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Lagrangian Framework for Systems Composed of High-Loss and Lossless Components

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Abstract

Using a Lagrangian mechanics approach, we construct a framework to study the dissipative properties of systems composed of two components one of which is highly lossy and the other is lossless. We have shown in our previous work that for such a composite system the modes split into two distinct classes, high-loss and low-loss, according to their dissipative behavior. A principal result of this paper is that for any such dissipative Lagrangian system, with losses accounted by a Rayleigh dissipative function, a rather universal phenomenon occurs, namely, *selective overdamping*: The high-loss modes are all overdamped, i.e., non-oscillatory, as are an equal number of low-loss modes, but the rest of the low-loss modes remain oscillatory each with an extremely high quality factor that actually increases as the loss of the lossy component increases. We prove this result using a new time dynamical characterization of overdamping in terms of a virial theorem for dissipative systems and the breaking of an equipartition of energy.

1 Introduction

In this paper we introduce a general Lagrangian framework to study the dissipative properties of two component systems composed of a high-loss and lossless components which can have gyroscopic properties. This framework covers any linear Lagrangian system provided it has a finite number of degrees of freedom, a nonnegative Hamiltonian, and losses accounted by the Rayleigh dissipative function, [Pars, Sec. 10.11, 10.12], [Gant, Sec. 8, 9, 46]. Such physical systems include, in particular, many different types of damped mechanical systems or electric networks.

Motivation. We are looking to design and study two-component composite dielectric media consisting of a high-loss and low-loss component in which the lossy component has a useful property (or functionality) such as magnetism. We want to understand what is the trade-off between the losses and useful properties inherited by the composite from its components. Our motivation comes from a major problem in the design of such structures where a component which carries a useful property, e.g., magnetism, has prohibitively strong losses in the frequency range of interest. Often this precludes the use of such a lossy component with otherwise excellent physically desirable properties. Then the question stands: Is it possible to design a composite system having a useful property at a level comparable to that of its lossy component but with significantly reduced losses over a broad frequency range?

An important and guiding example of a two-component dielectric medium composed of a high-loss and lossless components was constructed in [FigVit8]. The example was a simple layered structure which had magnetic properties comparable with a natural bulk material but with 100 times lesser losses in a wide frequency range. This example demonstrated that it is possible to design a composite material/system which can have a desired property comparable with a naturally occurring bulk substance but with significantly reduced losses.

In order to understand the general mechanism for this phenomenon, the authors in [FigWel1] considered a general dynamical system as the model for such a medium where the high-loss component of the medium was represented as a significant fraction of the entire system. We have found that for such a system the losses of the entire structure become small provided that the lossy component is sufficiently lossy. The general mechanism of this phenomenon, which we proved in [FigWel1], is the *modal dichotomy* when the entire set of modes of the system splits into two distinct classes, *high-loss* and *low-loss modes*, based on their dissipative properties. The higher loss modes for a wide range of frequencies contribute very little to losses and this is how the entire structure can be low loss, whereas the useful property is still present. In that work the way we accounted for the presence of a useful property was by simply demanding that the lossy component was a significant fraction of the entire composite structure without explicitly correlating the useful property and the losses.

In this paper, we continue our studies on the modal dichotomy and overdamping, which began in [FigWel1], but now we focus are attention on gyroscopic-dissipative systems. Although our results on modal dichotomy apply to the full generality of the dynamical systems considered here, our results on overdamping in this paper will be restricted to systems without gyroscopy. Consideration of overdamping in systems with gyroscopy will be considered in a future work.

Overview of results. Now in order to account in a general form for the physical properties of the two-component composite system with a high-loss and lossless components we introduce a Lagrangian framework with dissipation. This framework can be a basis for studying the interplay of physical properties of interest and

dissipation. These general physical properties include in particular gyroscopy (or gyrotropy), which is intimately related to magnetic properties, different symmetries and other phenomena such as overdamping that are not present in typical dynamical systems.

In the Lagrangian setting, the dissipation is taken into account by means of the Rayleigh dissipation function [Pars, Sec. 10.11, 10.12], [Gant, Sec. 8, 9, 46]. In the case of two-component composite with lossy and lossless components, the Rayleigh function is assumed to affect only a fraction $0 < \delta_R < 1$ of the total degrees of freedom of the system. In a rough sense, the loss fraction δ_R signifies the fraction of the degrees of freedom that are effected by losses.

We now give a concise qualitative description of the main results of this paper which are discussed next and organized under the following three topics: Overdamping and selective overdamping in Section 6; Virial theorem for dissipative systems and equipartition of energy in Section 4; Standard spectral theory vs. Krein theory in Sections 2 and 5.

Overdamping and selective overdamping. Overdamping is a regime in dissipative Lagrangian systems in which some of the systems eigenmodes are overdamped, i.e., have exactly zero frequency or, in other words, are non-oscillatory when losses are sufficiently large. The phenomenon of overdamping has been studied thoroughly but only in the case when the entire system is overdamped [Duff55], [Lan66, §7.6 & Chap. 9], [BarLan92] meaning all its modes are non-oscillatory. The Lagrangian framework and the methods we develop in this paper are applicable to the more general case allowing for only a fraction of the modes to be susceptible to overdamping, a phenomenon we call *selective overdamping*. Our analysis of overdamping and this selective overdamping phenomenon is carried out in Section 6. Our main results are Theorems 17, 19, 12, 20, and 26.

Our interest in selective overdamping is threefold. First, as we prove in this paper the selective overdamping of the system is a rather *universal phenomenon* that can occur for a two-component system when the losses of the lossy component are sufficiently large. Moreover, we show that *the large losses in the lossy component cause not only the modal dichotomy, but also results in overdamping of all of the high-loss modes while a positive fraction of the low-loss modes remain oscillatory*. Second, since mode excitation is efficient only when its frequency matches the frequency of the excitation force, if these high-loss modes go into the overdamping regime having exactly zero frequency these modes cannot be excited efficiently and therefore their associated losses are essentially eliminated. This explains the absorption suppression for systems composed of lossy and lossless components. Third, as we will show, as the losses in the lossy component increase the overdamped high-loss modes are more suppressed while all the low-loss oscillatory modes are more enhanced with increasingly high quality factor. *This provides a mechanism for selective enhancement of these high quality factor, low-loss oscillatory modes and selective suppression of the high-loss non-oscillatory modes.*

In fact, we show that *the fraction of all overdamped modes is exactly the loss fraction δ_R which satisfies $0 < \delta_R < 1$, and half of these overdamped modes are the high-loss modes. Thus it is exactly the positive fraction $1 - \delta_R > 0$ of modes which are low-loss oscillatory modes.* In addition, *these latter modes have extremely high quality factor which increases as the losses in the lossy component increase.* An example of this behavior using the electric circuit in Section 3 was shown numerically in [FigWel1].

Virial theorem for dissipative systems and equipartition of energy. Recall that the virial theorem from classical mechanics [Gold, pp. 83-86; §3.4] (which we review in Appendix 11) is about the oscillatory transfer of energy from one form to another and its equipartition. For conservative linear systems whose Lagrangian is the difference between the kinetic and potential energy, e.g., a spring-mass system or a parallel LC circuit, the virial theorem says that for any time-periodic state of the system, the time-averaged kinetic energy equals the time-averaged potential energy. This result is known as the equipartition of energy because of the fact that the total energy of the system, i.e., the Hamiltonian, is equal to the sum of the kinetic and potential energy. It is this result that we generalize in this paper for dissipative Lagrangian systems and their damped oscillatory modes.

In particular, we show that *for any damped oscillatory mode of the system the kinetic energy equals the potential energy and so there is an equipartition of the system energy for these damped oscillatory modes into kinetic and potential energy.* Our precise statements of these results can be found in Section 4 in Theorem 1 and Corollary 2.

We have also found that the transition to overdamping is characterized by a breakdown of the virial theorem. More specifically, as the losses of the lossy component increase some of the modes become overdamped with complete ceasing of any oscillations and with breaking of the equality between kinetic and potential energy. Such a *breaking of the equipartition of the energy can viewed as a dynamical characterization of the overdamped modes!* So we now have two different characterizations of overdamping: spectral and dynamical. The effectiveness of this dynamical characterization is demonstrated in our study of the selective overdamping phenomenon and in proving that a positive fraction of modes always remain oscillatory no matter how large the losses become in the lossy component of the composite system.

Standard spectral theory vs. Krein spectral theory. The study of the eigenmodes of a Lagrangian system often relies on the fact the system evolution can be transformed into the Hamiltonian form as the first-order linear differential equations. However, it is not emphasized enough that the Hamiltonian setting does not lead directly to the standard spectral theory of self-adjoint or dissipative operators but that only it might be reduced to one in some important and well known cases by a proper transformation as, for instance, in the case of a simple oscillator.

The general spectral theory of Hamiltonian systems is known as the *Krein spectral theory* [Arnold, §42], [Yak], [YakSta]. This theory is far more complex than the

standard spectral theory and it is much harder to apply. We have found, as discussed in Section 2, a general transformation that reduces the evolution equation to the standard spectral theory under the condition of positivity of the Hamiltonian. The standard spectral theory is complete, well understood, and allows for an elaborate and effective perturbation theory. We used that all in our analysis of the modal dichotomy and symmetries of the spectrum in Section 5 as well as the selective overdamping phenomenon in Section 6.

Our main results on spectral symmetries are Proposition 6 and Corollary 8. Our principal results on the modal dichotomy can be found in Theorems 9, 12 and Corollaries 4, 11. In fact, it is important to point out that the key result of this paper that leads to the modal dichotomy which is essential in the analysis of overdamping is Proposition 7 which relates to certain eigenvalue bounds. Moreover, these bounds and the dichotomy come from some general results that we derive in Appendix 9 on the perturbation theory for matrices with a large imaginary part.

Organization of the paper. The rest of the paper is organized as follows. Section 2 sets up and discusses the Lagrangian framework approach and model used in this paper to study the dissipative properties of two-component composite systems with a high-loss and a lossless component. In Section 3 we apply the developed approach to an electric circuit showing all key features of the method. Sections 4–6 are devoted to a precise formulation of all significant results in the form of Theorems, Propositions, and so on (with Section 7 providing the proofs of these results). More specifically, Section 4 discusses our extension of the virial theorem and equipartition of energy from classical mechanics for conservative Lagrangian systems to dissipative systems. Section 5 is on the spectral analysis of the system eigenmodes including spectral symmetries, modal dichotomy, and perturbation theory in the high-loss regime. Section 6 contains our analysis of the overdamping phenomena including selective overdamping. Appendices 8 and 9 contain results we need that we believe will be of use more generally in studying dissipative dynamical systems, but especially for composite systems with high-loss and lossless components. In particular, Appendix 9 is on the perturbation theory for matrices with a large imaginary part. Finally, Appendices 10 and 11 review the virial theorem from classical mechanics from the energetic point of view.

2 Model setup and discussion

2.1 Integrating the dissipation into the Lagrangian frameworks

The Lagrangian and Hamiltonian frameworks have numerous well known advantages in describing the evolution of physical systems. These advantages include, in particular, the universality of the mathematical structure, the flexibility in the choice of

variables and the incorporation of the symmetries with the corresponding conservation laws. Consequently, it seems quite natural and attractive to integrate dissipation into the Lagrangian and Hamiltonian frameworks. We are particularly interested in integrating dissipation into systems with gyroscopic features such as dielectric media with magnetic components and electrical networks.

We start with setting up the Lagrangian and Hamiltonian structures similarly to those in [FigVit1] and [FigSch2]. In particular, we use matrices to efficiently handle possibly large number of degrees of freedom. Since we are interested in systems evolving linearly the Lagrangian \mathcal{L} is assumed to be a quadratic function (bilinear form) of the coordinates $Q = [q_r]_{r=1}^N$ (column vector) and their time derivatives \dot{Q} , that is

$$\mathcal{L} = \mathcal{L}(Q, \dot{Q}) = \frac{1}{2} \begin{bmatrix} \dot{Q} \\ Q \end{bmatrix}^T M_L \begin{bmatrix} \dot{Q} \\ Q \end{bmatrix}, \quad M_L = \begin{bmatrix} \alpha & \theta \\ \theta^T & -\eta \end{bmatrix}, \quad (1)$$

where T denotes the matrix transposition operation, and α, η and θ are $N \times N$ -matrices with real-valued entries. In addition to that, we assume α, η to be symmetric matrices which are positive definite and positive semidefinite, respectively, and we assume θ is skew-symmetric, that is

$$\alpha = \alpha^T > 0, \quad \eta = \eta^T \geq 0, \quad \theta^T = -\theta. \quad (2)$$

Notice that only the skew-symmetric part of θ in (1) matters for the system dynamics because of the form of the Euler-Lagrange equations (5) and so without loss of generality we can assume as we have that the matrix θ is skew-symmetric. In fact, the symmetric part of the matrix θ if any alters the Lagrangian by a complete time derivative and consequently can be left out as we have done.

The Lagrangian (1) can be written in the energetic form

$$\mathcal{L} = \mathcal{T} - \mathcal{V}, \quad (3)$$

where

$$\mathcal{T} = \mathcal{T}(\dot{Q}, Q) = \frac{1}{2} \dot{Q}^T \alpha \dot{Q} + \frac{1}{2} \dot{Q}^T \theta Q, \quad \mathcal{V} = \mathcal{V}(\dot{Q}, Q) = \frac{1}{2} Q^T \eta Q - \frac{1}{2} \dot{Q}^T \theta Q \quad (4)$$

are interpreted as the kinetic and potential energy, respectively. By Hamilton's principle the dynamics of the system are governed by the Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{Q}} \right) - \frac{\partial \mathcal{L}}{\partial Q} = 0, \quad (5)$$

which are the following second-order ordinary differential equation (ODEs)

$$\alpha \ddot{Q} + 2\theta \dot{Q} + \eta Q = 0, \quad (6)$$

The above second-order ODEs can be turned in the first-order ODEs with help of the Hamiltonian function defined by the Lagrangian through the Legendre transformation

$$\mathcal{H} = \mathcal{H}(P, Q) = P^T \dot{Q} - \mathcal{L}(Q, \dot{Q}), \text{ where } P = \frac{\partial \mathcal{L}}{\partial \dot{Q}} = \alpha \dot{Q} + \theta Q. \quad (7)$$

Hence,

$$\dot{Q} = \alpha^{-1}(P - \theta Q) \quad (8)$$

and, consequently

$$\mathcal{H}(P, Q) = \frac{1}{2} \left[(P - \theta Q)^T \alpha^{-1} (P - \theta Q) + Q^T \eta Q \right] = \frac{1}{2} \dot{Q}^T \alpha \dot{Q} + \frac{1}{2} Q^T \eta Q. \quad (9)$$

In particular, the Hamiltonian is the sum of the kinetic and potential energy, that is

$$\mathcal{H} = \mathcal{T} + \mathcal{V}. \quad (10)$$

We remind that the Hamiltonian function $\mathcal{H}(P, Q)$ is interpreted as the system energy which is a conserved quantity, that is

$$\partial_t \mathcal{H}(P, Q) = 0. \quad (11)$$

The function $\mathcal{H}(P, Q)$ defined by (9) is a quadratic form associated with a matrix M_H through the relation

$$\mathcal{H}(P, Q) = \frac{1}{2} \begin{bmatrix} P \\ Q \end{bmatrix}^T M_H \begin{bmatrix} P \\ Q \end{bmatrix}, \quad (12)$$

where M_H is the $2N \times 2N$ matrix having the block form

$$M_H = \begin{bmatrix} \alpha^{-1} & -\alpha^{-1}\theta \\ -\theta^T \alpha^{-1} & \theta^T \alpha^{-1} \theta + \eta \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 \\ -\theta^T & \mathbf{1} \end{bmatrix} \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \eta \end{bmatrix} \begin{bmatrix} \mathbf{1} & -\theta \\ 0 & \mathbf{1} \end{bmatrix}, \quad (13)$$

where $\mathbf{1}$ is the identity matrix. Notice that the representations (12), (13) combined with the inequalities (2) imply

$$\mathcal{H}(P, Q) \geq 0 \text{ and } M_H = M_H^T \geq 0. \quad (14)$$

The Hamiltonian evolution equations then take the form

$$\partial_t \begin{bmatrix} P \\ Q \end{bmatrix} = J M_H \begin{bmatrix} P \\ Q \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix}, \quad (15)$$

Observe that the symplectic matrix J satisfies the following relations

$$J^2 = -\mathbf{1}, \quad J = -J^T. \quad (16)$$

The Rayleigh dissipation function. Up to now we have dealt with a conservative system satisfying the energy conservation (11). Let us introduce now dissipative forces using Rayleigh's method described in [Pars, Sec. 10.11, 10.12], [Gant, Sec. 8, 9, 46]. The *Rayleigh dissipation function* \mathcal{R} is defined as a quadratic function of the generalized velocities, namely

$$\mathcal{R} = \mathcal{R}(\dot{Q}) = \frac{1}{2} \dot{Q}^T \beta R \dot{Q}, \quad R \neq 0, \quad R = R^T \geq 0, \quad \beta \geq 0, \quad (17)$$

where the scalar β is a dimensionless *loss parameter* which we introduce to scale the intensity of dissipation. In particular, R is an $N \times N$ symmetric matrix with real-valued entries and, most importantly, is positive semidefinite.

The dissipation is then introduced through the following general Euler-Lagrange equations of motion with forces

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{Q}} \right) - \frac{\partial \mathcal{L}}{\partial Q} = - \frac{\partial \mathcal{R}}{\partial \dot{Q}} + F, \quad (18)$$

where $\frac{\partial \mathcal{R}}{\partial \dot{Q}}$ are generalized dissipative forces and $F = F(t)$ is an external force, yielding the following second-order ODEs

$$\alpha \ddot{Q} + (2\theta + \beta R) \dot{Q} + \eta Q = F. \quad (19)$$

2.2 The Lagrangian system and two-component composite model

Now by a linear (dissipative) *Lagrangian system* we mean a system whose state is described by a time-dependent $Q = Q(t)$ taking values in the Hilbert space \mathbb{C}^N with the standard inner product (\cdot, \cdot) (i.e., $(a, b) = a^* b$, where $*$ denotes the conjugate transpose, i.e., $a^* = \bar{a}^T$) whose dynamics are governed by the ODEs (19).

The energy balance equation for any such state, which follows from (19), is

$$\frac{d\mathcal{H}}{dt} = -2\mathcal{R} + \text{Re}(\dot{Q}, F), \quad (20)$$

where $\mathcal{H} = \mathcal{T} + \mathcal{V}$ as in (10) but now instead of (4) we have

$$\begin{aligned} \mathcal{T} &= \mathcal{T}(\dot{Q}, Q) = \frac{1}{2} (\dot{Q}, \alpha \dot{Q}) + \frac{1}{2} \text{Re}(\dot{Q}, \theta Q), \\ \mathcal{V} &= \mathcal{V}(\dot{Q}, Q) = \frac{1}{2} (Q, \eta Q) - \frac{1}{2} \text{Re}(\dot{Q}, \theta Q), \\ \mathcal{R} &= \mathcal{R}(\dot{Q}) = \frac{1}{2} (\dot{Q}, \beta R \dot{Q}). \end{aligned} \quad (21)$$

We continue to interpret \mathcal{T} , \mathcal{V} as the kinetic and potential energies, \mathcal{H} as the system energy, and $2\mathcal{R}$ as the dissipated power. The term $\text{Re}(\dot{Q}, F)$ is interpreted as the rate of work done by the force F .

