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The Carlson-Simpson Lemma in Reverse Mathematics

by

Julia Christina Erhard

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Theodore Slaman, Chair
Professor Leo Harrington
Professor Satish Rao

Spring 2013

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Abstract

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We examine the Carlson-Simpson Lemma ($VW(k,l)$), which is the combinatorial core of the Dual Ramsey Theorem, from the perspective of Reverse Mathematics. Our results include the following:

Working in the system $B\Sigma^0_2$, we carry out the construction of a failure of the ordered version of the Carlson-Simpson Lemma $OVW(k,l)$, which was introduced in [9]. This observation implies that we can construct such a recursive counterexample in the model of SRT^2_2 that was discussed in [13]. It follows that SRT^2_2 does not prove $OVW(k,l)$ over RCA_0.

We also show that the strength of the principle $VW(k,l)$ is independent of the number of colors l being used.

By proving that $VW(k,l)$ is not conservative over RCA_0 for arithmetical sentences, we conclude that $VW(k,l)$ is not provable from any theory that is conservative over RCA_0 for arithmetical sentences.

Contents

Acknowledgments

First and foremost, I would like to thank my advisor Theodore Slaman for introducing me to the topic of Reverse Mathematics, for his guidance, support and inspiration during the past four years. It was a great pleasure being a graduate student under his supervision and I cannot even put in words how much I learnt from him.

Secondly, I want to express my gratitude to Adam Day and Damir Dzhafarov for their valuable feedback on some details of my results, as well as to Antonio Montalban for financial support that allowed me to focus entirely on research during my final semester.

Thirdly, I want to thank the entire staff of the mathematics department, especially Barbara Peavy and Barb Waller for being infinitely helpful and approachable.

Last but not least, I would like to give my thanks to several people who have been supporting me to throughout the entire path of my education: To my parents for giving me the opportunity to pursue higher education. To Joel Ouaknine, James Worrell and Thomas Forster, who were my supervisors in previous institutions and without whom I would not have been able to even begin my degree at Berkeley. To my high school teachers Richard Urban and Hans Bussas for inspiring me to become a mathematician. To Hoi Wing Ng, for his immense encouragement and support in my education and career.

Chapter 1

Preliminaries

1.1 Reverse Mathematics

In ordinary mathematics, we usually specify first an axiom system and then state a theorem θ that we aim to prove from the axioms. If no axiom system is mentioned explicitly, Zermelo-Fraenkel Set Theory (ZFC) implicitly takes the role of this axiom system since almost all mathematics can be interpreted in it. The basic idea of Reverse Mathematics is to turn this question around and we ask

Which axioms are needed to prove the theorem θ?

Of course, this question demands some framework to be become non-trivial: We work in the very weak base theory RCA_0 (Recursive Comprehension axiom, cf. Definition 1.1.4), which is a subsystem of second-order arithmetic, and we attempt to prove in it the equivalence of θ with the axioms needed. Let us call these axioms ϕ. So the goal is to give two proofs in RCA_0, one of $\phi \to \theta$ and one of $\theta \to \phi$. If this can be accomplished, we know that ϕ is exactly the right amount of axioms we need to prove the theorem θ.

Definition 1.1.1: ($I\Sigma^0_1$)
*The Σ^0_1-**induction scheme** is the universal closure of*

$$(\phi(0) \wedge \forall n(\phi(n) \to \phi(n+1))) \to \forall n\, \phi(n)$$

where $\phi(x)$ is any Σ^0_1-formula, i.e. a formula that can be written with only one unbounded existential quantifier ranging over numbers.

Remark: The induction scheme can analogously be defined for many other classes of formulas, e.g. Σ^0_2-induction, *arithmetical induction* for formulas that can be written with any combination of number quantifiers and no set quantifiers, or *bounded induction* for formulas that can be expressed without unbounded quantifiers.

Definition 1.1.2:
*The Δ_1^0-**comprehension scheme** is the universal closure of*

$$\exists X \forall n (n \in X \leftrightarrow \phi(n))$$

where ϕ is any Δ_1^0-formula, i.e. a formula that can be written with only one unbounded existential number quantifier, and can also be written with one unbounded universal number quantifier.

Remark: The comprehension scheme can also be defined for other classes of formulas, e.g. Π_1^1-comprehension for formulas that can be written with just a single unbounded universal quantifier that ranges over sets and any combination of number quantifiers.

Definition 1.1.3:
P^- denotes the universal closure of the following basic statements of number theory, which characterize the basic properties of natural numbers

- $n + 1 \neq 0$
- $m + 1 = n + 1$ *implies* $m = n$
- $m + 0 = m$
- $(m + n) + 1 = m + (n + 1)$
- $m * 0 = 0$
- $m * (n + 1) = m * n + m$
- $\neg(m < 0)$
- $m < n + 1$ *if and only if* $(m < n$ *or* $m = n)$

Definition 1.1.4: (Recursive comprehension axiom)
RCA_0 is the subsystem of second-order arithmetic consisting P^- together with Σ_1^0-induction and Δ_1^0-comprehension.

The base theory RCA_0 is intentionally chosen to be very weak to allow a better separation of theorems. It is only strong enough to prove the existence of computable sets. However, in a model of RCA_0 non-computable sets may exist. Most of mathematics cannot be proven in RCA_0 since most mathematics will use sets that are not necessarily computable.

The minimum model of RCA_0 is the model $\mathcal{M} = (\mathbb{N}, REC)$, consisting of the non-negative integers and the sets of the model are the recursive sets. However, there are other models of RCA_0. For example, we can take some non-recursive set $A \subseteq \mathbb{N}$ and add all the sets recursive in A to the model $\mathcal{M}$, this will again be a model of RCA_0. Models of RCA_0

have the property that they are closed under Turing computability $\leq_T$ and join $\oplus$.

The idea of Reverse Mathematics dates back to Steel and Fridman in the 1970s (see [16]). It has been extensively studied by Simpson and his book [12] is a good collection of most results. Reverse Mathematics is a very fruitful approach to understand the connection and compare the strength of theorems from all parts of mathematics. The arguments are usually recursion theoretic. Several axiom systems show up repeatedly and form a nice hierarchy. Many theorems of any branch of mathematics are equivalent to one of these. We will define three of the most important ones in increasing order of strength. Each is a system is a subsystem of second-order arithmetic, including RCA_0 and strictly stronger than the previous one.

Definition 1.1.5: (Weak Koenig's Lemma)
WKL_0 is the system RCA_0 together with the statement "Every infinite binary tree has an infinite path".

Definition 1.1.6: (Arithmetical comprehension axiom)
ACA_0 is the system P^- together with the comprehension axiom for arithmetical formulas and the induction axiom

$$(0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n(n \in X)$$

Definition 1.1.7: (Π^1_1-comprehension axiom)
$\Pi^1_1 - CA$ is the system ACA_0 together with comprehension axiom for Π^1_1-formulas

Many of today's open questions in Reverse Mathematics originate in combinatorics. Some important combinatorial principles turned out to be rather difficult to classify and do not fit as nicely into this simple hierarchy as the theorem from other fields. This may be either because the Reverse Mathematics idea is more sensitive to distinguish combinatorial arguments, or because combinatorics is the only subject where truly new ideas are needed in every proof. Unsurprisingly, the result we are writing about is a statement of combinatorics and hence many arguments will have a discrete and pure flavor.

1.2 The Dual Ramsey Theorem

One of the most famous theorems in combinatorics is the infinite Ramsey Theorem and it is a very interesting example of Reverse Mathematics.

Definition 1.2.1:
*An n-**set** of natural numbers is a subset of $\mathbb{N}$ of size n. We write $[A]^n$ to denote the set of all $n-$sets of elements of a set A.*

Remark: For n-sets repetitions are not allowed. This should not be confused with our notation for sequences of elements of a set A of length n, which will be denoted throughout by A^n.

Theorem 1.2.2: (Ramsey Theorem)
*For every k-coloring $c : [\mathbb{N}]^n \to \{0, 1, \ldots, l-1\}$ of the n-sets of natural numbers, there is an infinite set $A \subseteq \mathbb{N}$ such that c takes the same color on every element of $A^{<\mathbb{N}}$, i.e. c restricted to A is monochromatic. Such a set A is called **homogeneous** for c.*

Let RT^n_l denote the principle of RCA_0 plus Ramsey's Theorem with the given parameters $l, n \geq 2$.

The mathematical dual theorem has been formulated and studied by Carlson and Simpson in [1]. The latter author continued with a recursion theoretic analysis of this theorem in [11]. 28 years later, very little is known about the strength of this principle and the combinatorial core. We will summarize the known Reverse Mathematical results about the Dual Ramsey Theorem in Section 1.4.

Definition 1.2.3:
*Given a finite alphabet A, e.g. $A = \{a, b, c\}$. An A-**partition** is a collection of pairwise disjoint non-empty subsets of $A \cup \mathbb{N}$, called **blocks**, whose union is $A \cup \mathbb{N}$ and no block contains more than one element of A.*

For example,

$$\{a, 0, 2, 4, 6, 8, \ldots\}, \{b, 3, 9, 27, \ldots\}, \{c\}, \{5, 25, \ldots\}\{1, 11\}, \{7, 13\}, \ldots$$

Blocks that contain no letter from the alphabet A, are called **free**. Note that, unlike in the combinatorial principles we discuss later, free blocks are allowed to be infinite here.

Definition 1.2.4:
Let $(\omega)^\omega_A$ denote the set of all A-partitions with infinitely many free blocks.
Let $(\omega)^k_A$ denote the set of all A-partitions with k free blocks.
If $A = \emptyset$, we omit the subscript A.

Definition 1.2.5:
*Let X and Y be A-partitions. We say Y is **coarser** than X if each block of X is contained in some block of Y. In other words, Y can be obtained from X by merging some of its blocks.*

Definition 1.2.6:
Let $X \in (\omega)^\omega_A$ be an A-partition with infinitely many free blocks. Then

$$(X)^k_A := \{Y \in (\omega)^k_A \mid Y \text{ is coarser than } X\ .\ \}$$

After fixing the notation, we are now ready to state the Dual Ramsey Theorem.

