris Zueger, George & Saira Mseis, Armon Mahajerin, Coleman Kronawitter, John Edmiston, Jeff Seifried, Tommy Cisneros, Michele Kotiuga, Adam Stooke, Andrew Myers, Chris "Ohio" Hogue, Andrew Stevens, Chuck Steadham, Hayley Currier, Adrienne Higa, Alex Hegyi, Rishi Nijhon and many others. You all helped make this an unforgettable experience and for that I am forever grateful.

Chapter 1

Introduction

1.1 Branched Tree-Like Structures

Tree-like structures formed by long flexible branches are ubiquitous in nature and the mechanics of these, and other plant structures, have been studied for centuries. Of particular interest is the use of rod theories to provide the mechanical models. These theories have been used by, among many others, Greenhill [20] to predict the maximum height of a tree, by McMillen & Goriely [39] to understand the mechanics of tendril perversion, and by Silk et al. [56] to model the evolution of a rice panicle. Many recent works which use rod theories to model tree-like structures have been motivated by the need to develop realistic computer graphics-based images of trees and forests. Incorporating biomechanical aspects of plant development, such as growth and the formation of residual stresses, makes this work particularly challenging. Despite these difficulties, many of the resulting images of trees exhibit extraordinary detail and realism (see the works of Prusinkiewicz and his coworkers [9, 14, 13, 24]).

One of the difficulties inherent in computing equilibrium configurations for tree-like structures modeled using elastic rods is the algebraic complexity of solving the coupled boundary-value problems. It is natural to wonder if multiple configurations are possible, and recently O'Reilly & Tresierras [45] have shown that this is indeed the case. For example, consider the three equilibrium configurations shown in figure 1.1. Here, two child branches are joined to a parent branch at a single point. All three branches are subject to terminal loadings, deform under their self-weight, and, for the parameter values selected to construct this example, three possible solutions to the boundary-value problem coexist.

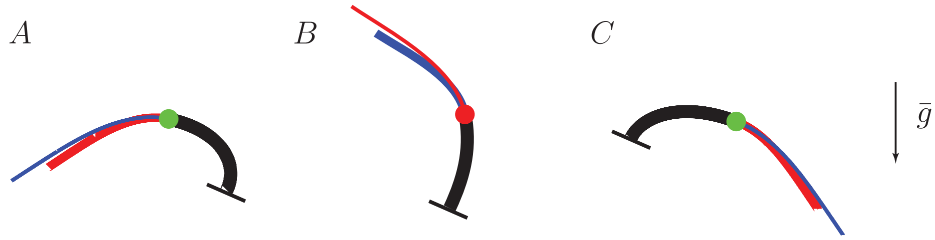

Figure 1.1 Three possible solutions to a boundary-value problem featuring three heavy elastic rods connected at a branching point. Solutions A and C satisfy the necessary condition L1 while B does not.

1.2 Buckling and Stability

The existence of the three solutions in figure 1.1 is indicative of the buckling phenomenon. It is worth recalling that buckling, as first elucidated by Euler in the mid-18$^{\text{th}}$ century, is caused when a bifurcation occurs in the solution to the equilibrium equations for a terminally loaded rod. As the load increases from zero, a critical value is reached whereby any further load produces additional configurations of the rod. Determining the trivial and buckled solutions of the rod is generally straightforward, as the governing equations lead to a two-point boundary value problem. However, establishing the nonlinear stability of the straight and buckled configurations is not a trivial problem. In fact, to this date, no stability criteria for a branched tree-like structure have been established, which is a problem this dissertation seeks to address.

1.3 Variational Approach

Our goal is to devise a methodology to formally analyze the nonlinear stability of a general elastic rod structure. The most appropriate metric for stability is to determine which configurations minimize the total energy of the rod. Therefore, the rich history of the calculus of variations provides the most useful tools for analysis. The main avenue can be traced back to Max Born's seminal thesis on equilibrium configurations of elastic rods in 1906 [5], and it involves using Jacobi's analysis of the second variation of the total energy to examine the possible existence of conjugate points. The non-existence of conjugate points can be considered as a necessary condition for the second variation to be non-negative, and consequently a necessary condition for stability. This has proven to be the most popular approach in modern analyses of elastic rod stability [23, 29, 32, 35, 37, 53]. Alternatively, there is a treatment of the second variation due to Legendre in 1786 that proved to be more fruitful in our work. This approach results in the necessary condition that a bounded solution to a Riccati differential equation must exist over the interval of interest. As discussed in two classic texts on calculus of variations [4, 15], the results from Legendre's treatment are equivalent to the Jacobi treatment for certain sets of boundary conditions. In addition, Legendre's treatment features prominently in the optimal control literature (see, e.g., [2, 6]), and therefore two different, but often equivalent, variational formulations of stability are possible for elastic rod problems.

For a single terminally loaded branch deforming under its own weight, the stability analysis consists of examining a scalar Riccati equation for bounded solutions. In our earlier work, O'Reilly & Peters [43], we established and verified these stability criteria for terminally loaded elastic struts. For this classical problem, the advantage is that the conditions for stability are both necessary and sufficient. In a more recent paper [42], we extend the stability criteria to tree-like structures composed of elastic rods. The resulting necessary condition, which we denote by L1, for the minimization of the second variation features a set of Riccati equations, one for each branch, which are coupled at the branching points. If a solution to each individual Riccati equation can be found, then L1 is satisfied. The condition includes the appealing result that a necessary condition for a branched structure to be stable is that each branch in the structure should be stable. Another set of new necessary conditions, which we denote by B3, is also established. These conditions pertain to the case where the abscissa of the branching point is variable. This variability is present in situations where the branches are held together by an adhesive at a branching point. Sufficient conditions for stability of branched tree-like structures remain to be found.

1.4 Outline of Dissertation

An outline of the dissertation is as follows: large portions of our papers ([42, 43]) concerning the necessary variational principles for single and branched structures are presented in Chapter 2. To put the conditions for stability in the context of a specific rod theory, the stability criteria are established for Euler's theory of the elastica in Chapter 3, along with some necessary kinematics. Chapter 4 contains several applications of the stability criteria to the classical problem of a thin strut deforming under terminal loading and self-weight, as well as an example of a human spine model with intrinsic curvature buckling under terminal loading (see Peters [48] and references therein for a detailed treatment). For a demonstration of the novel stability criteria for a tree-like structure, an illustrative example is shown in Chapter 5. With the intended goal of extending the stability criteria to more complex rod theories which can accommodate nonplanar deformations, torsion, shear and extension of the rod centerline, Chapters 6 & 7 present some background on the kinematics of two prominent theories (due to Green & Naghdi and Kirchhoff) and explain how the variational principles must be altered in order to establish second variation conditions for each theory. Finally, a detailed presentation of the alternative optimal control formulation of the stability criteria discussed in [43] is given in Appendix A.

Chapter 2

Variational Principles

2.1 Introduction

The calculus of variations has its origins in Johann Bernoulli's proposed brachistochrone problem (1696), and has since been richly developed by Euler, Lagrange, Legendre, Jacobi, Weierstrass, and many others. The central goal is to seek functions which extremize (i.e., maximize or minimize) a given functional. In mechanics, the functional of interest is typically the total energy, thus giving the extremizing functions a physical importance. When the total energy is minimized for a given solution, stability of that solution is confirmed (the same concept is true for maxima and instability). The extremals are found by examining the first variation of the functional and establishing necessary conditions that must be satisfied. In order to distinguish between a maximum and minimum of the functional, the second variation must be analyzed, giving additional necessary (and possibly sufficient) conditions.

The variational principles used in this chapter establish necessary and sufficient conditions for stability of a single rod structure in §2.2, and only necessary conditions for stability of a tree-like structure of branched rods in §2.3. The theory for branched structures also contains necessary conditions for the special case when the abscissa of the branching point is allowed to vary. These results are derived using Legendre's treatment, but there is a brief discussion of their equivalence to results using Jacobi's treatment at the end of §2.3.

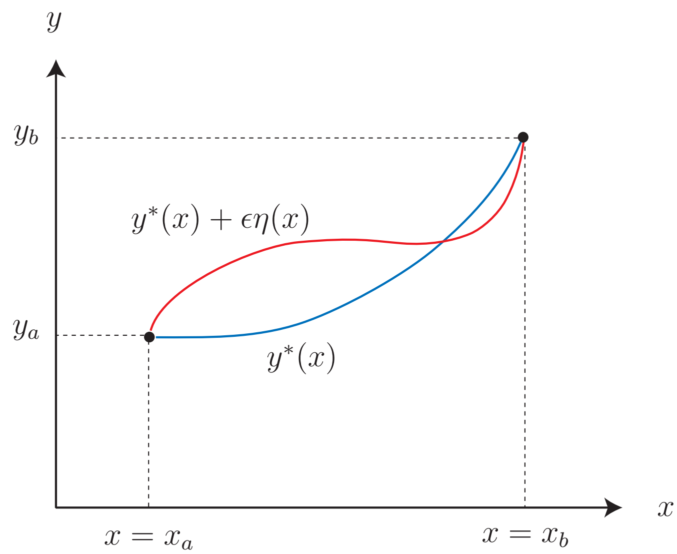

Figure 2.1 A schematic of the variation of a single extremal. In this example, $y(x = a)$ and $y(x = b)$ are prescribed, corresponding to fixed-fixed boundary conditions (see Chapter 4). There are also cases where either one or both of the endpoints is not prescribed.

2.2 Variational Principles for a Single Structure

According to the classical (indirect) method of the calculus of variations, we begin by considering a piecewise smooth real-valued function:

$$f = f\left(x, y, y^{'} = \frac{dy}{dx}\right). \tag{2.1}$$

Regularity of the function f is assumed:

$$\frac{\partial^2 f}{\partial y' \partial y'} > 0. \tag{2.2}$$

Here, we consider a functional of the form

$$I(y) = \int_a^b f\left(u, y(u), y^{'}(u)\right) du. \tag{2.3}$$

We seek an extremal $y^*(x)$ of I which satisfies the boundary conditions at $x = a$ and $x = b$:

$$\phi\left[y(a), y'(a)\right] = 0, \qquad \psi\left[y(b), y'(b)\right] = 0, \tag{2.4}$$

where ϕ and ψ are smooth scalar-valued functions. It will become clear in Chapter 4 §4.2 why the boundary conditions are defined in this manner.

2.2.1 First Variation and Necessary Conditions for an Extremal

We now wish to consider changes to I which arise when the function $y^*(x)$ is varied (see figure 2.1):

$$y(x,\epsilon) = y^*(x) + \epsilon\eta(x), \tag{2.5}$$

where ϵ is an arbitrary constant independent of x and y, and $\eta(x)$ is an arbitrary function of x which is independent of ϵ and satisfies the appropriate boundary conditions (from (2.4)):

$$\left.\frac{\partial\phi}{\partial y}\eta(a) + \frac{\partial\phi}{\partial y'}\eta'(a)\right|_{y=y^*(a)} = 0, \qquad \left.\frac{\partial\psi}{\partial y}\eta(b) + \frac{\partial\psi}{\partial y'}\eta'(b)\right|_{y=y^*(b)} = 0. \tag{2.6}$$

To obtain the first variation we compute $\frac{dI}{d\epsilon}$ with the help of the Leibniz rule, take the limit as $\epsilon \to 0$, and perform the standard integration by parts. As a result of these manipulations, we find that

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \left[\frac{\partial f}{\partial y'}\eta\right]_{x=a}^{x=b} + \int_a^b \left\{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right\}\eta dx. \tag{2.7}$$

In order for $y^*(x)$ to be an extremal, the right-hand side of (2.7) must vanish for all $\eta(x)$. Since $\eta(x)$ is an arbitrary function, this condition leads to the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \tag{2.8}$$

For a structure where y_a and y_b are prescribed, the first term on the right-hand side of (2.7) will vanish because $\eta(x=a) = \eta(x=b) = 0$. When y_a or y_b are not prescribed, the vanishing of the first variation yields the natural boundary conditions

$$\left.\frac{\partial f}{\partial y'}\right|_{x=a} = 0 \qquad \text{or} \left.\frac{\partial f}{\partial y'}\right|_{x=b} = 0. \tag{2.9}$$

2.2.2 Second Variation and a Stability Criterion for an Extremal

To establish a stability criterion for $y^*(x)$, we next examine the second variation $\delta^2 I$ of I. We shall assume that non-negativity of $\delta^2 I$ is necessary for stability of the extremal $y^*(x)$, while positive definiteness is sufficient for stability. A typical development of a criterion for positive-definiteness features the Jacobi condition and the search for conjugate points. Although the condition has been used with some success in models of elastic structures, we find it easier when dealing with branched structures to utilize a treatment originally proposed by Legendre in 1786, which leads to a Riccati differential equation. The existence of a bounded solution to this equation is one of the sought-after necessary conditions for $\delta^2 I \geq 0$. In fact, we are able to show that, for a single rod only, the existence of a bounded solution to the Riccati equation is also a sufficient condition for stability.

Necessary Conditions

We consider the same variations as those used in Section 2.2.1. Starting with the expression (2.7) for $\frac{dI}{d\epsilon}$, we differentiate once more with respect to ϵ and set ϵ to zero:

$$\delta^2 I = \left.\frac{d^2 I}{d\epsilon^2}\right|_{\epsilon=0} = \int_a^b \left\{P\eta^2 + 2Q\eta\eta' + R\eta'^2\right\} du, \tag{2.10}$$

where

$$P = P(x) = \frac{\partial^2 f}{\partial y \partial y}\left(x, y^*, y^{*'}\right), \quad Q = Q(x) = \frac{\partial^2 f}{\partial y \partial y'}\left(x, y^*, y^{*'}\right),$$

$$R = R(x) = \frac{\partial^2 f}{\partial y' \partial y'}\left(x, y^*, y^{*'}\right), \tag{2.11}$$

We now seek conditions for the non-negativity of $\delta^2 I$. Following Legendre, we add the following identity to the right-hand side of (2.10):

$$\int_a^b \frac{d}{du}\left(\eta^2 w\right) du - \left[\eta^2 w\right]_a^b = 0. \tag{2.12}$$

Thus, $\delta^2 I$ simplifies to

$$\delta^2 I = \int_a^b \left\{\left(P + w'\right)\eta^2 + 2\left(Q + w\right)\eta\eta' + R\eta'^2\right\} du - \left[\eta^2 w\right]_a^b, \tag{2.13}$$

provided we can find a function $w(x)$ which satisfies the following Riccati equation

$$w' + P - \frac{1}{R}\left(Q + w\right)^2 = 0. \tag{2.14}$$

The boundary condition on $w(x)$ is chosen so that

$$\eta^2(a)w(a) - \eta^2(b)w(b) = 0. \tag{2.15}$$

If a solution $w(x)$ to (2.14) can be found, then the resulting simplified expression for $\delta^2 I$ is non-negative:

$$\delta^2 I = \int_a^b R\left\{\eta' + \left(\frac{Q + w}{R}\right)\eta\right\}^2 du. \tag{2.16}$$

We refer to the existence of $w(x)$ as the necessary condition L1. Note that $R > 0$ (i.e., (2.23)) is used here to establish non-negativity. This necessary condition is also known as the strengthened Legendre condition.

Sufficient Conditions

To show how these conditions are also sufficient for stability in this special case, standard manipulations using integrations by parts can be used to establish several representations for $\delta^2 I$:

$$\begin{aligned}\delta^2 I &= A_2 = \int_0^L R\left\{\eta' + \left(\frac{Q+w}{R}\right)\eta\right\}^2 dx \\ &= B_2 = \int_0^L \eta\left\{P\eta + Q\eta' - \left(Q\eta + R\eta'\right)'\right\} dx + \left[\eta\left(Q\eta + R\eta'\right)\right]_0^L \\ &= C_2 = \int_0^L \left\{\eta' R\eta' + \left(P - Q'\right)\eta^2\right\} dx + \left[\eta\left(Q\eta\right)\right]_0^L . \end{aligned} \tag{2.17}$$

The expression A_2 follows from (2.16), the expression B_2 features the Jacobi differential operator $P\eta + Q\eta' - \left(Q\eta + R\eta'\right)'$, and the expression C_2 emphasizes the quadratic nature of $\delta^2 I$. We assume that the boundary conditions are such that

$$\eta(0) = 0, \qquad \eta Q\eta\big|_{x=L} = 0, \qquad \eta R\eta'\Big|_{x=L} = 0. \tag{2.18}$$

Justifying the counterparts to these conditions for branched structures was a significant impediment to establishing a sufficient condition for stability. We also take this opportunity to note that the boundary condition $\eta(0) = 0$ excludes the problematic free-free case $\eta'(0) = \eta'(L) = 0$ (see [35] for a discussion of this case).

To examine if $\delta^2 I$ is strictly positive, we first assume that a solution $w(x)$ to the Riccati equation associated with L1 exists $\forall x \in [0, L]$:

$$w' + P - \frac{1}{R}(Q+w)^2 = 0, \quad \left[\eta^2 w\right]_0^L = 0. \tag{2.19}$$

Now suppose $A_2 = 0$ (cf. $(2.17)_1$) for some function $\eta(x)$. Hence, η extremizes $\delta^2 I$ and consequently it must satisfy the Jacobi differential equation:

$$P\eta + Q\eta' - \left(Q\eta + R\eta'\right)' = 0. \tag{2.20}$$

However, as $R > 0$, we conclude by inspecting A_2 that

$$\eta' + \left(\frac{Q+w}{R}\right)\eta = 0. \tag{2.21}$$

Now $\eta(0) = 0$, so (2.21) implies that $\eta^{'}(0) = 0$. Invoking the existence and uniqueness theorem for solutions $\eta(x)$ to (2.20) with the initial conditions $\eta(0) = 0$ and $\eta^{'}(0) = 0$, we conclude that $\eta(x) = 0$ for all $x \in [0, L]$. Hence, $\delta^2 I$ is positive definite.

In conclusion, a sufficient condition, which we refer to as LS1, for $\delta^2 I$ to be positive definite in the case where there is a single branch and (2.18) holds is the existence of a solution $w(x)$ for all $x \in [0, L]$ to the Riccati equation (2.19). A closely related proof is presented in Gelfand & Fomin [15].

2.3 Variational Principles for a Branched Tree-Like Structure

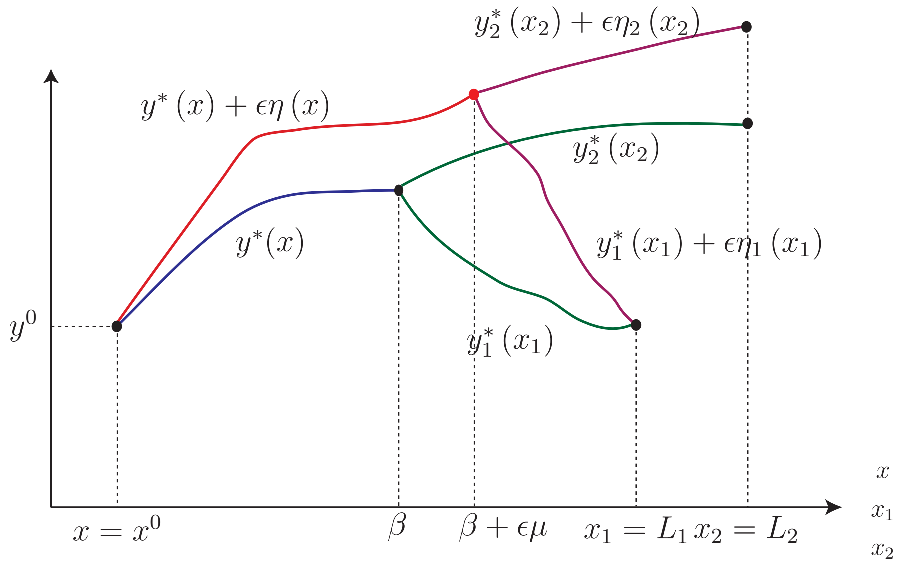

Figure 2.2 An example of the variations of the extremals and the branching point for a branched system with $N = 2$. In this example, $y_1(x_1 = L_1)$ is prescribed while $y_2(x_2 = L_2)$ is unspecified.

We now consider tree-like structures formed by the graphs of piecewise smooth functions (see figure 2.2). It suffices to consider the case where one branching point (or vertex) exists and to restrict attention to situations featuring the following set of piecewise smooth real-valued functions for each branch:

$$\begin{aligned} f &= f\left(x, y, y^{'} = \frac{dy}{dx}\right), \\ f_K &= f_K\left(x_K, y_K, y_K^{'} = \frac{dy_K}{dx_K}\right), \qquad (K = 1, ..., N). \end{aligned} \tag{2.22}$$

Regularity of the functions f and f_K is still assumed:

$$\frac{\partial^2 f}{\partial y' \partial y'} > 0, \qquad \frac{\partial^2 f_K}{\partial y_K' \partial y_K'} > 0, \qquad (K = 1, ..., N) . \tag{2.23}$$

It is convenient to define complementary functions for f and f_K:

$$\begin{aligned} g = g\left(x, y, y'\right) &= f - y' \frac{\partial f}{\partial y'}, \\ g_K = g_K\left(x_K, y_K, y_K'\right) &= f_K - y_K' \frac{\partial f_K}{\partial y_K'}. \end{aligned} \tag{2.24}$$

Referring to figure 2.2, we choose the coordinates x_K such that at a branching point $x = \beta$ and $x_K = \beta$. Compatibility of y and y_K is assumed:

$$y^- = y_K^+, \qquad (K = 1, ..., N) , \tag{2.25}$$

where we have introduced the abbreviations

$$y^- = y\left(\beta^-\right) = \lim_{x \nearrow \beta} y(x) , \qquad y_K^+ = y_K\left(\beta^+\right) = \lim_{x_K \searrow \beta} y_K(x_K) . \tag{2.26}$$

We now consider functionals of the form

$$I(y, y_1, \ldots, y_N) = \int_0^\beta f\left(u, y(u), y'(u)\right) du + \sum_{J=1}^{N} \int_\beta^{L_J} f_J\left(u, y_J(u), y_J'(u)\right) du. \tag{2.27}$$

We seek a set of extrema $\{y^*(x), y_K^*(x_K)\}$ of I which satisfy the fixed boundary condition at $x = 0$ and N compatibility conditions:

$$y(x = 0) = y^0, \qquad y^- = y_K^+. \tag{2.28}$$

We have elected to leave the N boundary conditions at $x_J = L_J$ unspecified.