The physical significance of the various energetic terms \mathcal{E} , where $\mathcal{E} \in \{\mathcal{H}, \mathcal{T}, \mathcal{V}, \mathcal{R}\}$, is that for a complex-valued state $Q = Q_1 + iQ_2$ with real-valued Q_1, Q_2 , F both Q_1 and Q_2 are also states representing physical solutions and $\mathcal{E}(\dot{Q}, Q) = \mathcal{E}(\dot{Q}_1, Q_1) + \mathcal{E}(\dot{Q}_2, Q_2)$. These latter two terms in the sum reduce to the previous definitions of the energetic term \mathcal{E} above in (4), (9), or (17) for real-valued vector quantities.

We now introduce an important quantity – the *loss fraction* δ_R . It is defined as the ratio of the rank of the matrix R to the total degrees of freedom N of the system

$$\text{loss fraction: } \delta_R = \frac{N_R}{N}, \quad N_R = \text{rank } R. \quad (22)$$

From our hypothesis $R \neq 0$ it follows that $0 < \delta_R \leq 1$. We consider the dissipative Lagrangian system to be a model of a two-component composite with a lossy and a lossless component whenever the loss fraction condition

$$\text{loss fraction condition: } 0 < \delta_R < 1 \quad (23)$$

is satisfied. We then associate the range of the operator R , i.e., $\text{Ran } R$, with the lossy component of the system and consider the lossy component to be highly lossy when $\beta \gg 1$, i.e., the high-loss regime. Our paper is focused on this case.

On the eigenmodes and the quality factor. To study the dissipative properties of the dissipative Lagrangian system (19) in the high-loss regime $\beta \gg 1$, one can consider the eigenmodes of the system and their quality factor in this regime.

An eigenmode of this system (19) is defined as a nonzero solution of the ODEs (19) with no forcing, i.e., $F = 0$, having the form $Q(t) = qe^{-i\zeta t}$, where $\text{Re } \zeta$ and $-\text{Im } \zeta$ are called its frequency and damping factor, respectively. Since the system is dissipative, i.e., dissipates energy as interpreted from the energy balance equation (20), the damping factor must satisfy

$$0 \leq 2\mathcal{R} = -\frac{d\mathcal{H}}{dt} = -2 \text{Im } \zeta \mathcal{H} \quad (24)$$

which implies that $-\text{Im } \zeta \geq 0$.

Thus, an eigenmode is a state of the dissipative Lagrangian system which is in damped harmonic motion and the motion is oscillatory provided $\text{Re } \zeta \neq 0$. An important quantity which characterizes the quality of such a damped oscillation is the quality factor Q_ζ (i.e., Q -factor) of the eigenmode which is defined as the reciprocal of the relative rate of energy dissipation per temporal cycle, i.e.,

$$Q_\zeta = 2\pi \frac{\text{energy stored in system}}{\text{energy lost per cycle}} = |\text{Re } \zeta| \frac{\mathcal{H}}{-\frac{d\mathcal{H}}{dt}}. \quad (25)$$

It follows immediately from this definition and (24) that Q_ζ depends only on the complex frequency ζ of the eigenmode, in particular,

$$Q_\zeta = -\frac{1}{2} \frac{|\text{Re } \zeta|}{\text{Im } \zeta}, \quad (26)$$

with the convention $Q_\zeta = +\infty$ if $\text{Im } \zeta = 0$.

2.3 The canonical system

Now the matrix Hamilton form of the Euler-Lagrange equation involving Rayleigh dissipative forces and external force (18) reads

$$\partial_t u = \left(J - \begin{bmatrix} \beta R & 0 \\ 0 & 0 \end{bmatrix} \right) M_H u + \begin{bmatrix} F \\ 0 \end{bmatrix}, \quad u = \begin{bmatrix} P \\ Q \end{bmatrix}. \quad (27)$$

It is important to recognize that the evolution equation (27) are not quite yet of the desired form which is the most suitable for the spectral analysis. Advancing general ideas of the Hamiltonian treatment of dissipative systems developed in [FigSch2] we can factor the matrix M_H as

$$M_H = K^T K, \quad (28)$$

where the matrix K is the block matrix

$$K = \begin{bmatrix} K_p & 0 \\ 0 & K_q \end{bmatrix} \begin{bmatrix} \mathbf{1} & -\theta \\ 0 & \mathbf{1} \end{bmatrix} = \begin{bmatrix} K_p & -K_p \theta \\ 0 & K_q \end{bmatrix}, \quad (29)$$

$$K_p = \sqrt{\alpha}^{-1}, \quad K_q = \sqrt{\eta} \quad (30)$$

which manifestly takes into account the gyroscopic term θ . Here $\sqrt{\alpha}$ and $\sqrt{\eta}$ denote the unique positive semidefinite square roots of the matrices α and η , respectively. In particular, it follows from the properties (2) and the proof of [ReSi1, §VI.4, Theorem VI.9] that K_p, K_q are $N \times N$ matrices with real-valued entries with the properties

$$K_p = K_p^T > 0, \quad K_q = K_q^T \geq 0. \quad (31)$$

The introduction of the matrix K according to [FigSch2] is intimately related to the introduction of force variables

$$v = K u. \quad (32)$$

Consequently, recasting the evolution equation (27) in terms of the force variables v yields the desired canonical form, namely

$$\partial_t v = -iA(\beta)v + f, \quad \text{where } A(\beta) = \Omega - i\beta B, \quad \beta \geq 0, \quad (33)$$

and

$$\Omega = iKJK^T = \begin{bmatrix} \Omega_p & -i\Phi^T \\ i\Phi & 0 \end{bmatrix}, \quad B = K \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} K^T = \begin{bmatrix} \tilde{R} & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} K_p F \\ 0 \end{bmatrix}, \quad (34)$$

$$\Omega_p = -i2K_p \theta K_p^T, \quad \Phi = K_q K_p^T, \quad \tilde{R} = K_p R K_p^T. \quad (35)$$

Observe that the $2N \times 2N$ matrices Ω , B in block form are Hermitian and positive semidefinite, respectively, that is

$$\Omega = \Omega^*, \quad B = B^* \geq 0. \quad (36)$$

Moreover, the matrix B does not have full rank and its rank is that of R , i.e.

$$\text{rank } B = \text{rank } R = N_R. \quad (37)$$

The *system operator* $A(\beta)$ has the following important properties

$$-\text{Im } A(\beta) = \beta B \geq 0, \quad \text{Re } A(\beta) = \Omega, \quad A(\beta)^* = -A(\beta)^T, \quad (38)$$

where the latter property comes from the fact that $\Omega^* = \Omega = -\Omega^T$ and $B^* = B = B^T$. Also, recall that for any square matrix M , one can write $M = \text{Re } M + i \text{Im } M$, where $\text{Re } M = \frac{M+M^*}{2}$ and $\text{Im } M = \frac{M-M^*}{2i}$ denote the real and imaginary parts of the matrix M , respectively.

Now by the *canonical system* we mean the system whose state is described by a time-dependent $v = v(t)$ taking values in the Hilbert space $H = \mathbb{C}^{2N}$ with the standard inner product (\cdot, \cdot) whose dynamics are governed by the ODEs (33). These ODEs will be referred to as the *canonical evolution equation*.

In this paper we will focus on the case when there is no forcing, i.e., $f = 0$ (or, equivalently, $F = 0$). In this case, the canonical system represents a dissipative dynamical system (since $-\text{Im } A(\beta) = \beta B \geq 0$) with the evolution governed by the semigroup $e^{-iA(\beta)t}$. As we shall see in this paper and as mentioned in the introduction, there are some serious advantages in studying this dissipative dynamical system, i.e., the canonical system, over the Lagrangian or Hamiltonian systems.

2.4 The energetic equivalence between the two systems

In a previous work [FigWell] of ours, we studied canonical systems whose states are solutions of a linear evolution equation in the canonical form (33) with system operator $A(\beta) = \Omega - i\beta B$, $\beta \geq 0$ having exactly the properties (36) in which the matrix B did not have full rank. In that paper, the canonical system was a simplified version of an abstract model of an oscillator damped retarded friction that modeled a two-component composite system with a lossy and lossless component. We showed that the state $v = v(t)$ of such a system satisfied the energy balance equation

$$\partial_t U[v(t)] = -W_{\text{dis}}[v(t)] + W[v(t)], \quad (39)$$

in which the system energy $U[v(t)]$, dissipated power $W_{\text{dis}}[v(t)]$, and $W[v(t)]$ the rate of work done by the force $f(t)$ were given by

$$U[v(t)] = \frac{1}{2}(v(t), v(t)), \quad W_{\text{dis}}[v(t)] = \beta(v(t), Bv(t)), \quad W[v(t)] = \text{Re}(v(t), f(t)). \quad (40)$$

An important result of this paper which follows immediately from the block form (34), (35) and the relation between the variables v, u, P, Q from (32), (27), (7)] is that a state Q of the Lagrangian system, whose dynamics are governed by second-order ODEs (19), and the corresponding state v of the canonical system, whose dynamics are governed by the canonical evolution equation (33), the energetics are equivalent in the sense

$$U[v] = \mathcal{T}(\dot{Q}, Q) + \mathcal{V}(\dot{Q}, Q), \quad W_{\text{dis}}[v] = 2\mathcal{R}(\dot{Q}), \quad W[v] = \text{Re}(\dot{Q}, F), \quad (41)$$

where, as you will recall from the energy balance equation (20), $\mathcal{H} = \mathcal{T} + \mathcal{V}$ was the system energy for the Lagrangian system.

On the eigenmodes and the quality factor. We will be interested in the eigenmodes of the canonical system (33) and their quality factor. An eigenmode of the canonical system is defined as a nonzero solutions of the ODEs (33) with no forcing, i.e., $f = 0$, having the form $v(t) = we^{-i\zeta t}$. The quality factor $Q[w]$ of such an eigenmode is defined as in [FigWel1] by

$$Q[w] = 2\pi \frac{\text{energy stored in system}}{\text{energy lost per cycle}} = |\text{Re } \zeta| \frac{U[v(t)]}{W_{\text{dis}}[v(t)]}. \quad (42)$$

By the energy balance equation (39) it follows that

$$Q[w] = -\frac{1}{2} \frac{|\text{Re } \zeta|}{\text{Im } \zeta}, \quad (43)$$

with the convention $Q[w] = +\infty$ if $\text{Im } \zeta = 0$.

Now given an eigenmode $Q(t) = qe^{-i\zeta t}$ of the Lagrangian system (19), it follows that the corresponding state v of the canonical system (33), related by (32), is an eigenmode of the form $v(t) = we^{-i\zeta t}$ (excluding the case $Ku = 0$, which can only occur if $\zeta = 0$). Importantly, the energetic equivalence (41) holds and their quality factors (26), (43) are equal, i.e.,

$$Q[w] = Q_{\zeta}. \quad (44)$$

A more detailed discussion on the eigenmodes of the two systems and the relationship between them can be found in Section 5.

3 Electric circuit example

One of the important applications of our methods described above is electric circuits and networks involving resistors representing losses. A general study of electric networks with losses can be carried out with the help of the Lagrangian approach, and that systematic study has already been carried out in this paper. For Lagrangian

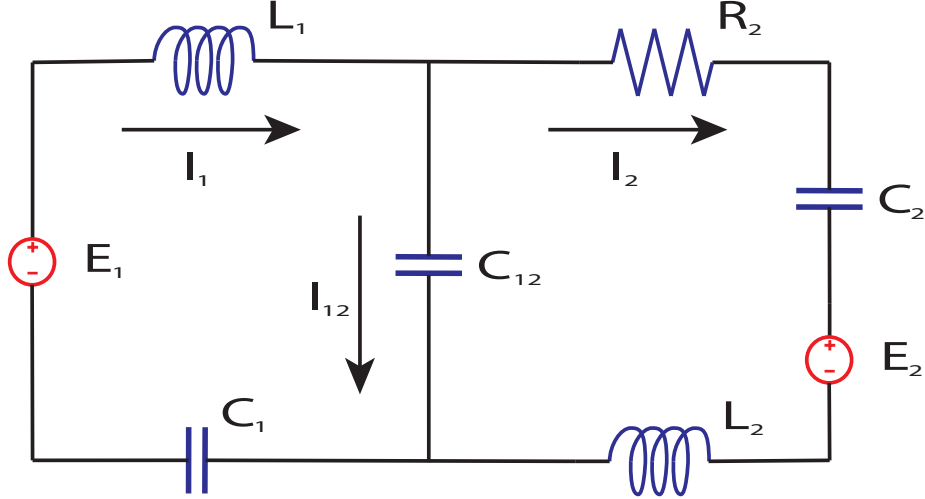


Figure 1: An electric circuit involving three capacitances C_1 , C_2 , C_{12} , two inductances L_1 , L_2 , a resistor R_2 , and two sources E_1 , E_2 . This electric circuit example fits within the framework of our model. Indeed, since resistors represent losses, this two component system consists of a lossy component and a lossless component – the right and left circuits, respectively.

treatment of electric networks and circuits we refer to [Gant, Sec. 9], [Gold, Sec. 2.5], [Pars].

We illustrate the idea and give a flavor of the efficiency of our methods by considering below a rather simple example of an electric circuit as in Fig. 1 with the assumptions

$$L_1, L_2, C_1, C_2, C_{12} > 0 \text{ and } R_2 \geq 0. \quad (45)$$

This example has the essential features of two component systems incorporating high-loss and lossless components.

The Lagrangian system. To derive evolution equations for the electric circuit in Fig. 1 we use a general method for constructing Lagrangians for circuits, [Gant, Sec. 9], that yields

$$\mathcal{T} = \frac{L_1}{2} \dot{q}_1^2 + \frac{L_2}{2} \dot{q}_2^2, \quad \mathcal{V} = \frac{1}{2C_1} q_1^2 + \frac{1}{2C_{12}} (q_1 - q_2)^2 + \frac{1}{2C_2} q_2^2, \quad \mathcal{R} = \frac{R_2}{2} \dot{q}_2^2, \quad (46)$$

where \mathcal{T} and \mathcal{V} are respectively the kinetic and the potential energies, $\mathcal{L} = \mathcal{T} - \mathcal{V}$ is the Lagrangian, and \mathcal{R} is the Rayleigh dissipative function. Notice that $I_1 = \dot{q}_1$ and $I_2 = \dot{q}_2$ are the currents. The general Euler-Lagrange equations of motion with forces are, [Gant, Sec. 8],

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{Q}} - \frac{\partial \mathcal{L}}{\partial Q} = -\frac{\partial \mathcal{R}}{\partial \dot{Q}} + F, \quad (47)$$

where Q are the charges and F the sources

$$Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad F = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad (48)$$

yielding from (45)–(48) the following second-order ODEs

$$\alpha \ddot{Q} + \beta R \dot{Q} + \eta Q = F, \quad (49)$$

with the dimensionless loss parameter

$$\beta = \frac{R_2}{\ell} \quad (\text{where } \ell > 0 \text{ is fixed and has same units as } R_2) \quad (50)$$

that scales the intensity of losses in the system, and

$$\alpha = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & \ell \end{bmatrix}, \quad \eta = \begin{bmatrix} \frac{1}{C_1} + \frac{1}{C_{12}} & -\frac{1}{C_{12}} \\ -\frac{1}{C_{12}} & \frac{1}{C_2} + \frac{1}{C_{12}} \end{bmatrix}. \quad (51)$$

Recall, the loss fraction δ_R defined in (22) is the ratio of the rank of the matrix R to the total degrees of freedom N of the system which in this case is

$$\text{loss fraction: } \delta_R = \frac{N_R}{N} = \frac{1}{2}, \quad N = 2, \quad N_R = \text{rank } R = 1. \quad (52)$$

Thus the Lagrangian system (49) has all the properties described in Sections 2.1, 2.2 and since the loss fraction condition (23), i.e., $0 < \delta_R < 1$, is satisfied then it is model of a two-component composite with a lossy and a lossless component.

The canonical system. Following the method in Section 2.3, we introduce the variables v defined by (7), (27), (32) in terms of the matrix K defined in (29), (30) as

$$v = Ku, \quad u = \begin{bmatrix} \alpha \dot{Q} \\ Q \end{bmatrix}, \quad (53)$$

$$K = \begin{bmatrix} K_p & 0 \\ 0 & K_q \end{bmatrix}, \quad K_p = \sqrt{\alpha}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{L_1}} & 0 \\ 0 & \frac{1}{\sqrt{L_2}} \end{bmatrix}, \quad K_q = \sqrt{\eta}.$$

As $\eta > 0$ is a 2×2 matrix in this case, then we can compute its square root explicitly using the formula

$$\sqrt{M} = \left(\sqrt{\text{Tr}(M) + 2\sqrt{\det(M)}} \right)^{-1} \left(\sqrt{\det(M)} \mathbf{1} + M \right), \quad (54)$$

where $\sqrt{\cdot}$ denotes the positive square root, which holds for any 2×2 matrix $M \geq 0$ with $M \neq 0$.

Consequently, in the Lagrangian system (49) in terms of the variable v is the desired canonical evolution equation from (33), namely,

$$\partial_t v = -iA(\beta)v + f, \quad \text{where } A(\beta) = \Omega - i\beta B, \quad \beta \geq 0, \quad (55)$$

and

$$\begin{aligned} \Omega &= \begin{bmatrix} 0 & -i\Phi^T \\ i\Phi & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \tilde{R} & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} K_p F \\ 0 \end{bmatrix}, \\ \Phi &= K_q K_p^T, \quad \tilde{R} = K_p R K_p^T = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\ell}{L_2} \end{bmatrix}. \end{aligned} \quad (56)$$

4 The virial theorem for dissipative systems and equipartition of energy

In this section we will introduce a new virial theorem for dissipative Lagrangian systems in terms of the eigenmodes of the system. This result generalizes the virial theorem from classical mechanics [Gold, pp. 83-86; §3.4], as discussed in Appendix 11, for conservative Lagrangian systems for these types of modes. Recall that the virial theorem is about the oscillatory transfer of energy from one form to another and its equipartition.

We can now state precisely and prove our generalization of the virial theorem to dissipative ($\beta \geq 0$) Lagrangian systems (19) and the equipartition of energy for the oscillatory eigenmodes. In fact, the proof in Section 7 shows that our theorem is true for more general system of ODEs in the form (19) since essentially all that really matters is that the energy balance equation (20) holds for the eigenmodes.

Theorem 1 (virial theorem) *If $Q(t) = qe^{-i\zeta t}$ is an eigenmode of the Lagrangian system (19) then the following identity holds if $\text{Re } \zeta \neq 0$:*

$$\mathcal{T}(\dot{Q}, Q) = \mathcal{V}(\dot{Q}, Q) - \left(\frac{\text{Im } \zeta}{\text{Re } \zeta} \right)^2 \text{Re}(\dot{Q}, \theta Q), \quad (57)$$

where $\mathcal{T}(\dot{Q}, Q)$ and $\mathcal{V}(\dot{Q}, Q)$ are the kinetic and potential energy, respectively, defined in (21). On the other hand, if $\text{Re } \zeta = 0$ then the identity (57) no longer holds and either $\zeta = 0$ or the identity $(\dot{Q}, \theta Q) = 0$ must hold.

