Lawrence Berkeley National Laboratory
Recent Work

Title
COMPUTATION OF INVARIANT TORI AND ACCELERATION OF THE K.A.M. ALGORITHM

Permalink
https://escholarship.org/uc/item/6kk385tn

Authors
Warnock, R.L.
Ruth, R.D.

Publication Date
1986-06-01

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Physics Division

Presented at the International Conference on the
Physics of Phase Space, College Park, MD,
May 20-23, 1986

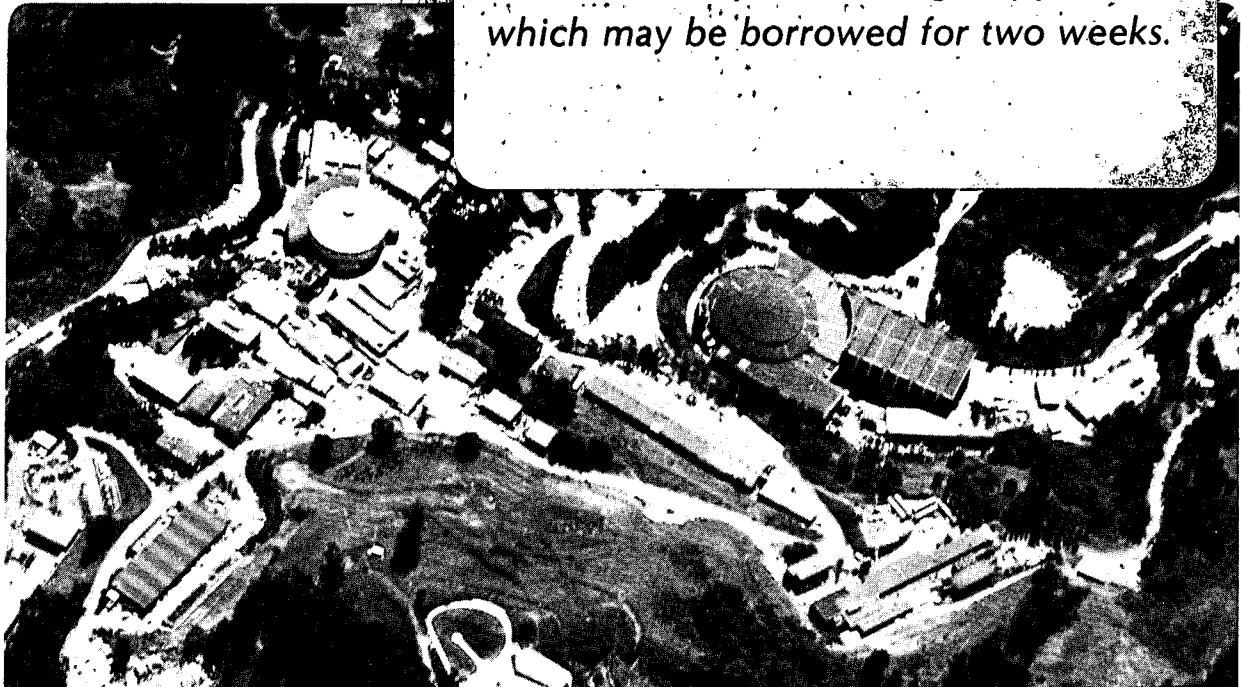COMPUTATION OF INVARIANT TORI AND ACCELERATION
OF THE K.A.M. ALGORITHM

R.L. Warnock and R.D. Ruth

June 1986

DISCLAIMER

COMPUTATION OF INVARIANT TORI AND ACCELERATION OF THE K.A.M. ALGORITHM[*]

