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Dependence of Portfolio Returns Over Time and the CAPM:

Diverse Holding Periods⁺

by

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Abstract

This paper shows that under some plausible assumptions about the distributions of returns and the utility functions of the investors the CAPM holds in every single period even if investors have multiperiod diverse investment horizons. This holds even when portfolio returns are dependent over time. The dependence over time is due to the fact that portfolio decisions of the investors at any given period, depend on the actual returns of previous periods via the wealth effect. Hence, the selected portfolios returns may be dependent over time even if the securities themselves are independent. It is shown that despite the above mentioned dependency the investors must choose, at each period, a portfolio from the mean-variance efficient frontier of that period, hence the CAPM holds in each period.

The paper, therefore, extends previous results of Levy and Samuelson [1992], Merton [1973] and Stapleton and Subrahmanyam [1978] that showed the existence of the CAPM in a multiperiod setting. Merton proved his result in a continuous time model, a framework that prohibits transaction costs and hence is quite restrictive. Levy and Samuelson overcame the problem of dependency of portfolios by assuming either a quadratic utility function or implicitly assuming a myopic utility function, whereas Stapleton and Subrahmanyam assumed an exponential utility and a common investment horizon. The above assumptions are quite restrictive, and therefore the model given in this paper provides a much broader scope to the CAPM. It allows discrete revisions, any risk averse preferences, diverse holding periods, dependency over time, and a wide class of distribution functions of returns.

Dependence of Portfolio Returns Over Time and the CAPM:

Diverse Holding Periods

Introduction

The Capital Asset Pricing Model (CAPM) developed by Sharpe [1964] and Lintner [1965] establishes the equilibrium risk return relationship in a single-period framework. It is the subject of numerous theoretical as well as empirical papers and is widely used by practitioners as well as academics. There are several extensions of the CAPM, which take into account taxes, transaction costs etc. Probably the most noticeable extensions of the CAPM are the continuous time risk-return equilibrium model developed by Merton [1973] and the segmented market equilibrium known as the General CAPM (GCAPM) developed by Levy [1978], Merton [1987] and Markowitz [1990].¹ The importance of extending the CAPM into a multiperiod framework lies beyond the necessity of theoretical consistency. A multiperiod CAPM is the foundation and justification for using risk adjusted discount rates in capital budgeting and it also provides the correct method of using these rates.² Moreover, if the CAPM holds only on the assumption that all investors have the same predetermined investment horizon, one cannot use the CAPM cost of equity (to calculate the weighted average cost of capital) to be employed in capital budgeting when the project induces

¹ For a comparison of these models see Sharpe [1991]. Merton [1973] does prove the existence of the CAPM with diverse holding periods. His assumption of continuous time with continuous revisions of portfolios prohibits the existence of transaction costs or projects with discrete cashflows, and hence is quite restrictive. For valuation of uncertain income streams in a discrete multi-period setting see Rubinstein [1976].

² See, e.g., Myers and Turnbull [1977], Fama [1977].

cashflows in several periods. Relaxing the one-period and common investment horizon assumptions thus provides justification for the use of the CAPM in many more applications.

In a recent paper, Levy and Samuelson [1992] show under plausible assumptions about the distribution of returns and utility functions of the investors, that the classic Sharpe-Lintner CAPM holds in much more cases than were hitherto believed. They demonstrate that the CAPM can be extended to a multiperiod discrete time model when investors have diverse investment horizons and are allowed to revise their portfolios each period. In their paper, however, they assume that the returns on the selected portfolios by any given investor are independent over time.³ The above assumption (which implies myopic decisions) however, may be quite restrictive since one may usually expect some dependency between the selected portfolios. Such dependence results due to the interaction between realized wealth at any period of time and its effect on future decisions, and hence on future returns.⁴ In this paper we prove that the Levy and Samuelson [1992] CAPM extensions indeed hold even when returns are dependent over time thus making the CAPM extensions more robust and therefore more significant.