Theorem 1.2.7: (Dual Ramsey Theorem)
For all $k, l \geq 2$, if $(\omega)^k = C_0 \cup C_1 \cup \cdots \cup C_{l-1}$ is an l-coloring of the set of all partitions of $\mathbb{N}$ with k blocks, where each C_i is Borel (cf. Definition 4.10 in [6]), then there exists a partition $X \in (\omega)^\omega$ such that $(X)^k$ is monochromatic, i.e. $(X)^k \subseteq C_i$ for some $i < l$.

Remark: Note that the Dual Ramsey Theorem is false, if we drop the Borel assumption. However, it has also been studied with different restrictions, e.g. in [9] with the requirement for the C_i to be open sets.

Let $DRT(k, l)$ denote RCA_0 together with the statement of the Dual Ramsey Theorem with the parameters k, l.

Theorem 1.2.8: (Open Dual Ramsey Theorem)
For all $k, l \geq 2$, if $(\omega)^k = C_0 \cup C_1 \cup \cdots \cup C_{l-1}$ is an l-coloring of the set of all partitions of $\mathbb{N}$ with k blocks, where each C_i is open, then there exists a partition $X \in (\omega)^\omega$ such that $(X)^k$ is monochromatic, i.e. $(X)^k \subseteq C_i$ for some $i < l$.

Let $ODRT(k, l)$ denote RCA_0 together with the statement of the Open Dual Ramsey Theorem for parameters k, l.

In the proof of the Dual Ramsey Theorem, Carlson and Simpson isolate the combinatorial core as a combinatorial lemma involving infinite sequences of letters from the alphabet A and an infinite set of variables. These sequences will be called infinite variable words (cf. Definition 1.3.1). We are interested in the strength of this key combinatorial lemma, to which we refer to as the Carlson-Simpson Lemma and which we denote by $VW(k, l)$.

1.3 Infinite variable words

Let us recall the most important terminology for studying the Dual Ramsey Theorem, for which we follow [9].

Throughout, A denotes a finite alphabet of letters with $|A| \geq 2$. Unless otherwise stated, we usually can safely assume $A = \{a, b\}$. Many arguments carry over straightforwardly to different parameters. Var denotes an infinite collection of variables disjoint from A.

Definition 1.3.1:
*An **infinite variable word** is an $\mathbb{N}$-sequence of elements of $A \cup Var$ in which infinitely many distinct variables occur, each finitely often.*
*A **finite variable word** is any proper initial segment of an infinite variable word.*

Remark: The letters of the alphabet A may occur any number of times, even infinitely often or never.

Definition 1.3.2:
*Let W be a (finite or infinite) variable word. A **substitution instance** of W is a word V of the same length as W in which all occurrences of some (possibly all) variables have been replaced by some letter from A. We do allow different variables to be substituted by different letters.*

For example,

$$U = a\ \ b\ \ a\ \ a\ \ x_1\ b\ \ b\ \ b\ \ x_1\ a$$

is a substitution instance of the finite variable word

$$W = a\ b\ x_0\ x_0\ x_1\ b\ x_2\ x_3\ x_1\ a$$

where we replaced all occurrences of the variable x_0 by the letter a, all occurrences of x_2 and x_3 by the letter b.

Definition 1.3.3:
*A substitution instance is **complete** if it contains no more variables.*
*A substitution instance is **infinite variable** if it contains infinitely many distinct variables.*

Given an infinite variable word W, we consider the set of finite words derived from it $W(A)$ consisting of all strings $\alpha \in A^{<\mathbb{N}}$ that are initial segments of a complete substitution instance ending just before the first occurrence of a new variable.

For example, if $W = a\ x_0\ x_0\ x_1\ b\ a\ b\ x_0 \ldots$ then a, abb, aaa are elements of $W(A)$, but $\emptyset, ab, aba, abba$ are not elements of $W(A)$.

Definition 1.3.4: (The Carlson-Simpson Lemma)
The principle $VW(k,l)$ is RCA_0 together with the the statement, that if $|A| = k$ and $c : A^{<\mathbb{N}} \to \{0, 1, \ldots, l-1\}$ is an l-coloring of the finite words in A, then there exists an infinite variable word W such that $W(A)$ is monochromatic, i.e. $W(A) \subseteq c^{-1}(j)$ for some $j < l$.
*We call such W **homogeneous for c**.*

In the definition of $W(A)$, it is essential for the truth of the statement that we cannot take all possible initial segments of W, only the ones ending before the first occurrence of a new variable. To see this, let us consider the following recursive coloring of $\{a, b\}^{<\mathbb{N}}$.

$$c(w) = \begin{cases} 1 & \text{if } w \text{ contains an odd number of } a's \\ 0 & \text{if } w \text{ contains an even number of } a's \end{cases}$$

We claim that there is no infinite variable word, such that every substitution instance of **every** initial segment receives the same color. Suppose W is an infinite variable word, whose first variable is x_0. Let W_0 be the initial segment cut off right after the first occurrence of x_0. Then the substitution instance of W_0 in which x_0 receives letter a and the substitution instance of W_0 in which x_0 receives letter b receive different colors by the choice of c. The parity of how many times the letter a occurs in the word changes.

However, when $W(A)$ only contains substitution instances of initial segments cut off before the first occurrence of a new variable, $V = x_0x_0x_1x_1x_2x_2x_3x_3\dots$ is homogeneous for c.

Proposition 1.3.5:
If W is an homogeneous infinite variable word for the coloring $c : A^{<\mathbb{N}} \to \{0, 1, \dots, l-1\}$, then any infinite variable substitution instance V of W is also homogeneous for c.

Proof. Let V be an infinite variable substitution instance of W. The set $V(A)$ is a subset of $W(A)$, since it contains all substitution instances of certain substitution instances of finite initial segments of W. So

$$V(A) \subseteq W(A) \subseteq c^{-1}(i) \text{ for some } i < l$$

i.e. $V(A)$ is monochromatic with the same color as $W(A)$. Thus V is also homogeneous for c. □

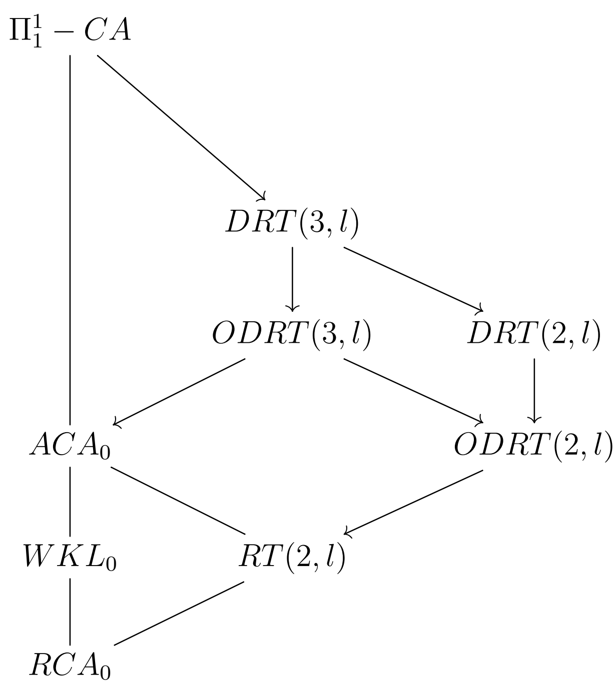
Definition 1.3.6:
*An infinite variable word is **ordered** if all occurrences of the n-th variable come before the first occurrence of the $(n+1)$-st variable.*

Definition 1.3.7: (Ordered Carlson-Simpson Lemma)
The principle $OVW(k,l)$ is RCA_0 together with the statement that for each l-coloring c of the finite words $A^{<\mathbb{N}}$ as above, there exists an ordered infinite variable word that is homogeneous for c.

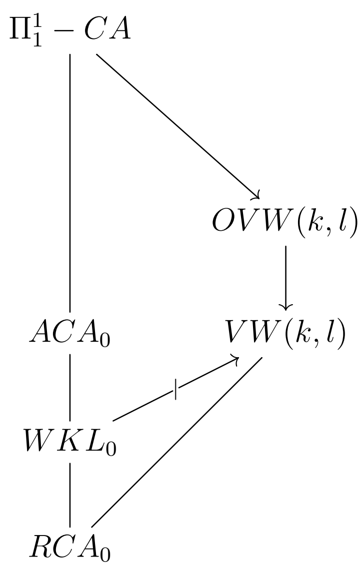
1.4 Reverse Mathematics diagrams

Just like for the Dual Ramsey Theorem, not much is known about the strength of these principles in terms of Reverse Mathematics. In this thesis, we investigate this question further, proving some results that will help to classify them.

The following diagrams summarize the previously known results regarding the principles of the Dual Ramsey Theorem and the Carlson-Simpson Lemma in comparison with selected other known subsystems of second-order arithmetic. Stronger systems are written above weaker systems. Straight lines indicate an extension, that is known to be strict. Arrows indicate an extension that may or may not be strict.



For the Carlson-Simpson Lemma:



The references for these are as follows: Placing $OVW(k,l)$ and $VW(k,l)$ between $\Pi^1_1 - CA$ and RCA_0 was observed in [11]. It was left open, whether this means the Dual Ramsey Theorem is also provable in $\Pi^1_1 - CA$ since multiple applications of the lemma are required and may push up the complexity. It was answered in [13], who proved that $DRT(k,l)$ lies between $\Pi^1_1 - CA$ and RCA_0.

The non-implication from WKL_0 was proved in [9]. In the same paper they prove that $ODRT(n+1,l) \to RT^n_l$. Since ACA_0 and RT^n_l for any $n \geq 3$, $l \geq 2$ have been established equivalent in [7], it follows that $ODRT(k,l) \to ACA_0$ for $k \geq 3$.

Overall, very little is known about the Reverse Mathematical strength of the Dual Ramsey Theorem and the Carlson-Simpson Lemma. The gap between $\Pi^1_1 - CA$ and WKL_0 is huge. Our goal is to uncover some more relationship between these principles, which hopefully will lead to an exact classification of them in the future.

Chapter 2

The Miller-Solomon example

2.1 Introduction

In [9], Miller and Solomon show that WKL_0 does not suffice to prove $OVW(2,2)$ by constructing a computable coloring $c : \{a,b\}^{<\mathbb{N}} \to \{0,1\}$ such that no Δ^0_2-definable ordered infinite variable word is homogeneous for c. Since there is an ω-model of WKL_0 in which all sets are low (see Corollary VIII.2.18 in [12]), $OVW(2,2)$ fails in this model and the result follows.