R. L. WARNOCK
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

R. D. RUTH
Stanford Linear Accelerator Center
Stanford University
Stanford, California 94305

We describe a method to compute invariant tori in phase space for classical non-integrable Hamiltonian systems. Our procedure is to solve the Hamilton-Jacobi equation stated as a system of equations for Fourier coefficients of the generating function. The system is truncated to a finite number of Fourier modes and solved numerically by Newton's method. The resulting canonical transformation serves to reduce greatly the non-integrable part of the Hamiltonian. Further transformations computed on progressively larger mode sets would lead to exact invariant tori, according to the argument of Kolmogorov, Arnold, and Moser (KAM)[1]. Our technique accelerates the original KAM algorithm, since each truncated Hamilton-Jacobi equation is solved accurately, rather than in lowest order. In examples studied to date the convergence properties of the method are excellent, even near chaotic regions and on the separatrices of isolated broad resonances. One can include enough modes at the first step to get accurate results with only one canonical transformation. A second transformation gives an estimate of error. We propose a criterion for breakup of a KAM torus, which arises naturally in the Hamiltonian-Jacobi formalism. We verify its utility in an example with $1\frac{1}{2}$ degrees of freedom and anticipate that it will be useful in systems of higher dimension as well.

We present results for a system with one degree of freedom having a periodic time-dependent Hamiltonian. In angle-action variables the Hamiltonian is

$$H(\phi, J, \theta) = H_0(J) + V(\phi, J, \theta), \tag{1}$$

where V has period 2π in the time variable θ. We seek a canonical transformation $(\phi, J) \mapsto (\psi, K)$ in the form

$$J = K + G_\phi(\phi, K, \theta), \tag{2}$$

$$\psi = \phi + G_K(\phi, K, \theta), \tag{3}$$

such that the new Hamiltonian becomes a function of K alone. The Hamilton-Jacobi equation to determine the generator G is the requirement that the new Hamiltonian H indeed depend only on K; namely

$$H_0(K + G_\phi) + V(\phi, K + G_\phi, \theta) + G_\theta = H_1(K). \tag{4}$$

We seek periodic solutions of (4) with the Fourier development

$$G(\phi, K, \theta) = \sum_{m,n} g_{mn}(K) e^{i(m\phi - n\theta)}. \tag{5}$$

We rearrange (4) by adding and subtracting terms so as to isolate terms linear in G_ϕ and G_θ. We then take the Fourier transform for $m \neq 0$ to cast Eq. (4) in the form

$$g = A(g), \tag{6}$$

where $g = [g_{mn}]$ is a vector of Fourier coefficients and

$$A_{mn}(g) = \frac{i}{(\omega_0 m - n)} \frac{1}{(2\pi)^2} \int\limits_0^{2\pi} \int\limits_0^{2\pi} d\phi d\theta e^{-i(m\phi - n\theta)} \left[H(\phi, K + G_\phi, \theta) - H_0(K) - \omega_0 G_\phi \right] \ , \ m \neq 0 \ ,$$

(7)

where $\omega_0(K) = \partial H_0 / \partial K$. To truncate the system (6) for numerical solution we restrict (m,n) to some bounded set B of integers, with $m \neq 0$, and put

$$G_\phi = \sum_{(m,n)\epsilon B} im g_{mn}(K) e^{i(m\phi - n\theta)} \ .$$

(8)

The set B is selected so that the only modes included are fairly close to resonance and are driven by the perturbation V (directly or through harmonics).

We show results from solving (6) by Newton's iteration, starting from $g = 0$. The action variable K is changed at each iteration in such a way as to make the final frequency $\omega = \partial H_1 / \partial K$ have a preassigned value. This is accomplished automatically by augmenting (6) with another equation to be iterated.