Corollary 2 (energy equipartition) *For systems with $\theta = 0$, if $Q(t) = qe^{-i\zeta t}$ is an eigenmode of the Lagrangian system (19) with $\text{Re } \zeta \neq 0$, then the following identity holds*

$$\mathcal{T}(\dot{Q}, Q) = \mathcal{V}(\dot{Q}, Q). \quad (58)$$

In other words, for the oscillatory eigenmodes there is an equipartition of the system energy, i.e., $\mathcal{H} = \mathcal{T} + \mathcal{V}$, between their kinetic energy \mathcal{T} and their potential energy \mathcal{V} .

5 Spectral analysis of the system eigenmodes

In this section we study time-harmonic solutions to the Euler-Lagrange equation (19) which constitutes a subject of the spectral theory. An important subject of this section is the study of the relations between standard spectral theory and the quadratic pencil formulation as it arises naturally as the time Fourier transformation of the Euler-Lagrange evolution equation.

5.1 Standard versus pencil formulations of the spectral problems

As introduced in Section 2.2, the eigenmodes of the Lagrangian system are nonzero solutions of the ODEs (19) with $F = 0$ having the form $Q(t) = qe^{-i\zeta t}$. For a fixed β , these modes correspond to the solutions of the quadratic eigenvalue problem (QEP)

$$C(\zeta, \beta)q = 0, \quad q \neq 0 \quad (59)$$

for the quadratic matrix pencil in ζ

$$C(\zeta, \beta) = \zeta^2\alpha + (2\theta + \beta R)i\zeta - \eta. \quad (60)$$

The set of eigenvalues (spectrum) of the pencil $C(\cdot, \beta)$ is the set

$$\sigma(C(\cdot, \beta)) = \{\zeta \in \mathbb{C} : \det C(\zeta, \beta) = 0\}, \quad (61)$$

which are exactly those values ζ for which a solution to the QEP (59) exists. The spectral theory of polynomial operator pencils [Mar88] can be applied to study the eigenmodes but it has its disadvantages such as being more complicated than standard spectral theory. Thus an alternative approach to the spectral theory is desirable.

Often the alternative approach is to use the Hamiltonian system and consider its eigenmodes, that is, the nonzero solutions of the Hamiltonian equations (27) with $F = 0$ having the form $u(t) = ue^{-i\zeta t}$. These modes correspond to the solutions of the eigenvalue problem

$$Mu = -i\zeta u, \quad u \neq 0 \quad (62)$$

for the matrix

$$M(\beta) = \left(J - \begin{bmatrix} \beta R & 0 \\ 0 & 0 \end{bmatrix} \right) M_H. \quad (63)$$

An advantage to this approach is the simple correspondence via (7), (27) between the set of modes of the two systems, namely, the eigenmodes of the Hamiltonian system are solutions of (27) having the block form $u(t) = \begin{bmatrix} p \\ q \end{bmatrix} e^{-i\zeta t}$ in which $Q(t) = qe^{-i\zeta t}$ an eigenmode of the Lagrangian system and $p = (-i\zeta\alpha + \theta)q$. In particular, this

means the matrix $iM(\beta)$ and the pencil $C(\cdot, \beta)$ have the same eigenvalues and hence the same spectrum, i.e.,

$$\sigma(iM(\beta)) = \sigma(C(\cdot, \beta)). \quad (64)$$

A major disadvantage of this approach is that $M(\beta)$ is a non-self-adjoint matrix such that the standard theory of self-adjoint or dissipative operators does not apply without further transformation of the system, even in the absence of losses, i.e., $\beta = 0$. Instead, in this case it is the Krein spectral theory [Arnold, §42], [Yak], [YakSta] that is often used. But this theory is far more complex than the standard spectral theory and much harder to apply.

Our approach to these spectral problems which overcomes the disadvantages in the pencil or Krein spectral theory is to use the canonical system and consider its eigenmodes, that is, the nonzero solutions of the canonical evolution equation (33) with $f = 0$ having the form $v(t) = we^{-i\zeta t}$. These modes correspond to the solutions of the eigenvalue problem

$$A(\beta)w = \zeta w, \quad w \neq 0 \quad (65)$$

for the system operator $A(\beta) = \Omega - i\beta B$ with Hermitian matrices Ω , B with $B \geq 0$. The advantage of this approach is that the standard spectral theory can be used since $-iA(\beta)$ is a dissipative operator and when losses are absent $A(0) = \Omega$ is self-adjoint. This is a serious advantage since the spectral theory is significantly simpler and allows the usage of the deep and effective results from perturbation theory such as those developed in [FigWell] to study the modes of such canonical evolution equations having the general form (33). This is the approach we take in this paper to study spectral symmetries and the dissipative properties of the eigenmodes of the Lagrangian system such as the modal dichotomy and overdamping phenomenon.

Correspondence between spectral problems. We now conclude this section by summarizing the correspondence between the two main spectral problems of this paper, namely, between the standard eigenvalue problem (65) and the quadratic eigenvalue problem (59). We do this in the next corollary which uses the following proposition that tells us the characteristic matrix of the system operator $\zeta\mathbf{1} - A(\beta)$ can be factored in terms of the quadratic matrix pencil $C(\zeta, \beta)$.

First we introduce some notation that will be useful. The Hilbert space $H = \mathbb{C}^{2N}$ with standard inner product (\cdot, \cdot) can be decomposed as $H = H_p \oplus H_q$ into the orthogonal subspaces $H_p = \mathbb{C}^N$, $H_q = \mathbb{C}^N$ with orthogonal matrix projections

$$P_p = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_q = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{bmatrix}. \quad (66)$$

In particular, the matrices Ω , B , and $A(\beta)$ defined in (33)–(35) are block matrices already partitioned with respect to the decomposition $H = H_p \oplus H_q$ and any vector $w \in H$ can be represented uniquely in the block form

$$w = \begin{bmatrix} \varphi \\ \psi \end{bmatrix}, \quad \text{where } \varphi = P_p w, \quad \psi = P_q w. \quad (67)$$

Then with respect to this decomposition we have the following results.

Proposition 3 *If $\zeta \neq 0$ then*

$$\zeta \mathbf{1} - A(\beta) = \begin{bmatrix} K_p & \zeta^{-1} \mathbf{i} \Phi^T \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \zeta^{-1} \mathbf{1} & 0 \\ 0 & \zeta \mathbf{1} \end{bmatrix} \begin{bmatrix} C(\zeta, \beta) & 0 \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} K_p^T & 0 \\ -\zeta^{-1} \mathbf{i} \Phi & \mathbf{1} \end{bmatrix}. \quad (68)$$

Corollary 4 (spectral equivalence) *For any $\zeta \in \mathbb{C}$,*

$$\det(\zeta \mathbf{1} - A(\beta)) = \frac{\det C(\zeta, \beta)}{\det \alpha}. \quad (69)$$

In particular, the system operator $A(\beta)$ and quadratic matrix pencil $C(\zeta, \beta)$ have the same spectrum, i.e.,

$$\sigma(A(\beta)) = \sigma(C(\cdot, \beta)). \quad (70)$$

Moreover, if $\zeta \neq 0$ then the following statements are true:

1. *If $A(\beta)w = \zeta w$ and $w \neq 0$ then*

$$w = \begin{bmatrix} -\mathbf{i} \zeta \sqrt{\alpha} q \\ \sqrt{\eta} q \end{bmatrix}, \quad \text{where } C(\zeta, \beta)q = 0, \quad q \neq 0. \quad (71)$$

2. *If $C(\zeta, \beta)q = 0$ and $q \neq 0$ then*

$$A(\beta)w = \zeta w, \quad \text{where } w = \begin{bmatrix} -\mathbf{i} \zeta \sqrt{\alpha} q \\ \sqrt{\eta} q \end{bmatrix} \neq 0. \quad (72)$$

5.2 On the spectrum of the system operator

In Section 5.1 it was shown that the study of the eigenmodes of the Lagrangian system (19) reduces to the quadratic eigenvalue problem (59) and this motivates a study of the spectrum $\sigma(C(\cdot, \beta))$ of the quadratic matrix pencil $C(\zeta, \beta)$ in (60). Corollary 4 makes it clear that we can instead study the eigenmodes of the canonical system (33) and, in particular, the system operator $A(\beta)$ spectrum satisfies $\sigma(A(\beta)) = \sigma(C(\cdot, \beta))$. The purpose of this section is to give a detailed analysis of the set $\sigma(A(\beta))$.

Recall, the system operator $A(\beta) = \Omega - \mathbf{i} \beta B$, $\beta \geq 0$ from (33) with $2N \times 2N$ matrices Ω , B has the fundamental properties (37), (38)

$$-\text{Im } A(\beta) = \beta B \geq 0, \quad \text{Re } A(\beta) = \Omega, \quad A(\beta)^* = -A(\beta)^T, \quad (73)$$

$$0 < \text{rank } B = N_R \leq N, \quad (74)$$

where $N_R = \text{rank } R$. These properties are particularly important in describing the spectrum $\sigma(A(\beta))$ of the system operator $A(\beta)$.

For instance, the next proposition on the spectrum for nondissipative ($\beta = 0$) Lagrangian systems (19) follows immediately from these properties which would otherwise not be exactly obvious for gyroscopic systems (i.e., $\theta \neq 0$).

Proposition 5 (real eigenfrequencies) *For nondissipative Lagrangian systems (19), that is when $\beta = 0$, all the eigenfrequencies ζ are real, and consequently any eigenmode evolution is of the form $Q(t) = qe^{-i\zeta t}$ with $\text{Im } \zeta = 0$.*

Proof. If $\beta = 0$ (i.e., no dissipation) then $A(0) = \Omega$ is a Hermitian matrix. Thus, the spectrum $\sigma(A(0))$ is a subset of \mathbb{R} . Hence, if $Q(t) = qe^{-i\zeta t}$ is an eigenmode of the Lagrangian systems (19) with $\beta = 0$ then by Corollary 4 we have $\zeta \in \sigma(A(0))$ and so $\text{Im } \zeta = 0$. This completes the proof. ■

In the next few sections we will give a deeper analysis of the spectral properties of $A(\beta)$ including spectral symmetries in Section 5.2.1 and the modal dichotomy in Sections 5.2.2, 5.2.3. In order to do so we must first introduce some notation. In the Hilbert space $H = \mathbb{C}^{2N}$, denote by b_j , $j = 1, \dots, N_R$ the nonzero eigenvalues of B (counting multiplicities) with the smallest denoted by

$$b_{\min} = \min_{1 \leq j \leq N_R} b_j. \quad (75)$$

In particular, the spectrum of B is

$$\sigma(B) = \{b_0, b_1, \dots, b_{N_R}\}, \quad (76)$$

where $b_0 = 0$.

Denote the largest eigenvalue of Ω by ω_{\max} . It follows from the fact that Ω is a Hermitian matrix which is skew-symmetric that

$$\omega_{\max} = \|\Omega\|, \quad (77)$$

where $\|\cdot\|$ denotes the operator norm on square matrices.

5.2.1 Spectral symmetry

The next proposition describes the spectral symmetries of the system operator A which follow from the property $A(\beta)^* = -A(\beta)^T$.

Proposition 6 (spectral symmetry) *The following statements are true:*

1. *The characteristic polynomial of $A(\beta)$ satisfies*

$$\overline{\det(-\bar{\zeta}\mathbf{1} - A(\beta))} = \det(\zeta I - A(\beta)) \quad (78)$$

for every $\zeta \in \mathbb{C}$. In particular, the spectrum $\sigma(A)$ of the system operator A has the symmetry

$$\sigma(A(\beta)) = \overline{-\sigma(A(\beta))}. \quad (79)$$

2. *If w is an eigenvector of the system operator A with corresponding eigenvalue ζ then \bar{w} is an eigenvector of A with corresponding eigenvalue $-\bar{\zeta}$.*
3. *If $\beta = 0$ (i.e., no dissipation) then $\det(-\zeta\mathbf{1} - A(0)) = \det(\zeta I - A(0))$ for every $\zeta \in \mathbb{C}$.*

5.2.2 Eigenvalue bounds and modal dichotomy

We will denote the discs centered at the eigenvalues of $-i\beta B$ with radius ω_{\max} by

$$D_j(\beta) = \{\zeta \in \mathbb{C} : |\zeta - (-i\beta b_j)| \leq \omega_{\max}\}, \quad 0 \leq j \leq N_R. \quad (80)$$

Two subsets of the spectrum $\sigma(A(\beta))$ of the system operator $A = \Omega - i\beta B$ which play a central role in our analysis are

$$\begin{aligned} \sigma_0(A(\beta)) &= \sigma(A(\beta)) \cap D_0(\beta), \\ \sigma_1(A(\beta)) &= \sigma(A(\beta)) \cap \bigcup_{j=1}^{N_R} D_j(\beta). \end{aligned} \quad (81)$$

Proposition 7 (eigenvalue bounds) *The following statements are true:*

1. *The eigenvalues of the system operator $A(\beta)$ lie in the union of the closed discs whose centers are the eigenvalues of $-i\beta B$ with radius ω_{\max} , that is,*

$$\sigma(A(\beta)) = \sigma_0(A(\beta)) \cup \sigma_1(A(\beta)). \quad (82)$$

2. *If $w \neq 0$ and $A(\beta)w = \zeta w$ then*

$$\operatorname{Re} \zeta = \frac{(w, \Omega w)}{(w, w)}, \quad -\operatorname{Im} \zeta = \beta \frac{(w, Bw)}{(w, w)} \geq 0. \quad (83)$$

3. *If ζ is an eigenvalue of $A(\beta)$ and $|\zeta| > \omega_{\max}$ then*

$$-\operatorname{Im} \zeta \geq \beta b_{\min} - \omega_{\max}. \quad (84)$$

Corollary 8 (spectral clustering) *The eigenvalues of the system operator $A(\beta) = \Omega - i\beta B$, $\beta \geq 0$ lie in the closed lower half of the complex plane, are symmetric with respect to the imaginary axis, and lie in the union of the closed discs whose centers are the eigenvalues of $-i\beta B$ with radius ω_{\max} . Moreover, if $\beta = 0$ (i.e., no dissipation) then the eigenvalues of $A(0) = \Omega$ are real and symmetric with respect to the origin.*

Theorem 9 (modal dichotomy I) *If $\beta > 2\frac{\omega_{\max}}{b_{\min}}$ then*

$$\sigma(A(\beta)) = \sigma_0(A(\beta)) \cup \sigma_1(A(\beta)), \quad \sigma_0(A(\beta)) \cap \sigma_1(A(\beta)) = \emptyset. \quad (85)$$

Furthermore, there exists unique invariant subspaces $H_{\ell\ell}(\beta)$, $H_{h\ell}(\beta)$ of the system operator $A(\beta) = \Omega - i\beta B$ with the properties

$$\begin{aligned} (i) \quad & H = H_{\ell\ell}(\beta) \oplus H_{h\ell}(\beta); \\ (ii) \quad & \sigma(A(\beta)|_{H_{\ell\ell}(\beta)}) = \sigma_0(A(\beta)), \quad \sigma(A(\beta)|_{H_{h\ell}(\beta)}) = \sigma_1(A(\beta)), \end{aligned} \quad (86)$$

where $H = \mathbb{C}^{2N}$. Moreover, the dimensions of these subspaces satisfy

$$\dim H_{h\ell}(\beta) = N_R, \quad \dim H_{\ell\ell}(\beta) = 2N - N_R. \quad (87)$$

Definition 10 (high-loss susceptible subspace) For the system operator $A(\beta) = \Omega - i\beta B$ with $\beta > 2\frac{\omega_{max}}{b_{min}}$ we will call its N_R -dimensional invariant subspace $H_{h\ell}(\beta)$ the high-loss susceptible subspace. We will call its $(2N - N_R)$ -dimensional invariant subspace $H_{\ell\ell}(\beta)$ the low-loss susceptible subspace.

Our reasoning for the definitions of these subspaces is clarified with the following corollary.

Corollary 11 (high-loss subspace: dissipative properties) If $\beta > 2\frac{\omega_{max}}{b_{min}}$ then

$$\begin{aligned}\sigma(A(\beta)|_{H_{\ell\ell}(\beta)}) &= \{\zeta \in \sigma(A(\beta)) : 0 \leq -\text{Im } \zeta \leq \omega_{max}\}, \\ \sigma(A(\beta)|_{H_{h\ell}(\beta)}) &= \{\zeta \in \sigma(A(\beta)) : -\text{Im } \zeta \geq \beta b_{min} - \omega_{max} > \omega_{max}\}.\end{aligned}\quad (88)$$

Furthermore, the quality factor (43) of any eigenmode of the canonical system (33) in the high-loss susceptible subspace $H_{h\ell}(\beta)$ satisfies

$$0 \leq \max_{\substack{w \text{ an eigenvector} \\ \text{of } A(\beta) \text{ in } H_{h\ell}(\beta)}} Q[w] \leq \frac{1}{2} \frac{\omega_{max}}{\beta b_{min} - \omega_{max}} < \frac{1}{2}.\quad (89)$$

In particular, as the losses go to ∞ the damping factor and quality factor of any such eigenmode goes to $+\infty$ and 0, respectively, that is,

$$\lim_{\beta \rightarrow \infty} \min_{\zeta \in \sigma(A(\beta)|_{H_{h\ell}(\beta)})} (-\text{Im } \zeta) = +\infty, \quad \lim_{\beta \rightarrow \infty} \max_{\substack{w \text{ an eigenvector} \\ \text{of } A(\beta) \text{ in } H_{h\ell}(\beta)}} Q[w] = 0.\quad (90)$$

We conclude this section with the following remarks. The spectrum of $A(\beta)$ restricted to the low-loss susceptible subspace $H_{\ell\ell}(\beta)$, that is, the set $\sigma(A(\beta)|_{H_{\ell\ell}(\beta)})$ in (88), is close to the real axis and actually coalesces to a finite set of real numbers as losses $\beta \rightarrow \infty$. This statement is made precise in the following section with Theorem 12 and the asymptotic expansions in (99). On the other hand, results on quality factor for the eigenmodes in $H_{\ell\ell}(\beta)$ is far more subtle than the results in Corollary 11 for the eigenmodes in the high-loss susceptible subspace $H_{h\ell}(\beta)$. For gyroscopic-dissipative systems considered in this paper, Corollary 11 above and Proposition 13 below give a partial description of the nature of the quality factor for large losses, i.e., $\beta \gg 1$, in the lossy component of the composite system. When gyroscopy is absent, i.e., $\theta = 0$, then a more complete analysis for quality factor can be carried out, which we have done in Section 6 of this paper in connection to our studies on the overdamping phenomenon. Consideration of quality factor and overdamping in dissipative systems with gyroscopy, however, requires a more subtle and detailed analysis that will be considered in a future work.

5.2.3 Modal dichotomy in the high-loss regime

We are interested in describing the spectrum $\sigma(A(\beta))$ of the system operator $A(\beta) = \Omega - i\beta B$, $\beta \geq 0$ in the high-loss regime, i.e., $\beta \gg 1$. We do this in this section by giving an asymptotic characterization, as $\beta \rightarrow \infty$, of the modal dichotomy as described in Theorem 9 and Corollary 11. In order to do so we need to give a spectral perturbation analysis of the matrix $A(\beta)$ as $\beta \rightarrow \infty$. Fortunately, this analysis has already been carried out in [FigWel1]. We now introduce the necessary notion and describe the results.