To compress several lengthy expressions, we define the following notation for a pair of functions $a(x)$ and $a_J(x_J)$ evaluated at a branching point:

$$\langle\langle a_J, a \rangle\rangle = a_J^+ - a^-, \qquad (J = 1, \ldots, N) . \tag{2.29}$$

We also employ three distinct abbreviations for jumps in functions which will play prominent roles in the sequel:

$$
\begin{aligned}
[\![f]\!]_B &= -\lim_{x\nearrow\beta} f\left(x, y^*(x), y^{*'}(x)\right) + \sum_{J=1}^{N} \lim_{x_J\searrow\beta} f_J\left(x_J, y_J^*(x_J), y_J^{*'}(x_J)\right), \\
[\![f_y]\!]_B &= -\lim_{x\nearrow\beta} \frac{\partial f}{\partial y}\left(x, y^*(x), y^{*'}(x)\right) + \sum_{J=1}^{N} \lim_{x_J\searrow\beta} \frac{\partial f_J}{\partial y_J}\left(x_J, y_J^*(x_J), y_J^{*'}(x_J)\right), \\
[\![f_r]\!]_B &= -\lim_{x\nearrow\beta} \frac{\partial f}{\partial y'}\left(x, y^*(x), y^{*'}(x)\right) + \sum_{J=1}^{N} \lim_{x_J\searrow\beta} \frac{\partial f_J}{\partial y_J'}\left(x_J, y_J^*(x_J), y_J^{*'}(x_J)\right).
\end{aligned}
\tag{2.30}
$$

2.3.1 Variations and Compatibility

As in the case of a single branch, we are now in a position to consider changes to I which arise when the functions $y^*(x)$ and $y_K^*(x_K)$ are varied. However, we now assume that the branching point β is also varied:

$$
\begin{aligned}
\beta &\rightarrow \beta + \epsilon\mu, \\
y(x,\epsilon) &= y^*(x) + \epsilon\eta(x), \\
y_K(x_K,\epsilon) &= y_K^*(x_K) + \epsilon\eta_K(x_K), \quad (K = 1, \dots, N),
\end{aligned}
\tag{2.31}
$$

where

$$
y(0,\epsilon) = y^0, \qquad y(\beta + \epsilon\mu, \epsilon) = y_K(\beta + \epsilon\mu, \epsilon). \tag{2.32}
$$

A set of $2N$ compatibility conditions are obtained by taking the first and second derivatives of $(2.32)_2$ with respect to ϵ and then setting $\epsilon \rightarrow 0$:

$$
\left(\mu y^{*'} + \eta\right)^- = \left(\mu y_K^{*'} + \eta_K\right)^+, \qquad \left(\mu^2 y^{*''} + 2\mu\eta'\right)^- = \left(\mu^2 y_K^{*''} + 2\mu\eta_K'\right)^+. \tag{2.33}
$$

Conditions of the form $(2.33)_1$ feature in adhesion problems where $N = 1$ (see, e.g., Majidi & Adams [33] or Seifert [55]).

2.3.2 Necessary Conditions for an Extremal

Here we follow O'Reilly & Tresierras [45] and establish necessary conditions for an extremal by examining the first variation of I. This is done by substituting (2.31) into I and proceeding in a manner identical to the method in §2.2.1. At the conclusion of these manipulations, we find that

$$\begin{aligned} \left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} &= -\mu\,[\![f]\!]_B + \left(\frac{\partial f}{\partial y'}\eta\right)^- - \sum_{J=1}^{N}\left(\frac{\partial f_J}{\partial y'_J}\eta_J\right)^+ + \sum_{J=1}^{N}\left.\left(\frac{\partial f_J}{\partial y'_J}\eta_J\right)\right|_{x_J=L_J} \\ &\quad + \int_0^\beta \left\{\frac{\partial f}{\partial y} - \frac{d}{du}\left(\frac{\partial f}{\partial y'}\right)\right\}\eta du + \sum_{J=1}^{N}\int_\beta^{L_J}\left\{\frac{\partial f_J}{d y_J} - \frac{d}{du}\left(\frac{\partial f_J}{\partial y'_J}\right)\right\}\eta_J du. \end{aligned} \tag{2.34}$$

With the help of (2.33), we can simplify the right-hand side of (2.34) by substituting for η_J^+. Then, using $(2.30)_2$, we can write (2.34) in another form:

$$\begin{aligned} \left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} &= -[\![f_r]\!]_B\,\eta^- - \mu\left([\![f]\!]_B - \sum_{J=1}^{N}\left(\frac{\partial f_J}{\partial y'_J}\right)^+ \left\langle\!\left\langle y_J^{*\prime}, y^{*\prime}\right\rangle\!\right\rangle\right) \\ &\quad + \sum_{J=1}^{N}\left.\left(\frac{\partial f_J}{\partial y'_J}\eta_J\right)\right|_{x_J=L_J} + \int_0^\beta \left\{\frac{\partial f}{\partial y} - \frac{d}{du}\left(\frac{\partial f}{\partial y'}\right)\right\}\eta du \\ &\quad + \sum_{J=1}^{N}\int_\beta^{L_J}\left\{\frac{\partial f_J}{\partial y_J} - \frac{d}{du}\left(\frac{\partial f_J}{\partial y'_J}\right)\right\}\eta_J du. \end{aligned} \tag{2.35}$$

It is illuminating to note that the first two terms on the right-hand side of this equation are obtained by varying $y(\beta)$ and β, respectively, of the branching point. For those branches where $y_J\left(x_J = L_J\right)$ are prescribed, the third term on the right hand side of (2.35) will vanish because $\eta_J\left(x_J = L_J\right) = 0$ for these branches.

In order for $\{y^*(x), y_K^*\left(x_K\right)\}$ to be extremals, the right-hand side of (2.35) needs to vanish for all μ, η^-, $\eta(x)$, and $\eta_J(x_J)$. The arbitrariness of $\eta(x)$ and $\eta_J(x_J)$ leads to the respective Euler-Lagrange equations:

$$\begin{aligned} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} &= 0, \\ \frac{d}{dx_K}\left(\frac{\partial f_K}{\partial y'_K}\right) - \frac{\partial f_K}{\partial y_K} &= 0, \qquad (K = 1, \ldots, N). \end{aligned} \tag{2.36}$$

For those branches where $y_J\left(x_J = L_J\right)$ is unspecified, the vanishing of the right-hand side of (2.35) also yields the natural boundary conditions

$$\frac{\partial f_J}{\partial y'_J}\left(L_J, y_J^*(L_J), y_J^{*\prime}(L_J)\right) = 0. \tag{2.37}$$

The peculiar feature of the branched system arises when we consider the first two terms on the right-hand side of (2.35). We suppose that we can vary η^- and μ in an arbitrary manner. This leads to a pair of conditions, which we refer to as the branching point conditions B1 and B2, respectively:

$$[\![f_r]\!]_B = 0, \qquad [\![g]\!]_B = 0. \tag{2.38}$$

These conditions are similar in form to the standard Erdmann-Weierstrass conditions which hold at a corner point of an extremal. The condition $[\![f_r]\!]_B = 0$ can also be related to the junction conditions discussed in works on multistructures and continuity of force and moments at a branching point (see Le Dret [27] and O'Reilly & Tresierras [45]). Furthermore, the relationship between the condition $[\![g]\!]_B = 0$ and material (or configurational) forces is discussed in Faruk Senan et al. [12] and O'Reilly & Tresierras [45].

Regularity of the functions $f, f_1, \ldots, f_N$ implies that the Legendre necessary condition for an extremal is identically satisfied. By fixing all but one of the functions $y, y_1, \ldots, y_N$, we can readily establish $N+1$ Weierstrass necessary conditions (featuring his excess function) and $2N+2$ Erdmann-Weierstrass corner conditions. In the applications considered in the sequel, these $3N+3$ conditions are identically satisfied and, in the interests of brevity, are not discussed further.

In summary, if $\{y^*(x), y_K^*(x_K)\}$ constitute a set of extremals for the variational problem, then (as expected) each element satisfies the appropriate Euler-Lagrange necessary condition (cf. (2.36)). If one end of an extremal is not specified, then the natural boundary condition (2.37) is assumed to hold there. The peculiar nature of branched structures manifests at the branching point. There, we have the pair of conditions (2.38). As emphasized in O'Reilly & Tresierras [45], apart from the condition $[\![g]\!]_B = 0$, the developments in this section can be considered as special cases of the results established by Ivanov & Tuzhilin [21].

2.3.3 Second Variation and a Stability Criterion for a Branched Structure

To establish a stability criterion for the set $\{y^*(x), y_K^*(x_K)\}$, we next examine the second variation $\delta^2 I$ of I. We shall assume that non-negativity of $\delta^2 I$ is necessary for stability of the set of extremals $\{y^*(x), y_K^*(x_K)\}$, while positive definiteness (i.e., $\delta^2 I > 0$) is sufficient for stability of this set. A typical development of a criterion for positive-definiteness features the Jacobi condition and the search for conjugate points. Although some work on generalizing the condition to networked structures has been presented in Pronin [49], it is not obvious how one can generalize the Jacobi condition to the structures of interest in this paper. Instead, we find it far more productive to extend a treatment originally proposed by Legendre in 1786 for a single extremal to branched structures. This Legendre-inspired treatment leads to a series of $N+1$ Riccati equations: one for each branch. The equations are coupled by a boundary condition at the branching point (which is easily implemented). The existence of a bounded solution to these equations is one of the sought-after necessary conditions for $\delta^2 I \geq 0$. We were unable to establish a condition for $\delta^2 I > 0$ for a branched structure, as we were able to do for a single branch.

We consider the same variations as those used in §2.3.1. Starting with the expression (2.27) for I and substituting the variations (2.31), we compute the second derivative of I with respect to ϵ and then set ϵ to zero:

$$\begin{aligned}
\left.\frac{d^2 I}{d\epsilon^2}\right|_{\epsilon=0} = \;& \mu^2\left(\left(\frac{df}{dx}\right)^- - \sum_{J=1}^{N}\left(\frac{df_j}{dx_j}\right)^+\right) \\
&+2\mu\left(\left(\frac{\partial f}{\partial y}\eta\right)^- - \sum_{J=1}^{N}\left(\frac{\partial f_J}{\partial y_J}\eta_J\right)^+\right) \\
&+2\mu\left(\left(\frac{\partial f}{\partial y'}\eta'\right)^- - \sum_{J=1}^{N}\left(\frac{\partial f_J}{\partial y'_J}\eta'_J\right)^+\right) \\
&+\int_0^\beta \left\{P\eta^2 + 2Q\eta\eta' + R\eta'\eta'\right\} du \\
&+\sum_{J=1}^{N}\int_\beta^{L_J}\left\{P_J\eta_J^2 + 2Q_J\eta_J\eta'_J + R_J\eta'_J\eta'_J\right\} du. \qquad (2.39)
\end{aligned}$$

In writing (2.39), we have used the standard abbreviations

$$P = P(x) = \frac{\partial^2 f}{\partial y \partial y}\left(x, y^*, y^{*'}\right), \quad Q = Q(x) = \frac{\partial^2 f}{\partial y \partial y'}\left(x, y^*, y^{*'}\right),$$

$$R = R(x) = \frac{\partial^2 f}{\partial y' \partial y'}\left(x, y^*, y^{*'}\right), \qquad (2.40)$$

and

$$\begin{aligned}
P_K = P_K(x_K) &= \frac{\partial^2 f_K}{\partial y_K \partial y_K}\left(x_K, y_K^*, {y_K^*}'\right), \\
Q_K = Q_K(x_K) &= \frac{\partial^2 f_K}{\partial y_K \partial y'_K}\left(x_K, y_K^*, {y_K^*}'\right), \\
R_K = R_K(x_K) &= \frac{\partial^2 f_K}{\partial y'_K \partial y'_K}\left(x_K, y_K^*, {y_K^*}'\right). \qquad (2.41)
\end{aligned}$$

Recalling the regularity assumptions (2.23), we conclude that the functions $R, R_1, \ldots, R_N$ are strictly positive.

With the help of the 2N compatibility conditions (2.33), we can simplify the right-hand side of (2.39) by substituting for η_J^+ and $\left(\eta'_J\right)^+$. With some rearranging we find that

$$\left(\frac{\partial f}{\partial y}\eta\right)^{-} - \sum_{J=1}^{N}\left(\frac{\partial f_J}{\partial y_J}\eta_J\right)^{+} = \left(\left(\frac{\partial f}{\partial y}\right)^{-} - \sum_{J=1}^{N}\left(\frac{\partial f_J}{\partial y_J}\right)^{+}\right)\eta^{-}$$
$$+\mu\sum_{J=1}^{N}\left(\frac{\partial f_J}{\partial y_J}\right)^{+}\left\langle\!\left\langle y_J^{*'}, y^{*'}\right\rangle\!\right\rangle$$
$$= -\left[\!\left[f_y\right]\!\right]_B \eta^{-} + \mu\sum_{J=1}^{N}\left(\frac{\partial f_J}{\partial y_J}\right)^{+}\left\langle\!\left\langle y_J^{*'}, y^{*'}\right\rangle\!\right\rangle, \qquad (2.42)$$

and

$$\mu\left(\frac{\partial f}{\partial y'}\eta'\right)^{-} - \mu\sum_{J=1}^{N}\left(\frac{\partial f_J}{\partial y_J'}\eta_J'\right)^{+} = -\mu\left[\!\left[f_r\right]\!\right]_B\left(\eta'\right)^{-}$$
$$+\frac{\mu^2}{2}\sum_{J=1}^{N}\left(\frac{\partial f_J}{\partial y_J'}\right)^{+}\left\langle\!\left\langle y_J^{*''}, y^{*''}\right\rangle\!\right\rangle$$
$$= \frac{\mu^2}{2}\sum_{J=1}^{N}\left(\frac{\partial f_J}{\partial y_J'}\right)^{+}\left\langle\!\left\langle y_J^{*''}, y^{*''}\right\rangle\!\right\rangle. \qquad (2.43)$$

We used condition B1, $\left[\!\left[f_r\right]\!\right]_B = 0$, to arrive at the final result in (2.43). With the help of $(2.38)_1$, we substitute (2.42) and (2.43) back into (2.39) to find that the second variation can be decomposed into two terms:

$$\left.\frac{d^2 I}{d\epsilon^2}\right|_{\epsilon=0} = I_\beta + I_2. \qquad (2.44)$$

Here, I_β is entirely associated with varying the branching point $x = \beta$ and the components of I_2 have a familiar form:

$$\begin{aligned} I_\beta &= \mu^2 e - 2\mu\eta^{-}\left[\!\left[f_y\right]\!\right]_B, \\ I_2 &= \int_0^\beta \left\{P\eta^2 + 2Q\eta\eta' + R\eta'\eta'\right\} du \\ &\quad + \sum_{J=1}^{N}\int_\beta^{L_J}\left\{P_J\eta_J^2 + 2Q_J\eta_J\eta_J' + R_J\eta_J'\eta_J'\right\} du, \end{aligned} \qquad (2.45)$$

where

$$e = \sum_{J=1}^{N}\left(2\left(\frac{\partial f_J}{\partial y_J}\right)^{+}\left\langle\!\left\langle y_J^{*'}, y^{*'}\right\rangle\!\right\rangle + \left(\frac{\partial f_J}{\partial y_J'}\right)^{+}\left\langle\!\left\langle y_J^{*''}, y^{*''}\right\rangle\!\right\rangle\right) - \left[\!\left[\frac{df}{dx}\right]\!\right]_B. \qquad (2.46)$$

We now turn to examining necessary conditions for the minimization of I.

Necessary Conditions

First, we examine the case where the branching point is fixed (i.e., $\mu = 0$). Referring to (2.44), for a minimum it is necessary that $I_2 \geq 0$, since $I_\beta = 0$ when the branching point is fixed. Following Legendre, we add the following identity to the right-hand side of $(2.45)_2$:

$$0 = \int_0^\beta \frac{d}{du}\left(\eta^2 w\right) du + \sum_{J=1}^{N} \int_\beta^{L_J} \frac{d}{dx_J}\left(\eta_J^2 w_J\right) dx_J \underbrace{- \left[\eta^2 w\right]_0^\beta - \sum_{J=1}^{N} \left[\eta_J^2 w_J\right]_\beta^{L_J}} . \tag{2.47}$$

After performing the usual manipulations of I_2, it becomes evident that we can dramatically simplify I_2 provided solutions to the following $N+1$ Riccati equations exist:

$$\begin{aligned} w^{'} + P - \frac{1}{R}\left(Q + w\right)^2 &= 0, \\ w_J^{'} + P_J - \frac{1}{R_J}\left(Q_J + w_J\right)^2 &= 0, \qquad (J = 1, \ldots, N). \end{aligned} \tag{2.48}$$

These solutions are subject to the conditions[1]

$$\begin{aligned} \eta^2(0) w(0) &\geq 0, \\ \eta_J^2\left(L_J\right) w_J\left(L_J\right) &\leq 0, \qquad (J = 1, \ldots, N), \\ \eta^- \eta^- \left[\!\left[w \right]\!\right]_B &\geq 0, \end{aligned} \tag{2.49}$$

which were obtained by examining the underbraced terms on the right-hand side of (2.47). We also note that for many applications, $(2.49)_3$ reduces to the condition $\left[\!\left[w \right]\!\right]_B = 0$. This condition couples the solutions of the Riccati equations at the branching point and is surprisingly easy to accommodate.

If bounded solutions $w, w_1, \ldots, w_N$ to (2.48) and (2.49) can be found, then the resulting expression for $I_2 = \hat{I}_2$ is non-negative:

$$\hat{I}_2 = \int_0^\beta R \left\{ \eta^{'} + \left(\frac{Q + w}{R} \right) \eta \right\}^2 dx + \sum_{J=1}^{N} \int_\beta^{L_J} R_J \left\{ \eta_J^{'} + \left(\frac{Q_J + w_J}{R_J} \right) \eta_J \right\}^2 dx_J \geq 0. \tag{2.50}$$

This expression is an expected generalization of a part of the second variation for problems involving multiple functionals. Conversely, by examining each of the branches separately, one can conclude using standard arguments from the calculus of variations

[1] In writing $(2.49)_3$ we have also used the simplification that $\eta^- = \eta_J^+$ at a branching point when $\mu = 0$ (cf. $(2.33)_1$).

that the existence of the bounded solutions $w, w_1, \ldots, w_N$ is also necessary for $I_2 \geq 0$. We are now in a position to state

CONDITION L1: *If the set of extremals $\{y^*(x), y_J^*(x_J)\}$ minimizes I then the solutions $\{w(x), w_J(x_J)\}$ to (2.48) which satisfy the boundary conditions (2.49) must remain bounded.*

We now turn our attention to the case when the branching point is allowed to vary. We again proceed by adding (2.47) to (2.44). After some rearranging and repeated use of the compatibility condition $(2.33)_1$, $\eta_J^+ = \eta^- - \mu \left\langle\left\langle y_J^{*\prime}, y^{*\prime} \right\rangle\right\rangle$, we find that

$$\begin{aligned} \left.\frac{d^2 I}{d\epsilon^2}\right|_{\epsilon=0} &= \eta^- \eta^- \, [\![w]\!]_B + \hat{I}_2 + \eta^2(0) w(0) - \sum_{J=1}^{N} \eta_J^2(L_J) w_J(L_J) \\ &\quad + \mu^2 \left(e + \sum_{J=1}^{N} \left(\left\langle\left\langle y_J^{*\prime}, y^{*\prime} \right\rangle\right\rangle \right)^2 w_J^+ \right) \\ &\quad - 2\mu\eta^- \left([\![f_y]\!]_B + \sum_{J=1}^{N} \left\langle\left\langle y_J^{*\prime}, y^{*\prime} \right\rangle\right\rangle w_J^+ \right). \end{aligned} \tag{2.51}$$

Noting that the variations η^- and μ are independent, we can conclude from (2.51) that a necessary condition for $\left.\frac{d^2 I}{d\epsilon^2}\right|_{\epsilon=0} \geq 0$ is that the condition L1 is satisfied and, in addition, the following matrix B is positive semi-definite:

$$\mathsf{B} = \begin{bmatrix} e + \sum_{J=1}^{N} \left(\left\langle\left\langle y_J^{*\prime}, y^{*\prime} \right\rangle\right\rangle \right)^2 w_J^+ & -\left([\![f_y]\!]_B + \sum_{J=1}^{N} \left\langle\left\langle y_J^{*\prime}, y^{*\prime} \right\rangle\right\rangle w_J^+ \right) \\ -\left([\![f_y]\!]_B + \sum_{J=1}^{N} \left\langle\left\langle y_J^{*\prime}, y^{*\prime} \right\rangle\right\rangle w_J^+ \right) & [\![w]\!]_B \end{bmatrix} \geq 0. \tag{B3}$$

Clearly, it is necessary to first solve the Riccati equations (2.48) in order to compute the boundary values w_J^+ featured in B3. We emphasize that the conditions B3 only apply to the case when the branching point is variable[2], while the condition L1 applies in all cases. Finally, it is worth mentioning that, while L1 is intimately related to the buckling stability of the individual branches, B3 pertain to the stability of the adhesion mechanism at the branching point. These conditions are used to examine the stability of a rod which is adhering to a rigid surface in [34].

2.3.4 A set of Jacobi transformations

The necessary condition L1 can also be expressed using an equivalent set of Jacobi equations. As is well-known (see Bolza [4] or Gelfand & Fomin [15]), each of the

[2] An example of such a situation can be found in O'Reilly & Tresierras [45].

Riccati equations in (2.48) can be transformed to a linear second-order ordinary differential equation using a Jacobi transformation. Thus, the transformations

$$w = -Q - R\frac{u^{'}}{u}, \qquad w_K = -Q_K - R_K\frac{u_K^{'}}{u_K}, \qquad (K = 1, \ldots, N)\,, \tag{2.52}$$

transform (2.48) to a set of $N+1$ Jacobi differential equations:

$$\begin{aligned} Pu + Qu^{'} - \left(Qu + Ru^{'}\right)^{'} &= 0, \\ P_K u_K + Q_K u_K^{'} - \left(Q_K u_K + R_K u_K^{'}\right)^{'} &= 0, \qquad (K = 1, \ldots, N)\,. \end{aligned} \tag{2.53}$$

As stated in Theorem 2.1 in Reid [50], bounded solutions to the Riccati equation for $w_K(x_K)$ on a given interval exist if, and only if, a solution $u_K\left(x_K\right)$ for the corresponding Jacobi differential equation exists on the same interval, where

$$u_K\left(x_K\right) \neq 0 \quad \text{and} \quad w_K = -\frac{Q_K u_K + R_K u_K^{'}}{u_K}. \tag{2.54}$$

This theorem can be used to establish the well-known equivalence of conjugate points for the solutions to (2.53) to blow up of the solutions to (2.48).

With the help of (2.52), we conclude that (2.53) could be investigated in place of (2.48); however, the coupling condition is more difficult to implement because it features $u, u_1, \ldots, u_N$ and their derivatives: $[\![w]\!]_B = \left[\!\left[Q + Ru^{'}/u\right]\!\right]_B$.

2.3.5 Comments on L1

In the discussion of the condition L1, we noted that the existence of solutions $\{w(x) \forall x \in [0, \beta], w_J\left(x_J\right) \forall x_J \in [\beta, L_K]\}$ to the Riccati equations (2.48) was necessary for $I_2 \geq 0$. The proof of this necessity can be achieved by appealing to Jacobi's necessary condition for a minimum.[3]

To elaborate, suppose that the Riccati equation on the branch $y^*_K\left(x_K\right)$ for some $K \in (1, \ldots, N)$ blows up for some $x^B_J \in [\beta, L_K]$. We now consider the variation η_K with all of the remaining N variations set to 0. It suffices to examine the accessory variational problem for $y^*_K\left(x_K\right)$ with $\eta^+_K = 0$ and $\eta^{'}_K\left(L_K\right) = 0$. With the help of Theorem 2.1 in Reid [50], the blow up of w_K at x^B_J can be related to the existence of a conjugate point x^B_J to L_K for u_K (see (2.52) and (2.53)). Appealing to Jacobi's necessary condition for a minimum, we can conclude that I is not minimized.

If $w(x)$ becomes unbounded, then we repeat the arguments in the previous paragraph for the variation $\eta(x)$ where $\eta^- = 0$ and $\eta\left(0\right) = 0$.

[3] In particular, see Fundamental Theorem III and its corollary in §of Bolza [4] or Theorem 2.10 in Ewing [11].

2.4 Closing Comments

We have established a set of conditions for the stability of a single branch or tree-like structure of branches. Once put into the context of a mathematical theory of elastic rods, the stability criteria will be verified for a series of classical problems and then applied to a simple tree-like structure of elastic rods. Stability can only be proven for the cases where sufficiency has been established; otherwise, we can only be certain of instability when a solution to the Riccati equations (2.48) fails to be found.