Moreover, from a homogeneous infinite variable word W, they derive a homogeneous ordered infinite variable word V computable in W'. Thus, any ω-model of $VW(2,2)$ which contains the computable sets must also contain non-low sets. Because otherwise, take the computable coloring c as in the main theorem and let W be an homogeneous infinite variable word of low Turing degree. Then the homogeneous ordered infinite variable word V derived from W is computable in W', hence it is Δ^0_2. Contradiction to the choice of c.

So WKL_0 does neither prove $VW(2,2)$ nor $OVW(2,2)$ and hence also not the stronger principles $VW(k,l)$ or $OVW(k,l)$ for any $k,l \geq 2$. We discuss the role of the parameters more in Chapter 4.

We will demonstrate that their proof for $OVW(2,2)$ can be carried out entirely in the system $B\Sigma^0_2$ (cf. Definition 2.2.1), not making use of $I\Sigma_2$ or even higher levels of the Bounding-Induction hierarchy. We will summarize the proof in the following section and point out why each step works in the system $B\Sigma^0_2$.

This means that this example of a failure of $OVW(2,2)$ can be constructed in the model that Slaman, Chong and Yang worked with in [2]. They construct a model of Stable Ramsey Theorem for pairs and $B\Sigma^0_2$, that is not a model of RT^2_2 nor of $I\Sigma^0_1$. Thus it follows that SRT^2_2 does not prove $OVW(2,2)$.

The main theorem that we are interested in [9] is

Theorem 2.1.1:
There is a computable two-coloring $c : A^{<\mathbb{N}} \to 2$ such that $W(A)$ is not monochromatic for any Δ^0_2-definable infinite variable word W.

Since there is an ω-model of WKL_0 in which all sets are low, $OVW(2,2)$ fails in this model.

2.2 The Miller-Solomon proof in $B\Sigma^0_2$

Definition 2.2.1:
$B\Sigma_2$ consists of all the sentences

$$\forall p \forall a((\forall x < a)(\exists y)\phi(x,y,p) \to (\exists b)(\forall x < a)(\exists y < b)\phi(x,y,p))$$

where ϕ is any Σ^0_2-definable formula, i.e. a formula that can be written with one unbounded existential quantifier followed by one unbounded universal quantifier only.

For more information on this principle, see [14]. All we need to know is that, working in the base theory $PA^- + I\Sigma_0 + exp$, $B\Sigma^0_2$ is weaker than induction for Σ^0_2-definable formulas, and equivalent to induction for Δ^0_2-definable formulas. This means that we need to convince ourselves that the only infinitary constructions used in the proof are no stronger than induction for Δ^0_2-definable formulas or the Bounding principle for Σ^0_2 or less complicated formulas.

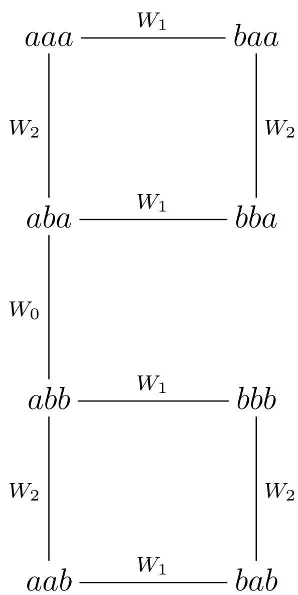
Definition 2.2.2:
*A finite set of finite variable words $W_0, W_1, \ldots, W_n$ with distinguished variables $x_0, x_1, \ldots, x_n$ respectively is **admissible** if the positions of first occurrences of the distinguished variable are pairwise distinct.*

Definition 2.2.3:
Let W be a variable word with x being one of its variables. Let V be a substitution instance of W with all variables replaced except for x. We write $V(x/a)$ to refer to the substitution instance of V in which all occurrences of x get replaced by the letter a.

An admissible set of finite variable words $W_0, W_1, \ldots, W_n$ induces a graph G as follows: Let $s \in \mathbb{N}$ be bigger than the length of each word, i.e. $s > |W_i|$ for all $i \leq n$. The vertices of the graph G are the elements of A^s. Two vertices v_1, v_2 are connected by an edge labelled W_i if there exists a string $\delta \in A^{s-|W_i|}$ such that the two vertices are the concatenation of an substitution instance V_i of all variables of W_i except the variable x_i with the string δ, i.e. $V_i(x_i/a) = v_1$ and $V_i(x_i/b) = v_2$ or vice versa.

We refer to a graph of this form as **induced graph**.

Let us consider an example: Let $W_0 = ab\mathbf{x_0}$, with distinguished variable x_0, $W_1 = \mathbf{x_0}$ with x_0 also, and $W_2 = x_0\mathbf{x_1}$ with distinguished variable x_1. The induced graph of this admissible set of variable words on A^3 is:



Looking at the first position of discrepancy between any two vertices and knowing that two vertices are adjacent if and only if they differ solely by the substitution of the distinguished variable. It is clear that in an induced graph there is at most one edge between any two vertices.

Definition 2.2.4:
*A **2-coloring** of a graph G with vertex set V is a map $c : V \to \{0, 1\}$ such that adjacent vertices receive different colors.*

Lemma 2.2.5:
Let $W_0, W_1, \ldots, W_n$ be an admissible set of finite variable words. Then the induced graph on A^s is two-colorable for any $s \geq max_{i \leq n}|W_i|$.

Moreover, this result holds in $B\Sigma_2$.

Proof. To see this, we first observe that an induced graph has no cycles of odd length. Pick any cycle in an induced graph. We claim, that each edge label must occur an even number of times as we traverse the cycle. This is because the first occurrences of the distinguished variables are all different.

More precisely, let the first position of the distinguished variable of W_i be position k_i.

By possible renaming of the indices, we assume

$$k_0 < k_1 < k_2 < \cdots < k_n$$

Fix any cycle in G and a vertex α in this cycle. We prove by induction, that each label occurs an even number of times on this cycle.

Consider W_1. Traverse around the cycle exactly once. Only when we cross an edge labelled W_1 the letter at position k_1 can change. So it must change an even number of times, before we reach the starting position.

Once we know that W_1 occurs on an even number of edges of the cycle, we continue with W_2. Let k_2 be the position of the distinguished variable of W_2. Then k_2 can only change if we cross an edge labelled W_2 or possibly W_1. We do not know whether it changes on W_1 edges or not, but if it changes once, it changes at all edges labelled W_1 of which there is an even number of times. The total number of times it changes must be even. Thus, since the number of times W_1 occurs is already known to be even, there must be an even number of W_2 labels also.

Continuing this way, we see that every label $W_1, W_2, \ldots, W_n$ occurs an even number of times on the cycle. No edge can have two labels and there are no un-labelled edges, thus every cycle in an induced graph is of even length.

Claim: Any graph without odd cycles is two-colorable.

We give a proof by induction on the number of edges: Let G be a graph on n vertices. If the graph has no edges, we can assign a randomly chosen colors to each vertex and we will have no conflict.

Assume now that every graph with only even cycles and fewer than m edges is 2-colorable. Let G be a graph with m edges that has only even cycles. Let $e = (u, v)$ be one of its edges. Consider the graph H obtained from G by deleting e. H has only even cycles and fewer than m edges, hence by assumption it is 2-colorable.

Let $c : V \to \{0, 1\}$ be a coloring of H. If u and v receive different colors in c, then c is also a 2-coloring of G and we are done. If u and v receive the same color, then there are no paths between u and v. If there was a path between u and v in H, pick the shortest such. The colors of the vertices on this path must alternate hence, knowing that u and v end up with the same color, this path must be of even length. Thus this path together with the edge e is an odd cycle in G contradicting our assumption. Thus there are no paths between u and v, so we can recolor the connected component of u and assign each of its vertices the opposite color. Then this modified coloring is a 2-coloring of G. This finishes the induction and hence the proof of the claim.

To see that this proof works in $B\Sigma_2$, we observe that it can be proved with bounded induction, since all quantifiers used can naturally be bounded by the fact that the problem is a finite graph with n vertices. The number of possible cycles is bounded above by $(n+1)!$, the number of possible colorings is bounded above by 2^n and so on. Hence we can express everything by formulas that use only bounded quantifiers. Thus the induction we do is a Σ^0_0 induction, which is a weaker principle than $B\Sigma^0_2$. Thus this result holds in the system $B\Sigma^0_2$. □

Lemma 2.2.6:
Let $c : A^{<\mathbb{N}} \to \{0, 1\}$ be a coloring. Let W be an ordered finite variable word, in which the variables $x_0, x_1, \ldots, x_e$ occur. Let $k_0 \in \mathbb{N}$ be such that for every $k > k_0$ there exists an index $i(k) \leq e$ such that for all substitution instances $\hat{W}$ of W of all variables except $x_{i(k)}$ and all strings $\alpha \in A^k$ we have

$$c(\hat{W}(x_{i(k)}/a)^\frown\alpha) \neq c(\hat{W}(x_{i(k)}/b)^\frown\alpha)$$

Then W is not the initial segment of an homogeneous infinite variable word U in which all occurrences of $x_0, x_1, \ldots, x_e$ in U occur in W.

Moreover, this result holds in $B\Sigma_2$.

Proof. Let W, k_0 and $i : \mathbb{N} \to \{0, 1, \ldots, e\}$ be as in the statement of the Lemma. Let U be any ordered infinite variable word such that $W \subset U$ so that all occurrences of $x_0, x_1, \ldots, x_e$ are in W.

Fix a variable x_m such that the position of the first occurrence of it is greater than $|W| + k_0$. Let k be the difference between this position and the length of W. Note that we have $k > k_0$. Choose any substitution instance $\hat{W}$ of W of all variables except the variable $x_{i(k)}$.

Pick a string $\alpha \in A^k$ such that the concatenation $\hat{W}^\frown\alpha$ is a possible substitution instance of an initial segment of U, i.e. places where U has occurrences of the same variable receive the same letter in $\hat{W}^\frown\alpha$. This substitution instance was cut off before the first occurrence of x_m. Filling in the variable $x_{i(k)}$ with a or with b must result in different colors since $k > k_0$ and by our assumption. Both these words are in $U(A)$ by our choice of k. Thus $U(A)$ is not monochromatic, thus U is not homogeneous for c.