The Hilbert space $H = \mathbb{C}^{2N}$ with standard inner product (\cdot, \cdot) is decomposed into the direct sum of orthogonal invariant subspaces of the operator B , namely

$$H = H_B \oplus H_B^\perp, \quad \dim H_B = N_R, \quad (91)$$

where $H_B = \text{Ran } B$ (the range of B) is the loss subspace of dimension $N_R = \text{rank } B$ with orthogonal projection P_B and its orthogonal complement, $H_B^\perp = \text{Ker } B$ (the nullspace of B), is the no-loss subspace of dimension $2N - N_R$ with orthogonal projection P_B^\perp .

The operators Ω and B with respect to the direct sum (91) are the 2×2 block operator matrices

$$\Omega = \begin{bmatrix} \Omega_2 & \Theta \\ \Theta^* & \Omega_1 \end{bmatrix}, \quad B = \begin{bmatrix} B_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad (92)$$

where $\Omega_2 = P_B \Omega P_B|_{H_B} : H_B \rightarrow H_B$ and $B_2 = P_B B P_B|_{H_B} : H_B \rightarrow H_B$ are restrictions of the operators Ω and B respectively to loss subspace H_B whereas $\Omega_1 = P_B^\perp \Omega P_B^\perp|_{H_B^\perp} : H_B^\perp \rightarrow H_B^\perp$ is the restriction of Ω to complementary subspace H_B^\perp . Also, $\Theta : H_B^\perp \rightarrow H_B$ is the operator $\Theta = P_B \Omega P_B^\perp|_{H_B^\perp}$ whose adjoint is given by $\Theta^* = P_B^\perp \Omega P_B|_{H_B} : H_B \rightarrow H_B^\perp$.

The perturbation analysis in the high-loss regime $\beta \gg 1$ for the system operator $A(\beta) = \Omega - i\beta B$ described in [FigWel1, §VI.A, Theorem 5 & Proposition 11] introduces an orthonormal basis $\{\dot{w}_j\}_{j=1}^{2N}$ diagonalizing the self-adjoint operators Ω_1 and $B_2 > 0$ from (92) with

$$B_2 \dot{w}_j = b_j \dot{w}_j \text{ for } 1 \leq j \leq N_R; \quad \Omega_1 \dot{w}_j = \rho_j \dot{w}_j \text{ for } N_R + 1 \leq j \leq 2N, \quad (93)$$

where

$$\begin{aligned} b_j &= (\dot{w}_j, B_2 \dot{w}_j) = (\dot{w}_j, B \dot{w}_j) \text{ for } 1 \leq j \leq N_R; \\ \rho_j &= (\dot{w}_j, \Omega_1 \dot{w}_j) = (\dot{w}_j, \Omega \dot{w}_j) \text{ for } N_R + 1 \leq j \leq 2N. \end{aligned} \quad (94)$$

Then for $\beta \gg 1$ the system operator $A(\beta)$ is diagonalizable with basis of eigenvectors $\{w_j(\beta)\}_{j=1}^{2N}$ satisfying

$$A(\beta) w_j(\beta) = \zeta_j(\beta) w_j(\beta), \quad 1 \leq j \leq 2N, \quad \beta \gg 1 \quad (95)$$

which split into two distinct classes

$$\begin{aligned} \text{high-loss: } & \zeta_j(\beta), w_j(\beta), \quad 1 \leq j \leq N_R; \\ \text{low-loss: } & \zeta_j(\beta), w_j(\beta), \quad N_R + 1 \leq j \leq 2N, \end{aligned} \quad (96)$$

with the following properties.

The high-loss class: the eigenvalues have poles at $\beta = \infty$ whereas their eigenvectors are analytic at $\beta = \infty$, having the asymptotic expansions

$$\zeta_j(\beta) = -ib_j\beta + \rho_j + O(\beta^{-1}), \quad b_j > 0, \quad \rho_j \in \mathbb{R}, \quad w_j(\beta) = \hat{w}_j + O(\beta^{-1}), \quad 1 \leq j \leq N_R. \quad (97)$$

The vectors \hat{w}_j , $1 \leq j \leq N_R$ form an orthonormal basis of the loss subspace H_B and

$$B\hat{w}_j = b_j\hat{w}_j, \quad \rho_j = (\hat{w}_j, \Omega\hat{w}_j), \quad \text{for } 1 \leq j \leq N_R. \quad (98)$$

In particular, b_j , $j = 1, \dots, N_R$ are all the nonzero eigenvalues of B (counting multiplicities).

The low-loss class: the eigenvalues and eigenvectors are analytic at $\beta = \infty$, having the asymptotic expansions

$$\begin{aligned} \zeta_j(\beta) &= \rho_j - id_j\beta^{-1} + O(\beta^{-2}), \quad \rho_j \in \mathbb{R}, \quad d_j \geq 0, \\ w_j(\beta) &= \hat{w}_j + O(\beta^{-1}), \quad N_R + 1 \leq j \leq 2N. \end{aligned} \quad (99)$$

The vectors \hat{w}_j , $N_R + 1 \leq j \leq 2N$ form an orthonormal basis of the no-loss subspace H_B^\perp and

$$B\hat{w}_j = 0, \quad \rho_j = (\hat{w}_j, \Omega\hat{w}_j), \quad d_j = (\hat{w}_j, \Theta^* B_2^{-1} \Theta \hat{w}_j) \quad \text{for } N_R + 1 \leq j \leq 2N. \quad (100)$$

By [FigWell, §VI.A, Proposition 7] we know that the asymptotic formulas for the real and imaginary parts of the complex eigenvalues $\zeta_j(\beta)$ as $\beta \rightarrow \infty$ are given by

$$\text{high-loss: } \quad \text{Re } \zeta_j(\beta) = \rho_j + O(\beta^{-2}), \quad \text{Im } \zeta_j(\beta) = -b_j\beta + O(\beta^{-1}), \quad 1 \leq j \leq N_R; \quad (101)$$

$$\text{low-loss: } \quad \text{Re } \zeta_j(\beta) = \rho_j + O(\beta^{-2}), \quad \text{Im } \zeta_j(\beta) = -d_j\beta^{-1} + O(\beta^{-3}), \quad N_R + 1 \leq j \leq 2N.$$

Observe that the expansions (101) imply

$$\lim_{\beta \rightarrow \infty} \text{Im } \zeta_j(\beta) = -\infty \quad \text{for } 1 \leq j \leq N_R; \quad \lim_{\beta \rightarrow \infty} \text{Im } \zeta_j(\beta) = 0 \quad \text{for } N_R + 1 \leq j \leq 2N, \quad (102)$$

justifying the names high-loss and low-loss.

The following theorem is the goal of this section. It characterizes the spectrum $\sigma(A(\beta))$ of the system operator $A(\beta) = \Omega - i\beta B$ and the modal dichotomy from Theorem 9 and Corollary 11 in the high-loss regime $\beta \gg 1$ in terms of the high-loss and low-loss eigenvectors.

Theorem 12 (modal dichotomy II) *For β sufficiently large, the modal dichotomy occurs as in Theorem 9 and Corollary 11 with the following equalities holding:*

$$\begin{aligned}\sigma(A(\beta)|_{H_{\ell\ell}(\beta)}) &= \{\zeta_j(\beta) : N_R + 1 \leq j \leq 2N\}, \\ \sigma(A(\beta)|_{H_{h\ell}(\beta)}) &= \{\zeta_j(\beta) : 1 \leq j \leq N_R\},\end{aligned}\tag{103}$$

and

$$\begin{aligned}H_{\ell\ell}(\beta) &= \text{span}\{w_j(\beta) : N_R + 1 \leq j \leq 2N\}, \\ H_{h\ell}(\beta) &= \text{span}\{w_j(\beta) : 1 \leq j \leq N_R\}.\end{aligned}\tag{104}$$

In particular, $H_{h\ell}(\beta)$ and $H_{\ell\ell}(\beta)$, the high-loss and low-loss susceptible subspaces of $A(\beta)$, respectively, have as a basis the high-loss eigenvectors $\{w_j(\beta)\}_{j=1}^{N_R}$ and the low-loss eigenvectors $\{w_j(\beta)\}_{j=N_R+1}^{2N}$, respectively.

In the next section we will study overdamping phenomena for Lagrangian systems (19). An important role in our analysis will be played by the eigenmodes $v_j(t, \beta) = w_j(\beta) e^{-i\zeta_j(\beta)t}$, $1 \leq j \leq 2N$ of the canonical system (33) which by the modal dichotomy split into the two distinct classes based on their dissipative properties:

$$\begin{aligned}\text{high-loss: } v_j(t, \beta) &= w_j(\beta) e^{-i\zeta_j(\beta)t}, \quad 1 \leq j \leq N_R; \\ \text{low-loss: } v_j(t, \beta) &= w_j(\beta) e^{-i\zeta_j(\beta)t}, \quad N_R + 1 \leq j \leq 2N.\end{aligned}\tag{105}$$

It should be emphasized here that the classification of these modes into high-loss and low-loss is based solely on the behavior of their damping factor in (102) as losses become large, i.e., as $\beta \rightarrow \infty$, and not necessarily on their quality factor which we will discuss at the end of this section. Moreover, the damping factor is related to energy loss by the energy balance equation (39), (40) satisfied by these modes which implies the dissipated power is

$$\begin{aligned}-\partial_t U[v_j(t, \beta)] &= -2 \text{Im} \zeta_j(\beta) U[v_j(t, \beta)], \quad 1 \leq j \leq 2N, \\ U[v_j(t, \beta)] &= \frac{1}{2} (w_j(\beta), w_j(\beta)) e^{2 \text{Im} \zeta_j(\beta)t}\end{aligned}\tag{106}$$

where $U[v_j(t, \beta)]$ defined in (40) was interpreted as the system energy. In particular, by (102), (106), and the fact $(\dot{w}_j, \dot{w}_j) = 1$ it follows that their system energy for any fixed $t > 0$ satisfies

$$\begin{aligned}\text{high-loss: } \lim_{\beta \rightarrow \infty} U[v_j(t, \beta)] &= 0, \quad 1 \leq j \leq N_R; \\ \text{low-loss: } \lim_{\beta \rightarrow \infty} U[v_j(t, \beta)] &= \frac{1}{2}, \quad N_R + 1 \leq j \leq 2N.\end{aligned}\tag{107}$$

This combined with (102), (106) is the justification for the names high-loss and low-loss modes.

Asymptotic formulas for the quality factor. The perturbation analysis in the high-loss regime $\beta \gg 1$ for the quality factor $Q[w_j(\beta)]$, $1 \leq j \leq 2N$ of the eigenmodes from (105) as given by (42), (43) has already been carried out in [FigWell, §IV A, Prop. 14]. We will now describe the results.

The quality factor $Q[w_j(\beta)]$, $1 \leq j \leq N_R$ for each high-loss eigenmode has a series expansion containing only odd powers of β^{-1} implying for $\beta \gg 1$ it is a nonnegative decreasing function in the loss parameter β which has the asymptotic formula as $\beta \rightarrow \infty$

$$\text{high-loss: } Q[w_j(\beta)] = \frac{1}{2} \frac{|\rho_j|}{b_j} \beta^{-1} + O(\beta^{-3}), \quad 1 \leq j \leq N_R. \quad (108)$$

It follows from this discussion that as $\beta \rightarrow \infty$

$$\text{high-loss: } Q[w_j(\beta)] \searrow 0, \quad 1 \leq j \leq N_R \quad (109)$$

(here we will use the notation $\searrow 0$ or $\nearrow +\infty$ to denote that a function of β is decreasing to 0 or increasing to $+\infty$, respectively, as $\beta \rightarrow \infty$).

The quality factor $Q[w_j(\beta)]$, $N_R + 1 \leq j \leq 2N$ for each low-loss eigenmode has two possibilities: (i) $Q[w_j(\beta)] = +\infty$ for $\beta \gg 1$; (ii) $Q[w_j(\beta)]$ has a series expansion containing only odd powers of β^{-1} implying either $Q[w_j(\beta)] \searrow 0$ or $Q[w_j(\beta)] \nearrow +\infty$ as $\beta \rightarrow \infty$. Finding necessary and sufficient conditions for when the cases occur is a subtle problem which is still open in general, but is solved completely in the next section on overdamping when $\theta = 0$ under the nondegeneracy condition $\text{Ker } \eta \cap \text{Ker } R = \{0\}$. We will now consider the cases $d_j \neq 0$ [cf. (110), (111)] or $\rho_j \neq 0$ (cf. Prop. 13). In the former case (which is the typical case see [FigWell, §IV A, Remark 9]), we have as $\beta \rightarrow \infty$ the asymptotic formula

$$\text{low-loss: } Q[w_j(\beta)] = \frac{1}{2} \frac{|\rho_j|}{d_j} \beta + O(\beta^{-1}), \quad \text{if } d_j \neq 0 \quad (110)$$

and so it follows in this case that as $\beta \rightarrow \infty$

$$\text{low-loss: } Q[w_j(\beta)] \searrow 0, \text{ if } \rho_j = 0 \text{ and } Q[w_j(\beta)] \nearrow +\infty, \text{ if } \rho_j \neq 0. \quad (111)$$

Asymptotic oscillatory low-loss modes. Now the low-loss modes from (105) with $\rho_j \neq 0$ will play a key role in the study of the selective overdamping phenomenon in Section 6.2. We call such modes the asymptotic oscillatory low-loss modes since if $\rho_j \neq 0$ then the limiting function $\lim_{\beta \rightarrow \infty} v_j(t, \beta) = \dot{w}_j e^{-i\rho_j t}$ (for fix t) is an oscillatory function in t with period $2\pi/\rho_j$. Thus as $\beta \rightarrow \infty$ we denote this by

$$\text{asymptotic oscillatory low-loss modes: } v_j(t, \beta) \sim \dot{w}_j e^{-i\rho_j t}, \quad \rho_j \neq 0. \quad (112)$$

From the quality factor discussion above for the low-loss modes the next proposition follows immediately, regardless of whether $d_j \neq 0$ or not.

Proposition 13 (Q-factor: asymptotic oscillatory low-loss modes) *Any low-loss eigenmodes from (105) with $\rho_j \neq 0$ has a quality factor $Q[w_j(\beta)]$ with the property that either $Q[w_j(\beta)] = +\infty$ for all $\beta \gg 1$ or $Q[w_j(\beta)] \nearrow +\infty$ as $\beta \rightarrow \infty$ (i.e., is an unbounded increasing function of the loss parameter β).*

6 Overdamping analysis

The phenomenon of *overdamping* (also called *heavy damping*) is best known for a simple damped oscillator. Namely, when the damping exceeds certain critical value all oscillations cease entirely, see, for instance, [Pain, Sec. 2]. In other words, if the damped oscillations are described by the exponential function $e^{-i\zeta t}$ with a complex constant ζ then in the case of overdamping (heavy damping) $\text{Re } \zeta = 0$. Our interest to overdamping is motivated by the fact that if an eigenmode becomes overdamped it can not resonate at any finite frequency. Consequently, the contribution of such a mode to losses at finite frequencies becomes minimal, and that provides a mechanism for the absorption suppression for systems composed of lossy and lossless components.

The treatment of overdamping for systems with many degrees of freedom involves a number of subtleties particularly in our case when the both lossy and lossless degrees of freedom are present, i.e., when the loss fraction condition (23) is satisfied. We will show that any Lagrangian system (19) with $\theta = 0$ can be overdamped with selective overdamping occurring whenever R does not have full rank, i.e., $N_R < N$, which corresponds to a model of a two-component composite with a high-loss and a low-loss component.

Let us be clear that we study overdamping in this paper only for the case $\theta = 0$ as the case $\theta \neq 0$ will differ significantly in the analysis and it is not expected that there exists such a critical value for damping when all oscillations cease entirely for any damped oscillation. A study of a proper generalization of overdamping in the case $\theta \neq 0$ will be carried out in a future publication.

We make here statements and provide arguments for overdamping for Lagrangian systems. We will study the phenomenon known as overdamping in this section under the following condition:

Condition 14 (no gyrotropy) *For our study of overdamping we assume henceforth that*

$$\theta = 0. \tag{113}$$

The following proposition is a key result that will be referenced often in our study of overdamping. It describes the spectrum of the $2N \times 2N$ matrices B, Ω in terms of the spectrum of the $N \times N$ matrices $\alpha^{-1}\eta, \alpha^{-1}R$.

Proposition 15 (spectra relations) For the $2N \times 2N$ matrices B, Ω and the $N \times N$ matrices α, η, R we have $\text{rank } B = \text{rank } R (= N_R)$ and

$$\begin{aligned} \det(\zeta \mathbf{1} - \Omega) &= \det(\zeta^2 \mathbf{1} - \alpha^{-1} \eta), \\ \det(\zeta \mathbf{1} - B) &= \zeta^N \det(\zeta \mathbf{1} - \alpha^{-1} R), \end{aligned} \quad (114)$$

for every $\zeta \in \mathbb{C}$. In particular, if b_{\min} and ω_{\max} denote the smallest nonzero eigenvalue and the largest eigenvalue of B and Ω , respectively, and ω_{\min} denotes the smallest positive eigenvalue of Ω then

$$\omega_{\max} = \sqrt{\max \sigma(\alpha^{-1} \eta)}, \quad \omega_{\min} = \sqrt{\min_{\lambda \in \sigma(\alpha^{-1} \eta), \lambda > 0} \lambda}, \quad (115)$$

and

$$b_{\min} = \min_{\lambda \in \sigma(\alpha^{-1} R), \lambda \neq 0} \lambda. \quad (116)$$

6.1 Complete and partial overdamping

To study whether overdamping is possible we need to determine conditions under which

$$\det(\zeta \mathbf{1} - A(\beta)) = 0, \quad \text{Re } \zeta = 0. \quad (117)$$

has a solution for the system operator $A(\beta) = \Omega - i\beta B$. But we know from Corollary 4 that $\det(\zeta \mathbf{1} - A(\beta)) = 0$ if and only if $\det C(\zeta, \beta) = 0$. Hence the solutions to (117) are the solution of

$$\det C(\zeta, \beta) = 0, \quad \zeta = -i\lambda, \quad \lambda \in \mathbb{R}. \quad (118)$$

Thus to study overdamping it suffices to study conditions under which the quadratic eigenvalue problem

$$C(-i\lambda, \beta)q = 0, \quad q \neq 0, \quad \lambda \in \mathbb{R}. \quad (119)$$

has a solution. To be explicit we have

$$C(-i\lambda, \beta) = -(\lambda^2 \alpha - \lambda \beta R + \eta). \quad (120)$$

As it turns out the key to studying overdamping will be to study zero sets of the family of quadratic forms

$$\mathbf{q}(q, \lambda) = (q, -C(-i\lambda, \beta)q) = \lambda^2 (q, \alpha q) - \lambda \beta (q, Rq) + (q, \eta q) \quad (121)$$

on certain subspaces of the Hilbert space $H_p = \mathbb{C}^N$. In particular, we find that

$$q \in \text{Ker } C(-i\lambda, \beta), \quad q \neq 0 \Rightarrow \mathbf{q}(q, \lambda) = 0 \text{ and} \quad (122)$$

$$\lambda = \frac{\beta (q, Rq)}{2 (q, \alpha q)} \pm \sqrt{\left[\frac{\beta (q, Rq)}{2 (q, \alpha q)} \right]^2 - \frac{(q, \eta q)}{(q, \alpha q)}}.$$

Thus we find that a necessary and sufficient condition for an eigenvalue ζ of the system operator $A(\beta)$ to satisfy $\text{Re } \zeta = 0$ is

$$\frac{\beta(q, Rq)}{2(q, \alpha q)} \geq \sqrt{\frac{(q, \eta q)}{(q, \alpha q)}}, \text{ for some } q \in \text{Ker } C(\zeta, \beta), q \neq 0. \quad (123)$$

Now there are some fundamental inequalities that play a key role in our over-damping analysis which are described in the following proposition. We will prove part of this proposition, the rest is proved in Section 7, since the proof is enlightening showing how the energy conservation law (20) and the virial theorem 1 can be used to derive these inequalities.