Chapter 3

Variational Principles and Euler's Theory of the Elastica

3.1 Introduction

With the stability criteria established, we now wish to provide a framework in which we can construct models of single or branched elastic rod structures. A suitable choice is Euler's theory of the elastica, formulated in 1744 (see Love [30] for a discussion of this theory). The elastica accounts for planar deformations of rods and can accommodate intrinsic curvature and changes in cross-sectional area. The former is used to model the "branch shape memory" exhibited by plant stems and also features prominently in models of the human spine (see [12, 16, 24, 44] and references therein for further details on the morphogenesis of branches).

3.2 Euler's Theory of the Elastica

Referring to figure 3.1, the position vector $\mathbf{r}$ of a material point labelled $\xi \in [0, L]$ on the inextensible centerline of $\mathcal{R}$ has the representation $\mathbf{r} = X\mathbf{E}_1 + Y\mathbf{E}_2$. Similarly, material points on the inextensible centerline of $\mathcal{R}_K$ where $K = 1, \ldots, N$ are identified using a coordinate $\xi_K \in [L, L_K]$ and their position vectors have the representation $\mathbf{r}_K = X_K\mathbf{E}_1 + Y_K\mathbf{E}_2$. The unit tangent vectors to the centerlines of $\mathcal{R}$ and $\mathcal{R}_K$ have

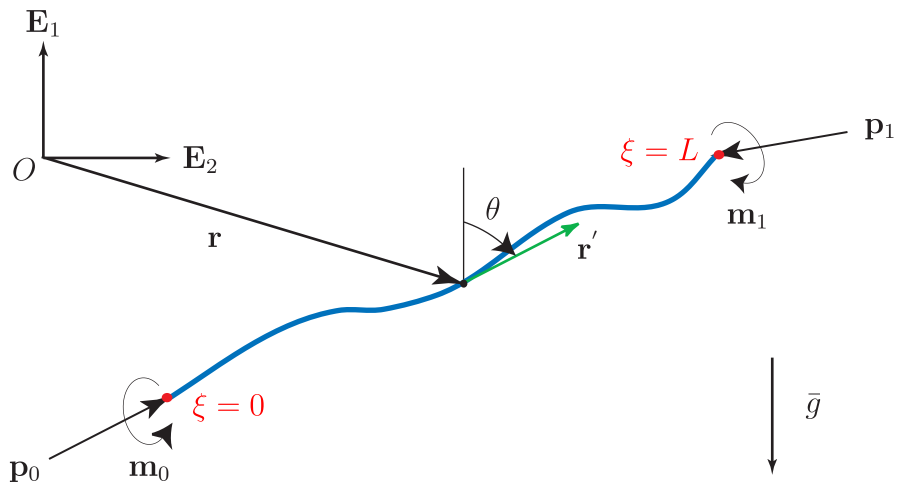

Figure 3.1 Schematic of an elastica which is subject to a terminal force $\mathbf{p}_0$ and terminal moment $\mathbf{m}_0$ at $\xi = 0$ and a terminal force $\mathbf{p}_1$ and terminal moment $\mathbf{m}_1$ at the end $\xi = L$. In addition, a gravitational loading $-\rho_0\bar{g}\mathbf{E}_1$ acts on the rod.

the respective representations

$$\begin{aligned} \mathbf{r}' = \frac{\partial \mathbf{r}}{\partial \xi} &= \cos(\theta)\,\mathbf{E}_1 + \sin(\theta)\,\mathbf{E}_2, \\ \mathbf{r}'_K = \frac{\partial \mathbf{r}_K}{\partial \xi_K} &= \cos(\theta_K)\,\mathbf{E}_1 + \sin(\theta_K)\,\mathbf{E}_2. \end{aligned} \tag{3.1}$$

The curvature κ of the centerline of $\mathcal{R}$ is defined by the relation $\kappa = \frac{\partial \theta}{\partial \xi}$. When this centerline is unloaded it relaxes into a centerline with a curvature κ_g. The curvature κ_g is known as the intrinsic curvature. Related remarks pertain to the two other rods.

In addition to a gravitational force $-\bar{g}\mathbf{E}_1$, we assume that the tips of the rods $\mathcal{R}_K$ are subject to the respective terminal dead loadings $-\mathbf{F}_K$. These forces have the representations

$$\mathbf{F}_K = F_{K_1}\mathbf{E}_1 + F_{K_2}\mathbf{E}_2, \qquad (K = 1, \ldots, N)\,. \tag{3.2}$$

We assume that the rod $\mathcal{R}_K$ has a flexural stiffness $EI_K = EI_K(\xi_K)$, a length $L_K - L$, an intrinsic curvature $\kappa_{g_K} = \kappa_{g_K}(\xi_K)$, and a mass per unit length $\rho_{0_K} = \rho_{0_K}(\xi_K)$, where $K = 1, \ldots, N$. The rods $\mathcal{R}_1, \ldots, \mathcal{R}_N$ are connected at $\xi_K = L$ to the rod $\mathcal{R}$ which has a length L, flexural rigidity EI, and mass density per unit length $\rho_0 = \rho_0(\xi)$. The bending moment in each elastica is prescribed by a classic constitutive equation:

$$\mathbf{M} = EI\left(\frac{\partial \theta}{\partial \xi} - \kappa_g\right)\mathbf{E}_3, \qquad \mathbf{M}_K = EI_K\left(\frac{\partial \theta_K}{\partial \xi_K} - \kappa_{g_K}\right)\mathbf{E}_3. \tag{3.3}$$

More general constitutive equations are possible, but (3.3) suffice to illustrate the stability criterion.

The total energy of the rods consists of the sum of the strain energies, gravitational potential energies, and the potential energy of the terminal loads:

$$
\begin{aligned}
V \;=\; & \int_{\xi=0}^{\xi=L} \frac{EI}{2}\left(\frac{\partial\theta}{\partial\xi} - \kappa_g\right)^2 d\xi + \sum_{K=1}^{N} \int_{L}^{L_K} \frac{EI_K}{2}\left(\frac{\partial\theta_K}{\partial\xi_K} - \kappa_{g_K}\right)^2 d\xi_K \\
& + \int_0^L \bar{g}\rho_0 X(\xi) d\xi + \sum_{K=1}^{N} \int_L^{L_K} \bar{g}\rho_{0_K} X_K\left(\xi_K\right) d\xi_K \\
& + \sum_{K=1}^{N} \mathbf{F}_K \cdot \mathbf{r}_K\left(L_K\right), \qquad (K = 1, \ldots, N).
\end{aligned} \tag{3.4}
$$

This representation of the energy is not particularly useful. Thus, with the assistance of (3.1), we proceed to express the components of $\mathbf{r}$ and $\mathbf{r}_K$ in integral form. For example,

$$
X(\xi) = \int_0^{\xi} \cos(\theta(x)) dx, \qquad X_K(\xi_K) = \int_0^L \cos(\theta(x)) dx + \int_L^{\xi_K} \cos\left(\theta_K(x)\right) dx. \tag{3.5}
$$

It is also convenient to define the masses

$$
m(\xi) = \int_{\xi}^{L} \rho_0(u) du, \qquad m_K\left(\xi_K\right) = \int_{\xi_K}^{L_K} \rho_{0_K}(u) du. \tag{3.6}
$$

Note that $m(0)$ and $m_K(L)$ are the total masses of the rods $\mathcal{R}$ and $\mathcal{R}_K$, respectively.

With the help of (3.5) and (3.6) and some additional manipulations, we arrive at an equivalent expression for the total energy:

$$
\begin{aligned}
V \;=\; & \int_0^L \left\{ \frac{EI}{2}\left(\frac{\partial\theta}{\partial\xi} - \kappa_g\right)^2 + \left(\mathbf{F} + \mathbf{W} + m(\xi)\bar{g}\mathbf{E}_1\right) \cdot \mathbf{r}' \right\} d\xi \\
& + \sum_{K=1}^{N} \int_L^{L_K} \left\{ \frac{EI_K}{2}\left(\frac{\partial\theta_K}{\partial\xi_K} - \kappa_{g_K}\right)^2 + \left(\mathbf{F}_K + m_K\left(\xi_K\right)\bar{g}\mathbf{E}_1\right) \cdot \mathbf{r}_K' \right\} d\xi_K,
\end{aligned} \tag{3.7}
$$

where $\mathbf{F}$ and $\mathbf{W}$ are the combined terminal force and weight, respectively, of the rods $\mathcal{R}_1$ to $\mathcal{R}_N$:

$$
\mathbf{F} = \sum_{K=1}^{N} \mathbf{F}_K, \qquad \mathbf{W} = \sum_{K=1}^{N} \mathbf{W}_K = \sum_{K=1}^{N} m_K\left(L\right)\bar{g}\mathbf{E}_1. \tag{3.8}
$$

We have expressed (3.7) in a form which allows it to be identified with the functional

I: $y = \theta$, $y_K = \theta_K$, $x = \xi$, $x_K = \xi_K$, etc. The resulting expression also makes it transparent that the rod $\mathcal{R}$ must support the weight and terminal loading of the branches $\mathcal{R}_1, \ldots, \mathcal{R}_N$. It thus facilitates the extension of this work to situations featuring an arbitrary number of heavy terminally loaded rods.

3.3 First Variation

The Euler-Lagrange equations (2.36) associated with the functional V are

$$\begin{aligned}\frac{\partial}{\partial \xi}\left(EI\left(\frac{\partial \theta^*}{\partial \xi} - \kappa_g\right)\right) &= \left(\mathbf{F} + \mathbf{W} + m(\xi)\bar{g}\mathbf{E}_1\right) \cdot \left(-\sin\left(\theta^*\right)\mathbf{E}_1 + \cos\left(\theta^*\right)\mathbf{E}_2\right), \\ \frac{\partial}{\partial \xi_K}\left(EI_K\left(\frac{\partial \theta_K^*}{\partial \xi_K} - \kappa_{g_K}\right)\right) &= \left(\mathbf{F}_K + m_K\left(\xi_K\right)\bar{g}\mathbf{E}_1\right) \cdot \left(-\sin\left(\theta_K^*\right)\mathbf{E}_1 + \cos\left(\theta_K^*\right)\mathbf{E}_2\right).\end{aligned} \tag{3.9}$$

The solutions θ^* and θ_K^* (i.e., the extremals) to these equations satisfy the boundary conditions

$$\theta^*(0) = \theta_0, \qquad \frac{\partial \theta_K^*}{\partial \xi_K}\left(\xi_K = L_K\right) = \kappa_{g_K}\left(\xi_K = L_K\right), \qquad (K = 1, \ldots, N). \tag{3.10}$$

Finally, as the branching angles are prescribed, we have the auxiliary conditions

$$\theta^+ - \theta_1^- = \chi_1, \qquad \theta^+ - \theta_2^- = \chi_2, \qquad \theta_2^- - \theta_1^- = \chi_3. \tag{3.11}$$

Because the branching angles are prescribed and the branch point is fixed, the branch conditions B1 and B2 are not applicable to this problem. It is worth noting that equations (3.9) can also be obtained by performing the familiar balances of linear and angular momenta for Euler's elastica.

3.4 Second Variation

Prior to examining the second variation of V, we recall the definitions (2.40) and (2.41). For the branched rod structure of interest, it is straightforward to show that

$$\begin{aligned} P &= -\left(\mathbf{F} + \mathbf{W} + m(\xi)\bar{g}\mathbf{E}_1\right) \cdot \left(\cos\left(\theta^*\right)\mathbf{E}_1 + \sin\left(\theta^*\right)\mathbf{E}_2\right), \\ P_K &= -\left(\mathbf{F}_K + m_K\left(\xi_K\right)\bar{g}\mathbf{E}_1\right) \cdot \left(\cos\left(\theta_K^*\right)\mathbf{E}_1 + \sin\left(\theta_K^*\right)\mathbf{E}_2\right), \end{aligned} \tag{3.12}$$

and

$$Q = 0, \qquad R = EI, \qquad Q_K = 0, \qquad R_K = EI_K. \tag{3.13}$$

The Riccati equations we need to solve can be inferred from (2.48):

$$\begin{aligned} \frac{\partial w}{\partial \xi} + P - \frac{w^2}{EI} &= 0, \\ \frac{\partial w_K}{\partial \xi_K} + P_K - \frac{w_K^2}{EI_K} &= 0, \qquad (K = 1, \ldots, N). \end{aligned} \tag{3.14}$$

The desired solutions to these equations need to satisfy the boundary conditions (from (2.49))[1]

$$w_K\left(L_K\right) = 0, \qquad w^- = \sum_{K=1}^{N} w_K^+, \qquad (K = 1, \ldots, N). \tag{3.15}$$

To see if (3.14) has solutions, we integrate $(3.14)_2$ backwards from $x_K = L_K$ using the conditions $(3.15)_1$. At the branching point $\xi_K = L^+$, we use $(3.15)_2$ to deduce the boundary condition for the remaining Riccati equation $(3.14)_1$. If solutions to all $N + 1$ Riccati equations can be found in this manner, then we say that $L1$ is satisfied and, thus, so too is a necessary condition for stability.

[1]The condition $\eta^2(0)w(0) = 0$ is identically satisfied because $\eta(0) = 0$ due to the fixed boundary condition at $\xi = 0$.

Chapter 4

Application to a Single Rod

4.1 Introduction

There are many examples of problems utilizing a single elastic rod to which we could apply our stability criteria. It is helpful to first choose a classical problem with a well-known result in order to verify the efficacy of our approach. Such an example is the terminally loaded strut, which has an exact closed-form solution. This example is analyzed for three sets of boundary conditions, followed by a terminally loaded strut also under gravitational loading. The final example presented is a simplified planar model of the human spine with intrinsic curvature. Therefore, each application contains a prominent feature of Euler's elastica theory: planar deformations, self-weight and intrinsic curvature.

4.2 Buckling of a Thin Strut Under Thrust

The first example is the classical problem of a terminally loaded thin strut (as detailed in Love [30]), with differences in boundary conditions presenting three distinct buckling problems. The analysis is a portion of the work that was published in O'Reilly & Peters [43].

The problems of interest feature the deformed planar shape of a long, slender, uniform rod which has a flexural rigidity of EI and a fixed length ℓ (cf. Figure 4.1). The ends of the rod are defined by the material coordinates $\xi = 0$ and $\xi = \ell$, and, in its fixed reference configuration, the centerline of the rod is parallel to the constant unit vector $\mathbf{E}_1$: $\mathbf{R} = \xi\mathbf{E}_1$, where $\mathbf{R}$ is the position vector of a point on the centerline. A constant force $P\mathbf{E}_1$ is applied at $\xi = 0$ and a force $-P\mathbf{E}_1$ is applied at $\xi = \ell$.

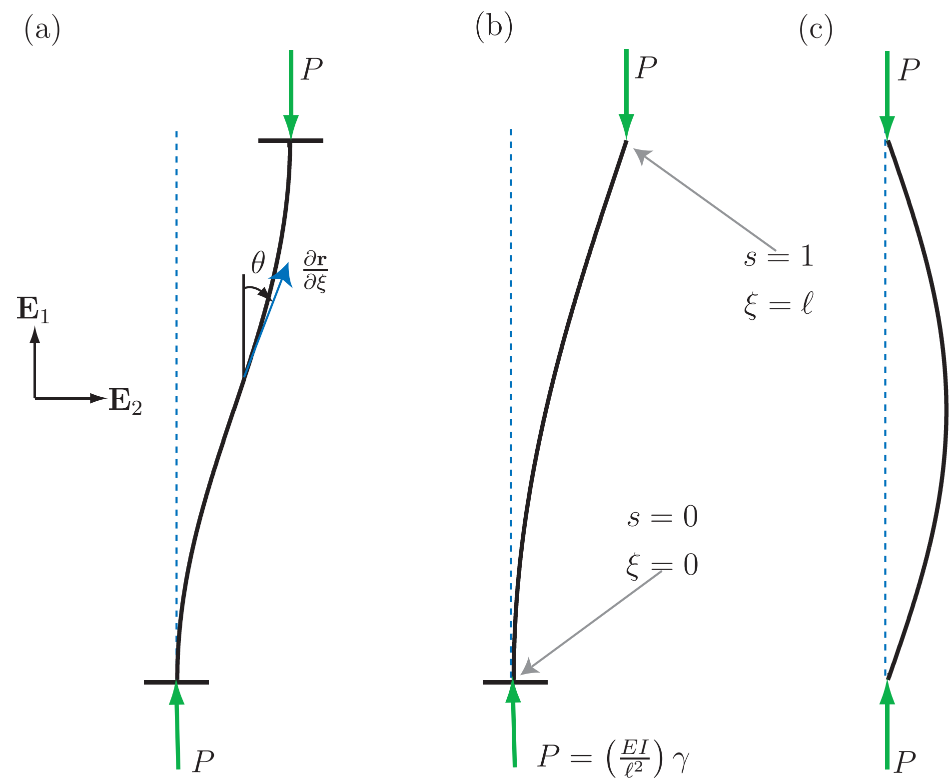

Figure 4.1 Schematic of an elastic rod under dimensionless axial load γ for three sets of boundary conditions: (a) fixed-fixed (also known as Dirichlet-Dirichlet), (B) fixed-free (also known as Dirichlet-Neumann), and (c) free-free (also known as Neumann-Neumann).

The deformed configuration of the rod is defined by the vector-valued function $\mathbf{r}(\xi) = X\mathbf{E}_1 + Y\mathbf{E}_2$. As before, the unit tangent vector to the deformed centerline has the representations

$$\frac{\partial \mathbf{r}}{\partial \xi} = \frac{\partial X}{\partial \xi}\mathbf{E}_1 + \frac{\partial Y}{\partial \xi}\mathbf{E}_2 = \cos(\theta)\mathbf{E}_1 + \sin(\theta)\mathbf{E}_2. \tag{4.1}$$

The bending moment $\mathbf{m} = M\mathbf{E}_3$ in the rod is proportional to the curvature $\frac{\partial \theta}{\partial \xi}$:

$$M = EI\frac{\partial \theta}{\partial \xi}. \tag{4.2}$$

For a terminally loaded rod, the equations governing the elastica reduce to

$$\mathbf{n} = -P\mathbf{E}_1, \qquad \frac{\partial M}{\partial \xi} + ((\cos(\theta)\mathbf{E}_1 + \sin(\theta)\mathbf{E}_2) \times \mathbf{n}) \cdot \mathbf{E}_3 = 0, \tag{4.3}$$

where $\mathbf{n}$ is the contact force. This pair of equations can be reduced to a second-order ordinary differential equation for $\theta(\xi)$:

$$EI\frac{\partial^2\theta}{\partial\xi^2} + P\sin(\theta) = 0. \tag{4.4}$$

Thus, they need to be supplemented by boundary conditions on θ and $\frac{\partial\theta}{\partial\xi}$. Analytical expressions for $\theta(s)$, $X(s)$, and $Y(s)$ for all the solutions to (4.4) that are discussed here can be found in Love [30, Section 263].

By summing the potential energies of the terminal loads and the strain energy of the rod, the following expression for the total energy of the rod can be computed:

$$E = \int_{\xi=0}^{\xi=\ell} \frac{EI}{2}\left(\frac{\partial\theta}{\partial\xi}\right)^2 + P\cos(\theta)\,d\xi. \tag{4.5}$$

It is well known that (4.4) are the Euler-Lagrange differential equations associated with seeking extremizers of E.

It is convenient to non-dimensionalize the governing equations (4.4) with the help of the coordinate s and parameter γ:

$$s = \frac{\xi}{\ell}, \qquad \gamma = \frac{P\ell^2}{EI}. \tag{4.6}$$

The ends of the rod can now be denoted by $s = s_0$ and $s = s_f$ where $s_0 = 0$ and $s_f = 1$. Further, (4.4) simplifies to

$$\theta'' + \gamma\sin(\theta) = 0, \tag{4.7}$$

where the prime denotes the partial derivative with respect to s.

The dimensionless form of the total energy functional (4.5) can also be determined:

$$I = \int_0^1 \left(\frac{1}{2}(\theta')^2 + \gamma\cos(\theta)\right) ds. \tag{4.8}$$

With the help of (2.14), the Riccati equation for a thin strut is

$$w' = w^2 + \gamma\cos(\theta^*). \tag{4.9}$$

Referring to Figure 4.1, we are interested in three sets of boundary conditions in the sequel. In the first case, the ends of the rod are clamped. This case is called the fixed-fixed case or the Dirichlet-Dirichlet (DD) case. For the second case, which is called the fixed-free case or the Dirichlet-Neumann (DN) case, the end $s = 0$ is clamped while the end at $s = 1$ is free of applied moments. Finally, in the third case, which we refer to as the free-free case or the Neumann-Neumann (NN) case, both ends of the rod are free of applied moments. In summary,

$$\begin{aligned}
&\mathrm{DD}: \quad \theta\,(s=0)=0, \quad \text{and} \quad \theta\,(s=1)=0,\\
&\mathrm{DN}: \quad \theta\,(s=0)=0, \quad \text{and} \quad \theta'\,(s=1)=0,\\
&\mathrm{NN}: \quad \theta'\,(s=0)=0, \quad \text{and} \quad \theta'\,(s=1)=0.
\end{aligned} \tag{4.10}$$

As defined in (2.4), we consider the solution to (4.7) subject to boundary conditions of the form

$$\phi\left[\theta\left(s_0\right), \theta'\left(s_0\right)\right]=0, \qquad \psi\left[\theta\left(s_f\right), \theta'\left(s_f\right)\right]=0, \tag{4.11}$$

where ϕ and ψ are smooth scalar-valued functions. Clearly (4.11) encompass the three sets of conditions listed in (4.10).

4.2.1 The Fixed-Fixed Strut

For the case where Dirichlet boundary conditions are applied at both ends of the rod (see $(4.10)_1$), the boundary conditions for equations (4.7) are determined from the boundary conditions

$$\phi=\theta(s_0), \qquad \psi=\theta(s_f), \tag{4.12}$$

with the help of (4.11).

A bifurcation diagram showing the behavior of all possible equilibria (i.e., solutions to (4.7) subject to (4.12)) is presented in Figure 4.2. In the bifurcation diagram, the dimensionless energy $\bar{E}$ is given by (4.8). The displacement y is the signed maximum lateral distance that the rod is displaced when it deforms to either postbuckled state and is defined by

$$y_m=\operatorname{sgn}(\mathrm{Y})\left(\max|Y/\ell|\right), \tag{4.13}$$

where Y is the Cartesian coordinate defined by $(Y/\ell)'=\sin(\theta)$.