When returns are independent over time one can use the standard multi-variate formulas for the multi-period mean and variance as suggested by Tobin [1965] to prove the existence of the CAPM in each single period. Indeed this technique has been employed by Levy and Samuelson [1992]. When returns are dependent over time, however, these multi-variate formulas cannot be employed anymore, since the variance of the terminal wealth

³ Levy and Samuelson [1992] conjectured however that this assumption can be relaxed and that the results hold also when the selected portfolios are dependent over time. Stapleton and Subrahmanyam [1978] however conjectured the opposite.

⁴ Stapleton and Subrahmanyam [1978] in a similar model overcome the problem of dependency by assuming an exponential utility function. They also assumed a common investment horizon, and that returns are multivariate normal.

becomes a function of previous results and decisions. Dynamic programming techniques must, therefore, be used to prove that the CAPM holds in every single period.

Thus, the main contribution of this paper is by showing that the CAPM holds in every single period even when the parameters change every period and that the selected portfolio returns are dependent over time (via the wealth effect), with no need to assume a myopic utility function. This makes the CAPM theoretical foundation much stronger, and provides a justification to employ the CAPM for capital budgeting.

In the following section the existence of the CAPM will be proved under four alternative sets of assumptions (cases) concerning the distribution functions of returns and/or the utility functions of the investors. In Section II concluding remarks are provided.

I. Existence of the Multi-period CAPM when Investors have diverse holding periods

In this section it is shown that the multiperiod CAPM holds under the following four sets of assumptions (cases) regarding the distribution function of returns and/or the utility functions of the investors: 1) Investors who have a quadratic utility function; no restrictions on the distribution functions of returns; 2) The distribution function of final wealth is normal; 3) The multiperiod distribution of terminal wealth is lognormal (see Aitchison and Brown [1963] and Dexter, Yu and Ziemba [1980]); and 4) each single-period distribution of returns is normal, but no constraints are put on the multi-period distribution (which is the product of the single distributions, and hence is not normal).⁵ In the last three cases no restrictions are imposed on the utility function of the investor apart from risk aversion. In all cases, investors have diverse investment horizons (except case 2) and they are allowed to revise

⁵ The classical single-period CAPM is a special case of the above cases 1,2, and 4 when there is only one period and no revisions are allowed.

their portfolios each period. Each investor strives to maximize the expected utility of final wealth.

Denote by T_k the number of investment periods of the k^{th} investor. Also denote by $T = \max_k \{T_k\}$. We show that the CAPM holds for the decision makers at all periods. The proof of existence consists of showing that at each period, regardless of his or her holding period, T , the investor will choose a portfolio that is mean-variance (M-V) efficient for that period.⁶ This implies that all investors choose M-V efficient portfolios, i.e., a linear combination of the market portfolio and the riskless asset and therefore the CAPM holds at each period. The methodology of proof is by induction. There are T periods and T investment decisions. At $T-1$ a portfolio is selected and the returns are obtained at period T . Similarly, the selected portfolio at $T-2$ yields return at period $T-1$, etc. It is first shown that at the last investment decision, i.e., at $T-1$, the investor chooses a portfolio from the M-V efficient set applicable to the last period. Based on that it is then shown that also in the decision before last, $T-2$, it is never optimal to choose a portfolio which does not lie on the efficient set for that period. It hence follows that in all periods the investor will choose a portfolio from the M-V efficient set which implies the CAPM in each period.⁷

The investor makes a portfolio investment decision at $T-2$ yielding wealth \tilde{W}_{T-1} and an investment decision at $T-1$ yielding a terminal wealth of \tilde{W}_T (where a tilde denotes a random variable). It can be shown that for all four cases given above, and for any given realized wealth W_{T-1} , the final portfolio revision at $T-1$ will be done by selecting a mean-

⁶ Actually at any time some investors may enter the market and some may leave. The index k may be omitted, however, without loss of generality.