Proposition 16 (fundamental inequalities) *The following statements are true.*

1. For any $q \in \mathbb{C}^N$ with $q \neq 0$,

$$\omega_{max} \geq \sqrt{\frac{(q, \eta q)}{(q, \alpha q)}}. \quad (124)$$

Moreover, if η^{-1} exists then

$$\sqrt{\frac{(q, \eta q)}{(q, \alpha q)}} \geq \omega_{min}. \quad (125)$$

2. If the matrix R has full rank, i.e., $N_R = N$, then for any $q \in \mathbb{C}^N$ with $q \neq 0$,

$$\frac{\beta(q, Rq)}{2(q, \alpha q)} \geq \frac{\beta}{2} b_{min}. \quad (126)$$

3. For any $q \in \text{Ker } C(\zeta, \beta)$, $q \neq 0$ with $\text{Re } \zeta \neq 0$,

$$\frac{\beta(q, Rq)}{2(q, \alpha q)} = -\text{Im } \zeta < |\zeta| = \sqrt{\frac{(q, \eta q)}{(q, \alpha q)}} \leq \omega_{max}. \quad (127)$$

4. If $\det C(\zeta, \beta) = 0$ [or equivalently, $\zeta \in \sigma(A(\beta))$] then $\text{Re } \zeta = 0$ whenever any one of the following two inequalities is satisfied: (i) $-\text{Im } \zeta \geq \omega_{max}$ or (ii) η^{-1} exists and $|\zeta| < \omega_{min}$.

5. If $Q(t) = qe^{-i\zeta t}$ is an eigenmode of the Lagrangian system (19) then energy equipartition, i.e., $\mathcal{T}(\dot{Q}, Q) = \mathcal{V}(\dot{Q}, Q)$, can only hold if $|\zeta| \leq \omega_{max}$ or, in the case η^{-1} exists, if $\omega_{min} \leq |\zeta| \leq \omega_{max}$.

Proof. The first two statements are proved in Section 7. The third statement will now be proved using the energy conservation law (20) and the virial theorem 1, or more specifically, Corollary 2 on the equipartition of energy. Suppose $q \in \text{Ker } C(\zeta, \beta)$, $q \neq 0$ with $\text{Re } \zeta \neq 0$. Then $Q(t) = qe^{-i\zeta t}$ is an eigenmode of the Lagrangian system (19). It follows from (20) and Corollary 2 that for the system energy $\mathcal{H} = \mathcal{T}(\dot{Q}, Q) + \mathcal{V}(\dot{Q}, Q)$ we have

$$2\mathcal{R}(\dot{Q}) = -\frac{d}{dt}\mathcal{H} = -2\text{Im } \zeta \mathcal{H}, \quad (128)$$

$$\mathcal{T}(\dot{Q}) = \mathcal{V}(Q), \quad (129)$$

where $\mathcal{T}(\dot{Q}, Q) = \frac{1}{2}(\dot{Q}, \eta\dot{Q})$ and $\mathcal{V}(\dot{Q}, Q) = \frac{1}{2}(Q, \eta Q)$ are the kinetic and potential energy, respectively, of this state Q and $2\mathcal{R}(\dot{Q}) = (\dot{Q}, \beta R\dot{Q})$ is the dissipated power. Notice that for $\mathcal{E} \in \{\mathcal{H}, \mathcal{T}, \mathcal{V}, \mathcal{R}\}$ we have $\mathcal{E}(t) = e^{2\text{Im } \zeta t} \mathcal{E}(0)$. It now follows from this and (128), (129) that

$$\frac{\beta(q, Rq)}{2(q, \alpha q)} = \frac{\mathcal{R}(\dot{Q})}{2\mathcal{T}(\dot{Q}, Q)} = \frac{-\text{Im } \zeta \mathcal{H}}{\mathcal{H}} = -\text{Im } \zeta, \quad (130)$$

$$\sqrt{\frac{(q, \eta q)}{(q, \alpha q)}} = \sqrt{|\zeta|^2 \frac{\mathcal{V}(\dot{Q}, Q)}{\mathcal{T}(\dot{Q}, Q)}} = |\zeta|. \quad (131)$$

The proof of the third statement now follows from these two equalities and the inequality (124). The fourth statement follows immediately from the inequality (127) and the inequality (125).

The fifth statement follows from the fact that if $\mathcal{T}(\dot{Q}, Q) = \mathcal{V}(\dot{Q}, Q)$ then the equality (131) holds, regardless of whether $\text{Re } \zeta \neq 0$ or not as long as $\zeta \neq 0$, and so the result follows immediately from the inequalities (124) and (125) if $\zeta \neq 0$. And since the existence of η^{-1} prohibits $\zeta = 0$, the proof now follows. This completes the proof. ■

From the inequalities (126), (127) one can see clearly that if the matrix R has full rank then all the eigenvalues of $A(\beta)$ are purely imaginary once

$$\beta \geq 2 \frac{\omega_{\max}}{b_{\min}}. \quad (132)$$

In other words, (132) is a sufficient condition for all of the eigenmodes of the canonical system (33) [or, equivalently, the Lagrangian system (19)] to be overdamped, provided R has full rank.

But it is also clear that this argument must be refined if R does not have full rank, i.e., $N_R < N$, since then $\det R = 0$ and hence $\min \sigma(\alpha^{-1}R) = 0$. The refinement we use is perturbation theory. In particular, we use our results in Sections 5.2.1, 5.2.3, 4 on the modal dichotomy and the virial theorem to derive our main results on overdamping. Our argument is essentially based on determining which of the eigenmodes cannot maintain the energy equipartition in Corollary 2, i.e., the equality between the kinetic and potential energy, when β is sufficiently large by either using the inequalities in Corollary 11 or using the asymptotic expansions in Section 5.2.3 for the eigenmodes in (105). This is where the fifth statement in Proposition 16 is relevant.

We derive the following results on overdamping using the inequalities in Proposition 16.

Theorem 17 (complete overdamping) *If $\beta \geq 2\frac{\omega_{max}}{b_{min}}$ and R has full rank, i.e., $N_R = N$, then all the eigenmodes of the canonical system (33) are overdamped, i.e., for the system operator $A(\beta) = \Omega - i\beta B$ we have*

$$\sigma(A(\beta)) = \{\zeta \in \sigma(A(\beta)) : \operatorname{Re} \zeta = 0\}. \quad (133)$$

Example 18 (overdamped regime is optimal) *The following example shows for the class of system operators $A = \Omega - i\beta B$ under consideration in this paper, the regime $\beta > 2\frac{\omega_{max}}{b_{min}}$ is optimal for guaranteeing overdamping in the sense that we can always find an example of a system operator satisfying $\beta < 2\frac{\omega_{max}}{b_{min}}$ with β as close to $2\frac{\omega_{max}}{b_{min}}$ as we like such that the set $\{\zeta \in \sigma(A(\beta)) : \operatorname{Re} \zeta = 0\}$ is empty. In particular, it is proven using the following family of system operators*

$$\begin{aligned} \Omega &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, & B &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ A(\beta) &= \Omega - i\beta B, & \beta &\geq 0, \end{aligned} \quad (134)$$

derived from the system with one degree of freedom having Lagrangian and Rayleigh dissipation function

$$\begin{aligned} \mathcal{L} = \mathcal{L}(Q, \dot{Q}) &= \frac{1}{2} \begin{bmatrix} \dot{Q} \\ Q \end{bmatrix}^T M_L \begin{bmatrix} \dot{Q} \\ Q \end{bmatrix}, & M_L &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ R &= \frac{1}{2} (\dot{Q}, \beta R \dot{Q}), & R &= 1, \end{aligned} \quad (135)$$

which is just is the Lagrangian for a one degree-of-freedom spring-mass-damper system with mass 1, spring constant 1, and viscous damping coefficient β . Then we have

$$\begin{aligned} b_{min} &= 1, & \omega_{max} &= 1, \\ \sigma(A(\beta)) &= \left\{ \xi_{\pm}(\beta) = -i\frac{\beta}{2} \pm \sqrt{1 - \left(\frac{\beta}{2}\right)^2} \right\}, \end{aligned} \quad (136)$$

and therefore

$$\begin{aligned}
&\text{if } \beta > 2 \text{ then } \{\zeta \in \sigma(A(\beta)) : \text{Re } \zeta = 0\} = \sigma(A(\beta)), \\
&\text{if } \beta = 2 \text{ then } \sigma(A(\beta)) = \{-i\}, \\
&\text{if } \beta < 2 \text{ then } \{\zeta \in \sigma(A(\beta)) : \text{Re } \zeta = 0\} = \emptyset.
\end{aligned} \tag{137}$$

In this case, at the boundary of the overdamping regime, where $\beta = 2\frac{\omega_{max}}{\nu_{min}}$, we have critical damping of the system.

Now we want to answer the question of which modes of the canonical system (33) are overdamped in the case when R does not have full rank, i.e., $N_R < N$. The next theorem gives us a partial answer to the question.

Theorem 19 (partial overdamping) *If $\beta > 2\frac{\omega_{max}}{\nu_{min}}$ and $N_R < N$ then the modal dichotomy occurs as in Theorem 9 and Corollary 11. In addition, any eigenmode of the canonical system (33) in the invariant subspace $H_{h\ell}(\beta)$ (the high-loss susceptible subspace) is overdamped, i.e.,*

$$\sigma(A(\beta)|_{H_{h\ell}(\beta)}) \subseteq \{\zeta \in \sigma(A(\beta)) : \text{Re } \zeta = 0\}. \tag{138}$$

Now we ask the important and natural question: Can any of the modes of the system operator $A(\beta)$ in the low-loss susceptible subspace $H_{\ell\ell}(\beta)$ be overdamped when R from the Rayleigh dissipative function \mathcal{R} in (17) does not have full rank, i.e., in the case when the Lagrangian system is a model of a two-component composite with a lossy and a lossless component? This question will be investigated in the next section.

6.2 Selective overdamping

We are interested in solving the problem of describing all the modes of the canonical system (33) [or, equivalently, the Lagrangian system (19)], with system operator $A = \Omega - i\beta B$, that are overdamped when losses are sufficiently large, i.e., $\beta \gg 1$. In the previous section we solved the problem for Lagrangian systems with complete Rayleigh dissipative, i.e., R has full rank, for then Theorem 17 tells us that all the modes will become overdamped. On the other hand, for Lagrangian systems with incomplete Rayleigh dissipative, i.e., R does not have full rank, then Theorem 19 is a partial solution to the problem since it tells us that, due to sufficiently large losses, modal dichotomy occurs splitting the modes into the two $A(\beta)$ -invariant subspaces $H_{\ell\ell}(\beta)$ and $H_{h\ell}(\beta)$ (the high-loss and low-loss susceptible subspaces, respectively) with the modes belonging to $H_{h\ell}(\beta)$ overdamped. Thus to solve the problem completely we need to describe which of the modes in the low-loss susceptible subspace $H_{\ell\ell}(\beta)$ can be overdamped when losses are sufficiently large. In this section we solve the problem asymptotically as losses become extremely large, i.e., as $\beta \rightarrow \infty$, using

the virial theorem 1 and, in particular, energy equipartition 2 for the eigenmodes and the perturbation theory developed in [FigWel1] which we described in Section 5.2.3.

Our description of the selective overdamping phenomenon is based on the modal dichotomy in the high-loss regime $\beta \gg 1$ as described in Section 5.2.3. In particular, we use Theorem 12 to characterize overdamping in terms of the high-loss and low-loss eigenmodes $v_j(t, \beta) = w_j(\beta) e^{-i\zeta_j(\beta)t}$, $1 \leq j \leq 2N$ of the canonical system (33) which split into the two distinct classes based on their dissipative properties (105).

We say that such an eigenmode $v_j(t, \beta)$ is *overdamped* for $\beta \gg 1$ if $\text{Re } \zeta_j(\beta) = 0$ for all β sufficiently large. We say that such an eigenmode *remains oscillatory* for $\beta \gg 1$ if $\text{Re } \zeta_j(\beta) \neq 0$ for all β sufficiently large.

6.2.1 On the overdamping of the high-loss eigenmodes

Our first result tells us that all of the high-loss eigenmodes $v_j(t, \beta) = w_j(\beta) e^{-i\zeta_j(\beta)t}$, $1 \leq j \leq N_R$ from (105) are overdamped regardless of whether R has full rank or not. It is an immediate corollary of Theorem 19 and Theorem 12.

Theorem 20 (overdamped high-loss modes) *All of the high-loss eigenmodes from (105) are overdamped for $\beta \gg 1$.*

6.2.2 On the overdamping of the low-loss eigenmodes

Now we will discuss and give a solution to the problem of determining which of the low-loss eigenmodes $v_j(t) = w_j(\beta) e^{-i\zeta_j(\beta)t}$, $N_R + 1 \leq j \leq 2N$ in (105) will be overdamped. In order to do this we work under the following nondegeneracy condition:

Condition 21 (nondegeneracy) *For our study of the overdamping of the low-loss modes in this section, we assume henceforth that*

$$\text{Ker } \eta \cap \text{Ker } R = \{0\}. \quad (139)$$

We have two main goals in this section. Our first is to find necessary and sufficient conditions to have $\text{Re } \zeta_j(\beta) = 0$ for $\beta \gg 1$. Let κ be the number of low-loss eigenmodes that are overdamped for $\beta \gg 1$. Our second goal is to calculate this number κ .

We begin by recalling some of the important properties of the low-loss eigenpairs $\zeta_j(\beta)$, $w_j(\beta)$, $N_R + 1 \leq j \leq 2N$ in (96) for the system operator $A(\beta) = \Omega - i\beta B$ that we will need in this section. By (99) and (101) the limits

$$\lim_{\beta \rightarrow \infty} w_j(\beta) = \hat{w}_j, \quad \lim_{\beta \rightarrow \infty} \text{Re } \zeta_j(\beta) = \rho_j, \quad \lim_{\beta \rightarrow \infty} \text{Im } \zeta_j(\beta) = 0 \quad (140)$$

exist and from (93), (100) the limiting vectors $\{\hat{w}_j\}_{j=N_R+1}^{2N}$ are an orthonormal basis for $\text{Ker } B$ which also diagonalize the self-adjoint operator Ω_1 from (92) satisfying

$$B\hat{w}_j = 0, \quad \Omega_1\hat{w}_j = \rho_j\hat{w}_j, \quad N_R + 1 \leq j \leq 2N. \quad (141)$$

Overview of main results. Before we proceed, let us now summarize the results in this section. First, it's obvious from this discussion that a necessary condition for the low-loss eigenmode $v_j(t, \beta)$ to be overdamped for $\beta \gg 1$ is $\rho_j = 0$ (even though it does so with a slow damping of order β^{-1} , as seen in (140) and (99)), but what is not obvious is that this is in fact a sufficient condition. Although in the case η^{-1} exists (i.e., $\text{Ker } \eta = \{0\}$), it just follows from the fourth statement in Proposition 16 since if $\rho_j = 0$ then by (140) we have $\zeta_j(\beta) \rightarrow 0$ which implies $|\zeta_j(\beta)| < \omega_{\min}$ for $\beta \gg 1$ and hence $\text{Re } \zeta_j(\beta) = 0$ for $\beta \gg 1$. But this argument fails when $\text{Ker } \eta \neq \{0\}$. Thus we must find an alternative method to show that $\rho_j = 0$ is equivalent to $\text{Re } \zeta_j(\beta) = 0$ for $\beta \gg 1$. The key idea that leads to an alternative method is to realize that by the virial theorem 1 and the fifth statement in Proposition 16, when η^{-1} exists the modes with $\rho_j = 0$ are exactly the modes that break the energy equipartition.

Hence our general method is to show, even in the case $\text{Ker } \eta = \{0\}$, that for a low-loss modes with $\rho_j = 0$, there is a breakdown of the virial theorem 1 since as losses increase, i.e., as $\beta \rightarrow \infty$, the equality between the kinetic and potential energy (see Corollary 2) of the corresponding eigenmode of the Lagrangian system (19) can no longer be maintained and must eventually break which forces the modes to be non-oscillatory for $\beta \gg 1$, i.e., overdamped. This proves that $\rho_j = 0$ is equivalent to $\text{Re } \zeta_j(\beta) = 0$ for $\beta \gg 1$.

Thus it follows that the low-loss eigenmodes in (105) of the canonical system (33) which are overdamped for $\beta \gg 1$, their limiting vectors (satisfy $\lim_{\beta \rightarrow \infty} v_j(t, \beta) = \dot{w}_j \in \text{Ker } \Omega_1$ for t fixed) form an orthonormal basis for $\text{Ker } \Omega_1$. This implies of course that $\kappa = \dim \text{Ker } \Omega_1$ is the number of low-loss eigenmodes that are overdamped for $\beta \gg 1$. A major result of this section is that in fact $\kappa = N_R$, where $N_R = \text{rank } B = \text{rank } R$. But N_R is also the number of high-loss eigenpairs in (96). Thus we conclude using these results and Theorem 20 that the number of eigenmodes in (105) which are overdamped in the high-loss regime $\beta \gg 1$ is exactly $2N_R$ with N_R of these being low-loss eigenmodes and the rest being all the high-loss eigenmodes. Moreover, in the case $N_R < N$, the remaining $2N - 2N_R > 0$ modes are low-loss oscillatory modes with an extremely high quality factor that actually increases as the losses increase, i.e., as $\beta \rightarrow \infty$.

Main results. We now give the statements of our main results. The proof of the first theorem we give in this section as it is enlightening. The proofs of the rest of the statements below are found in Section 7.

Theorem 22 (overdamped low-loss modes) *A necessary and sufficient condition for a low-loss eigenmode $v_j(t, \beta) = w_j(\beta) e^{-i\zeta_j(\beta)t}$ from (105) to be overdamped for $\beta \gg 1$ is $\rho_j = 0$ (or equivalently, $\dot{w}_j \in \text{Ker } \Omega_1$).*

Proof. Let $v_j(t, \beta) = w_j(\beta) e^{-i\zeta_j(\beta)t}$ be a low-loss eigenmode from (105). If it is overdamped for $\beta \gg 1$ then $\text{Re } \zeta_j(\beta) = 0$ for $\beta \gg 1$ (i.e., for all β sufficiently large) and so by (140), (141) we have $\rho_j = 0$ and $\dot{w}_j \in \text{Ker } \Omega_1$ (the nullspace of Ω_1).