For the Dirichlet-Dirichlet case, the boundary conditions on the variations are determined with the help of (2.6):

$$\eta\left(s_0\right)=0, \qquad \eta\left(s_f\right)=0. \tag{4.14}$$

Therefore, the boundary conditions (2.15) are trivially satisfied and $w(s)$ can take on any value at the endpoints s_0 and s_f. It remains to find a solution to the Riccati equation (4.9) where $w(s)$ is finite. Restricting attention to the straight strut (i.e., $\cos\left(\theta^*\right)=1$), we find that a solution

$$w(s)=\sqrt{\gamma}\tan\left(\sqrt{\gamma}\,(s-0.5)\right) \tag{4.15}$$

exists provided $\gamma<\pi^2$. When $\gamma\geq\pi^2$, the bounded solution for all $s\in[0,1]$ does not exist. Consequently, the sufficient condition recovers the classical result that a strut with fixed-fixed boundary conditions is stable provided $\gamma<\pi^2$. These results agree with those obtained using Jacobi's necessary condition, as shown by Born [5], among

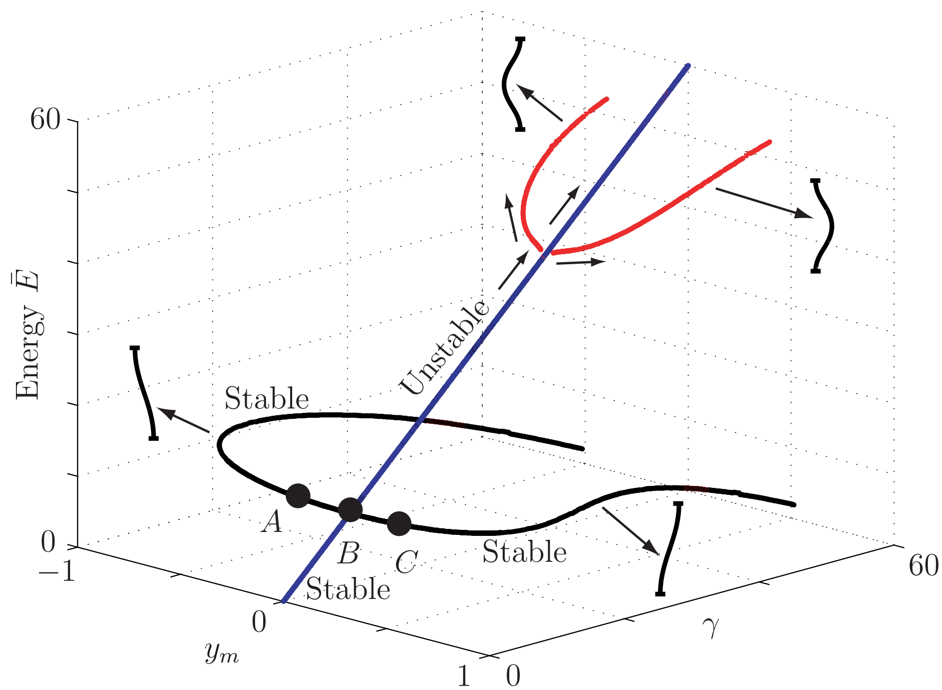

Figure 4.2 Bifurcation diagram showing the behavior of all equilibrium solutions for the Dirichlet-Dirichlet case over the region $\gamma = [0, 60]$.

others (see, e.g., [35]).

For the buckled configurations (i.e., solutions along the branch containing points A and C in Figure 4.2), the Riccati equation (4.9) must be solved numerically. A variety of buckled configurations, along with their respective Riccati solutions, are graphically represented in Figure 4.3. The key feature in Figure 4.3(b) is that each solution $w(s)$ is finite. Hence, by invoking condition LS1, we conclude that every buckled solution featured in Figure 4.3(a) is stable. For the straight strut and all subsequent branches beyond point B in Figure 4.2, no solutions to the Riccati equation can be computed, and so these configurations are unstable.

4.2.2 The Fixed-Free Strut

For the case where there is a Dirichlet boundary condition at the beginning of the rod and a Neumann boundary condition at the end (i.e., fixed-free), we have

$$\phi = \theta(s_0), \qquad \psi = \theta'(s_f). \tag{4.16}$$

As in the previous case, we obtain solutions to (4.7) subject to (4.16) and plot the resulting bifurcation diagram in Figure 4.4.

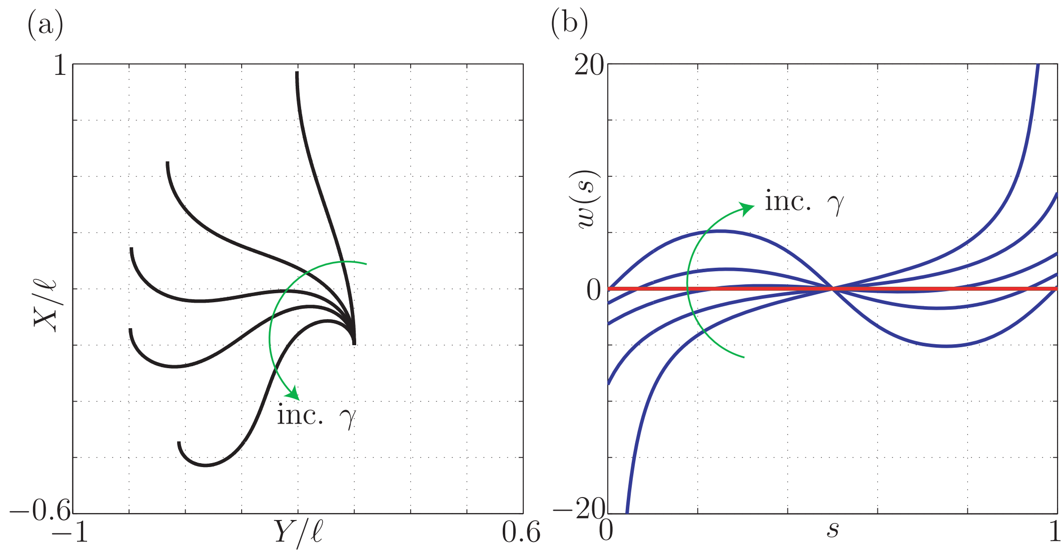

Figure 4.3 (a) Buckled solutions to the state equations, which reside on branch A-C in Figure 4.2; (b) Solutions to the Riccati equation (4.9) for each buckled configuration. In both (a) and (b), $\gamma = 10, 12, 15, 20$, and 40.

For the Dirichlet-Neumann case, the boundary conditions on the variations are

$$\eta\left(s_0\right) = 0, \qquad \eta'\left(s_f\right) = 0. \tag{4.17}$$

Consequently, the conditions (2.15) simplify to

$$\eta^2(s_f)w(s_f) = 0. \tag{4.18}$$

Therefore, we find that this condition is satisfied if we can find a solution to the Riccati equation where $w(s_f) = 0$ and $w(s)$ is finite. Restricting attention to the straight strut, we find that a solution

$$w(s) = \sqrt{\gamma}\tan\left(\sqrt{\gamma}\,(s-1)\right) \tag{4.19}$$

exists provided $\gamma < \frac{\pi^2}{4}$. When $\gamma \geq \frac{\pi^2}{4}$, the bounded solution for all $s \in [0,1]$ does not exist. Consequently, the L1 condition recovers the classical result that a strut with fixed-free boundary conditions is stable provided $\gamma < \frac{\pi^2}{4}$. This result can be found Born [5, Sect. 9], who used Jacobi's necessary condition to establish the bound on γ.

Just as in the fixed-fixed case, solutions to the Riccati equation (4.9) are graphically represented in Figure 4.5 for the branch of solutions labeled A and C in Figure 4.4. As before, the solutions in Figure 4.5(b) are finite. Therefore, since condition LS1 is satisfied, the configurations in Figure 4.5(a) are stable for the fixed-free case. Similarly, no Riccati solutions could be computed for the straight strut and subsequent

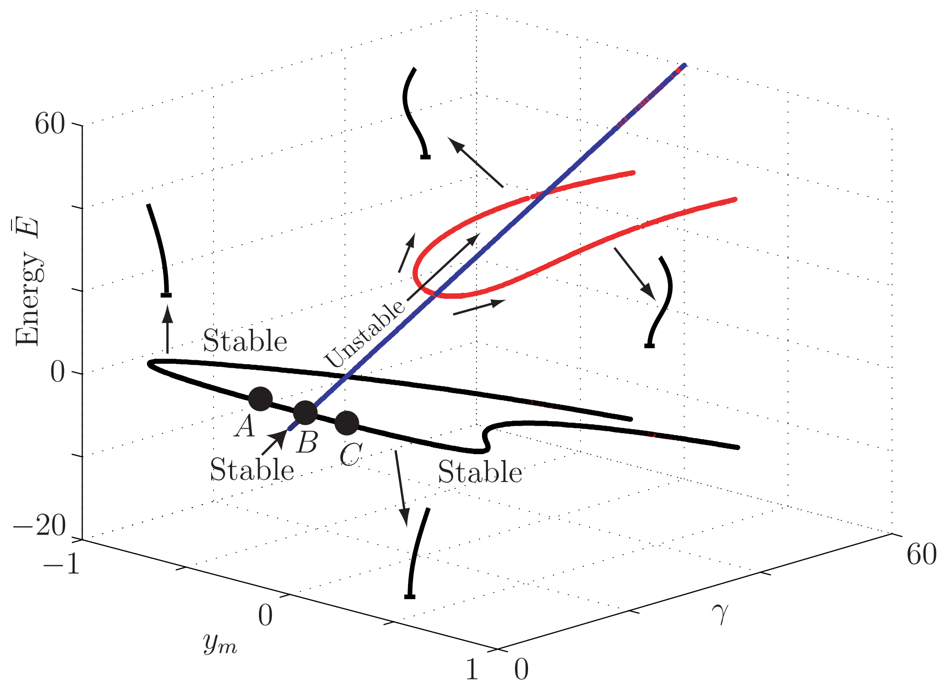

Figure 4.4 Bifurcation diagram showing solution behavior for the Dirichlet-Neumann case over the region $\gamma = [0, 60]$.

branches beyond point B in Figure 4.4, so these rod configurations are unstable.

4.2.3 The Free-Free Strut

The final case of interest arises when both boundary conditions are Neumann (free):

$$\phi = \theta'(s_0), \qquad \psi = \theta'(s_f). \tag{4.20}$$

The resulting bifurcation diagram is shown in Figure 4.6 and bears a striking similarity to the fixed-fixed case (Fig. 4.2).

For the Neumann-Neumann strut, the boundary conditions on the variations are

$$\eta'(s_0) = 0, \qquad \eta'(s_f) = 0. \tag{4.21}$$

As in the previous two cases, the conditions (2.15) simplify considerably:

$$w(s_0) = 0, \qquad w(s_f) = 0. \tag{4.22}$$

These boundary conditions imply that we require $w(s) = 0$. However, even for the trivial case where the rod is straight and $\cos(\theta) = 1$, $w(s) = 0$ does not satisfy (4.9). Consequently, the necessary condition L1 cannot be applied to the free-free

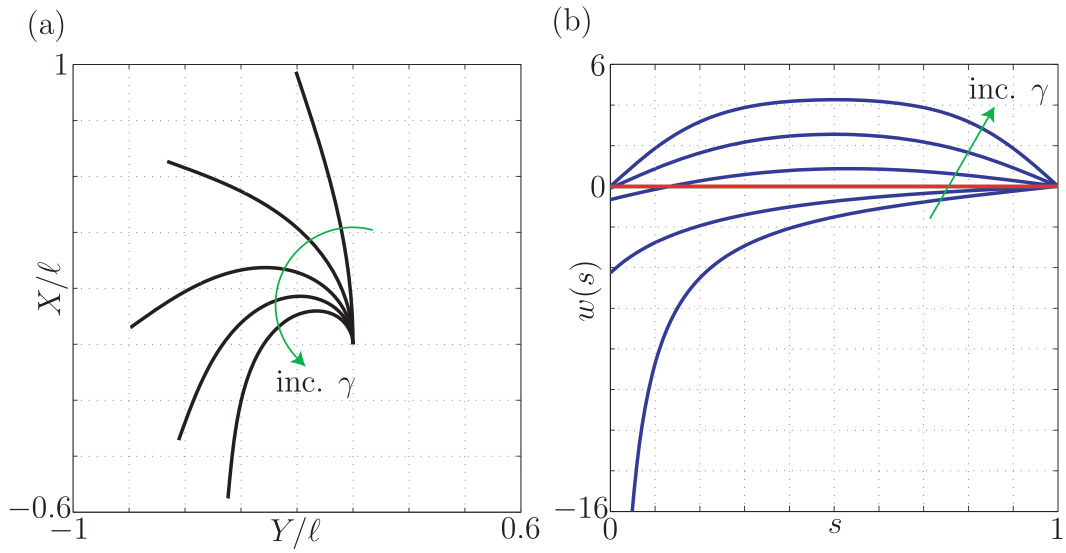

Figure 4.5 (a) Buckled solutions to the state equations, which reside on branches A & C in Figure 4.4; (b) Solutions to the Riccati equation (4.9) for each buckled configuration. In both (a) and (b), $\gamma = 2.5, 3, 5, 10,$ and 20.

beam. As shown by Manning [35], application of Jacobi's necessary condition to the free-free strut is not valid. However, the Riccati equation illustrates this in a transparent manner. An alternate examination of this result using optimal control theory is discussed in Appendix A.

4.3 Buckling of a Heavy Strut Under Terminal Load

The uniform strut has a mass density per unit length of ρ_0, a flexural rigidity EI, and a length L. As can be seen in figure 4.7(a), the rod is clamped at $\xi = 0$ and is subject to a vertical force $-F\mathbf{E}_1$ at the free end $\xi = L$. The Euler-Lagrange and Riccati equations for this problem can be deduced from $(3.9)_1$ and $(3.14)_1$:

$$\begin{aligned} EI\frac{\partial^2\theta^*}{\partial\xi^2} &= -\left(F + \rho_0\bar{g}\left(L-\xi\right)\right)\sin\left(\theta^*\right), \\ \frac{\partial w}{\partial \xi} - \frac{w^2}{EI} &= \left(F + \rho_0\bar{g}\left(L-\xi\right)\right)\cos\left(\theta^*\right). \end{aligned} \tag{4.23}$$

The solutions to these equations are subject to the boundary conditions

$$\theta^*\left(\xi = 0\right) = 0, \qquad \frac{\partial\theta^*}{\partial\xi}\left(\xi = L\right) = 0, \qquad w\left(\xi = L\right) = 0. \tag{4.24}$$

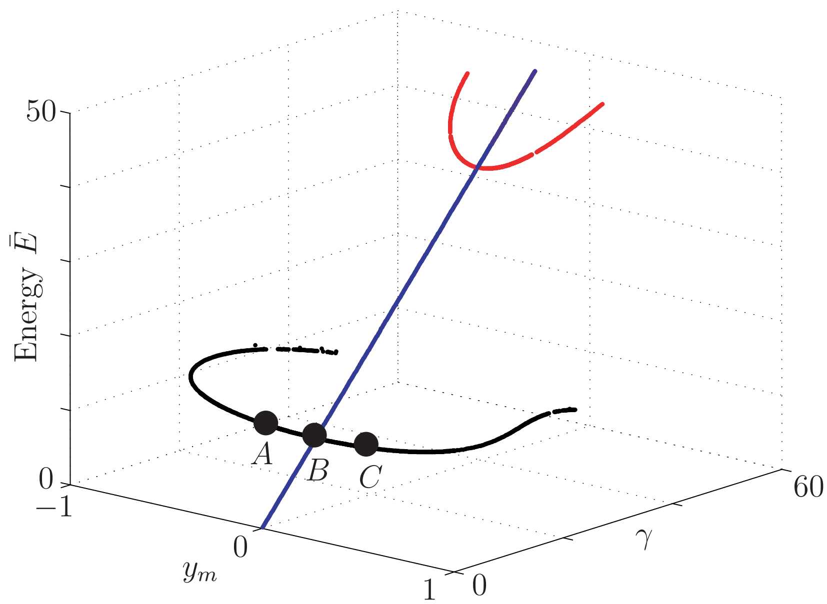

Figure 4.6 Bifurcation diagram showing solution behavior for the Neumann-Neumann case over the region $\gamma = [0, 60]$.

In the sequel, we shall fix the dimensionless weight parameter $\alpha = \frac{\rho_0 \bar{g} L^3}{EI}$ and vary the terminal load parameter $\gamma = \frac{FL^2}{EI}$.

Examining the solution to the Euler-Lagrange equation $(4.23)_1$, we observe that the straight strut $\theta^* = 0$ is a solution for all F and $\rho_0 \bar{g}$. Assuming that α is sufficiently small, we find that the Riccati equation for the straight strut,

$$\frac{\partial w}{\partial \xi} - \frac{w^2}{EI} = \left(F + \rho_0 \bar{g}\,(L - \xi)\right), \tag{4.25}$$

has a bounded solution provided γ is smaller than a critical value γ_{crit}.[1] A representative sample of w for varying γ is shown in figure 4.7(b). When $\gamma \geq \gamma_{\text{crit}}$, then (4.25) does not have a solution. We thus conclude that the straight strut is stable for $\gamma < \gamma_{\text{crit}}$.

For $\gamma > \gamma_{\text{crit}}$, the Euler-Lagrange equation $(4.23)_1$ admits two non-trivial solutions (or buckled states) which are mirror images of each other. The evolution of these solutions as F is increased is shown in figure 4.7(a). We note that, as γ is increased, it shows considerable deflection from the vertical. It suffices to examine a single Riccati equation $(4.23)_2$ to determine the stability of both buckled solutions. Referring to figure 4.7(c), we find that $(4.23)_2$ possess bounded solutions for each of the pair of buckled states and conclude that the buckled states are stable.

[1] An analytic expression, featuring Airy functions, for $w(\xi)$ can be established, but we do not pursue this here.

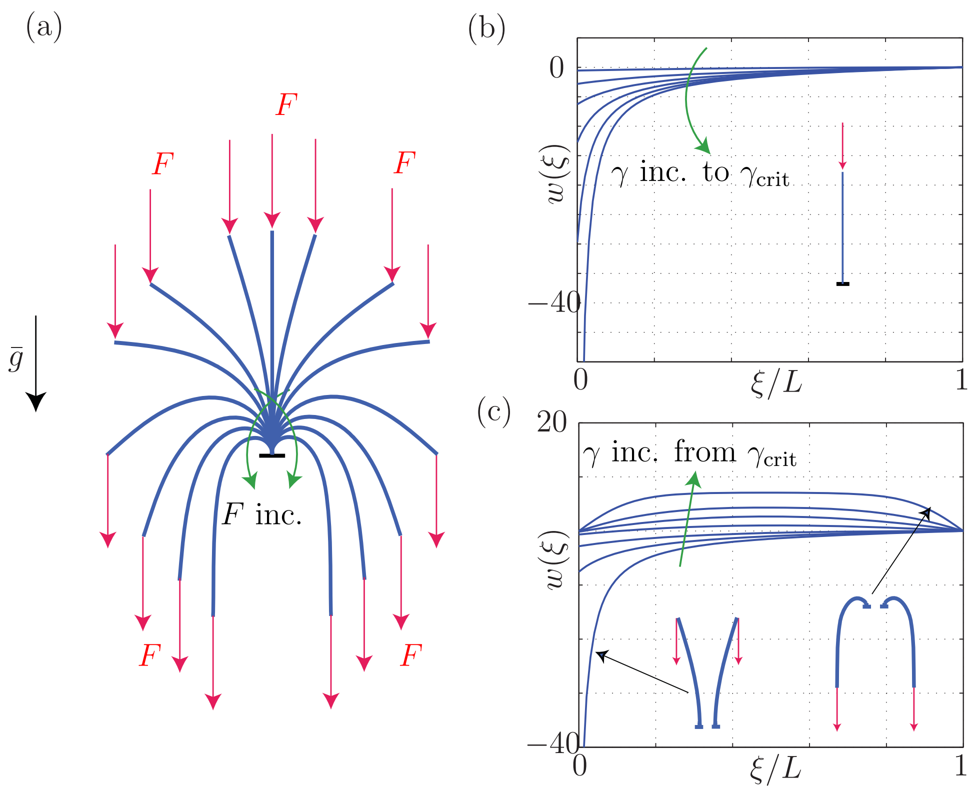

Figure 4.7 (a) Schematic of the straight and buckled states of a heavy terminally loaded strut elastica which is subject to a terminal load $-F\mathbf{E}_1$. (b) The solution of (4.25) for the straight strut for γ increasing from 0 to γ_{crit}. (c) The solution of $(4.23)_2$ for the buckled strut for γ increasing from γ_{crit}. For the examples shown, $\alpha = 1.0$, $\gamma_{\text{crit}} \approx 2.16$, and the values of γ used in (a) and (c) are $2.2, 2.5, 3, 5, 10, 20,$ and 50.

4.4 Buckling of the Human Spine

One of the more fascinating applications of elastic rod theory is the modeling of the human spine. The seminal work by Lucas & Bresler [31] in 1961 laid the foundations for the use of Euler buckling theory in the analysis of spinal stability. Their result was that the isolated spine has a buckling load of 21 N, which is far lower than what the trunk is capable of supporting *in vivo*. This implies an impressive stabilizing mechanism by the surrounding back muscles, ligaments, and tissues. Many researchers have subsequently used the Euler buckling approach to advance the body of knowledge regarding spinal stability [3, 10, 40, 46, 54].

With this literature in mind, we wish to consider a continuous elastic rod model of the spine using the elastica (see Chapter 3). The model will be restricted to the sagittal plane, where the spine has a known intrinsic curvature $\kappa_g = \kappa_g(\xi)$. In addition to a gravitational force $-\bar{g}\mathbf{E}_1$, we assume that the tip of the spine is subject to a terminal dead loading $-\mathbf{F} = F\mathbf{E}_1$. A fixed-free set of boundary conditions will be imposed, such that the base of the spine is fixed and the tip is free (see §4.2.2 for more details on these boundary conditions). Therefore, from (3.9) and (3.14), the Euler-Lagrange and Riccati equations for the spine model are

$$\frac{\partial}{\partial \xi}\left(EI\left(\frac{\partial \theta^*}{\partial \xi} - \kappa_g\right)\right) = \left(\mathbf{F} + m(\xi)\bar{g}\mathbf{E}_1\right)\cdot\left(-\sin\left(\theta^*\right)\mathbf{E}_1 + \cos\left(\theta^*\right)\mathbf{E}_2\right), \tag{4.26}$$

$$\frac{\partial w}{\partial \xi} + P - \frac{w^2}{EI} = 0, \tag{4.27}$$

where $m(\xi) = \int_\xi^L \rho_0(u)du$, as defined in (3.6), and

$$P = -\left(\mathbf{F} + m(\xi)\bar{g}\mathbf{E}_1\right)\cdot\left(\cos\left(\theta^*\right)\mathbf{E}_1 + \sin\left(\theta^*\right)\mathbf{E}_2\right). \tag{4.28}$$

The desired solutions to these equations need to satisfy the boundary conditions (from (3.10) and (3.15))

$$\theta^*(0) = \theta_0, \qquad \frac{\partial \theta^*}{\partial \xi}\left(\xi = L\right) = \kappa_g\left(\xi = L\right), \qquad w\left(L\right) = 0. \tag{4.29}$$

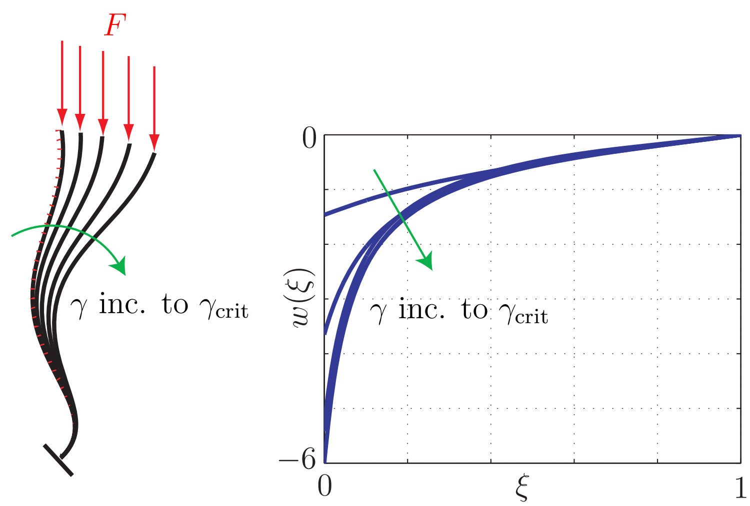

Figure 4.8 Schematic of an unbuckled terminally loaded spine (left) and the corresponding solutions to the Riccati equation (right). For the examples shown, $\alpha = 0$, $\gamma_{\text{crit}} \approx 2.53$, and γ values used are $1, 2, 2.25, 2.4$ and 2.5.