Again, the statement is just finite combinatorics, so it even holds in RCA_0 plus bounded induction. □

Now we will give a proof of Theorem 2.1.1.

Proof. We will construct the desired coloring c of $A^{<\mathbb{N}}$ in stages. At each stage s we will color the words in A^s.

The requirements to diagonalize against all Δ^2_0 ordered infinite variable words are as follows:

> R_e: If $U_e(n) = \lim_s \phi_{e,s}(n, s)$ is total and codes an ordered infinite variable word, then U_e is not homogeneous for c.

At stage s, let $U_{e,s}$ be the approximation to U_e, that is the longest converging initial segment with computations of at most s steps. If U_e is an ordered infinite variable word, we will see longer correct initial segments over time.

Color the empty sequence with 0.

At stage s: Consider the set

$$S_s = \{V_{e,s} | V_{e,s} \text{ is the initial segment of } U_{e,s} \text{ with variables } x_0, x_1, \ldots x_e \text{ that is cut off just before the first occurrence of } x_{e+1}\}$$

If a word has fewer variables, ignore it at this stage. We assume that the indices of the variables are ordered by the positions of their first occurrences.

Pick a pivot variable for each word in this set so that the positions of their first occurrences are pairwise distinct. This is possible, since we can proceed in order and each chosen pivot variable rules out at most one variable of the later words. So they have enough variables remaining. Construct the induced graph for the set S_s on A^s and 2-color the words in A^s accordingly.

It can be verified that the so constructed coloring meets all the requirements: If the limit of $U_{e,s}$ as s approaches infinity represents an ordered infinite variable world, then the correct initial segment with e distinct variables cut before the $e+1$st variable will eventually be in the set S_s at every stage, so we make sure to rule out all further positions for possible places where new variables can occur by assigning different colors to some substitution instance in which only one variable of $x_0, x_1, \ldots, x_{e-1}$ is switched, as in Lemma 2.2.6. This finishes the proof of Theorem 2.1.1

□

Theorem 2.2.7:
The proof of Theorem 2.1.1 can be carried out in the system $B\Sigma_2$.

Proof. The construction is effective and all sets involved can be defined by formulas with bounded quantifiers. The only place where we need to be careful is the claim that we will see longer correct initial segments. To know that we will eventually see all positions of correct initial segments, we require $B\Sigma_2$. □

2.3 $OVW(k,l)$ and Stable Ramsey Theorem

Definition 2.3.1:
*A coloring $c : [\mathbb{N}]^2 \to \{0,1\}$ of the pairs of natural numbers is **stable** if for every $x \in \mathbb{N}$ there exists $y \in \mathbb{N}$ such that $\{x,z\}$ receives the same color for all $z > y$.*

Definition 2.3.2: (Stable Ramsey Theorem)
The system SRT^2_2 is RCA_0 together with the statement that Ramsey's Theorem restricted to stable colorings holds.

This system is clearly extended by RT^2_2, but for a long time it was not known whether SRT^2_2 is strictly weaker than RT^2_2. In a recent result [2], Chong, Slaman and Yang answer this question by constructing a model of $RCA_0 + B\Sigma^0_2 + SRT^2_2$ in which RT^2_2 fails. This model is non-standard, so it is not an ω-model. Moreover, Σ^0_2-induction does not hold in this model.

Since the construction of the counterexample to $OVW(k,l)$ of [9], which we presented in the previous section does only use principles weaker than or equal to bounding for Σ^0_2-definable formulas, one can carry out this construction in model of [2], obtaining a model of SRT^2_2 in which $OVW(k,l)$ fails. Thus we have the following Reverse Mathematics result:

Corollary 2.3.3:
Over RCA_0, SRT^2_2 does not prove $OVW(k,l)$ for any $k,l \geq 2$.

Unfortunately, the reduction from $OVW(k,l)$ to $VW(k,l)$ given in [9] uses Σ^0_2-induction, so we do not get the same result for $VW(k,l)$.

Chapter 3

Ordered versus unordered infinite variable words

3.1 Introduction

We are interested in investigating the difference between the Carlson-Simpson Lemma $VW(k,l)$ and the ordered version $OVW(k,l)$, which was introduced in [9], in more depth from a recursion theoretic point of view.

Clearly, an homogeneous ordered infinite variable word is an homogeneous infinite variable word. Thus $OVW(k,l)$ is the stronger principle out of the two; i.o.w. $OVW(k,l)$ proves $VW(k,l)$ over RCA_0. It is not known, whether this implication is reversible or whether $OVW(k,l)$ is strictly stronger than $VW(k,l)$.

At first glance it may seem easy to revert: Given a coloring $c : A^{<\mathbb{N}} \to \{0,1,\ldots,l-1\}$ and an homogeneous infinite variable word W, we can always convert it to an ordered infinite variable word as follows:

Substitute all other variables that have an occurrence between the first and the last occurrence of x_0 by letters. Make sure to substitute all occurrences of these variables, including the ones occurring after the last occurrence of the variable we keep.

Every variable may only occur finitely often, so there is only finitely many substitutions made and infinitely many variables left. Then take the next variable after the last occurrence of x_0 that has not been substituted yet and continue to fill in all variables between its first and last occurrence and so on. Call this word V. We know that $V(A) \subseteq W(A)$, since V is a substitution instance, c.f. Proposition 1.3.5. Hence $V(A)$ is monochromatic for the same color and by construction V is ordered.

For example, let

$$W = x_0\ x_1\ x_1\ x_2\ b\ a\ x_0\ x_2\ x_3\ x_1\ x_4\ x_4\ x_3\ \ldots$$

be the beginning of an homogeneous infinite variable word and assume that x_0 occurs exactly twice. Then

$$V = x_0\ \ a\ \ a\ \ a\ \ b\ \ a\ \ x_0\ \ a\ \ x_3\ \ a\ \ b\ \ b\ \ x_3\ \ldots$$

would be the the beginning of the homogeneous ordered substitution instance. All occurrences of x_1 and x_2 were filled in by the letter a, making x_3 the second variable that we keep. Occurrences of x_4 were filled in with the letter b. Note that it does not matter which letter we pick to substitute variables that we do not keep. We could always use a.

However, this construction is not recursive. It uses $0'$ as an oracle since we need to obtain the information to find the last occurrence of the variables that we keep. The question we want to answer is whether there is a more clever construction of an infinite variable ordered substitution instance that avoids having $0'$ as an oracle.

We answer this negatively as follows: We will construct a recursive infinite variable word such that every ordered infinite variable substitution instance computes $\emptyset'$. This implies that that the reduction from the unordered case to the ordered case recursive in $\emptyset'$ is best possible, i.e. it cannot be done without using an oracle that is computationally at least as powerful as $\emptyset'$.

Moreover, it follows that the statement "Every infinite variable word has an ordered infinite variable substitution instance" implies ACA_0.

3.2 Necessity of the oracle

We prove the following result:

Theorem 3.2.1:
There exists a recursive infinite variable word W such that every ordered infinite variable substitution instance of W computes $\emptyset'$.

Proof. Construction of W:

The variable x_i is assigned to the computation of $\phi_i(i)$. The positions of the occurrences of this variable will give us an upper bound on the time it takes $\phi_i(i)$ to converge, if it converges at all.

The construction is a finite extension argument: At each stage s, we specify a finite initial segment of W of length at least $s+1$, denoted W_s, compatible with all W_t for $t < s$. Since the length increases at each stage, the so constructed word is infinite. Then we let $W = \bigcup_{s\in\mathbb{N}} W_s$.

At stage 0: Let $W_0 = x_0$.
At stage $s+1$: So far W_s contains the variables $x_0, x_1, \ldots, x_s$. Consider the set of diagonal computations below $s+1$ whose convergence we first see after s steps

$$S_{s+1} = \{e \leq s | \phi_{e,s}(e) \downarrow \text{ and } e \notin S_t \text{ for any } t \leq s\}$$

Let

$$W_{s+1} = \begin{cases} W_s^\frown x_{s+1} & \text{if } S_{s+1} \text{ is empty;} \\ W_s^\frown x_s x_{s-1} \ldots x_i^\frown x_{s+1} & \text{if } i \text{ is the least element of } S_{s+1}. \end{cases}$$

This concludes the construction of $W = \bigcup_{s\in\mathbb{N}} W_s$.

Claim 1: W is a infinite variable word.

Since we append a new variable x_{s+1} to W_{s+1} at each step, W contains infinitely many variables and hence it is also an infinite sequence of elements of $Var \cup A$, in fact only Var. Every variable occurs at most finitely often, since x_s is written down once at stage s and at most $s+1$ times thereafter, namely once for every converging computation of $\phi_t(t)$ for $t \leq s$. So a variable x_s is written down no more than $s+2$ times. Thus W is an infinite variable word according to Definition 1.3.1.

Claim 2: W is recursive.

The set S_{s+1} only depends on the finite approximations to finitely many diagonal computations and finitely many previously computed finite sets, hence the construction is effective. Thus W is recursive.

The result will follow once we establish the truth of the next lemma.

□

Lemma 3.2.2:
Let W be as constructed in Theorem 3.2.1. Then every ordered infinite variable substitution instance of W computes $\emptyset'$.

Proof. In any ordered infinite variable substitution instance V, the last occurrence of the $n+1$-st variable gives an upper bound on when $\phi_n(n)$ converges, if ever. To decide whether $n \in \emptyset'$, we look at the position of the last occurrence of the $n+1$-st variable in V. As V is ordered and has infinitely many variables, we can recognize effectively the last occurrence of the $n+1$-st variable by waiting for the first occurrence of the $n+2$-nd variable. Let

$y_0, y_1, y_2, \ldots$ be the variables of V in order.

For example:

$$V = a\ b\ b\ a\ y_0\ a\ a\ b\ y_0\ a\ a\ a\ b\ y_1\ a \ldots$$

Let us assume that W's variables are $x_0, x_1, x_2, \ldots$ and V's variables are $y_0, y_1, y_2, \ldots$ ordered by their first occurrences.

The variable y_0 corresponds to some variable x_i in V. In our example, we do not know without further knowledge whether it corresponds to x_4 (which happens if $W = x_0x_1x_2x_3x_4\ldots$) or to x_3 (if $W = x_0x_1x_1x_2x_3\ldots$) or to x_2 (if $W = x_0x_1x_1x_0x_2\ldots$). It all depends on when we observe certain converging diagonal computations. The good news is that we do not really need to know.