Lets now prove the converse. Suppose $\rho_j = 0$. Then we must show that $\text{Re } \zeta_j(\beta) = 0$ for $\beta \gg 1$. Suppose that this was not true. Then since $\zeta_j(\beta)$ was analytic at $\beta = \infty$ we must have $\text{Re } \zeta_j(\beta) \neq 0$ for $\beta \gg 1$. Thus by Corollary 4 there is a corresponding eigenmode $Q_j(t, \beta) = q_j(\beta) e^{-i\zeta_j(\beta)t}$ of the Lagrangian system (19) such that with respect to the block representation (67) the vector $w_j(\beta)$ is

$$w_j(\beta) = \begin{bmatrix} -i\zeta_j(\beta) \sqrt{\alpha} q_j(\beta) \\ \sqrt{\eta} q_j(\beta) \end{bmatrix}. \quad (142)$$

In particular, v_j and Q_j are related by (32), namely,

$$v_j = K u_j, \text{ where } u_j = \begin{bmatrix} P_j \\ Q_j \end{bmatrix}, \quad P_j = \alpha \dot{Q}_j. \quad (143)$$

By Section 2.4 on the energetic equivalence between the canonical system (33) and the Lagrangian system (19) it follows that

$$U[v_j] = \mathcal{T}(\dot{Q}, Q) + \mathcal{V}(\dot{Q}, Q). \quad (144)$$

The term on the left is the system energy of the eigenmode v_j of the canonical system defined in (40) whereas the sum on the right is the system energy of the eigenmode Q_j of the Lagrangian system (19), i.e., the sum of its kinetic and potential energy. As proved in Section 7.3.2, for any fixed time t we have

$$\begin{aligned} \frac{1}{2} &= \lim_{\beta \rightarrow \infty} U[v_j(t, \beta)] = \lim_{\beta \rightarrow \infty} \mathcal{V}(\dot{Q}_j(t, \beta), Q_j(t, \beta)), \\ &\lim_{\beta \rightarrow \infty} \mathcal{T}(\dot{Q}_j(t, \beta), \dot{Q}_j(t, \beta)) = 0. \end{aligned} \quad (145)$$

This proves that

$$\mathcal{T}(\dot{Q}, Q) \neq \mathcal{V}(\dot{Q}, Q) \text{ for } \beta \gg 1 \quad (146)$$

and, consequently, by the energy equipartition (see Corollary 2) that

$$\text{Re } \zeta_j(\beta) = 0 \text{ for } \beta \gg 1. \quad (147)$$

This completes the proof. ■

Corollary 23 *The collection of limiting values as $\beta \rightarrow \infty$ (with t fixed) of the low-loss eigenmodes in (105) of the canonical system (33) which are overdamped for $\beta \gg 1$ is $\{\dot{w}_j\}_{j=N_R+1}^{2N} \cap \text{Ker } \Omega_1$ and is an orthonormal basis for $\text{Ker } \Omega_1$.*

Proposition 24 *Let κ denote the number low-loss eigenmodes in (105) of the canonical system (33) which are overdamped for $\beta \gg 1$. Then*

$$\kappa = \dim \text{Ker } \Omega_1 = N_R, \quad (148)$$

where $N_R = \text{rank } R = \text{rank } B$.

Theorem 25 (number of overdamped modes) *The total number of eigenmodes of the canonical system (33) from (105) which are overdamped for $\beta \gg 1$ is $2N_R$, where $N_R = \text{rank } R = \text{rank } B$. Furthermore, N_R of these are low-loss eigenmodes and the rest are all of the high-loss eigenmodes. Moreover, if $N_R < N$ then the $2N - 2N_R$ modes which are not overdamped as $\beta \gg 1$ are low-loss eigenmodes which remain oscillatory for $\beta \gg 1$.*

Theorem 26 (selective overdamping) *If $N_R < N$ then there is exactly $2N - 2N_R$ low-loss eigenmodes of the canonical system (33) from (105) which remain oscillatory for $\beta \gg 1$ (i.e., $\text{Re } \zeta_j(\beta) \neq 0$ for $\beta \gg 1$) and their quality factors as given by (42), (43) have the following property*

$$\text{low-loss oscill. modes: } Q[w_j(\beta)] = +\infty \text{ for } \beta \gg 1 \text{ or } Q[w_j(\beta)] \nearrow +\infty \text{ as } \beta \rightarrow \infty. \quad (149)$$

7 Proof of results

This section contains the proofs of the results from Sections 4, 5 and 6. Proofs are organized in the order in which statements appeared and grouped according to the sections and subsections in which they are located in the paper. All assumptions, notation, and conventions used here will adhere to that previously introduced in this paper.

7.1 The virial theorem for dissipative systems and equipartition of energy

The proofs of statements in Section 4 are contained here.

Proof of Theorem 1. For $Q(t) = qe^{-i\zeta t}$ an eigenmode of the Lagrangian system (19), we consider the virial

$$G = (\alpha\dot{Q}, Q). \quad (150)$$

In our proof we will use the real quantities $\mathcal{T}_0(\dot{Q}) = \frac{1}{2}(\dot{Q}, \alpha\dot{Q})$ and $\mathcal{V}_0(Q) = \frac{1}{2}(Q, \eta Q)$. If $\zeta = 0$ we are done. Thus, suppose that $\zeta \neq 0$. The virial (150) satisfies

$$G = \frac{2}{-i\zeta} \mathcal{T}_0(\dot{Q}); \quad \dot{G}(t) = e^{2\text{Im}\zeta t} G(0). \quad (151)$$

Taking the time-derivative of (151) and using (19) yields the identity

$$4 \frac{\text{Im}\zeta}{-i\zeta} \mathcal{T}_0(\dot{Q}) = \dot{G} = 2\mathcal{T}_0(\dot{Q}) - 2\mathcal{V}_0(Q) - \frac{2}{-i\zeta} \mathcal{R}(\dot{Q}) - (2\theta\dot{Q}, Q), \quad (152)$$

where $\mathcal{R}(\dot{Q}) = \frac{1}{2}(\dot{Q}, \beta R \dot{Q})$. The energy balance equation (20) implies

$$-2\mathcal{R}(\dot{Q}) = \partial_t (\mathcal{T}_0(\dot{Q}) + \mathcal{V}_0(Q)) = 2\text{Im} \zeta (\mathcal{T}_0(\dot{Q}) + \mathcal{V}_0(Q)). \quad (153)$$

Combining the identities (152) and (153) yields immediately the identity

$$\frac{\text{Re} \zeta}{\zeta} \mathcal{T}_0(\dot{Q}) = \frac{\text{Re} \zeta}{\zeta} \mathcal{V}_0(Q) + (\theta \dot{Q}, Q). \quad (154)$$

From (154) and since $\theta^T = -\theta$ we derive the identity

$$\mathcal{T}_0(\dot{Q}) = \mathcal{V}_0(Q) - \frac{\zeta}{\text{Re} \zeta} (\dot{Q}, \theta Q) \quad (155)$$

if $\text{Re} \zeta \neq 0$ or $(\dot{Q}, \theta Q) = 0$ if $\text{Re} \zeta = 0$. Suppose now that $\text{Re} \zeta \neq 0$. Then, by taking the imaginary and real parts of the identity (155), we find that

$$\mathcal{T}_0(\dot{Q}) = \mathcal{V}_0(Q) - \text{Re}(\dot{Q}, \theta Q) - \left(\frac{\text{Im} \zeta}{\text{Re} \zeta} \right)^2 \text{Re}(\dot{Q}, \theta Q). \quad (156)$$

The rest of the proof of Theorem 1 now immediately follows from (156) and (21). ■

Proof of Corollary 2. Corollary 2 follows immediately from the virial theorem 1. ■

7.2 Spectral analysis of the system eigenmodes

The proofs of statements in Section 5 are contained here.

7.2.1 Standard versus pencil formulations of the spectral problems

Proof of Proposition 3. The factorization (68) follows from formula (173) in Appendix 8 and (30), (31), (33)–(35), (60) since

$$\begin{aligned} \zeta I - A(\beta) &= \begin{bmatrix} \zeta \mathbf{1} - \Omega_p + i\tilde{R} & i\Phi^T \\ -i\Phi & \zeta \mathbf{1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1} & \zeta^{-1}i\Phi^T \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \zeta \mathbf{1} - \Omega_p + i\tilde{R} - \zeta^{-1}i\Phi^T(-i\Phi) & 0 \\ 0 & \zeta \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ -\zeta^{-1}i\Phi & \mathbf{1} \end{bmatrix} \\ &= \begin{bmatrix} K_p & \zeta^{-1}i\Phi^T \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \zeta^{-1}\mathbf{1} & 0 \\ 0 & \zeta \mathbf{1} \end{bmatrix} \begin{bmatrix} C(\zeta, \beta) & 0 \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} K_p^T & 0 \\ -\zeta^{-1}i\Phi & \mathbf{1} \end{bmatrix}. \end{aligned}$$

This completes the proof. ■

Proof of Corollary 4. The proof of this corollary follows immediately from the factorization (68) using the facts $K_p^T = K_p = \sqrt{\alpha}^{-1}$, $\Phi\sqrt{\alpha} = \sqrt{\eta}$ and

$$\begin{bmatrix} K_p^T & 0 \\ -\zeta^{-1}i\Phi & \mathbf{1} \end{bmatrix}^{-1} = \begin{bmatrix} \sqrt{\alpha} & 0 \\ \zeta^{-1}i\Phi\sqrt{\alpha} & \mathbf{1} \end{bmatrix}.$$

■

7.2.2 On the spectrum of the system operator

Proof of Proposition 6. The proof of this proposition follows immediately from elementary properties of the determinant and using the fundamental property of the system operator $A(\beta) = \Omega - i\beta B$ in (73), namely, $A(\beta)^* = -A(\beta)^T$ which implies $-A(\beta) = A(\beta)$. ■

7.2.3 Eigenvalue bounds and modal dichotomy

Proof of Proposition 7. The second statement follows immediately from (73) since if $A(\beta)w = \zeta w$ and $w \neq 0$ then

$$\operatorname{Re} \zeta = \frac{\operatorname{Re}(w, A(\beta)w)}{(w, w)} = \frac{(w, \Omega w)}{(w, w)}, \quad -\operatorname{Im} \zeta = -\frac{\operatorname{Im}(w, A(\beta)w)}{(w, w)} = \frac{(w, \beta B w)}{(w, w)} \geq 0.$$

From this and using (73)–(77), the first and third statements follow immediately from Proposition 27 in Appendix 9. This completes the proof. ■

Proof of Corollary 8. The proof of this corollary follows immediately from Propositions 6 and 7. ■

Proof of Theorem 9. Suppose that $\beta > 2\frac{\omega_{\max}}{b_{\min}}$. It follows from Theorem 29 in Appendix 9 that the $2N \times 2N$ matrix $A(\beta)$ satisfies the hypotheses of the theorem and together with (73), (74) implies $\operatorname{rank}(\operatorname{Im} A(\beta)) = \operatorname{rank}(B) = N_R$ and the sets $\sigma_0(A(\beta)), \sigma_1(A(\beta))$ in (81) satisfy

$$\sigma(A(\beta)) = \sigma_0(A(\beta)) \cup \sigma_1(A(\beta)), \quad \sigma_0(A(\beta)) \cap \sigma_1(A(\beta)) = \emptyset.$$

Moreover, Theorem 29 gives the existence of unique projection matrices $P_0(\beta), P_1(\beta)$ such that in the space $H = \mathbb{C}^{2N}$ the subspaces $H_{h\ell}(\beta) = \operatorname{Ran} P_1(\beta)$ and $H_{\ell\ell}(\beta) = \operatorname{Ran} P_0(\beta)$ have exactly the desired properties in Theorem 9 with their dimensions satisfying

$$\dim H_{h\ell}(\beta) = \operatorname{rank} P_1(\beta) = N_R, \quad \dim H_{\ell\ell}(\beta) = \operatorname{rank} P_0(\beta) = 2N - N_R.$$

This completes the proof. ■

Proof of Corollary 11. First, if w is an eigenvector of $A(\beta)$ with corresponding eigenvalue ζ then Proposition 7 and (77) imply

$$-\operatorname{Im} \zeta \geq 0, \quad |\operatorname{Re} \xi| = \left| \frac{(w, \Omega w)}{(w, w)} \right| \leq \|\Omega\| = \omega_{\max}. \quad (157)$$

Assume now that $\beta > 2\frac{\omega_{\max}}{b_{\min}}$. Then by (73)–(77), (157), and Theorem 9 the proof of the first part of Corollary 11 follows immediately from Corollary 30 in Appendix 9. The rest of the proof now follows immediately from (157), formula (43), and the first part of Corollary 11. This completes the proof. ■

Proof of Theorem 12. Consider the high-loss and low-loss eigenpairs $\zeta_j(\beta)$, $w_j(\beta)$, $1 \leq j \leq 2N$ for $\beta \gg 1$ introduced in Section 5.2.3. They have the properties that $\{w_j(\beta)\}_{j=1}^{2N}$ are a basis of eigenvectors for the system operator $A(\beta)$ satisfying

$$A(\beta) w_j(\beta) = \zeta_j(\beta) w_j(\beta), \quad 1 \leq j \leq 2N;$$

$$\lim_{\beta \rightarrow \infty} -\operatorname{Im} \zeta_j(\beta) = \infty, \quad 1 \leq j \leq N_R; \quad \lim_{\beta \rightarrow \infty} -\operatorname{Im} \zeta_j(\beta) = 0, \quad N_R + 1 \leq j \leq 2N.$$

The proof of Theorem 12 follows from these facts, Theorem 9, and Corollary 11. ■

7.3 Overdamping analysis

The proofs of statements in Section 6 are contained here. Recall, that we are assuming from now on that Condition 14 is true i.e., $\theta = 0$.

Proof of Proposition 15. Using the block representation of the matrices Ω , B in (34) and the result (176) in Appendix 8 on the determinant of block matrices with commuting block entries we find that for any $\zeta \in \mathbb{C}$ we have

$$\det(\zeta \mathbf{1} - \Omega) = \det \left(\begin{bmatrix} \zeta \mathbf{1} & i\Phi^T \\ -i\Phi & \zeta \mathbf{1} \end{bmatrix} \right) = \det(\zeta^2 \mathbf{1} - \Phi^T \Phi) = \det(\zeta^2 \mathbf{1} - \alpha^{-1} \eta),$$

$$\det(\zeta \mathbf{1} - B) = \det \left(\begin{bmatrix} \zeta \mathbf{1} - \tilde{R} & 0 \\ 0 & \zeta \mathbf{1} \end{bmatrix} \right) = \zeta^N \det(\zeta \mathbf{1} - \tilde{R}) = \zeta^N \det(\zeta \mathbf{1} - \alpha^{-1} R_H),$$

since by (30), (31), (35) we have $\Phi^T \Phi = \sqrt{\alpha}^{-1} \eta \sqrt{\alpha}^{-1}$ and $\tilde{R} = \sqrt{\alpha}^{-1} R_H \sqrt{\alpha}^{-1}$. The proof now follows from these facts. ■

7.3.1 Complete and partial overdamping

Proof of Proposition 16. The last three statements have already been proved. The first two statements are proved using Proposition 15 and the following series of inequalities which just use elementary properties of the operator norm $\|\cdot\|$ for Hermitian matrices

$$0 \leq \frac{(q, \eta q)}{(q, \alpha q)} = \frac{(\sqrt{\alpha} q, \sqrt{\alpha}^{-1} \eta \sqrt{\alpha}^{-1} \sqrt{\alpha} q)}{(\sqrt{\alpha} q, \sqrt{\alpha} q)} \leq \left\| \sqrt{\alpha}^{-1} \eta \sqrt{\alpha}^{-1} \right\| = \max \sigma(\alpha^{-1} \eta), \quad (158)$$

$$\frac{(q, M q)}{(q, \alpha q)} = \frac{(\sqrt{\alpha} q, \sqrt{\alpha}^{-1} M \sqrt{\alpha}^{-1} \sqrt{\alpha} q)}{(\sqrt{\alpha} q, \sqrt{\alpha} q)} \geq \left\| \left(\sqrt{\alpha}^{-1} M \sqrt{\alpha}^{-1} \right)^{-1} \right\|^{-1} = \min \sigma(\alpha^{-1} M), \quad (159)$$

where $0 \leq M = \eta$ or R and the inequality (159) requires the hypothesis that η or R , respectively, be invertible so that $M > 0$. This completes the proof. ■

Proof of Theorem 17. Suppose $\beta \geq 2\frac{\omega_{\max}}{b_{\min}}$ and $N_R = N$, where $N_R = \text{rank } R$. If ζ was an eigenvalue of $A(\beta)$ with $\text{Re } \zeta \neq 0$ then by Proposition 16 we have

$$-\text{Im } \zeta = \frac{\beta(q, Rq)}{2(q, \alpha q)} \geq \frac{\beta}{2} b_{\min} \geq \omega_{\max} \geq |\zeta| > -\text{Im } \zeta,$$

a contradiction. Therefore, all the eigenvalues of the system operator $A(\beta)$ are real. This completes the proof. ■

Proof of Theorem 19. Suppose $\beta \geq 2\frac{\omega_{\max}}{b_{\min}}$ and $N_R < N$. Then by Theorem 9 and Corollary 11, if $\zeta \in \sigma(A(\beta)|_{H_{h\ell}(\beta)})$ then $-\text{Im } \zeta > \omega_{\max}$ and so by Proposition 16 we must have $\text{Re } \zeta = 0$. Therefore, all the eigenvalues of $A(\beta)$ in $\sigma(A(\beta)|_{H_{h\ell}(\beta)})$ are real. ■

7.3.2 Selective overdamping

Proof of Theorem 20. If $v_j(t, \beta) = w_j(\beta) e^{-i\zeta_j(\beta)t}$ is a high-loss eigenmode from (105) then by the modal dichotomy (12) we must have $\zeta_j(\beta) \in \sigma(A(\beta)|_{H_{h\ell}(\beta)})$ for $\beta \gg 1$ (i.e., for β sufficiently large). By Theorem 19 it follows that $\text{Re } \zeta_j(\beta) = 0$ for $\beta \gg 1$. Therefore, $v_j(t, \beta)$ is overdamped for $\beta \gg 1$. This completes the proof. ■

We will now assume in the rest of the proofs that Condition 21 is true i.e., $\text{Ker } \eta \cap \text{Ker } R = \{0\}$.

Proof of Theorem 22. The proof of this theorem was started in Section 6.2.2. It remains to show that (145) holds. First, we have that $\lim_{\beta \rightarrow \infty} w_j(\beta) = \hat{w}_j$ with $\Omega_1 \hat{w}_j = 0$ and $\|\hat{w}_j\| = 1$ by the properties of the low-loss eigenmodes in (105) and since (140), (141) is true for these modes together with the hypothesis $\rho_j = 0$.