The intrinsic curvature κ_g must be prescribed in a way that accurately represents the lordotic and kyphotic curvatures of the lumbar and thoracic regions of the spine.

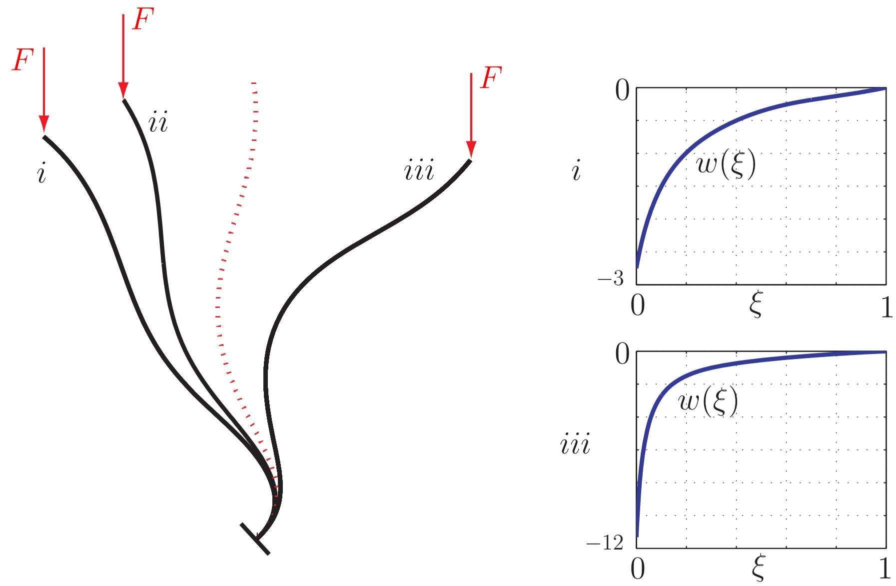

Figure 4.9 Schematic of three configurations for a buckled spine (left) and the Riccati solutions for the two stable configurations, i and iii (right). For the example shown, $\alpha = 0$ and $\gamma = 2.75$.

Using a polynomial curve fitted to radiographic measurements of the spine provided by Yang et al. [59], we obtain a general prescription of intrinsic curvature. In order to get this prescription into a more tractable form, a transformation of variables is required so that κ_g can be written as a function of ξ (see Peters [48] for a detailed description of this procedure).

With the gravity dependence suppressed, we can solve the Euler-Lagrange equation (4.26) subject to an increasing terminal force F (as in §4.3 we use the dimensionless load parameter γ). As figure 4.8 shows, the unbuckled rod configurations no longer remain undeformed as in the case where the intrinsic curvature is absent. Due to this change in the nature of buckling, it becomes especially important to examine solutions of the Riccati equation (4.27) to identify the configurations that are unstable. When γ is increased up to a critical value γ_{crit}, all of the solutions to (4.27) are bounded, as the right side of figure 4.8 shows. Therefore, despite the deformations that occur in the pre-buckled configurations, they are all stable and no other configurations are possible.

For $\gamma > \gamma_{\text{crit}}$, the Euler-Lagrange equation (4.26) admits two additional solutions, which are no longer symmetrical about the vertical axis. Since there is no trivial solution, as in the classical case (§4.2), solving the Riccati equation becomes imperative for identifying the unstable configuration because it is not obvious *a priori* which of the three solutions to (4.26) fails to minimize the total energy. This is illustrated in figure 4.9 for a specific example where $\gamma > \gamma_{\text{crit}}$. Here, we find that configurations i

and *iii* possess bounded solutions to (4.27), while configuration *ii* does not; hence, we conclude, with the help of L1, that configuration *ii* is unstable, and with the help of LS1, that configurations *i* and *iii* are stable.

Chapter 5

Application to Branched Rods

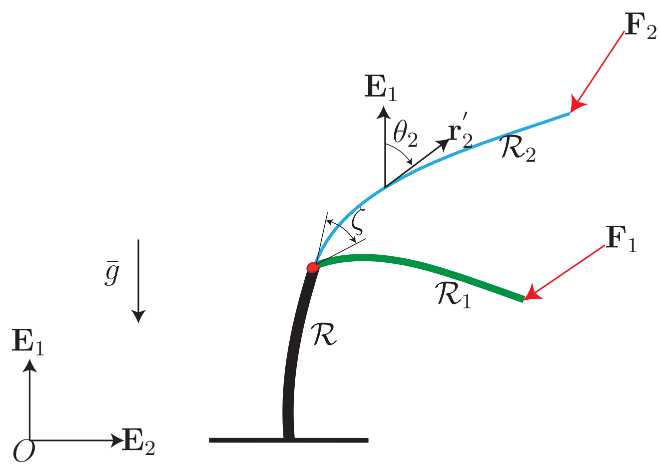

Figure 5.1 The configuration of a tree-like structure composed of three rods with a branching angle ζ. The rods $\mathcal{R}_1$ and $\mathcal{R}_2$ model child branches while $\mathcal{R}$ models the parent branch.

5.1 Introduction

As we showed in O'Reilly & Peters [42], we wish to verify the stability criteria established in §2.3 for a simple illustrative example. This is the first use of a second variation analysis to examine the nonlinear stability of a branched tree-like structure. The example shown is the simplest case of a three-branch structure, but it is straightforward to extend the stability analysis to more complex structures.

5.2 Buckling and Stability for a Tree-Like Structure

We now consider the tree-like structure shown in figure 5.1, consisting of three rods which are connected at a branching point. We assume that the base of the parent branch $\mathcal{R}$ is clamped while the tips of the child branches, $\mathcal{R}_1$ and $\mathcal{R}_2$, are free from external loading. To explore the wealth of solutions possible for a branched structure of this type we vary the angle θ_b at the base of $\mathcal{R}$ and see how the resulting strain κ_b at the base changes as a function of θ_b:

$$\theta_b = \lim_{\xi \searrow 0} \theta(\xi), \qquad \kappa_b = \lim_{\xi \searrow 0} \frac{\partial \theta}{\partial \xi}(\xi). \tag{5.1}$$

As emphasized in O'Reilly & Tresierras [45] the resulting function $\kappa_b(\theta_b)$ clearly indicates the presence of multiple solutions.[1]

The parameter values for the example discussed in the present section are

$$\begin{aligned}
\mathcal{R} &: \quad L = 1.0, \quad EI = 5, \quad \rho_0 = 7, \quad \xi \in [0, L)\,; \\
\mathcal{R}_1 &: \quad L_1 = 1.5, \quad EI_1 = 1.0, \quad \rho_{0_1} = 4.5, \quad \xi_1 \in [L, L + L_1)\,; \\
\mathcal{R}_2 &: \quad L_2 = 1.15, \quad EI_2 = 5, \quad \rho_{0_2} = 7, \quad \xi_2 \in [L, L + L_2)\,.
\end{aligned} \tag{5.2}$$

The branching angles (see (3.11)) for the example are

$$\chi_1 = 0, \qquad \chi_2 = 0, \qquad \chi_3 = 0. \tag{5.3}$$

The strain κ_b as a function of θ_b is shown in figure 5.2. Observe that for $\theta_b \in (-30.37°, 30.37°)$, three distinct solutions to the Euler-Lagrange equations (3.9) are possible.

To characterize the stability of the solutions to the boundary-value problem, the Riccati equations (3.14) are examined. An example of such a calculation is shown in figure 5.3 for the case where $\theta_b = 20°$. As expected, two of the solutions (A and C) satisfy the necessary condition L1, while a bounded solution to the Riccati equation $(3.14)_1$ for w is non-existent for B. Combining the results for all values of θ_b, we conclude that the necessary condition L1 is satisfied except for the dashed portion of the graph of $\kappa_b(\theta_b)$ in figure 5.2(a). For the portion of the graph of $\kappa_b(\theta_b)$ where L1 is not satisfied, we can conclude that the tree-like structures for these cases are unstable. In these unstable cases, the parent branch has buckled under the weight of the child branches.

[1] In O'Reilly & Tresierras [45], a situation of a branched structure of three rods (similar to that shown in figure 5.1) with 9 possible configurations is presented.

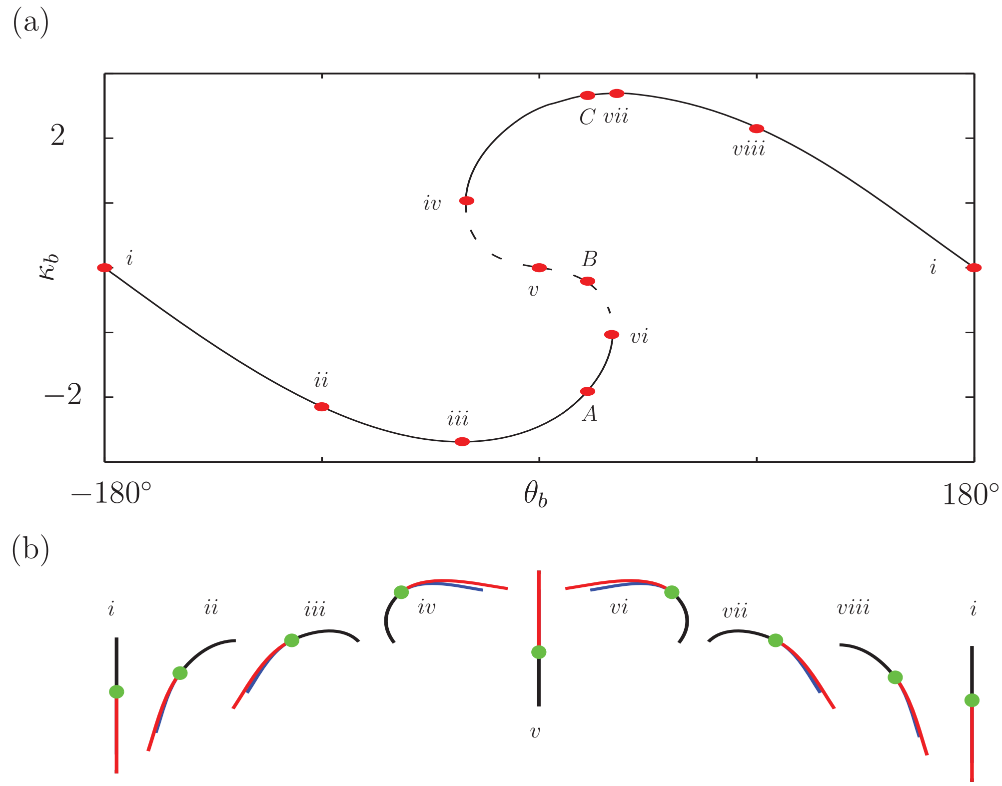

Figure 5.2 (a) The strain κ_b at the base of the parent branch as a function of the angle θ_b for a tree-like structure composed of three rods. (b) Selected configurations of the three rods. In this figure, the labels $i - vii$ correspond to the base angles $-180°, -90°, -32°, -30°, 0°, 30°, 32°, 90°$ respectively. The points labeled A, B, and C correspond to $\theta_b = 20°$. Further results for these three solutions are presented in figures 1.1 and 5.3.

5.3 A Tree of Riccati Equations

It is illuminating to apply the condition L1 to branched structures such as those shown in figure 5.4. This structure consists of seven branches joined at three branching points. The branches have respective lengths, arc length parameters, and flexural rigidities of $L, L_1, \ldots, L_6$, $\xi, \xi_1, \ldots, \xi_6$, and $EI, EI_1, \ldots, EI_6$. We assume that the equilibrium configuration of the structure has been computed. Then, in order to determine if the total energy of the structure satisfies the necessary condition for a minimum, we need to find a bounded set of solutions $w, w_1, \ldots, w_6$ to a set of Riccati equations. The boundary conditions for $w_3, \ldots, w_6$ and the jump conditions that are used to prescribe w_1^-, w_2^- and w^- are easily inferred from our earlier developments (cf.

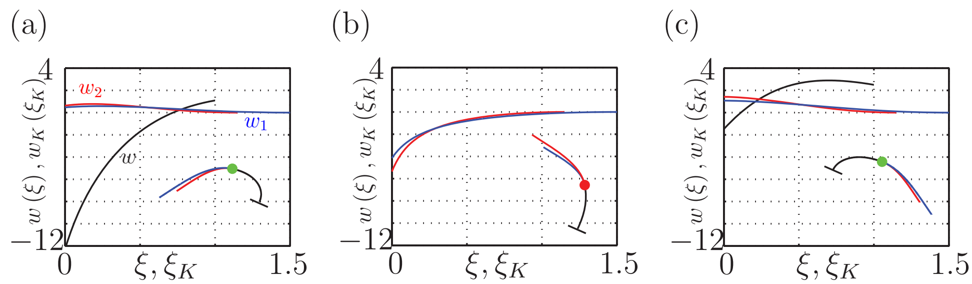

Figure 5.3 Solutions $w(\xi)$, $w(\xi_1)$, and $w_2(\xi_2)$ to the Riccati equations (3.14) for the solutions to the boundary-value problem that are shown in figure 1.1. For these solutions, $\theta_b = 20°$: (a) configuration A, (b) configuration B, and (c) configuration C shown in figure 5.2. For configuration B, a bounded solution $w(\xi)$ does not exist.

(2.49) and (3.15)). As a result, a set of seven Riccati equations can be established:

$$\begin{aligned} \frac{\partial w}{\partial \xi} + P - \frac{w^2}{EI} &= 0, \\ \frac{\partial w_K}{\partial \xi_K} + P_K - \frac{w_K^2}{EI_K} &= 0, \qquad (K = 1, \ldots, 6). \end{aligned} \tag{5.4}$$

subject to the boundary conditions

$$\begin{gathered} w_K(L_K) = 0, \quad (K = 3, 4, 5, 6), \\ w_1^- = w_3^+ + w_4^+, \quad w_2^- = w_5^+ + w_6^+, \quad w^- = w_1^+ + w_2^+. \end{gathered} \tag{5.5}$$

We refer to the set of Riccati equations (5.4) as a *tree of Riccati equations*. Alternatively, using the transformations (2.52), a tree of Jacobi differential equations of the form (2.53) could also be established. With the help of L1, either tree of equations will be useful in distinguishing unstable solutions in computer-generated images of trees which are based on rod models.

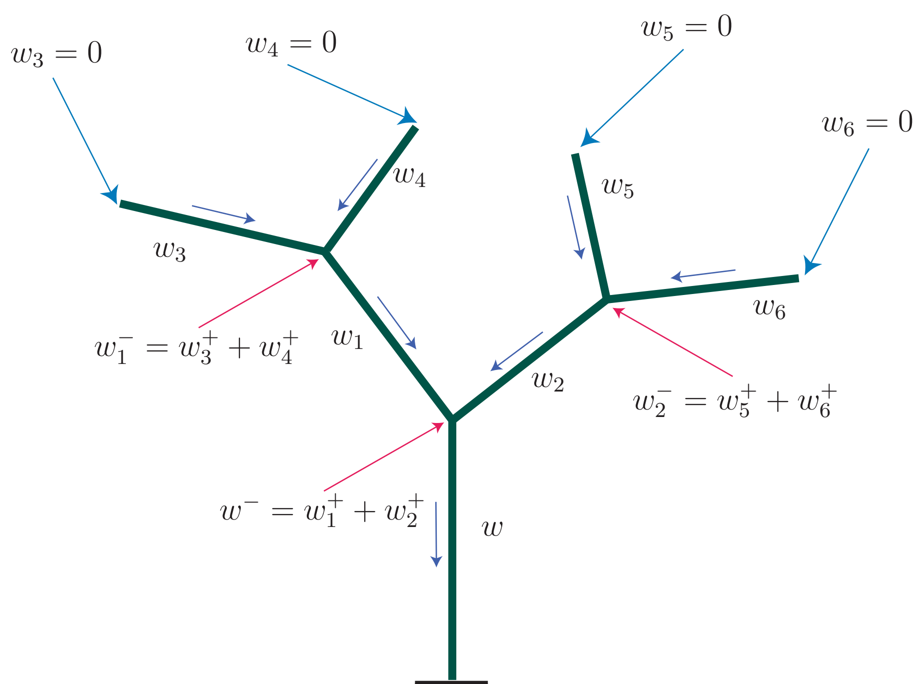

Figure 5.4 Schematic of the solution procedure for the set of Riccati equations for a tree-like structure featuring seven branches and three branching points.

Chapter 6

Stability Criteria for Green and Naghdi's Rod Theory

6.1 Introduction

The rod theory discussed in this chapter originated in the work by Green and Laws [17] and was later developed in a series of papers by Green, Naghdi and several of their co-workers [18, 19, 51]. We can consider the theory discussed here as one which subsumes the earlier theory of Euler's elastica and Kirchhoff rod theory (see Chapter 7). In this theory, the material curve (centerline) is extensible, and the directors $\mathbf{d}_\alpha$ can change their length and relative orientation. Thus, the unusual feature of their theory is that the directors can deform in an arbitrary manner. We will show in this chapter that this feature of the theory allows, in a fairly straightforward manner, for the establishment of stability criteria based on variational principles.

6.2 Kinematics

In Green and Naghdi's rod theory, there are two directors, denoted by $\mathbf{d}_1$ and $\mathbf{d}_2$, which describe the behavior of the directed curve for the rod, and the tangent vector is denoted by $\frac{\partial \mathbf{r}}{\partial \xi} = \mathbf{d}_3$ (see figure 6.1). It is also important to note that the material curve associated with the directed curve is assumed to be extensible: $\frac{\partial \mathbf{r}}{\partial \xi} = \mu \mathbf{e}_t$, where μ is the stretch. We define a fixed reference configuration of the directed curve by the vector fields $\mathbf{R} = \mathbf{R}(\xi)$ and $\mathbf{D}_\alpha = \mathbf{D}_\alpha(\xi)$. The referential vectors $\mathbf{D}_\alpha$ are defined

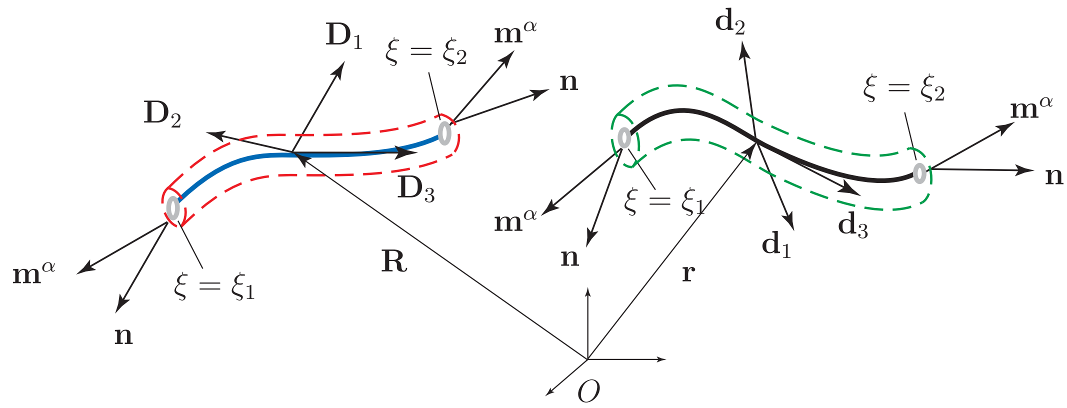

Figure 6.1 Schematic of a Green-Naghdi rod with position vector $\mathbf{r}(\xi)$ and directors $\mathbf{d}_i$. The rod is subject to a contact force $\mathbf{n}$ and contact director force $\mathbf{m}^\alpha$ at both ends, $\xi = \xi_1$ and $\xi = \xi_2$. The reference configuration is also shown in this figure.

by the linear transformation $\mathbf{F}_0$:

$$\mathbf{F}_0 = \mathbf{D}_1 \otimes \mathbf{E}_1 + \mathbf{D}_2 \otimes \mathbf{E}_2 + \mathbf{D}_3 \otimes \mathbf{E}_3. \tag{6.1}$$

That is, $\mathbf{D}_i = \mathbf{F}_0\mathbf{E}_i$. For many reference configurations, we can choose $\mathbf{D}_i$ such that $\mathbf{F}_0 = \mathbf{I}$.

Under a motion of the directed curve, the vectors $\mathbf{d}_i$ can change their relative orientation and magnitude. Consequently,

$$\mathbf{d}_i = \mathbf{F}_1\mathbf{D}_i, \quad (i = 1, 2, 3), \tag{6.2}$$

where $\mathbf{F}_1 = \mathbf{F}_1(\xi, t)$ is a linear transformation [1]. It should be clear that

$$\mathbf{d}_\alpha = \mathbf{F}_1\mathbf{F}_0\mathbf{E}_\alpha, \quad \frac{\partial \mathbf{r}}{\partial \xi} = \mathbf{F}_1\frac{\partial \mathbf{R}}{\partial \xi}, \quad (\alpha = 1, 2). \tag{6.3}$$

The total energy of a rod can be written as

$$I = \int_{\xi_0}^{\xi_1} \left(\rho_0\psi\left(\mathbf{r}^{'}, \mathbf{d}_\alpha, \mathbf{d}_\alpha^{'}\right) + U\left(\mathbf{r}, \mathbf{d}_\alpha\right)\right) d\xi, \quad (\alpha = 1, 2), \tag{6.4}$$

where $\rho_0\psi$ denotes the strain energy function, U denotes the potential function, and the prime $'$ denotes $\frac{\partial}{\partial \xi}$.

[1] In Green and Naghdi's rod theory, the directors $\mathbf{d}_\alpha$ are allowed to deform in an arbitrary manner. If $\mathbf{F}_1$ and $\mathbf{F}_0$ are restricted to be orthogonal tensors, then the model is reduced to simpler rod theories, such as Kirchhoff's rod theory (see Chapter 7).

Green and Naghdi's rod theory also has 12 strains, which are defined as follows:

$$\begin{aligned} \gamma_{ik} &= \mathbf{d}_i \cdot \mathbf{d}_k - \mathbf{D}_i \cdot \mathbf{D}_k, \\ \kappa_{\alpha k} &= \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \cdot \mathbf{d}_k - \frac{\partial \mathbf{D}_\alpha}{\partial \xi} \cdot \mathbf{D}_k. \end{aligned} \tag{6.5}$$

The directed curve in this theory resists bending, extension, lateral contractions and expansions, and torsion. Its strain energy function per unit mass is a function of the strains γ_{ik} and $\kappa_{\alpha k}$:

$$\psi = \tilde{\psi}\left(\gamma_{ik}, \kappa_{\alpha k}\right). \tag{6.6}$$

We see from equations (6.5) that the strain energy also depends on the Cartesian components of the directors $x_i, d_{\alpha j}$. Therefore, by combining (6.6) with a potential energy function, we can write the total energy as

$$I = \int_{\xi_0}^{\xi_1} \left(\rho_0 \psi\left(x_i^{'}, d_{\alpha i}, d_{\alpha i}^{'}\right) + U\left(x_i, d_{\alpha i}\right)\right) d\xi, \tag{6.7}$$

where

$$x_i = \mathbf{r} \cdot \mathbf{E}_i, \qquad d_{\alpha i} = \mathbf{d}_\alpha \cdot \mathbf{E}_i, \qquad (\alpha = 1, 2), \qquad (i = 1, 2, 3). \tag{6.8}$$

It will become clear in the following sections why this form of the total energy is advantageous.