⁷ The efficient set in each period is the capital market line (CML), namely the line rising from the risk-free interest rate and tangent to the efficient frontier composed of risky assets.

variance efficient portfolio. Whereas it is quite straightforward to show that at T-1 the investor will choose their portfolios from the M-V efficient set or from the Capital Market Line (CML), the extension of this result to period T-2 is not straightforward.⁸ The difficulty arises since expected utility is defined on terminal wealth and not on W_{T-1} . For example, suppose that there are two portfolios yielding random wealths \tilde{W}_{T-1}^* and \tilde{W}_{T-1}^{**} , respectively, and it is given that $E(\tilde{W}_{T-1}^*) > E(\tilde{W}_{T-1}^{**})$ and $Var(\tilde{W}_{T-1}^*) < Var(\tilde{W}_{T-1}^{**})$, where E and Var denote the expectation and variance operators, respectively. To be more specific assume that $E(\tilde{W}_{T-1}^*)$ is on the T-1 period CML and $E(\tilde{W}_{T-1}^{**})$ is below this CML. It is not obvious that all investors who maximize expected utility defined on terminal wealth, $EU(\tilde{W}_T)$, would prefer at period T-2 to invest in portfolio yielding \tilde{W}_{T-1}^* rather than \tilde{W}_{T-1}^{**} . The reason is that the realized wealth (or the return) at T-1, affects the choice in period T-1, hence \tilde{W}_{T-1}^* (and \tilde{W}_{T-1}^{**}) and the returns in the final period are dependent, even though the returns on each security are independent over time. Consequently, it appears that a higher mean and a lower variance at period T-1 do not guarantee that the product of returns at (T-1) and T will also provide higher mean and lower variance. It may therefore seem, due to the selected portfolio returns dependence, that at T-1 an interior portfolio may be preferred and the CAPM would not hold at that period.

The dependency over time of the portfolios returns may be described more specifically as follows: Suppose that at T-1 the CAPM holds, and the investor selects a linear combination $\alpha \tilde{r}_m + (1-\alpha) r$ at that time, where α is the proportion of wealth invested in the

⁸ This applies to investors who are still left in the market; recall that some may liquidate their portfolios earlier and some new investors may join the market at T-1. Thus, the number of investors in each period does not have to be constant.

market portfolio whose return is \bar{r}_m (i.e., $1 + \text{rate of return}$) and r stands for the riskless return.⁹ The investment proportions α , however, are functions of the realized wealth in period T-1, and hence the terminal wealth \tilde{W}_T is given by

$$\tilde{W}_T = \tilde{W}_{T-1} [\alpha(\tilde{W}_{T-1}) \bar{r}_m + (1 - \alpha(\tilde{W}_{T-1})) r]$$

where the terminal wealth of period T-1 is the initial wealth at the beginning of period T. Thus, even if returns of each security (as well as the efficient frontiers) are independent over time, we have a dependency between the returns in period T-1 and period T since $\alpha(W_{T-1})$ is a function of W_{T-1} . If the utility function is myopic (See Merton and Samuelson [1974] and Samuelson [1990]) then $\alpha(W_{T-1}) = \alpha$ and the returns on the selected portfolios are independent over time, and the CAPM holds in each period for all the above four cases (see Levy and Samuelson [1992]). Nevertheless, we show below that the CAPM holds in every period in all the above four cases even when this type of dependency prevails, i.e., there is no need to confine ourselves to a myopic utility function.¹⁰

The main theorem of the paper is proved next:

Theorem 1 Under the four cases above the investor chooses each period a portfolio from the CML corresponding to that period. The CAPM, therefore, holds in each single period. This is valid even when the parameters as well as the risk free interest rate varies from one period to another.

⁹ The return on the riskless asset and the distribution of returns may be changing over time without affecting the results of this paper. For ease of notation we did not assign these variables a time index.

¹⁰ Levy and Samuelson [1992] show that the every-single period CAPM holds with dependency for a quadratic utility function. We show in this paper that only risk aversion is needed.