It is clear, that each variable y_n in V corresponds to a variable x_j in W with $j \geq n$, so it belongs to a computation $\phi_j(j)$ with $j \geq n$. This follows from the fact that the first occurrences of the variables in V occur in order. Substituting some variables with letters, may shift the indices to correspond to diagonal computations of larger index, but not smaller ones.

Suppose the last occurrence of the $(n+1)$-st variable, y_n, happens at the s-th position of V. Then we claim that

$$n \in \emptyset' \text{ if and only if } \phi_{n,s}(n) \text{ converges.}$$

If $\phi_n(n)$ never halts, then $\phi_{n,s}(n)$ will not halt for any choice of s, so we will draw the correct conclusion, namely that $n \notin \emptyset'$, regardless of which position the $(n+1)$-st variable occurs at.

Otherwise, suppose $\phi_n(n)$ converges first at stage t. Our goal is to establish, that $t \leq s$.

Then at stage $t+1$ the set S_{t+1} was not empty since it contains n. So

$$W_{t+1} = W_t^\frown x_t x_{t-1} \ldots x_{n+1} x_n^\frown x_{s+1}$$

or, more likely,

$$W_{t+1} = W_t^\frown x_t x_{t-1} \ldots x_{n+1} x_n x_{n-1} \ldots x_i^\frown x_{s+1}$$

depending on whether n was the least element of S_{t+1} or not. In either case, the variable x_n was written down at this stage and all variables with indices between n and t were written down also. Observe that $|W_t| \geq t$. Recall that y_n corresponds to a computation with index $j \geq n$.

We distinguish two cases:

Case 1: x_j is one of the variables $x_n, x_{n+1}, \ldots, x_{t-1}x_t$. Then it was written down at stage $t+1$, so its last occurrence happens at a position $s \geq t$.

Case 2: $j > t$. Then all its occurrences, in particular the last occurrence, happens after stage $t+1$, hence $s > t$.

□

Corollary 3.2.3:
Lemma 3.3 in [9] is best possible, i.e. the bound $V \leq_T W'$ is tight.

Proof. In Theorem 3.2.1 we have shown that in general we cannot do better than using the jump as an oracle when trying to pass from an infinite variable word to an ordered infinite variable word. □

The observation that the use of $\emptyset'$ cannot be avoided when introducing order into infinite variable words does not however imply that $OVW(k,l)$ is strictly stronger than $VW(k,l)$, because given a coloring $c : A^{<\mathbb{N}} \to \{0, 1, \ldots, l-1\}$ and an homogeneous unordered infinite variable word W, it may be that there is an homogeneous ordered infinite variable word V unrelated to W, i.e one that is not an substitution instance of W at all.

So, the Reverse Mathematics relationship between $OVW(k,l)$ and $VW(k,l)$ is still open.

3.3 Relativizing the result

Theorem 3.3.1:
For every set X. There exists a X-computable infinite variable word W such that for every ordered infinite variable substitution instance V of W

$$X' \leq_T V$$

Proof. Relativizing the proof to computations with oracle X gives immediately that

$$X' \leq_T V \oplus X.$$

However, we can drop the use of X on the right hand side by observing, that we can code X into W:

Since our alphabet has at least two letters, take $a, b \in A$ and let a denote 0, b denote 1. The odd positions of W will be reserved for the code of X using the letters a and b. When we take a substitution instance of W, the odd positions have not been changed and contain all information we need to recover X. Thus V itself computes X, so we can drop the use of X on the right hand side, yielding the stronger result. □

Corollary 3.3.2:
Over RCA_0, the statement "For every infinite variable word, there is an ordered infinite variable substitution instance" implies ACA_0.

Proof. By Theorem 3.3.1, every model of RCA_0 in which this statement holds must be closed under the Turing jump, hence it is a model of ACA_0 (cf. [12]). □

Chapter 4

On the parameters

4.1 Introduction

The Carlson-Simpson Lemma depends on two parameters: The size of the alphabet k and the number of colors l. So for each choice of $k, l \geq 1$ we have a corresponding subsystem $VW(k, l)$. We are interested in finding out whether any of these principles are equivalent.

If $l = 1$, i.e. if there is only one color, then the principle is trivial for any choice of $k \geq 1$, since all words in $A^{<\mathbb{N}}$ get the same color, hence any infinite variable word W is homogeneous for such a coloring. This is clearly provable in RCA_0.

If $k = 1$, i.e. if the alphabet has only one letter, it is a little less obvious. Let us assume $A = \{a\}$. Then $A^{<\mathbb{N}} = \{\emptyset, a, aa, aaa, aaaa, \dots\}$. If $c : A^{<\mathbb{N}} \to \{0, 1, \dots, l-1\}$ is a coloring, we know by the infinite pigeon hole principle that there must be some color that is used infinitely often, let's say $S := c^{-1}(i)$ is infinite. From this set we can easily construct an infinite variable word W putting first occurrences of new variables at positions n such that the string in $A^{<\mathbb{N}}$ of length $n-1$ is in S. So in this case we can always construct a homogeneous infinite variable word from c. So in this case it is also effectively true.

For $k, l \geq 2$, the Carlson-Simpson Lemma is not provable in RCA_0, which follows for example from Theorem 2.1.1. It is unknown, whether some of the principles are equivalent. To our knowledge, there has never been a written prove that

$$VW(k+1, l) \to VW(k, l)$$

or

$$VW(k, l+1) \to VW(k, l+1)$$

although this is expected. We will give a proof of these two results in RCA_0 and we also prove the reverse direction for the number of colors l, showing that this parameter does not influence the strength of the principles $VW(k, l)$ when $l \geq 2$.

4.2 Number of colors

We will give a direct proof, that the number of colors does not influence the strength of the Carlson-Simpson Lemma.

Theorem 4.2.1:
$RCA_0 \vdash VW(k, l+1) \leftrightarrow VW(k, l)$.

Proof. Given an alphabet A of size k and assume that $VW(k, l+1)$ holds. Given a coloring $c : A^{<\mathbb{N}} \to l$. We need to exhibit an infinite variable word U such that $U(A)$ is monochromatic.

Since $VW(k, l+1)$ holds and c can be viewed as an $(l+1)$-coloring with one color not being used, we can take the U that is homogeneous for c and whose existence is guaranteed by $VW(k, l+1)$.

Conversely, given the same alphabet A and assume that $VW(k, l)$ holds. Given a coloring $c : A^{<\mathbb{N}} \to l+1$, we need to find an infinite variable word U such that $U(A)$ is monochromatic with respect to c.

Identify two colors as one color. For example, if blue and green are two of the colors used and turquoise is a color not in the range of c, we will define c' by just sending all blue or greens to turquoise and otherwise assign the same colors as c. Now this identification induces a coloring $c' : A^{<\mathbb{N}} \to l$ with l colors. By $VW(k, l)$, there exists an infinite variable word U homogeneous for c'.

If $U(A)$ is of a color other than turquoise, we are done, as this case U is also homogeneous for c. If $U(A)$ is turquoise, construct a two-coloring $d : A^{<\mathbb{N}} \to \{$ blue, green $\}$ as follows.

The color of a word w of length k is the same as the finite variable word obtained by cutting U just before the first occurrence of the $(k+1)$-st variable and substituting the variables according to the letter of the word. More precisely, let us assume the variables of U are $x_0, x_1, x_2, \ldots$ when ordered by the positions of their first occurrence. Then all occurrences of the first variable x_0 in U is replaced by the first letter of w, all occurrences of the second variable x_1 will be replaced by the second letter etc.

For example, consider the word $w = abb$, then

$$d(abb) = c(\text{ the initial segment of U cut off before the first occurrence of } x_3 \\ \text{with the substitutions } x_0 \to a, x_1 \to b, x_2 \to b)$$

Observe that this is a well-defined 2-coloring of $A^{<\mathbb{N}}$: It is defined on strings of every length, since U has infinitely many distinct variables. Because $U(A)$ is turquoise, the coloring d only assigned the colors blue and green to those words, thus it is a two-coloring.

Also note that d is defined recursively in the infinite variable word U and the $l+1$-coloring c.

Since $2 \leq l$, we can apply $VW(k, 2)$ again to obtain an infinite variable word W homogeneous for d. From U and W we will construct an infinite variable word U' such that U' is homogeneous for c.

The infinite variable word U' will be obtained from U by substituting some of its variables by letters of A and renaming some variables. Assume that $x_0, x_1, x_2, \ldots$ are the variables of U sorted by the position of their first occurrence. W is like a recipe how we need to modify U to become homogeneous for c.

Proceed as follows: The n-th symbol of W will tell us what we need to do with the n-th variable x_{n-1}. If the n-th symbol is a letter, then we fill in all occurrences of x_{n-1} in U by this letter. If the n-th symbol is a variable that is not equal to any variables in any positions we have seen so far, then we keep x_{n-1}. If the n-th symbol is a variable that we have seen before, then replace all occurrences of x_{n-1} in U by x_{j-1}, where j is the first position at which the variable was seen in W.

Let us do an example. Suppose our words starts like

$$W = a\ b\ y_0\ y_1\ y_0\ b\ y_2 \ldots$$

then in U we replace all occurrences of x_0 by the letter a, all occurrences of x_1 by b, we keep the variables x_2 and x_3, we replace all occurrences of x_4 by x_2, we replace all occurrences of x_5 by b and so on.

Claim 1: U' is an infinite variable word.

U and W both contain infinitely many distinct variables each used finitely often by Definition 1.3.1. So there is infinitely many variables in U that will not be replaced, because W instructs us to keep the variables around. Out of those infinitely many variables, finitely many will be merged; leaving still infinitely many variables in U'.

Claim 2: U' is homogeneous for c.

The set of words $U'(A)$ is a subset of $U(A)$, thus c assigns every element of $U'(A)$ either one of the colors blue and green by assumption. The definition of d tells us exactly which substitution instances end up colored blue and which substitution instances end up being colored green. W is a strategy how to make sure to get a monochromatic set $U'(A)$.

More precisely, if W is homogeneous for the color *blue*, we claim that all words in $U'(A)$ receive color *blue* from c.