6.3 First Variation and Necessary Conditions for an Extremal

To extend the stability criteria established in Chapter 2 to Green and Naghdi's theory of rods, it is helpful to consider a functional of the form

$$I\left(y_1, y_2, \ldots, y_N\right) = \int_a^b f\left(u, y_1(u), \ldots, y_N(u), y_1^{'}(u), \ldots, y_N^{'}(u)\right) du, \tag{6.9}$$

where $y_1(x), \ldots, y_N(x)$ are scalar functions of the independent variable x.

We seek extremals $y_1^*(x), \ldots, y_N^*(x)$ of I which satisfy the boundary conditions at $x = a$ and $x = b$:

$$y_1(a) = y_{1a}, \ldots, y_N(a) = y_{Na}, \qquad y_1(b) = y_{1b}, \ldots, y_N(b) = y_{Nb}. \tag{6.10}$$

We now wish to consider changes to I which arise when the functions $y_1^*(x), \ldots, y_N^*(x)$ are varied:

$$y_1(x, \epsilon) = y_1^*(x) + \epsilon\eta_1(x), \ldots, y_N(x, \epsilon) = y_N^*(x) + \epsilon\eta_N(x), \tag{6.11}$$

where ϵ and $\eta_1(x), \ldots, \eta_N(x)$ are defined as in §2.2.1 of Chapter 2.

Following a similar analysis, the first variation becomes

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_a^b \left\{ \frac{\partial f}{\partial y_1}\eta_1 + \frac{\partial f}{\partial y_1^{'}}\eta_1^{'} + \ldots + \frac{\partial f}{\partial y_N}\eta_N + \frac{\partial f}{\partial y_N^{'}}\eta_N^{'} \right\} du. \tag{6.12}$$

In order for $y_1^*(x), \ldots, y_N^*(x)$ to be extremals, the right-hand side of (6.12) must vanish for all $\eta_1(x), \ldots, \eta_N(x)$. If we assume that $\eta_1(x)$ is arbitrary, then $\eta_2(x) = \ldots = \eta_N(x) = 0$ are valid choices for the remaining variations. Therefore, substituting these variations into (6.12) and integrating by parts yields the first Euler-Lagrange equation:

$$\frac{\partial f}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_1^{'}}\right) = 0, \qquad \left[\frac{\partial f}{\partial y_1^{'}}\eta_1\right]_a^b = 0. \tag{6.13}$$

Repeating this process for all N variations yields the remaining $N-1$ Euler-Lagrange equations:

$$\frac{\partial f}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_2^{'}}\right) = 0, \quad \ldots, \quad \frac{\partial f}{\partial y_N} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_N^{'}}\right) = 0, \tag{6.14}$$

subject to the boundary conditions

$$\left[\frac{\partial f}{\partial y_2^{'}}\eta_2\right]_a^b = 0, \ldots, \left[\frac{\partial f}{\partial y_N^{'}}\eta_N\right]_a^b = 0. \tag{6.15}$$

6.3.1 First Variation Conditions for a Green-Naghdi Rod

We now consider a total energy functional for a Green-Naghdi rod as given by (6.4). The variations for the directors and their derivatives are defined as follows:

$$\mathbf{r} \quad = \mathbf{r}^* + \epsilon\mathbf{v}_3, \quad \mathbf{r}^{'} \quad = \mathbf{r}^{*'} + \epsilon\mathbf{v}_3^{'}, \tag{6.16}$$

$$\mathbf{d}_\alpha \quad = \mathbf{d}_\alpha^* + \epsilon\mathbf{v}_\alpha, \quad \mathbf{d}_\alpha^{'} \quad = \mathbf{d}_\alpha^{*'} + \epsilon\mathbf{v}_\alpha^{'}, \quad (\alpha = 1, 2) \tag{6.17}$$

Substituting in these variations and computing the first variation, we obtain the Euler-Lagrange equations

$$\frac{d}{d\xi}\left(\rho_0\frac{\partial\psi}{\partial\mathbf{r}'}\right) - \frac{\partial U}{\partial\mathbf{r}} = \mathbf{0}, \quad \frac{d}{d\xi}\left(\rho_0\frac{\partial\psi}{\partial\mathbf{d}_\alpha'}\right) - \rho_0\frac{\partial\psi}{\partial\mathbf{d}_\alpha} - \frac{\partial U}{\partial\mathbf{d}_\alpha} = \mathbf{0}. \tag{6.18}$$

We can write the Euler-Lagrange equations in a more concise form by using the following constitutive relations:

$$\mathbf{n} = \rho_0\frac{\partial\psi}{\partial\mathbf{r}'}, \quad \mathbf{k}^\alpha = \rho_0\frac{\partial\psi}{\partial\mathbf{d}_\alpha}, \quad \mathbf{m}^\alpha = \rho_0\frac{\partial\psi}{\partial\mathbf{d}_\alpha'},$$
$$\rho_0\mathbf{f} = -\frac{\partial U}{\partial\mathbf{r}}, \quad \rho_0\mathbf{l}^\alpha = -\frac{\partial U}{\partial\mathbf{d}_\alpha}. \tag{6.19}$$

Therefore, the final form of the Euler-Lagrange equations are as follows:

$$\begin{aligned} \frac{d\mathbf{n}}{d\xi} + \rho_0 \mathbf{f} &= \mathbf{0}, \\ \frac{d\mathbf{m}^\alpha}{d\xi} - \mathbf{k}^\alpha + \rho_0 \mathbf{l}^\alpha &= \mathbf{0}, \quad (\alpha = 1, 2). \end{aligned} \tag{6.20}$$

These are the familiar balances of linear and director momenta for the Green-Naghdi rod theory.

6.4 Second Variation and Stability Criteria for Extremals

For conciseness we now wish to express functionals of the form (6.9) as

$$I = \int_a^b f(\mathsf{y}, \mathsf{y}')du, \tag{6.21}$$

where $\mathsf{y} = [y_1(x), \ldots, y_N(x)]^T$ and $\mathsf{y}' = \left[y_1'(x), \ldots, y_N'(x)\right]^T$.

Following a similar analysis to the one featured in Chapter 2 §2.2.2 yields the second variation:

$$\delta^2 I = \left.\frac{d^2 I}{d\epsilon^2}\right|_{\epsilon=0} = \int_a^b \left\{ \boldsymbol{\eta} \cdot \mathsf{P}\boldsymbol{\eta} + 2\boldsymbol{\eta} \cdot \mathsf{Q}\boldsymbol{\eta}' + \boldsymbol{\eta}' \cdot \mathsf{R}\boldsymbol{\eta}' \right\} du, \tag{6.22}$$

where

$$\mathsf{P} = \mathsf{P}(x) = \frac{\partial^2 f}{\partial \mathsf{y} \partial \mathsf{y}} \left(x, \mathsf{y}^*, \mathsf{y}^{*'}\right), \quad \mathsf{Q} = \mathsf{Q}(x) = \frac{\partial^2 f}{\partial \mathsf{y} \partial \mathsf{y}'} \left(x, \mathsf{y}^*, \mathsf{y}^{*'}\right),$$

$$\mathsf{R} = \mathsf{R}(x) = \frac{\partial^2 f}{\partial \mathsf{y}' \partial \mathsf{y}'} \left(x, \mathsf{y}^*, \mathsf{y}^{*'}\right). \tag{6.23}$$

We now add the following Legendre identity to the right-hand side of (6.22):

$$\int_a^b \left(\boldsymbol{\eta} \cdot \mathsf{S}\boldsymbol{\eta}\right)' du - \left[\boldsymbol{\eta} \cdot \mathsf{S}\boldsymbol{\eta}\right]_a^b = 0, \tag{6.24}$$

where $\mathsf{S}(x) = \mathsf{S}^T(x)$ is a symmetric N-dimensional matrix-valued function of x.

Thus, $\delta^2 I$ simplifies to

$$\delta^2 I = \int_a^b \left\{ \left(\mathsf{P} + \mathsf{S}'\right) \boldsymbol{\eta} \cdot \boldsymbol{\eta} + 2\left(\mathsf{Q} + \mathsf{S}\right) \boldsymbol{\eta} \cdot \boldsymbol{\eta}' + \boldsymbol{\eta}' \cdot \mathsf{R}\boldsymbol{\eta}' \right\} du. \tag{6.25}$$

We now choose $\mathsf{S}(x)$ to satisfy the following matrix Riccati equation:

$$\mathsf{S}' + \mathsf{P} - \left(\mathsf{Q} + \mathsf{S}\right)^T \mathsf{R}^{-1} \left(\mathsf{Q} + \mathsf{S}\right) = \mathbf{0}, \tag{6.26}$$

subject to the boundary conditions on S,

$$\boldsymbol{\eta}(b) \cdot \mathsf{S}(b)\boldsymbol{\eta}(b) - \boldsymbol{\eta}(a) \cdot \mathsf{S}(a)\boldsymbol{\eta}(a) \leq 0. \tag{6.27}$$

If a solution $\mathsf{S}(x)$ to (6.26) can be found, then the resulting simplified expression for $\delta^2 I$ is non-negative:

$$\delta^2 I = \int_a^b \left\| \mathsf{R}\boldsymbol{\eta}' + (\mathsf{Q} + \mathsf{S})\,\boldsymbol{\eta} \right\|_{\mathsf{R}^{-1}}^2 du, \tag{6.28}$$

where

$$\|\mathsf{a}\|_{\mathsf{R}^{-1}}^2 = \mathsf{a} \cdot \mathsf{R}^{-1}\mathsf{a}.$$

It should be clear that the following condition is implied:

$$\mathsf{R}^{-1} > 0. \tag{6.29}$$

This is the matrix analogue to the classical Legendre necessary condition for stability of extremals ($R > 0$).

6.4.1 Second Variation Conditions for a Green-Naghdi Rod

In order to compute the second variation and obtain the same form as (6.22), the total energy must be written as (6.7). If we let the director components $x_i, d_{\alpha j}$ fill out a 9−dimensional vector d with variations v, such that

$$\mathsf{d} = \begin{bmatrix} d_{11} \\ \vdots \\ d_{13} \\ d_{21} \\ \vdots \\ d_{23} \\ x_1 \\ \vdots \\ x_3 \end{bmatrix}, \quad \mathsf{d}' = \begin{bmatrix} d_{11}' \\ \vdots \\ d_{13}' \\ d_{21}' \\ \vdots \\ d_{23}' \\ x_1' \\ \vdots \\ x_3' \end{bmatrix}, \quad \mathsf{v} = \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{E}_1 \\ \vdots \\ \mathbf{v}_1 \cdot \mathbf{E}_3 \\ \mathbf{v}_2 \cdot \mathbf{E}_1 \\ \vdots \\ \mathbf{v}_2 \cdot \mathbf{E}_3 \\ \mathbf{v}_3 \cdot \mathbf{E}_1 \\ \vdots \\ \mathbf{v}_3 \cdot \mathbf{E}_3 \end{bmatrix} \tag{6.30}$$

then the second variation can be expressed as (6.22), where

$$\mathsf{P} = \mathsf{P}(\xi) = \frac{\partial^2 f}{\partial \mathsf{d} \partial \mathsf{d}} \left(\xi, \mathsf{d}^*, \mathsf{d}^{*'} \right), \quad \mathsf{Q} = \mathsf{Q}(\xi) = \frac{\partial^2 f}{\partial \mathsf{d} \partial \mathsf{d}'} \left(\xi, \mathsf{d}^*, \mathsf{d}^{*'} \right),$$

$$\mathsf{R} = \mathsf{R}(\xi) = \frac{\partial^2 f}{\partial \mathsf{d}' \partial \mathsf{d}'} \left(\xi, \mathsf{d}^*, \mathsf{d}^{*'} \right). \tag{6.31}$$

Note that the functional for Green and Naghdi's theory is $f = \rho_0 \psi + U$.

6.4.2 Stability Criteria

Consider a Green-Naghdi rod and solve the boundary value problem for an equilibrium solution. Then, if a solution $\mathsf{S}(\xi)$ to (6.26) and subject to the boundary conditions

$$\mathsf{v}(\xi_1) \cdot \mathsf{S}(\xi_1)\mathsf{v}(\xi_1) - \mathsf{v}(\xi_0) \cdot \mathsf{S}(\xi_0)\mathsf{v}(\xi_0) \leq 0. \tag{6.32}$$

can be found, the solution to the boundary value problem satisfies a necessary condition for stability.

It is also clear that, if we wish to extend these results to a tree-like structure of branched Green-Naghdi rods, equation (6.32) easily allows for the establishment of a branching condition similar to $(2.49)_3$:

$$[\![\mathsf{S}]\!]_B = 0. \tag{6.33}$$

Chapter 7

Stability Criteria for Kirchhoff's Rod Theory

7.1 Introduction

The theory discussed in this chapter originated with the theory first presented by Kirchhoff in 1859 [26] that was capable of modeling bending and torsion. In the early 20$^{\text{th}}$ century, the Cosserat brothers re-formulated Kirchhoff's rod theory while introducing the concept of directors [7, 8]. Modern treatments of the theory can be found in works by Antman [1] and Rubin [51]. The use of directors is similar to that of Green and Naghdi's theory, except that the centerline in Kirchhoff's theory is inextensible and the cross sections remain plane and normal to the centerline. That is, the directors are constrained to deform rigidly and retain their orientation relative to the tangent vector $\mathbf{e}_t$ to the material curve. As a result of these constraints, some modifications are necessary in order to extend the previous stability criteria to this rod theory.

Kirchhoff's rod theory is the most popular among modern researchers for application to 3-dimensional problems modeled using elastic rods. With the advent of numerical methods, and applications to DNA modeling in particular, the number of papers on this rod theory has soared. Stability criteria for Kirchhoff's theory do exist, as shown by Born [5], Maddocks [32] and Manning et al. [36], but no attempt has yet been made to extend the resulting stability conditions to branched tree-like structures.

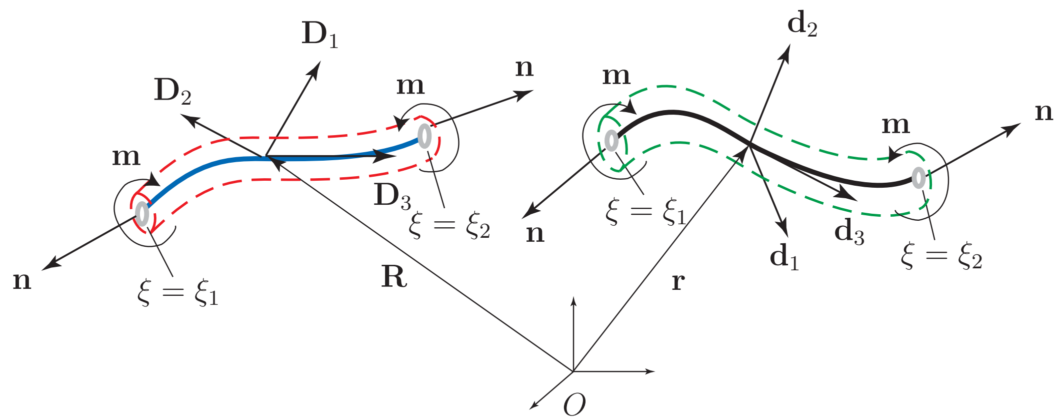

Figure 7.1 Schematic of a Kirchhoff rod with position vector $\mathbf{r}(\xi)$ and directors $\mathbf{d}_i$. The rod is subject to a contact force $\mathbf{n}$ and contact moment $\mathbf{m}$ at both ends, $\xi = \xi_1$ and $\xi = \xi_2$. The reference configuration is also shown in this figure.

7.2 Kinematics

In Kirchhoff's rod theory, we have the two directors $\mathbf{d}_1$ and $\mathbf{d}_2$, and the tangent vector denoted by $\frac{\partial \mathbf{r}}{\partial \xi}$ (see figure 7.1). Since the centerline is assumed to be inextensible for this theory, $\frac{\partial \mathbf{r}}{\partial \xi} = \mathbf{e}_t$. To ensure that the cross-sections remain plane and retain their orientation relative to the centerline, we assume that $\mathbf{D}_1, \mathbf{D}_2$ and $\mathbf{D}_3$ define a right-handed orthonormal basis at each ξ:

$$[\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3] = 1. \tag{7.1}$$

If $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is a fixed right-handed basis, then we can define a rotation tensor $\mathbf{P}_0$:

$$\mathbf{P}_0 = \mathbf{D}_1 \otimes \mathbf{E}_1 + \mathbf{D}_2 \otimes \mathbf{E}_2 + \mathbf{D}_3 \otimes \mathbf{E}_3. \tag{7.2}$$

That is, $\mathbf{D}_i = \mathbf{P}_0 \mathbf{E}_i$. For many reference configurations, we can choose $\mathbf{D}_i$ such that $\mathbf{P}_0 = \mathbf{I}$; some exceptions are for rods with intrinsic curvature.

Under a motion of the directed curve, the vectors $\mathbf{d}_i$ retain their relative orientation and magnitude. These restrictions are equivalent to

$$\mathbf{d}_i \cdot \mathbf{d}_j = \delta_{ij}, \qquad \mathbf{d}_i \cdot \mathbf{d}_j \times \mathbf{d}_k = e_{ijk}, \tag{7.3}$$

where δ_{ij} is the Kronecker delta and e_{ijk} is the permutation symbol. Consequently,

$$\mathbf{d}_i = \mathbf{P}\mathbf{D}_i, \qquad (i = 1, 2, 3), \tag{7.4}$$

where $\mathbf{P}$ is a rotation tensor. Rotation tensors are by definition proper-orthogonal tensors and therefore have the properties that $\mathbf{P}\mathbf{P}^T = \mathbf{I}$ and $\det(\mathbf{P}) = 1$. This is a reduction of the more general case seen in Chapter 6. It also should be clear that

$$\mathbf{d}_\alpha = \mathbf{P}\mathbf{P_0}\mathbf{E}_\alpha, \qquad \frac{\partial \mathbf{r}}{\partial \xi} = \mathbf{P}\frac{\partial \mathbf{R}}{\partial \xi}, \qquad (\alpha = 1, 2). \tag{7.5}$$

These two equations define the constraints on the rod and are often known as "Kirchhoff's constraints." The most popular method to parameterize $\mathbf{P}$ in rod theory is to use Euler angles.

The constraints (7.5) imply that

$$\mathbf{d}_i^{'} = \boldsymbol{\kappa} \times \mathbf{d}_i. \tag{7.6}$$

Here, $\boldsymbol{\kappa} = \kappa_i \mathbf{d}_i$ is the axial vector of the skew-symmetric tensor:

$$\boldsymbol{\kappa} = \frac{1}{2}\epsilon[\boldsymbol{\Omega}], \quad \boldsymbol{\Omega} = \Omega_{ij}\mathbf{d}_i \otimes \mathbf{d}_j, \quad \Omega_{ij} = \mathbf{d}_i \cdot \mathbf{d}_j^{'}. \tag{7.7}$$

The strains in Kirchhoff's rod theory are defined as

$$\boldsymbol{\kappa} = \kappa_1\mathbf{d}_1 + \kappa_2\mathbf{d}_2 + \kappa_3\mathbf{d}_3, \tag{7.8}$$

where κ_i are the same as the Cartesian components of the strain, $\kappa_i = \boldsymbol{\kappa} \cdot \mathbf{E}_i$. We can write the strain energy as a function of the strains: $\psi = \psi(\kappa_1, \kappa_2, \kappa_3)$. Since the strains are determined by the vectors $\mathbf{d}_i$ and $\mathbf{d}_i^{'}$, the total energy can be written in the form

$$I = \int_{\xi_0}^{\xi_1} \left(\rho_0 \Psi\left(\mathbf{d}_i, \mathbf{d}_i^{'}\right) + U\left(\mathbf{r}, \mathbf{d}_i\right)\right) d\xi, \quad (i = 1, 2, 3). \tag{7.9}$$

We note that Ψ is invariant under superimposed rigid body motions of the rod (i.e., Ψ is properly invariant).

7.3 First Variation and Necessary Conditions for an Extremal

Following the developments in Chapter 6, we consider a functional of the form

$$I\left(y_1, y_2, \ldots, y_N\right) = \int_a^b f\left(u, y_1(u), \ldots, y_N(u), y_1^{'}(u), \ldots, y_N^{'}(u)\right) du, \tag{7.10}$$

where $y_1(x), \ldots, y_N(x)$ are scalar functions of the independent variable x.

We seek extremals $y_1^*(x), \ldots, y_N^*(x)$ of I which achieve prescribed values at $x = a$ and $x = b$ and which satisfy the k given equations:

$$G_j\left(y_1, \ldots, y_N, y_1^{'}, \ldots, y_N^{'}, x\right) = 0, \qquad (j = 1, 2, \ldots, k < N). \tag{7.11}$$

We consider the same variations to $y_1^*(x), \ldots, y_N^*(x)$ as in Chapter 6:

$$y_1(x,\epsilon) = y_1^*(x) + \epsilon\eta_1(x), \quad \ldots, \quad y_N(x,\epsilon) = y_N^*(x) + \epsilon\eta_N(x). \tag{7.12}$$

Substituting in these variations and evaluating $I'(0)$, we get

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_a^b \left\{ \frac{\partial f}{\partial y_1}\eta_1 + \frac{\partial f}{\partial y_1'}\eta_1' + \ldots + \frac{\partial f}{\partial y_N}\eta_N + \frac{\partial f}{\partial y_N'}\eta_N' \right\} du. \tag{7.13}$$

We can also differentiate the k equations (7.11) with respect to ϵ (and set $\epsilon \to 0$) as follows:

$$\frac{\partial G_j}{\partial y_1}\eta_1 + \frac{\partial G_j}{\partial y_1'}\eta_1' + \ldots + \frac{\partial G_j}{\partial y_N}\eta_N + \frac{\partial G_j}{\partial y_N'}\eta_N' = 0, \quad (j = 1, 2, \ldots, k). \tag{7.14}$$

These are the linearized constraints.