Proof: Cases 1,2 and 3: The theorem is first proved for cases 1,2, and 3. In these cases we can safely rely on the terminal wealth mean and variance, since we have either a quadratic utility function, normal distribution or lognormal distribution. In case 4, we cannot rely only on the mean and variance of the terminal wealth since the terminal wealth distribution is unknown. (terminal wealth is the product of some normal distribution functions and hence is not necessarily normal). In the first three cases it is therefore sufficient to show that the CAPM holds in each single period, if for any given desired expected mean of terminal mean the variance of the terminal wealth is minimized. In cases 1,2, and 3 it is easy to verify that at T-1 (the last decision) the investor will choose a portfolio from the CML corresponding to the expected returns to be obtained at T. This is so since in case 1 the utility function is quadratic, and in case 2 the distribution is normal for period T, and it is well known (see e.g., Markowitz [1952], [1991] and Tobin [1958]) that in these cases all utility maximizers choose their portfolios from the M-V efficient set. In case 3 (lognormal distribution of returns for period T), the investor chooses his or her portfolio according to mean and coefficient of variation of \tilde{W}_T (for the proof see Levy [1973], [1991]). It hence follows that for any given desired expected returns at period T, the investor will choose at T-1 a portfolio that provides the lowest standard deviation, i.e., again, a portfolio on the CML. Thus, it is straightforward for cases 1,2 and 3 that for the last period the CAPM holds.

The main contribution of this Theorem is by showing that also at T-2 the investor will choose a portfolio on the CML corresponding to returns obtained at T-1. Since at T-1 the investor chooses a portfolio on the CML, the wealth at the end of period T, given the realized wealth in period T-1 is given by

$$\tilde{W}_T \mid W_{T-1} = W_{T-1} [\alpha (W_{T-1}) \tilde{r}_m + (1 - \alpha (W_{T-1})) r] \quad (1)$$

The expected conditional wealth and the conditional standard deviations of wealth are given by¹¹

$$E (\tilde{W}_T | W_{T-1}) = W_{T-1} [\alpha(W_{T-1}) \mu_m + (1 - \alpha(W_{T-1})) r] \quad (2)$$

$$SD (\tilde{W}_T | W_{T-1}) = W_{T-1} \alpha(W_{T-1}) \sigma_m \quad (3)$$

where $E (\tilde{r}_m) = \mu_m$

$$\text{Var} (\tilde{r}_m) = \sigma_m^2$$

Suppose the investor wishes to choose a portfolio with expected wealth of M at period T. Namely, M is the expected value of the terminal wealth. This can be achieved by choosing $\alpha(W_{T-1})$ so as to solve

$$W_{T-1} [\alpha(W_{T-1}) \mu_m + (1 - \alpha(W_{T-1})) r] = M \quad (4)$$

or

$$W_{T-1} [\alpha(W_{T-1}) (\mu_m - r) + r] = M \quad (4')$$

That is, for any realized wealth W_{T-1} , $\alpha(W_{T-1})$ is adjusted to obtain a fixed expected value of terminal wealth. Then from equation (4'), $\alpha(W_{T-1})$ is given by,

$$\alpha(W_{T-1}) = \{ (M / W_{T-1}) - r \} / (\mu_m - r) \quad (5)$$

Substitute $\alpha(W_{T-1})$ from eq. (5) in eq. (3) to obtain the conditional standard deviation (SD) of \tilde{W}_T :

$$SD (\tilde{W}_T | W_{T-1}) = \sigma_m \left[\frac{M}{(\mu_m - r)} - \frac{W_{T-1} r}{(\mu_m - r)} \right]$$

¹¹ Note that | means 'conditional', hence \tilde{W}_T is a random variable but W_{T-1} is given (hence it is without the tilde notation).

and the conditional variance is,

$$Var(\tilde{W}_T | W_{T-1}) = [\sigma_m^2 / (\mu_m - r)^2] (M - W_{T-1}r)^2 \quad (6)$$

Note that in the variance calculation, W_{T-1} is treated as a constant since it is the conditional variance of \tilde{W}_T given the realized wealth W_{T-1} .