The example chosen is the non-integrable two-resonance Hamiltonian

$$H = \nu J + \frac{1}{2}\alpha J^2 + \varepsilon_1 J^{5/2} \cos(5\phi - 3\theta) + \varepsilon_2 J^2 \cos(8\phi - 3\theta),$$

(9)

where $\nu, \alpha, \varepsilon_1, \varepsilon_2$ are constants. For small $\varepsilon_1, \varepsilon_2$ we compute a KAM curve at a frequency equal to the golden mean $\omega_* = (\sqrt{5} - 1)/2$, which is between the two resonances, and explore its breakup as ε_1 and ε_2 increase to critical values. With $\nu = 0.5$, $\alpha = 0.1$, we find an apparently solid KAM curve for $\varepsilon_1 = 2\varepsilon_2 = 6 \times 10^{-5}$, for which case the resonance widths are $\Delta J_1 = 0.049$, $\Delta J_2 = 0.054$, as compared to the resonance separation $J_{r1} - J_{r2} = 0.25$. The curve $J(\phi, \theta = 0)$ shown in Fig. 1 was computed in 4 Newton iterations with 40 modes in the set B. It agrees well with results from direct integration of Hamilton's ordinary differential equations, and the corresponding canonical transformation leaves a very small residual perturbation. The average of the absolute value of the residual perturbation divided by a similar average of the original perturbation is 1.1×10^{-5}. Expanding the mode set to 77 modes and doing further iterations, we reduce this ratio to 6.4×10^{-8}.

To identify the breakup of the KAM curve ("transition to chaos") as the $\varepsilon's$ are increased, we propose the criterion that the Jacobian of Eq. (3) vanish at some (ϕ, θ):

$$\partial\psi / \partial\phi = 1 + G_{K\phi} = \partial J / \partial K = 0 \ .$$

(10)