Now, following a treatment given in Weinstock [58], we multiply the jth equation of the system (7.14) by the unspecified function $\mu_j(x)$ (i.e., Lagrange multiplier), for all $j = 1, 2, \ldots, k$, and we add these terms to (7.13) to obtain

$$\begin{aligned} \delta I = \int_a^b \Bigg\{ & \left[\frac{\partial f}{\partial y_1} + \sum_{j=1}^k \mu_j \frac{\partial G_j}{\partial y_1} \right] \eta_1 + \left[\frac{\partial f}{\partial y_1'} + \sum_{j=1}^k \mu_j \frac{\partial G_j}{\partial y_1'} \right] \eta_1' + \ldots \\ & + \left[\frac{\partial f}{\partial y_N} + \sum_{j=1}^k \mu_j \frac{\partial G_j}{\partial y_N} \right] \eta_N + \left[\frac{\partial f}{\partial y_N'} + \sum_{j=1}^k \mu_j \frac{\partial G_j}{\partial y_N'} \right] \eta_N' \Bigg\} du \\ = \int_a^b & \left\{ \frac{\partial F}{\partial y_1}\eta_1 + \frac{\partial F}{\partial y_1'}\eta_1' + \ldots + \frac{\partial F}{\partial y_N}\eta_N + \frac{\partial F}{\partial y_N'}\eta_N' \right\} du = 0, \end{aligned} \tag{7.15}$$

where we define

$$F = f + \sum_{j=1}^k \mu_j(x) G_j. \tag{7.16}$$

Integrating by parts the second, fourth, …, $2N$th terms of (7.15), we get

$$\begin{aligned} \delta I = \int_a^b & \left\{ \left[\frac{\partial F}{\partial y_1} - \frac{d}{du}\left(\frac{\partial F}{\partial y_1'} \right) \right] \eta_1' + \ldots + \left[\frac{\partial F}{\partial y_N} - \frac{d}{du}\left(\frac{\partial F}{\partial y_N'} \right) \right] \eta_N' \right\} du \\ & + \left[\frac{\partial F}{\partial y_1'}\eta_1 \right]_a^b + \ldots + \left[\frac{\partial F}{\partial y_N'}\eta_N \right]_a^b = 0. \end{aligned} \tag{7.17}$$

Due to the set of k constraints (7.11), the variations $\eta_1, \eta_2, \ldots, \eta_N$ cannot be arbitrarily chosen, as they could in Chapter 6. To mitigate this, we assign the unspecified functions $\mu_j(x)$ to be any set of k functions which make vanish the coefficients of $\eta_1, \eta_2, \ldots, \eta_N$ in (7.17). That is, if $u_1, u_2, \ldots, u_k$ denote the first k functions of

$y_1, y_2, \ldots, y_N$, then the functions $\mu_j(x)$ are chosen so that the following equations are satisfied:

$$\frac{\partial F}{\partial u_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_i'}\right) = 0, \quad (i = 1, 2, \ldots, k). \tag{7.18}$$

With the functions $\mu_j(x)$ fixed in this manner, the first variation (7.17) becomes

$$\delta I = \int_a^b \left\{ \left[\frac{\partial F}{\partial u_{k+1}} - \frac{d}{du}\left(\frac{\partial F}{\partial u_{k+1}'}\right)\right]\eta_{k+1} + \ldots + \left[\frac{\partial F}{\partial u_N} - \frac{d}{du}\left(\frac{\partial F}{\partial u_N'}\right)\right]\eta_N \right\} du = 0, \tag{7.19}$$

where $u_{k+1}, \ldots, u_N = y_N$ denote the final $(N-k)$ functions of $y_1, y_2, \ldots, y_N$. Since the variations $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_N$ are arbitrary, we may follow the same approach as in §6.3. Therefore, we have

$$\frac{\partial F}{\partial u_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_i'}\right) = 0, \quad (i = k+1, k+2, \ldots, N). \tag{7.20}$$

Thus, by combining (7.18) and (7.20), we see that all N functions $y_j(x)$ are accounted for, and we obtain the N Euler-Lagrange equations

$$\frac{\partial F}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right) = 0, \quad \frac{\partial F}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'}\right) = 0, \quad \ldots, \quad \frac{\partial F}{\partial y_N} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_N'}\right) = 0. \tag{7.21}$$

7.3.1 First Variation Conditions for a Kirchhoff Rod

For Kirchhoff's rod theory, we present a formulation of the necessary conditions for stability as given by Steigmann & Faulkner [57]. First we consider a total energy functional of the form

$$\begin{aligned} I(\mathbf{r}, \mathbf{d}_i) &= \int_{\xi=0}^{\xi=L} \rho_0 \Psi\left(\mathbf{d}_i, \mathbf{d}_i'\right) d\xi - U(\mathbf{r}, \mathbf{d}_i) \\ &= \int_0^L \left[\rho_0 \Psi\left(\mathbf{d}_i, \mathbf{d}_i'\right) - \mathbf{b}\cdot\mathbf{r}\right] d\xi - \mathbf{f}\cdot\mathbf{r}(L), \end{aligned} \tag{7.22}$$

where U is a potential energy function, $\mathbf{f}$ is a terminal force applied at $\xi = L$, and $\mathbf{b}$ is a distributed force per unit length of the rod.

The variations for the directors and their derivatives are defined as follows:

$$\mathbf{r} = \mathbf{r}^* + \epsilon\mathbf{u}, \quad \mathbf{r}' = \mathbf{r}^{*'} + \epsilon\mathbf{u}', \tag{7.23}$$

$$\mathbf{d}_i = \mathbf{d}_i^* + \epsilon\mathbf{v}_i, \quad \mathbf{d}_i' = \mathbf{d}_i^{*'} + \epsilon\mathbf{v}_i', \quad (i = 1, 2, 3) \tag{7.24}$$

We admit only the functions $\mathbf{d}_i$ that comply with the constraints (7.3). This results in the restriction

$$\mathbf{d}_i \cdot \mathbf{v}_j + \mathbf{d}_j \cdot \mathbf{v}_i = 0 \tag{7.25}$$

on admissible variations $\mathbf{v}_i$. This implies that there is an axial vector of a skew-symmetric tensor $\boldsymbol{\alpha} = \alpha_{ij}\mathbf{d}_i \otimes \mathbf{d}_j$ such that

$$\mathbf{v}_i = \mathbf{a} \times \mathbf{d}_i. \tag{7.26}$$

Moreover, this result can be used with (7.23) to find that

$$\mathbf{u}^{'} = \mathbf{a} \times \mathbf{e}_t. \tag{7.27}$$

In addition, we add the vector of Lagrange multipliers $\mathbf{L}(\xi)$ to the functional (7.22), yielding

$$\hat{I}(\mathbf{r}, \mathbf{d}_i) = I(\mathbf{r}, \mathbf{d}_i) + \int_0^L \mathbf{L} \cdot \left(\mathbf{r}^{'} - \mathbf{e}_t\right) d\xi. \tag{7.28}$$

Substituting in the variations and evaluating $\hat{I}^{'}(0) = 0$, we get

$$\left.\frac{d\hat{I}}{d\epsilon}\right|_{\epsilon=0} = \sum_{i=1}^{3} \int_0^L \left\{\mathbf{v}_i \cdot \rho_0 \frac{\partial \Psi}{\partial \mathbf{d}_i} + \mathbf{v}_i^{'} \cdot \rho_0 \frac{\partial \Psi}{\partial \mathbf{d}_i^{'}} - \mathbf{b} \cdot \mathbf{u} + \mathbf{L} \cdot \left(\mathbf{u}^{'} - \mathbf{a} \times \mathbf{e}_t\right) - \mathbf{f} \cdot \mathbf{u}^{'}\right\} d\xi. \tag{7.29}$$

After integrating by parts, invoking (7.26) and using the constitutive relation for the contact moment

$$\mathbf{M}(\xi) = \sum_{i=1}^{3} \mathbf{d}_i \times \rho_0 \frac{\partial \Psi}{\partial \mathbf{d}_i^{'}}, \tag{7.30}$$

the first variation becomes

$$\begin{aligned} \left.\frac{d\hat{I}}{d\epsilon}\right|_{\epsilon=0} &= [\mathbf{a} \cdot \mathbf{M}]_0^L + [\mathbf{u} \cdot (\mathbf{L} - \mathbf{f})]_0^L \\ &\quad + \sum_{i=1}^{3} \int_0^L \mathbf{a} \cdot \left\{\mathbf{d}_i \times \frac{\partial \Psi}{\partial \mathbf{d}_i} - \mathbf{d}_i \times \left(\frac{\partial \Psi}{\partial \mathbf{d}_i^{'}}\right)^{'} + \mathbf{L} \times \mathbf{e}_t\right\} d\xi \\ &\quad - \int_0^L \mathbf{u} \cdot \left(\mathbf{L}^{'} + \mathbf{b}\right) d\xi. \end{aligned} \tag{7.31}$$

Therefore, we obtain the Euler-Lagrange equations

$$\begin{aligned} \mathbf{L} \times \mathbf{e}_t &= \sum_{i=1}^{3} \mathbf{d}_i \times \left(\frac{\partial \Psi}{\partial \mathbf{d}_i^{'}}\right)^{'} - \sum_{i=1}^{3} \mathbf{d}_i \times \frac{\partial \Psi}{\partial \mathbf{d}_i}, \\ \mathbf{L}^{'} + \mathbf{b} &= 0, \end{aligned} \tag{7.32}$$

and the natural boundary conditions

$$[\mathbf{a} \cdot \mathbf{M}]_0^L = 0, \quad [\mathbf{u} \cdot (\mathbf{L} - \mathbf{f})]_0^L = 0. \tag{7.33}$$

Note that $\mathbf{L} = \mathbf{n}$, so the contact force is completely prescribed by the Lagrange multipliers. For strain energy functions which are invariant under superimposed rigid body motions, $(7.32)_1$ becomes[1]

$$\mathbf{L} \times \mathbf{e}_t = \mathbf{M}'. \tag{7.34}$$

It should not come as a surprise that equations $(7.32)_2$ and (7.34) are the familiar balances of linear and director momenta for Kirchhoff's rod theory. While in vector form, equations $(7.32)_2$ and (7.34) also happen to be equivalent to the scalar form of the Euler-Lagrange equations (7.21).

7.4 Parametrization of the Rotation by Euler Angles

We now consider a method wherein the strains are parameterized using a set of Euler angles ψ, θ, ϕ. As is customary in the theory of motion for a rigid body, we can envision the origin of the director frame moving along the centerline of the rod with unit velocity. Therefore, the strains κ_1, κ_2, κ_3 are the components of the angular velocity resolved along the directions of the directors $\mathbf{d}_i$ (see Love [30, Section 253]). The advantage in this context is that the use of Euler angles will eliminate the need to consider the constraints on the director components in the variational formulation.

7.4.1 3-2-1 Euler Angles

There are many suitable choices for a combination of Euler angles. A popular choice, especially in aerospace and automotive applications, is the 3-2-1 set, which we will use here[2]. Thus, all possible deformations can easily be accounted for, where κ_1 and κ_2 are the flexural strains of the rod, and κ_3 is the torsion.

The Euler angles can be viewed as three successive rotations about a different axis. The overall rotation can be expressed (using Euler's representation) as

$$\mathbf{P} = \mathbf{H}(\phi, \mathbf{t}_1)\mathbf{H}(\theta, \mathbf{t}_2')\mathbf{H}(\psi, \mathbf{E}_3), \tag{7.35}$$

where each $\mathbf{H} = \mathbf{H}(\chi, \mathbf{h})$ defines a rotation χ about the unit vector $\mathbf{h}$, and $\mathbf{t}$, $\mathbf{t}'$ and

[1]Equation $(7.32)_1$ can be written as $\mathbf{L} \times \mathbf{e}_t = \mathbf{M}' - \sum_{i=1}^{3} \left(\mathbf{d}_i \times \partial\Psi/\partial\mathbf{d}_i + \mathbf{d}_i' \times \partial\Psi/\partial\mathbf{d}_i'\right)$, where the term in parentheses is identically zero (see Steigmann & Faulkner [57] for a derivation of this result).

[2]The 3-2-3 set of Euler angles is used in Love [30], but the difference is only a matter of personal preference.

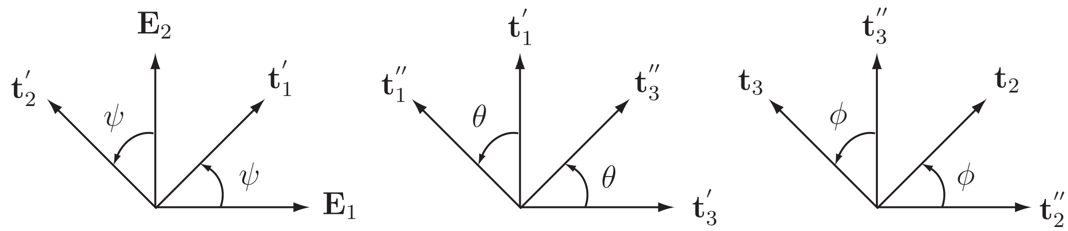

Figure 7.2 The transformations of various basis vectors for each individual rotation in a set of 3-2-1 Euler angles.

$\mathbf{t}''$ define separate sets of basis vectors (see figure 7.2). Here,

$$\mathbf{t}_i = \mathbf{H}(\phi, \mathbf{t}_1)\mathbf{t}_i'', \quad \mathbf{t}_i'' = \mathbf{H}(\theta, \mathbf{t}_2'' = \mathbf{t}_2')\mathbf{t}_i', \quad \mathbf{t}_i' = \mathbf{H}(\psi, \mathbf{t}_3' = \mathbf{E}_3)\mathbf{E}_i. \tag{7.36}$$

This formulation is restricted to the case where the rod has no intrinsic curvature. Therefore, the reference directors $\mathbf{D}_i = \mathbf{E}_i$; in other words, the tensor $\mathbf{P}_0$ described by (7.2) is equal to the identity tensor $\mathbf{I}$.

As shown in O'Reilly [41], the basis vectors can be expressed as linear combinations of each other:

$$\begin{bmatrix}\mathbf{t}_1'\\ \mathbf{t}_2'\\ \mathbf{t}_3'\end{bmatrix} = \begin{bmatrix}\cos(\psi) & \sin(\psi) & 0\\ -\sin(\psi) & \cos(\psi) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix},$$
$$\begin{bmatrix}\mathbf{t}_1''\\ \mathbf{t}_2''\\ \mathbf{t}_3''\end{bmatrix} = \begin{bmatrix}\cos(\theta) & 0 & -\sin(\theta)\\ 0 & 1 & 0\\ \sin(\theta) & 0 & \cos(\theta)\end{bmatrix}\begin{bmatrix}\mathbf{t}_1'\\ \mathbf{t}_2'\\ \mathbf{t}_3'\end{bmatrix},$$
$$\begin{bmatrix}\mathbf{t}_1\\ \mathbf{t}_2\\ \mathbf{t}_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\phi) & \sin(\phi)\\ 0 & -\sin(\phi) & \cos(\phi)\end{bmatrix}\begin{bmatrix}\mathbf{t}_1''\\ \mathbf{t}_2''\\ \mathbf{t}_3''\end{bmatrix}. \tag{7.37}$$

The relationships (7.37) can be combined to express $\mathbf{t}_i$ (i.e., the directors $\mathbf{d}_i$) in terms of the fixed Cartesian basis vectors $\mathbf{E}_i$:

$$\begin{bmatrix}\mathbf{d}_1\\ \mathbf{d}_2\\ \mathbf{d}_3\end{bmatrix} = \begin{bmatrix}\cos(\psi)\cos(\theta) & \sin(\psi)\cos(\theta) & -\sin(\theta)\\ -\sin(\psi)\cos(\phi) + \cos(\psi)\sin(\theta)\sin(\phi) & \cos(\psi)\cos(\phi) + \sin(\psi)\sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi)\\ \sin(\psi)\sin(\phi) + \cos(\psi)\sin(\theta)\cos(\phi) & -\cos(\psi)\sin(\phi) + \sin(\psi)\sin(\theta)\cos(\phi) & \cos(\theta)\cos(\phi)\end{bmatrix}\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix}. \tag{7.38}$$

A concise representation of the vectors $\{\mathbf{g}_i\}$ about which the Euler angles rotate is known as the Euler basis, and it can be expressed in terms of the Cartesian basis vectors as

$$\begin{bmatrix}\mathbf{g}_1\\ \mathbf{g}_2\\ \mathbf{g}_3\end{bmatrix} = \begin{bmatrix}\mathbf{E}_3\\ \mathbf{t}_2^{'}\\ \mathbf{d}_1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ -\sin(\psi) & \cos(\psi) & 0\\ \cos(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) & -\sin(\theta)\end{bmatrix}\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix}. \tag{7.39}$$

Singularities

A problem inherent in using Euler angles is the singularity which occurs for some values of the second angle, θ. For the 3-2-1 set, this singularity occurs when $\theta = \pm\frac{\pi}{2}$. One of the easiest ways to show this fact is to look at how the Euler basis is affected at these values of θ. When $\theta = \pm\frac{\pi}{2}$, $\mathbf{g}_1 = \mathbf{E}_3 = \pm\mathbf{g}_3$, and the Euler basis fails to span $\mathbb{E}^3$. To avoid this singularity, it is necessary to restrict the second Euler angle such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The other two angles, ψ and ϕ, are free to range from 0 to 2π.

7.4.2 Definition of Strains

From equation (7.8) and the angular velocity vector associated with the 3-2-1 Euler angles, the strains have several representations:

$$\begin{aligned}\boldsymbol{\kappa} &= \kappa_1\mathbf{d}_1 + \kappa_2\mathbf{d}_2 + \kappa_3\mathbf{d}_3\\ &= \psi^{'}\mathbf{E}_3 + \theta^{'}\mathbf{t}_2^{'} + \phi^{'}\mathbf{d}_1\\ &= \left(-\psi^{'}\sin(\theta) + \phi^{'}\right)\mathbf{d}_1 + \left(\psi^{'}\sin(\phi)\cos(\theta) + \theta^{'}\cos(\phi)\right)\mathbf{d}_2\\ &\quad + \left(\psi^{'}\cos(\phi)\cos(\theta) - \theta^{'}\sin(\phi)\right)\mathbf{d}_3.\end{aligned} \tag{7.40}$$

Therefore, the strains can be described entirely in terms of the Euler angles ψ, θ and ϕ. The Euler angles can vary arbitrarily, thereby automatically satisfying the constraints on the director components and mitigating any difficulty in formulating the second variation of the energy functional in component form. Alternatively, the strain components can be obtained from $(7.7)_1$ and the derivative of the constraints on the directors $(7.3)_1$:

$$\kappa_1 = \mathbf{d}_2^{'}\cdot\mathbf{d}_3, \quad \kappa_2 = \mathbf{d}_3^{'}\cdot\mathbf{d}_1, \quad \kappa_3 = \mathbf{d}_1^{'}\cdot\mathbf{d}_2. \tag{7.41}$$

7.5 Second Variation and Stability Criteria for Extremals

An analysis of the second variation for a functional of the form (7.10) exactly parallels the analysis described in §6.4, and so for conciseness we only reproduce the main results here. If a solution $\mathsf{S}(x)$ to the matrix Riccati equation

$$\mathsf{S}' + \mathsf{P} - (\mathsf{Q} + \mathsf{S})^T \, \mathsf{R}^{-1} \, (\mathsf{Q} + \mathsf{S}) = \mathbf{0}, \tag{7.42}$$

subject to the boundary conditions on S

$$\boldsymbol{\eta}(b) \cdot \mathsf{S}(b)\boldsymbol{\eta}(b) - \boldsymbol{\eta}(a) \cdot \mathsf{S}(a)\boldsymbol{\eta}(a) \leq 0 \tag{7.43}$$

can be found, then the resulting simplified expression for $\delta^2 I$ is non-negative:

$$\delta^2 I = \int_a^b \left\| \mathsf{R}\boldsymbol{\eta}' + (\mathsf{Q} + \mathsf{S})\, \boldsymbol{\eta} \right\|^2_{\mathsf{R}^{-1}} du, \tag{7.44}$$

where

$$\|\mathsf{a}\|^2_{\mathsf{R}^{-1}} = \mathsf{a} \cdot \mathsf{R}^{-1}\mathsf{a} \quad \text{and} \quad \mathsf{R}^{-1} > 0.$$

7.5.1 Second Variation Conditions for a Kirchhoff Rod

To use the results given in §6.4 and 7.5, the strain energy for a Kirchhoff rod must be written in the form $\Psi = \Psi(\kappa_1, \kappa_2, \kappa_3, \xi)$. As shown in section 7.4 the strains can be expressed in terms of the Euler angles and their derivatives. For the functionals considered here, the potential energy must also be restricted to functions which depend solely on the Euler angles and their derivatives[3]. Therefore, the total energy is assumed to have the form

$$I = \int_0^L \left[\rho_0 \Psi \left(\psi, \theta, \phi, \psi', \theta', \phi', \xi \right) + U \left(\psi, \theta, \phi, \psi', \theta', \phi', \xi \right) \right] d\xi. \tag{7.45}$$

If we let the Euler angles ψ, θ, ϕ fill out a vector d, there would be 3 total independent components. Therefore, we define d with variations v such that[4]

$$\mathsf{d} = \begin{bmatrix} \psi \\ \theta \\ \phi \end{bmatrix}, \quad \mathsf{d}' = \begin{bmatrix} \psi' \\ \theta' \\ \phi' \end{bmatrix}, \quad \mathsf{v} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \tag{7.46}$$

and the second variation can be expressed as (6.22), where

$$\mathsf{P} = \mathsf{P}(\xi) = \frac{\partial^2 f}{\partial \mathsf{d} \partial \mathsf{d}} \left(\xi, \mathsf{d}^*, \mathsf{d}^{*'} \right), \quad \mathsf{Q} = \mathsf{Q}(\xi) = \frac{\partial^2 f}{\partial \mathsf{d} \partial \mathsf{d}'} \left(\xi, \mathsf{d}^*, \mathsf{d}^{*'} \right),$$

$$\mathsf{R} = \mathsf{R}(\xi) = \frac{\partial^2 f}{\partial \mathsf{d}' \partial \mathsf{d}'} \left(\xi, \mathsf{d}^*, \mathsf{d}^{*'} \right). \tag{7.47}$$

[3] This is clearly a restriction on the potential applications of this theory. However, we have found it difficult to relax these assumptions and obtain a stability criterion.

[4] The variations of the Euler angles ψ, θ, ϕ are prescribed by η_1, η_2, η_3, respectively.

Note that the functional for Kirchhoff's theory is $f = \rho_0 \Psi + U$. Therefore, if the 3×3 matrix $\mathsf{S}(\xi)$ satisfying (7.42) and subject to the boundary conditions

$$\mathsf{v}(\xi_1) \cdot \mathsf{S}(\xi_1)\mathsf{v}(\xi_1) - \mathsf{v}(\xi_0) \cdot \mathsf{S}(\xi_0)\mathsf{v}(\xi_0) \leq \mathbf{0}. \tag{7.48}$$

has a finite solution, then the equilibrium solutions satisfy a necessary condition for stability.

7.6 Extension to Branched Rods

Parallelling the developments in Chapters 2 and 3, the second variation stability conditions can easily be extended to accommodate tree-like structures of branched Kirchhoff rods. It is clear that equation (7.48) allows for the establishment of a branching condition similar to $(2.49)_3$ and (6.33):

$$[\![\mathsf{S}]\!]_B = 0. \tag{7.49}$$

This concludes our presentation on Kirchhoff rods.

Chapter 8

Closing Comments and Future Work

The work presented in this dissertation provides a concise methodology for determining when $\delta^2 I \geq 0$ for a functional I defined on a simple elastic rod structure. Most importantly, this work provides the first such criteria for a branched tree-like structure. One of these criteria is that a bounded set of Riccati solutions must exist for a tree-like structure. The appealing feature is that none of the individual branches can be unstable, otherwise the entire tree-like structure becomes unstable.

The final two chapters seek to address the problem of extending the stability criteria to more complex rod theories accommodating deformations in all three dimensions. Further work remains on extending the stability criteria to accommodate tree-like structures using Kirchhoff's rod theory, in particular. One notable application would be to use these conditions in conjunction with the algorithms developed by Prusinkiewicz and his coworkers [9, 14, 13, 24] to compute the equilibrium configurations of branched tree-like structures. With a suitable stability criterion in place, physically unrealizable configurations of the tree models can be systematically eliminated. Other applications include problems involving the adhesion of elastic rods to surfaces and contributing to evolving mechanical theories of plant growth.

Finally, an important component of the stability criteria that is lacking are conditions proving sufficiency for stability in the case of branched structures. This problem will need to be addressed for models using the elastica, Green-Naghdi and Kirchhoff theories.

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Appendix A

An Optimal Control Formulation

A.1 Introduction

An alternative variational formulation for the equilibrium configurations of the elastic strut can be found when the problem of determining these configurations is formulated as an optimal control problem. In contrast to the related control-based formulation of Bryson and Ho [6, page 191] and Sachkov [52, 53], the resulting formulation results in a singular control problem. However, with the significant assistance of works by Bell and Jacobson [2], we are able to show the equivalence of this formulation to the one presented in Chapter 2 §2.2.2. Specifically, this formulation corresponds to the example of a thin strut under terminal load presented in Chapter 4 §4.2. Furthermore, one of the necessary conditions for optimality is identical to the Riccati equation (4.9) associated with the condition L1 discussed in Chapter 2.