Since the expected terminal wealth M is fixed for all possible investment strategies, the investor should choose the optimum investment strategy which will minimize the variance of the terminal wealth \tilde{W}_T . However, it is well known¹² that for any two random variables $\tilde{W}_T, \tilde{W}_{T-1}$

$$Var(\tilde{W}_T) = E_{W_{T-1}} [Var(\tilde{W}_T | \tilde{W}_{T-1})] + Var_{W_{T-1}} [E(\tilde{W}_T | \tilde{W}_{T-1})].$$

However since by construction $E(\tilde{W}_T | W_{T-1}) = M$ for all W_{T-1} , the second term vanishes and the above equation reduces to

$$Var(\tilde{W}_T) = E_{W_{T-1}} [Var(\tilde{W}_T | \tilde{W}_{T-1})] \quad (7)$$

Using (7) and (6) one obtains:

$$Var(\tilde{W}_T) = [\sigma_m^2 / (\mu_m - r)^2] E_{W_{T-1}} (M - \tilde{W}_{T-1}r)^2$$

i.e.,

$$Var(\tilde{W}_T) = [\sigma_m^2 / (\mu_m - r)^2] [M^2 - 2MrE(\tilde{W}_{T-1}) + r^2 E(\tilde{W}_{T-1}^2)] \quad (8)$$

Since the terminal wealth is fixed at M , the optimum investment strategy at $T-2$ is the one which will minimize $Var(\tilde{W}_T)$ given by eq. (7). It is next shown that for any M-V interior (i.e., not on the CML) portfolio, there is an investment portfolio on the CML at period $T-1$

¹² See Ross [1983].

which dominates it (recall that the decisions corresponding to the CML of T-1 are made at T-2). To see this consider portfolio m_1 in Figure 1 (which presents the efficient line at period T-1) that is an interior portfolio in the M-V space. Consider also portfolio m_2 which lies on the horizontal line emanating from m_1 but is on the CML. Obviously, by construction

$$E_{m_1}(\tilde{W}_{T-1}) = E_{m_2}(\tilde{W}_{T-1}) \quad \text{since they are both on the same horizontal line.}$$

Hence, the portfolio which will minimize $E(\tilde{W}_{T-1}^2)$ (see eq. 8) is the best portfolio for the given M in the multiperiod framework. However, from the relationship

$$E(\tilde{W}_{T-1})^2 = \text{Var}(\tilde{W}_{T-1}) + [E(\tilde{W}_{T-1})]^2$$

and using the fact that m_2 is left to m_1 on the same horizontal line and the fact that $\text{Var}(\tilde{W}_{T-1})$ corresponding to portfolio m_2 is smaller than the one corresponding to portfolio m_1 , we conclude that $E(\tilde{W}_{T-1}^2)$ and hence $\text{Var}(\tilde{W}_{T-1})$ will be smaller for portfolio m_2 .

Thus, for any interior portfolio like m_1 there is a portfolio on the T-1 period capital market line which dominates it in the sense of maximizing $EU(\tilde{W}_T)$, (recall, cases 1,2 and 3 rely solely on the mean and variance of terminal wealth), hence all investors irrespective of their preferences will choose at T-2 the investment strategy which is located on the T-1 period capital market line, hence we have a separation property and the CAPM holds also at T-1.¹³

We started the proof by arbitrarily fixing the expected terminal wealth at M. However, since the proof holds for any value M, it holds for all investors provided the assumptions of cases 1,2, and 3 hold. Since at T-2 the investors choose the portfolio on the CML, it follows by induction they do so for any decision taking place at t , $t = T-3$,

¹³ Recall that at period T-1 the distributions of returns are not necessarily normal. Moreover, nothing is known about these distributions and they may even not belong to the same family. Also, nothing is known on the correlation between these distributions (see Cases 1 and 4). Yet we claim that the CAPM holds also for period T-1.

$T-4, \dots, 1$.

The theorem is next proved for case 4.