Suppose

$$W = x_0\ a\ x_1\ x_0\ b\ x_2 \dots$$

Then, since W is homogeneous for d, we know d colors the following words all blue:

$$\emptyset, aabab, babbb, aaaab, baabb, aa, ba$$

By construction of W, this means that the substitution instances of U cut off just before the first occurrence of x_5 in which all letters are substituted according to one of the first four letters of W receive *blue* by the coloring c, as well as the substitution instances of U cut off just before the first occurrence of x_2, as long as we substitute all occurrences of x_1 by the letter a.

This is exactly the same as allowing all elements of $U(A)$ as long as x_1 gets substituted by a, x_4 gets substituted by b, x_0 and x_3 get the same color. This is achieved by merging x_0 and x_3 to be one variable in U and replacing all occurrences of x_1 by the letter a, all occurrences of x_4 by the letter b.

Thus the construction of U' is exactly what we need to do to make sure all words in $W(U')$ will be colored with the same color. Thus we have constructed a homogeneous infinite variable word for the $(l+1)$-coloring using only $VW(k,l)$.

So in RCA_0 we have that

$$VW(k,l) \leftrightarrow VW(k,l+1)$$

□

4.3 Alphabet size

In the previous section, we saw that the parameter l does not matter in the principle $VW(k,l)$. For the alphabet size k, it is very easy to verify that $VW(k+1,l) \rightarrow VW(k,l)$. It is open whether this can be reversed.

We include our proof of the easy direction for completeness:

Proposition 4.3.1:
$RCA_0 \vdash VW(k+1,l) \rightarrow VW(k,l)$

Proof. Suppose $VW(k+1,l)$ holds. Let $|A| = k$ and let $c : A^{<\mathbb{N}} \rightarrow \{0,1,\dots,l-1\}$ be an l-coloring. We need to find an infinite variable word W that is homogeneous for c.

Extend the alphabet by a new symbol, i.e. let $B = A \cup \{*\}$, where $*$ is a letter not contained in A.

Define an l-coloring d on $B^{<\mathbb{N}}$ as follows: The color of a word $w \in B^{<\mathbb{N}}$ is the same as the color of it where all $*$ have been deleted. For example

$$d(* * * \; b \; a \; *) = c(b \; a)$$

This is a well-defined coloring recursive in c. By $VW(k+1, l)$ there is an infinite variable word U homogeneous for d. Since we extended the alphabet, U may contain the special character $*$.

Let W be U with all special characters $*$ deleted. Notice that W still has infinitely many variables, thus it is an infinite variable word. Moreover, W is homogeneous for c by construction and it is obtained from U in a recursive way. Thus $VW(k, l)$ holds. $\square$

Chapter 5

Forcing notions

5.1 Introduction

In this chapter, we consider the Carlson-Simpson Lemma together with the method of forcing. We will work with the ordered version $OVW(k,l)$. Our goal is to find a coloring c without homogeneous ordered infinite variable word, i.e. an instance of failure of $OVW(k,l)$, that continues to fail when we add generic reals.

Forcing is a powerful method developed in set theory to prove the independence of the continuum hypothesis; see [3] for details. Later it was simplified [4] and adapted to forcing in arithmetic, which is what we will be concerned with here.

The basic idea is to expand the universe to contain more sets than before. While doing this, we ensure some properties that we want the new set to have, are met. A condition specifies some finite piece of information about the set. If every real G that is compatible with the condition has some property, we say the condition **forces** the property, i.e. this partial information is sufficient to guarantee the property. The set G that we construct, is the set that is compatible with all the countably many conditions and hence will have all these properties we want. G is called a **generic real**.

5.2 Cohen forcing

Let us recall the basic definitions of Cohen forcing. We will mostly follow the notation of [5].

Definition 5.2.1:
*Let $(P, \preccurlyeq)$ be a partial order, called a **notion of forcing**. The elements of P are called **conditions**.*

In Cohen forcing, the conditions p are just finite partial functions from $\mathbb{N}$ to the set $\{0,1\}$. A condition is meant to describe a set of objects G that are compatible with it, in this case the object G is a total functions from $\mathbb{N}$ to $\{0,1\}$ that agree with the partial function p on its domain $dom(p)$.

Definition 5.2.2:
*Let $p, q \in P$ be two conditions. We say p **extends** q and we write $p \preccurlyeq q$ if $p \supseteq p$ as subsets of $\mathbb{N} \times \{0,1\}$.*

The idea is that a condition p that extends q contains the same information and possibly more, hence narrowing down the possible choices of the real G that we are constructing. Thus the counterintuitive direction of the inequality sign.

Definition 5.2.3:
*A subset $D \subseteq P$ of conditions is **dense** if for every $p \in P$ there is an element of D that extends p.*

Intuitively, a set D is dense if it describes a property that, independent of what finite amount of G we have specified so far, it is still possible for G to have this property, i.e. to **meet** D.

Definition 5.2.4:
*A non-empty subset $F \subseteq P$ is a **filter**, if the following two properties hold*

$$p \in F \wedge p \preccurlyeq q \rightarrow q \in F \text{ and}$$

$$p, q \in F \rightarrow \exists r \in F (r \preccurlyeq p \wedge r \preccurlyeq q).$$

*Given a collection $\mathcal{D}$ of dense subsets of P. A filter F is $\mathcal{D}$-**generic** if it meets every element of $\mathcal{D}$, i.e. for every $D \in \mathcal{D}$, $D \cap F \neq \emptyset$. When $\mathcal{D}$ is the collection of all definable dense sets, we will drop the $\mathcal{D}$ and call it a **generic filter**.*

We often identify a filter F with the object G that it defines. For Cohen forcing, any generic filter defines a total function from $\mathbb{N}$ into $\{0,1\}$. Total, because the sets $E_n = \{p \in P | p(n) \text{ is defined}\}$ are definable dense sets.

The key fact about forcing is stated in the following proposition.

Proposition 5.2.5: (Forcing)
Let $(P, \preccurlyeq)$ be a notion of forcing, let $\mathcal{D}$ be a countable collection of dense subsets of P and let $p \in P$. Then there is a $\mathcal{D}$-generic filter containing p. [5]

The method of forcing is really an immediate consequence of the Baire Category Theorem, cf. [**set theory**].

Theorem 5.2.6: (Baire Category Theorem)
The system BCT is RCA_0 together with the statement

"If $D_0, D_1, D_2, \ldots$ are a definable collection of dense open sets of reals, then the intersection $D = \bigcap_{n\in\mathbb{N}} D_n$ is dense in $\mathbb{R}$.

Later we will use the fact that a model is closed under Cohen-generics if and only if it is a model of BCT.

5.3 OVW(k,l) and Cohen forcing

First we consider the notion of Cohen forcing together with $OVW(k,l)$. We will conclude that the Baire Category Theorem does not imply $OVW(k,l)$ over RCA_0.

More precisely, we start with a model in which $OVW(k,l)$ fails, we pick an instance of failure of $OVW(k,l)$ i.e. a coloring c with no homogeneous ordered infinite variable word in the model, and show that we can add Cohen generic reals for any given family of dense sets without adding an infinite variable word homogeneous for this instance c.

Let $\mathcal{M} \models WKL_0 + \neg OVW(k,l)$ as discussed in Chapter 2. Let $c : \{a,b\}^< \to \{0,1\}$ be a (recursive) coloring that has no homogeneous infinite variable word computable in the model $\mathcal{M}$. Note that the same proof works for any choice of $k,l \geq 2$.

First, we need to extend our definition of $W(A)$ to finite variable word.

Definition 5.3.1:
For any finite variable word W, let $W(A)$ be the set of all complete substitution instances that are cut off just before the first occurrence of a new variable.

The following is a trivial observation, which we use later.

Proposition 5.3.2:
If W is a finite initial segment of an infinite variable word U, then $W(A) \subseteq U(A)$

Throughout we fix some effective coding of sequences of elements of in $A \cup Var$ by reals, i.e. subsets of $\mathbb{N}$.

Lemma 5.3.3:
Let c be a coloring as above. Let $e \in \mathbb{N}$ be fixed. For every set Y in the model $\mathcal{M}$, the set of forcing conditions p such that

$$p \Vdash \Phi_e^{G\oplus Y} \text{ does not code a homogeneous infinite variable word for } c$$

is dense in $(P, \preccurlyeq)$.

Proof. Given a forcing condition $q \in P$. We need to show that it can be extended to a condition $p \preccurlyeq q$ such that $p \Vdash \Phi_e^{G \oplus Y}$ does not code an homogeneous ordered infinite variable word for c.

First, we use $\emptyset'$ to decide, whether there exists a number $n \in \mathbb{N}$ and a condition $p \in P$ such that $p \preccurlyeq q$ and $p \Vdash \Phi_{e,n}^{G \oplus Y}$ is unordered or inhomogeneous for c.

More precisely, let $W_{p,n}$ be the finite word that is the longest converging sequence computed with oracle $p \oplus Y$ by the e-th Turing machine after n steps. $W_{p,n}$ is an initial segment of $W_G := \Phi_e^{G \oplus Y}$.

So, if the variables in $W_{e,n}$ are not ordered, then W_G cannot be ordered either. We will refer to this situation as **unorder**.

Consider $W_{p,n}(A)$. This is a finite set and it is a subset of $W_G(A)$. If this set is not monochromatic in c, then we know W_G will not be homogeneous for c either. We will refer to this situation as an **inhomogeneity**.

We can recognize the unorder of the variables and the inhomogeneity of $W_{e,n}$ effectively, hence $\emptyset'$ is sufficient as an oracle.

If $\emptyset'$ tells us that such n and p exist, start looking for such a pair in an effective way and take the first such p as our extension of q. Then we are done.

Otherwise, if $\emptyset'$ tells us that there is no such n and $p \in P$, then we proceed as follows:

Start extending

$$q = p_0 \succ p_1 \succ p_2 \succ \cdots \succ p_i \succ \dots$$

where p_{i+1} is the first extension of p_i that we see where

$$\Phi_e^{G \oplus Y} \text{ is defined on } \{0, 1, \dots, i\} \text{ and}$$
$$\Phi_e^{G \oplus Y} \text{ has at least } i \text{ distinct variables}$$

This process must stop. Suppose it keeps going forever, then it is a way to construct from q and Y recursively an homogeneous ordered infinite variable word. Note that we have already ensured that there is no extension that would give us inhomogeneity for c or unorder in the word, so at this stage we only need to find extensions to make the word longer and have more and more variables. So we have a contradiction to the assumption that c has no homogeneous ordered infinite variable word computable in the model.