Now we use the representation (142) and (166) for $w_j(\beta)$ and $\text{Ker } \Omega_1$, respectively, to imply that as $\beta \rightarrow \infty$

$$w_j(\beta) = \begin{bmatrix} -i\zeta_j(\beta) \sqrt{\alpha} q_j(\beta) \\ \sqrt{\eta} q_j(\beta) \end{bmatrix} \rightarrow \hat{w}_j = \begin{bmatrix} 0 \\ \hat{\psi} \end{bmatrix}, \quad -i\zeta_j(\beta) \sqrt{\alpha} q_j(\beta) \rightarrow 0, \quad \sqrt{\eta} q_j(\beta) \rightarrow \hat{\psi} \quad (160)$$

for some $\hat{\psi} \in \mathbb{C}^N$. It follows from (160) and the fact $\lim_{\beta \rightarrow \infty} \text{Im } \zeta_j(\beta) = 0$ that for any fixed time t we have as $\beta \rightarrow \infty$

$$\begin{aligned} \mathcal{T}(\dot{Q}_j(t, \beta), Q_j(t, \beta)) &= \frac{1}{2} (\dot{Q}_j(t, \beta), \alpha \dot{Q}_j(t, \beta)) \\ &= \frac{1}{2} (-i\zeta_j(\beta) \sqrt{\alpha} q_j(\beta), -i\zeta_j(\beta) \sqrt{\alpha} q_j(\beta)) e^{2\text{Im } \zeta_j(\beta)t} \rightarrow 0, \\ \mathcal{V}(\dot{Q}_j(t, \beta), Q_j(t, \beta)) &= \frac{1}{2} (Q_j(t, \beta), \eta Q_j(t, \beta)) \\ &= \frac{1}{2} (\sqrt{\eta} q_j(\beta), \sqrt{\eta} q_j(\beta)) e^{2\text{Im } \zeta_j(\beta)t} \rightarrow \frac{1}{2} (\hat{\psi}, \hat{\psi}), \\ U[v_j(t, \beta)] &= \frac{1}{2} (w_j(\beta), w_j(\beta)) e^{2\text{Im } \zeta_j(\beta)t} \rightarrow \frac{1}{2} (\hat{w}_j, \hat{w}_j) = \frac{1}{2}. \end{aligned}$$

This proves (145) since by (160) we have $(\dot{w}_j, \dot{w}_j) = (\dot{\psi}, \dot{\psi})$. This completes the proof. ■

Proof of Corollary 23. This corollary follows immediately from Theorem 22 and the fact that the $\{\dot{w}_j\}_{j=N_R+1}^{2N}$ are an orthonormal basis for $\text{Ker } B$ which diagonalize the self-adjoint operator Ω_1 such that (141) is satisfied. ■

Proof of Theorem 25. This theorem follows immediately from Theorems 20 and 22, Corollary 23, and Proposition 13 (which is proved below). ■

Proof of Theorem 26. This theorem follows immediately from Theorems 25, 22 and Proposition 13. ■

Proof of Proposition 24. Let κ denote the number of low-loss eigenmodes of the canonical system (33) in (105) which are overdamped for $\beta \gg 1$. Then by Corollary 23 we know that

$$\kappa = \dim \text{Ker } \Omega_1.$$

We will now prove that

$$\kappa = N_R \tag{161}$$

where by (37), $N_R = \text{rank } R = \text{rank } B$. We will use the elementary facts from linear algebra that since $\eta, R \geq 0$ are $N \times N$ matrices which are positive semidefinite then

$$\mathbb{C}^N = \text{Ker } \eta \oplus \text{Ran } \eta, \quad \text{Ker } \eta = \text{Ker } \sqrt{\eta}, \quad \text{Ran } \eta = \text{Ran } \sqrt{\eta}, \tag{162}$$

$$(\text{Ran } R)^\perp = \text{Ker } R \text{ and } (\text{Ran } \eta)^\perp = \text{Ker } \eta \tag{163}$$

Another elementary fact from linear algebra which we need is that if S_1 and S_2 are subspaces in the same Hilbert space then

$$(S_1 \cap S_2)^\perp = S_1^\perp + S_2^\perp,$$

where $S_1^\perp + S_2^\perp = \{u + v : u \in S_1^\perp \text{ and } v \in S_2^\perp\}$. Thus with $S_1 = \text{Ran } \eta, S_2 = \text{Ran } R$ we have

$$(\text{Ran } \eta \cap \text{Ran } R)^\perp = \text{Ker } \eta \oplus \text{Ker } R, \tag{164}$$

where the sum is direct by Condition 21, i.e., the hypothesis $\text{Ker } \eta \cap \text{Ker } R = \{0\}$. In particular, this implies

$$\dim (\text{Ran } \eta \cap \text{Ran } R) = N - \dim \text{Ker } \eta - \dim \text{Ker } R = N_R - \dim \text{Ker } \eta. \tag{165}$$

The proof of (161) will be carried out in a series of steps which we outline now before we begin. First, we will prove that

$$\text{Ker } \Omega_1 = \left\{ \begin{bmatrix} 0 \\ \psi \end{bmatrix} : \sqrt{\eta}\psi \in \text{Ran } \eta \cap \text{Ran } R \right\}, \tag{166}$$

where the block form is with respect to the decomposition $H = H_p \oplus H_q$, $H = \mathbb{C}^{2N}$, $H_p = H_q = \mathbb{C}^N$ described in Section 5.1, i.e., the form (67). Next, it follows immediately from (162) that

$$\{\psi \in \mathbb{C}^N : \sqrt{\eta}\psi \in \text{Ran } \eta \cap \text{Ran } R\} = \text{Ker } \eta \oplus \{\phi \in \text{Ran } \eta : \sqrt{\eta}\phi \in \text{Ran } \eta \cap \text{Ran } R\}. \tag{167}$$

Finally, we prove that

$$\dim \{ \phi \in \text{Ran } \eta : \sqrt{\eta} \phi \in \text{Ran } \eta \cap \text{Ran } R \} = N_R - \dim \text{Ker } \eta. \quad (168)$$

The proof of (161) follows immediately from (166)–(168).

We begin by computing the representation (166) for $\text{Ker } \Omega_1$. First, by its definition from (92), Ω_1 is the restriction of the operator Ω to the no-loss subspace $H_B^\perp = \text{Ker } B$, that is,

$$\Omega_1 = P_B^\perp \Omega P_B^\perp |_{H_B^\perp} : H_B^\perp \rightarrow H_B^\perp, \quad (169)$$

where P_B^\perp is the orthogonal projection onto H_B^\perp in the Hilbert space $H = \mathbb{C}^{2N}$ with standard inner product (\cdot, \cdot) . Denote by $P_{\tilde{R}}^\perp$ the orthogonal projection onto $\text{Ker } \tilde{R}$ in the Hilbert space $H_p = \mathbb{C}^N$ with the standard inner product. Then it follows from the block representation of Ω and B in (92) with respect to the decomposition $H = H_p \oplus H_q$ that P_B^\perp and $P_B^\perp \Omega P_B^\perp$ with respect to this decomposition are the block operators

$$P_B^\perp = \begin{bmatrix} P_{\tilde{R}}^\perp & 0 \\ 0 & \mathbf{1} \end{bmatrix}, \quad P_B^\perp \Omega P_B^\perp = \begin{bmatrix} P_{\tilde{R}}^\perp & 0 \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} 0 & -i\Phi^T \\ i\Phi & 0 \end{bmatrix} \begin{bmatrix} P_{\tilde{R}}^\perp & 0 \\ 0 & \mathbf{1} \end{bmatrix} = \begin{bmatrix} 0 & -iP_{\tilde{R}}^\perp \Phi^T \\ i\Phi P_{\tilde{R}}^\perp & 0 \end{bmatrix}. \quad (170)$$

From (169), (170) it follows that

$$\text{Ker } \Omega_1 = \left\{ \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \text{Ker } \tilde{R} \oplus H_q : -iP_{\tilde{R}}^\perp \Phi^T \psi = 0 \text{ and } i\Phi P_{\tilde{R}}^\perp \varphi = 0 \right\}.$$

But if $i\Phi P_{\tilde{R}}^\perp \varphi = 0$ with $\varphi \in \text{Ker } \tilde{R}$ then $P_{\tilde{R}}^\perp \varphi = \varphi$ by definition of $P_{\tilde{R}}^\perp$ and hence $\tilde{R}\varphi = 0$ and $\Phi\varphi = 0$ which implies $\eta K_p^T \varphi = 0$ and $RK_p^T \varphi = 0$ since by (35) and (30) we have $\Phi = K_q K_p^T$, $\tilde{R} = K_p R K_p^T$, $K_p = \sqrt{\alpha}^{-1}$, and $K_q = \sqrt{\eta}$. By the hypothesis of Condition 21 this implies that $\varphi = 0$. Also, since \tilde{R} is a Hermitian matrix it follows that $I - P_{\tilde{R}}^\perp$ is the orthogonal projection onto the $\text{Ran } \tilde{R}$ so that $P_{\tilde{R}}^\perp \Phi^T \psi = 0$ is equivalent to $\Phi^T \psi \in \text{Ran } \tilde{R}$ and this is equivalent to $\sqrt{\eta} \psi \in \text{Ran } R$ by properties (31). But $\sqrt{\eta} \psi \in \text{Ran } R$ is equivalent to $\sqrt{\eta} \psi \in \text{Ran } R \cap \text{Ran } \sqrt{\eta}$ which by (162) is equivalent to $\sqrt{\eta} \psi \in \text{Ran } R \cap \text{Ran } \eta$. This proves (166).

Finally, we will now prove (168). It follows from the fact that η is a Hermitian matrix and (162) that the restriction of the operator $\sqrt{\eta}$ to $\text{Ran } \eta$, i.e., $\sqrt{\eta}|_{\text{Ran } \eta} : \text{Ran } \eta \rightarrow \text{Ran } \eta$, is an invertible operator and that

$$\{ \phi \in \text{Ran } \eta : \sqrt{\eta} \phi \in \text{Ran } \eta \cap \text{Ran } R \} = \sqrt{\eta}|_{\text{Ran } \eta}^{-1} (\text{Ran } \eta \cap \text{Ran } R). \quad (171)$$

From this it follows that

$$\dim \{ \phi \in \text{Ran } \eta : \sqrt{\eta} \phi \in \text{Ran } \eta \cap \text{Ran } R \} = \dim (\text{Ran } \eta \cap \text{Ran } R).$$

The proof of (168) immediately follows from this fact and (165). This completes the proof of the theorem. ■

8 Appendix: Schur complement and the Aitken formula

Let

$$M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \quad (172)$$

be a square matrix represented in block form where P and S are square matrices with the latter invertible, that is, $\|S^{-1}\| < \infty$. Then the following *Aitken block-diagonalization formula* holds [Zhang, §0.1], [Aitken, §29],

$$M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} \mathbf{1} & QS^{-1} \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} P - QS^{-1}R & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ S^{-1}R & \mathbf{1} \end{bmatrix}, \quad (173)$$

where the matrix

$$M/S = P - QS^{-1}R \quad (174)$$

is known as the *Schur complement* of S in M . The Aitken formula (173) readily implies

$$\det M = \det S \det (P - QS^{-1}R). \quad (175)$$

From this we can conclude that

$$\begin{aligned} \text{If } QS = SQ \text{ then } \det M &= \det (SP - QR); \\ \text{If } RS = SR \text{ then } \det M &= \det (PS - QR). \end{aligned} \quad (176)$$

But the latter statement is true even if $\det S = 0$ which is proved using a limiting argument. Indeed, by substituting for S in (172) the small perturbation $S_\delta = S + \delta I$ which is invertible for $0 < |\delta| \ll 1$ and commutes with any matrix that S commutes with then statement 176 is true for S_δ and the limit as $\delta \rightarrow 0$ yields the desired result.

9 Appendix: Eigenvalue bounds and dichotomy for non-Hermitian matrices

In this appendix we discuss results pertaining to bounds on the eigenvalues of non-Hermitian matrices in terms of their real and imaginary parts. We also discuss a phenomenon, which we call *modal dichotomy*, that occurs for any non-Hermitian matrix with a non-invertible imaginary part which is large in comparison to the real part of the matrix. The bounds and dichotomy described here are extremely useful tools in studying the spectral properties of dissipative systems especially for composite systems with high-loss and lossless components.

Recall, any square matrix M can be written as the sum of a Hermitian matrix and a skew-Hermitian matrix,

$$M = \operatorname{Re} M + i \operatorname{Im} M, \quad \operatorname{Re} M = \frac{M + M^*}{2}, \quad \operatorname{Im} M = \frac{M - M^*}{2i}, \quad (177)$$

the real and imaginary parts of M , respectively, and M^* is the conjugate transpose of the matrix M .

For the set of $n \times n$ matrices, denote by $\|\cdot\|$ the operator norm

$$\|M\| = \sup_{x \neq 0} \frac{\|Mx\|_2}{\|x\|_2}, \quad (178)$$

where $\|\cdot\|_2 = \sqrt{(\cdot, \cdot)}$ and (\cdot, \cdot) denotes the standard Euclidean inner product on $n \times 1$ column vectors with entries in \mathbb{C} . Denote the spectrum of a square matrix M by $\sigma(M)$. Recall in the case M is a Hermitian matrix, i.e., $M^* = M$, or a normal matrix, i.e., $M^*M = MM^*$, that

$$\|M\| = \max_{\zeta \in \sigma(M)} |\zeta|, \quad \|(\zeta I - M)^{-1}\|^{-1} = \text{dist}(\zeta, \sigma(M)), \quad (179)$$

where $\text{dist}(\zeta, \sigma(M)) = \inf_{\lambda \in \sigma(M)} |\zeta - \lambda|$ and by convention $\|(\zeta I - M)^{-1}\|^{-1} = 0$ if $\zeta \in \sigma(M)$.

The first result which we will prove here is the following proposition which gives bounds on the eigenvalues of a matrix M in terms of $\text{Re } M$ and $\text{Im } M$.

Proposition 27 (eigenvalue bounds) *Let M be a square matrix. Then*

$$\sigma(M) \subseteq \{\zeta \in \mathbb{C} : \text{dist}(\zeta, \sigma(i \text{Im } M)) \leq \|\text{Re } M\|\} \quad (180)$$

In other words, the eigenvalues of M lie in the union of the closed discs whose centers are the eigenvalues of $i \text{Im } M$ with radius $\|\text{Re } M\|$. Moreover, the imaginary part of any eigenvalue ζ of M satisfies the inequality

$$\|\text{Im } M\| \geq |\text{Im } \zeta| \geq |\gamma| - \|\text{Re } M\| \quad (181)$$

for some $\gamma \in \sigma(\text{Im } M)$ depending on ζ .

Before we proceed with the proof we will need the following perturbation result from [Kato, Sec. V.4, p. 291, Prob. 4.8] for non-Hermitian matrices which we prove here for completeness.

Lemma 28 *If $M = M_0 + E$ is a square matrix with M_0 normal then*

$$\sigma(M) \subseteq \{\zeta \in \mathbb{C} : \text{dist}(\zeta, \sigma(M_0)) \leq \|E\|\} \quad (182)$$

Proof. Let $M = M_0 + E$ be a square matrix with M_0 normal. Then

$$\sigma(M) \cap \sigma(M_0) \subseteq \{\zeta \in \mathbb{C} : \text{dist}(\zeta, \sigma(M_0)) \leq \|E\|\}.$$

Let $\rho(M_0) = \{\zeta \in \mathbb{C} : \zeta \notin \sigma(M_0)\}$, i.e., the resolvent set of M_0 . Then for any $\zeta \in \rho(M_0)$ we have

$$\zeta I - M = \zeta I - M_0 - (M - M_0) = (I - E(\zeta I - M_0)^{-1})(\zeta I - M_0).$$

Thus if $\|E(\zeta I - M_0)^{-1}\| < 1$ then the matrix $(I - E(\zeta I - M_0)^{-1})$ is invertible since the Neumann series $\sum_{j=0}^{\infty} T^j$ converges to $(I - T)^{-1}$ for any square matrix T satisfying $\|T\| < 1$. This implies

$$\sigma(M) \cap \rho(M_0) \subseteq \{\zeta \in \rho(M_0) : \|E(\zeta I - M_0)^{-1}\| \geq 1\}.$$

But the inequality $\|E(\zeta I - M_0)^{-1}\| \leq \|E\| \|(\zeta I - M_0)^{-1}\|$ and the fact M_0 is a normal matrix implies

$$\begin{aligned} \sigma(M) \cap \rho(M_0) &\subseteq \left\{ \zeta \in \rho(M_0) : \|(\zeta I - M_0)^{-1}\|^{-1} \leq \|E\| \right\} \\ &\subseteq \{\zeta \in \mathbb{C} : \text{dist}(\zeta, \sigma(M_0)) \leq \|E\|\}. \end{aligned}$$

Therefore,

$$\sigma(M) = \sigma(M) \cap \sigma(M_0) \cup \sigma(M) \cap \rho(M_0) \subseteq \{\zeta \in \mathbb{C} : \text{dist}(\zeta, \sigma(M_0)) \leq \|E\|\}.$$

This completes the proof. ■

Proof of Proposition 27. Let M be any square matrix. Then $M = -i \text{Im } M + \text{Re } M$ and $-i \text{Im } M$ is a normal matrix. By the previous lemma this implies

$$\sigma(M) \subseteq \{\zeta \in \mathbb{C} : \text{dist}(\zeta, \sigma(i \text{Im } M)) \leq \|\text{Re } M\|\}.$$

Now let ζ be any eigenvalue of M . Then for any corresponding eigenvector x of unit norm

$$|\text{Im } \zeta| = |\text{Im}(x, Mx)| = |(x, \text{Im } Mx)| \leq \|\text{Im } M\|. \quad (183)$$

Moreover, we know there exists an eigenvalue γ of $\text{Im } M$ (depending on ζ) such that

$$|\zeta - i\gamma| = \text{dist}(\zeta, \sigma(i \text{Im } M)) \leq \|\text{Re } M\|.$$

Hence

$$\|\text{Re } M\| \geq |\zeta - i\gamma| \geq |\text{Im}(\zeta - i\gamma)| = |\text{Im } \zeta - \gamma| \geq |\gamma| - |\text{Im } \zeta|. \quad (184)$$

The proof now follows from (183) and (184). ■

The next result we prove is a proposition on dichotomy of the spectrum of a non-Hermitian matrix M which we mentioned in the introduction of this appendix. In particular, the following proposition tells us that the spectrum of a matrix M will split into two disjoint parts when the imaginary part $\text{Im } M$ is non-invertible and “large” in comparison to the real part $\text{Re } M$. The term “large” means that the bottom of the nonzero spectrum of $|\text{Im } M|$ must be greater than twice the top of the spectrum

of $|\operatorname{Re} M|$ (where for a square matrix A , $|A| = \sqrt{A^*A}$ as defined in [ReSi1, §V1.4, p. 196]), that is,

$$\min_{\gamma \in \sigma(\operatorname{Im} M), \gamma \neq 0} |\gamma| > 2 \max_{\lambda \in \sigma(\operatorname{Re} M)} |\lambda|. \quad (185)$$

This of course implies $\|\operatorname{Im} M\|$ is “large” in comparison to $\|\operatorname{Re} M\|$ since it follows from this inequality that

$$\|\operatorname{Im} M\| > 2 \|\operatorname{Re} M\|. \quad (186)$$

Theorem 29 (modal dichotomy) *Let M be any $n \times n$ matrix which is non-Hermitian such that its imaginary part $\operatorname{Im} M$ is non-invertible with rank $m = \operatorname{rank}(\operatorname{Im} M)$. Denote by γ_j , $j = 1, \dots, m$ the nonzero eigenvalues of $\operatorname{Im} M$. If $\min_{1 \leq j \leq m} |\gamma_j| > 2 \|\operatorname{Re} M\|$ then*

$$\sigma(M) = \sigma_0(M) \cup \sigma_1(M), \quad \sigma_0(M) \cap \sigma_1(M) = \emptyset, \quad (187)$$

where

$$\begin{aligned} \sigma_0(M) &= \{\zeta \in \sigma(M) : |\zeta| \leq \|\operatorname{Re} M\|\}, \\ \sigma_1(M) &= \{\zeta \in \sigma(M) : |\zeta - i\gamma_j| \leq \|\operatorname{Re} M\| \text{ for some } j\}. \end{aligned} \quad (188)$$

Furthermore, there exists unique matrices P_0, P_1 with the properties

$$\begin{aligned} (i) \quad \mathbf{1} &= P_0 + P_1, \quad P_i P_j = \delta_{ij} P_j; \\ (ii) \quad MP_0 &= P_0 M P_0, \quad MP_1 = P_1 M P_1; \\ (iii) \quad \sigma(MP_0|_{\operatorname{Ran} P_0}) &= \sigma_0(M), \quad \sigma(MP_1|_{\operatorname{Ran} P_1}) = \sigma_1(M), \end{aligned} \quad (189)$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$ ($i, j = 1, 2$). Moreover, these projection matrices have rank satisfying

$$\operatorname{rank} P_1 = m, \quad \operatorname{rank} P_0 = n - m. \quad (190)$$

Proof. All the eigenvalues of $\operatorname{Im} M$ by hypothesis are γ_j , $j = 1, \dots, m$ and $\gamma_0 = 0$. It follows from Proposition 27 that

$$\sigma(M) \subseteq \bigcup_{j=0}^n \{\zeta \in \mathbb{C} : |\zeta - i\gamma_j| \leq \|\operatorname{Re} M\|\}$$

and the imaginary part of any eigenvalue ζ of M satisfies the inequality

$$\|\operatorname{Im} M\| \geq |\operatorname{Im} \zeta| \geq |\gamma_{j'}| - \|\operatorname{Re} M\|$$

for some $j' \in \{1, \dots, n\}$ depending on ζ . In particular, it follows that

$$\sigma(M) = \sigma_0(M) \cup \sigma_1(M)$$

where

$$\begin{aligned}\sigma_0(M) &= \{\zeta \in \sigma(M) : |\zeta| \leq \|\operatorname{Re} M\|\}, \\ \sigma_1(M) &= \{\zeta \in \sigma(M) : |\zeta - i\gamma_j| \leq \|\operatorname{Re} M\| \text{ for some } j \neq 0\}.\end{aligned}$$

Now suppose that $\min_{1 \leq j \leq m} |\gamma_j| > 2\|\operatorname{Re} M\|$. We will now show that $\sigma_0(M) \cap \sigma_1(M) = \emptyset$. Suppose this were not true then we could find $\zeta \in \sigma_0(M) \cap \sigma_1(M)$ which would imply for some $\gamma_{j'}$ with $j' \neq 0$ we have

$$\|\operatorname{Re} M\| \geq |\zeta - i\gamma_{j'}| \geq |\gamma_{j'}| - \|\operatorname{Re} M\| \geq \min_{1 \leq j \leq m} |\gamma_j| - \|\operatorname{Re} M\| > \|\operatorname{Re} M\|,$$

a contradiction. Thus $\sigma_0(M) \cap \sigma_1(M) = \emptyset$.