Case 4

This case needs a special proof since investors do not necessarily make their decisions based solely on the mean and variance of terminal wealth. The reason why the above proof cannot be used for case 4 is that if X_1 and X_2 , the returns in any two periods, say period 1 and 2, are normally distributed then $X_1 X_2$ (terminal wealth) is not normally distributed. Hence expected utility maximizers cannot examine only the mean and variance of terminal wealth as done in the previous cases and the whole distribution should be analyzed. A different technique is therefore needed to prove that the CAPM holds in each period also in case 4.

At $T-1$, given the realized wealth W_{T-1} all risk averse investors select a portfolio which lies on the CML, namely a linear combination of r_m and r . (Recall given W_{T-1} the distribution in the last period is normal, hence the M-V rule is optimal).

It is shown that also in this case at $T-2$ investors would also select a portfolio located on the CML, hence the CAPM holds also for $T-2$. To see this, consider portfolio m_1 , given in Figure 1. It is claimed that there is at least one portfolio located on the $T-2$ capital market line which dominates portfolio m_1 in terms of expected utility defined on terminal wealth.

In this specific case, portfolio m_3 (vertical comparison!) will lead to a higher expected utility in comparison to m_1 . To see this note that both portfolios m_1 and m_3 have the same variance and m_3 has a higher mean, and both are normally distributed. Thus, if m_1 is characterized by $X_{m_1} \sim N(\mu, \sigma)$, then m_3 is given by $X_{m_3} \sim N(\mu + a, \sigma)$ where $a > 0$.

Suppose that the return on the last period investment is normal given by the random variable \tilde{Y} . The terminal wealth if \tilde{X}_1 is selected at $T-2$ is $\tilde{X}_1 \cdot \tilde{Y}$ and the terminal wealth

if \tilde{X}_3 is selected at that period T-2 is $\tilde{X}_3\tilde{Y}$. For this comparison the dominance of \tilde{X}_3 over \tilde{X}_1 is not revealed. However, consider the following investment policy. For each realized return, X_3 , on m_3 , deposit \$a at the risk free asset and invest the rest in \tilde{Y} . Thus, if m_3 is selected at T-2 then the terminal wealth will be $\tilde{Y}(\tilde{X}_1 + a)$ under this policy¹⁴ which dominates $\tilde{Y}\tilde{X}_1$ (which is the return if m_1 is selected) by First Degree Stochastic Dominance (see Hadar and Russell (1969); Hanoch and Levy (1969); Rothschild and Stiglitz [1971], and for a review article on Stochastic Dominance see Levy [1992]).¹⁵

Thus, for every interior portfolio (like m_1) at period T-2, there is a portfolio like m_3 on the CML which dominates it in terms of $EU(\tilde{W}_T)$. Therefore, also at T-2 all investors select their portfolio from the CML, hence the separation theorem and the CAPM holds also in period T-2. This concludes the proof of case 4. In the above analysis, for all cases, the risk free rate and the parameters of the distribution functions could vary in time, without altering the result. This therefore concludes the proof.

Q.E.D.

Concluding Remarks

The Sharpe-Lintner CAPM assumes either normal distribution and risk aversion or a quadratic utility function. All investors are assumed to have the same investment horizon. This assumption however is quite confining and unrealistic.

Levy and Samuelson [1992] show that when investors have diverse holding periods and portfolio revisions are allowed then the CAPM holds in every revision period also in the

¹⁴ Recall $\tilde{X}_3 = \tilde{X}_1 + a$ since both are normal and differ by a constant shift.

¹⁵ Of course normal distribution assumption is not consistent with the fact that stock price cannot be negative (limited liability) but this deficiency hold also in the Sharpe-Lintner one-period CAPM.

following four cases:

- 1) Quadratic utility function defined on terminal wealth.
- 2) The terminal wealth is normally distributed but there is no constraint on each period distribution.
- 3) Terminal wealth is lognormally distributed but there are no restrictions on the return distributions in each period.
- 4) Returns in each revision period are normal but the terminal wealth distribution is not normal.

Thus, not only the constant investment horizon assumption is relaxed but also a wide range of conditions either on preference or on the distribution of returns are allowed.