Let $p = p_i$ so that p_i cannot be extended in the described way. Any filter G compatible with this p will be partial or have finitely many distinct variables.

For any condition q we found an extension p that forces $\Phi_e^{G\oplus Y}$ does not code an homogeneous infinite variable word for c. Thus the set of such conditions is dense. □

We would like to point out that this Lemma cannot straightforwardly adapted to the $VW(k,l)$ case, because of the requirement that variables in an infinite variable word must occur finitely often.

If we are trying to construct an ordered infinite variable word and we know that it is not possible to extend to an unordered word, then waiting for a new variable automatically ensures that variables occur only finitely often. However, if we are working with $VW(k,l)$, the infinite word constructed by the p_n may have some variables occur infinitely often and at no finite stage can we make sure this does not happen.

Even though this proof cannot be adapted to $VW(k,l)$, we will obtain the result that being closed under Cohen-generic reals does not imply $VW(k,l)$ as a consequence of the next chapter.

The argument would naturally carry over to the principle $VWI(k,l)$ (cf. [9] Chapter 4) which is the version of the Carlson-Simpson Lemma, in which variables are allowed to occur infinitely often. Except for the fact that to our knowledge it is unknown whether there exist any models of RCA_0 in which $VWI(k,l)$ fails. Thus we cannot get our construction started. However, if one succeeds to prove that $VWI(k,l)$ is strictly stronger than RCA_0, then our argument would immediately imply that $VWI(k,l)$ cannot be proved from the Baire Category Theorem.

Theorem 5.3.4:
For any $\mathcal{M}$-definable family of dense sets E_i there is a real $G \in \bigcap_{n\in\mathbb{N}} E_n$ such that $\mathcal{M}[G]$ does not compute an homogeneous infinite variable word for c.

Proof. Let $D_e \subseteq P$ be the set of conditions such that $p \in D_e$ implies

$$p \Vdash \Phi_e(G) \text{ is not an homogeneous ordered infinite variable word for } c$$

This set is dense, by Lemma 5.3.3. Let $C_n = \{p : |dom(p)| \geq n\}$. These sets are also dense. Let $\mathcal{D}$ be the collection of D_e, E_n and C_n. Let G be a $\mathcal{D}$-generic filter. Let $G = \bigcup_{p\in G} p$. Then $M[G]$ computes no homogeneous infinite variable word for c. □

Corollary 5.3.5:
$BCT \nvdash OVW(k,l)$

Proof. Start with our model of $WKL_0 + \neg OVW(k,l)$ and let c be an instance of a failure of $OVW(k,l)$. We have shown that for any definable family of dense sets in the model, we can add a Cohen generic while preserving the fact that c has no homogeneous infinite variable word computable in the model. Repeat the process to add generics for all (countably many) definable families of dense sets. Now we have a model of BCT where $OVW(k,l)$ fails for the coloring c. □

Corollary 5.3.6:
Closure under Cohen-generic reals, does not prove $OVW(k,l)$.

5.4 OVW(k,l) and Mathias forcing

The same argument works for adding Mathias-generic reals.

In Mathias forcing, a condition $p = (s, X) \in P$ consists of two parts, a finite set of natural numbers s and an infinite set of natural numbers X definable in the model such that

$$\max s < \min X$$

Definition 5.4.1: (extensions in Mathias forcing)
*Let $p_1 = (s_1, X_1) \in P$ and $p_2 = (s_2, X_2) \in P$ be two conditions. We say p_1 **extends** p_2 and we write $p_1 \preccurlyeq p_2$ if*

$$s_1 \supseteq s_2$$
$$X_1 \subseteq X_2$$
$$s_1 \cap \bar{s}_2 \subset X_2$$

In other words, a conditions p_1 is an extension of p_2 if s_1 is obtained from s_2 by adding some elements from X_2 to the finite part, and X_1 is X_2 with some elements removed. So we extend the finite part s_2 with elements taken from the infinite part X_2 and we "thin" the infinite part.

We can carry out the same argument as for Cohen forcing. The only difference is that the process must terminate, otherwise there is an X-computable homogeneous infinite variable word in the model already.

Lemma 5.4.2:
Let c be a coloring as above. Let $e \in \mathbb{N}$ be fixed. For every set Y in the model $\mathcal{M}$, the set of forcing conditions p such that

$$p \Vdash \Phi_e^{G \oplus Y} \textit{ does not code a homogeneous infinite variable word for } c$$

is dense in $(P, \preccurlyeq)$.

Proof. Given a forcing condition $q = (s, X) \in P$. We need to show that it can be extended to a condition $p \preccurlyeq q$ such that $p = (t, Z) \Vdash \Phi_e^{G \oplus Y}$ does not code an homogeneous ordered infinite variable word for c.

First, we use $\emptyset'$ to decide, whether there exists a number $n \in \mathbb{N}$ and an extension t with elements of X such that $p = (t, X - \{x \in X | x > \max t\})$ and $p \Vdash \Phi_{e,n}^{G \oplus Y}$ is unordered or inhomogeneous for c.

More precisely, let $W_{p,n}$ be the finite word that is the longest converging sequence computed with oracle $s \oplus Y$ by the e-th Turing machine after n steps. $W_{p,n}$ is an initial segment of $W_G := \Phi_e^{G \oplus Y}$.

So if the variables in $W_{e,n}$ are not ordered, then W_G cannot be ordered either.

Consider $W_{p,n}(A)$. This is a finite set and it is a subset of $W_G(A)$. If this set is not monochromatic in c, then we know W_G will not be homogeneous for c either.

We can recognize the unorder of the variables and the inhomogeneity of $W_{e,n}$ effectively, hence $\emptyset'$ is sufficient as an oracle.

If $\emptyset'$ tell us that such n and p exist, start looking for them in an effective way and take the first such p as our extension of q. Then we are done.

Otherwise, if $\emptyset'$ tells us that there is no such n and $p \in P$, then we proceed as follows:

Start extending with elements from X

$$q = p_0 \succ p_1 \succ p_2 \succ \cdots \succ p_i \succ \ldots$$

where $p_{i+1} = (s_{i+1}, X_{i+1})$ is the first extension of p_i that we see where

$$\Phi_e^{G \oplus Y} \text{ is defined on } \{0, 1, \ldots, i\} \text{ and}$$
$$\Phi_e^{G \oplus Y} \text{ has at least } i \text{ distinct variables}$$

For each extension we only thin the set X_i from below, making sure to throw out elements smaller than the maximum element of s_{i+1}.

This process must stop. Suppose it keeps going forever, then it is a way to construct from q, X and Y recursively an homogeneous ordered infinite variable word. Note that we have already ensured that there is no extension that would give us inhomogeneity for c or unorder in the word, so at this stage we only need to find extensions to make the word grow longer and have more and more variables. If this process does not terminate, we have a contradiction to the assumption that c has no homogeneous ordered infinite variable word computable in the model. Recall that Y and X are both sets definable in the model.

Let $p = p_i$ so that p_i cannot be extended in the described way. Any filter G compatible with this p will be partial or have finitely many distinct variables.

For any condition q we found an extension p that forces $\Phi_e^{G\oplus Y}$ does not code an homogeneous infinite variable word for c. Thus the set of such conditions is dense. $\square$

Theorem 5.4.3:
For any $\mathcal{M}$-definable family of dense sets E_i there is a real $G \in \bigcap_{n\in\mathbb{N}} E_n$ such that $\mathcal{M}[G]$ does not compute an homogeneous infinite variable word for c.

Proof. Let $D_e \subseteq P$ be the set of conditions such that $p \in D_e$ implies

$$p \Vdash \Phi_e(G) \text{ is not an homogeneous ordered infinite variable word for } c$$

This set is dense, by Lemma 5.4.2. Let $C_n = \{p : |dom(p)| \geq n\}$. These sets are also dense. Let $\mathcal{D}$ be the collection of D_e, E_n and C_n. Let G be a $\mathcal{D}$-generic filter. Let $G = \bigcup_{p\in G} p$. Then $M[G]$ computes no homogeneous infinite variable word for c. $\square$

Corollary 5.4.4:
Closure under Mathias-generic reals, does not prove $OVW(k,l)$.

Chapter 6

Non-conservation for arithmetical sentences

6.1 Introduction

Definition 6.1.1:
*Let T_2 be an extension of a theory T_2. We say T_2 is **conservative over** T_1 **for arithmetical sentences**, if every arithmetical sentence provable in T_2 is also provable in T_1. In other words, T_1 and T_2 prove the same arithmetical sentences.*

From a Reverse Mathematics point of view, if we can prove that T_2 is not conservative over T_1 for two subsystems of second-order arithmetic T_1, T_2, then we know that T_1 and T_2 are not the same system in our Reverse Mathematics chart.

Definition 6.1.2:
*[8] A **cut** of a model of arithmetic is a non-empty proper initial segment that has no maximum element. We denote cuts by $I \subseteq \mathcal{M}$.*

In this section, we work with a particular strong failure of the Σ_2^0 bounding principle. We consider the first order model $\mathcal{M}$ of $P^- + I\Sigma_1^0$ in which there is a Δ_2^0-definable injection π of the whole model into a proper cut $I \subset \mathcal{M}$ (cf. Lemma 3.4 in [10]).

$$\mathcal{M} \models P^- + I\Sigma_1^0 + \exists\Delta_2^0 \text{ injection } \pi : \mathcal{M} \to I$$

We will prove that the principle $VW(k,l)$ cannot hold in any second-order extension of this model.

From this it follows that $VW(k,l)$ is not conservative for arithmetical sentences over RCA_0 since the non-existence of such a Δ_2^0 injection is an arithmetical sentence that is not provable in RCA_0.

Let us give an outline and some notation first:

By the Shoenfield Limit Lemma (cf. Lemma III.3.3 in [15]), a set X is Δ^0_2 if and only if it can be computably approximated, i.e. X is the limit of a computable sequence of finite sets.