A.2 Singular Optimal Control Theory

Proceeding with the formulation, we seek to compute the control u which extremizes the cost functional

$$J = F[\mathsf{x}(s_0)] + G[\mathsf{x}(s_f)] + \int_{s_0}^{s_f} L(\mathsf{x}, u, s) ds \tag{A.1}$$

subject to the following conditions:

$$\mathsf{x}' = f(\mathsf{x}, u, s), \qquad \phi[\mathsf{x}(s_0)] = 0, \qquad \psi[\mathsf{x}(s_f)] = 0. \tag{A.2}$$

The state vector x is n-dimensional, u is a scalar control variable, ϕ and ψ are smooth scalar functions of the initial and final states $\mathsf{x}(s_0)$ and $\mathsf{x}(s_f)$, respectively, and F and G are scalar-valued functions representing the initial and final costs. The initial constraint ϕ is not typical in optimal control formulations but is necessary in the problems of interest here because the components of $\mathsf{x}(s_0)$ are generally not all specified.

We can adjoin equations (A.2) to the cost functional with Lagrange multipliers $\lambda(s)$, ν_0, and ν_f. This yields the augmented cost functional $\bar{J}$:

$$\bar{J} = F[\mathsf{x}(s_0)] + \nu_0\phi[\mathsf{x}(s_0)] + G[\mathsf{x}(s_f)] + \nu_f\psi[\mathsf{x}(s_f)] + \int_{s_0}^{s_f} [H(\mathsf{x}, u, \lambda, s) - \lambda(s)\cdot\mathsf{x}']\,ds, \tag{A.3}$$

where the Hamiltonian

$$H = L + \lambda \cdot f. \tag{A.4}$$

In the elastic strut problem, the two state variables are θ and θ':

$$x_1 = \theta, \qquad x_2 = \theta'. \tag{A.5}$$

The dimensionless arc-length s becomes the independent variable such that $s_0 = 0$ and $s_f = 1$. Therefore, the problem is to minimize

$$J = \int_0^1 \left(\frac{1}{2}x_2^2 + \beta\cos(x_1)\right) ds, \tag{A.6}$$

subject to

$$x_1' = x_2, \qquad x_2' = u, \tag{A.7}$$

and appropriate boundary conditions (4.11) on $\mathsf{x}(s)$. Note that the terminal costs F and G are generally identically zero for the elastica problems of interest here. The Hamiltonian for the problem is defined as

$$H(\mathsf{x}, u, \lambda, s) = \frac{1}{2}x_2^2 + \beta\cos(x_1) + \lambda_1 x_2 + \lambda_2 u. \tag{A.8}$$

Comparing (4.7) to (A.7), it is important to note that we have chosen to specify the vector field f as $\mathsf{f} = \begin{bmatrix} x_2 & u \end{bmatrix}^T$ rather than $\mathsf{f} = \begin{bmatrix} x_2 & -\beta\sin(x_1) \end{bmatrix}^T$. It will shortly become apparent that the control variable u corresponds to the (dimensionless) moment of the applied force P.

A.3 First Variation

We now follow a standard procedure (see, e.g., [6]) and consider the first variation $\delta\bar{J}$ of $\bar{J}$ with changes in the control u for fixed endpoints s_0 and s_f. The change in u

causes changes in x and consequently

$$\delta \bar{J} = \left[\left(F_{\mathsf{x}} + \nu_0 \phi_{\mathsf{x}} + \lambda \right) \delta \mathsf{x} \right]_{s=s_0} + \left[\left(G_{\mathsf{x}} + \nu_f \psi_{\mathsf{x}} - \lambda \right) \delta \mathsf{x} \right]_{s=s_f} + \int_{s_0}^{s_f} \left[\left(H_{\mathsf{x}} + \lambda' \right) \delta \mathsf{x} + H_u \delta u \right] ds. \tag{A.9}$$

The subscript x and u on a function denote partial derivatives of the function with respect to these variables. By requiring that $\delta \bar{J}$ vanish for arbitrary δu, we obtain necessary conditions for optimality:

$$\begin{gathered} \lambda' = -H_{\mathsf{x}}, \qquad H_u = 0, \\ \lambda(s_0) = -\left(F_{\mathsf{x}} + \nu_0 \phi_{\mathsf{x}} \right)_{s=s_0}, \qquad \lambda(s_f) = \left(G_{\mathsf{x}} + \nu_f \psi_{\mathsf{x}} \right)_{s=s_f}. \end{gathered} \tag{A.10}$$

These equations are supplemented by the state equations and boundary conditions (A.2):

$$\mathsf{x}' = H_\lambda, \qquad \phi[\mathsf{x}(s_0)] = 0, \qquad \psi[\mathsf{x}(s_f)] = 0. \tag{A.11}$$

The identity $H_u = 0$ generally serves to define the optimal control u^*, while the differential equations $\mathsf{x}' = H_\lambda$ and $\lambda' = -H_{\mathsf{x}}$ are used to compute the optimal solutions $\lambda^*(s)$ and $\mathsf{x}^*(s)$. In the sequel, we avoid ornamenting the optimal solution with an asterix unless it is necessary to avoid confusion. The boundary conditions on $\lambda(s)$ and the boundary conditions on $\mathsf{x}(s)$ (which are presented in $(\text{A.10})_{3,4}$ and $(\text{A.11})_{2,3}$) total $2n$ in number and are intended to make the differential equations $\lambda' = -H_{\mathsf{x}}$ and $\mathsf{x}' = H_\lambda$ well-posed.[1]

It is important to note that the Hamiltonian defined in equation (A.8) is linear in the control u. This implies that the optimal control problem is singular, meaning that the standard optimal control formulation where Pontryagin's Minimum Principle is invoked to determine the optimal control u^* is of no assistance in finding the optimal control u^*. For this reason, starting in the 1960s new necessary conditions for singular optimal control problems were established by a number of researchers (see [2, 6, 22, 38] and references therein).

The first of the aforementioned necessary conditions is known as the generalized Legendre-Clebsch condition and is defined as

$$(-1)^q \frac{\partial}{\partial u} \left[\left(\frac{d^{2q}}{ds^{2q}} \right) H_u \right] \geq 0, \tag{A.12}$$

where $2q$ is the lowest-order derivative of H_u in which u appears explicitly. The second

[1] For further details on the role of the transversality conditions in prescribing boundary conditions, the reader is referred to [2, 28, 47].

necessary condition is known as the Jacobson condition [22], and it requires that

$$\begin{aligned} \mathsf{f}_u \cdot \left[F_{\mathsf{xx}} + (\nu_0 \phi_{\mathsf{x}})_{\mathsf{x}}\right] + H_{u\mathsf{x}}\big|_{(\mathsf{x}=\mathsf{x}^*(s_0),u=u^*(s_0))} &= 0, \\ \mathsf{f}_u \cdot \left[G_{\mathsf{xx}} + (\nu_f \psi_{\mathsf{x}})_{\mathsf{x}}\right] + H_{u\mathsf{x}}\big|_{\left(\mathsf{x}=\mathsf{x}^*\left(s_f\right),u=u^*\left(s_f\right)\right)} &= 0. \end{aligned} \tag{A.13}$$

Finally, to determine the optimal control in singular problems, a series of partial derivatives of H are computed until u^* can be prescribed:[2]

$$H_u = 0, \qquad H_u' = 0, \qquad H_u'' = 0, \ldots \tag{A.14}$$

A.3.1 Application to the Elastica

Applying the necessary conditions (A.10) to the elastic strut problem, we obtain the state equations (A.7) as well as

$$\lambda_1' = -\frac{\partial H}{\partial x_1} = \beta \sin(x_1), \qquad \lambda_2' = -\frac{\partial H}{\partial x_2} = -x_2 - \lambda_1. \tag{A.15}$$

The state and costate equations, (A.7) and (A.15), require four boundary conditions in order to obtain a solution to the optimal control problem. There are two boundary conditions given on the states, which must be supplemented by two transversality conditions on the costates.

For our elastic strut problem, $q = 1$ and upon examining the Legendre-Clebsch condition we find that

$$\begin{aligned} H_u = \lambda_2, \qquad H_u' = \lambda_2' = -x_2 - \lambda_1, \qquad H_u'' = -x_2' - \lambda_1' = -u - \beta\sin(x_1), \\ \frac{\partial}{\partial u}\left(H_u''\right) = \frac{\partial}{\partial u}\left(-\beta\sin(x_1) - u\right) = -1. \end{aligned} \tag{A.16}$$

Hence, the condition is satisfied for all solutions, regardless of the boundary conditions used. Secondly, as $F_{\mathsf{xx}} = G_{\mathsf{xx}} = (\nu_0\phi_{\mathsf{x}})_{\mathsf{x}} = (\nu_f\psi_{\mathsf{x}})_{\mathsf{x}} = H_{u\mathsf{x}} = 0$, the Jacobson condition is trivially satisfied. Finally, we can determine that the optimal control $u = u^*$ from (A.14) with the help of the intermediate results $(\text{A.16})_{1,2,3}$:

$$u^* = -\beta \sin(x_1). \tag{A.17}$$

This prescription for the optimal control is valid regardless of the boundary conditions imposed on the strut.

In summary, the necessary conditions for an extremal $(\mathsf{x}^*, \lambda^*, u^* = -\beta\sin(x_1))$ of J are the satisfaction of

$$\lambda_1' = \beta\sin(x_1), \qquad \lambda_2' = -x_2 - \lambda_1, \qquad x_1' = x_2, \qquad x_2' = -\beta\sin(x_1), \tag{A.18}$$

[2] For other examples featuring these conditions, the interested reader is referred to [6, 25].

subject to the boundary and transversality conditions

$$\phi\left[x_1\left(s_0\right), x_2\left(s_0\right)\right]=0, \qquad \psi\left[x_1\left(s_f\right), x_2\left(s_f\right)\right]=0, \tag{A.19}$$

and

$$\lambda\left(s_0\right)=\nu_0 \left.\phi_{\mathsf{x}}\right|_{\mathsf{x}=\mathsf{x}^*\left(s_0\right)}, \qquad \lambda\left(s_f\right)=\nu_f \left.\psi_{\mathsf{x}}\right|_{\mathsf{x}=\mathsf{x}^*\left(s_f\right)}, \tag{A.20}$$

respectively. Note that (A.19) and (A.20) constitute six equations. Two of these equations are used to determine ν_0 and ν_f and the other four provide boundary conditions for (A.18).

A.4 The Second Variation

Prior to computing the second variation, it is convenient to define notations for the first derivatives of f and second derivatives of L evaluated on the extremal:

$$\mathsf{A}=\mathsf{f}_{\mathsf{x}}\left(\mathsf{x}^*, u^*\right), \qquad \mathsf{B}=\mathsf{f}_u\left(\mathsf{x}^*, u^*\right), \tag{A.21}$$

$$\mathsf{Q}=L_{\mathsf{xx}}\left(\mathsf{x}^*, u^*\right), \qquad \mathsf{C}^T=L_{u\mathsf{x}}\left(\mathsf{x}^*, u^*\right), \qquad \mathsf{R}=L_{uu}\left(\mathsf{x}^*, u^*\right)=\mathsf{0}. \tag{A.22}$$

In addition, the derivatives of the initial and terminal costs and the transversality conditions will play a key role in the sequel:

$$\begin{aligned}
\mathsf{Q}_0 &= \left.\left(F+\nu_0\phi\right)_{\mathsf{xx}}\right|_{\left(\mathsf{x}=\mathsf{x}^*\left(s_0\right), u=u^*\left(s_0\right)\right)}, \\
\mathsf{Q}_f &= \left.\left(G+\nu_f\psi\right)_{\mathsf{xx}}\right|_{\left(\mathsf{x}=\mathsf{x}^*\left(s_f\right), u=u^*\left(s_f\right)\right)}.
\end{aligned} \tag{A.23}$$

The second variation of the amended functional $\bar{J}$ can be expressed as

$$\begin{aligned}
\delta^2\bar{J} &= \frac{1}{2}\int_{s_0}^{s_f} \delta\mathsf{x}\cdot\left(\mathsf{Q}\delta\mathsf{x}\right)+2\delta u\cdot\left(\mathsf{C}\delta\mathsf{x}\right) ds \\
&\quad +\frac{1}{2}\delta\mathsf{x}\left(s_0\right)\cdot\left(\mathsf{Q}_0\delta\mathsf{x}\left(s_0\right)\right)+\frac{1}{2}\delta\mathsf{x}\left(s_f\right)\cdot\left(\mathsf{Q}_f\delta\mathsf{x}\left(s_f\right)\right),
\end{aligned} \tag{A.24}$$

where the variations $\delta\mathsf{x}$, δu, and $\delta\lambda$ satisfy the differential equations and boundary conditions

$$\begin{aligned}
\delta\mathsf{x}' &= \mathsf{A}\delta\mathsf{x}+\mathsf{B}\delta u, \\
\delta\lambda' &= -\mathsf{Q}\delta\mathsf{x}-\mathsf{C}^T\delta u+\mathsf{A}^T\delta\lambda, \\
\left.\phi_{\mathsf{x}}\delta\mathsf{x}\left(s_0\right)\right|_{\left(\mathsf{x}=\mathsf{x}^*\left(s_0\right), u=u^*\left(s_0\right)\right)} &= 0, \\
\left.\psi_{\mathsf{x}}\delta\mathsf{x}\left(s_f\right)\right|_{\left(\mathsf{x}=\mathsf{x}^*\left(s_f\right), u=u^*\left(s_f\right)\right)} &= 0.
\end{aligned} \tag{A.25}$$

As a final preliminary, we define the linearized Hamiltonian h:

$$h = \frac{1}{2}\delta\mathsf{x}\cdot(\mathsf{Q}\delta\mathsf{x}) + \delta u\cdot(\mathsf{C}\delta\mathsf{x}) + \delta\lambda\cdot(\mathsf{A}\delta\mathsf{x}+\mathsf{B}\delta u)\,. \tag{A.26}$$

Following the classical treatment by Legendre [4], it is standard procedure to add an identity to $\delta^2\bar{J}$:

$$\begin{aligned} 0 = & \frac{1}{2}\int_{s_0}^{s_f}\frac{d}{ds}\left(\delta\mathsf{x}\cdot(\mathsf{S}(s)\delta\mathsf{x})\right)ds + \frac{1}{2}\delta\mathsf{x}\left(s_0\right)\cdot\left(\mathsf{S}\left(s_0\right)\delta\mathsf{x}\left(s_0\right)\right) \\ & -\frac{1}{2}\delta\mathsf{x}\left(s_f\right)\cdot\left(\mathsf{S}\left(s_f\right)\delta\mathsf{x}\left(s_f\right)\right). \end{aligned} \tag{A.27}$$

The matrix $\mathsf{S}=\mathsf{S}^T$ is assumed to be a differentiable function of s. Evaluating $\delta\mathsf{x}'$ using $(\text{A.25})_{1,2}$, and combining terms, we find that $\delta^2\bar{J}$ transforms to

$$\begin{aligned} \delta^2\bar{J} = & \frac{1}{2}\int_{s_0}^{s_f}\delta\mathsf{x}\cdot\left(\left(\mathsf{S}'+\mathsf{Q}+\mathsf{SA}+\mathsf{A}^T\mathsf{S}\right)\delta\mathsf{x}\right) + 2\delta u\cdot\left(\left(\mathsf{C}+\mathsf{B}^T\mathsf{S}\right)\delta\mathsf{x}\right)ds \\ & +\frac{1}{2}\delta\mathsf{x}\left(s_0\right)\cdot\left(\left(\mathsf{Q}_0+\mathsf{S}\left(s_0\right)\right)\delta\mathsf{x}\left(s_0\right)\right) \\ & +\frac{1}{2}\delta\mathsf{x}\left(s_f\right)\cdot\left(\left(\mathsf{Q}_f-\mathsf{S}\left(s_f\right)\right)\delta\mathsf{x}\left(s_f\right)\right). \end{aligned} \tag{A.28}$$

It follows that a sufficient condition for positive semi-definiteness (or non-negativity) of $\delta^2\bar{J}$ is[3]

$$\begin{aligned} \mathsf{S}'+\mathsf{Q}+\mathsf{SA}+\mathsf{A}^T\mathsf{S} &\geq 0, \\ \mathsf{C}+\mathsf{B}^T\mathsf{S} &= 0, \\ \delta\mathsf{x}\left(s_0\right)\cdot\left(\left(\mathsf{Q}_0+\mathsf{S}\left(s_0\right)\right)\delta\mathsf{x}\left(s_0\right)\right)+\delta\mathsf{x}\left(s_f\right)\cdot\left(\left(\mathsf{Q}_f-\mathsf{S}\left(s_f\right)\right)\delta\mathsf{x}\left(s_f\right)\right) &\geq 0. \end{aligned} \tag{A.29}$$

We note that the variations $\delta\mathsf{x}\left(s_{0,f}\right)$ in $(\text{A.29})_3$ also need to satisfy the linearized transversality conditions $(\text{A.25})_{3,4}$. This significantly reduces the number of restrictions on the matrices $\mathsf{Q}_0+\mathsf{S}\left(s_0\right)$ and $\mathsf{Q}_f-\mathsf{S}\left(s_f\right)$.

It has been shown by Bell and Jacobson [2] that a matrix S which satisfies $(\text{A.29})_1$ solves the Riccati differential equation[4]

$$\mathsf{S}'+\mathsf{Q}+\mathsf{SA}+\mathsf{A}^T\mathsf{S}+\mathsf{M}^T\left(\frac{\partial h''_u}{\partial u}\right)^{-1}\mathsf{M}=0, \tag{A.30}$$

[3] We use the standard notation to show that a matrix P is positive semi-definite by the expression $\mathsf{P}\geq 0$.

[4] An alternative prescription for a Riccati equation of this type was presented in McDanell and Powers [38]. For the elastica problem of interest in the present paper, the Riccati equations proposed in [2] and [38] are identical.

where

$$\begin{aligned}
\frac{\partial h_u''}{\partial \delta u} &= \frac{\partial}{\partial \delta u}\left(\frac{\partial^3 h}{\partial s \partial s \partial \delta u}\right) \\
&= \mathsf{CAB} + \mathsf{B}^T \mathsf{A}^T \mathsf{C}^T + \mathsf{C}'\mathsf{B} - \left(\mathsf{B}'\right)^T \mathsf{C}^T - \mathsf{B}^T \mathsf{QB}, \\
\mathsf{M} &= \left(\mathsf{AB} - \mathsf{B}'\right)^T \mathsf{S} + \mathsf{B}^T \mathsf{Q} - \mathsf{CA} - \mathsf{C}'.
\end{aligned} \tag{A.31}$$

The boundary conditions on S in the Riccati differential equation (A.30) are obtained by examining $(\text{A.29})_{2,3}$. To this end, we first note that Bell and Jacobson [2] also showed that for $(\text{A.29})_2$ to be satisfied, it is necessary and sufficient that

$$\mathsf{C}\left(s_f\right) + \mathsf{B}^T\left(s_f\right)\mathsf{S}\left(s_f\right) = 0. \tag{A.32}$$

We will use $(\text{A.29})_3$ and (A.32) to determine the boundary conditions for solutions to (A.30).

A.4.1 Application to the Elastica

For the problem of interest, many of the matrices featuring in the second variation are simple:

$$\mathsf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathsf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathsf{C} = 0, \qquad \mathsf{Q} = \begin{bmatrix} -\beta\cos\left(x_1\right) & 0 \\ 0 & 1 \end{bmatrix}. \tag{A.33}$$

In addition,

$$\frac{\partial h_u''}{\partial \delta u} = -\mathsf{B}^T\mathsf{QB} = -1, \qquad \mathsf{M} = \begin{bmatrix} S_{11} & S_{12}+1 \end{bmatrix}. \tag{A.34}$$

For all the problems of interest, (A.32) implies that

$$S_{12}\left(s_f\right) = 0, \qquad S_{22}\left(s_f\right) = 0. \tag{A.35}$$

The remaining boundary value is obtained from the condition $(\text{A.29})_3$. These conditions depend on the boundary conditions for the elastic strut, and we defer imposing them until later.

With the help of (A.33) and (A.34), the Riccati differential equations can be computed from (A.30):

$$S_{11}' = S_{11}^2 + \beta\cos(x_1), \qquad S_{12}' = S_{11}S_{12}, \qquad S_{22}' = S_{12}^2, \tag{A.36}$$

Subject to (A.35), these equations have the solution

$$S_{11}(s) = r(s, c, \beta), \qquad S_{12}(s) = 0, \qquad S_{22}(s) = 0, \tag{A.37}$$

where c is a constant. Remarkably then, the issue of the minimization of the second variation boils down to a single Riccati equation (A.36)$_1$, which is identical to (4.9) that we found earlier in Chapter 4 §4.2 using a different variational principle. Thus, if we can find a bounded solution $S_{11}(s)$ to (A.36)$_1$, then we will have established sufficient conditions for the energy I (which is equivalent to E) to be minimized and necessary conditions for J to be minimized.

In the special case where $\cos(x_1) = 0$, the function $r(s, c, \beta)$ can be easily computed:

$$r(\tau, c, \beta) = \sqrt{\beta} \tan\left(\sqrt{\beta}\,(\tau - c)\right). \tag{A.38}$$

This solution is shown in Figure A.1 for two cases. Of particular importance in the results shown in this figure are the unboundedness of r when β has certain critical values:

$$\lim_{x\to\pm 0.5} r\left(x, 0.5, \beta = \pi^2\right) = \pm\infty, \qquad \lim_{x\to 0} r\left(x, 1.0, \beta = \frac{\pi^2}{4}\right) = -\infty. \tag{A.39}$$

The solutions shown in Figure A.1(a) feature in the stability analysis of the straight fixed-fixed strut, while those shown in Figure A.1(b) feature in the stability analysis of the straight fixed-free strut.

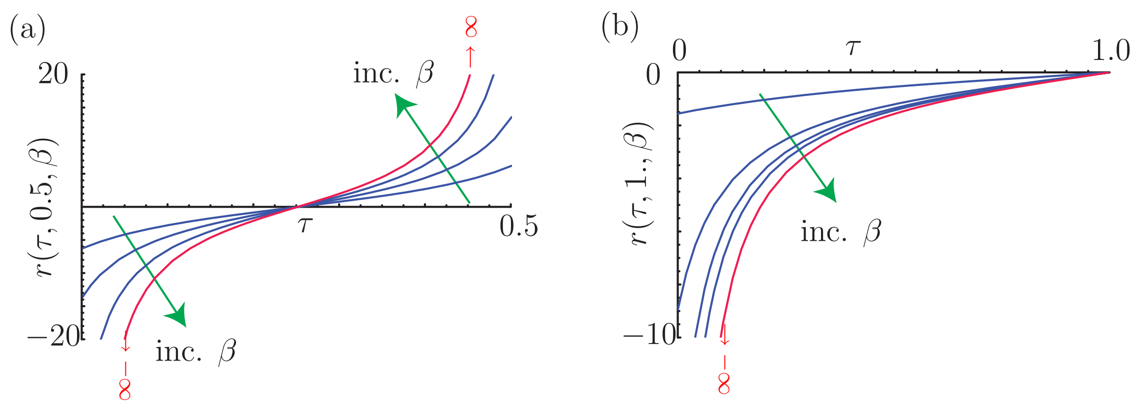

Figure A.1 Solutions (A.38) to the Riccati equation (A.37) when $\cos(x_1) = 1.0$ for (a) the case where $c = 0.5$ and $\tau \in [-0.5, 0.5]$, and (b) $c = 1.0$ and $\tau \in [0, 1.0]$. In (a), $\beta = 4, 6, 8$, and π^2. In (b), $\beta = 1, 2, 2.2, 2.3$, and $\frac{\pi^2}{4}$.