Levy and Samuelson [1992] prove these extensions for the case where the mean-variance efficient frontiers are independent over time, and assume that the returns on the selected portfolios are also independent over time. The latter independence follows for a myopic utility function, but does not hold in general since the portfolio choice at any period is a function of the realized return from the investment in the previous period, and hence a dependency may be created even if the returns on each asset are independent over time.

While Levy and Samuelson conjecture that their results hold also for the dependent case (and prove it for case 1), it is proved in this paper that the four extensions hold for the dependent case as well, and there is no need to confine the result to a myopic utility function.

These results are important not only from a theoretical point of view but also from a practical point of view. If the CAPM holds only for one-period decision making, one cannot employ it (to determine the discount rates) in capital budgeting which requires by its nature a multiperiod framework. Thus, the implication of these results is that one can use the CAPM cost of capital in each single period and apply it for discounting the project's

cashflow. This is true even when the securities parameters (e.g., β and the interest rate) vary from one period to another.

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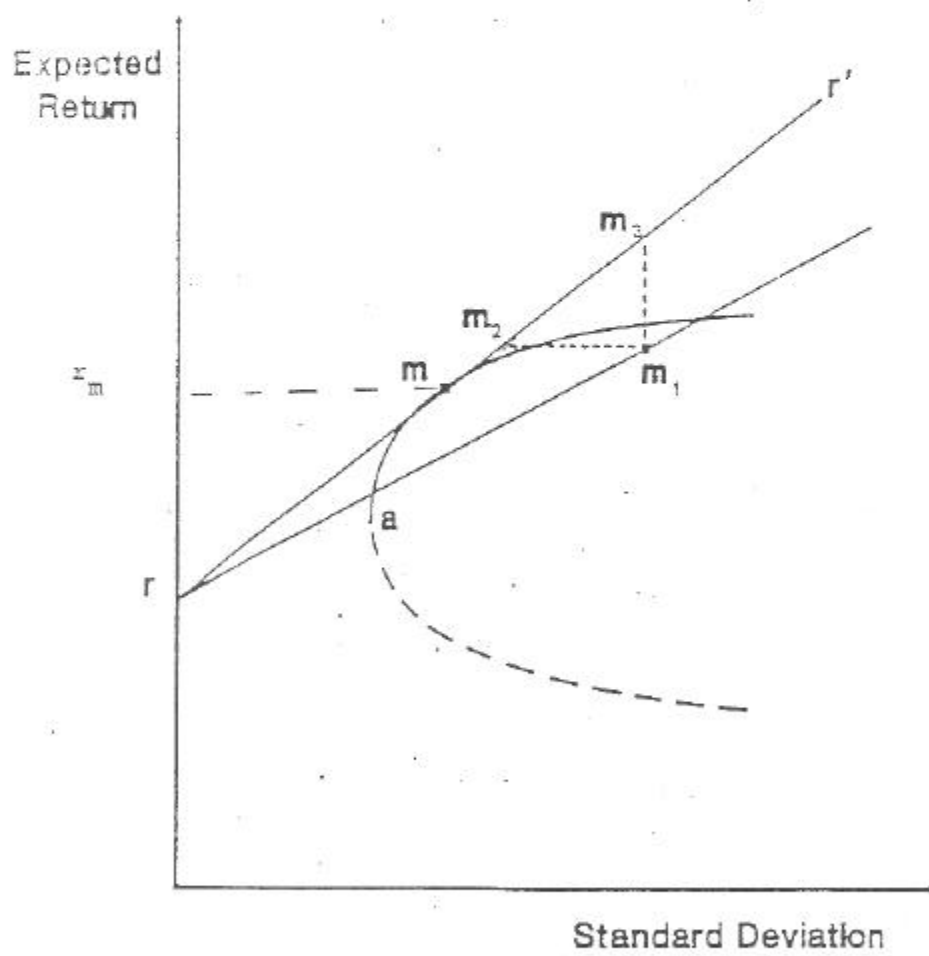


FIGURE 1: THE ONE-PERIOD EFFICIENT FRONTIER