At every stage s in the construction, we have an approximation of the injection π, given by sets of possible pre-images of each number. Let F_i^s be the set of possible pre-images of the number i at stage s.

For each non-empty F_i^s, pick the smallest element to be its representative a_i^s at this stage. We think of these numbers to code finite variable words w_i^s for some effective coding of variable words, which is fixed throughout the construction.

Using this approximation, we can effectively diagonalize against all sets that are coded in the model by making sure our coloring is not monochromatic for any words coded.

In order to make the coloring well defined, we take substitution instances of the words coded, ensuring that they are incompatible. To do that, we require them to have sufficiently many variables.

Definition 6.1.3:
*Two finite variable words V and W are **compatible**, if there exists a variable word U (possibly using only letters) of length*

$$|U| \geq \max\{|V|, |W|\}$$

*whose initial segment of length $|V|$ is a substitution instance of V and whose initial segment of length $|W|$ is a substitution instance of W. Otherwise call them **incompatible**.*

For example $V = x_0\ a\ b\ x_0\ x_0$ and $W = a\ a\ x_1\ b$ are incompatible since x_0 must be replaced by a in U because of position 0, yielding an contradiction at position 3. $V = x_0\ x_0$ and $W = x_1\ b$ are compatible, as witnessed by $U = b\ b$.

Proposition 6.1.4:
Let V and W be finite variable words. If there exists a position $n \leq \min\{|V|, |W|\}$ such that $V(n)$ and $W(n)$ are two distinct elements of A, then V and W are incompatible.

Proof. A substitution instance can only fill in variables and not change the letters of a word. Thus any substitution instance of V must have the same letter at position n, hence a different letter than any substitution instance of W has at position n. □

6.2 The main result

We will fix an alphabet $A = \{a, b\}$, but we note that the same proof works if the alphabet of size $|A| \geq 2$.

Theorem 6.2.1:
There is a coloring $c : A^{<\mathbb{N}} \to \{0, 1\}$, recursive in the model $\mathcal{M}$, such that if W is a infinite set in the model $\mathcal{M}$ and $\mathcal{M}[W] \models I\Sigma_1(W)$, then W does not code an infinite variable word homogeneous for c.

Proof. Let $I \subseteq \mathcal{M}$ be the cut and $\pi : \mathcal{M} \to I$ be the Δ^0_2-definable injection as described above.

Construction of c:
Fix some number $t_0 \in \mathcal{M} - I$ outside of the cut I for the rest of the construction. At each stage s, we will color all words in A^s.

At stage 0: Color the empty string red.

At stage $s > 0$: Consider the set

$$S_s = \{w_i^s \mid i < t_0 \text{ and } w_i^s \text{ codes a finite variable word of length } \leq s \text{ and } w_i^s \text{ contains at least } t_0 \text{ distinct variables}\}$$

If S_s is empty, color A^s arbitrary and move on to the next stage. Otherwise, for simplicity of notation we will drop the superscript, so let w_i for $i \leq t_0$ be the words that elements of S_s code.

For each pair of indices $i < j \leq t_0$, make w_i and w_j incompatible as follows:

Determine the position n of the first occurrence of the first variable in either w_i or w_j (which ever occurs first), y say. Without loss of generality, assume that it occurs in w_i; the other case being identical with the roles of i and j reversed.

We distinguish three cases:

If $w_j(n) = a$, substitute all occurrences of y in w_i by b.

If $w_j(n)$ is some letter $\neq a$, substitute all occurrences of y in w_i by a.

If $w_j(n)$ is a variable z, substitute all occurrences of y in w_i by a and all occurrences of z in w_j by b.

These are the only possible cases, since $w_j(n)$ cannot be undefined by our assumption that both w_i and w_j have enough variables.

After this substitution both words w_i and w_j may have one less variable than before. This is why we require them to have t_0 many variables to begin with. Moreover, w_i and w_j are now guaranteed to be incompatible because at position n one of them has letter a, the other one has some other letter.

Continue with the next pair of indices.

Having replaced at most $t_0 - 1$ variables by letters in each word w_i (one for each index $j \neq i < t_0$), we are now left with a set of pairwise incompatible words w_i', each containing at least one variable.

For $i \leq t_0$, color all the words in A^s that are extensions of some complete substitution instance of w_i' as follows:

Pick the first remaining variable y in w_i'. Take all complete substitution instances of w_i' in which y is substituted by a, concatenated with any string of length $s - |w_i'|$ and color them red. Color all other complete substitution instances of w_i' concatenated with any string of length $s - |w_i'|$ and color them blue.

Having done so for all $i \leq t_0$, color all remaining words in A^s red.

This coloring is well-defined, since the words w_i' are pairwise incompatible, thus for any two distinct such words, the sets of their substitution instances are disjoint. This follows immediately from Definition 6.1.3.

The construction is effective since the set S_s is bounded. Thus the coloring c is recursive.

Claim: If W is an infinite subset of $\mathcal{M}$ as in the statement of the theorem that codes an infinite variable word, then W is not homogeneous for c.

Let $W \subseteq \mathcal{M}$ be a subset such that $\mathcal{M}[W] \models I\Sigma_1^0$. Then by bounded induction, we know that for every $l \in \mathcal{M}$, the restriction of W to length l, $W|_l$, is coded in $\mathcal{M}$. By Σ_1^0 induction, we can prove that if W is an infinite variable word, then there exists some l such that $W|_l$ mentions t_0 distinct variables.

Let w_0 be the first such initial segment that mentions t_0 many variables. Let a_0 be its code. Then $\pi(a_0) = k$ for some $k \leq t_0$ since $t_0 \notin I$. As π is a Δ_2^0 injection, from some point onwards, say from stage s_0, $\pi^{-1}(k)$ will always consist only of a_0 and thus, w_0 will be one of the words in our set S_s for all stages $s > s_0$.

Thus we diagonalize against w_0 at every stage $s > s_0$. More precisely, at every stage $s > s_0$, if $s + 1$ is the location of a first occurrence of a new variable, then $W(A)$ will not be monochromatic since at stage s the color of words compatible with w_0 changes with the distinguished variable chosen in the construction; this means the color is different for different substitution instances of w_0, hence of initial segments of W. Since W has infinitely many distinct variables, there must be infinitely many places that are locations of first occurrences of a variable. Thus, there must be one beyond s_0. Hence $W(A)$ is not monochromatic. Thus W is not homogeneous for c. □

6.3 Reverse Mathematics consequences

We proved that that

$$RCA_0 + VW(k,l) \vdash \neg\exists\Delta^0_2 \text{ injection of the model into a proper cut}$$

and since RCA_0 does not prove the non-existence of such a cut, we conclude that $VW(k,l)$ is not conservative for arithmetical sentences over RCA_0.

Let us summarize these consequences in a number of corollaries.

As we noted earlier, the same proof works for more than 2 colors and bigger alphabet size. Thus we have proved the following result.

Corollary 6.3.1:
The model $\mathcal{M}$ cannot be extended by adding reals to obtain a second-order model $(\mathcal{M}, S) \models RCA_0 + VW(k,l)$

Corollary 6.3.2:
$RCA_0 + VW(k,l) \vdash \neg\exists\Delta^0_2$ *injection of the model into a proper cut.*

Corollary 6.3.3:
$VW(k,l)$ is not conservative for arithmetical sentences over RCA_0.

Corollary 6.3.4:
If T is a theory that is conservative over RCA_0 for arithmetical sentences, then T does not prove $VW(k,l)$ for any $k, l \geq 2$.

Proof. The non-existence of a Δ^0_2-definable injection of the entire model into a proper cut is not provable from RCA_0, since the model $\mathcal{M}$ can be extended to a model of RCA_0 by adding the reals recursive in $\mathcal{M}$.

So, $VW(k,l)$ is not conservative over RCA_0 for arithmetical sentences, hence every theory that is conservative over RCA_0 for arithmetical sentences cannot prove $VW(k,l)$ for any $k,l \geq 2$. □

Examples of such theories include COH, WKL_0, closure under 1-generics and closure under Mathias generics.

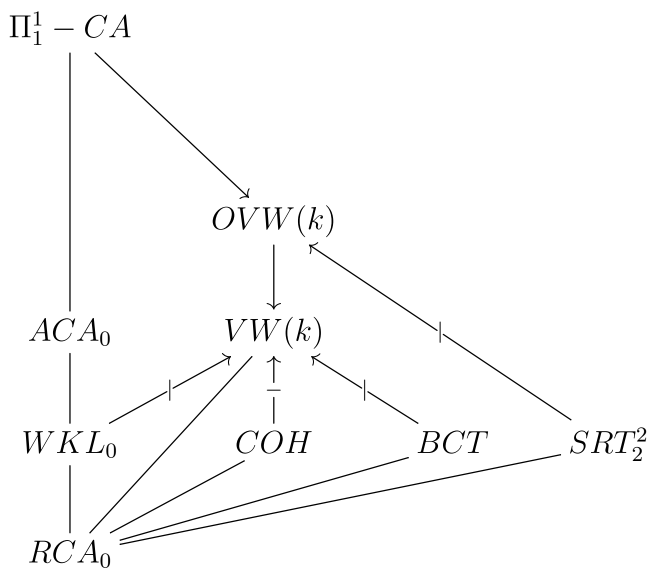
This Reverse Mathematics non-implication was left open in Chapter 5, where we proved it for $OVW(k,l)$, so we specifically mention it again here.

Corollary 6.3.5:
Closure under Mathias generics does not prove $VW(k,l)$.

Chapter 7

Reverse Mathematics diagram revisited

Putting all the Reverse Mathematics results of this thesis into our Reverse Mathematics diagram, we have a slightly better idea of the strength of the Carlson-Simpson Lemma. Symbolic for our result that the number of colors does not make a difference, we drop the l in the Carlson-Simpson principles.



In this thesis, we proved that $VW(k,l)$ proves that there is no Δ^0_2-injection of the model into a proper cut. To our knowledge this is the first example in recursion theory, where something has been proven **from** the Carlson-Simpson Lemma.

A lot of work still has to be done to classify the Carlson-Simpson Lemma completely in Reverse Mathematics. For the interested reader, we suggest to try to establish any implications regarding $VW(k,l)$ and RT^l_k or $VW(k,l)$ and ACA_0. It is feasible that just like the Dual Ramsey Theorem these principles are actually stronger than ACA_0.

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