Now it follows from the spectral theory of matrices that the finite-dimensional vector space \mathbb{C}^n of $n \times 1$ column vectors with entries in \mathbb{C} can be written as the direct sum of two invariant subspaces for the matrix M ,

$$\mathbb{C}^n = H_0 \oplus H_1, \quad H_0 = \cup_{\zeta \in \sigma_0(M)} \operatorname{Ker}(M - \zeta \mathbf{1})^n, \quad H_1 = \cup_{\zeta \in \sigma_1(M)} \operatorname{Ker}(M - \zeta \mathbf{1})^n.$$

That is, H_0 is the union of the generalized eigenspaces of M corresponding to eigenvalues in $\sigma_0(M)$ and H_1 is the union of generalized eigenspaces of M corresponding to the eigenvalues in $\sigma_1(M) = \sigma(M) \setminus \sigma_0(M)$. Denote the projection matrix onto H_0 along H_1 by P_0 , that is, the matrix satisfying $P_0^2 = P_0$, $\operatorname{Ran} P_0 = H_0$, and $\operatorname{Ran}(\mathbf{1} - P_0) = H_1$. Then it follows from the spectral theory of matrices that P_0 and $P_1 = \mathbf{1} - P_0$ are the unique matrices with the properties

$$\begin{aligned}(i) \quad & \mathbf{1} = P_0 + P_1, \quad P_i P_j = \delta_{ij} P_j; \\ (ii) \quad & M P_0 = P_0 M P_0, \quad M P_1 = P_1 M P_1; \\ (iii) \quad & \sigma(M P_0|_{\operatorname{Ran} P_0}) = \sigma_0(M), \quad \sigma(M P_1|_{\operatorname{Ran} P_1}) = \sigma_1(M).\end{aligned}$$

Now it follows that

$$n = \dim H_1 + \dim H_0 = \dim \operatorname{Ran} P_1 + \dim \operatorname{Ran} P_0 = \operatorname{rank} P_1 + \operatorname{rank} P_0$$

so that to complete the proof of this proposition we need only show that

$$\operatorname{rank} P_0 = n - m.$$

In order to prove this we begin by giving an explicit representation of the matrix P_0 in terms of the resolvent of M which is defined as

$$R(\zeta) = (\zeta \mathbf{1} - M)^{-1}, \quad \zeta \notin \sigma(M).$$

To do this we define

$$r_0 = \frac{1}{2} \min_{1 \leq j \leq m} |\gamma_j|, \quad D(0, r_0) = \{\zeta \in \mathbb{C} : |\zeta| \leq r_0\}.$$

We will now prove that

$$\sigma_0(M) \subseteq D(0, r_0), \quad \sigma_1(M) \cap D(0, r_0) = \emptyset.$$

First, we notice that

$$r_0 = \|\operatorname{Re} M\| + \frac{1}{2} \left(\min_{1 \leq j \leq m} |\gamma_j| - 2 \|\operatorname{Re} M\| \right) > \|\operatorname{Re} M\|$$

which proves $\sigma_0(M) \subseteq D(0, r_0)$. Next, we have that if $\zeta \in D(0, r_0)$ then for any γ_j with $j \neq 0$,

$$|\zeta - i\gamma_j| \geq |\gamma_j| - |\zeta| \geq \min_{1 \leq j \leq m} |\gamma_j| - |\zeta| \geq \min_{1 \leq j \leq m} |\gamma_j| - r_0 = \frac{1}{2} \min_{1 \leq j \leq m} |\gamma_j| > \|\operatorname{Re} M\|$$

implying that $\zeta \notin \sigma_1(M)$. This proves $\sigma_1(M) \cap D(0, r_0) = \emptyset$. It now follows from the Cauchy-Riesz functional calculus [Bau85] that

$$P_0 = \frac{1}{2\pi i} \int_{|\zeta|=r_0} R(\zeta) d\zeta$$

where the contour integral is over the simply closed positively oriented path $\zeta(\theta) = r_0 e^{i\theta}$, $0 \leq \theta \leq 1$.

Now we consider the family of matrices

$$M(t) = (1-t) \operatorname{Re} M + i \operatorname{Im} M, \quad 0 \leq t \leq 1.$$

Notice that for $0 \leq t \leq 1$ we have

$$\begin{aligned} M(0) &= M, & M(1) &= i \operatorname{Im} M, \\ \operatorname{Im} M(t) &= \operatorname{Im} M, & \operatorname{Re} M(t) &= (1-t) \operatorname{Re} M, \\ \min_{1 \leq j \leq m} |\gamma_j| &> 2 \|\operatorname{Re} M\| &\geq 2 \|\operatorname{Re} M(t)\|. \end{aligned}$$

It follows from this that the results proven so far in this theorem for the matrix M apply to the matrix $M(t)$ for each $t \in [0, 1]$. In particular, for each $t \in [0, 1]$ we have

$$\sigma(M(t)) = \sigma_0(M(t)) \cup \sigma_1(M(t)), \quad \sigma_0(M(t)) \cap \sigma_1(M(t)) = \emptyset,$$

where

$$\begin{aligned} \sigma_0(M(t)) &= \{\zeta \in \sigma(M(t)) : |\zeta| \leq \|\operatorname{Re} M(t)\|\}, \\ \sigma_1(M(t)) &= \{\zeta \in \sigma(M(t)) : |\zeta - i\gamma_j| \leq \|\operatorname{Re} M(t)\| \text{ for some } j\}. \end{aligned}$$

Furthermore, there exists unique matrices $P_0(t)$, $P_1(t)$ with the properties

- (i) $\mathbf{1} = P_0(t) + P_1(t)$, $P_i(t)P_j(t) = \delta_{ij}P_j(t)$;
- (ii) $M(t)P_0(t) = P_0(t)M(t)P_0(t)$, $M(t)P_1(t) = P_1(t)MP_1(t)$;
- (iii) $\sigma(M(t)P_0(t)|_{\text{Ran } P_0(t)}) = \sigma_0(M(t))$, $\sigma(M(t)P_1(t)|_{\text{Ran } P_1(t)}) = \sigma_1(M(t))$.

Moreover,

$$\begin{aligned} r_0 &= \frac{1}{2} \min_{1 \leq j \leq m} |\gamma_j|, \quad D(0, r_0) = \{\zeta \in \mathbb{C} : |\zeta| \leq r_0\}, \\ \sigma_0(M(t)) &\subseteq D(0, r_0), \quad \sigma_1(M(t)) \cap D(0, r_0) = \emptyset, \\ P_0(t) &= \frac{1}{2\pi i} \int_{|\zeta|=r_0} R(\zeta, t) d\zeta, \quad R(\zeta, t) = (\zeta \mathbf{1} - M(t))^{-1}. \end{aligned}$$

It follows from these facts that $P_0(t)$ is a continuous matrix projection-valued function of t in $[0, 1]$ and

$$\text{tr } P_0(t) = \text{rank } P_0(t), \quad 0 \leq t \leq 1,$$

where $\text{tr}(\cdot)$ denotes the trace of a square matrix. But $\text{tr } P_0(t)$ is a continuous function of t since $P_0(t)$ is a continuous matrix-valued function implying $\text{rank } P_0(t)$ is a continuous function of t taking values in the nonnegative integers only. From this we conclude $\text{rank } P_0(t)$ is constant for $t \in [0, 1]$. In particular,

$$\text{rank } P_0 = \text{rank } P_0(0) = \text{rank } P_0(1).$$

Moreover,

$$\begin{aligned} P_0(1) &= \frac{1}{2\pi i} \int_{|\zeta|=r_0} R(\zeta, 1) d\zeta, \quad R(\zeta, 1) = (\zeta \mathbf{1} - i \text{Im } M)^{-1}, \\ \sigma(i \text{Im } M) &= \sigma_0(M(1)) = \{\zeta \in \sigma(M(1)) : |\zeta| \leq \|\text{Re } M(1)\|\} = \{0\} \\ \text{Ran } P_0(1) &= \cup_{\zeta \in \sigma_0(M(1))} \text{Ker}(M(1) - \zeta \mathbf{1})^n = \text{Ker}(i \text{Im } M)^n = \text{Ker}(\text{Im } M). \end{aligned}$$

But this implies that

$$\text{rank } P_0 = \text{rank } P_0(1) = \dim \text{Ran } P_0(1) = \dim \text{Ker}(\text{Im } M) = n - \text{rank}(\text{Im } M) = n - m.$$

This completes the proof. ■

We conclude this appendix by giving an alternative characterization of the subsets $\sigma_0(M)$ and $\sigma_1(M)$ from Theorem 29 in terms of the magnitude of the imaginary parts of the eigenvalues of M . In particular, the next corollary tells us that if the $\text{Im } M$ is associated with losses and if the imaginary part of the eigenvalues of M are associated with damping then only the eigenmodes with eigenvalues in $\sigma_1(M)$ are susceptible to large damping, relative to the norm of $\text{Re } M$, when losses are large.

Corollary 30 *If $\min_{1 \leq j \leq m} |\gamma_j| > 2 \|\operatorname{Re} M\|$ then*

$$\begin{aligned}\sigma_0(M) &= \{\zeta \in \sigma(M) : |\operatorname{Im} \zeta| \leq \|\operatorname{Re} M\|\}, \\ \sigma_1(M) &= \left\{ \zeta \in \sigma(M) : |\operatorname{Im} \zeta| \geq \min_{1 \leq j \leq m} |\gamma_j| - \|\operatorname{Re} M\| \right\}.\end{aligned}\tag{191}$$

Proof. By Theorem 29 we know that

$$\sigma(M) = \sigma_0(M) \cup \sigma_1(M), \quad \sigma_0(M) \cap \sigma_1(M) = \emptyset,$$

where

$$\begin{aligned}\sigma_0(M) &= \{\zeta \in \sigma(M) : |\zeta| \leq \|\operatorname{Re} M\|\}, \\ \sigma_1(M) &= \{\zeta \in \sigma(M) : |\zeta - i\gamma_j| \leq \|\operatorname{Re} M\| \text{ for some } j\},\end{aligned}$$

and γ_j , $j = 1, \dots, m$ are the nonzero eigenvalues of $\operatorname{Im} M$. In particular, these facts imply immediately that

$$\sigma_1(M) \supseteq \{\zeta \in \sigma(M) : |\operatorname{Im} \zeta| > \|\operatorname{Re} M\|\}, \quad \sigma_0(M) \subseteq \{\zeta \in \sigma(M) : |\operatorname{Im} \zeta| \leq \|\operatorname{Re} M\|\}.$$

Now if $\zeta \in \sigma_1(M)$ then there exists a nonzero eigenvalue γ of $\operatorname{Im} M$ such that

$$\|\operatorname{Re} M\| \geq |\zeta - i\gamma| = |\operatorname{Re} \zeta + i(\operatorname{Im} \zeta - \gamma)| \geq |\operatorname{Im} \zeta - \gamma| \geq |\gamma| - |\operatorname{Im} \zeta| \geq \min_{1 \leq j \leq m} |\gamma_j| - |\operatorname{Im} \zeta|$$

which by the hypothesis $\min_{1 \leq j \leq m} |\gamma_j| > 2 \|\operatorname{Re} M\|$ implies

$$\sigma_1(M) \subseteq \left\{ \zeta \in \sigma(M) : |\operatorname{Im} \zeta| \geq \min_{1 \leq j \leq m} |\gamma_j| - \|\operatorname{Re} M\| \right\} \subseteq \{\zeta \in \sigma(M) : |\zeta| > \|\operatorname{Re} M\|\}.$$

The corollary follows immediately now from these facts. ■

10 Appendix: Energetics

The term of *energetics* refers to a fundamental set up for a system evolution when two energies are defined for any system configuration, namely the *kinetic energy* and the *potential energy*. Here is a concise description of the energetics given by H. Poincaré, [Poincaré FS, p. 115-116]:

“The difficulties inherent in the classic mechanics have led certain minds to prefer a new system they call *energetics*.

Energetics took its rise as an outcome of the discovery of the principle of the conservation of energy. Helmholtz gave it its final form.

It begins by defining two quantities which play the fundamental role in this theory. They are *kinetic energy*, or *vis viva*, and *potential energy*.

All the changes which bodies in nature can undergo are regulated by two experimental laws:

1°. The sum of kinetic energy and potential energy is constant. This is the principle of the conservation of energy.

2°. If a system of bodies is at A at the time t_0 and at B at the time t_1 , it always goes from the first situation to the second in such a way that the mean value of the difference between the two sorts of energy, in the interval of time which separates the two epochs t_0 and t_1 , may be as small as possible.

This is Hamilton's principle, which is one of the forms of the principle of least action."

In other words according to energetics the kinetic and potential energies \mathcal{T} and \mathcal{V} that are related to the Lagrangian \mathcal{L} and the total energy \mathcal{H} by the following relations

$$\mathcal{L} = \mathcal{T} - \mathcal{V}, \quad \mathcal{H} = \mathcal{T} + \mathcal{V}, \quad (192)$$

implying

$$\mathcal{T} = \frac{1}{2}(\mathcal{L} + \mathcal{H}), \quad \mathcal{V} = \frac{1}{2}(\mathcal{H} - \mathcal{L}). \quad (193)$$

We assume the total energy to be equal to the Hamiltonian, that is to be defined by the Lagrangian through the Legendre transformation

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{Q}} \dot{Q} - \mathcal{L}. \quad (194)$$

Then we arrive at the following uniquely defined expressions for the kinetic and potential energies

$$\mathcal{T} = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial \dot{Q}} \dot{Q}, \quad \mathcal{V} = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial Q} Q - \mathcal{L}. \quad (195)$$

We are particularly interested in the case of the quadratic Lagrangian

$$\mathcal{L} = \mathcal{L}(Q, \dot{Q}) = \frac{1}{2} \dot{Q}^T \alpha \dot{Q} + \dot{Q}^T \theta Q - \frac{1}{2} Q^T \eta Q, \quad (196)$$

where α and η are real symmetric matrices which are positive definite and positive semidefinite, respectively, and θ is a real skew-symmetric matrix, that is

$$\alpha^T = \alpha > 0, \quad \eta^T = \eta \geq 0, \quad \theta^T = -\theta, \quad (197)$$

Then according to (194) the total energy \mathcal{H} takes the form

$$\mathcal{H} = \frac{1}{2} \dot{Q}^T \alpha \dot{Q} + \frac{1}{2} Q^T \eta Q \geq 0. \quad (198)$$

Consequently, as it follows from (193) the kinetic and the potential energies are defined by

$$\mathcal{T} = \frac{1}{2}\dot{Q}^T\alpha\dot{Q} + \frac{1}{2}\dot{Q}^T\theta Q, \quad \mathcal{V} = \frac{1}{2}Q^T\eta Q - \frac{1}{2}\dot{Q}^T\theta Q. \quad (199)$$

The Euler-Lagrange equations corresponding to the quadratic Lagrangian (196) are

$$\alpha\ddot{Q} + 2\theta\dot{Q} + \eta Q = 0. \quad (200)$$

Multiplying the above equation by Q^T from the left yields

$$Q^T\alpha\ddot{Q} + 2Q^T\theta\dot{Q} + Q^T\eta Q = 0, \quad (201)$$

and this equation can be recast as

$$\partial_t \left(Q^T\alpha\dot{Q} \right) - \dot{Q}^T\alpha\dot{Q} - 2\dot{Q}^T\theta Q + Q^T\eta Q = 0. \quad (202)$$

Combining the above equation with (193) and (196) we arrive at the following important identity that holds for any solution Q to the Euler-Lagrange equation (200)

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = \frac{1}{2}\partial_t G, \quad G = Q^T\alpha\dot{Q}, \quad (203)$$

where the term G is often referred to as the *virial*.

11 Appendix: Virial theorem

Let us introduce now the time average $\langle f \rangle$ for a time dependent quantity $f = f(t)$ defined by

$$\langle f \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) dt, \quad (204)$$

and observe that if $f(t)$ is bounded for $-\infty < t < \infty$ then

$$\langle \partial_t f \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} (f(T) - f(0)) = 0. \quad (205)$$

Notice now that in the case of interest when $\alpha > 0, \eta \geq 0$ and consequently $\mathcal{H} \geq 0$, all eigenfrequencies of the system are real and hence all solutions to the Euler-Lagrange equation (201) are bounded. Consequently, applying the relation (205), using the identity (203) we obtain the virial theorem:

$$\langle \mathcal{L} \rangle = \langle \mathcal{T} \rangle - \langle \mathcal{V} \rangle = \frac{1}{2} \langle \partial_t G \rangle = 0, \quad (206)$$

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