At such a point it may be impossible to solve uniquely for ϕ in terms of ψ. Fig. 2 shows $\partial\psi / \partial\phi$ corresponding to the case of Fig. 1. When $\varepsilon_1 = 2\varepsilon_2$ is increased to 1.4×10^{-4}, we get J and $\partial\psi / \partial\phi$ as shown in Figures 3 and 4, respectively. The anticipated zeros of $\partial\psi / \partial\phi$ appear in Fig. 4; however, the behavior of $\partial\psi / \partial\phi$ near transition is rather sensitive to the number of modes included. Judging from numerical integration of Hamilton's equations, we believe that the case of $\varepsilon_1 = 2\varepsilon_2 = 1.4 \times 10^{-4}$ is actually a little beyond transition.
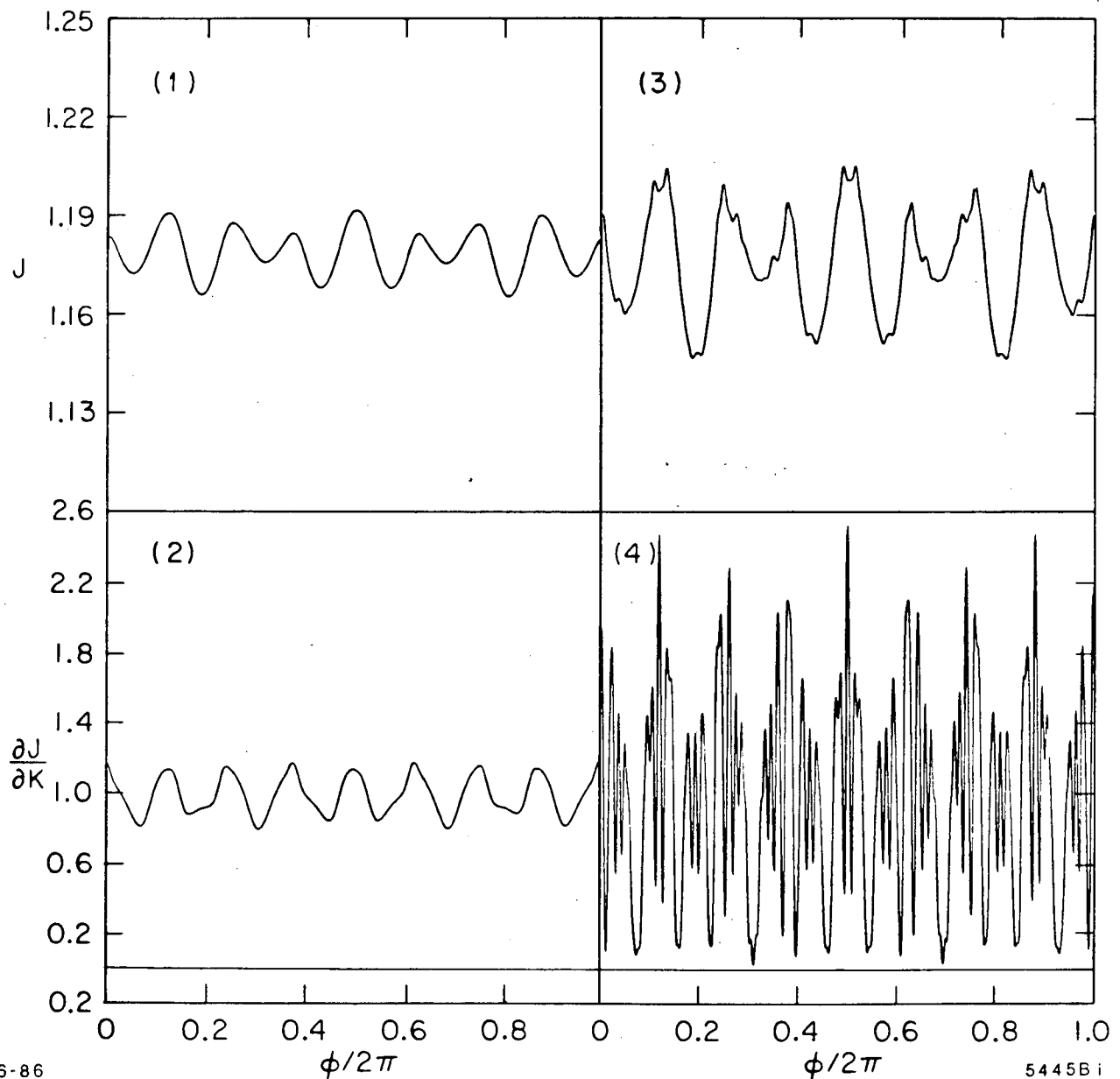
As the transition to chaos is approached, it becomes more difficult to expand the mode set. If too many modes are included, convergence of the Newton iteration suffers, and there is little if any reduction in the residual perturbation beyond that obtained with about 100 modes. Nevertheless, with 100 modes the ratio of residual to original perturbation is small; even at $\varepsilon_1 = 2\varepsilon_2 = 1.2 \times 10^{-4}$ this ratio is 1.5×10^{-4}. This suggests that further canonical transformations, computed on progressively larger mode sets, would in fact yield an exact invariant torus. To date we have

computed the second canonical transformation only in lowest order. The average absolute value of the torus distortion from the second transformation, divided by that from the first, varies from 2.8×10^{-6} at $\varepsilon_1 = 2\varepsilon_2 = 6 \times 10^{-5}$ to 4.1×10^{-3} at $\varepsilon_1 = 2\varepsilon_2 = 1.2 \times 10^{-4}$.

We conclude that the method provides a promising alternative to canonical perturbation theory and its modern variants. Unlike perturbation theory, its algebraic complexity does not increase as more accuracy is demanded, and the required computer programs are quite simple. The fact that the method is effective near chaotic regions is of great interest for applications. The generalization of (10) to higher dimensions, namely $\det(1 + G_{K\phi}) = 0$, may provide a useful criterion for the breakup of KAM surfaces in complicated systems of interest. We give an extended account of this work in Ref. 2.

REFERENCES

1. V. I. Arnold, "Mathematical Methods of Classical Mechanics", Springer, Berlin, 1978.

2. R.L. Warnock and R.D. Ruth, "Invariant Tori Through Direct Solution of the Hamilton-Jacobi Equation", SLAC-PUB-3865, LBL-21709, to be published.

6-86

$\phi/2\pi$

$\phi/2\pi$

5445Bi